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Tesis Doctoral

GRADINGS ON STRUCTURABLE ALGEBRAS RELATED TO AN HERMITIAN FORM

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UNIVERSIDAD DE ZARAGOZA
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DOCTORAL THESIS

Gradings on structurable algebras related to an hermitian form

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Contents

Resumen

Este trabajo es la memoria de una tesis doctoral en matemáticas cuyo tema principal son las graduaciones de grupo. El objetivo de esta tesis es la clasificación de las graduaciones de grupo en las álgebras estructurables relacionadas con una forma hermítica salvo isomorfismo. Comenzamos escribiendo una generalización de la teoría de graduaciones que aparece en [EK13] y aplicamos esta teoría a la clasificación de graduaciones en r-fold cross products. Después encontramos una construcción de un álgebra asociativa con una 3-graduación y una involución a partir de nuestras álgebras estructurables. Finalmente, usamos la construcción anterior para clasificar graduaciones salvo isomorfismo en álgebras estructurables relacionadas con una forma hermítica.

Abstract

This work is a doctoral thesis in mathematics whose main subject are group gradings. The aim of this thesis is to classify group gradings on structurable algebras related to a hermitian form up to isomorphism. We start writing a generalization of the theory of gradings which appears in [EK13], and we apply this theory to the classification of gradings on r-fold cross products. Afterwards, we find a construction of an associative algebra with a 3-grading and an involution from our structurable algebras. Finally, we use the previous construction to classify gradings up to isomorphism on structurable algebras related to a hermitian form.

Introducción

En el contexto de las álgebras de Líe, las graduaciones por grupos abelianos aparecen naturalemente en distinots contextos. Uno de ellos es la graduación de cartan, es decir, la graduación

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha \right)$$

donde Φ es un sistema de raices. Esta graduación se puede ver como una graduación por el retículo de raices $\langle \Phi \rangle$. Otro ejemplo es la 3-graduación

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

inducida por un álgebra de Jordan via la construcción de Tits-Kantor-Koecher (ver [Tit62],[Kan64] y [Koe67]). Una generalización de esto es la 5-graduación

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$$

inducida por la construcción de Allison-Kantor a partir de un álgebra estructurable.

El estudio sistemático de graduaciones en álgebras de Lie empezó en 1984 por Patera y Zassenhaus [PZ89]. Este estudio fue continuado por distintos autores como Bahturin, Draper, Elduque o Kotchetov (consultar el monográfico [EK13]). Aún quedan abiertos algunos problemas como la clasificación salvo isomorfismo de las graduaciones en álgebras de Lie simples en característica 0, o incluso la clasificación de graduaciones finas salvo equivalencia en cuerpos de característica prima.

Una forma de atacar el problema consiste en estudiar las graduaciones del álgebra 5-graduada inducida por la construcción de Allison-Kantor clasificando salvo isomorfismo las graduaciones en álgebras estructurables centrales y simples. Hay una clasificación en seis clases de álgebras estructurables centrales y simples por Allison, que fue completada por Smirnov.

Theorem 0.0.1 ([Ali78],[Smi90b])**.** *Sea* $(\mathcal{A}, -)$ *un álgebra estructurable central y simple sobre un cuerpo de característica distinta de* 2, 3 *y* 5, *entonces,* $(\mathcal{A}, -)$ *es isomorfa a un álgebra en una de las siguientes clases:*

(1) Un álgebra asociativa central y simple con involución.

(2) un álgebra de Jordan central y simple con la identidad como involución,

(3) un álgebra construida a partir de una forma hermítica h, en un módulo $\mathcal{W}$ sobre un álgebra asociativa con involución y unidad central y simple $(\mathcal{E}, -)$,

(4) Una forma torcida del producto tensorial de álgebras de composición. a twisted form of a tensor product of composition algebras,

(5) Una forma torcida de un álgebra de matrices 2×2 construida a partir de una forma cúbica admisible y no degenerada con punto base c y un escalar $0 \neq \theta \in \mathbb{F}$, ó

(6) el álgebra $T(\mathcal{C})$ para un álgebra de Cayley $\mathcal{C}$.

Sobre cuerpos algebráicamente cerrados, una clasificación completa salvo isomorfismo graduaciones por grupos abelianos está terminada para las clases (1), (2), (4) y (6) (ver [EK13] y [AC20]). El propósito principal de esta tesis es la clasificación de graduaciones salvo isomorfismo en álgebras estructurables relacionadas con una forma hermítica, es decir, las de la clase 3.

En esta memoria, $\mathbb{F}$ denotará un cuerpo, $\overline{\mathbb{F}}$ su clausura algebráica y asumiremos que $0 \in \mathbb{N}$. La estructura es la siguiente:

En el capítulo 1, repasaremos las herramientas que aparecen en el primer capítulo de [EK13]. Por tanto, explicaremos cómo las graduaciones están relacionadas con quasitoros en el grupo de automorfismo cuando trabajamos sobre cuerpos algebráicamente cerrados de característica 0, y con representaciones de esquemas afines en grupo diagonalizables en cuerpos arbitrarios. Sin embargo, en el monográfico, los resultados se dan en términos de álgebras, pero como vamos a necesitar distintas estructuras algebráicas, generalizaremos esto a Ω-álgebras $\mathbb{F}$-lineales, lo que nos permitirá generalizar los teoremas de transferencia.

En el capítulo 2, introducimos los r-fold vector cross products, una generalización del producto vectorial en $\mathbb{R}^3$. Este capítulo sirve como ejemplo de coómo usar las técnicas del tema anterior. El resultado principal es la clasificación salvo isomorfismo de graduaciones por grupos abelianos en cada una de las clases de isomorfismos de cada r-fold cross products (ver los teoremas 2.6.1, 2.6.4 y 2.6.11).

En el capítulo 3, introducimos las álgebras estructurables. Comenzamos con una motivación a partir de las álgebras de Jordan, definimos las álgebras estructurables, damos los principales ejemplos y la clasificación de las álgebras estructurables centrales y simples. Finalmente, introducimos la construcción de Állison-Kantor y explicamos cómo el grupo de estructura de un álgebra estructurable es isomorfo al grupo de isomorfismos del álgebra de Lie 5-graduada itroducida a partir de la construcción de Allison-Kantor.

En el capítulo 4, dedicamos nuestro estudio al álgebras estructurables centrales y simples relacionadas con una forma hermítica y lo relacionamos con otras estructuras algebráicas. Enseñamos que hay una equivalencia de categorías entre la categoría de sistemas triples asociativos centrales y simples del segundo tipo y una categoría plena de la categoría de álgebras asociativas 3-graduadas con una involución (ver el corolario 4.1.21), damos una definición de graduación estructurable, que es una graduación que existe si y sólo si el álgebra con involución es un álgebra estructurable relacionada con una forma hermítica, y enseñamos que hay una equivalencia de categorías entre la categoría de los sistemas triples asociativos del segundo tipo centrales y simples y la categoría de álgebras con involución centrales y simples con una graduación estructurable (ver proposición 4.2.29). Finalmente, enseñamos que todas las álgebras estructurables relacionadas con una forma hermítica centrales y simples tiene una única graduación estructurable excepto por un álgebra especial llamada la split quartic Cayley algebra (ver las proposiciones 4.3.3, 4.3.5, 4.3.9 y el lema 4.3.12).

Finalmente, el capítulo 5 está dedicado al estudio de los esquemas de automorfismos de las álgebras estructurables relacionadas con una forma hermítica centrales y simple sobre un cuerpo algebráicamente cerrado (ver el teorema 5.1.7 y el teorema 5.1.13). Y usamos esto para clasificar las graduaciones salvo isomorfismo. Los resultados principales son los teoremas 5.2.63 y 5.2.65.

Introduction

In the context of Lie algebras, gradings by abelian groups appear naturally in different contexts. One of those is the Cartan grading, i.e., the grading:

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha \right)$$

where Φ is a root system. This grading can be seen as a grading by the root latice $\langle \Phi \rangle$. Another example, is the 3-grading

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

induced from a Jordan algebra through the Tits-Kantor-Koecher construction (see [Tit62],[Kan64] and [Koe67]). A generalization of this is the 5-grading

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$$

induced by the Allison-Kantor construction from a structurable algebra.

The systematic study of gradings on Lie algebras started in 1989 by Patera and Zassenhaus [PZ89]. This study was carried out by other different authours like Bahturin, Draper, Elduque or Kotchetov (see [EK13] for a survey). Still, some problems like the complete classification up to isomorphism of gradings on simple Lie algebra is opened in characteristic 0, and even the classification of fine gradings up to equivalence over fields of prime characteristic is an open problem.

An approach to understand the problem is to study the gradings of the 5-graded Lie algebras induced by the Allison-Kantor construction by classifying up to isomorphism the gradings on central-simple structurable algebras. There is a classification into six classes of central-simple structurable algebras carried out by Allison and Smirnov.

Theorem 0.0.2 ([Ali78],[Smi90b]). *Let $(\mathcal{A}, -)$ be a finite-dimensional central simple structurable algebra over a field $\mathbb{F}$ of characteristic different from 2, 3 and 5. Then, $(\mathcal{A}, -)$ is isomorphic to one of the following algebras:*

(1) A central simple associative algebra with involution,

(2) a central simple Jordan algebra with the identity involution,

(3) an algebra constructed from a nondegenerate hermitian form h on a module $\mathcal{W}$ over a unital central simple associative algebra with involution $(\mathcal{E}, -)$,

(4) a twisted form of a tensor product of composition algebras,

(5) a twisted form of an algebra of 2×2 matrices constructed from an admisible nondegenerate cubic form N with base point c and scalar $0 \neq \theta \in \mathbb{F}$ or

(6) the algebra $T(\mathcal{C})$ for a Cayley algebra $\mathcal{C}$.

Over algebraically closed fields, a classification up to isomorphism of gradings by abelian groups is completed for the classes (1), (2), (4) and (6) (see [EK13] and [AC20]). The main purpose of this thesis is to classify the gradings up to isomorphism on structurable algebras related to a hermitian form, i.e., those on class (3).

On this thesis, $\mathbb{F}$ will denote a field, $\overline{\mathbb{F}}$ its algebraic closure and we will assume that $0 \in \mathbb{N}$. The structure is the following:

In Chapter 1, we review the tools given in the first chapter of [EK13]. Thus, we are going to show how gradings are related to quasitori on the automorphism groups on algebraically closed fields of characteristic 0, and to representations of diagonalizable affine group schemes over fields of prime characteristic. However, in the monograph, the results are given in terms of algebras, but since we are going to need various algebraic structures, we will generalize it to $\mathbb{F}$-linear Ω-algebras, which will allow us to generalize the transfer theorem.

In Chapter 2, we introduce r-fold vector cross products, a generalization of the vector cross product on $\mathbb{R}^3$. This chapter works as an example of how to use this techniques. The main result is the classification up to isomorphism of gradings by abelian groups on each isomorphism class of r-fold vector cross products (see Theorems 2.6.1, 2.6.4 and 2.6.11).

In Chapter 3, we introduce structurable algebras. We start with a motivation arising from Jordan algebras, we define structurable algebras, we give the main examples and the classification of central-simple structurable algebras. Finally, we introduce the Allison-Kantor construction and show how the structure group of a structurable algebra is isomorphic to the isomorphism group of the 5-graded Lie algebra induced by its Allison-Kantor construction.

In Chapter 4, we devote our study to central-central simple structurable algebras related to an hermitian form and we relate it with other algebraic structures. We show that there is an equivalence of categories between the category of central simple associative triple system of the second kind and a full subcategory of the cateogory of 3-graded associative algebras with involution (see Corollary 4.1.21), we give a definition of structurable grading, which is a kind of grading which exists if and only if the algebra with involution is a structurable algebra related to a hermitian form, and we show that there is an equivalence of categories between the category of central-simple associative triple systems of the second kind and the categrory of central simple algebras with involution and a structurable grading (see Proposition 4.2.29). Finally, we show that all the central simple structurable algebras with a hermitian form have only one structurable grading except for a special one called the split quartic Cayley algebra (see Propositions 4.3.3, 4.3.5, 4.3.9 and Lemma 4.3.12).

Finally, Chapter 5 is devoted to study the automorphism group scheme of central-simple structurable algebras related to an hermitian form over an algebraically closed field, (see Theorem 5.1.7 and Theorem 5.1.13). And we use this to classify the gradings up to isomorphism. Our main results are Theorems 5.2.63 and 5.2.65.

Chapter 1

Preliminaries

The main purpose of this thesis is the study of gradings by abelian groups on structurable algebras. The main reference for the study of gradings is the first chapter of the monograph on gradings on simple Lie algebras [EK13]. The most useful result found there is the duality between gradings by abelian groups and actions. This leads, for example, to prove the transfers theorems, which give a correspondence of group gradings up to isomorphism in any two algebras with the same automorphism group scheme. However, sometimes, in order to classify gradings it is more useful to work in different algebraic structures, e.g., algebras with involutions, triple systems or graded algebras. The theory has been adapted to other contexts for example in [DET20]. This chapter attempts to write a generalization of this results to a wider context. In section 1.1, we introduce the concept of $\mathbb{F}$-linear Ω-algebras and give examples. In section 1.2, we introduce the concept of algebraic group and in section 1.3, we introduce a generalization: affine group schemes. In section 1.4, we give the main definitions of gradings in terms of $\mathbb{F}$-linear Ω-algebras, and show that a graded $\mathbb{F}$-linear Ω-algebra is an Ω-algebra, and finally, in section 1.5, we give the main tools relating gradings with derivations, the automorphism group and the automorphism group scheme.

1.1 Algebraic structures

In this section we will review the concept of Ω-algebra introduced in [Coh81] and we will adapt it to the concept of $\mathbb{F}$-linear Ω-algebras.

Definition 1.1.1. An **operator domain** is a set Ω with a map $a\colon \Omega \to \mathbb{N}$. The elements of Ω are called **operators** and for every operator $\omega \in \Omega$ we say that $a(\omega)$ is the **arity** of the operator (or that ω is $a(\omega)$-**ary**).

Given an operator domain as before, we denote $\Omega(n) = \{\omega \in \Omega \mid a(\omega) = n\}$

Definition 1.1.2. Let A be a set. An **Ω-algebra structure** on A is a family of maps:

$$\Omega(n) \to A^{A^n}$$

for $n \in \mathbb{N}$, where A^0 denotes the set with one element. The set A with this structure is called an **Ω-algebra**. In case A is an $\mathbb{F}$-vector space and in case the maps factor through $\mathrm{Hom}_{\mathbb{F}}(A^{\otimes^n}, A)$ for $n \geq 1$ and considering, for the case $n = 0$, a map $\Omega(0) \to \mathrm{Hom}_{\mathbb{F}}(\mathbb{F}, \mathcal{A})$, we call it an **$\mathbb{F}$-linear Ω-algebra**. We will denote it by (A, Ω), or $(\mathcal{A}, \omega_1, ..., \omega_n)$ if $\Omega = \{\omega_1, ..., \omega_n\}$ or simply A in case Ω is clear.

If the context is clear we will say Ω-algebra instead of $\mathbb{F}$-linear Ω-algebra.

For an Ω-algebra (A, Ω) and an n-ary operator ω, for the image of ω applied to the n-uple $(a_1, ..., a_n) \in A^n$, we write $\omega(a_1, ..., a_n)$.

Remark 1.1.3. If we have an $\mathbb{F}$-linear Ω-algebra $(\mathcal{A}, \Omega)$ and a field extension $\mathbb{K}/\mathbb{F}$, we can define a $\mathbb{K}$-linear Ω-algebra over the vector space $\mathcal{A} \otimes_{\mathbb{F}} \mathbb{K}$ such that every $\omega \in \Omega(n)$ defines an operator $\omega_{\mathbb{K}}$ by $\omega_{\mathbb{K}}(a_1 \otimes 1, ..., a_n \otimes 1) = \omega(a_1, ..., a_n) \otimes 1$ for all $a_1, ..., a_n$ for $n \geq 1$ and every $\omega \in \Omega(0)$ defines an operator $\omega_{\mathbb{K}}$ by $\omega_{\mathbb{K}}(1) = \omega(1)$. We denote this Ω-algebra by $(\mathcal{A}_{\mathbb{K}}, \Omega_{\mathbb{K}})$

Remark 1.1.4. We can notice that the definition of Ω-algebra and $\mathbb{F}$-linear Ω-algebra are the same but in the first case we work in the monoidal category of sets and in the second case we work in the monoidal category of vector spaces as defined in [EGNO15, Example 2.3.1] and in [EGNO15, Example 2.3.2].

Given two ($\mathbb{F}$-linear) Ω-algebras A and B, we say that B is a **subalgebra** of A if $B \subseteq A$ and for any $n \in \mathbb{N}$ and $\omega \in \Omega(n)$, which defines operators ω_A and ω_B respectively, satisfies $\omega_A \mid_{B^n} = \omega_B$.

Given two ($\mathbb{F}$-linear) Ω-algebras A and B, an $\mathbb{F}$-linear map $f \colon A \to B$ is a **homomorphism** if for every $\omega \in \Omega(0)$, $f \circ \omega_A = \omega_B$ and for every $n \in \mathbb{Z}_{>0}$, $\omega \in \Omega(n)$ and $a_1, ..., a_n \in A$, the following identity holds:

$$f(\omega_A(a_1, ..., a_n)) = \omega_B(f(a_1), ..., f(a_n)). \tag{1.1.1}$$

A homomorphism $f \colon \mathcal{A} \to \mathcal{B}$ is a **monomorphism** if for every Ω-algebra $\mathcal{C}$ and every two homomorphisms $g, h \colon \mathcal{C} \to \mathcal{A}$ $f \circ g = f \circ h$ implies $g = h$, an **epimorphism** if for every Ω-algebra $\mathcal{C}$ and every two homomorphisms $g, h \colon \mathcal{B} \to \mathcal{C}$ $g \circ f = h \circ f$ implies $g = h$, an **isomorphism** if it has an inverse, and an **automorphism** if it is an isomorphism and $\mathcal{A} = \mathcal{B}$. The group of automorphisms will be denoted by $\mathrm{Aut}(\mathcal{A}, \Omega)$. We say that a subspace $\mathcal{I}$ of $\mathcal{A}$ is an **ideal** if and only if there is an Ω-algebra $\mathcal{B}$ and a homomorphism $f\colon \mathcal{A} \to \mathcal{B}$ such that $\mathcal{I}$ is the kernel of φ.

Remark 1.1.5. From the definition it follows that any ideal $\mathcal{I}$ of $\mathcal{A}$ satisfies that for all $n > 0$ and $\omega \in \Omega(n)$, $\omega(\mathcal{I}, \mathcal{A}, ..., \mathcal{A}) + \omega(\mathcal{A}, \mathcal{I}, ..., \mathcal{A}) + ... + \omega(\mathcal{A}, \mathcal{A}, ..., \mathcal{I}) \subseteq \mathcal{I}$. Moreover, given a subspace $\mathcal{I}$ satisfying the previous identities, we can define an Ω algebra on $\mathcal{A}/\mathcal{I}$ by passing to the quotient, and the morphism sending a to $a + \mathcal{I}$ for all $a \in \mathcal{A}$ has kernel $\mathcal{I}$. Hence, $\mathcal{I}$ is an ideal.

A **derivation** $d\colon \mathcal{A} \to \mathcal{A}$ on an $\mathbb{F}$-linear Ω-algebra is an $\mathbb{F}$-linear map satisfying that for any $\omega \in \Omega(0)$ the identity $d \circ \omega = 0$ holds and for any $n \geq 1$, $\omega \in \Omega(n)$ and $a_1, ..., a_n \in \mathcal{A}(n)$, the following identity holds:

$$d(\omega(a_1, a_2, ..., a_n)) = \omega(d(a_1), a_2..., a_n) + \omega(a_1, d(a_2)..., a_n) + ... + \omega(a_1, ..., d(a_n)). \quad (1.1.2)$$

The set of derivations of an Ω-algebra $\mathcal{A}$, has a structure of Lie algebra with the product given by $[d_1, d_2] = d_1 \circ d_2 - d_2 \circ d_1$. We denote this algebra as $\mathrm{Der}(\mathcal{A}, \Omega)$.

We will introduce some examples of $\mathbb{F}$-linear Ω-algebras:

Examples 1.1.6. (a) A vector space is just an $\mathbb{F}$-linear $\emptyset$-algebra.

(b) A **n-ary algebra** $\mathcal{A}$ is a vector space $\mathcal{A}$ and Ω consists of one element m of arity n. In case $n = 2$ we just call it an algebra and in case $n = 3$ we call it a triple system.

(c) An **algebra with involution** is a vector space $\mathcal{A}$ with Ω consisting of one element m of arity 2 and one element τ of arity 1 such that:

   (1) $m(\tau(a_1), \tau(a_2)) = \tau(m(a_2, a_1))$ for all $a_1, a_2 \in \mathcal{A}$, and
   (2) $\tau^2(a) = a$ for all $a \in \mathcal{A}$.

(d) Given a vector space $\mathcal{A}$ and a bilinear form $b\colon \mathcal{A} \times \mathcal{A} \to \mathbb{F}$ we have the Ω-algebra $\mathcal{A}$ where Ω consists of one element which we denote ω_b of arity 3 defined by $\omega(a_1, a_2, a_3) = b(a_1, a_2)a_3$.

   We can describe the automorphisms of this $\mathbb{F}$-linear Ω-algebra. Indeed, $\varphi \in \mathrm{Aut}(\mathcal{A}, \Omega)$ if and only if it has an inverse and for all $a_1, a_2, a_3 \in \mathcal{A}$,

$$\varphi(\omega_b(a_1, a_2, a_3)) = \omega_b(\varphi(a_1), \varphi(a_2), \varphi(a_3)).$$

Equivalently, this happens if and only if for all $a_1, a_2, a_3 \in \mathcal{A}$,

$$b(a_1, a_2)\varphi(a_3) = b(\varphi(a_1), \varphi(a_2))\varphi(a_3).$$

This shows that φ is an automorphism if and only if

$$b(a_1, a_2) = b(\varphi(a_1), \varphi(a_2)).$$

For that reason, if for instance, b is a nondegenerate symmetric bilinear form we get that

$$\mathrm{Aut}(\mathcal{A}, \Omega) = \mathrm{O}(\mathcal{A}, b)$$

and in case it is a nondegenerate symplectic bilinear form we get that

$$\mathrm{Aut}(\mathcal{A}, \Omega) = \mathrm{Sp}(\mathcal{A}, b).$$

In the subsequent sections and chapters we will introduce some more examples.

Let $(\mathcal{A}, \Omega)$ be an Ω-algebra. For every $n \geq 1$, $a_1, ..., a_{n-1} \in \mathcal{A}$, $\omega \in \Omega(n)$ and $1 \leq i \leq n$, we denote by $\omega^i_{(a_1,...,a_{n-1})}$ the endomorphism of the vector space $\mathcal{A}$ defined by $\omega^i_{(a_1,...,a_{n-1})}(x) = \omega(a_1, ..., a_{i-1}, x, a_i, ..., a_{n-1})$.

Definition 1.1.7. Let $(\mathcal{A}, \Omega)$ be an Ω-algebra. We define its **multiplication algebra** $\mathcal{I}(\mathcal{A}, \Omega)$ as the subalgebra of $\mathrm{End}_{\mathbb{F}}(\mathcal{A})$ generated by the elements of the form $\omega^i_{(a_1,...,a_{n-1})}$ for every $\omega \in \Omega$ of arity $n \geq 1$ and for every $i \in \{1, ..., n\}$. We say that $(\mathcal{A}, \Omega)$ is **simple** if $\mathcal{I}(\mathcal{A}, \Omega) \neq 0$ and its only ideals are $\mathcal{A}$ and 0.

Remark 1.1.8. Notice that a subspace of $\mathcal{A}$ is an ideal of $(\mathcal{A}, \Omega)$ if and only if it is a submodule for $\mathcal{I}(\mathcal{A}, \Omega)$. Therefore, $(\mathcal{A}, \Omega)$ is simple if and only if $\mathcal{I}(\mathcal{A}, \Omega) \neq 0$ and $\mathcal{A}$ is an irreducible module of $\mathcal{I}(\mathcal{A}, \Omega)$.

Definition 1.1.9. Let $(\mathcal{A}, \Omega)$ be an Ω-algebra. We define its **centroid** as:

$$\mathcal{C}(\mathcal{A}, \Omega) = \{f \in \mathrm{End}_{\mathbb{F}}(\mathcal{A}) \mid f \circ \varphi = \varphi \circ f \ \forall \varphi \in \mathcal{I}(\mathcal{A}, \Omega)\}.$$

We say that $(\mathcal{A}, \Omega)$ is **central** if $\mathcal{C}(\mathcal{A}, \Omega) = \mathbb{F}\mathrm{id}$. We say that it is **central simple** if it is central and simple.

