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On the total positivity of q-Bernstein mass matrices and their accurate computations[☆]

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ABSTRACT

In this paper the total positivity of the Gramian (mass) matrices of q-Bernstein bases is analyzed. Furthermore, we provide an efficient method to obtain a bidiagonal decomposition of these mass matrices allowing us to calculate their singular values and inverses to high relative accuracy. Numerical examples are provided to illustrate the high accuracy of the performed computations using the proposed decompositions.

1. Introduction

Quantum calculus (see [1]) uses q-integers, q-binomial coefficients, and other q-analogues of classical calculus. In particular, it includes the q-Bernstein bases of polynomial spaces, for values $0 \leq q \leq 1$ (see [2]). The q-Bernstein bases have interesting applications in several areas, such as Computer-Aided Geometric Design (see [3,4] and references in there) and Approximation Theory (see [5]). When $q = 1$, q-Bernstein bases coincide with Bernstein bases, which are the most often used bases in Computer-Aided Geometric Design.

An important topic in Numerical Linear Algebra is the design and analysis of algorithms adapted to the structure of totally positive matrices, that is, matrices whose minors are nonnegative, and allowing the resolution of related algebraic problems to high relative accuracy (HRA). An algorithm is said to be performed to HRA if the relative error in the computations is of the order of the unit round-off (or machine precision). It is well known that a sufficient condition so that an algorithm can be carried out to HRA is the non-inaccurate cancellation condition, also called NIC condition, which holds if the algorithm only uses products, quotients, and sums of numbers with the same sign (see page 52 in [6]). Moreover, if the floating-point arithmetic is well-implemented the subtraction of initial data can also be allowed without losing HRA (see page 53 in [6]).

Algorithms to HRA have been achieved only for a few classes of matrices. Among these classes, we can mention the Hilbert matrices, which are the Gramian matrices of the monomial bases (see [7]). Furthermore, computations to HRA for Gramian matrices

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of other relevant polynomial bases, such as the geometric, Poisson and Bernstein bases, have been shown in [8,9], and for Gramian matrices of non polynomial bases in [10]. In all these cases, the total positivity of the considered matrix has played a key role because it leads to a bidiagonal factorization and, when this factorization can be performed to HRA, the algorithms of [11] (see also [7,12]) can be applied to compute to HRA the remaining mentioned algebraic computations.

Bernstein bases on the interval $[0,1]$ are B-bases of the generated polynomial spaces. As a consequence, any polynomial totally positive basis on this interval can be obtained as the product of a Bernstein basis and a totally positive matrix. In this paper, this property is used to analyze the total positivity of the q-Bernstein mass matrices, that is, the Gramian matrices of q-Bernstein bases, and to exploit it to obtain factorizations providing algebraic calculations to HRA when computing their singular values, which coincide with their eigenvalues due to the symmetry, or their inverses.

This paper is organized as follows, in Section 2, we recall some notations, concepts and auxiliary results related to the theory of Total Positivity, Neville elimination and the bidiagonal decomposition of totally positive matrices. In Section 3, we deduce and analyze the change of basis matrix between Bernstein and q-Bernstein bases. The bidiagonal factorization of matrices of Section 3 is obtained in Section 4. This result is applied in Section 5 to prove that q-Bernstein mass matrices are totally positive and derive accurate computations when solving algebraic problems with these matrices. Finally, numerical experiments confirming the accuracy of the proposed algorithms are included in Section 6.

2. Notations and auxiliary results

Let us recall that a real matrix is said to be totally positive (TP) when all its minors are nonnegative and strictly totally positive (STP) when all its minors are positive. Moreover, a system $(u_0,\dots,u_n)$ of functions defined on $I\subseteq\mathbb{R}$ is said to be totally positive or TP if, for any sequence $t_1<\cdots<t_{n+1}$ in the domain, the collocation matrix $\big(u_{j-1}(t_i)\big)_{i,j=1,\dots,n+1}$ is TP. On the other hand, if the functions of a TP system sum up to one, that is,

$$\sum_{i=0}^{n} u_i(t)=1,\quad t\in I,$$

the system $(u_0,\dots,u_n)$ is said to be normalized totally positive (NTP). The class of NTP bases is very important in Computer-Aided Geometric Design (CAGD) because it provides shape preserving representations. This property means that the shape of the parametric curves $\gamma(t)=\sum_{i=0}^{n}P_iu_i(t)$, $t\in I$, imitates the shape of their control polygon $P_0\cdots P_n$. It is well known that, among all NTP bases of a given space of functions with shape preserving representations, there exists a unique NTP basis, which has optimal shape preserving properties (cf. [13]).

As a consequence of Corollary 3.10 and Proposition 3.11 of [13], B-bases can be characterized as follows.

Theorem 1. *A TP basis $(u_0,\dots,u_n)$ is a B-basis of a vector space of functions U if and only if, for any other TP basis $(v_0,\dots,v_n)$ of U, the change of basis matrix A such that $(v_0,\dots,v_n)=(u_0,\dots,u_n)A$ is TP.*

Nowadays, bidiagonal factorizations are very useful to achieve accurate algorithms for performing computations with TP matrices. In fact, the parameterization of TP matrices leading to HRA algorithms is provided by their bidiagonal factorization, which is in turn closely related to the Neville elimination (cf [14–16]). The essence of this procedure is to make zeros in a column of a given matrix A by adding to each row an appropriate multiple of the previous one. In particular, it consists of n major steps, providing the following sequence of matrices

$$A^{(1)}:=A\to A^{(2)}\to\cdots\to A^{(n+1)},\tag{1}$$

such that, $A^{(r)}=(a_{i,j}^{(r)})_{1\le i,j\le n+1}$, $2\le r\le n+1$, satisfies

$$a_{i,j}^{(r)}=0,\quad j=1,\dots,r-1,\quad i=j+1,\dots,n+1,\tag{2}$$

and so, $A^{(n+1)}$ is upper triangular. In more detail, the matrix $A^{(r+1)}$ is computed from the matrix $A^{(r)}$ according to the following formula

$$a_{i,j}^{(r+1)}:=\begin{cases}a_{i,j}^{(r)}, & \text{if } 1\le i\le r,\\ a_{i,j}^{(r)}-\frac{a_{i,r}^{(r)}}{a_{i-1,r}^{(r)}}a_{i-1,j}^{(r)}, & \text{if } r+1\le i,j\le n+1, \text{ and } a_{i-1,j}^{(r)}\neq 0,\\ a_{i,j}^{(r)}, & \text{if } \ r+1\le i\le n+1, \text{ and } a_{i-1,r}^{(r)}=0.\end{cases}\tag{3}$$

The entry

$$p_{i,j}:=a_{i,j}^{(j)},\quad 1\le j\le i\le n+1,\tag{4}$$

is the (i,j) pivot of the Neville elimination of the matrix A and $p_{i,i}$ is called the ith diagonal pivot.

The Neville elimination of A can be done without row exchanges if all the pivots are nonzero. Then,

$$m_{i,j}:=\begin{cases}a_{i,j}^{(j)}/a_{i-1,j}^{(j)}=p_{i,j}/p_{i-1,j}, & \text{if } a_{i-1,j}^{(j)}\neq 0,\\ 0, & \text{if } a_{i-1,j}^{(j)}=0,\end{cases}\quad 1\le j<i\le n+1.\tag{5}$$

is called the (i, j) multiplier.

The complete Neville elimination of A consists of performing the Neville elimination to obtain the upper triangular matrix $U = A^{(n+1)}$ (see (1) and (2)) and next, the Neville elimination of the lower triangular matrix U^T. If in the complete process no row exchanges are needed, we say that the complete Neville elimination can be performed with no row and column exchanges. In this case, the multipliers of the complete Neville elimination of A (resp., A^T) are the multipliers of the Neville elimination of A (resp. of A^T) if $i \ge j$ (resp., $j \ge i$) (see [16]).

The Neville elimination procedure of a nonsingular matrix is illustrated with the following example:

$$A = A^{(1)} = \begin{pmatrix} 2 & 6 & 24 \\ 10 & 36 & 198 \\ 20 & 114 & 950 \end{pmatrix} \to A^{(2)} = \begin{pmatrix} 2 & 6 & 24 \\ 0 & 6 & 78 \\ 0 & 42 & 554 \end{pmatrix} \to A^{(3)} = \begin{pmatrix} 2 & 6 & 24 \\ 0 & 6 & 78 \\ 0 & 0 & 8 \end{pmatrix}. \tag{6}$$

The multipliers and the diagonal pivots are $m_{2,1} = 5$, $m_{3,1} = 2$, $m_{3,2} = 7$, $p_{1,1} = 2$, $p_{2,2} = 6$, and $p_{3,3} = 8$.

Neville elimination is a nice and efficient tool to analyze the total positivity of a given matrix. This fact, is shown in the following characterization, which can be derived from Theorem 4.1, Corollary 5.5 of [14] and the arguments of p. 116 of [16].