Remark 1.1.10. The centroid of an Ω-algebra $(\mathcal{A}, \Omega)$ is a unital subalgebra of $\mathrm{End}_{\mathbb{F}}(\mathcal{A})$.

For an algebra with involution $(\mathcal{A}, -)$, we define its center as:

$$\mathcal{Z}(\mathcal{A}, -) = \{x \in \mathcal{A} \mid \overline{x} = x, xy = yx \quad \forall x, y \in \mathcal{A}\}.$$

The following is a known result but the author is not aware of a reference for this specific case.

Lemma 1.1.11. *A unital algebra with involution $(\mathcal{A}, -)$ is central if and only if $\mathcal{Z}(\mathcal{A}, -) = \mathbb{F}1$ where 1 is the unit of $\mathcal{A}$.*

Proof. Denote $[x, y] = xy - yx$ and $[x, y, z] = (xy)z - x(yz)$ for all $x, y, z \in \mathcal{A}$.

We know that $\mathbb{F}1 \subseteq \mathcal{Z}(\mathcal{A}, -)$ and that if $x \in \mathcal{Z}(\mathcal{A}, -)$, $L_x \in \mathcal{C}(\mathcal{A}, -)$. Moreover, if $L_x = L_{\lambda 1}$, that implies that $x = \frac{1}{\lambda}1$ for any $\lambda \in \mathbb{F}^\times$. Hence, if $\mathcal{C}(\mathcal{A}, -) = \mathbb{F}\mathrm{id}$, $\mathcal{Z}(\mathcal{A}, -) = \mathbb{F}1$. Now, assume that $\mathcal{Z}(\mathcal{A}, -) = \mathbb{F}1$. Let $f \in \mathcal{C}(\mathcal{A}, -)$. Then, $f(x) = f(1x) = f(1)x$. Therefore, $f = L_{f(1)}$. We have to prove that $f(1) \in \mathcal{Z}(\mathcal{A}, -)$. For any $x \in \mathcal{A}$, $f(1)x = f(1x) = f(x1) = xf(1)$. Hence $[f(1), \mathcal{A}] = 0$. That means that $L_{f(1)} = f = R_{f(1)}$. Now, for any $x, y \in \mathcal{A}$, $f(xy) = f(x)y$. Hence, $f(1)(xy) = (f(1)x)y$. Thus, $[f(1), \mathcal{A}, \mathcal{A}] = 0$. Also, $(xf(1))y = f(x)y = xf(y) = x(f(1)y)$. Therefore, $[\mathcal{A}, f(1), \mathcal{A}] = 0$. Similarly, we get $[\mathcal{A}, \mathcal{A}, f(1)] = 0$. Finally, $\overline{f(1)} = f(\overline{1}) = f(1)$ which proves $f(1) \in \mathcal{Z}(\mathcal{A}, -)$ so $\mathcal{C}(\mathcal{A}, -) = \mathbb{F}\mathrm{id}$. □

Proposition 1.1.12. *The centroid of a simple Ω-algebra, is a division ring. Moreover, if it has a nonzero operator of arity greater or equal to 2, it is a field.*

Proof. Let $(\mathcal{A}, \Omega)$ be a simple Ω algebra. Take $0 \neq f \in \mathcal{C}(\mathcal{A}, \Omega)$. $f(\mathcal{A}) \neq 0$ is a submodule of $\mathcal{A}$. Therefore, $f(\mathcal{A}) = \mathcal{A}$. Since $\ker f$ is another submodule of $\mathcal{A}$ under the action of $\mathcal{I}(\mathcal{A}, \Omega)$ and $\ker f \neq \mathcal{A}$, then $\ker f = 0$. Therefore f is an isomorphism of vector spaces. $f^{-1} \in \mathcal{C}(\mathcal{A}, \Omega)$ due to the fact that for an operator ω of arity $n \geq 1$, $a_1, ..., a_{n-1} \in \mathcal{A}$ and $i \in \{1, ..., n\}$, we have:

$$\begin{aligned} f^{-1}(\omega^i_{(a_1,...,a_{n-1})}(x)) &= f^{-1}(\omega^i_{(a_1,...,a_{n-1})}(f(f^{-1}(x)))) \\ &= f^{-1}(f(\omega^i_{(a_1,...,a_{n-1})}(f^{-1}(x)))) = \omega^i_{(a_1,...,a_{n-1})}(f^{-1}(x)). \end{aligned}$$

Hence, $\mathcal{C}(\mathcal{A}, \Omega)$ is a division algebra. If there is a nonzero operator ω of arity $n \geq 2$, we can take $a_1, a_2, ..., a_n$ such that $\omega(a_1, a_2, ..., a_n) \neq 0$. Then, given $0 \neq f, g \in \mathcal{C}(\mathcal{A}, \Omega)$:

$$\begin{aligned} f(g(\omega(a_1, a_2, ..., a_n))) &= f(\omega^1_{(a_2,...,a_n)}(g(a_1))) \\ &= \omega^2_{(g(a_1),a_3...,a_n)}(f(a_2)) \\ &= g(\omega^1_{(f(a_2),...,a_n)}(a_1)) \\ &= g(f(\omega^2_{(a_1,a_3...,a_n)}(a_2))) \\ &= g(f(\omega(a_1, a_2, ..., a_n))). \end{aligned}$$

This implies that $\omega(a_1, a_2, ..., a_n) \in \ker(fg-gf)$. Moreover, since $\ker(fg-gf)$ is a submodule of $\mathcal{A}$, we get that $\ker(fg-gf) = \mathcal{A}$. Hence, $fg - gf = 0$ and this implies that $\mathcal{C}(\mathcal{A}, \Omega)$ is commutative. $\square$

Proposition 1.1.13. *Let $(\mathcal{A}, \Omega)$ be an Ω-algebra. $(\mathcal{A}, \Omega)$ is central simple if and only if for every field extension $\mathbb{K}/\mathbb{F}$, the Ω-algebra $(\mathcal{A}_{\mathbb{K}}, \Omega_{\mathbb{K}})$ is simple.*

Proof. Assume first that $(\mathcal{A}_{\mathbb{K}}, \Omega_{\mathbb{K}})$ is simple for every field extension $\mathbb{K}/\mathbb{F}$. Then $(\mathcal{A}, \Omega) = (\mathcal{A}_{\mathbb{F}}, \Omega_{\mathbb{F}})$ is simple. If $(\mathcal{A}, \Omega)$ is not central, take $f \in \mathcal{C}(\mathcal{A}, \Omega) \setminus \mathbb{F}\mathrm{id}$. Then, $\mathbb{K} = \mathbb{F}(f)$ is a field due to Proposition 1.1.12, and $\mathcal{A}$ is a $\mathbb{K}$-linear Ω-algebra. Thus, $\mathcal{A} \otimes_{\mathbb{F}} \mathbb{K} \cong \mathcal{A} \otimes_{\mathbb{K}} (\mathbb{K} \otimes_{\mathbb{F}} \mathbb{K})$. Since $\mathbb{K} \otimes_{\mathbb{F}} \mathbb{K}$ is not a simple algebra (due to the fact that the algebra homomorphism $\mathbb{K} \otimes_{\mathbb{F}} \mathbb{K} \to \mathbb{K}$ sending $a \otimes b$ to ab has nontrivial kernell), taking a proper nontrivial ideal I, we have that $\mathcal{A} \otimes_{\mathbb{K}} I$ is an ideal of $(\mathcal{A}_{\mathbb{K}}, \Omega_{\mathbb{K}})$.

Finally, if $(\mathcal{A}, \Omega)$ is central simple and $\mathbb{K}/\mathbb{F}$ is a field extension, a nonzero element of $x \in \mathcal{A} \otimes_{\mathbb{F}} \mathbb{K}$ can be written as $x = a_1 \otimes \alpha_1 + ... + a_k \otimes \alpha_k$ for linearly independent $a_1, ..., a_k \in \mathcal{A} \setminus \{0\}$ and $\alpha_1, ..., \alpha_k \in \mathbb{K}^{\times}$. Since $\mathcal{A}$ is an irreducible $\mathcal{I}(\mathcal{A}, \Omega)$ module, by Jacobson density, there is $\varphi \in \mathcal{I}(\mathcal{A}, \Omega)$ such that $\varphi(a_1) = a_1$ and $\varphi(a_i) = 0$ for $i = 1, ..., k$. Denote by $\varphi_{\mathbb{K}}$ its linear extension to $\mathcal{A} \otimes_{\mathbb{F}} \mathbb{K}$. Thus, $\varphi_{\mathbb{K}}(x) = a_1 \otimes 1$ and $\varphi_{\mathbb{K}} \in \mathcal{I}(\mathcal{A}_{\mathbb{K}}, \Omega_{\mathbb{K}})$, which implies that $\mathcal{A} \otimes_{\mathbb{F}} \mathbb{K} = \mathrm{Ideal}\langle\{a_1 \otimes 1\}\rangle \subseteq \mathrm{Ideal}\langle\{x\}\rangle$ where $\mathrm{Ideal}\langle A\rangle$ means the ideal generated by the set A. Thus, $\mathrm{Ideal}\langle\{x\}\rangle = \mathcal{A} \otimes_{\mathbb{F}} \mathbb{K}$, which implies that $(\mathcal{A}_{\mathbb{K}}, \Omega_{\mathbb{K}})$ is simple. $\square$

1.2 Algebraic groups

In this section we are going to review the concept of algebraic groups. An introduction to algebraic groups can be found in [Hum75].

Definition 1.2.1. Let $f_1, ..., f_k \in \mathbb{F}[x_1, ..., x_n]$. We denote the set of zeros of these polynomials by:

$$\mathcal{V}(f_1, ..., f_k) = \{x \in \mathbb{F}^n \mid f_1(x) = ... = f_k(x) = 0\}.$$

We say that a subset $X \subseteq \mathbb{F}^n$ is an **affine algebraic variety** if there are $f_1, ..., f_k \in \mathbb{F}[x_1, ..., x_n]$ such that $X = \mathcal{V}(f_1, ..., f_k)$.

There is a topology we can induce in an affine variey X in which the closed subsets are the subsets of the form $X \cap \mathcal{V}(g_1, ..., g_l)$ for $g_1, ..., g_l \in \mathbb{F}[x_1, ..., x_n]$. This is called the **Zariski topology**.

Let $X \subseteq \mathbb{F}^n$ and $Y \subseteq \mathbb{F}^m$ be two affine varieties. A map $\varphi\colon X \to Y$ is a **morphism** of affine varieties if there are $\psi_1, ..., \psi_m \in \mathbb{F}[x_1, .., x_m]$ such that $\varphi(x) = (\psi_1(x), ..., \psi_m(x))$.

Remark 1.2.2. If X and Y are affine varieties, the cartesian product $X \times Y$ is also an affine variety [Hum75].

Definition 1.2.3. An **affine algebraic group** is an affine variety G together with two morphisms of affine varieties:

$$m\colon G \times G \to G$$
$$\iota\colon G \to G$$

such that ι has an inverse ι^{-1}, which is a morphism of affine varieties, and such that they satisfy the axioms for the multiplication and the inverse in a group.

Some examples of algebraic groups are the following:

Examples 1.2.4. (1) The set $\mathbb{F}^\times$ of units of the field can be seen as an algebraic variety since there is a bijection $\mathbb{F}^\times \to \mathcal{V}(xy-1)$ given by $x \mapsto (x, x^{-1})$. The multiplicative group $G_m = \mathbb{F}^\times$, with product and inverse given by $m(x,y) = xy$ and $\iota(x) = x^{-1}$ for all $x, y \in G_m$, is an algebraic group. Indeed, the corresponding multiplication morphism in $V = \mathcal{V}(xy-1)$, denoted by $m\colon V \times V \to V$, is given by $m((x_1,y_1),(x_2,y_2)) = (x_1x_2, y_2y_1)$.

(2) The general linear group $\mathrm{GL}(n,\mathbb{F}) = \{A \in \mathcal{M}_n(\mathbb{F}) \mid A \text{ is invertible}\}$, with product and inverse given by $m(A,B) = AB$ and $\iota(A) = A^{-1}$ for all $A, B \in \mathrm{GL}(n,\mathbb{F})$, can be embedded on the algebraic variety $\mathcal{V}(\det(x_{i,j})t - 1) \subseteq \mathbb{F}^{n\times n+1}$ via the map

$$A = (a_{i,j}) \mapsto (a_{1,1}, ..., a_{1,n}, ..., a_{n,1}, ..., a_{n,n}, \det(A)^{-1}).$$

This is an algebraic group [Hum75, 7.1]. Moreover, the group G_m is the particular case with $n = 1$.

(3) The diagonal group $\mathrm{D}(n,\mathbb{F})$ is the subgroup of $\mathrm{GL}(n,\mathbb{F})$ consisting on invertible diagonal matrices. This is an algebraic group which is isomorphic to the direct product of n copies of G_m. [Hum75, 7.1].

(4) The group $\mu_n = \mathcal{V}(x^n - 1) \subseteq \mathbb{F}$ with multiplication given by $m(x,y) = xy$ and inverse given by $\iota(x) = x^{-1}$ is also an algebraic group. Note that if $x^n - 1$ has n different roots in $\mathbb{F}$, then μ_n is the cyclic group of order n.

(5) Given two affine algebraic groups G and H where we denote its multiplication by juxtaposition and we denote by 1 the unit of G, its direct product $G \times H$ is an affine algebraic group. Moreover, if we have a morphism $\varphi\colon G \times H \to H$ of algebraic groups satisfying:

$$\varphi(g_1, \varphi(g_2, h_1)) = \varphi(g_1 g_2, h) \text{ for all } g_1, g_2 \in G,\ h \in H,$$
$$\varphi(1, h) = h \text{ for all } h \in H,$$

and satisfying that for all $g \in G$ the map $h \mapsto \varphi(g, h)$ is an automorphism of H, then we can define a product on $H \times G$ by $(h_1, g_1)(h_2, g_2) = (h_1\varphi(g_1, h_2), g_1 g_2)$. This is an algebraic group called the semidirect product and denoted by $H \rtimes_\varphi G$ (or simply $H \rtimes G$ if φ is not known or it is known from the context) [Hum75, 8.4].

(6) For an $\mathbb{F}$-linear Ω-algebra $(\mathcal{A}, \Omega)$ over a finite dimensional vector space, its group of automorphisms $\mathrm{Aut}(\mathcal{A}, \Omega)$ is a subgroup of $\mathrm{GL}(n, \mathbb{F})$ where n is the dimension of $\mathcal{A}$. The equations come from the identities (1.1.1), and the fact that any multilinear map can be written as a homogeneous polynomial.

Definition 1.2.5. A **torus** is an affine algebraic group isomorphic to $\mathrm{D}(n, \mathbb{F})$ for some n. A **quasitorus** (or equivalently d-**group** [Hum75, 6.2]) is an affine algebraic group which is isomorphic to the direct product of a torus and a finite abelian algebraic group. A **diagonalizable group** is an algebraic group which is isomorphic to a closed subgroup of $\mathrm{D}(n, \mathbb{F})$ for some n.

1.3 Affine group schemes

We will assume that the most basic definitions for categories, as well as Yoneda's lemma, are known. For a reference, the reader can refer to [Mac98]. We will denote by Set, Grp and $\mathrm{Alg}_\mathbb{F}$ the categories of sets, groups and unital, commutative and associative algebras over the field $\mathbb{F}$.

Affine algebraic groups sometimes don't grasp all the meaning of the equations they are dealing with. For instance, if we work in $\mathbb{Q}$, since the algebraic variety $\mathcal{V}(x^3 - 1) \subseteq \mathbb{Q}$ just contains the unit element, μ_3 is just the trivial group. In order to overcome this circumstance, we can approach this problem from a functorial point of view. Given a set of polynomials $S \subseteq \mathbb{F}[x_1, ..., x_n]$, we can define the functor $\mathcal{V}(S)\colon \mathrm{Alg}_\mathbb{F} \to \mathrm{Set}$ by sending any $\mathbb{F}$-algebra R to $\mathcal{V}(S) = \{x \in R^n \mid f(x) = 0 \text{ for all } f \in S\}$, and which sends any morphism of $\mathbb{F}$-algebras $\varphi\colon R \to T$, to the morphism $\mathcal{V}(S)(\varphi)$ which acts componentwise. In this case, $\mathcal{V}(x^3 - 1)$ is not the same as $\mathcal{V}(x - 1)$.

Moreover, if we denote by $\mathcal{I}$, the ideal generated by the elements of S and denote $\mathcal{A} = \mathbb{F}[x_1, ..., x_n]/\mathcal{I}$, there is a natural isomorphism θ between $\mathcal{V}(S)$ and $\mathrm{Hom}_{\mathrm{Alg}_{\mathbb{F}}}(\mathcal{A}, \cdot)$, defined by $\theta_R(r_1, ..., r_s)(\varphi(x_1, ..., x_n) + \mathcal{I}) \mapsto \varphi(r_1, ..., r_n)$ for any $\mathbb{F}$-algebra R. Using this functorial approach we define the concept of affine group schemes. A good reference for this topic is [Wat79]. Here we will introduce the definition and the most useful results and will refer to this last reference for the proofs.

Definition 1.3.1. An **affine group scheme** over $\mathbb{F}$ is a functor $\mathbf{G}\colon \mathrm{Alg}_{\mathbb{F}} \to \mathrm{Grp}$ such that its composition with the forgetful functor $\mathbf{Forgetful}\colon \mathrm{Grp} \to \mathrm{Set}$ is representable (i.e. the functor $\mathbf{Forgetful} \circ \mathbf{G}$ is naturally isomorphic to the functor $\mathrm{Hom}_{\mathrm{Alg}_{\mathbb{F}}}(\mathcal{A}, \cdot)$ for some algebra $\mathcal{A}$). The morphisms of affine group schemes are just the natural transformations.

Remark 1.3.2. If a representable functor $\mathbf{F}$ from $\mathrm{Alg}_{\mathbb{F}}$ to Set is isomorphic to $\mathrm{Hom}(\mathcal{A}, \cdot)$, we say that $\mathbf{F}$ is represented by $\mathcal{A}$ or that $\mathcal{A}$ represents $\mathbf{F}$. In the context of the previous definition we can say that $\mathbf{G}$ is represented by $\mathcal{A}$ and write $\mathcal{A} = \mathbb{F}[\mathbf{G}]$, which makes sense because as a consequence of Yoneda's lemma, $\mathbf{G}$ is represented by a unique algebra up to isomorphism.

The main affine group schemes we will need to use are given in the following examples:

Examples 1.3.3. (1) Let G be an abelian group, we will denote by

$$G^D\colon \mathrm{Alg}_{\mathbb{F}} \to \mathrm{Grp}$$

the functor given by $G^D(R) = \mathrm{Hom}_{\mathrm{Grp}}(G, R^\times)$. Moreover for any morphism of algebras $f\colon R \to S$, $G^D(f)(\varphi) = f \circ \varphi$ for all $\varphi \in G^D(R)$. This is an affine group scheme represented by the group algebra $\mathbb{F}G$ [Wat79].

(2) Let V be a finite dimensional vector space. Then, we define $\mathbf{GL}(V)$ as the affine group scheme, such that for any algebra R, its group of R-points $\mathbf{GL}(V)(R)$ consists on the invertible R-linear transformations of $V \otimes_{\mathbb{F}} R$. Given morphism of algebras $f\colon R \to S$, in order to define $\mathbf{GL}(V)(f)$, we notice that S is an R-module by defining $rs = f(r)s$ for any $r \in R$, $s \in S$. Moreover, the morphism $\theta\colon (V \otimes_{\mathbb{F}} R) \otimes_R S \to V \otimes_{\mathbb{F}} S$ given by $\theta(v \otimes r \otimes s) = v \otimes f(r)s$ for all $v \in V$, $r \in R$ and $s \in S$ is an isomorphism with inverse induced by $\theta^{-1}(v \otimes s) = v \otimes 1 \otimes s$ for all $v \in V$, $s \in S$. $\mathbf{GL}(V)(f)$ sends $\varphi \in \mathbf{GL}(V)(R)$ to $\theta \circ (\varphi \otimes \mathrm{id}) \circ \theta^{-1} \in \mathbf{GL}(V)(S)$.

(3) Let m be a natural number. We denote by $\boldsymbol{\mu}_m$ the affine group scheme whose group of R-points is $\boldsymbol{\mu}_m(R) = \{r \in R^\times \mid r^m = 1\}$. Its representing algebra is $\mathbb{F}[x]/(x^m - 1)$.

(4) Let $(\mathcal{A}, \Omega)$ be a finite dimensional $\mathbb{F}$-linear Ω-algebra. Given a unital, commutative and associative algebra R, we denote by $(\mathcal{A}_R, \Omega_R)$ the Ω algebra in which $\mathcal{A}_R = \mathcal{A} \otimes_{\mathbb{F}} R$ and for any $n \in \mathbb{N}$, an element $\omega \in \Omega(n)$ defines a multilinear operator ω_R by $\omega(a_1 \otimes r_1, ..., a_n \otimes r_n) = \omega(a_1, ..., a_n) \otimes r_1 \cdots r_n$ for all $a_1, ..., a_n \in \mathcal{A}$ and $r_1, ..., r_n \in R$. We define $\mathbf{Aut}(\mathcal{A}, \Omega)$ as the affine group scheme whose group R-points is $\mathbf{Aut}(\mathcal{A}, \Omega)(R) = \mathrm{Aut}_R(\mathcal{A}_R, \Omega_R)$ (here $\mathrm{Aut}_R(\mathcal{A}_R, \Omega_R)$ is the subgroup of R-linear automorphisms of $\mathrm{Aut}(\mathcal{A}_R, \Omega_R)$). Its representing algebra is

$$\mathbb{F}[x_{1,1}, ..., x_{1,n}, ..., x_{n,1}, ..., x_{n,n}, t]/\mathcal{I}$$

where $\mathcal{I}$ is generated the polynomials which are 0 due to the identities (1.1.1) and $\det(x_{i,j})t - 1$.

(5) Given a vector space V and a quadratic form $q \colon V \to \mathbb{F}$, we define the **orthogonal group scheme**, denoted as $\mathbf{O}(V, q)$, the affine group scheme whose group of R-points is $\mathbf{O}(V, q)(R) = \{\alpha \in \mathbf{GL}(V)(R) \mid q_R(\alpha(v)) = v \ \forall v \in V \otimes_{\mathbb{F}} R\}$. Notice that if we are working over a field of characteristic different from 2, q is determined by its polar form $q(x, y) = q(x + y) - q(x) - q(y)$. If the characteristic of the field is 2, we could also define q_R by $q_R(v_1 \otimes r_1 + ... + v_k \otimes r_k) = q(v_1)r_1^2 + ... + q(v_k)r_k^2 + \sum_{1 \le i < j \le k} q(v_i, v_j)r_i r_j$. If the characteristic is not 2, we define the affine group scheme $\mathbf{O}^+(V, q)$ as the affine group scheme whose group of R-points is $\mathbf{O}^+(V, q)(R) = \{\alpha \in \mathbf{O}(V, q)(R) \mid \det(\alpha) = 1\}$ and we call it the **special orthogonal group scheme**.

In case we have a bilinear form b, we define the affine group schemes $\mathbf{O}(V, b)$ and $\mathbf{O}^+(V, b)$ in the analogous way.

(6) There is the trivial affine group scheme $\mathbf{1}$ whose group of R-points is the trivial group. Its representing algebra is just the field $\mathbb{F}$.

(7) Let G be a finite group. Consider the algebra $\mathrm{Map}(G, \mathbb{F})$ of maps from G to $\mathbb{F}$. For $g \in G$ let e_g be the map defined as $e_g(g) = 1$ and $e_g(h) = 0$ for all $h \neq g$. With this notation, $\mathrm{Map}(G, \mathbb{F}) = \bigoplus_{g \in G} \mathbb{F}e_g$. This is a Hopf algebra with the coproduct given by $\Delta(e_g) = \sum_{hf=g} e_h \otimes e_f$, the counit given by $\epsilon(e_g) = e_{g^{-1}}$. Then, we define the **constant group scheme** G as the affine group scheme represented by this Hopf algebra (see 1.3.6 below).

Definition 1.3.4. We say that an affine group scheme $\mathbf{G}$ is **diagonalizable** if there is an abelian group G such that $\mathbf{G}$ is isomorphic to G^D.

Remark 1.3.5. Notice that for an $\mathbb{F}$-linear Ω-algebra $\mathcal{A}$, there is a monomorphism $i\colon \mathbf{Aut}(\mathcal{A},\Omega) \to \mathbf{GL}(\mathcal{A})$ induced by the usual inclusions

$$i_R\colon \mathbf{Aut}(\mathcal{A},\Omega)(R) \to \mathbf{GL}(\mathcal{A})(R)$$

for every R in $\mathrm{Alg}_{\mathbb{F}}$.

Definition 1.3.6. A (commutative) **Hopf algebra** is a unital commutative and associative algebra $(\mathcal{A}, m)$ with two homomorphisms of unital algebras $\Delta\colon \mathcal{A} \to \mathcal{A} \otimes_{\mathbb{F}} \mathcal{A}$ and $\epsilon\colon \mathcal{A} \to \mathbb{F}$ called **comultiplication** and **counit** respectively and a linear map $S\colon \mathcal{A} \to \mathcal{A}$ called **antipode** such that if $\mu\colon \mathbb{F} \to \mathcal{A}$ is the unit map the following identities hold:

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta \text{ (coassociativity)}$$
$$(\epsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \epsilon) \circ \Delta \text{ (counit axiom)}$$
$$m \circ (S \otimes \mathrm{id}) \circ \Delta = \mu \circ \epsilon = m \circ (\mathrm{id} \otimes S) \circ \Delta \text{ (antipode axiom)}$$

Any affine group scheme $\mathbf{G}$ has three natural transformations denoted by $\mathrm{mult}\colon \mathbf{G} \times \mathbf{G} \to \mathbf{G}$, $\mathrm{inv}\colon \mathbf{G} \to \mathbf{G}$ and $\mathrm{unit}\colon \mathbf{1} \to \mathbf{G}$ corresponding to the multiplication, the inverse morphism and the unit morphism. If $\mathcal{A}$ is the representing algebra of $\mathbf{G}$, $\mathcal{A} \otimes_{\mathbb{F}} \mathcal{A}$ is the representing algebra of $\mathbf{G} \times \mathbf{G}$. Applying Yoneda's lemma, the previous natural transformations give rise to three morphisms denoted $\Delta\colon \mathcal{A} \to \mathcal{A} \otimes_{\mathbb{F}} \mathcal{A}$, $S\colon \mathcal{A} \to \mathcal{A}$ and $\epsilon\colon \mathcal{A} \to \mathbb{F}$. Then, $\mathcal{A}$ with Δ, ϵ and S as comultiplication, counit and antipode respectively is a Hopf algebra. Moreover, given a commutative Hopf algebra we can induce an affine group scheme using Yoneda's lemma [Wat79, 1.4]. Since the inverse and the unit maps in a group are defined from the multiplication, if we have the comultiplication, the counit and the antipode are uniquely determined.

Example 1.3.7. For an abelian group G, the affine group scheme G^D is represented by the group algebra $\mathbb{F}G$. The comultiplication is induced by $\Delta(g) = g \otimes g$, $\epsilon(g) = 1$ for all $g \in G$.

Definition 1.3.8. Let $\mathbf{G}$ be an affine group scheme. Denote by $\mathbb{F}[\tau]$ the 2-dimensional algebra satisfying $\tau^2 = 0$. Consider the morphism of algebras $\pi\colon \mathbb{F}[\tau] \to \mathbb{F}$ sending τ to 0. We denote the kernel of $\mathbf{G}(\pi)$ by $\mathrm{Lie}(\mathbf{G})$ and call it the Tangent Lie algebra of $\mathbf{G}$.

The name of Lie algebra is due to the fact that it can be endowed with a Lie bracket. In order to define the bracket, we need to identify $\mathbf{G}(\mathbb{F}[\tau])$ with $\mathrm{Hom}_{\mathrm{Alg}_{\mathbb{F}}}(\mathbb{F}[\mathbf{G}], \mathbb{F}[\tau])$. Any $x \in \mathrm{Lie}(\mathbf{G})$ is of the form $x = \epsilon + d\tau$ for some $d\colon \mathbb{F}[\mathbf{G}] \to \mathbb{F}$. For any $x_1 = \epsilon + d_1\tau$ and $x_2 = \epsilon + d_2\tau$, we define their bracket by $[x_1, x_2] = \epsilon + (d_1 \otimes d_2 - d_2 \otimes d_1) \circ \Delta\tau$ (see [Wat79, 12.2]).

Given a morphism $\theta\colon \mathbf{G} \to \mathbf{H}$ between two affine group schemes, we denote by $d\theta\colon \mathrm{Lie}(\mathbf{G}) \to \mathrm{Lie}(\mathbf{H})$ its restriction to the tangent Lie algebras.

Remark 1.3.9. The Lie algebra of an affine group scheme can be endowed with a canonical structure of Lie algebra if the characteristic of $\mathbb{F}$ is 0 as above, and of a restricted Lie algebra if the characteristic is $p > 0$ (see for instance [EK13, A.5]).

Example 1.3.10. Let $\mathcal{A}$ be an Ω-algebra and $\mathbf{G} = \mathbf{Aut}(\mathcal{A}, \Omega)$. An element $\varphi + d\tau \in \mathbf{G}(\mathbb{F}[\tau])$ with $\varphi, d \in \mathrm{End}_{\mathbb{F}}(\mathcal{A})$ is in the kernel of $\mathbf{G}(\pi)$ if and only if $\varphi = \mathrm{id}$. Let $d \in \mathrm{End}_{\mathbb{F}}(\mathcal{A})$. $\mathrm{id} + d\tau \in \mathbf{G}(\mathbb{F}[\tau])$ if and only if for every $n \in \mathbb{N}$, $\omega \in \Omega(n)$ and $a_1, ..., a_n \in \mathcal{A}$ $(\mathrm{id} + d\tau)\omega(a_1 \otimes 1, ..., a_n \otimes 1) = \omega(\mathrm{id}(a_1) + d(a_1)\tau, ..., \mathrm{id}(a_n) + d(a_n)\tau)$. Since $\tau^2 = 0$ this happens if and only if $n \geq 1$ and $d(\omega(a_1, ..., a_n)) = \omega(d(a_1), a_2, ..., a_n) + \omega(a_1, d(a_2), ..., a_n) + \omega(a_1, a_2, ..., d(a_n))$ or $n = 0$ and $\varphi \circ \omega = 0$. Hence $\mathrm{Lie}(\mathbf{Aut}(\mathcal{A}, \Omega)) = \mathrm{Der}(\mathcal{A}, \Omega)$.

1.4 Gradings: Definitions

In this section we are going to rewrite the theory of gradings in [EK13] in terms of Ω-algebras. Most of the proofs are equivalent and so we are going to refer to the monograph.

Definition 1.4.1. Let G be a group. A G-**grading** Γ on an $\mathbb{F}$-linear Ω-algebra $\mathcal{A}$ is a vector space decomposition:

$$\Gamma \colon \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$$

satisfying that for every $\omega \in \Omega(0)$, $\omega(1) \in \mathcal{A}_e$ and that for every $\omega \in \Omega(n)$, $g_1, ..., g_n \in G$, and every $a_i \in \mathcal{A}_{g_i}$ for $i \in \{1, ..., n\}$:

$$\omega(a_1, ..., a_n) \in \mathcal{A}_{g_1 \cdots g_n}.$$

We call the subspaces $\mathcal{A}_g$ the **homogeneous components of the grading**.

Given a group element $g \in G$, we say that an element $0 \neq a \in \mathcal{A}$ is **homogeneous** of degree g (and we write $\deg(a) = g$) if $a \in \mathcal{A}_g$.

We define the **support** of the grading Γ as $\mathrm{Supp}(\Gamma) = \{g \in G \mid \mathcal{A}_g \neq 0\}$.

If we fix the grading we say that $\mathcal{A}$ is a G-**graded** Ω**-algebra**.

Remark 1.4.2. Given a G-grading

$$\Gamma \colon \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$$

on an $\mathbb{F}$-linear Ω-algebra $\mathcal{A}$ we can denote by Ω' the operator domain in which $\Omega'(n) = \Omega(n)$ for $n \neq 1$ and $\Omega'(1) = \Omega(1) \cup \{\pi_g \mid g \in G\}$. With this notation, the G-graded Ω-algebra $\mathcal{A}$ can be equivalently defined as the Ω'-algebra $\mathcal{A}$ where for every $g \in G$, π_g defines the 1-ary operator given by $\pi_g(a) = a_g$ where a_g is the projection of a on $\mathcal{A}_g$ with respect to the decomposition given by Γ.

Proposition 1.4.3. *Given an operator domain Ω and a group G, denote Ω' as in Remark 1.4.2. There is a correspondence between G-graded Ω-algebras and $\mathbb{F}$-linear Ω' algebras satisfying:*

(1) $\pi_g \circ \pi_g = \pi_g$ *for all* $g \in G$ *and* $\pi_g \circ \pi_h = 0$ *for all* $g \neq h \in G$.

(2) $\sum_{g\in G} \pi_g = \mathrm{id}$

(3) For all $\omega \in \Omega'(0)$ *the identity* $\pi_e \circ \omega = \omega$ *holds.*

(4) For all $n \geq 1$, $\omega \in \Omega'(n)$ *and* $g_1, ..., g_n \in G$, *the following diagram commutes:*

$$\begin{array}{ccccc} \mathcal{A} \times ... \times \mathcal{A} & \xrightarrow{\pi_{g_1} \times ... \times \pi_{g_n}} & & & \mathcal{A} \times ... \times \mathcal{A} \\ {\scriptstyle \pi_{g_1} \times ... \times \pi_{g_n}} \big\downarrow & & & & \big\downarrow {\scriptstyle \omega} \\ \mathcal{A} \times ... \times \mathcal{A} & \xrightarrow{\omega} & \mathcal{A} & \xrightarrow{\pi_{g_1 \cdots g_n}} & \mathcal{A}. \end{array}$$

Proof. Due to Remark 1.4.2, if we have a G-graded Ω-algebra, we can define an $\mathbb{F}$-linear Ω' algebra satisfying $(1), (2), (3)$ and (4).

If we have an Ω' algebra satisfying $(1), (2), (3)$ and (4), define $\mathcal{A}_g$ as the image of π_g. We need to prove that $\mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$ and that this direct sum is a G-grading.

In order to prove that $\mathcal{A}$ is the direct sum of the subspaces $\mathcal{A}_g$ for all $g \in G$, denote by $i_g \colon \mathcal{A}_g \to \mathcal{A}$ the inclusion and $p_g \colon \mathcal{A} \to \mathcal{A}_g$ the projection induced by π_g. If $a \in \mathcal{A}$, (2) implies

$$a = \sum_{g\in G} \pi_g(a).$$

Thus, $\mathcal{A} = \sum_{g\in G} \mathcal{A}_g$. Moreover, if $0 = \sum_{g\in G} a_g$ with $a_g \in \mathcal{A}_g$ for all $g \in G$, and if $b_g \in \mathcal{A}$ is such that $\pi_g(b_g) = a_g$, due to (1) we have that

$$0 = \pi_h \left(\sum_{g\in G} \pi_g(b_g) \right) = \sum_{g\in G} \pi_h \pi_g(b_g) = \pi_h(\pi_h(b_h)) = \pi_h(b_h) = a_h.$$

Hence the sum is a direct sum, i.e., $\mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$.

Now, (3) implies that for all $\omega \in \Omega(0)$, $\omega(1) \in \mathcal{A}_e$ and (4) implies that for all $n \geq 1$, $\omega \in \Omega(n)$, $g_1, .., g_n \in G$ and every $a_i \in \mathcal{A}_{g_i}$ for $i \in \{1, ..., n\}$, $\omega(a_1, ..., a_n) \in \mathcal{A}_{g_1,...,g_n}$. □

Definition 1.4.4. Given a G-grading $\Gamma\colon \mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$ on an Ω-algebra $\mathcal{A}$ and an H-grading $\Gamma'\colon \mathcal{B} = \bigoplus_{h\in H} \mathcal{B}_h$ on an Ω-algebra $\mathcal{B}$, we say that they are **equivalent** if there is an **equivalence of graded Ω-algebras**, i.e. an isomorphism $\varphi\colon \mathcal{A} \to \mathcal{B}$ such that there exist a bijection $\alpha\colon \operatorname{Supp}\Gamma \to \operatorname{Supp}\Gamma'$ such that for all $g \in \operatorname{Supp}\Gamma$, $\varphi(\mathcal{A}_g) = \mathcal{B}_{\alpha(g)}$. In this case we say that Γ and Γ' are **equivalent gradings**.

Definition 1.4.5. Given an operator domain Ω and a group G. Given two $\mathbb{F}$-linear Ω-algebras $\mathcal{A}$ and $\mathcal{B}$ with G-gradings $\Gamma\colon \mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$ and $\Gamma'\colon \mathcal{B} = \bigoplus_{g\in G} \mathcal{B}_g$, a **homomorphism of G-graded algebras** is a homomorphism $\varphi\colon \mathcal{A} \to \mathcal{B}$ of $\mathbb{F}$-linear Ω-algebras such that for every $g \in G$, $\varphi(\mathcal{A}_g) \subseteq \mathcal{B}_g$. We say that φ is an **isomorphism of G-graded algebras** if φ is an isomorphism of Ω-algebras and a homomorphism of G-graded algebras, in which case we say that $\mathcal{A}$ and $\mathcal{B}$ are isomorphic G-graded Ω-algebras.

Remark 1.4.6. Given an operator domain Ω and a group G. Denote Ω' as in Remark 1.4.2. Given two G-graded Ω-algebras $\mathcal{A}$ and $\mathcal{B}$, a homomorphism between the G-graded Ω-algebras $\mathcal{A}$ and $\mathcal{B}$ is the same as a homomorphism between the corresponding $\mathbb{F}$-linear Ω'-algebras (notice that $\varphi(\mathcal{A}_g) \subseteq \mathcal{B}_g$ is equivalent to $\varphi\circ\pi_g = \pi_g\circ\varphi$). The same thing happens with the isomorphisms.

Definition 1.4.7. Given two groups G and H, a G-grading $\Gamma\colon \mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$ on some Ω-algebra $\mathcal{A}$, and group homomorphism $\alpha\colon G \to H$, we can define the grading:

$$^{\alpha}\Gamma\colon \mathcal{A} = \bigoplus_{h\in H} \mathcal{A}_h$$

where for every $h \in H$, $\mathcal{A}_h = \bigoplus_{g\in G\colon \alpha(g)=h} \mathcal{A}_g$.

We say that Γ is **weakly isomorphic** to an H-grading $\Gamma'\colon \mathcal{A} = \bigoplus_{h\in H} \mathcal{A}_h$ if there is an isomorphism $\alpha\colon G \to H$ such that $^{\alpha}\Gamma$ is isomorphic to Γ'.

Definition 1.4.8. Given a G-grading $\Gamma\colon \mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$ of an Ω-algebra $\mathcal{A}$ and an element $h \in G$, the **shift of Γ by h** is the grading

$$\Gamma^{[h]}\colon \mathcal{A} = \bigoplus_{g\in G} \mathcal{A}'_g$$

of the vector space $\mathcal{A}$, where $\mathcal{A}'_{hg} = \mathcal{A}_g$ for all $h \in G$. A **graded homomorphism** between two graded vector spaces $\mathcal{A}$ y $\mathcal{B}$ with gradings $\Gamma_{\mathcal{A}}\colon \mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$ and $\Gamma_{\mathcal{B}}\colon \mathcal{B} = \bigoplus_{g\in G} \mathcal{B}_g$ is a homomorphism $\varphi\colon \mathcal{A} \to \mathcal{B}$ such that there is an element h for which $\varphi(\mathcal{A}_g) \subseteq \mathcal{B}_{hg}$ for all $g \in G$. In this case, we say that it has degree h. We denote the space of graded homomorphisms of degree h as $\operatorname{Hom}_{\mathbb{F}}(\mathcal{A}, \mathcal{B})_h$ and we denote:

$$\mathrm{Hom}^{\mathrm{gr}}_{\mathbb{F}}(\mathcal{A},\mathcal{B}) = \bigoplus_{g\in G} \mathrm{Hom}_{\mathbb{F}}(\mathcal{A},\mathcal{B})_h.$$

We define **graded homomorphism** between two graded Ω-algebras as a homomorphism between the Ω-algebras which is also a graded homomorphism between the two graded vector spaces.

Remark 1.4.9. The kernel of a graded homomorphism $\varphi\colon \mathcal{A}\to\mathcal{B}$ of G-graded Ω-algebras is a graded subspace of $\mathcal{A}$. Indeed, if $x=\sum_{g\in G} x_g$ with $g\in G$, is such that $\varphi(x)=0$, since there is some $h\in G$ such that $\varphi(x_g)\in\mathcal{B}_{hg}$ for all $g\in G$, we have that $\varphi(x_g)=0$ and so $x_g\in\ker\varphi$ for all $g\in G$. Similarly, we can prove that the image is a graded subspace.

Definition 1.4.10. Given an operator domain Ω, an Ω-algebra $\mathcal{A}$, two groups G and H, a G-grading $\Gamma\colon \mathcal{A}=\bigoplus_{g\in G}\mathcal{A}_g$ and an H-grading $\Gamma\colon \mathcal{A}=\bigoplus_{h\in H}\mathcal{A}'_h$, we say that Γ is a **refinement** of Γ' (or that Γ' is a **coarsening** of Γ) if for any $g\in G$ there is $h\in H$ such that $\mathcal{A}_g\subseteq\mathcal{A}'_h$. If the inclusion is strict for some $g\in G$ we say that the refinement (or coarsening) is **proper**. If a grading has no proper refinement we say that it is **fine**.

We are going to finish the section by introducing some groups and group schemes related to a grading.

Let $(\mathcal{A},\Omega)$ be an Ω-algebra. Let G be a group and $\Gamma\colon\mathcal{A}=\bigoplus_{g\in G}\mathcal{A}_g$ a grading on $(\mathcal{A},\Omega)$. The **diagonal group scheme Diag**(Γ) is the group scheme whose group of R-points for an $\mathbb{F}$-algebra R is:

$$\mathbf{Diag}(\Gamma)(R)=\{\varphi\in\mathbf{Aut}(\mathcal{A},\Omega)(R)\mid \varphi_{|\mathcal{A}_g\otimes_{\mathbb{F}}R}\in R^{\times}\mathrm{id}_{\mathcal{A}_g\otimes_{\mathbb{F}}R}\,\forall g\in G\}.$$

The **stabilizer group scheme Stab**(Γ) of Γ is the group scheme of automorphisms of the Ω' algebra defined as in Remark 1.4.2. We denote by $\mathrm{Stab}(\Gamma)=\mathbf{Stab}(\Gamma)(\mathbb{F})$ and we call it the **stabilizer group** of Γ. We denote by $\mathrm{Aut}(\Gamma)$ the group of self equivalences of Γ. Finally, the quotient $\mathrm{Aut}(\Gamma)/\mathrm{Stab}(\Gamma)$ is called the **Weyl group** and denoted by $W(\Gamma)$.

1.5 Gradings: derivations, automorphisms and affine group schemes

In this last section we are going to assume that Ω is an operator domain and $\mathcal{A}$ a finite dimensional $\mathbb{F}$-linear Ω-algebra. If Γ is a grading by an abelian group G, by taking a smaller group we can assume that G is generated by the support of Γ. Hence, throughout this section, G will denote a finitely generated abelian group.

1.5.1 Derivations

Let $\mathbb{F}$ be a field of characteristic 0. Let $\Gamma\colon \mathcal{A} = \bigoplus_{i\in\mathbb{Z}} \mathcal{A}_i$ be a $\mathbb{Z}$-grading of $\mathcal{A}$. We can define a linear map d_Γ induced by:

$$d_\Gamma(x) = ix \text{ for all } x \in \mathcal{A}_i, \text{ for all } i \in \mathbb{Z}. \tag{1.5.1}$$

Lemma 1.5.1. *With the previous notation, d_Γ is a derivation of the Ω-algebra $\mathcal{A}$. Moreover, any diagonalizable derivation of the Ω-algebra $\mathcal{A}$ with integral eigenvalues induces a $\mathbb{Z}$-grading $\Gamma_d\colon \mathcal{A} = \bigoplus_{i\in\mathbb{Z}} \mathcal{A}_i$ on $\mathcal{A}$ given by $\mathcal{A}_i = \{x \in \mathcal{A} \mid d(x) = ix\}$.*

Proof. Let d_Γ be the linear map defined previously. If $\omega \in \Omega(0)$, $\omega(\lambda) \in \mathcal{A}_0$ for all $\lambda \in \mathbb{F}$. Hence, $d_\Gamma \circ \omega(\lambda) = 0$ implying $d_\Gamma \circ \omega = 0$. If $n \geq 1$ and $\omega \in \Omega(n)$, take $i_1, .., i_n \in \mathbb{Z}$ and $a_k \in \mathcal{A}_{i_k}$ for each $k \in \{1, ..., n\}$. Then, $\omega(a_1, ..., a_n) \in \mathcal{A}_{i_1+...+i_n}$. Hence:

$$\begin{aligned} d_\Gamma(\omega(a_1, ..., a_n)) &= (i_1 + ... + i_n)\omega(a_1, ..., a_n) \\ &= \omega(i_1 a_1, a_2, ..., a_n) + \omega(a_1, i_2 a_2, ..., a_n)... + \omega(a_1, ..., i_n a_n) \\ &= \omega(d_\Gamma(a_1), a_2, ..., a_n) + \omega(a_1, d_\Gamma(a_2), ..., a_n) + \\ &\quad ... + \omega(a_1, a_2, ..., d_\Gamma(a_n)) \end{aligned}$$

which implies that the identity (1.1.2) is satisfied by the elements of a basis and this implies that it is satisfied for all elements in $\mathcal{A}$.

Conversely, if d is a derivation as in the statement of the proposition and denote $\mathcal{A}_i = \{x \in \mathcal{A} \mid d(x) = ix\}$, we will show that Γ is a $\mathbb{Z}$-grading of the Ω-algebra. If $\omega \in \Omega(0)$, the identity $d(\omega(1)) = 0$ implies that $\omega(1) \in \mathcal{A}_0$. For $n \geq 1$, $\omega \in \Omega(n)$, $i_1, .., i_n \in \mathbb{Z}$ and $a_k \in \mathcal{A}_{i_k}$ for each $k \in \{1, ..., n\}$, taking into account that $d(\omega(a_1, ..., a_n)) = \omega(d(a_1), a_2, ..., a_n) + \omega(a_1, d(a_2), ..., a_n) + ... + \omega(a_1, a_2..., d(a_n))$, and arguing as before, we can show that $d(\omega(a_1, ..., a_n)) = (i_1 + ... + i_n)\omega(a_1, ..., a_n)$ which implies that $\omega(a_1, ..., a_n) \in \mathcal{A}_{i_1+...+i_n}$. □

Remark 1.5.2. Notice that the proof of Lemma 1.5.1 doesn't require $\mathcal{A}$ to be finite dimensional.

In [Smi97] it is is shown that any $\mathbb{Z}$-grading $\Gamma\colon \mathcal{A} = \bigoplus_{k=-n}^{n} \mathcal{A}_i$ on a simple associative algebra, not necessarily finite dimensional, arises from a Peirce decomposition with respect to a complete system of idempotents $e_1, ..., e_n$ in the following way:

$$\mathcal{A}_k = \bigoplus_{i=1}^{n} e_i \mathcal{A} e_{k-i}$$

for $k \in -n, ..., n$. The previous lemma allows us to show in a different way a finite dimensional version of this result.

Proposition 1.5.3. *Let $\mathcal{A}$ be a simple associative algebra over an algebraically closed field $\mathbb{F}$ of characteristic 0. Then $\mathcal{A}$ is isomorphic to $\mathcal{M}_n(\mathbb{F})$ and up to isomorphism any $\mathbb{Z}$-grading, is given by a n-uple $(\lambda_1, ...\lambda_n) \in \mathbb{Z}^n$ by setting $\deg(E_{i,j}) = \lambda_i - \lambda_j$.*

Proof. The fact that $\mathcal{A}$ is isomorphic to $\mathcal{M}_n(\mathbb{F})$ is just [HER68, Theorem 4.1.3]. Let $\Gamma\colon \mathcal{A} = \bigoplus_{i\in\mathbb{Z}} \mathcal{A}_i$ be a $\mathbb{Z}$-grading on $\mathcal{A}$. Denote d_Γ the derivation defined as in (1.5.1). Due to the fact that every derivation on a simple finite dimensional associative algebra is inner, there is a matrix $a \in \mathcal{A}$ such that $d(x) = \mathrm{ad}_a(x) = ax - xa$.