Theorem 2. *A given matrix A is STP (resp. nonsingular TP) if and only if its complete Neville elimination can be performed without row and column exchanges, the multipliers of the Neville elimination of A and A^T are positive (resp. nonnegative), and the diagonal pivots of the Neville elimination of A are positive.*

Furthermore, as a consequence of Theorem 4.2 and the arguments of p.116 of [16], we know that a nonsingular TP matrix A admits a decomposition of the form

$$A = F_n F_{n-1} \cdots F_1 D G_1 G_2 \cdots G_n, \tag{7}$$

where F_i (resp. G_i) is the TP, lower (resp. upper) triangular bidiagonal matrix given by

$$F_i = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & m_{i+1,1} & 1 & & \\ & & & \ddots & \ddots & \\ & & & & m_{n+1,n+1-i} & 1 \end{pmatrix}, \quad G_i^T = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & \widetilde{m}_{i+1,1} & 1 & & \\ & & & \ddots & \ddots & \\ & & & & \widetilde{m}_{n+1,n+1-i} & 1 \end{pmatrix}, \tag{8}$$

and D is the diagonal matrix whose diagonal elements are the diagonal pivots, $p_{i,i} > 0$, $i = 1, \dots, n+1$, of the Neville elimination of A (see (4)). The elements $m_{i,j}$ of the lower triangular bidiagonal matrix F_i in (8) are the multipliers of the Neville elimination of A. Furthermore, the entries $\widetilde{m}_{j,i}$ of the upper triangular bidiagonal matrix G_i in (8) are the multipliers of the Neville elimination of A^T. If, in addition, the elements m_{ij}, $\widetilde{m}_{ij}$ satisfy the following properties,

$$m_{ij} = 0 \quad \Rightarrow \quad m_{hj} = 0,\ \forall\, h > i \quad \text{and} \quad \widetilde{m}_{ij} = 0 \quad \Rightarrow \quad \widetilde{m}_{ik} = 0,\ \forall\, k > j, \tag{9}$$

we can guarantee that the decomposition (7) of the matrix A is unique.

The transpose of a TP matrix is another TP matrix and then we can write

$$A^T = G_n^T G_{n-1}^T \cdots G_1^T D F_1^T F_2^T \cdots F_n^T,$$

where F_i (resp., G_i) is the lower (resp., upper) triangular bidiagonal matrix given in (8). If, in addition, A is symmetric ($A = A^T$), conditions (9) holds and then $G_i = F_i^T$. Then we have

$$A = F_n F_{n-1} \cdots F_1 D F_1^T F_2^T \cdots F_n^T. \tag{10}$$

By defining the following matrix $BD(A) = (BD(A)_{i,j})_{1 \le i,j \le n+1}$, such that

$$BD(A)_{i,j} := \begin{cases} m_{i,j}, & \text{if } i > j, \\ p_{i,i}, & \text{if } i = j, \\ \widetilde{m}_{j,i}, & \text{if } i < j, \end{cases} \tag{11}$$

the bidiagonal factorization (7) of a $(n+1) \times (n+1)$ nonsingular TP matrix A can be represented (cf. [17]).

Let us observe that, for the matrix A in (6), the bidiagonal decomposition (7) can be written as follows:

$$\begin{pmatrix} 2 & 6 & 24 \\ 10 & 36 & 198 \\ 20 & 114 & 950 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 7 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 8 \end{pmatrix} \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 9 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}.$$

Moreover, since the diagonal pivots and multipliers of the Neville elimination of A and A^T are all positive, we conclude that A is strictly totally positive. The above factorization can be represented in the following matrix form:

$$BD(A) = \begin{pmatrix} 2 & 3 & 4 \\ 5 & 6 & 9 \\ 2 & 7 & 8 \end{pmatrix}.$$

Let us recall that if the entries of $BD(A)$ can be computed to HRA, using the algorithms raised in [7], problems such as the computation of A^{-1}, of the singular values of A, as well as the resolution of linear systems of equations $Ax = b$, for vectors b whose entries have alternating signs, can be performed to HRA. One can find the implementation of those algorithms through the link [11]. The name of the corresponding functions is `TNSingularValues`, `TNInverseExpand` (applying the algorithm proposed in [12]) and `TNSolve`, respectively. All these functions require the matrix $BD(A)$ as input argument.

3. Matrix conversion between q-Bernstein and Bernstein bases

The Bernstein polynomials of degree n on $[0,1]$ are defined by

$$B_k^n(t) := \binom{n}{k} t^k (1-t)^{n-k}, \quad k = 0, \dots, n. \tag{12}$$

Bernstein bases are the polynomial bases most used in CAGD because they have optimal shape preserving and stability properties (see [18,19]). These nice properties are related to the fact that $(B_0^n, \dots, B_n^n)$ is the normalized B-basis of the space $\mathbf{P}^n[0,1]$ of polynomials of degree no greater than n (cf. [13,20]) and then, any TP basis $(p_0^n, \dots, p_n^n)$ of $\mathbf{P}^n[0,1]$ can be written in terms of the B-basis as follows,

$$(p_0^n, \dots, p_n^n) = (B_0^n, \dots, B_n^n) A_n, \tag{13}$$

where A_n is nonsingular and TP (see Theorem 1). Moreover, if $(p_0^n, \dots, p_n^n)$ is normalized then A_n is stochastic (see Theorem 4.3 of [13]).

The q-Bernstein polynomials of degree n on $[0,1]$ are defined as

$$Q_k^n(t) := \begin{bmatrix} n \\ k \end{bmatrix}_q t^k \prod_{r=0}^{n-k-1} (1 - q^r t), \quad k = 0, \dots, n, \tag{14}$$

where, for $q > 0$, the q-binomial coefficients $\begin{bmatrix} n \\ k \end{bmatrix}_q$, $k = 0, \dots, n$, are given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q!\,[n-k]_q!}$$

and, for any non-negative integer n, the q-factorial $[n]_q!$ is defined by

$$[n]_q! := [n]_q\,[n-1]_q \cdots [1]_q,$$

and the q-integer $[n]_q$ by

$$[n]_q := \begin{cases} 1 + q + \cdots + q^{n-1} = \dfrac{1-q^n}{1-q}, & \text{if } q \neq 1 \\ n, & \text{if } q = 1. \end{cases} \tag{15}$$

Clearly, $[n]_q$ is a polynomial in q and, less obviously, the q-binomial coefficients $\begin{bmatrix} n \\ k \end{bmatrix}_q$, $k = 0, \dots, n$ are also polynomials in q with integer polynomials, known as Gaussian polynomials. For the particular case $q = 1$, the q-Bernstein basis (14) coincides with the Bernstein basis (12).

By Corollary 3.3 of [13], for any $q \in (0,1]$, the basis $(Q_0^n, \dots, Q_n^n)$ is TP on the interval $[0,1]$ and STP on $(0,1)$. Moreover, the partition of unity property satisfied by $(Q_0^n, \dots, Q_n^n)$ can be deduced using Proposition 5.2 of [21]. Then we can guarantee that the change of basis matrix $\widetilde{A}_n$ such that

$$(Q_0^n, \dots, Q_n^n) = (B_0^n, \dots, B_n^n) \widetilde{A}_n, \tag{16}$$

is nonsingular, stochastic and TP. This section is devoted to deriving the explicit expression of the elements of $\widetilde{A}_n = (\widetilde{a}_{i,j})_{1 \le i,j \le n+1}$.

First, let us define

$$q_k^n(t) := Q_k^n(t) / \begin{bmatrix} n \\ k \end{bmatrix}_q, \quad b_k^n(t) := B_k^n(t) / \binom{n}{k}, \quad k = 0, \dots, n,$$

and the $(n+1) \times (n+1)$ matrix $A_n = (a_{i,j})_{1 \le i,j \le n+1}$ such that

$$(q_0^n, \dots, q_n^n) = (b_0^n, \dots, b_n^n) A_n. \tag{17}$$

Clearly,

$$\widetilde{A}_n = \widetilde{D}_n^{-1} A_n D_n, \tag{18}$$

where $D_n := \operatorname{diag}\left(\begin{bmatrix} n \\ k-1 \end{bmatrix}_q \right)_{1 \le k \le n+1}$, $\widetilde{D}_n := \operatorname{diag}\left(\binom{n}{k-1} \right)_{1 \le k \le n+1}$ and then

$$\widetilde{a}_{i,j} = \frac{\begin{bmatrix} n \\ j-1 \end{bmatrix}_q}{\binom{n}{i-1}} a_{i,j}, \quad 1 \le i, j \le n+1. \tag{19}$$

The following result provides a recursive formula satisfied by the entries of A_n, $n > 1$.