The matrix a decomposes uniquely into $a = s + n$ where s is diagonalizable, n is nilpotent and $[s, n] = sn - ns = 0$ (Jordan decomposition). As it is shown in the proof of [Eld13, 2.6(i)], $\mathrm{ad}_a = \mathrm{ad}_s + \mathrm{ad}_n$ is the Jordan decomposition of ad_a. Since ad_a is diagonalizable, it is also its Jordan decomposition, and so, $\mathrm{ad}_n = 0$, implying that n is in the center of $\mathcal{A}$. Therefore, $n = 0$ due to the fact that it is nilpotent. Thus, a is diagonalizable.

Since a is diagonalizable, applying a suitable automorphism of $\mathcal{A}$, we can assume that a is the diagonal matrix $\mathrm{diag}(\lambda_1, ..., \lambda_n)$. Moreover, since $\mathrm{ad}_a = \mathrm{ad}_{a-\lambda_1 \mathrm{id}}$, we can assume that $\lambda_1 = 0$. Hence, the elements $E_{i,j}$ are a basis of $\mathcal{A}$ such that $d_\Gamma(E_{i,j}) = \mathrm{ad}_a(E_{i,j}) = (\lambda_i - \lambda_j)E_{i,j}$. Since the eigenvalues of d_Γ are integers and $\lambda_1 = 0$, then, each λ_i for $\imath \in \{1, ..., n\}$ is an integer and due to Lemma 1.5.1 we get the result.

□

1.5.2 Automorphisms

Let $\mathbb{F}$ be an algebraically closed field of characteristic 0. Let G be a finitely generated abelian groups and $\widehat{G} = \mathrm{Hom}_{\mathrm{Grp}}(G, \mathbb{F}^\times)$ its group of characters. Let $\mathcal{A}$ be a finite dimensional $\mathbb{F}$-linear Ω-algebra and $\Gamma\colon \mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$ a G-grading. We can define a homomorphism $\eta_\Gamma\colon \widehat{G} \to \mathrm{Aut}(\mathcal{A}, \Omega)$ by:

$$\eta_\Gamma(\chi)(x) = \chi(g)x \quad \text{for all } x \in \mathcal{A}_g,\ g \in G \text{ and } \chi \in \widehat{G}. \tag{1.5.2}$$

Proposition 1.5.4. *Given an operator domain Ω, an algebraically closed field $\mathbb{F}$ of characteristic 0, a finitely generated abelian group G and a finite dimensional $\mathbb{F}$-linear Ω-algebra $\mathcal{A}$, the G-gradings on $\mathcal{A}$ are in one-to-one correspondence with the homomorphisms of algebraic groups $\widehat{G} \to \mathrm{Aut}(\mathcal{A}, \Omega)$. Two G-gradings are isomorphic if and only if the corresponding homomorphisms are conjugate by an element of $\mathrm{Aut}(\mathcal{A}, \Omega)$. The weak isomorphism* *classes of gradings on $\mathcal{A}$ with the property that the support generates the grading group are in one-to-one correspondence with the conjugacy classes of quasitori in* $\mathrm{Aut}(\mathcal{A},\Omega)$*.*

Proof. This proof is similar to the proof of [EK13, Proposition 1.28.] □

Proposition 1.5.5. *With the previous notation, the equivalence classes of fine gradings on the Ω-algebra $\mathcal{A}$ are in one-to-one correspondence with conjugacy classes of maximal quasitori in* $\mathrm{Aut}(\mathcal{A},\Omega)$*.*

Proof. This proof is similar to the proof of [EK13, Proposition 1.32.] □

1.5.3 Automorphism group schemes

Now we will not assume any hypotheses on the field $\mathbb{F}$. Let G be a finitely generated abelian group. Let $\mathcal{A}$ be an $\mathbb{F}$-linear Ω-algebra. The group algebra $\mathbb{F}G$ has a structure of Hopf algebra as in Example (1.3.7) with coproduct, $\Delta(g) = g \otimes g$, antipode $S(g) = g^{-1}$ and counit $\epsilon(g) = 1$. In [EK13, 1.3.] it is explained that a G-grading $\Gamma\colon \mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$ on the vector space $\mathcal{A}$ is equivalent to an $\mathbb{F}G$-comodule given by the coaction $\rho_\Gamma\colon \mathcal{A} \to \mathcal{A} \otimes_{\mathbb{F}} \mathbb{F}G$:

$$\rho_\Gamma(a) = a \otimes g \quad \text{for all } a \in \mathcal{A}_g,\ g \in G.$$

Given a coaction $\rho\colon \mathcal{A} \to \mathcal{A} \otimes_{\mathbb{F}} \mathbb{F}G$ it is also shown that we can conversely define a G-grading on $\mathcal{A}$ by:

$$\mathcal{A}_g = \{a \in \mathcal{A} \mid \rho(a) = a \otimes g\} \quad \text{for all } g \in G.$$

In a similar way as in [EK13], the comodule ρ induces a G-grading on the Ω-algebra $\mathcal{A}$ if and only if ρ is a morphism of Ω-algebras. Hence, the coaction ρ_Γ induced by a grading Γ induces an automorphism $\varphi_\Gamma \in \mathbf{Aut}(\mathcal{A},\Omega)(\mathbb{F}G)$ satisfying:

$$\varphi_\Gamma(a \otimes 1) = \rho_\Gamma(a) \tag{1.5.3}$$

for all $a \in \mathcal{A}$. Moreover, using Yoneda's Lemma, a comodule $\rho\colon \mathcal{A} \to \mathcal{A} \otimes_{\mathbb{F}} \mathbb{F}G$ is equivalent to a representation of the diagonal group scheme G^D (i.e. a morphism $\theta\colon G^D \to \mathbf{GL}(\mathcal{A})$). This allows us to prove the following results:

Proposition 1.5.6. *The G-gradings on the Ω-algebra $\mathcal{A}$ are in one-to-one correspondence with the morphisms of affine group schemes $G^D \to \mathbf{Aut}(\mathcal{A},\Omega)$. Two G-gradings are isomorphic if and only if the corresponding morphisms are conjugate by an element of* $\mathrm{Aut}(\mathcal{A},\Omega)$*. The weak isomorphism classes of*

$\mathcal{A}$ with the property that the support generates the grading group are in one-to-one correspondence with the $\mathrm{Aut}(\mathcal{A}, \Omega)$*-orbits of diagonalizable subgroup schemes in* $\mathbf{Aut}(\mathcal{A}, \Omega)$.

Proof. The proof is similar to the proof of [EK13, Proposition 1.36.]. □

Proposition 1.5.7. *Let Ω and Ω' be two operator domains. Let $\mathcal{A}$ be an Ω algebra and $\mathcal{B}$ an Ω' algebra. Assume that we have a morphism $\theta\colon \mathbf{Aut}(\mathcal{A}, \Omega) \to \mathbf{Aut}(\mathcal{B}, \Omega')$. Then, for any abelian group G, we have a mapping $\Gamma \to \theta(\Gamma)$ from G-gradings on $(\mathcal{A}, \Omega)$ to G-gradings on $(\mathcal{B}, \Omega')$. If Γ and Γ' are isomorphic (respectively weakly isomorphic), then $\theta(\Gamma)$ is isomorphic (weakly isomorphic) to $\theta(\Gamma')$.*

Proof. The proof is equivalent to he proof of [EK13, Theorem 1.38]. □

Remark 1.5.8. With the notation of Proposition 1.5.7, if θ is an isomorphism, the correspondence $\Gamma \to \theta(\Gamma)$ is a one-to-one correspondence between G-gradings on $(\mathcal{A}, \Omega)$ and G-gradings on $(\mathcal{B}, \Omega')$.

Remark 1.5.9. The Hopf algebra $\mathbb{F}G$ satisfies that an element $x \in \mathbb{F}G$ is in the group G if and only if $\Delta(x) = x \otimes x$. Hence, every automorphism of the Hopf algebra $\mathbb{F}G$ induces an automorphism in G and conversely, an automorphism of G induces an automorphism of $\mathbb{F}G$ since $\mathbb{F}G$ is spanned by the elements of G (see [Wat79, 2.2]).

Chapter 2

First example: Cross Products.

This chapter is devoted to explaining the results in [DET20], i.e., give a classification of gradings on r-fold cross products, which is a good way to show how to use the results in the previous chapter. In this chapter $\mathbb{F}$ will always be an arbitrary field of characteristic different from 2 unless it is stated otherwise.

In Section 2.1, we give the definition of r-fold cross product. In Section 2.2, we give some relevant properties of composition algebras. In Section 2.3 we give a classification of r-fold cross products determining their possible bilinear forms (see Proposition 2.3.10). Section 2.4 is another theoretical interlude in order to introduce the Clifford algebra and some relevant groups for the chapter like the spin group. In Section 2.5 we give a description of the automorphism group scheme for each isomorphism class of r-fold cross product (see Corollary 2.5.2, Theorem 2.5.3, Theorem 2.5.8 and Theorem 2.5.12). Finally, in Section 2.6, we give a classification up to isomorphism of the gradings (see Theorems 2.6.1, 2.6.4 and 2.6.11).

2.1 Definition: r-fold cross products

The usual vector cross product $\times$ on $\mathbb{R}^3$ is an antisymmetric product given by the linearization of $e_i \times e_{i+1} = e_{i+2}$ for $i \in \{1, 2, 3\}$ taking the indices modulo 3 where $\{e_1, e_2, e_3\}$ is the standard basis of $\mathbb{R}^3$. However, if we denote by $\langle \cdot , \cdot \rangle$ the usual scalar product in $\mathbb{R}^3$ and by $\| \cdot \|$ the norm, this vector cross product is determined (up to sign) by the following two properties:

(1) For all $x, y \in \mathbb{R}^3$, $\langle x \times y, x\rangle = \langle x \times y, y\rangle = 0$.

(2) For all $x, y \in \mathbb{R}^3$, $\|x \times y\| = \|x\|\|y\| \sin(\theta)$ where θ is the angle between x and y.

We can notice that squaring the identity in (2) we get that $\langle x\times y, x\times y\rangle = \langle x,x\rangle\langle y,y\rangle(1-\cos^2\theta) = \langle x,x\rangle\langle y,y\rangle - \langle x,y\rangle^2$. Hence, (2) is equivalent to:

(2') For all $x, y \in \mathbb{R}^3$, $\langle x \times y, x \times y\rangle = \det \begin{pmatrix} \langle x,x\rangle & \langle x,y\rangle \\ \langle y,x\rangle & \langle y,y\rangle \end{pmatrix}$

Using these properties, a generalization of the vector cross product on a vector space with a nondegenerate symmetric bilinear form over a field of arbitrary characteristic different from 2 is given in [Eck43] and [BG67]. Since the nondegenerate bilinear form might not be uniquely determined by the vector cross product, we introduce a slightly different definition:

Definition 2.1.1 ([DET20])**.** Let V be a finite-dimensional vector space, of dimension n, over a field $\mathbb{F}$ of characteristic different from 2, and let $1 \leq r \leq n$ be a natural number. An **r-fold cross product X on V** is a multilinear map

$$X : V^r \to V$$

such that there is a nondegenerate symmetric bilinear form $b : V \times V \to \mathbb{F}$ satisfying that for all $v_1, ..., v_r \in V$:

$$b(X(v_1, ..., v_r), v_i) = 0 \quad \text{for all } i \in \{1, ..., n\}, \tag{2.1.1}$$
$$b(X(v_1, ..., v_r), X(v_1, ..., v_r)) = \det(b(v_i, v_j)). \tag{2.1.2}$$

We can say that (V, X) (or $(V, X, b) := (V, X, \omega_b)$ if b is fixed) is an **(r-fold) cross product**.

Remark 2.1.2. Notice that (V, X, b) can be seen as the Ω-algebra (V, X, ω_b) where ω_b is the 3-ary operator defined by $\omega_b(u, v, w) = b(u, v)w$ for all $u, v, w \in V$.

Remark 2.1.3. Note that if (V, X, b) is an r-fold cross product and $\alpha, \beta \in \mathbb{F}$ are nonzero scalars satisfying $\beta^{r-1} = \alpha^2$, then, equations (2.1.1) and (2.1.2) are satisfied for αX and βb. Therefore, $(V, \alpha X, \beta b)$ is an r-fold cross product.

Remark 2.1.4. Notice that using what is shown in Example 1.1.6 (d), it follows that $\mathrm{Aut}(V, X, b) = \mathrm{Aut}(V, X) \cap \mathrm{O}(V, b)$ or more generally, in terms of affine group schemes, $\mathbf{Aut}(V, X, b) = \mathbf{Aut}(V, X) \cap \mathbf{O}(V, b)$.

2.2 Interlude: Composition algebras

Composition algebras as well as the gradings on the octonion algebra will play an important role in this chapter. For that reason, we will add a little survey on composition algebras. The interested reader can refer to [ZSSS82] or [Eld20] for more information.

Recall that given a vector space V, a **quadratic form** $n\colon V \to \mathbb{F}$ is a map satisfying $n(\alpha x) = \alpha^2 n(x)$ for all $\alpha \in \mathbb{F}$ and $x \in V$, and that its polar form, $n\colon V \times V \to \mathbb{F}$ defined by $n(x, y) = n(x + y) - n(x) - n(y)$ (which we will also denote by n) is a bilinear form. We say that the quadratic form is **nonsingular** if its polar form is nondegenerate.

Definition 2.2.1. A **composition algebra** over $\mathbb{F}$ is a triple $(\mathcal{A}, \cdot\,, n)$ (or just $(\mathcal{A}, n)$ if the product is clear from the context) where $(\mathcal{A}, \cdot)$ is a nonassociative algebra and $n\colon \mathcal{A} \to \mathbb{F}$ is a nonsinglar quadratic form, called the **norm**, which is multiplicative, i.e. for every $x, y \in \mathcal{A}$, the following identity holds:

$$n(xy) = n(x)n(y).$$

Although there is not a complete classification of these algebras, there are some which are well known:

Definition 2.2.2. A **Hurwitz algebra** is a unital composition algebra. If $(\mathcal{A}, n)$ is a Hurwitz algebra, we call the map $t\colon \mathcal{A} \to \mathbb{F}$ defined by $t(a) = n(a, 1)$, its **trace form**.

Proposition 2.2.3. *If $(\mathcal{A}, n)$ is a Hurwitz algebra:*

(1) The map $-\colon \mathcal{A} \to \mathcal{A}$ defined by $\overline{a} = t(a)1 - a$ is an involution, where t is the trace form.

(2) The identity $n(x, z\overline{y}) = n(xy, z) = n(y, \overline{x}z)$ holds for every $x, y, z \in \mathcal{A}$.

*(3) Any $x \in \mathcal{A}$ satisfies the **Cayley-Hamilton equation**:*

$$x^2 - t(x)x + n(x)1 = 0.$$

(4) $\mathcal{A}$ is an alternative algebra, i.e. $x^2 y = x(xy)$ and $yx^2 = (yx)x$ for all $x, y \in \mathcal{A}$.

Proof. The proof is in [Eld20, Proposition 2.2.] □

Remark 2.2.4. Notice that for any $x \in \mathcal{C}$, $t(x) = 0$ is equivalent to $\overline{x} = -x$.

Remark 2.2.5. Using the Cayley-Hamilton equation it can be shown that $n(x) = x\overline{x}$ and that $n(x) = n(\overline{x})$.

Remark 2.2.6. The Cayley-Hamilton equation implies that for any $\varphi \in \mathrm{Aut}(\mathcal{C})$ and $x \in \mathcal{C}$, $n(\varphi(x)) = n(x)$. Moreover, linearizing, this implies that for any $x, y \in \mathcal{C}$, $n(\varphi(x), \varphi(y)) = n(x, y)$. Therefore, $\mathrm{Aut}(\mathcal{C}) = \mathrm{Aut}(\mathcal{C}, n)$. This argument is functorial, so in general we have $\mathbf{Aut}(\mathcal{C}) = \mathbf{Aut}(\mathcal{C}, n)$.

A way to find examples is to construct them inductively. In order to do so, we are going to introduce the Cayley-Dickson doubling process.

Definition 2.2.7. Let $\mathcal{A}$ be an algebra with unit 1, and an involution $-\colon \mathcal{A} \to \mathcal{A}$ satisfying that for any $a \in \mathcal{A}$, $a + \overline{a} \in \mathbb{F}1$ and $a\overline{a} \in \mathbb{F}1$. Given $\mu \in \mathbb{F}^\times$, we say that the algebra $\mathcal{B} = \mathcal{A} \oplus \mathcal{A}$ with product and involution given by:

$$
\begin{aligned}
&(a_1, a_2)(a_3, a_4) = (a_1 a_3 + \mu a_4 \overline{a}_2, \overline{a}_1 a_4 + a_3 a_2), \text{ and} \\
&\overline{(a_1, a_2)} = (\overline{a}_1, a_2)
\end{aligned}
$$

is an algebra with involution obtained by the **Cayley-Dickson doubling process** from $(\mathcal{A}, -)$ and μ and we denote it by $\mathfrak{CD}(\mathcal{A}, -, \mu)$ or just $\mathfrak{CD}(\mathcal{A}, \mu)$.

Remark 2.2.8. Sometimes we will just say that $\mathcal{B} = \mathcal{A} \oplus w\mathcal{A}$ with $w^2 = \mu$.

Let $(\mathbb{K}, -)$ be either $\mathbb{F} \oplus \mathbb{F}$ with the exchange involution or a separable quadratic extension of $\mathbb{F}$ where the involution is the only automorphism of order 2. An algebra with involution obtained by the Cayley-Dickson doubling process from $(\mathbb{K}, -)$ is called a **quaternion algebra** and an algebra with involution obtained by the Cayley-Dickson doubling process from a quaternion algebra is called an **octonion algebra** or a **Cayley algebra**.

Theorem 2.2.9 (Hurwitz)**.** *Any Hurwitz algebra over $\mathbb{F}$ is finite dimensional and isomorphic to one of the following:*

(1) $\mathbb{F}$.

(2) $\mathbb{F} \oplus \mathbb{F}$.

(3) A separable quadratic field over $\mathbb{F}$.

(4) A quaternion algebra.

(5) A Cayley algebra.

Moreover, $n\colon \mathcal{C} \to \mathbb{F}$ is given by the Cayley-Hamilton equation.

Proof. The proof is in [Sha66, Theorem 3.25] □

The Cayley algebras are interesting for the study of Lie groups and Lie algebras. Indeed we have the following result.

Proposition 2.2.10. *Let $\mathcal{C}$ be a Cayley algebra.* $\mathrm{Der}(\mathcal{C})$ *is a Lie algebra of type* $\mathfrak{g}_2$ *and* $\mathbf{Aut}(\mathcal{C})$ *is an affine group scheme of type* G_2*.*

Proof. See [SV00, Theorem 2.3.5, Proposition 2.4.5]. □

The **(split) Cayley algebra** $\mathcal{C}$ is the algebra obtained by taking $\mathbb{K} = \mathbb{F} \oplus w_1\mathbb{F}$ with $w_1^2 = 1$, $\mathbb{H} = \mathbb{K} \oplus w_2\mathbb{K}$ with $w_2^2 = 1$ and $\mathcal{C} = \mathbb{H} \oplus w_3\mathbb{H}$ with $w_3^2 = 1$.

The gradings by abelian groups on the split Cayley algebra have been classified over arbitrary algebraically closed fields in [Eld98] and in [EK12]. Given an abelian group G, we can define two kind of gradings on $\mathcal{C}$ up to isomorphism.

In order to introduce the first kind of gradings, we need to introduce the Cartan basis. We give a construction which is shown in full detail in [EK13, Chapter 4], where the reader should refer for the proof. To begin with we take $x = \frac{1}{2}(1 + w_1)$. Take $y \in \mathcal{C}$ such that $n(x, \overline{y}) = 1$. Denote $e_1 = xy$ and $e_2 = 1 - e_1 = \overline{e}_1$. Define the following subspaces of $\mathcal{C}$:

$$\mathcal{U} = \{x \in \mathcal{C} \mid e_1x = x = xe_2,\ e_2x = 0 = xe_1\},$$
$$\mathcal{V} = \{x \in \mathcal{C} \mid e_2x = x = xe_1,\ e_1x = 0 = xe_2\}.$$

We can find a basis $\{u_1, u_2, u_3\}$ of $\mathcal{U}$ and a basis $\{v_1, v_2, v_3\}$ such that

$$n(u_i, v_i) = 1$$

and the norm in the rest of the combinations of elements is 0. This basis is called a **good basis** of $\mathcal{C}$ or a **Cartan basis** of $\mathcal{C}$. The multiplication is given by table 2.1.

Now, let $\gamma = (g_1, g_2, g_3)$ be a triple of elements on G such that $g_1g_2g_3 = e$. Denote by $\Gamma^1_{\mathcal{C}}(G, \gamma)$ the grading induced by

$$\deg(e_1) = e = \deg(e_2) \text{ and}$$
$$\deg(u_i) = g_i = \deg(v_i)^{-1},$$

for every $i \in \{1, 2, 3\}$.

Given another such triple $\gamma' = (g'_1, g'_2, g'_3)$, we denote $\gamma \sim \gamma'$ if and only if there is $\pi \in \mathrm{Sym}_3$ such that $g_{\pi(i)} = g'_i$ for all $i \in \{1, 2, 3\}$ or there is $\pi \in \mathrm{Sym}_3$ such that $g^{-1}_{\pi(i)} = g'_i$ for all $i \in \{1, 2, 3\}$.

	e_1	e_2	u_1	u_2	u_3	v_1	v_2	v_3
e_1	e_1	0	u_1	u_2	u_3	0	0	0
e_2	0	e_2	0	0	0	v_1	v_2	v_3
u_1	0	u_1	0	v_3	$-v_2$	$-e_1$	0	0
u_2	0	u_2	$-v_3$	0	v_1	0	$-e_1$	0
u_3	0	u_3	v_2	$-v_1$	0	0	0	$-e_1$
v_1	v_1	0	$-e_2$	0	0	0	u_3	$-u_2$
v_2	v_2	0	0	$-e_2$	0	$-u_3$	0	u_1
v_3	v_3	0	0	0	$-e_2$	u_2	$-u_1$	0

Table 2.1: Multiplication table of the Cartan basis on the split Cayley algebra.

We will define another basis called the Cayley-Dickson basis $\{x_i\}_{i=0}^{7}$ by $x_0 = 1$, $x_1 = w_1$, $x_2 = w_2$, $x_3 = w_2w_1$, $x_4 = w_3$, $x_5 = w_3w_1$, $x_6 = w_3w_2$ and $x_7 = w_3(w_2w_1)$.

Let $H \subseteq G$ be a subgroup isomorphic to $\mathbb{Z}_2^3$. Let $\alpha\colon \mathbb{Z}_2^3 \to H$ be an isomorphism. Denote:

$$c_1 = \alpha(\bar{1},\bar{0},\bar{0}),\ c_2 = \alpha(\bar{0},\bar{1},\bar{0}),\ c_3 = \alpha(\bar{0},\bar{0},\bar{1})$$

We denote by $\Gamma^2_{\mathcal{C}}(G,H)$ the grading induced by $\deg(w_i) = c_i$ for $i \in \{1,2,3\}$. This is a grading as shown in [EK13, (4.13)] and for a fixed subgroup H, different choices of α give isomorphic gradings since we can use the elements of degrees c_1, c_2, c_3 (for a fixed algebra) to generate $\mathcal{C}$ via the Cayley-Dickson process. Finally we have the following theorem:

Theorem 2.2.11 ([EK12]). *Let $\mathcal{C}$ be the Cayley algebra over an algebraically closed field and let G be an abelian group. Then, any G-grading on $\mathcal{C}$ is isomorphic to some $\Gamma^1_{\mathcal{C}}(G,\gamma)$ or some $\Gamma^2_{\mathcal{C}}(G,H)$, but not both. Also,*

(1) $\Gamma^1_{\mathcal{C}}(G,\gamma)$ is isomorphic to $\Gamma^1_{\mathcal{C}}(G,\gamma')$ if and only if $\gamma \sim \gamma'$.

(2) $\Gamma^2_{\mathcal{C}}(G,H)$ is isomorphic to $\Gamma^2_{\mathcal{C}}(G,H')$ if and only if $H = H'$.

Proof. The proof is in [EK12, Theorem 3.8]. □

Let's now introduce a new class of composition algebras:

Definition 2.2.12. Let $(S,\star,n)$ be a composition algebra. We say that it is a **symmetric composition algebra** if $n(x\star y,z) = n(x,y\star z)$ for all $x,y,z \in \mathcal{C}$.

Theorem 2.2.13. *Let* $(S, \star, n)$ *be a composition algebra.* $(S, \star, n)$ *is symmetric if and only if for any* $x, y \in S$, $(x \star y) \star x = n(x)y = x \star (y \star x)$.