Proposition 3. *Let $A_n = (a_{i,j}^{(n)})_{1\le i,j\le n+1}$, $n \ge 1$, be the matrices satisfying (17). Then $A_1 = I_2$, where I_2 denotes the 2×2 identity matrix. Moreover, for $n > 1$,*

$$a_{i,j}^{(n)} = a_{i,j}^{(n-1)} + [n-j]_q(1-q)\,a_{i-1,j}^{(n-1)}, \quad 1\le i\le n+1, \quad 1\le j\le n,$$
$$a_{i,n+1}^{(n)} = \delta_{i,n+1}, \quad 1\le i\le n+1, \tag{20}$$

with the convention $a_{0,j}^{(n-1)} := 0$ and $a_{n+1,j}^{(n-1)} := 0$.

Proof. For $n=1$, $q_i^1(t) = b_i^1(t)$, $i=0,1$, and then $A_1 = I_2$. For $n>1$, by (14), we can write

$$q_j^n(t) = t^j \prod_{r=0}^{n-j-1}(1-q^r t) = (1-q^{n-j-1}t)\,q_j^{n-1}(t)$$
$$= (1-t+(1-q^{n-j-1})t)\sum_{i=0}^{n-1} a_{i+1,j+1}^{(n-1)} b_i^{n-1}(t), \quad 0\le j\le n-1,$$

where $A_{n-1} = \left(a_{i,j}^{(n-1)}\right)_{1\le i,j\le n}$ is the matrix such that

$$(q_0^{n-1},\dots,q_n^{n-1}) = (b_0^{n-1},\dots,b_n^{n-1})A_{n-1}.$$

Then we have

$$\sum_{i=0}^{n} a_{i+1,j+1}^{(n)} b_i^n(t) = \sum_{i=0}^{n-1} a_{i+1,j+1}^{(n-1)}\left(b_i^n(t) + (1-q^{n-j-1})b_{i+1}^n(t)\right)$$
$$= \sum_{i=0}^{n}\left(a_{i+1,j+1}^{(n-1)} + (1-q^{n-j-1})a_{i,j+1}^{(n-1)}\right) b_i^n(t),$$

and deduce the following identities

$$a_{i,j}^{(n)} = a_{i,j}^{(n-1)} + (1-q^{n-j})a_{i-1,j}^{(n-1)}, \quad 1\le i\le n+1, \quad 1\le j\le n.$$

Moreover, since $q_n^n(t) = b_n^n(t) = t^n$, we have

$$a_{i,n+1}^{(n)} = \delta_{i,n+1}, \quad 1\le i\le n+1.$$

Finally, using formula (15), we can write

$$1-q^{n-j} = [n-j]_q(1-q),$$

and the result follows. □

The recursion provided by Proposition 3 allows us to deduce the explicit expression of the entries of the matrix (17), as it is shown in the following result.

Theorem 4. *The matrix $A_n = (a_{i,j}^{(n)})_{1\le i,j\le n+1}$ satisfying (17) is lower triangular and*

$$a_{i,j}^{(n)} = \begin{cases} (1-q)^{i-j}\displaystyle\sum_{\alpha_1<\cdots<\alpha_{i-j}\subset \mathcal{I}_{n-j}}\prod_{k=1}^{i-j}[\alpha_k]_q, & \text{if } 1\le j<i\le n,\\ 1, & \text{if } 1\le i=j\le n+1,\\ 0, & \text{elsewhere},\end{cases} \tag{21}$$

where $\mathcal{I}_k := \{1,\dots,k\}$ for $k\in\mathbb{N}$.

Proof. Identities (21) are going to be proved using induction on $n\in\mathbb{N}$. By Proposition 3, $A_1 = I_2$, where I_2 denotes the 2×2 identity matrix and so, (21) holds for $n=1$. Now, let us suppose that (21) holds for $n\in\mathbb{N}$. Using the recursion (20), we immediately deduce that $a_{i,j}^{(n+1)} = 0$, for $i<j$, and $a_{i,i}^{(n+1)} = 1$, $i=1,\dots,n+2$. Moreover, for $1\le i<j\le n+1$ we can write

$$a_{i,j}^{(n+1)} = a_{i,j}^{(n)} + [n+1-j]_q(1-q)\,a_{i-1,j}^{(n)}$$
$$= (1-q)^{i-j}\left(\sum_{\alpha_1<\cdots<\alpha_{i-j}\subset\mathcal{I}_{n-j}}\prod_{k=1}^{i-j}[\alpha_k]_q + \sum_{\alpha_1<\cdots<\alpha_{i-j-1}\subset\mathcal{I}_{n-j}}[n+1-j]_q\prod_{k=1}^{i-j-1}[\alpha_k]_q\right)$$
$$= (1-q)^{i-j}\sum_{\alpha_1<\cdots<\alpha_{i-j}\subset\mathcal{I}_{n+1-j}}\prod_{k=1}^{i-j}[\alpha_k]_q.$$

So, we conclude that formula (21) is also satisfied for $n+1$. □

Now, using (19) and Theorem 4, the matrix $\widetilde{A}_n = (\widetilde{a}_{i,j}^{(n)})_{1\le i,j\le n+1}$ satisfying (16) can be described by

$$\widetilde{a}_{i,j}^{(n)} = \begin{cases} \dfrac{\begin{bmatrix} n \\ j-1 \end{bmatrix}_q}{\binom{n}{i-1}}(1-q)^{i-j} \displaystyle\sum_{\alpha_1<\cdots<\alpha_{i-j}\subset \mathcal{I}_{n-j}} \prod_{k=1}^{i-j} [\alpha_k]_q, & \text{if } 1\le j<i\le n, \\ \dfrac{\begin{bmatrix} n \\ i-1 \end{bmatrix}_q}{\binom{n}{i-1}}, & \text{if } 1\le i=j\le n+1, \\ 0, & \text{elsewhere}, \end{cases} \tag{22}$$

with the previous notation $\mathcal{I}_k := \{1,\dots,k\}$, for $k\in\mathbb{N}$ (see also Section 2 of [22]). The entries (22) are illustrated with the following example.

Example 1. The matrix $\widetilde{A}_4$ such that $(Q_0^4,\dots,Q_4^4) = (B_0^4,\dots,B_4^4)\widetilde{A}_4$ is

$$\widetilde{A}_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{4}(1-q)\Big([1]_q+[2]_q+[3]_q\Big) & \frac{1}{4}[4]_q & 0 & 0 & 0 \\ \frac{1}{6}(1-q)^2\Big([1]_q[2]_q+[1]_q[3]_q+[2]_q[3]_q\Big) & \frac{1}{6}[4]_q(1-q)\Big([1]_q+[2]_q\Big) & \frac{1}{6}[6]_q & 0 & 0 \\ \frac{1}{4}(1-q)^3[1]_q[2]_q[3]_q & \frac{1}{4}[4]_q(1-q)^2[1]_q[2]_q & \frac{1}{4}[6]_q(1-q)[1]_q & \frac{1}{4}[4]_q & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

4. Factorizations of the change of basis matrix between Bernstein and q-Bernstein bases

First, let us recall that the converse of a matrix $A = (a_{i,j})_{1\le i,j\le n}$ is defined as

$$A^{\#} = (a_{i,j}^{\#})_{1\le i,j\le n} := (a_{n+1-i,n+1-j})_{1\le i,j\le n}. \tag{23}$$

Clearly, the converse $A^{\#}$ can be written as $A^{\#} = RAR$, where R is the $n\times n$ matrix obtained by reversing the order of the rows of the identity $n\times n$ matrix.

In order to achieve a suitable factorization of the matrix $\widetilde{A}_n$ satisfying

$$(Q_0^n,\dots,Q_n^n) = (B_0^n,\dots,B_n^n)\widetilde{A}_n,$$

let us define the matrix $\widetilde{M}_n := \widetilde{A}_n^{\#}$, that can be considered as the change of basis matrix such that $(Q_n^n,\dots,Q_0^n) = (B_n^n,\dots,B_0^n)\widetilde{M}_n$. We also define the matrix $M_n := A_n^{\#}$, where A_n is the matrix satisfying

$$(q_0^n,\dots,q_n^n) = (b_0^n,\dots,b_n^n)A_n,$$

with $q_k^n(t) := Q_k^n(t)/\begin{bmatrix} n \\ k \end{bmatrix}_q$ and $b_k^n(t) := B_k^n(t)/\binom{n}{k}$, for $k=0,\dots,n$ (see (17)). Clearly,

$$\widetilde{M}_n = \widetilde{D}_n^{-1} M_n D_n, \tag{24}$$

with $D_n = \operatorname{diag}\left(\begin{bmatrix} n \\ n-k+1 \end{bmatrix}_q\right)_{1\le k\le n+1}$ and $\widetilde{D}_n = \operatorname{diag}\left(\binom{n}{n-k+1}\right)_{1\le k\le n+1}$ and so,

$$\widetilde{m}_{i,j} = \frac{\begin{bmatrix} n \\ n-j+1 \end{bmatrix}_q}{\binom{n}{n-i+1}} m_{i,j}, \quad 1\le i,j\le n+1, \tag{25}$$

where $M_n = (m_{i,j}^{(n)})_{1\le i,j\le n+1}$ and $\widetilde{M}_n = (\widetilde{m}_{i,j}^{(n)})_{1\le i,j\le n+1}$.

From (23) and Theorem 4, we can deduce that $M_n = (m_{i,j}^{(n)})_{1\le i,j\le n+1}$ is an upper triangular matrix such that

$$m_{i,j}^{(n)} = \begin{cases} (1-q)^{j-i} \displaystyle\sum_{\alpha_1<\cdots<\alpha_{j-i}\subset \mathcal{I}_{j-2}} \prod_{k=1}^{j-i} [\alpha_k]_q, & \text{if } 2\le i<j\le n+1, \\ 1, & \text{if } 1\le i=j\le n+1, \\ 0, & \text{elsewhere}, \end{cases} \tag{26}$$

where $\mathcal{I}_k := \{1,\dots,k\}$ for $k\in\mathbb{N}$. Moreover, using (24) and (26), we deduce the following identities

$$\widetilde{m}_{i,j}^{(n)} = \begin{cases} \dfrac{\begin{bmatrix} n \\ n-j+1 \end{bmatrix}_q}{\binom{n}{n-i+1}}(1-q)^{j-i} \displaystyle\sum_{\alpha_1<\cdots<\alpha_{j-i}\subset \mathcal{I}_{j-2}} \prod_{k=1}^{j-i} [\alpha_k]_q, & \text{if } 2\le i<j\le n+1, \\ \dfrac{\begin{bmatrix} n \\ n-i+1 \end{bmatrix}_q}{\binom{n}{n-i+1}}, & \text{if } 1\le i=j\le n+1, \\ 0, & \text{elsewhere}, \end{cases} \tag{27}$$

where $\mathcal{I}_k := \{1, \ldots, k\}$ for $k \in \mathbb{N}$.

The following result obtains the pivots and the multipliers of the Neville elimination of M_n, providing its bidiagonal factorization (7).

Theorem 5. *For $q \in (0,1]$, the matrix $M_n = (m_{i,j}^{(n)})_{1\le i,j\le n+1}$ satisfying (26) is TP and*

$$M_n = G_1 \cdots G_n, \tag{28}$$

where G_i, $i = 1,\ldots,n$, are upper triangular bidiagonal matrices whose structure is given in (8), that can be computed to HRA. Their off-diagonal entries $\widetilde{m}_{i,j}$, $1 \le j < i \le n+1$, are given by

$$\widetilde{m}_{i,j} = (1-q)[i-j]_q = 1 - q^{i-j}, \quad 1 < j < i \le n+1, \qquad \widetilde{m}_{i,1} = 0, \quad 2 \le i \le n+1. \tag{29}$$

Proof. First, let us notice that the matrix M_n is upper triangular and define $L_n := M_n^T$ such that $L_n = (l_{i,j}^{(n)})_{1\le i,j\le n+1}$, with

$$l_{i,j}^{(n)} = \begin{cases} (1-q)^{i-j} \displaystyle\sum_{\alpha_1<\cdots<\alpha_{i-j}\subset\mathcal{I}_{i-2}} \prod_{k=1}^{i-j} [\alpha_k]_q, & \text{if } 2 \le j < i \le n+1, \\ 1, & \text{if } 1 \le i = j \le n+1, \\ 0, & \text{elsewhere}, \end{cases} \tag{30}$$

where $\mathcal{I}_k := \{1, \ldots, k\}$ for $k \in \mathbb{N}$.

Since $l_{i,1}^{(n)} = 0$ for $i = 2, \ldots, n+1$, we can deduce that the multipliers of the Neville elimination of L_n satisfy $m_{i,1} = 0$ for $i = 2, \ldots, n+1$.

Let $L^{(1)} := L_n$ and $L^{(r)} = (l_{ij}^{(r)})_{1\le i,j\le n+1}$, $r = 2, \ldots, n$, be the matrices obtained after $r-1$ steps of the Neville elimination of L_n. Now, by induction on $r \in \{1, \ldots, n+1\}$, we shall deduce the following identities

$$l_{i,j}^{(r)} = (1-q)^{i-j} \sum_{\alpha_1<\cdots<\alpha_{i-j}\subset\mathcal{I}_{i-r}} \prod_{k=1}^{i-j} [\alpha_k]_q, \quad 2 \le j < i \le n+1. \tag{31}$$

Taking into account (30), identities (31) are obtained for $r = 1$. On the other hand, if (31) holds for some $r \in \{1, \ldots, n\}$, we can write

$$\frac{l_{i,r}^{(r)}}{l_{i-1,r}^{(r)}} = (1-q)\frac{\sum_{\alpha_1<\cdots<\alpha_{i-r}\subset\mathcal{I}_{i-r}} \prod_{k=1}^{i-r}[\alpha_k]_q}{\sum_{\alpha_1<\cdots<\alpha_{i-r-1}\subset\mathcal{I}_{i-r-1}} \prod_{k=1}^{i-r-1}[\alpha_k]_q} = (1-q)\frac{\prod_{k=1}^{i-r}[\alpha_k]_q}{\prod_{k=1}^{i-r-1}[\alpha_k]_q} = (1-q)[i-r]_q. \tag{32}$$

Since $l_{i,j}^{(r+1)} = l_{i,j}^{(r)} - \left(l_{i,r}^{(r)} / l_{i-1,r}^{(r)} \right) l_{l-1,j}^{(r)}$, from (31) and (32), we obtain

$$\begin{aligned} l_{i,j}^{(r+1)} &= (1-q)^{i-j} \left(\sum_{\alpha_1<\cdots<\alpha_{i-j}\subset\mathcal{I}_{i-r}} \prod_{k=1}^{i-j}[\alpha_k]_q - \sum_{\alpha_1<\cdots<\alpha_{i-j-1}\subset\mathcal{I}_{i-r-1}} [i-r]_q \prod_{k=1}^{i-j-1}[\alpha_k]_q \right) \\ &= \sum_{\alpha_1<\cdots<\alpha_{i-j}\subset\mathcal{I}_{i-r-1}} \prod_{k=1}^{i-j}[\alpha_k]_q. \end{aligned}$$

By considering identities (4) and (31), we can deduce that the pivots $p_{i,j}$ of the Neville elimination of L_n satisfy

$$p_{i,j} = l_{i,j}^{(j)} = (1-q)^{i-j} \sum_{\alpha_1<\cdots<\alpha_{i-j}\subset\mathcal{I}_{i-j}} \prod_{k=1}^{i-j}[\alpha_k]_q = (1-q)^{i-j} \prod_{k=1}^{i-j}[\alpha_k]_q, \tag{33}$$

and, for $i = j$, the diagonal pivots are $p_{i,i} = 1$ for $i = 1, \ldots, n+1$. Furthermore, the multipliers of the Neville elimination of L_n satisfy

$$m_{i,j} = \frac{p_{i,j}}{p_{i-1,j}} = (1-q)\,[i-j]_q, \quad 2 \le j < i \le n+1. \tag{34}$$

Then, the bidiagonal factorization (7) of $L_n = M_n^T$ can be written as follows

$$L_n = F_n \cdots F_1, \tag{35}$$

and the off-diagonal entries $m_{i,j}$ of the bidiagonal matrices are given by $m_{i,1} = 0$ for $i = 2, \ldots, n+1$ and (34) when $2 \le j < i \le n+1$. Taking into account that $M_n = L_n^T$, we have

$$M_n = F_1^T \cdots F_n^T,$$

and defining $G_i := F_i^T$, $i = 1, \ldots, n$, the factorization (28) for M_n is obtained. Taking into account (34), formula (29) for the off diagonal entries $\widetilde{m}_{i,j}$ is deduced.

Finally, we can deduce that M_n is TP for $q \in (0,1]$, since the multipliers $\widetilde{m}_{i,j}$ in (29) are not negative. Furthermore, $\widetilde{m}_{i,j}$ can be clearly computed to HRA. □

Taking into account (24), Theorem 1 of [23] and the bidiagonal decomposition (7) of M_n, the bidiagonal factorization corresponding to $\widetilde{M}_n$ can be deduced, as it is illustrated in the following result.