Proof. See for example [Eld20, Theorem 3.2]. □

Some examples of symmetric composition algebras are the following:

Example 2.2.14 (Para-Hurwitz algebras)**.** Let $(\mathcal{C}, \cdot, n)$ be a Hurwitz algebra. Define a new composition algebra $(\mathcal{C}, \bullet, n)$ by defining the product $\bullet$ as

$$x \bullet y = \overline{x} \cdot \overline{y}.$$

This is a symmetric composition algebra as shown in [Eld20].

Example 2.2.15 (Okubo algebras in the case $\mathrm{char}\mathbb{F} \neq 3$)**.** Assume that $\mathrm{char}\mathbb{F} \neq 3$. Let $\omega \in \mathbb{F}$ be a primitive cubic root of 1. Take a central simple associative algebra of degree 3 with trace $\mathrm{tr}\colon \mathcal{A} \to \mathbb{F}$. Let $\mathcal{S} = \{x \in \mathcal{A} \mid \mathrm{tr}(x) = 0\}$. Define a multiplication $\star$ and a norm n on $\mathcal{S}$ by:

$$\begin{aligned} &x \star y = \omega xy - \omega^2 yx - \frac{\omega - \omega^2}{3}\mathrm{tr}(xy)1, \text{ and} \\ &n(x) = -\frac{1}{2}\mathrm{tr}(x^2). \end{aligned}$$

Then $(\mathcal{S}, \star, n)$ is a symmetric composition algebra. See for example [Eld20] for a proof.

2.3 Cross Products: Classification

We still have on hold the question of which cross products exist up to isomorphism. The classification is already made in [Eck43] and [BG67] but here we are going to sketch the proof and we are going to tackle the problem of the classification of bilinear forms (see Proposition 2.3.10).

Lemma 2.3.1. *Let* $\mathbb{F}$ *be an algebraically closed field. Let* (V, X, b) *be an* r*-fold cross product over an* n*-dimensional* $\mathbb{F}$*-vector space* V *with* $r > 1$*. Let* $v \in V$ *be such that* $b(v, v) = 1$*. Denote* $W = \{w \in V \mid b(v, w) = 0\}$ *and define* $\overline{X}\colon W^{r-1} \to W$ *by* $\overline{X}(v_1, .., v_{r-1}) = X(v, v_1, .., v_{r-1})$*. Then,* $\overline{X}$ *is an* $(r-1)$*-fold cross product.*

Proof. This is a trivial result which can be found in [Eld04]. □

The previous result shows us that we can start the classification in low values of r. The proof of existence of 2-fold products is based on the following result:

Proposition 2.3.2 ([BG67]). *Let (V, X, b) be a 2-fold cross product (denote $u \times v = X(u, v)$ for all $u, v \in V$). Define an algebra on $\mathcal{A} = \mathbb{F}1 \oplus V$ and a quadratic form $q\colon \mathcal{A} \to \mathbb{F}$ by:*

$$\begin{aligned}&(\alpha 1 + u)(\beta 1 + v) = (\alpha\beta - b(u, v))1 + (\alpha v + \beta u + u \times v), \text{ and}\\ &q(\alpha 1 + v) = \alpha^2 + b(v, v).\end{aligned} \tag{2.3.1}$$

With this notation $(\mathcal{A}, q)$ is a Hurwitz algebra.

Proof. The proof is in [BG67] □

The previous proposition and Theorem 2.2.9 show that there are only 2-fold cross products in the cases where $n = 3$ and where $n = 7$. This together with Lemma 2.3.1 implies that there can be r-fold cross products on dimension n in the cases where: $r = 1$, $r = n - 1$ and $r = n - 5$. We will introduce an algebra we need and we will give examples of cross products:

Definition 2.3.3. Let V be a finite dimensional vector space. The **exterior algebra** $\bigwedge V$ is the quotient of the tensor algebra $T(V)$, i.e., the algebra whose underlying vector space is

$$T(V) = \mathbb{F}1 \oplus \bigoplus_{i=1}^{\infty} \underbrace{V \otimes_{\mathbb{F}} \ldots \otimes_{\mathbb{F}} V}_{i \text{ times}}$$

and its multiplication is given by $(v_1 \otimes_{\mathbb{F}} \ldots \otimes_{\mathbb{F}} v_i)(w_1 \otimes_{\mathbb{F}} \ldots \otimes_{\mathbb{F}} w_j) = v_1 \otimes_{\mathbb{F}} \ldots \otimes_{\mathbb{F}} v_i \otimes_{\mathbb{F}} w_1 \otimes_{\mathbb{F}} \ldots \otimes_{\mathbb{F}} w_j$ for every $v_1, \ldots, v_i, w_1, \ldots, w_j \in V$ and $1 \leq i, j$, with the ideal I generated by the elements of the form $v_1 \otimes v_2 + v_2 \otimes v_1$ for $v_1, v_2 \in V$. We denote the image of the element $v_1 \otimes \ldots \otimes v_k \in T(V)$ as $v_1 \wedge \ldots \wedge v_k$ for any $v_1, \ldots, v_k \in V$, $k \geq 1$. The natural $\mathbb{Z}$-grading on $T(V)$ induces a $\mathbb{Z}$-gradings on $\bigwedge V$ (i.e. the elements of the form $v_1 \wedge \ldots \wedge v_p$ have degree p). We denote by $\bigwedge^p V$ the subspace of elements of degree p.

Remark 2.3.4. Given a linear map $\varphi\colon V \to V$ over a vector space of dimension n, an equivalent definition (see [Coh91, 2.4]) of its determinant $\det \varphi \in \mathbb{F}$, is the constant satisfying that for any $v_1, \ldots, v_n \in V$, the following equation holds:

$$\varphi(v_1) \wedge \ldots \wedge \varphi(v_1) = (\det \varphi)(v_1 \wedge \ldots \wedge v_n).$$

Example 2.3.5. Let $b\colon V\times V\to\mathbb{F}$ be a nondegenerate, symmetric, bilinear form of determinant 1, i.e., such that there exists a basis $v_1,..,v_n$ of V satisfying that $\det(b(v_i,v_j))\in\mathbb{F}^{\times 2}$. Consider the exterior algebra $\bigwedge V$. We are going to denote by $\iota_p\colon V^p\to\bigwedge^p V$ the map given by $\iota_p(v_1,...,v_p)=v_1\wedge...\wedge v_p$. We extend b to $\bigwedge V$ by the following formula:

$$b(v_1\wedge...\wedge v_p, w_1\wedge...\wedge w_q)=\begin{cases}\det(b(v_i,w_j)) & \text{if } p=q\\ 0 & \text{otherwise.}\end{cases}$$

This extension of b is symmetric since if we have $2p$ vectors denoted by $v_1,...,v_p,w_1,...,w_p$, the matrix $A=(b(v_i,w_j))$ is the transpose of the matrix $B=(b(w_i,v_j))$, and it is nondegenerate since if $v_1,...,v_n$ is an orthogonal basis in V, the set of vectors $v_{\lambda_1}\wedge...\wedge v_{\lambda_p}$ for $1\le\lambda_1<..<\lambda_p\le n$ and $1\le p\le n$ is an orthogonal basis of $\bigwedge V$.

If we choose an element $w\in\bigwedge^n V$ such that $b(w,w)=1$, which exists since the determinant of b is 1, we define the star operator $*\colon\bigwedge V\to\bigwedge V$ to be the linear extension of the map satisfying that for any $1\le p\le n$, $v\in\bigwedge^p V$ and $u\in\bigwedge^{n-p} V$:

$$b({}^*v,u)=b(v\wedge u,w).$$

With this notation, the multilinear map defined as

$$\begin{array}{rcc} X\colon V^{n-1} & \longrightarrow & V\\ (v_1,...,v_{n-1}) & \longmapsto & {}^*(v_1\wedge...\wedge v_{n-1})\end{array}$$

is an $(n-1)$-fold cross product on V relative to b (see [BG67, Theorem 3.3]).

Example 2.3.6. Let $(\mathcal{C},n)$ be a composition algebra of dimension greater that 2. Denote by $\mathcal{C}_0$ the subspace of trace 0 elements, i.e, the orthogonal elements to the unit 1 with respect to the norm:

$$\mathcal{C}_0=\{x\in\mathcal{C}\mid t(x)=n(x,1)=0\}.$$

We define a bilinear form $b_n\colon\mathcal{C}_0\times\mathcal{C}_0\to\mathbb{F}$ and a product $\times$ on $\mathcal{C}_0$ by means of:

$$xy=-b_n(x,y)1+x\times y. \tag{2.3.2}$$

They are uniquely defined since $\mathcal{C}=\mathbb{F}1\oplus\mathcal{C}_0$. Due to the Cayley-Hamilton equation, we get that $x\times x=0$ and that $b_n(x,x)=n(x)=\frac{1}{2}n(x,x)$ for all $x\in\mathcal{C}$. Linearizing, we see that $b_n(x,y)=\frac{1}{2}n(x,y)$ for all $x,y\in\mathcal{C}_0$. Applying the involution defined in Proposition 2.2.3 it is easy to see that $\times$ is anticommutative and b_n is symmetric. Hence, $x\times y=\frac{1}{2}(xy-yx)$ and

$b_n(x,y)1 = -\frac{1}{2}(xy+yx)$. Moreover, using that $\mathcal{C}$ is alternative, and a few basic properties we can show that it satisfies

$$(x \times y) \times y = b_n(x,y)y - b_n(y,y)x \tag{2.3.3}$$

for all $x, y \in C_0$ (see [EK13, Theorem 4.23]).

X is a 2-fold cross product (see [BG67, Theorem 4.1]). Concretely, if $\mathcal{C}$ is a Cayley algebra, we get a 2-fold cross product on a vector space of dimension 7, which is not considered in Example 2.3.5. In this case, we will denote the product by $(\mathcal{C}_0, X^{\mathcal{C}_0}, b_n)$.

Example 2.3.7. Let $\mathcal{C}$ be a Cayley algebra and denote $b_n(x,y) = \frac{1}{2}n(x,y)$ for every $x, y \in \mathcal{C}$. Define two trilinear maps $X^{\mathcal{C}}_\epsilon \colon \mathcal{C} \times \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ for $\epsilon = \pm 1$ by:

$$X^{\mathcal{C}}_1(x,y,z) = (x\overline{y})z - b_n(x,y)z - b_n(y,z)x + b_n(x,z)y, \text{ and} \tag{2.3.4}$$
$$X^{\mathcal{C}}_{-1}(x,y,z) = x(\overline{y}z) - b_n(x,y)z - b_n(y,z)x + b_n(x,z)y. \tag{2.3.5}$$

In each case $(\mathcal{C}, X^{\mathcal{C}}_\epsilon, b_n)$ is a 3-fold cross product (see [BG67, Theorem 5.1]). Moreover, the following holds for all $u_i, v_i \in \mathcal{C}$, $1 \leq i \leq 3$:

$$\begin{aligned} b_n(X^{\mathcal{C}}_\epsilon(u_1,u_2,u_3), X^{\mathcal{C}}_\epsilon(v_1,v_2,v_3)) &= \det(b_n(u_i,v_j)) \\ &+ \epsilon \sum_{\substack{\sigma\in S_3,\\ \text{even}}} \sum_{\substack{\tau\in S_3,\\ \text{even}}} b_n(u_{\sigma(1)}, v_{\tau(1)}) b_n(u_{\sigma(2)}, X^{\mathcal{C}}_\epsilon(u_{\sigma(3)}, v_{\tau(2)}, v_{\tau(3)})). \end{aligned} \tag{2.3.6}$$

See [Eld96, Proposition 3].

Given a vector space V and a linear operator $X \colon V \to V$, denote by $\operatorname{tr}(X)$ its trace. Now we can review the classification of cross products.

Theorem 2.3.8. *Let X be an r-fold cross product on a vector space V of dimension n, $1 \leq r \leq n$. Then, one and only one of the following conditions holds:*

(1) n is even, $r = 1$, $X^2 = -\mathrm{id}$ and $\operatorname{tr}(X) = 0$.

(2) $n \geq 3$, $r = n-1$ and:

$$X(v_1, .., v_r) = {}^*(v_1 \wedge ... \wedge v_r)$$

for all $v_1, .., v_r$ as in Example 2.3.5.

(3) $n = 7$, $r = 2$ *and* (V, X) *is isomorphic to* $(\mathcal{C}_0, X^{\mathcal{C}_0})$ *for a Cayley algebra* $\mathcal{C}$.

(4) $n = 8$, $r = 3$ *and* (V, X) *is isomorphic to* $(\mathcal{C}, \alpha X^{\mathcal{C}}_{\epsilon})$ *for a Cayley algebra* $\mathcal{C}$, *a nonzero* $\alpha \in \mathbb{F}$, *and* $\epsilon = \pm 1$.

Conversely, all the pairs (V, X) *in items* $(1) - (4)$ *are cross products.*

Proof. Everything is proved in [BG67] except for the description of case $r = 1$, where it is shown that n is even, and a few more details on the case $r = 3$, $n = 8$. The full complete proof for this case is in [DET20]. We are going to review it here.

If $r = 1$ and X is a 1-fold cross product relative to a bilinear form b, then X satisfies: (1) $b(X(u), u) = 0$ and (2) $b(X(u), X(v)) = b(u, v)$ (i.e, the linearization of $b(X(u), X(u)) = b(u, u)$) for all $u, v \in V$. (2) and the fact that b is nondegenerate implies that X is an isomorphism and that its inverse is its adjoint X^* with respect to b. Hence, $XX^* = \mathrm{id}$.

Take $u, v \in V$. From $b(X(u + v), u + v) = 0$ we get $b(X(u), v) + b(X(v), u) = 0$. Therefore, the fact that b is symmetric implies that $b(X(u) + X^*(u), v) = 0$. Since b is nondegenerate, $X = -X^*$. Finally, since $\mathrm{tr}(X) = \mathrm{tr}(X^*)$ and $\mathrm{char}(\mathbb{F}) \neq 2$, we get that $\mathrm{tr}(X) = 0$.

Conversely, let $X : V \to V$ be an endomorphism such that $X^2 = -\mathrm{id}$ and $\mathrm{tr}(X) = 0$. If $-1 \in \mathbb{F}^2$, take $i \in \mathbb{F}$ such that $i^2 = -1$. We can split V as $V = V_+ \oplus V_-$ where $V_\sigma = \{v \in V \mid X(v) = \sigma i v\}$ for $\sigma = \pm$. Due to the fact that $\mathrm{tr}(X) = 0$, $\dim V_+ = \dim V_-$ so n is even. Take a nonsingular symmetric bilinear form such that $b(V_+, V_+) = 0$ and $b(V_-, V_-) = 0$. If $v \in V_\sigma$ and $u \in V_{-\sigma}$ for $\sigma = \pm$, $b(X(v), v) = \sigma i b(v, v) = 0$ and $b(X(u), X(v)) = -\sigma\sigma i^2 b(u, v) = b(u, v)$. Hence, X is a 1-fold cross product with respect to b.

If $-1 \notin \mathbb{F}^2$, the subalgebra $\mathbb{F} \oplus \mathbb{F}X \subseteq \mathrm{End}(V)$, is a quadratic field extension of $\mathbb{F}$ and V is a vector space over $\mathbb{K}$. If $\{v_1, ..., v_n\}$ is a $\mathbb{K}$-basis of V, $\{v_1, X(v_1), ..., v_n, X(v_n)\}$ is an $\mathbb{F}$-basis. Define b to be a symmetric bilinear form such that the previous $\mathbb{F}$ basis is orthonormal and then it follows that X is a 1-fold cross product on V.

For $n = 8$, $r = 3$, if (V, X, b) is a 3-fold cross product on a vector space of dimension 8, due to [Eld96, Proposition 3] we have (2.3.6) for either $\epsilon = 1$, which we call type I, or for $\epsilon = -1$, which we call type II. In case it is of type II, we can take $(V, X, -b)$ instead and we get a 3-fold cross product of type I. Hence, using [BG67, Theorem 5.1] and [Eld96, Proposition 3], we get the result. □

Now, we can tackle the question of determining the bilinear forms. We already have at least one bilinear form for each r-fold cross product. The problem is solved in the following two results.

Lemma 2.3.9. *Let X be a 1-fold cross product on a vector space V of dimension $2n$ relative to the nondegenerate symmetric bilinear form b. Then, there is a basis $\{v_1, X(v_1), ..., v_n, X(v_n)\}$ of V and $\lambda_1, ..., \lambda_n \in \mathbb{F}$, all different from 0, such that $b(v_i, v_i) = \lambda_i = b(X(v_i), X(v_i))$ and the basis is orhogonal. Moreover, all the bilinear forms adapted to X are of this form.*

Proof. This is a direct consequence of [BG67, Theorem 2.2] and its proof. □

Proposition 2.3.10 ([DET20])**.** *Let X be an r-fold cross product on a vector space V of dimension n relative to the nondegenerate symmetric bilinear forms b and b'.*

(1) If $r = 2$ and $n = 7$, then $b = b'$.

(2) If $n \geq 3$ and $r = n-1$, then there is a scalar $\mu \in \mathbb{F}$ with $\mu^{n-2} = 1$ such that $b' = \mu b$. In particular, for $n = 3$, b is uniquely determined.

(3) If $n = 8$ and $r = 3$, then b' is either b or $-b$. In particular, if we assume (V, X, b) to be of type I, b is uniquely determined.

Proof. The proof is in [DET20] but for completeness we show it here.

(1) Equation (2.3.2) shows that in this case the bilinear form is uniquely determined.

(2) Let $n \geq 3$ and $r = n - 1$. After an extension of scalars two different bilinear forms are still different. Therefore, without loss of generality we can assume that $\mathbb{F}$ is algebraically closed. Let $\{v_1, ..., v_n\}$ be an orhonormal basis relative to b.

The fact that b satisfies (2.1.1), implies that there is $\lambda \in \mathbb{F}$ such that $X(v_1, ..., v_{n-1}) = \lambda v_n$. The fact that b satisfies (2.1.2), implies that $\lambda^2 = b(X(v_1, ..., v_{n-1}), X(v_1, ..., v_{n-1})) = \det(b(v_i, v_j)) = 1$. Hence $\lambda = \pm 1$. Taking $-v_n$ instead of v_n if necesary, we can assume that $X(v_1, ..., v_{n-1}) = v_n$.

Denote by $\Phi \colon V^n \to \mathbb{F}$ the map defined by

$$\Phi(u_1, ..., u_n) = b(X(u_1, ..., u_{n-1}), u_n) \tag{2.3.7}$$

for all $u_1, ..., u_{n-1} \in V$. This is a multilinear form. Moreover (2.1.1) implies that Φ is alternating. This implies that for any permutation σ, the following equation folds:

$$X(v_{\sigma(1)}, ..., v_{\sigma(n-1)}) = (-1)^{\sigma} v_{\sigma(n)}. \tag{2.3.8}$$

Similarly, we can define $\Phi'\colon V^n \to \mathbb{F}$ by

$$\Phi'(u_1, ..., u_n) = b'(X(u_1, ..., u_{n-1}), u_n)$$

and show that it is also alternating. The fact that Φ and Φ' are alternating, implies that there are $\psi, \psi'\colon \bigwedge^n V \to \mathbb{F}$ such that $\Phi = \psi \circ \iota_p$ and $\Phi' = \psi' \circ \iota_p$ with the notation of Example 2.3.5 (see [MB79, Chapter XVI]). Since $\bigwedge^n V$ has dimension 1, it implies that there is a nonzero scalar $\mu \in \mathbb{F}$ such that $\Phi' = \mu\Phi$. Equation (2.3.8) implies that for every permutation σ and $i \in \{1, ..., n\}$:

$$\begin{aligned} b'(v_{\sigma(n)}, v_i) &= (-1)^\sigma \Phi'(v_{\sigma(1)}, ..., v_{\sigma(n-1)}, v_i) \\ &= (-1)^\sigma \mu\Phi(v_{\sigma(1)}, ..., v_{\sigma(n-1)}, v_i) = \mu b(v_{\sigma(n)}, v_i), \end{aligned}$$

so $b' = \mu b$ and substituting in (2.1.2) we deduce that $\mu^{n-2} = 1$. It is clear that the converse holds.

(3) In this case, we can assume again that $\mathbb{F}$ is algebraically closed. Let $\Phi, \Phi'\colon V^4 \to \mathbb{F}$ be the maps defined by

$$\Phi(v_1, v_2, v_3, v_4) = b(X(v_1, v_2, v_3), v_4)$$

and

$$\Phi'(v_1, v_2, v_3, v_4) = b'(X(v_1, v_2, v_3), v_4)$$

for every $v_1, v_2, v_3, v_4 \in V$. They are multilinear and alternating as in the previous case. Linearizing in v_3 in (2.1.2), we get:

$$\begin{aligned} \begin{vmatrix} b(x,x) & b(x,y) & b(x,v) \\ b(y,x) & b(y,y) & b(y,v) \\ b(u,x) & b(u,y) & b(u,v) \end{vmatrix} &= b(X(x,y,u), X(x,y,v)) \\ &= \Phi(x,y,u,X(x,y,v)) \\ &= -\Phi(x,y,X(x,y,v),u) \\ &= -b(X(x,y,X(x,y,v)),u). \end{aligned}$$

On the other hand:

$$\begin{aligned} &\begin{vmatrix} b(x,x) & b(x,y) & b(x,v) \\ b(y,x) & b(y,y) & b(y,v) \\ b(u,x) & b(u,y) & b(u,v) \end{vmatrix} \\ &= b\left(\begin{vmatrix} b(x,y) & b(x,v) \\ b(y,y) & b(y,v) \end{vmatrix} x - \begin{vmatrix} b(x,x) & b(x,v) \\ b(y,x) & b(y,v) \end{vmatrix} y + \begin{vmatrix} b(x,x) & b(x,y) \\ b(y,x) & b(y,y) \end{vmatrix} v, u\right). \end{aligned}$$

Since b is non degenerate, we get that:

$$X(x,y,X(x,y,v)) = \begin{vmatrix} b(x,x) & b(x,v) \\ b(y,x) & b(y,v) \end{vmatrix} y - \begin{vmatrix} b(x,y) & b(x,v) \\ b(y,y) & b(y,v) \end{vmatrix} x - \begin{vmatrix} b(x,x) & b(x,y) \\ b(y,x) & b(y,y) \end{vmatrix} v.$$

Therefore, applying $X(x,y,\cdot)$ again we get

$$X(x,y,X(x,y,X(x,y,v))) = - \begin{vmatrix} b(x,x) & b(x,y) \\ b(y,x) & b(y,y) \end{vmatrix} X(x,y,v)$$

for all $x,y,v \in V$, i.e.:

$$X(x,y,\cdot)^3 = - \begin{vmatrix} b(x,x) & b(x,y) \\ b(y,x) & b(y,y) \end{vmatrix} X(x,y,\cdot).$$

The same equation is obtained substituting b by b'. Hence, for all $x,y \in V$ satisfying $X(x,y,\cdot) \neq 0$, we get that:

$$b(x,x)b(y,y) - b(x,y)^2 = b'(x,x)b'(y,y) - b'(x,y)^2. \tag{2.3.9}$$

Since the set of such x,y is a Zariski dense subset, we get that (2.3.9) is satisfied for any $x,y \in V$.

Given $x \in V$ with $b(x,x) \neq 0$, we denote

$$S_x = \{v \in V \mid b(x,v) = 0 = b'(x,v)\}.$$

Since it is defined by two linear equations, this is a subspace of dimension at least 6. Since the maximal dimension of a totally isotropic subspace of a bilinear form in an 8-dimensional vector space is 4, S_x is not totally isotropic for neither b nor b'. Therefore, for any $v \in S_x$, equation (2.3.9) implies:

$$b(v,v) = \frac{b'(x,x)}{b(x,x)} b'(v,v).$$

Calling $\mu = \frac{b'(x,x)}{b(x,x)}$, and linearizing the previous equation we get that $b(u,v) = \mu b'(u,v)$ for all $u,v \in S_x$. Substituting in (2.3.9), we get

$$(1-\mu^2) \begin{vmatrix} b(u,u) & b(u,v) \\ b(v,u) & b(v,v) \end{vmatrix} = 0$$

for every $u, v \in S_x$. The fact that S_x is not totally isotropic implies that we can find $u, v \in S_x$ such that $b(u,u) \neq 0 \neq b(v,v)$ and $b(u,v) = 0$. Thus, due to the previous equation, $1 = \mu^2$. Therefore, $b(x,x) = \pm b'(x,x)$ for every x such that $b(x,x) \neq 0$. Thus, the polynomial map

$$(b(x,x) - b'(x,x))(b(x,x) + b'(x,x)) \tag{2.3.10}$$

is 0 for every x such that $b(x,x) \neq 0$. Since this is a Zariski open set, the map is 0 for every $x \in V$. Hence, either $b = b'$ or $b = -b'$.