Corollary 6. *For $q \in (0,1]$, the matrix $\widetilde{M}_n = (\widetilde{m}_{i,j}^{(n)})_{1\le i,j\le n+1}$ satisfying (27) is TP and*

$$\widetilde{M}_n = DG_1 \cdots G_n, \tag{36}$$

where G_i, $i = 1,\dots,n$, are upper triangular bidiagonal matrices whose structure is given in (8) and D is the diagonal matrix $D = diag\left(\left[{n \atop n-i+1}\right]_q / \binom{n}{n-i+1} \right)_{1\le i\le n+1}$. The off-diagonal elements $\widetilde{m}_{i,j}$, $2 \le j < i \le n+1$, are given by

$$\widetilde{m}_{i,j} = \frac{\left[{n \atop n-i+1}\right]_q}{\left[{n \atop n-i+2}\right]_q}(1-q)[i-j]_q = \frac{[n-i+2]_q}{[i-1]_q}(1-q^{i-j}), \quad 2 \le j < i \le n+1,$$
$$\widetilde{m}_{i,1} = 0, \quad 2 \le i \le n+1.$$

This factorization can be computed to HRA.

The bidiagonal decomposition in (7) for the change of basis matrix $\widetilde{M}_n$, described by (36), can be represented through $BD(\widetilde{M}_n) = \left(BD(\widetilde{M}_n)_{i,j} \right)_{1\le i,j\le n+1}$, with

$$BD(\widetilde{M}_n)_{i,j} = \begin{cases} \frac{[n-j+2]_q}{[j-1]_q}(1-q^{j-i}), & \text{if } 2 \le i < j \le n+1, \\ \frac{\left[{n \atop n-i+1}\right]_q}{\binom{n}{n-i+1}}, & \text{if } 1 \le i = j \le n+1, \\ 0, & \text{elsewhere.} \end{cases} \tag{37}$$

This matrix form of the bidiagonal factorization of $\widetilde{M}_n$ is illustrated with the following example.

Example 2. Taking into account (37), the bidiagonal factorization (7) of $\widetilde{M}_4$ can be represented by means of the following matrix

$$BD(\widetilde{M}_4) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{[4]_q}{4} & \frac{[3]_q}{[2]_q}(1-q) & \frac{[2]_q}{[3]_q}(1-q^2) & \frac{[1]_q}{[4]_q}(1-q^3) \\ 0 & 0 & \frac{[6]_q}{6} & \frac{[2]_q}{[3]_q}(1-q) & \frac{[1]_q}{[4]_q}(1-q^2) \\ 0 & 0 & 0 & \frac{[4]_q}{4} & \frac{[1]_q}{[4]_q}(1-q) \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

In order to encourage the understanding of the numerical experimentation carried out in Section 6, we present the pseudocode of an algorithm for the computation of $BD(\widetilde{M}_n)$ (see (37)). We can observe that Algorithm 1 has a computational cost of $O(n^2)$ arithmetic operations.

5. HRA computations with q-Bernstein mass matrices

Bernstein polynomials are square integrable functions under the following inner product

$$\langle f, g \rangle_{\alpha,\beta} := \int_0^1 t^\alpha (1-t)^\beta f(t) g(t)\, dt, \quad \alpha, \beta > -1. \tag{38}$$

The Bernstein basis $(B_0^n, \dots, B_n^n)$ is not orthogonal and then it is often transformed into another orthogonal basis of $\mathbf{P}^n[0,1]$ by means of a Gramian matrix.

The Gramian (mass) matrix of $(B_0^n, \dots, B_n^n)$ with respect to the inner product (38) is a symmetric matrix $G^{\alpha,\beta} = (g_{i,j}^{\alpha,\beta})_{1\le i.j\le n+1}$ where

$$g_{i,j}^{\alpha,\beta} := \left\langle B_{i-1}^n, B_{j-1}^n \right\rangle_{\alpha,\beta} = \binom{n}{i-1}\binom{n}{j-1} \frac{\Gamma(i+j+\alpha-1)\Gamma(2n-i-j+\beta+3)}{\Gamma(2n+\alpha+\beta+2)}, \tag{39}$$

for $1 \le i, j \le n+1$, and $\Gamma(x)$ is the Gamma function (see [8]).

If $\alpha = 0$ and $\beta = 0$, the Gramian matrix (39) is known as Bernstein mass matrix. Many nice properties and applications of Bernstein mass matrices can be seen in [24–26].

The following result proves that q-Bernstein mass matrices are also TP matrices.

Corollary 7. *The Gramian matrix $\widetilde{G}^{\alpha,\beta}$, $\alpha, \beta > -1$, with respect to the inner product (38), of the q-Bernstein basis given in (14) is TP.*

Algorithm 1 Computation to HRA of the matrix $BD(\widetilde{M}_n)$, for $\widetilde{M}_n$ given by (27)

Require: q, n $\quad (q \in (0,1])$
Ensure: BDM bidiagonal decomposition (11) of $\widetilde{M}_n$ to HRA
$q1 = \texttt{zeros}(1,n)$; $q2 = \texttt{zeros}(1,n)$; $q3 = \texttt{zeros}(1,n)$;
$BDM = \texttt{zeros}(n+1)$
$q1(1) = 1$; $q2(1) = 1$; $q3(1) = 1$;
$BDM(1,1) = 1$;
$BDM(n+1,n+1) = 1$
  for $i = 2 : n$
    $q1(i) = q1(i-1)q$
    $q2(i) = q2(i-1) + q1(i)$
    $q3(i) = q2(i)q3(i-1)$
  end
  for $i = 2 : n+1$
    for $j = i+1 : n+1$
      $BDM(i,j) = \frac{q2(n-j+2)}{q2(j-1)}(1 - q1(j-i+1))$
    end
  end
  for $i = 2 : ceil((n+1)/2)$
    $BDM(i,i) = \frac{q3(n)}{q3(n-i+1)q3(i-1)\binom{n}{n-i+1}}$
    $BDM(n+2-i,n+2-i) = \frac{q3(n)}{q3(n-i+1)q3(i-1)\binom{n}{n-i+1}}$
  end

Proof. Clearly,

$$\widetilde{G}^{\alpha,\beta} = \widetilde{A}_n G^{\alpha,\beta} \widetilde{A}_n^T,$$

where $\widetilde{A}_n$ is the change of basis matrix such that $(Q_0^n, \dots, Q_n^n) = (B_0^n, \dots, B_n^n)\widetilde{A}_n$. Since the Bernstein basis $(B_0^n, \dots, B_n^n)$ is the normalized B-basis of $\mathbf{P}^n[0,1]$, we deduce that $\widetilde{A}_n$ is nonsingular, stochastic and TP. Hence $\widetilde{A}_n^T$ is also TP. On the other hand, Theorem 2 of [8] proves that $G^{\alpha,\beta}$ is STP for any $\alpha, \beta > -1$. So, we can deduce that $\widetilde{G}^{\alpha,\beta}$ is a TP matrix because it is obtained as the product of TP matrices (see Theorem 3.1 of [27]). □

The following auxiliary result relates q-Bernstein mass matrices and their converses and will be used to derive accurate computations.

Lemma 8. *Let B_i^n, $i = 0, \dots, n$, be the Bernstein polynomials of degree n on the interval $[0,1]$ and $G_{0,n}^{\alpha,\beta}$, $G_{n,0}^{\alpha,\beta}$ the Gramian matrix of the basis $(B_0^n, \dots, B_n^n)$ and $(B_n^n, \dots, B_0^n)$, respectively, with respect to the inner product (38). Then,*

$$G_{0,n}^{\alpha,\beta} = G_{n,0}^{\beta,\alpha}. \tag{40}$$

Proof. Taking into account (38), (12) and using the change of variable $\tau = 1 - t$, we can write

$$\begin{aligned}\left\langle B_i^n, B_j^n \right\rangle_{\alpha,\beta} &= \binom{n}{i}\binom{n}{j} \int_0^1 t^{\alpha+i+j}(1-t)^{\beta+2n-i-j}\, dt \\ &= \binom{n}{n-i}\binom{n}{n-j} \int_0^1 (1-\tau)^{\alpha+i+j} \tau^{\beta+2n-i-j}\, d\tau = \left\langle B_{n-i}^n, B_{n-j}^n \right\rangle_{\beta,\alpha}.\end{aligned}$$

From the previous identity, (40) follows. □

Using of the previous results, we shall derive a procedure to achieve computations to HRA when considering q-Bernstein mass matrices.

Theorem 9. *Let Q_i^n, $i = 0, \dots, n$, be the q-Bernstein polynomials of degree n on the interval $[0,1]$ in (14) and $\widetilde{G}_{0,n}^{\alpha,\beta}$, $\widetilde{G}_{n,0}^{\alpha,\beta}$ the Gramian matrix of $(Q_0^n, \dots, Q_n^n)$ and $(Q_n^n, \dots, Q_0^n)$, respectively, with respect to the inner product (38). Then, for any $\alpha, \beta > -1$ such that $\Gamma(\alpha)$, $\Gamma(\beta)$ and $\Gamma(\alpha+\beta)$ can be computed to HRA, the matrix $\widetilde{G}_{0,n}^{\alpha,\beta}$ can be computed to HRA. Furthermore, $\widetilde{G}_{n,0}^{\alpha,\beta}$ and its bidiagonal decomposition (7) can be computed to HRA.*

Proof. First, let R_n be the matrix obtained by reversing the order of the rows of the $(n+1) \times (n+1)$ identity matrix. Clearly,

$$\widetilde{G}_{0,n}^{\alpha,\beta} = R_n \widetilde{G}_{n,0}^{\alpha,\beta} R_n.$$

Now, let $\widetilde{M}_n$ be the change of basis matrix such that

$$(Q_n^n, \dots, Q_0^n) = (B_n^n, \dots, B_0^n)\widetilde{M}_n.$$

Using Lemma 8, we can write

$$\widetilde{G}_{n,0}^{\alpha,\beta} = \widetilde{M}_n^T\, G_{n,0}^{\alpha,\beta}\, \widetilde{M}_n = \widetilde{M}_n^T\, G_{0,n}^{\beta,\alpha}\, \widetilde{M}_n, \tag{41}$$

where $G_{0,n}^{\alpha,\beta}$ and $G_{n,0}^{\alpha,\beta}$ denote the Gramian matrix of the Bernstein bases $(B_0^n, \dots, B_n^n)$ and $(B_n^n, \dots, B_0^n)$, respectively, with respect to the inner product (38).

Let us notice that, by Corollary 6, the bidiagonal factorization (7) of $\widetilde{M}_n$ can be computed to HRA (see (37) providing $BD(\widetilde{M}_n)$). Clearly, the bidiagonal factorization (7) of $\widetilde{M}_n^T$ can also be computed to HRA. In fact, $BD(\widetilde{M}_n^T) = BD(\widetilde{M}_n)^T$.

On the other hand, taking into account Theorem 2 of [8], we can deduce that $G_{0,n}^{\beta,\alpha}$ is STP and $BD(G_{0,n}^{\beta,\alpha}) = (BD(G_{0,n}^{\beta,\alpha})_{i,j})_{1\le i,j\le n+1}$ is described by

$$BD(G_{0,n}^{\beta,\alpha})_{i,j} := \begin{cases} \dfrac{(n-i+2)(i+\beta-1)(2n-i+\alpha+3)}{(i-1)(2n-i-j+\alpha+3)(2n-i-j+\alpha+4)}, & \text{if } i > j, \\ \dbinom{n}{i-1}^2 \dfrac{\Gamma(i+\beta)\Gamma(2n-2i+\alpha+3)}{\Gamma(2n-i+\alpha+\beta+3)\binom{2n-i+\alpha+2}{i-1}}, & \text{if } i = j, \\ \dfrac{(n-j+2)(j+\beta-1)(2n-j+\alpha+3)}{(j-1)(2n-i-j+\alpha+3)(2n-i-j+\alpha+4)}, & \text{if } i < j. \end{cases} \tag{42}$$

Let us observe that, since $\alpha, \beta > -1$, the off-diagonal entries of $BD(G_{0,n}^{\beta,\alpha})$ are positive and can be computed to HRA. Moreover, if $\Gamma(\alpha)$, $\Gamma(\beta)$ and $\Gamma(\alpha+\beta)$ can be computed to HRA, taking into account that for $n \in \mathbb{N}$, $\Gamma(x+n) = \Gamma(x)\prod_{k=0}^{n-1}(x+k)$, we deduce that $BD(G_{0,n}^{\beta,\alpha})_{i,i}$ can also be computed to HRA.

Finally, let us recall that if the bidiagonal decomposition (7) of two nonsingular TP matrices is provided to HRA, Algorithm 5.1 of [7] obtains to HRA the bidiagonal decomposition (7) of the product. Consequently, $\widetilde{G}_{n,0}^{\alpha,\beta}$ and its bidiagonal factorization (7) can also be computed to HRA. Finally, $R\widetilde{G}_{n,0}^{\alpha,\beta} R$ can be computed to HRA by an appropriated change of the position of the entries of $\widetilde{G}_{n,0}^{\alpha,\beta}$. □

Let us observe that the diagonal entries $BD(G^{\alpha,\beta})_{i,i}$, $1 \le i \le n+1$, can be easily computed since they satisfy

$$BD(G^{\alpha,\beta})_{1,1} = \frac{\Gamma(\alpha+1)\Gamma(2n+\beta+1)}{\Gamma(2n+\alpha+\beta+2)},$$
$$BD(G^{\alpha,\beta})_{i+1,i+1} = \frac{(n-i+1)^2(i+\alpha)(2n-i+\alpha+\beta+2)(2n-i+\beta+2)}{i(2n-2i+\beta+1)(2n-2i+\beta+2)^2(2n-2i+\beta+3)} BD(G^{\alpha,\beta})_{i,i}, \tag{43}$$

for $1 \le i \le n$.

Now, we provide the pseudocode of Algorithm 2 to get the bidiagonal decomposition (7), in the matrix form (11), of the Gramian Matrix of the Bernstein basis $(B_0^n, \dots, B_n^n)$ to HRA, whenever $\Gamma(\alpha)$, $\Gamma(\beta)$ and $\Gamma(\alpha+\beta)$ can be evaluated to HRA. We can observe that Algorithm 2 has a computational cost of $O(n^2)$ arithmetic operations. Then, Algorithm 3 computes the bidiagonal decomposition (11) of the Gramian matrix $\widetilde{G}_{n,0}^{\alpha,\beta} = \widetilde{M}_n^T\, G_{0,n}^{\beta,\alpha}\, \widetilde{M}_n$ to HRA.

Finally, let us notice that, given $\alpha, \beta > -1$ such that $\Gamma(\alpha)$, $\Gamma(\beta)$ and $\Gamma(\alpha+\beta)$ can be evaluated to HRA, using $BD(\widetilde{G}_{n,0}^{\alpha,\beta})$ as input argument, the Matlab functions in Koev's routines available in [11] compute the solution of several fundamental problems in Linear Algebra related to $\widetilde{G}_{n,0}^{\alpha,\beta}$ to HRA. Let us see that, taking into account that $\widetilde{G}_{0,n}^{\alpha,\beta} = R\widetilde{G}_{n,0}^{\alpha,\beta} R$, these problems can be also solved to HRA when considering the q-Bernstein mass matrices $\widetilde{G}_{0,n}^{\alpha,\beta}$.

– Given $B := BD(A)$, computed to HRA, the Matlab function `TNSingularValues(B)` computes the singular values of a matrix A to HRA. The computational cost of this function is $O(n^3)$ (see [17]). Since R a unitary matrix, that is, $R^{-1} = R$, the singular values of $\widetilde{G}_{0,n}^{\alpha,\beta}$ and $\widetilde{G}_{n,0}^{\alpha,\beta}$ coincide.

– Given $B := BD(A)$, computed to HRA, the Matlab function `TNInverseExpand`(B) returns A^{-1} to HRA. In this case, the computational cost of the functions is $O(n^2)$ arithmetic operations (see [12]). The inverse matrix $(\widetilde{G}_{0,n}^{\alpha,\beta})^{-1} = R(\widetilde{G}_{n,0}^{\alpha,\beta})^{-1}R$ can be obtained to HRA by reversing the order of the entries of $(\widetilde{G}_{n,0}^{\alpha,\beta})^{-1}$, obtained to HRA.

– Given $B := BD(A)$, computed to HRA, for a vector d with alternating signs, the Matlab function `TNSolve`(B, d) returns the solution of the linear system $Ac = d$ to HRA. It requires $O(n^2)$ arithmetic operations (see [11]). The linear system of equations $\widetilde{G}_{0,n}^{\alpha,\beta} x = b$ is clearly equivalent to $\widetilde{G}_{n,0}^{\alpha,\beta}\tilde{x} = \tilde{b}$, with $\tilde{x} := Rx$ and $\tilde{b} = Rb$. If the vector b has alternating sings, then so has the vector $\tilde{b}$. Therefore $\tilde{x}$ can be computed to HRA and x obtained reversing the order of the entries of $\tilde{x}$.

Section 6 illustrates the accurate results obtained using the proposed algorithms.