□

2.4 Clifford algebra and groups therein

In the following section we will deal with the group Spin(7) (more concretely with the corresponding affine group scheme). This however, will not be our only encounter with spin groups and such kind of groups. Thus, we will use this section to review the main necessary theory.

Let (V, q) be a **nonsingular quadratic space**, i.e., a pair where V is a finite dimensional vector space V with a non singular quadratic form $q\colon V \to \mathbb{F}$. We will also denote by q its linearization given by $q(x,y) = q(x+y) - q(x) - q(y)$ for all $x, y \in V$. We have a natural $\mathbb{Z}$-grading in the tensor algebra $T(V)$ induced by $\deg(v_1 \otimes_{\mathbb{F}} ... \otimes_{\mathbb{F}} v_i) = i$, $v_1, ..., v_i \in V$ and $1 \leq i$. This induces a $\mathbb{Z}_2$-grading by taking quotient on the degrees. We denote by $T_{\bar{0}}(V)$ the subspace of degree $\bar{0}$ (even part) and by $T_{\bar{1}}(V)$ the subspace of degree $\bar{1}$ (odd part).

Definition 2.4.1. Denote by $I(q)$ the ideal generated by the elements of the form $v \otimes v - q(v)$ for every $v \in V$. We define the **Clifford algebra** $\mathfrak{Cl}(V, q)$ as the following quotient:

$$\mathfrak{Cl}(V,q) = T(V)/I(q).$$

Since $I(q)$ is a graded ideal, the $\mathbb{Z}_2$-grading in $T(V)$ induces a $\mathbb{Z}_2$-grading in $\mathfrak{Cl}(V,q)$:

$$\mathfrak{Cl}(V,q) = \mathfrak{Cl}_{\bar{0}}(V,q) \oplus \mathfrak{Cl}_{\bar{1}}(V,q).$$

We call the subalgebra $\mathfrak{Cl}_{\bar{0}}(V,q)$ the **even Clifford algebra**.

In both cases we will denote the multiplication with a dot, i.e., for any two elements $x, y \in \mathfrak{Cl}(V,q)$, we denote their product as $x \cdot y$. Note that $V \cap I(q) = 0$. Thus, V embeds naturally in $\mathfrak{Cl}(V,q)$ (concretely, it embeds in $\mathfrak{Cl}_{\bar{1}}(V,q)$).

Remark 2.4.2. For $x, y \in V$, linearizing the identity $x \cdot x = q(x)1$, we get that $x \cdot y + y \cdot x = q(x, y)1$.

These algebras satisfy that $\dim \mathfrak{Cl}(V, q) = 2^{\dim V}$ and $\dim \mathfrak{Cl}_{\bar{0}}(V, q) = 2^{\dim V - 1}$ (see [Knu91, (1.5.2)]).

Lemma 2.4.3 (Universal property)**.** *Let (V, q) be a nonsingular quadratic space. Let $\mathcal{A}$ be a unital associative algebra with unit $1_{\mathcal{A}}$. Let $f \colon V \to \mathcal{A}$ be an $\mathbb{F}$ linear map satisfying $f(v)^2 = q(v)1_{\mathcal{A}}$. Then, there is a unique algebra homomorphism $h \colon \mathfrak{Cl}(V, q) \to \mathcal{A}$ whose restriction to V is f.*

Proof. See the proof of [Knu91, Chapter IV (1.1.2)]. □

Theorem 2.4.4. *Let V be a vector space and q a nonsingular quadratic form over V:*

(1) If $\dim V = 2m + 1$ is odd, then $\mathfrak{Cl}_{\bar{0}}(V, q)$ is a central simple $\mathbb{F}$-algebra of degree 2^m.

(2) If $\dim V = 2m$ is even, then the center of $\mathfrak{Cl}_{\bar{0}}(V, q)$ is an étale quadratic $\mathbb{F}$-algebra $\mathcal{Z}$. If $\mathcal{Z}$ is a field, then $\mathfrak{Cl}_{\bar{0}}(V, q)$ is a central simple $\mathcal{Z}$-algebra of degree 2^{m-1}. If $\mathcal{Z} \cong \mathbb{F} \oplus \mathbb{F}$, then $\mathfrak{Cl}_{\bar{0}}(V, q)$ is the direct product of two central simple $\mathbb{F}$-algebras of degree 2^{m-1}. Moreover, the center $\mathcal{Z}$ is isomorphic to $\mathbb{F}[x]/\langle x^2 - \mu \rangle$ where $\mu \in \mathbb{F}^\times$ is a representative of the discriminant of q.

Proof. The proof can be found in [Knu91, Chapter IV]. □

Definition 2.4.5. The identity map in V extends to an involution σ on $T(V)$ given by $\sigma(v_1 \otimes ... \otimes v_k) = v_k \otimes ... \otimes v_1$ for every $v_1, ..., v_k \in V$, and $k \geq 1$. Given a nonsingular quadratic form $q \colon V \to \mathbb{F}$, this involution preserves $x \otimes x - q(x)1$; thus, it preserves the ideal $I(q)$. Therefore, σ induces an involution, which we also denote σ, on the Clifford algebra $\mathfrak{Cl}(V, q)$. We call it the **canonical involution on** $\mathfrak{Cl}(V, q)$. This involution restricts to an involution $\sigma_{\bar{0}}$ on $\mathfrak{Cl}_{\bar{0}}(V, q)$. This involution is called the **canonical involution on** $\mathfrak{Cl}_{\bar{0}}(V, q)$.

Proposition 2.4.6. *Let $(S, \star, n)$ be a symmetric composition algebra of dimension 8. Denote by $r_x, l_x \in \mathrm{End}(S)$ the endomorphisms given by $r_x(y) = y \star x = l_y(x)$ for all $x, y \in S$. The linear map $\Phi \colon S \to \mathrm{End}_{\mathbb{F}}(S \oplus S)$ given by:*

$$x \mapsto \begin{pmatrix} 0 & l_x \\ r_x & 0 \end{pmatrix}$$

satisfies that $\Phi(x)^2 = n(x)\mathrm{id}$ for all $x \in S$. Moreover, the morphism of algebras $\Phi \colon \mathfrak{Cl}(S, n) \to \mathrm{End}_{\mathbb{F}}(S \oplus S)$ (also denoted by Φ) that it induces by *the universal property of the Clifford algebra is an isomorphism. If we denote by* $\tilde{n}\colon (S\oplus S)\times(S\oplus S)\to\mathbb{F}$ *the bilinear form given by* $\tilde{n}((x_1,y_1),(x_2,y_2)) = n(x_1,x_2)+n(y_1,y_2)$ *and by* τ *the adjoint map with respect to this norm, (i.e.* $\tilde{n}(A(x),y)=\tilde{n}(x,\tau(A)y)$*), then* $\Phi(\sigma(x))=\tau(\Phi(x))$ *for all* $x\in\mathfrak{Cl}(S,n)$ *where we denote by* σ *the canonical involution of* $\mathfrak{Cl}(S,n)$*.*

Proof. Using Theorem 2.2.13 it is easy to check that $\Phi(x)^2 = n(x)\mathrm{id}$. The rest of the proof is [KMRT, (35.1)]. ☐

The algebra $\mathrm{End}_{\mathbb{F}}(S\oplus S)$ has a $\mathbb{Z}_2$-grading given by:

$$\begin{aligned}\mathrm{End}_{\mathbb{F}}(S\oplus S)_{\bar 0} &= \left\{\begin{pmatrix} x & 0\\ 0 & y\end{pmatrix} \mid x,y\in\mathrm{End}_{\mathbb{F}}(S)\right\},\ \text{and}\\ \mathrm{End}_{\mathbb{F}}(S\oplus S)_{\bar 1} &= \left\{\begin{pmatrix} 0 & x\\ y & 0\end{pmatrix} \mid x,y\in\mathrm{End}_{\mathbb{F}}(S)\right\}.\end{aligned} \tag{2.4.1}$$

Φ is a morphism of graded algebras. Therefore, there are $\rho_+,\rho_-\colon \mathfrak{Cl}_{\bar 0}(S,n)\to \mathrm{End}_{\mathbb{F}}(S)$ satisfying that for all $x\in\mathfrak{Cl}_{\bar 0}(S,n)$:

$$\Phi(x) = \begin{pmatrix}\rho^+(x) & 0\\ 0 & \rho^-(x)\end{pmatrix}. \tag{2.4.2}$$

Over the Clifford algebra we can define some groups, related with the orthogonal groups. We will see their definition.

Definition 2.4.7. Given a nonsingular quadratic space (V,q), we define the **special Clifford group** $\Gamma^+(V,q)$ by:

$$\Gamma^+(V,q) = \{c\in\mathfrak{Cl}_{\bar 0}(V,q)^\times \mid c\cdot V\cdot c^{-1} = V\}.$$

Similarly, we can define the **even Clifford group** of (V,q) as the affine group scheme $\mathbf{\Gamma}^+(V,q)$, whose R-points are:

$$\mathbf{\Gamma}^+(V,q)(R) = \{c\in\mathfrak{Cl}_{\bar 0}(V,q)_R^\times \mid c\cdot V_R\cdot c^{-1} = V_R\}.$$

Notice that for any $v\in V$ and $c\in\mathfrak{Cl}_{\bar 0}(V,q)$, we get that:

$$q(c\cdot v\cdot c^{-1}) = (c\cdot v\cdot c^{-1})^2 = c\cdot v\cdot v\cdot c^{-1} = q(v).$$

Therefore, we can define a map:

$$\begin{aligned}\chi\colon \Gamma^+(V,q) &\longrightarrow \mathrm{O}(V,q)\\ c &\longmapsto \chi_c\colon v\mapsto c\cdot v\cdot c^{-1}.\end{aligned}$$

χ is called the **vector representation of the special Clifford group**. The map, χ can be extended to a morphism of affine group schemes

$$\chi\colon \mathbf{\Gamma}^+(V,q) \to \mathbf{O}(V,q),$$

which we call the **vector representation of the even Clifford group**. As in [KMRT, 23.A], we have an exact sequence:

$$1 \to \mathbf{G}_{\mathrm{m}} \to \mathbf{\Gamma}^+(V,q) \xrightarrow{\chi} \mathbf{O}^+(V,q) \to 1$$

which is used to show that $\mathbf{\Gamma}^+(V,q)$ is smooth. The Lie algebra of this affine group scheme is

$$\mathrm{Lie}(\mathbf{\Gamma}^+(V,q)) = V \cdot V.$$

Therefore, since a basis of $V \cdot V$ is $\{1\} \cup \{v_i \cdot v_j\}_{1\le i<j\le n}$ where $v_1, ..., v_n$ is a basis of V, we get that

$$\dim \mathrm{Lie}(\mathbf{\Gamma}^+(V,q)) = \frac{(\dim V)(\dim V-1)}{2} + 1.$$

Let σ be the canonical involution of $\mathfrak{Cl}(V,q)$. Define the spinor norm homomorphism $\mathrm{Sn}\colon \mathbf{\Gamma}^+(V,q) \to \mathbf{G}_{\mathrm{m}}$ by $x \mapsto x \cdot \sigma(x)$.

Definition 2.4.8. We define the **spin group scheme of** (V,q) $\mathbf{Spin}(V,q)$ as the kernel of the spinor homomorphism. The group of rational points, denoted by $\mathrm{Spin}(V,q)$, is called the **spin group of** (V,q).

The fact that $\mathbf{Spin}(V,q)$ is the kernel of the spinor norm morphism implies that the R points are:

$$\mathbf{Spin}(V,q)(R) = \{c \in \mathfrak{Cl}_{\overline{0}}(V,q)_R^\times \mid c \cdot V_R \cdot c = V_R,\, c\sigma(c) = 1\}.$$

As it is shown in [KMRT, 23.A]:

$$\mathrm{Lie}(\mathbf{Spin}(V,q)) = \{v \in V \cdot V \mid v + \sigma(v) = 0\}.$$

It is also shown that there is an exact sequence:

$$1 \to \mathbf{Spin}(V,q) \to \mathbf{\Gamma}^+(V,q) \to \mathbf{G}_m \to 1.$$

Therefore, [KMRT, 21.5] and [KMRT, 22.11] imply that

$$\dim \mathrm{Lie}(\mathbf{Spin}(V,q)) = \frac{(\dim V)(\dim V-1)}{2}. \tag{2.4.3}$$

We call the restriction to $\mathrm{Spin}(S,n)$ of ρ^+ and ρ^- as defined in (2.4.2) the **half spin representations of** $\mathrm{Spin}(S,n)$ and we will write ρ^+_x and ρ^-_x instead of using parentheses. Moreover, $\rho^+_x, \rho^-_x \in \mathrm{O}(S,n)$. If we extend it in the same way to the context of affine group schemes, we call them the **half spin representations of Spin**(S,n).

Remark 2.4.9. From the definition, $\mathbf{Spin}(V,q)$ is a subgroup scheme of $\mathbf{\Gamma}^+(V,q)$. Denote the inclusion by $\iota\colon \mathbf{Spin}(V,q) \to \mathbf{\Gamma}^+(V,q)$. The vector representation of the even Clifford group restricts to a morphism $\chi \circ \iota$ which we also denote by χ and which we call the **vector representation of the spin group scheme**. We can do the same argument for the context of groups where we would call the morphisms the **vector representation of the spin group**. Notice that from the definition of the spin group we have that $\chi_c(u) = c \cdot u \cdot \sigma(c)$ for all $c \in \mathbf{Spin}(V,q)(R)$ and $u \in V_R$.

Remark 2.4.10. Notice that the group of R-points of $\mathbf{Spin}(V,q)$ is

$$\mathbf{Spin}(V,q)(R) = \{c \in \mathfrak{Cl}_{\bar{0}}(V,q)_R \mid c \cdot V_R \cdot c^{-1} = V_R,\ c\sigma(c) = 1\}.$$

Moreover, this affine group scheme is smooth (see [KMRT, 23.A]).

Let $(S,\star,n)$ be a symmetric composition algebra of dimension 8.

Definition 2.4.11. A triple $(f_0,f_1,f_2) \in \mathbf{O}(S,n)^3(R)$ for some $\mathbb{F}$-algebra R is called a **related triple** if for any $x,y \in S_R$, $f_0(x\star y) = f_1(x)\star f_2(y)$.

Proposition 2.4.12. *There is an isomorphism from* $\mathrm{Spin}(S,n)$ *to the group* $\mathcal{T}(S,n) = \{(f_0,f_1,f_2) \in \mathrm{O}^+(S,n)^3 \mid f_0(x\star y) = f_1(x)\star f_2(y)\}$.

Proof. This is proved in [KMRT, (35.8)]. However, for completeness and concretion, we are going to sketch it.

For $x \in S$ and $c \in \mathrm{Spin}(S,n)$ we have that $\chi_c(x) = c\cdot x\cdot c^{-1}$ or equivalently, we get that $c \cdot x = \chi_c(x)\cdot c$. Since the morphism Φ, defined in Proposition 2.4.6, is an isomorphism, this is equivalent to the identity we get after applying it, i.e:

$$\begin{pmatrix} \rho_c^+ & 0 \\ 0 & \rho_c^- \end{pmatrix}\begin{pmatrix} 0 & l_x \\ r_x & 0 \end{pmatrix} = \begin{pmatrix} 0 & l_{\chi_c(x)} \\ r_{\chi_c(x)} & 0 \end{pmatrix}\begin{pmatrix} \rho_c^+ & 0 \\ 0 & \rho_c^- \end{pmatrix}.$$

If we apply both sides of the equality to $(y,y) \in S$, we get that $\rho_c^+(x\star y) = \chi_c(x)\star\rho_c^-(y)$ and $\rho_c^-(y\star x) = \rho_c^+(y)\star\chi_c(x)$. [KMRT, (35.4)] implies that $\chi_c(x\star y) = \rho_c^-(x)\star\rho_c^+(y)$.

Finally, in [KMRT, (35.7)] it is shown that the morphism:

$$\begin{array}{ccc} \mathrm{Spin}(S,n) & \to & \mathcal{T}(S,n) \\ c & \to & (\chi_c, \rho_c^-, \rho_c^+) \end{array} \tag{2.4.4}$$

is a well defined isomorphism. $\square$

Remark 2.4.13. The morphism in (2.4.4) extends to an isomorphism of affine group schemes $\mathbf{Spin}(S,n) \cong \mathbf{T}(S,n)$:

$$\begin{array}{ccc} \mathbf{Spin}(S,n) & \to & \mathbf{T}(S,n) \\ c & \to & (\chi_c, \rho_c^-, \rho_c^+). \end{array} \tag{2.4.5}$$

where we denote $\mathbf{T}(S,n)(R) = \{(f_0, f_1, f_2) \in \mathbf{O}^+(S,n)(R) \mid f_0(x \star y) = f_1(x) \star f_2(y) \, \forall x, y \in S\}$ [KMRT, (35.8)].

2.5 Automorphism group schemes

In this section we will calculate the automorphism group schemes of the r-fold cross products (V, X) . We will divide it into four subsections in which we will tackle four different cases of $n = \dim V$ and r (see Corollary 2.5.2, Theorem 2.5.3, Theorem 2.5.8 and Theorem 2.5.12).

2.5.1 n even, $r = 1$

Let X be a 1-fold cross product on a vector space of even dimension n relative to a bilinear form b. Denote $\mathbb{K} = \mathbb{F}\mathrm{id} \oplus \mathbb{F}X$ as in the proof of Theorem 2.3.8. $\mathbb{K}$ has a canonical involution sending X to $-X$.

Denote by $\mathbf{Cent}_{\mathbf{GL}(V)}(X)$ the group scheme whose group of R-points is given by the elements of $\mathbf{GL}(V)(R)$ which commute with X_R.

Theorem 2.5.1 ([DET20]). *Let $X\colon V \to V$ be a 1-fold cross product on the even-dimensional vector space V, relative to a nondegenerate symmetric bilinear form b.*

(1) $\mathbf{Aut}(V, X) = \mathbf{Cent}_{\mathbf{GL}(V)}(X)$.

(2) $\mathbf{Aut}(V, X, b) = \mathbf{U}(V, h)$ *where h is the hermitian nondegenerate form given by:*

$$\begin{aligned} h\colon V \times V &\longrightarrow \mathbb{K} \\ (u, v) &\longmapsto b(u, v)\mathrm{id} - b(X(u), v)X \end{aligned}$$

for any $u, v \in V$ and $\mathbf{U}(V, h)$ is the corresponding unitary group scheme whose group of R-points is given by the elements $\varphi \in \mathbf{GL}(V)(R)$ satisfying that $h_R(\varphi(u), \varphi(v)) = h_R(u, v)$ for all $u, v \in V_R$.

Proof. (1) is by definition.

Let's prove (2). h is clearly bilinear. In order to show that h is hermitian, we just need to show that for all $u, v \in V_R$, $h(u, v) = \overline{h(v, u)}$ and that $h(X(u), v) = Xh(u, v)$. Indeed, $h(X(u), v) = Xh(u, v)$ follows from substituntion in the expresion of h since $X^2 = -1$, and as it is shown in the proof of Theorem 2.3.8 $b(X(u), u) = 0$ for all $u \in V_R$. Linearizing, we obtain $b(X(u), v) + b(u, X(v)) = 0$ for all $u, v \in V_R$. Therefore, the fact that b is symmetric, implies that $h(u, v) = \overline{h(v, u)}$.

In order to show that both affine group schemes are the same, we first take $\varphi \in \mathbf{Aut}(V, X, b)(R)$. By definition, for all $u, v \in V_R$, we get that $b(\varphi(u), \varphi(v)) = b(u, v)$ and $b(X_R(\varphi(u)), \varphi(v)) = b(\varphi(X_R(u)), \varphi(v)) = b(X_R(u), v)$. Therefore, $\varphi \in \mathbf{U}(V, h)(R)$. Conversely, if $\varphi \in \mathbf{U}(V, h)(R)$, since id and X are linearly independent, we get that $b(X_R(\varphi(u)), \varphi(v)) = b(X_R(u), v) = b(\varphi(X_R(u)), \varphi(v))$. Since b is nondegenerate, we get that φ commutes with X_R. Hence $\varphi \in \mathbf{Aut}(V, X, b)(R)$. □

If $-1 \in \mathbb{F}^2$, then, we can decompose V into a direct sum of its eigenspaces. A morphism commutes with X if and only if it preserves its eigenspaces. Hence, we get the following corollary:

Corollary 2.5.2 ([DET20])**.** *Let $X: V \to V$ be a 1-fold cross product on a vector space V of even dimension, relative to a bilinear form b over a field containing a square root i of -1. Then:*

(1) $\mathbf{Aut}(V, X)$ is isomorphic to $\mathbf{GL}(V_+) \times \mathbf{GL}(V_-)$, where $V_\sigma = \{v \in V \mid X(v) = \sigma i v\}$ for $\sigma = \pm$.

(2) $\mathbf{Aut}(V, X, b)$ is isomorphic to $\mathbf{GL}(V_+)$.

Proof. For the second part, we just need to recall that $b(V_+, V_+) = b(V_-, V_-) = 0$. Hence, V_+ is paired with V_- and the action of the automorphism on V_+ determines its action on V_-. □

2.5.2 $n \geq 3$, $r = n - 1$

Given a vector space V with a nondegenerate symetric bilinear form b, we can define an affine subgroup scheme $\widetilde{\mathbf{O}}(V, b)$ of $\mathbf{GL}(V)$ whose R points are:

$$\widetilde{\mathbf{O}}(V, b)(R) = \{\varphi \in \mathbf{GL}(V)(R) \mid b_R(\varphi(u), \varphi(v)) = \det(\varphi) b(u, v)\, \forall u, v \in V_R\}.$$

Theorem 2.5.3 ([DET20])**.** *Let $X: V^{n-1} \to V$ be an $(n-1)$-fold cross product on the n-dimensional vector space V relative to the bilinear form b. Then the following two equations hold:*

(1) $\mathbf{Aut}(V,X) = \widetilde{\mathbf{O}}(V,b)$, *and*

(2) $\mathbf{Aut}(V,X,b) = \mathbf{O}^+(V,b)$.

Proof. We may extend scalars and, with the same arguments as in Proposition 2.3.10, assume that there is a basis $v_1, ..., v_n$ satisfying Equation (2.3.8). Let $\varphi \in \mathbf{Aut}(V)(R)$. Then, $\varphi \in \mathbf{Aut}(V)(R)$ if and only if:

$$X_R(\varphi(v_{\sigma(1)}), ..., \varphi(v_{\sigma(n-1)})) = \varphi(X_R(v_{\sigma(1)}, ..., v_{\sigma(n-1)})) = (-1)^\sigma \varphi(v_{\sigma(n)})$$

for any permutation σ. Define Φ_R as in (2.3.7). Since b is nondegenerate, this happens if and only if:

$$\begin{aligned} b_R(\varphi(v_{\sigma(n)}), \varphi(v_{\sigma(i)})) &= \Phi_R(\varphi(v_{\sigma(1)}), ..., \varphi(v_{\sigma(n-1)}), \varphi(v_{\sigma(i)})) \\ &= \det(\varphi)\Phi_R(v_{\sigma(1)}, .., v_{\sigma(n-1)}, v_{\sigma(i)}) \ \ \text{(see Remark 2.3.4)} \\ &= \begin{cases} \det(\varphi) & \text{if } i = n \\ 0 & \text{otherwise} \end{cases} \\ &= \det(\varphi) b_R(v_{\sigma(n)}, v_{\sigma(i)}) \end{aligned}$$

where Φ is the alternating form defined in Proposition 2.3.10. Therefore, φ is an automorphism if and only if

$$b_R(\varphi(u), \varphi(v)) = \det(\varphi) b_R(u, v) \tag{2.5.1}$$

for all $u, v \in V_R$. The second part is a direct consequence of Remark 2.1.4. □

Proposition 2.5.4 ([DET20]). *Let $X\colon V^{n-1} \to V$ be an $(n-1)$-fold cross product in the n-dimensional vector space V relative to the nondegenerate symmetric bilinear form $b\colon V \times V \to \mathbb{F}$. The determinant provides a short exact sequence:*

$$1 \to \mathbf{O}^+(V,b) \to \widetilde{\mathbf{O}}(V,b) \overset{\det}{\to} \boldsymbol{\mu}_{n-2} \to 1.$$

Proof. For any element $\varphi \in \widetilde{\mathbf{O}}(V,b)(R)$, we will prove first that $(\det \varphi)^{n-2} = 1$. After extending scalars, we can take an orthonormal basis $v_1, ..., v_n$ satisfying Equation (2.3.8). We have the following:

$$\begin{aligned} \det \varphi &= b(\varphi(v_n), \varphi(v_n)) \ \text{(using (2.5.1))} \\ &= b(\varphi(X(v_1, ..., v_{n-1})), \varphi(X(v_1, ..., v_{n-1})) \\ &= b(X(\varphi(v_1), ..., \varphi(v_n)), X(\varphi(v_1), ..., \varphi(v_n)))) \\ &= \det(b(\varphi(v_i), \varphi(v_j))) \\ &= (\det \varphi)^{n-1}. \end{aligned}$$

Hence, we have proven $(\det \varphi)^{n-2} = 1$.