6. Numerical experiments

To test our algorithms, we have solved the above mentioned algebraic problems when considering q-Bernstein mass matrices with dimension $n + 1 = 4, 5, \dots, 20$. We have taken several $q \in (0, 1]$ and real values α, β satisfying conditions given in Theorem 9

Algorithm 2 Computation of $BD(G^{\alpha,\beta})$ for the Gramian matrix $G^{\alpha,\beta}$ of the Bernstein basis $(B_0^n,\dots,B_n^n)$

Require: $\alpha,\ \beta,\ n$ (such that $\alpha,\ \beta > -1$)
Ensure: $BDGB$ bidiagonal decomposition (11) of $G^{\alpha,\beta}$

$BDGB = \texttt{zeros}(n+1)$
$c1 = \texttt{zeros}(1,n+1)$; $c2 = \texttt{zeros}(1,n+1)$; $c3 = \texttt{zeros}(1,n+1)$; $c4 = \texttt{zeros}(1,n+1)$
$BDGB(1,1) = \frac{\Gamma(\alpha+1)\Gamma(2n+\beta+1)}{\Gamma(2n+\alpha+\beta+2)}$
$c2(1) = 2n-1+\beta+3$
for $i = 2 : n+1$
  $c1(i) = i-1$
  $c2(i) = c2(i-1)-1$
  $c3(i) = c2(i)+1$
  $c4(i) = (n-i+2)(i+\alpha-1)c2(i)$
  $BDGB(i,i) = \frac{c4(i)(n-i+2)(c2(i)+\alpha)}{c1(i)(c2(i)-i)(c3(i)-i)^2(c3(i)-i+1)} BDGB(i-1,i-1)$
  for $j = 1 : i-1$
    $res = \frac{c4(i)}{c1(i)(c2(i)-j)(c3(i)-j)}$
    $BDGB(i,j) = res$
    $BDGB(j,i) = res$
  end
end

Algorithm 3 Computation of $BD(\widetilde{G}_{n,0}^{\alpha,\beta})$ for the Gramian matrix $\widetilde{G}_{n,0}^{\alpha,\beta}$ of the basis $(Q_n^n,\dots,Q_0^n)$

Require: $\alpha,\ \beta,\ q,\ n$ (such that $\alpha,\ \beta > -1$ and $q \in (0,1)$)
Ensure: $BDGQB$ bidiagonal decomposition (11) of $\widetilde{G}_{n,0}^{\alpha,\beta}$ to HRA

$BDM = zeros(n+1)$
$BDGB = zeros(n+1)$
$BDGQB = zeros(n+1)$
$B1 = zeros(n+1)$
$BDM = \texttt{BDM}(q,n)$
$BDGB = \texttt{BDGB}(\beta,\alpha,n)$
$B1 = \texttt{TNProduct}((BDM)^T, BDGB)$
$BDGQB = \texttt{TNProduct}(B1,(BDM))$

to guarantee that their bidiagonal decomposition can be computed to HRA. To calculate the relative errors, we have obtained all the solutions in Mathematica using a 100-digit arithmetic and we have considered these computations as the exact solutions of the proposed algebraic problems.

We have also obtained the 2-norm condition number of all considered matrices, $G = \widetilde{G}_{n,0}^{\alpha,\beta}$. For this purpose we have used the Mathematica command `Norm[G,2]· Norm[Inverse[G],2]`. In Fig. 1 this conditioning is depicted. It can be easily observed that the conditioning drastically increases with the size of the matrices, indicating that they are nearly singular. Consequently, standard routines implementing best numerical methods fail to solve accurately usual algebraic problems. In contrast, the accurate algorithms that we have developed in this paper exploit the structure of the considered matrices obtaining, as we will see, numerical results to HRA.

Computation of the singular values of q-Bernstein mass matrices. Algorithm 4 uses the bidiagonal decomposition of the Gramian matrix $\widetilde{G}_{n,0}^{\alpha,\beta}$, provided by Algorithm 3, to compute its singular values to HRA.

We have compared the lowest singular value obtained using Algorithm 4 and the Matlab command `svd` for all considered matrices. The relative error of the approximations is illustrated in Fig. 2. Looking at the results, we notice that our approach computes accurately the lowest singular value regardless of the ill-conditioning of the considered Gramian matrices. In contrast, the Matlab command `svd` returns results that become not accurate when the dimension of the Gramian matrices increases.

Algorithm 4 Computation of the singular values of q-Bernstein mass matrices to HRA

Require: $\alpha,\ \beta,\ q,\ n$ ($\alpha,\ \beta > -1$ and $q \in (0,1]$)
Ensure: $\mathbf{v} \in \mathbb{R}^{n+1}$ containing the singular values of $\widetilde{G}_{0,n}^{\alpha,\beta}$

$BDGQB = \texttt{zeros}(n+1)$
$BDGQB = \texttt{BDGQB}(\alpha,\beta,q,n)$
$\mathbf{v} = \texttt{TNSingularValues}(BDGQB)$

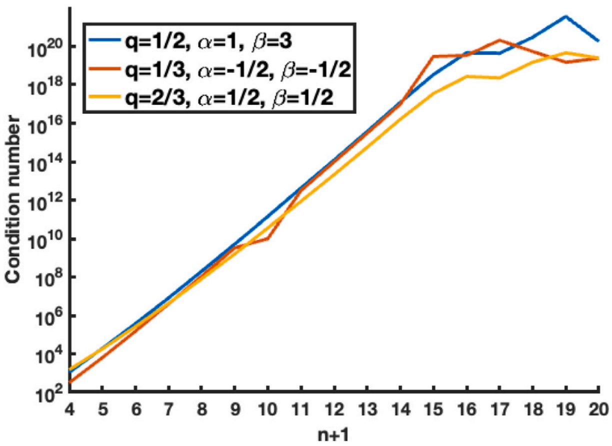

Fig. 1. The 2-norm conditioning of the of q-Bernstein mass matrices $\widetilde{G}_{0,n}^{\alpha,\beta}$.

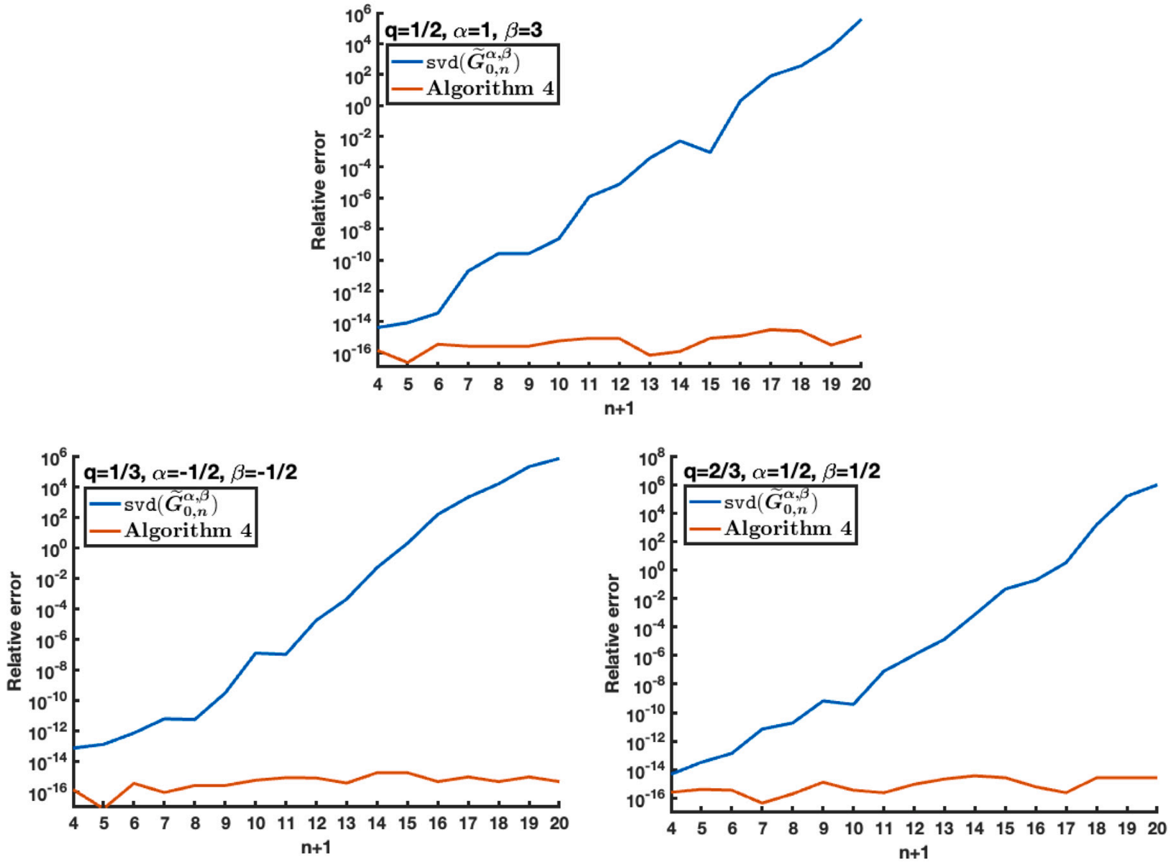

Fig. 2. Relative error in the approximations to the lowest singular value of q-Bernstein mass matrices.