Now we have to show that $\det\colon \widetilde{\mathbf{O}}(V,b) \to \boldsymbol{\mu}_{n-2}$ gives a quotient map.

Let R be a unital, associative and commutative algebra. Let $r \in \boldsymbol{\mu}_{n-2}(R)$. Consider the degree 2 extension $S = R[T]/(T^2 - r)$. Denote by t the class of T in S. The algebra S is a free R-module of rank 2. Thus, in view of [Wat79, 13.1], the extension S/R is faithfully flat.

Take an orthogonal basis $v_1, ..., v_n$ and define an automorphism $\varphi \in \mathrm{GL}(V)(S)$ by:

$$\varphi(v_i) = \begin{cases} tv_i & \text{for all } 1 \le i < n \\ t^{n-1}v_i & \text{if } i = n. \end{cases}$$

Notice that:

$$\det \varphi = t^{2(n-1)} = r^{n-1} = r. \tag{2.5.2}$$

And notice that $t^2 = r = r^{n-1} = t^{2(n-1)}$. Therefore, $\varphi \in \widetilde{\mathbf{O}}(V,b)(S)$. Now, Equation (2.5.2) and [Wat79, Theorem 15.5] imply that this is a quotient map. Since its kernel is $\mathbf{O}^+(V,b)$, the proposition holds. □

Remark 2.5.5. $\boldsymbol{\mu}_1$ is the trivial group. Therefore, in Proposition 2.5.4, the exactness of the sequence in the case $n = 3$ implies that $\widetilde{\mathbf{O}}(V,b) = \mathbf{O}^+(V,b)$. Hence, $\mathbf{Aut}(V,X) = \mathbf{Aut}(V,X,b) = \mathbf{O}^+(V,b)$.

Remark 2.5.6. $\mathbf{O}^+(V,b)$ is smooth (see [KMRT, 23.A]). Thus, Proposition 2.5.4 and [KMRT, (22.12)] imply that if $\boldsymbol{\mu}_{n-2}$ is smooth, then $\widetilde{\mathbf{O}}(V,b)$ is smooth. Moreover, due to [KMRT, (22.4)], if $\widetilde{\mathbf{O}}(V,b)$ is smooth, $\boldsymbol{\mu}_{n-2}$ is smooth.

$\mathbb{F}[\boldsymbol{\mu}_k] = \mathbb{F}[x]/(x^k - 1)$ for any $k \ge 1$. This is a separable algebra whenever the characteristic of $\mathbb{F}$ doesn't divide k. Hence, it is reduced and $\boldsymbol{\mu}_k$ is smooth. If the characteristic of $\mathbb{F}$ is p and $k = pm$ for some $m > 1$, we get that $(x^m - 1)^p = 0$. Therefore, the algebra is not reduced and this implies that $\boldsymbol{\mu}_k$ is not smooth. Therefore $\widetilde{\mathbf{O}}(V,b)$ is smooth if and only if the characteristic of $\mathbb{F}$ doesn't divide $n - 2$.

The Lie algebra of $\widetilde{\mathbf{O}}(V,b)$ is:

$$\mathrm{Lie}(\widetilde{\mathbf{O}}(V,b)) = \{f \in \mathrm{End}_{\mathbb{F}}(V) \mid b(f(u),v) + b(u,f(v)) = \mathrm{tr}(f)b(u,v)\ \forall u,v \in V\}.$$

Clearly, $\mathrm{Lie}(\widetilde{\mathbf{O}}(V,b)) \cap \mathfrak{sl}(V) = \mathfrak{so}(V,b)$ If we extend scalars and assume that the field is algebraically closed, we can take an orthonormal basis $v_1, ..., v_n$ of V. We get that for all $f \in \mathrm{Lie}(\widetilde{\mathbf{O}}(V,b))$

$$2\mathrm{tr}(f) = \sum_{i=1}^{n} (b(f(v_i), v_i) + b(v_i, f(v_i))) = \sum_{i=1}^{n} \mathrm{tr}(f) b(v_i, v_i) = n\mathrm{tr}(f),$$

which implies that $(n-2)\mathrm{tr}(f) = 0$. In case the characteristic of $\mathbb{F}$ doesn't divide $(n-2)$, that means that $\mathrm{tr}(f) = 0$. In case, the characteristic of $\mathbb{F}$ divides $(n-2)$, we get that $f - \frac{\mathrm{tr}(f)}{n}\mathrm{id} \in \mathrm{Lie}(\widetilde{\mathbf{O}}(v, b))$. Therefore we get:

$$\mathrm{Lie}(\widetilde{\mathbf{O}}(V, b)) = \begin{cases} \mathfrak{so}(V, b) & \text{if char}\mathbb{F} \text{ does not divide } n-2 \\ \mathfrak{so}(V, b) \oplus \mathbb{F}\mathrm{id} & \text{if char}\mathbb{F} \text{ divides } n-2. \end{cases}$$

2.5.3 $n = 7$, $r = 2$

According to Theorem 2.3.8, if (V, X) is a 2-fold cross product on a vector space V of dimension 7, then, it is isomorphic to $(\mathcal{C}_0, X^{\mathcal{C}_0})$ for some Cayley algebra $\mathcal{C}$. Proposition 2.3.10 shows that the bilinear form is uniquely determined and is b_n as defined in Example 2.3.6.

The author doesn't know of any source with a detailed proof of the following result, hence we give it here:

Proposition 2.5.7. *The morphism*

$$\begin{array}{rcl} \theta\colon \mathbf{Aut}(\mathcal{C}) & \to & \mathbf{Aut}(\mathcal{C}_0, X^{\mathcal{C}_0}) \\ f & \mapsto & f_{|\mathcal{C}_0} \end{array}$$

given by the restriction of f to the subspace of trace 0 elements, is an isomorphism of affine group schemes.

Proof. Let n be the norm of $\mathcal{C}$ and $b_n(x, y) = \frac{1}{2}(n(x+y) - n(x) - n(y))$. Define the morphism $\tau\colon \mathbf{Aut}(\mathcal{C}_0, X^{\mathcal{C}_0}) \to \mathbf{Aut}(\mathcal{C})$ by

$$\tau_R(f)(1 \otimes r + x) = 1 \otimes r + f(x)$$

for every $r \in R$ and $x \in \mathcal{C}_0 \otimes_{\mathbb{F}} R$. In case τ is well defined, it is clear that $\theta \circ \tau = \mathrm{id}$. Moreover, it is clear that every $f \in \mathbf{Aut}(\mathcal{C})(R)$ satisfy that $f(1) = 1$. Since $\mathcal{C} = \mathbb{F}1 \oplus \mathcal{C}_0$, then, $\tau_R \theta_R(f) = f$. Hence, $\tau \circ \theta = \mathrm{id}$. Which implies that $\tau = \theta^{-1}$. We just need to show that τ is well defined, i.e., that for every $f \in \mathbf{Aut}(\mathcal{C}_0, X^{\mathcal{C}_0})(R)$, $\tau(f) \in \mathbf{Aut}(\mathcal{C})(R)$.

Take $x, y \in \mathcal{C}_0 \otimes R$ and $r_1, r_2 \in R$. Using Equation (2.3.1), we get:

$$\begin{aligned}\tau(f)((1\otimes r_1+x)(1\otimes r_2+y)) &= \tau(f)(1\otimes((r_1r_2)-b_{nR}(x,y))\\ &\quad+(1\otimes r_1)x+(1\otimes r_2)y+X^{\mathcal{C}_0}(x,y)\\ &= 1\otimes((r_1r_2)-b_{nR}(x,y))+(1\otimes r_1)\tau(f)(x)\\ &\quad+(1\otimes r_2)\tau(f)(y)+X^{\mathcal{C}}(\tau(f)(x),\tau(f)(y)).\end{aligned}$$

This shows, due to Equation (2.3.1), that $\tau_R(f)$ is an automorphism if and only if

$$b_{nR}(x,y)=b_{nR}(\tau_R(f)(x),\tau_R(f)(y))=b_{nR}(f(x),f(y)) \tag{2.5.3}$$

for all $x,y\in\mathcal{C}_{0R}$. But this is true because of Equation (2.3.3).

□

Having this result, the following is easy.

Theorem 2.5.8 ([DET20])**.** *Let (V,X,b) be a 2-fold cross product on a seven-dimensional vector space V relative to the bilinear form b. Then*

$$\mathbf{Aut}(V,X)=\mathbf{Aut}(V,X,b)$$

and it is a simple affine group scheme of type G_2.

Proof. $\mathbf{Aut}(V,X)=\mathbf{Aut}(V,X,b)$ is a consequence of Equation (2.5.3). The fact that it is a simple affine group scheme of type G_2 is due to the fact that it is isomorphic to $\mathbf{Aut}(\mathcal{C})$ and Proposition 2.2.10. □

2.5.4 $n=8$, $r=3$

Due to Theorem 2.3.8, we know that a 3-fold cross product on a vector space of dimension 8 is isomorphic to $(\mathcal{C},\alpha X_1^{\mathcal{C}})$ for a Cayley algebra $\mathcal{C}$, a nonzero scalar α and $X_1^{\mathcal{C}}$ as in Equation (2.3.4). Since $\mathbf{Aut}(\mathcal{C},X_1^{\mathcal{C}})=\mathbf{Aut}(\mathcal{C},\alpha X_1^{\mathcal{C}})$ for all nonzero $\alpha\in\mathbb{F}$, we can assume that $\alpha=1$.

We consider the following triple product.

$$\{x,y,z\}=(x\bar{y})z. \tag{2.5.4}$$

We call this a $3C$-product and the pair $(\mathcal{C},\{\cdots\})$ a $3C$-algebra. In view of (2.3.4), we have that $\mathbf{Aut}(\mathcal{C},X_1^{\mathcal{C}},b_n)=\mathbf{Aut}(\mathcal{C},\{\cdots\},n)$. On the other hand, since $\mathcal{C}$ is an alternative algebra, if $n\colon\mathcal{C}\to\mathbb{F}$ denotes its norm, we have:

$$\{x,x,y\}=n(x)y=\{y,x,x\}. \tag{2.5.5}$$

In view of (2.5.5), we get that $\mathbf{Aut}(\mathcal{C}, \{\cdots\}) = \mathbf{Aut}(\mathcal{C}, \{\cdots\}, n)$. Therefore, in general we have:

$$\mathbf{Aut}(\mathcal{C}, X_1^{\mathcal{C}}, b_n) = \mathbf{Aut}(\mathcal{C}, \{\cdots\}) = \mathbf{Aut}(\mathcal{C}, \{\cdots\}, n). \tag{2.5.6}$$

In [Eld96, Theorem 10] it is shown that the automorphism group of $(\mathcal{C}, \{\cdots\})$, i.e, $\mathbf{Aut}(\mathcal{C}, \{\cdots\})(\mathbb{F})$, is isomorphic to the group $\mathrm{Spin}(\mathcal{C}_0, n)$. Similar arguments can be used in order to calculate the automorphism group scheme.

Let $\mathcal{C}$ be a Cayley algebra with norm n. Consider the para-Cayley algebra $(\mathcal{C}, \bullet, n)$ as in Example 2.2.14. Define the maps $l_x, r_x\colon \mathcal{C} \to \mathcal{C}$ as $l_x(y) = x \bullet y = \bar{x}\bar{y} = r_y(x)$ for all $x, y \in \mathcal{C}$. Consider the isomorphism

$$\Phi\colon \mathfrak{Cl}(\mathcal{C}, n) \to \mathrm{End}_{\mathbb{F}}(\mathcal{C} \oplus \mathcal{C})$$

as in Proposition 2.4.6. Consider the isomorphism of affine group schemes $\mathbf{Spin}(\mathcal{C}, n) \cong \mathbf{T}(\mathcal{C}, n)$ as in Remark 2.4.13 and denote by e_0 the unit in the Cayley algebra $\mathcal{C}$ in order not to confuse it with the unity of the Clifford algebra.

Proposition 2.5.9 ([DET20]). *For $(\mathcal{C}, \bullet, n)$ and e_0 as in the previous paragraph, the isomorphism $\mathbf{Spin}(\mathcal{C}, n) \cong \mathbf{T}(\mathcal{C}, n)$ given in (2.4.5) induces an isomorphism*

$$\begin{aligned}\mathbf{Spin}(\mathcal{C}_0, -n) \cong \{&(f_0, f_1, f_2) \in \mathbf{O}^+(\mathcal{C}, n)^3 \mid \\ &f_0(x \bullet y) = f_1(x) \bullet f_2(y), \text{ and } f_0(e_0) = e_0 \,\forall x, y \in \mathcal{C}\}.\end{aligned}$$

Proof. The proof is in [DET20, Proposition 3.7.] but for completeness we write it here. Consider the map:

$$\begin{array}{rcl}\mathcal{C}_0 & \to & \mathfrak{Cl}(\mathcal{C}, n) \\ x & \mapsto & e_0 \cdot x.\end{array}$$

Notice that $(e_0 \cdot x)^2 = -(e_0 \cdot e_0) \cdot (x \cdot x) + n(e_0, x) e_0 \cdot x$ due to Remark 2.4.2. Moreover, since $x \in \mathcal{C}_0$, we get that $(e_0 \cdot x)^2 = -n(e_0)n(x)1 = -n(x)1$. The universal property of the Clifford algebras and the fact that for all $x \in \mathfrak{Cl}_0$, $e_0 \cdot x \in \mathfrak{Cl}_0(\mathcal{C}, n)$ imply that this linear map induces an embedding

$$\Psi\colon \mathfrak{Cl}(\mathcal{C}_0, -n) \;\to\; \mathfrak{Cl}_0(\mathcal{C}, n). \tag{2.5.7}$$

We know that $\dim \mathfrak{Cl}(\mathcal{C}_0, -n) = 2^{\dim \mathcal{C}_0} = 2^{\dim \mathcal{C} - 1} = \dim \mathfrak{Cl}_0(\mathcal{C}, n)$ (see Section 2.4). Therefore, Ψ is an isomorphism.

Let σ be the canonical involution on $\mathfrak{Cl}(\mathcal{C}, n)$. Denote by σ' the composition of the canonical involution in $\mathfrak{Cl}(\mathcal{C}_0, -n)$ with the automorphism induced by the map $\mathcal{C}_0 \to \mathfrak{Cl}(\mathcal{C}_0, -n)$ sending x to $-x$. It is clear that in this case, the restriction of σ' to the even part is the same as the one of the canonical involution.

Let $x \in \mathcal{C}_0$. We get:

$$\Psi(\sigma'(x)) = -\Psi(x) = -e_0 \cdot x = x \cdot e_0 = \sigma(x) \cdot \sigma(e_0) = \sigma(e_0 \cdot x) = \sigma(\Psi(x)).$$

Therefore, it follows

$$\Psi\sigma' = \sigma\Psi. \tag{2.5.8}$$

The elements of $\mathcal{C}_0$ anticommute with e_0 due to Remark 2.4.2. Therefore

$$\begin{aligned}\Psi(\mathfrak{Cl}(\mathcal{C}_0, -n)_{\bar{0}}) &= \{a \in \mathfrak{Cl}(\mathcal{C}, n)_{\bar{0}} \mid e_0 \cdot a = a \cdot e_0\}, \text{ and} \\ \Psi(\mathfrak{Cl}(\mathcal{C}_0, -n)_{\bar{1}}) &= \{a \in \mathfrak{Cl}(\mathcal{C}, n)_{\bar{0}} \mid e_0 \cdot a = -a \cdot e_0\}.\end{aligned}$$

Concretely, for any $u \in \mathrm{Spin}(\mathcal{C}_0, -n)$ we get that $\Psi(u) \cdot e_0 = e_0 \cdot \Psi(u)$. Hence, for any $x \in \mathcal{C}_0$ and $u \in \mathrm{Spin}(\mathcal{C}_0, -n)$ we have:

$$\Psi(u \cdot x \cdot u^{-1}) = \Psi(u) \cdot e_0 \cdot x \cdot \Psi(u^{-1}) = e_0 \cdot \Psi(u) \cdot x \cdot \Psi(u^{-1}).$$

Multiplying by e_0 and noting that $e_0^{\cdot 2} = 1$ and that $u \cdot x \cdot u^{-1} \in \mathcal{C}_0$, we get

$$\Psi(u) \cdot x \cdot \Psi(u)^{-1} = e_0 \cdot \Psi(u \cdot x \cdot u^{-1}) = e_0 \cdot e_0 \cdot \chi_u(x) = \chi_u(x) \in \mathcal{C}_0.$$

Since $\Psi(u) \cdot e_0 \cdot \Psi(u)^{-1} = e_0$, it implies that $\Psi(u) \in \mathrm{Spin}(\mathcal{C}, n)$. Conversely, if $v \in \mathrm{Spin}(\mathcal{C}, n)$ such that $e_0 \cdot v = v \cdot e_0$, then there is $u \in \mathfrak{Cl}(\mathcal{C}_0, n)_{\bar{0}}$ such that $v = \Psi(u)$. Since $1 = v \cdot \sigma(v) = \Psi(u \cdot \sigma'(u))$ due to (2.5.8), the fact that Ψ is an isomorphism implies that $u \cdot \sigma'(u) = 1$. Thus, for any $x \in \mathcal{C}_0$,

$$\Psi(u \cdot x \cdot u^{-1}) = v \cdot e_0 \cdot x \cdot v^{-1} = e_0 \cdot \chi_v(x) = \Psi(\chi_u(x)) \in \Psi(\mathcal{C}_0).$$

The fact that Ψ is an isomorphism implies that $\chi_u(x) \in \mathcal{C}_0$. Hence $u \in \mathrm{Spin}(\mathcal{C}_0, -n)$.

Therefore, we can restrict Ψ to a group isomorphism:

$$\mathrm{Spin}(\mathcal{C}_0, -n) \cong \mathrm{Cent}_{\mathrm{Spin}(\mathcal{C}, n)}(e_0)$$

where $\mathrm{Cent}_{\mathrm{Spin}(\mathcal{C},n)}(e_0)$ is the centralizer of e_0 in $\mathrm{Spin}(\mathcal{C},n)$, i.e., those elements $u \in \mathrm{Spin}(\mathcal{C},n)$ such that $\chi_u(e_0) = e_0$. Therefore, we get that $\mathrm{Spin}(\mathcal{C}_0, -n) \cong \{u \in \mathrm{Spin}(\mathcal{C},n) \mid \chi_u(e_0) = e_0\}$. Since the arguments are functorial, Ψ induces an isomorphism of affine group schemes $\mathbf{Spin}(\mathcal{C}_0, -n) \cong \{u \in \mathbf{Spin}(\mathcal{C},n) \mid \chi_u(e_0) = e_0\}$. Composing with the isomorphism, (2.4.5) we get

$$\begin{array}{rcl} \mathbf{Spin}(\mathcal{C}_0, -n) & \to & \{(f_0, f_1, f_2) \in \mathbf{T}(\mathcal{C},n) \mid f_0(e_0) = e_0\} \\ u & \mapsto & (\chi_{\Psi(u)}, \rho^-_{\Psi(u)}, \rho^+_{\Psi(u)}). \end{array}$$

□

Remark 2.5.10. The fact that $\mathcal{C}$ is a Cayley algebra implies that it has a unit e_0. Given a triple $(f_0, f_1, f_2) \in \mathbf{T}(\mathcal{C},n)$ such that $f_0(e_0) = e_0$, the fact that $f_2(e_0 \bullet y) = f_0(e_0) \bullet f_1(y)$ for all $y \in \mathcal{C}$, implies that $f_1(y) = \overline{f_2(\overline{y})}$ for all $y \in \mathcal{C}$.

Conversely, if we have $(f_0, f_1, f_2) \in \mathbf{T}(\mathcal{C},n)$, with $f_1(y) = \overline{f_2(\overline{y})}$ for all $y \in \mathcal{C}$, the fact that $f_2(x) \bullet e_0 = \overline{f_2(x)} = f_1(\overline{x}) = f_1(x \bullet e_0) = f_2(x) \bullet f_0(e_0)$ implies that $f_0(e_0) = e_0$.

Since f_0, and f_1 are determined by f_2, we obtain an injective homomorphism:

$$\begin{array}{rcl} \theta\colon \mathbf{Spin}(\mathcal{C}_0, -n) & \to & \mathbf{O}^+(\mathcal{C},n) \\ u & \mapsto & \rho^+_{\Psi(u)} \end{array} \tag{2.5.9}$$

with Ψ as in (2.5.7).

We would like to show that the image of θ is the automorphism group scheme of $(\mathcal{C}, X_1^{\mathcal{C}})$.

Remark 2.5.11. Notice that for every $x \in \mathcal{C}_0$,

$$\begin{pmatrix} \rho^+_{\Psi(x)} & 0 \\ 0 & \rho^-_{\Psi(x)} \end{pmatrix} = \Phi(\Psi(x))$$

where Φ is defined as in Propositon 2.4.6. On the other hand, since Φ is a morphism of algebras, we get:

$$\Phi(\Psi(x)) = \Phi(e_0 \cdot x) = \begin{pmatrix} 0 & l_{e_0} \\ r_{e_0} & 0 \end{pmatrix} \begin{pmatrix} 0 & l_x \\ r_x & 0 \end{pmatrix} = \begin{pmatrix} l_{e_0} r_x & 0 \\ 0 & r_{e_0} l_x \end{pmatrix}.$$

Since for all $y \in \mathcal{C}$ we have that $l_{e_0} r_x(y) = \overline{e_0}\overline{(\overline{y}\overline{x})} = xy$ and similarly, $r_{e_0} l_x(y) = yx$ then, if we define for all $x \in \mathcal{C}$, the linear morphisms $L_x, R_x \in \mathrm{End}_{\mathbb{F}}(\mathcal{C})$ by $L_x(y) = xy$ and $R_x(y) = yx$, we get that $\rho^+_{\Psi(x)} = L_x$ and $\rho^-_{\Psi(x)} = R_x$. Since for any $u \in \Gamma^+(\mathcal{C}_0, -n)$ there is a natural number s and non isotropic elements $u_1, ..., u_{2s} \in \mathcal{C}_0$ such that $u = u_1 \cdots u_{2s}$ (see [JAC89, Theorem 4.15]) and since for $u_1, ..., u_{2s} \in \mathcal{C}_0$ we have that $\sigma(u_1 \cdots u_n) = u_{2s} \cdots u_1$, we have that $u\sigma(u) = 1$ if and only if $n(u_1) \cdots n(u_{2s}) = 1$. Hence, the image of θ of $\mathrm{Spin}(\mathcal{C}_0, -n)$ is the group of elements of the form $L_{u_1} \cdots L_{u_{2s}}$ with $u_1, ..., u_{2s} \in \mathcal{C}_0$ and $n(u_1) \cdots n(u_{2s}) = 1$. It is shown in [Eld96] that this group is $\mathrm{Aut}(\mathcal{C}, \{\cdots\})$.

We will extend this result to the setup of affine group schemes.

Theorem 2.5.12 ([DET20]). *The group scheme of automorphisms* $\mathbf{Aut}(\mathcal{C}, \{\cdots\})$ *is isomorphic to* $\mathbf{Spin}(\mathcal{C}_0, -n)$ *via* θ *as defined in* (2.5.9)

Proof. For completeness we write the proof here.