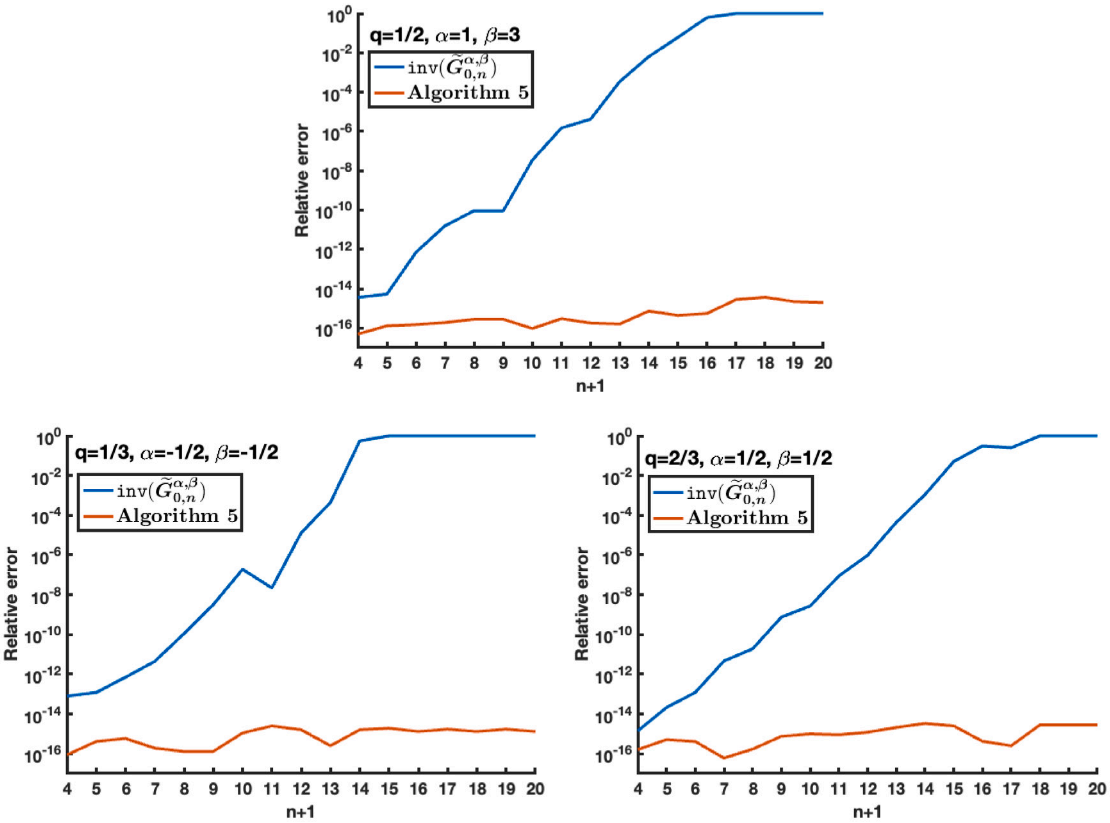

Fig. 3. Relative error in the computation of the inverse of q-Bernstein mass matrices.

Computation of the inverse of q-Bernstein mass matrices. Algorithm 5 uses the bidiagonal decomposition of the Gramian matrix $\widetilde{G}^{\alpha,\beta}_{n,0}$, provided by Algorithm 3, to compute the inverse of the q-Bernstein mass matrix $\widetilde{G}^{\alpha,\beta}_{0,n}$ to HRA.

For all considered Gramian matrices, we have compared the inverses obtained using Algorithm 5 and the Matlab command `inv`. The achieved relative errors are shown in Fig. 3. We observe that our algorithm provides very accurate results in contrast to the inaccurate results obtained when using the Matlab command `inv`.

Algorithm 5 Computation of the inverse of q-Bernstein mass matrices to HRA

Require: $\alpha,\ \beta,\ q,\ n \qquad (\alpha,\ \beta > -1$ and $q \in (0,1])$
Ensure: A matrix $InvGQB$ which is the inverse of $\widetilde{G}^{\alpha,\beta}_{0,n}$
$BDGQB = \mathtt{zeros}(n+1)$
$InvGQB = \mathtt{zeros}(n+1)$
$R = \mathtt{zeros(n+1)}$
for $i = 1 : n+1$
$R(i, n+2-i) = 1$
end
$BDGQB = \mathtt{BDGQB}(\alpha, \beta, q, n)$
$InvGQB = R \cdot \mathtt{TNInverseExpand}(BDGQB) \cdot R$

Resolution of linear systems with q-Bernstein mass matrices. Algorithm 6 takes the bidiagonal decomposition of the Gramian matrix $\widetilde{G}^{\alpha,\beta}_{n,0}$, provided by Algorithm 3, to compute the solution of $\widetilde{G}^{\alpha,\beta}_{0,n} c = d$ to HRA.

For all considered Gramian matrices, we have compared the solution obtained using Algorithm 6 and the Matlab command \. In Fig. 4 we show the relative errors. We clearly see that, in spite of the dimension of the problem, the proposed algorithm preserves the accuracy as opposed to the results obtained with the Matlab command \.

Algorithm 6 Resolution of linear systems of equations $\widetilde{G}_{0,n}^{\alpha,\beta} c = d$ to HRA

Require: $\alpha,\ \beta,\ q,\ n \quad (\alpha,\ \beta > -1$ and $q \in (0,1])$
Ensure: $c \in \mathbb{R}^{n+1}$ containing the solution of the linear system $\widetilde{G}_{0,n}^{\alpha,\beta} c = d$
$R = \mathtt{zeros}(n+1)$
$BDGQB = \mathtt{zeros}(n+1)$
$c = \mathtt{zeros}(n+1,1)$
  for $i = 1 : n+1$
    $R(i, n+2-i) = 1$
  end
$BDGQB = \mathtt{BDGQB}(\alpha,\beta,q,n)$
$c = R \cdot \mathtt{TNSolve}(BDGQB, R \cdot d)$

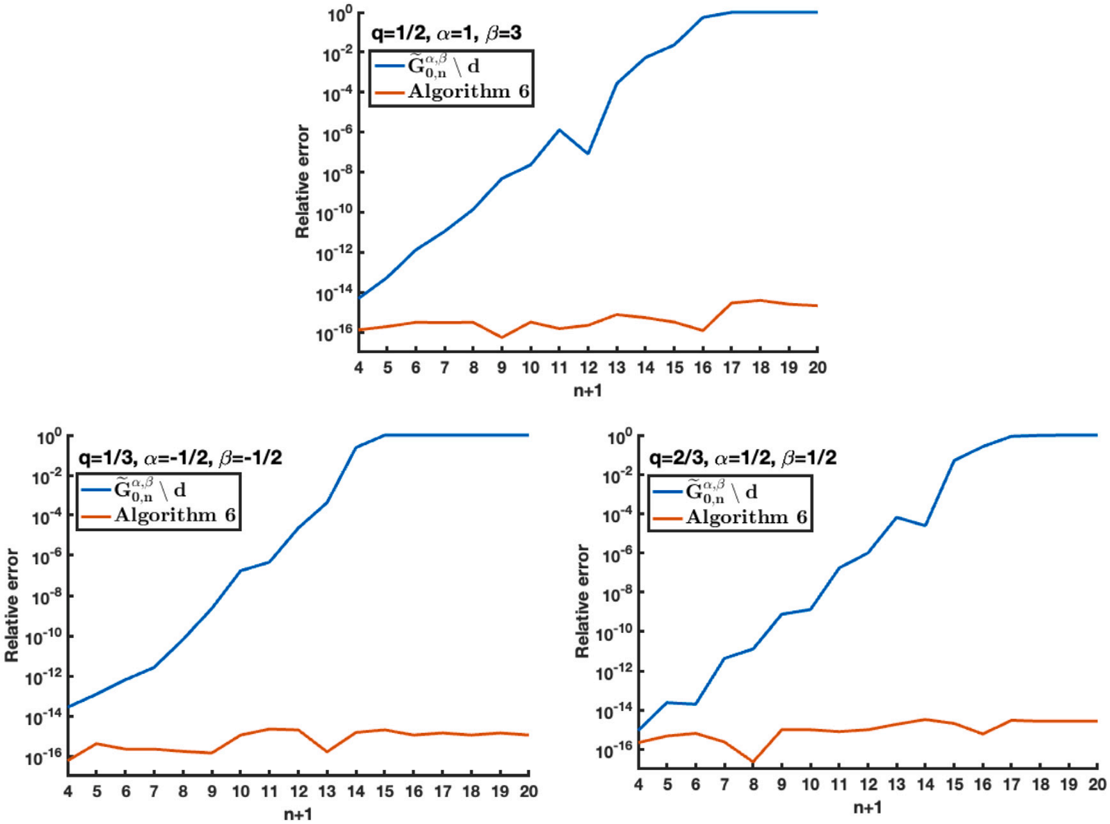

Fig. 4. Relative error of the approximations to the solution of the linear systems $\widetilde{G}_{0,n}^{\alpha,\beta} c = d$, where d is a vector of random values with alternating signs.

Data availability

Data will be made available on request.

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