Given an algebra R, and a related triple $(f_0, f_1, f_2) \in (\mathbf{O}^+(\mathcal{C}, n)(R))^3$ such that $f_0(e_0) = e_0$, we have that for any $x, y, z \in \mathcal{C}$:

$$
\begin{aligned}
f_2(\{x,y,z\}) &= f_2((x\overline{y})z) \\
&= f_2((y\overline{x}) \bullet \overline{z}) \\
&= f_0(y\overline{x}) \bullet f_1(\overline{z}) \\
&= f_0(\overline{y} \bullet x) \bullet \overline{f_2(z)} \quad \text{(see Remark 2.5.10)} \\
&= (f_1(\overline{y}) \bullet f_2(x)) \bullet \overline{f_2(z)} \\
&= (\overline{f_2(y)} \bullet f_2(x)) \bullet \overline{f_2(z)} \\
&= (f_2(y)\overline{f_2(x)}) \bullet \overline{f_2(z)} \\
&= (f_2(x)\overline{f_2(y)})f_2(z) \\
&= \{f_2(x), f_2(y), f_2(z)\}.
\end{aligned}
$$

Therefore, θ factors through $\mathbf{Aut}(\mathcal{C}, \{\cdots\})$. Due to the arguments shown in Remark 2.5.11, $\theta_{\overline{\mathbb{F}}}$ is bijective. Due to Example 1.3.10, $\mathrm{Lie}(\mathbf{Aut}(\mathcal{C}, \{\cdots\})) = \mathrm{Der}(\mathcal{A}, \{\cdots\})$. It is shown in [Eld96, Theorem 12] that $\mathrm{Der}(\mathcal{A}, \{\cdots\})$ is isomorphic to the orthogonal Lie algebra $\mathfrak{so}(\mathcal{C}_0, -n)$, which has dimension 21. Since θ is injective, the differential $d\theta$ is also injective and since due to (2.4.3) the dimension of $\mathrm{Lie}(\mathbf{Spin}(\mathcal{C}_0, -n))$ is 21, $d\theta$ is bijective. Finally, since $\mathbf{Spin}(\mathcal{C}_0, -n)$ is smooth as shown in Remark 2.4.10, [EK13, Theorem A.50] implies that θ is an isomorphism.

□

Proposition 2.5.13. *For any Cayley algebra* $\mathcal{C}$*, and* b_n *as in Example 2.3.7, let* $\iota\colon \mathbf{Aut}(\mathcal{C}, X_1^{\mathcal{C}}, b_n) \to \mathbf{Aut}(\mathcal{C}, X_1^{\mathcal{C}})$ *be the inclusion. Then* ι *is an isomorphism.*

Proof. Assume that (V, X, b) is a type I 3-fold cross product over a vector space of dimension 8. Consider the inclusion $\iota\colon \mathbf{Aut}(V, X, b) \to \mathbf{Aut}(V, X)$.

Let R be a field extension of $\mathbb{F}$ or $\mathbb{F}[\tau]/\langle\tau^2\rangle$ (which we just denote $\mathbb{F}[\tau]$). If we have a $\varphi \in \mathbf{Aut}(V, X, b)(R)$, and we define

$$X'_R(u, v, w) = \varphi^{-1} X_R(\varphi(u), \varphi(v), \varphi(w))$$

and

$$b'_R(u, v) = b_R(\varphi(x), \varphi(y))$$

for all $u, v, w \in V \otimes_{\mathbb{F}} R$, then (V_R, X'_R, b'_R) satisfies (2.1.1), (2.1.2), (2.3.4) and it satisfies that b'_R is nondegenerate. We identify V with $V \otimes 1$. Since in $\mathbb{F}[\tau]$, $\mu^2 = 1$ if and only if $\mu = \pm 1$, with the same arguments as in Proposition 2.3.10, we can show that

$$b_R(x, x) = \pm b'_R(x, x)$$

for all x such that $b_R(x, x) \neq 0$. Therefore,

$$(b_R(x, x) - b'_R(x, x))(b_R(x, x) - b'_R(x, x)) = 0$$

in the Zariski open set $U = \{x \in V \mid b_R(x, x) \neq 0\}$ of V. Since $b'_R(x, x) = \pm b_R(x, x) \in \mathbb{F}$ for all $x \in U$, then either $b_R(x, x) - b'_R(x, x) = 0$ for all $x \in U$ or $b_R(x, x) + b'_R(x, x) = 0$ for all $x \in U$. Thus, as in Proposition 2.3.10, either $b_R(u, v) = b'_R(u, v)$ or $b_R(u, v) = -b'_R(u, v)$ for all $u, v \in V$. Thus, $b_R = \pm b'_R$.

Since φ is an automorphism, $X'_R = X_R$. Since (V, X', b') satisfies (2.3.4) and $b'_R = \pm b_R$, we get that $b'_R = b_R$. Therefore, $\varphi \in \mathrm{Aut}(V, X, b)(R)$. This implies that ι_R is an isomorphism for all R as before. Concretely, for $R = \mathbb{F}$, $R = \overline{\mathbb{F}}$ and $R = \mathbb{F}[\tau]$.

Now, since letting $\pi\colon \mathbb{F}[\tau] \to \mathbb{F}$ be the morphism of algebras with unit sending τ to 0, the following diagram commutes:

$$\begin{array}{ccc} \mathbf{Aut}(V, X, b)(\mathbb{F}[\tau]) & \xrightarrow{\iota_{\mathbb{F}}} & \mathbf{Aut}(V, X)[\mathbb{F}[\tau]] \\ {\scriptstyle \mathbf{Aut}(V,X,b)(\pi)}\downarrow & & \downarrow{\scriptstyle \mathbf{Aut}(V,X)(\pi)} \\ \mathbf{Aut}(V, X, b)(\mathbb{F}) & \xrightarrow{\iota_{\mathbb{F}[\tau]}} & \mathbf{Aut}(V, X)(\mathbb{F}) \end{array}$$

and due to the fact that $\iota_{\mathbb{F}}$ and $\iota_{\mathbb{F}[\tau]}$ are isomorphisms, $\iota_{\mathbb{F}}(\ker \mathbf{Aut}(V, X, b)(\pi)) = \ker \mathbf{Aut}(V, X)(\pi)$. Thus $d\iota$ is an isomorphism.

Finally, due to (2.5.6) and Theorem 2.5.12, $\mathbf{Aut}(V, X, b)$ is smooth. Therefore, [EK13, Theorem A.50] implies that ι is an isomorphism. $\square$

2.6 Gradings

In this section we will assume that $\mathbb{F}$ is algebraically closed unless otherwise stated. Every grading group will be abelian and finitely generated. We are going to classify up to isomorphism all the group gradings on r-cross products (V, X) and up to equivalence all the fine gradings. We are going to divide this section into different subsections in which we deal with different values of $n = \dim V$ and r (see Theorems 2.6.1, 2.6.4 and 2.6.11).

2.6.1 n even, $r = 1$

Let (X, V) be a 1-fold cross product on an even dimensional vector space V of dimension $n = 2s$. A grading by an abelian group G is a vector space decomposition $\Gamma\colon V = \bigoplus_{g\in G} V_g$ such that $X(V_g) \subseteq V_g$. Since the field is algebraically closed and $X^2 = -\mathrm{id}$, take $i \in \mathbb{F}$ such that $i^2 = -1$ and denote $V_\sigma = \{v \in V \mid X(v) = \sigma i v\}$. Then, $V = V_+ \oplus V_-$ and for every $g \in G$, $V_g = (V_g \cap V_+) \oplus (V_g \cap V_-)$. Therefore, we obtain a grading of the vector space V_+ and a grading of the vector space V_-. Conversely, any pair of gradings of the vector spaces V_+ and V_- induce a grading on (V, X).

Given a basis $v_1, ..., v_s$ of V_+, a basis $w_1, ..., w_s$ of V_- and elements of the group $g_1, ..., g_s, h_1, ..., h_s \in G$, we denote by $\Gamma(G, V, X, (g_1, ..., g_s), (h_1, ..., h_s))$ the grading on (V, X) such that $\deg v_i = g_i$ and $\deg w_i = h_i$.

Since a G-grading on a vector space is a vector space decomposition with subspaces indexed by G and since $\mathbf{Aut}(V, X) = \mathbf{GL}(V_+) \times \mathbf{GL}(V_-)$, we get the following result.

Theorem 2.6.1. *Let (X, V) be a 1-fold cross product on an even dimensional vector space V of dimension $n = 2s$ over an algebraically closed field $\mathbb{F}$. The only gradings up to isomorphism on (X, V) are $\Gamma(G, V, X, (g_1, ..., g_s), (h_1, ..., h_s))$ for $g_1, ..., g_s, h_1, ..., h_s \in G$. Moreover, if we have $2s$ elements $g'_1, ..., g'_s, h'_1, ..., h'_s \in G$, the grading $\Gamma(G, V, X, (g_1, ..., g_s), (h_1, ..., h_s))$ is isomorphic to the grading $\Gamma(G, V, X, (g'_1, ..., g'_s), (h'_1, ..., h'_s))$ if and only if there are permutations $\sigma, \tau \in \mathrm{Sym}_n$ such that $(g_1, ..., g_s) = (g'_{\sigma(1)}, ..., g'_{\sigma(s)})$ and $(h_1, ..., h_s) = (h'_{\tau(1)}, ..., h'_{\tau(s)})$.*

Similarly, we can get the following corollary.

Corollary 2.6.2. *Up to equivalence, there is a unique fine grading Γ with universal group $\mathbb{Z}^n$, i.e., the grading $\Gamma(\mathbb{Z}^n, V, X(\epsilon_1, ..., \epsilon_s), (\epsilon_{s+1}, ..., \epsilon_n))$ where $\epsilon_1, ..., \epsilon_n$ is the canonical basis of $\mathbb{Z}^n$.*

Those previous results are a restatement of the results obtained in [DET20, 4.1].

If we denote $\Gamma = \Gamma(\mathbb{Z}^n, V, X, (\epsilon_1, ..., \epsilon_s), (\epsilon_{s+1}, ..., \epsilon_n))$, we get that for any $\mathbb{F}$-algebra R,

$$\begin{aligned}\mathbf{Diag}(\Gamma)(R) = \{\varphi \in \mathbf{GL}(V) \mid \exists r_1, ..., r_s, t_1, ..., t_s \in R^\times, \text{ such that} \\ \varphi(v_i) = r_i v_i,\ \varphi(w_i) = s_i w_i,\ \forall i \in \{1, ..., s\}\}.\end{aligned}$$

Thus,

$$\mathbf{Diag}(\Gamma(\mathbb{Z}^n, V, X, (\epsilon_1, ..., \epsilon_s), (\epsilon_{s+1}, ..., \epsilon_n))) \cong \mathbf{G}_{\mathrm{m}}^n.$$

The fact that the homogeneous components of Γ are one dimensional imply that $\mathbf{Stab}(\Gamma) = \mathbf{Diag}(\Gamma)$. We denote by Sym_s the symmetric group of s elements. For any two permutations $\sigma, \tau \in \mathrm{Sym}_s$ we denote by $f^\Gamma_{\sigma,\tau} \colon V \to V$ the map induced by $f^\Gamma_{\sigma,\tau}(v_i) = v_{\sigma(i)}$ and $f^\Gamma_{\sigma,\tau}(w_i) = w_{\tau(i)}$. Consider the morphism:

$$\begin{aligned}\eta\colon \mathrm{Stab}(\Gamma) \times \mathrm{Sym}_s \times \mathrm{Sym}_s &\to \mathrm{Aut}(\Gamma) \\ (\varphi, \sigma, \tau) &\mapsto \varphi \circ f^\Gamma_{\sigma,\tau}.\end{aligned}$$

It is not hard to show that η is well defined and that it is an isomorphism. Hence, we can show that:

$$\mathrm{W}(\Gamma(\mathbb{Z}^n, V, X, (\epsilon_1, ..., \epsilon_s), (\epsilon_{s+1}, ..., \epsilon_n))) \cong \mathrm{Sym}_s \times \mathrm{Sym}_s.$$

Remark 2.6.3. The previous arguments work for any field in which -1 is a square.

If $-1 \notin \mathbb{F}^2$, $\mathbb{K} = \mathbb{F}\mathrm{id} \oplus \mathbb{F}X$ is a field. Moreover, given a G-grading $\Gamma\colon V = \bigoplus_{g\in G} V_g$ of (V, X). Since for any $g \in G$ $X(V_g) \subseteq V_g$, it is a grading of the $\mathbb{K}$-vector space. In particular, if we take a $\mathbb{K}$-basis, there is only one fine grading up to equivalence, with universal group $\mathbb{Z}^s$ and assigning, for a $\mathbb{K}$-basis $v_1, ..., v_s$ of V, degrees $\deg v_i = \epsilon_i$.

2.6.2 $n \geq 3$, $r = n - 1$

In this case we are going to consider an r-fold cross product (V, X) over an $r+1$ dimensional vector space relative to a nondegenerate symmetric bilinear form b.

Let G be an abelian group and $\Gamma\colon V = \bigoplus_{g\in G} V_g$ a G-grading of (V, X). We can induce an automorphism $\varphi_\Gamma \in \mathbf{Aut}(V, X)(\mathbb{F}G) = \widetilde{\mathbf{O}}(V, X)(\mathbb{F}G)$ as in (1.5.3). If $v, w \in V$ are homogeneous elements, φ_Γ is in $\widetilde{\mathbf{O}}(V, X)(\mathbb{F}G)$ if and only

$$b(v, w) \deg v \deg w = b(\varphi_\Gamma(v), \varphi_\Gamma(w)) = (\det \varphi_\Gamma) b(v, w).$$

Therefore, if $v, w \in V$ we get

$$b(v,w) = 0 \text{ unless } (\deg v)(\deg w) = \det \varphi_\Gamma. \tag{2.6.1}$$

Let us denote $h = \det \varphi_\Gamma$. We can define a map:

$$\begin{array}{rcl} \delta\colon G & \to & \mathbb{Z}_{\geq 0} \\ g & \mapsto & \delta(g) := \dim_{\mathbb{F}} V_g. \end{array}$$

Since V is a direct sum of its homogeneous components,

$$\sum_{g\in G} \delta(g) = n. \tag{2.6.2}$$

Since for a homogeneous basis $v_1, ..., v_n$, $\varphi_\Gamma(v_i \otimes 1) = v_i \otimes \deg v_i$ for all $i \in \{1, ..., n\}$, we get that $\det \varphi_\Gamma = \prod_{i=1}^n \deg v_i$. Therefore,

$$h = \det \varphi_\Gamma = \prod_{g\in G} g^{\delta(g)}. \tag{2.6.3}$$

Finally, due to (2.6.1) and the fact that b is nondegenerate, we can take a basis of homogeneous elements $u_1, ..., u_s, v_1, w_1, ..., v_r, w_r$ such that $\deg(u_i)^2 = h = \deg(v_i)\deg(w_i)$ and $b(u_i, u_i) = 1 = b(v_i, w_i)$ and any other combination is 0. Hence, the elements of V_g and $V_{g^{-1}h}$ are in duality by b, so we get that for all $g \in G$:

$$\delta(g) = \delta(g^{-1}h). \tag{2.6.4}$$

Proposition 2.5.4 implies that if e is the neutral element of the group:

$$h^{n-2} = e. \tag{2.6.5}$$

Conversely, assume that we have a map $\delta\colon G \to \mathbb{Z}_{\geq 0}$ satisfying (2.6.2), (2.6.3) and (2.6.4), we can take a basis $\{v_1^g, ..., v_{\delta(g)}^g \mid g \in G, \delta(g) > 0\}$ of V satisfying

$$\begin{array}{l} b(v_i^g, v_i^g) = 1 \text{ for every } g \in G \text{ such that } g^2 = h \text{ and } 1 \leq i \leq \delta(g), \\ b(v_i^{g_1}, v_i^{g_2}) = 1 \text{ for every } g_1, g_2 \in G \text{ such that } g_1^2 \neq h \neq g_2^2 \text{ and } g_1 g_2 = h \end{array} \tag{2.6.6}$$

and the rest of the combinations is 0. This is a grading since the map φ_Γ defined by $\varphi_\Gamma(v_i^g \otimes 1) = v_i^g \otimes g$ is in $\widetilde{\mathbf{O}}(\mathcal{V}, b)$ which implies that, the map ρ_Γ defined as in (1.5.3) is a comodule map. We denote this grading by $\Gamma(G, \delta)$. This proves the following theorem:

Theorem 2.6.4 ([DET20]). *Let $X\colon V^{n-1} \to V$ be an $(n-1)$-fold cross product on the n-dimensional vector space V with $n \geq 3$ over an algebraically closed field $\mathbb{F}$, relative to a nondegenerate symmetric bilinear form b. Let G be an abelian group.*

Then, any G-grading on (V, X) is isomorphic to a grading $\Gamma(G, \delta)$ for a unique map $\delta\colon G \to \mathbb{Z}_{\geq 0}$ satisfying:

(1) $\sum_{g\in G} \delta(g) = n$,

(2) $\delta(g) = \delta(g^{-1}h)$ *where* $h = \prod_{g\in G} g^{\delta(g)}$

Given a map δ satisfying the restriction in Theorem 2.6.4, for $\Gamma = \Gamma(G, \delta)$, we can take a basis as in (2.6.6). Denoting $u_1, ..., u_p$ the elements of the basis such that $b(u_i, u_i) = 1$, choosing pairs $v_1, w_1, ..., v_q, w_q$ of elements of the basis such that $b(v_i, w_i) = 1$ and denoting $\deg(u_i) = g_i$, $\deg(v_j) = g'_j$ and $\deg(w_j) = g''_j$ for every $1 \leq i \leq p$ and $1 \leq j \leq q$, we can check that these elements satisfy:

$$g_i^2 = h = g'_j g''_j \text{ where } h = g_1 \cdots g_p g'_1 g''_1 \ldots g'_q g''_q$$

Moreover, $n = p + 2q$. This allow us to find a refinement in which all the homogeneous components have dimension 1. In order to do so, consider the group

$$U = \langle x_1, ..., x_p, y_1, ..., y_q, z_1, ..., z_q \mid x_1^2 = ... = x_p^2 = y_1z_1 = ... = y_qz_q \\ = x_1 \cdots x_p y_1 z_1 \cdots y_q z_q \rangle,$$

i.e., the group generated by elements $x_1, ..., x_p, y_1, ..., y_q$, $z_1, ..., z_q$ subject to the relations

$$x_1^2 = ... = x_p^2 = y_1z_1 = ... = y_qz_q = x_1 \cdots x_p y_1 z_1 \cdots y_q z_q. \tag{2.6.7}$$

The possibilities for U depend on p and q.

(p=0) In the case where $p = 0$, $n = 2q$ and since $n \geq 3$, we have that $q \geq 2$ and n is even. Moreover, the group is:

$$U = \langle y_1, ..., y_q, z_1, ..., z_q \mid y_1z_1 = ... = y_qz_q = y_1z_1 \cdots y_qz_q \rangle$$

Denote $u = y_1z_1 \cdots y_qz_q$ since $y_jz_j = u$ for all $1 \leq j \leq q$ we get that $u^q = u$ and $z_j = uy_j^{-1}$. Hence, U is generated by $y_1, ..., y_q, u$ with the relation $u^q = u$, which implies that there is an isomorphism:

$$\begin{array}{rcl} U & \to & \mathbb{Z}^q \times \mathbb{Z}/(q-1) \\ y_i & \mapsto & (0,...,1,...0;\overline{0}) \\ z_i & \mapsto & (0,...,-1,...0;\overline{1}). \end{array}$$

This also shows that the elements $y_1,...,y_q,z_1,...,z_q$ are all different.

(p=1) In this case, since $n=1+2q$ and $n\geq 3$, we get that $q\geq 1$ and n is odd. Moreover, the group is:

$$U=\langle x_1,y_1,...,y_q,z_1,...,z_q \mid x_1^2=y_1z_1=...=y_qz_q=x_1(y_1z_1\cdots y_qz_q)\rangle.$$

In this case, since $y_jz_j=x_1^2$ for all $1\leq j\leq q$, we get that $x_1^2=x_1(x_1^2)^q=x_1^n$ and $z_j=x_1^2y_j^{-1}$. Hence, U is generated by $x_1,y_1,...,y_q$ with the relation $x_1^n=x_1^2$. Therefore, there is an isomorphism

$$\begin{array}{rcl} U & \to & \mathbb{Z}^q \times \mathbb{Z}/(2q-1) \\ y_i & \mapsto & (0,...,1,...0;\overline{0}) \\ z_i & \mapsto & (0,...,-1,...0;\overline{2}) \\ x_1 & \mapsto & (0,...,0;\overline{1}). \end{array}$$

which also implies that the elements $x_1,y_1,z_1,...,y_q,z_q$ are all different.

($p\geq 2$) In that case $n=p+2q$ so $q\geq 1$ in case $p=2$ and it is arbitrary otherwise. For any $2\leq i\leq p$, there is an element $t_i\in U$ such that $x_it_i=x_1$. Moreover, since $x_1^2=x_i^2$, we have that $t_i^2=e$. Due to (2.6.7), we have that $x_1^2=x_1\cdots x_py_1z_1\cdots y_qz_q=x_1^{p+2q}t_2\cdots t_p$. Hence, $t_p=t_2\cdots t_{p-1}x_1^{p+2q-2}=t_2\cdots t_{p-1}x_1^{n-2}$ and squaring this identity, we get $x_1^{2n-4}=e$. Moreover, $z_j=x_1^2y_j^{-1}$ for all $1\leq j\leq q$. Hence, U is generated by the elements $x_1,t_2,...,t_{p-1},y_1,...,y_q$ with the relations $x_1^{2n-4}=t_2^2=...=t_{p-1}^2=e$. Therefore, we have an isomorphism:

$$\begin{array}{rcl} U & \to & \mathbb{Z}^q \times \mathbb{Z}/(2n-4)\times(\mathbb{Z}/(2))^{p-2} \\ x_1 & \mapsto & (0,...,0;\overline{1};\overline{0},...,\overline{0}) \\ x_i \text{ with } 2\leq i\leq p-1 & \mapsto & (0,...,0;\overline{1};\overline{0},...,\overline{1},...,\overline{0}) \\ x_p & \mapsto & (0,...,0;\overline{n-2};\overline{1},...,\overline{1}) \\ y_j & \mapsto & (0,...,1,...,0;\overline{0};\overline{0},...,\overline{0}) \\ z_j & \mapsto & (0,...,-1,...,0;\overline{2};\overline{0},...,\overline{0}) \end{array}$$

which also implies that the elements $x_1,...,x_p,y_1,z_1,...,y_q,z_q$ are all different

Define $\delta_U : U \to \mathbb{Z}_{\geq 0}$ by $\delta_U(x_i) = \delta_U(y_j) = \delta_U(z_j) = 1$ for all $1 \leq i \leq p$ and $1 \leq j \leq q$ and $\delta_U(u) = 0$ for the rest. This is well defined because all those elements are different and it satisfies the conditions of Theorem 2.6.4 because $n = p+2q$ and because of (2.6.7). The fact that all the homogeneous components have dimension 1 imply that $\Gamma(U, \delta_U)$ is a fine grading which refines $\Gamma(G, \delta)$. Since the relations are the only ones needed to obtain a grading, U is its universal group.

From this arguments it follows:

Corollary 2.6.5 ([DET20]). *Let $X : V^{n-1} \to V$ be an $(n-1)$-fold cross product on the n-dimensional vector space V with $n \geq 3$ over an algebraically closed field $\mathbb{F}$, relative to a nondegenerate symmetric bilinear form b.*

Up to equivalence, the fine gradings on (V, X) are the gradings $\Gamma(U, \delta_U)$, where U is the abelian group

$$U = \langle x_1, ..., x_p, y_1, ..., y_q, z_1, ..., z_q \,|x_1^2 = ... = x_p^2 \\ = y_1 z_1 = ... = y_q z_q = x_1 \cdots x_p \cdots y_1 z_1 \cdots y_q z_q \rangle$$

with $p + 2q = n$ and $\delta_U : U \to \mathbb{Z}_{\geq 0}$ is the map given by:

$$\delta_U(x_i) = \delta_U(y_j) = \delta_U(z_j) = 1$$

for all $i \in \{1, ..., p\}$ and $j \in \{1, ..., q\}$, and $\delta_U(u) = 0$ for a different $u \in U$.

Moreover, U is, up to isomorphism the universal grading group of $\Gamma(U, \delta_U)$. U is described up to isomorphism by one of these conditions:

- *If $p = 0$, U is isomorphic to $\mathbb{Z}^q \times \mathbb{Z}/(q-1)$,*
- *If $p = 1$, U is isomorphic to $\mathbb{Z}^q \times \mathbb{Z}/(2q-1)$,*
- *If $p > 1$, U is isomorphic to $\mathbb{Z}^q \times \mathbb{Z}/(2n-4) \times (\mathbb{Z}/(2))^{p-2}$.*

Let $\sigma \in \mathrm{Sym}_p$, $\tau \in \mathrm{Sym}_q$, and $\lambda = (\overline{i_1}, ..., \overline{i_p}) \in \mathbb{Z}_2$. Fix a basis $u_1, ..., u_p, v_1, w_1, ..., v_q, w_q$ as before, we define morphisms f_σ, g_τ, and h_λ by:

$$\begin{array}{rcl} f_\sigma : V & \to & V \\ u_i & \mapsto & u_{\sigma(i)} \\ v_j & \mapsto & v_j \\ w_j & \mapsto & w_j \end{array},$$

$$\begin{array}{rcl} g_\tau\colon V & \to & V \\ u_i & \mapsto & u_i \\ v_j & \mapsto & v_{\tau(j)} \\ w_j & \mapsto & w_{\tau(j)} \end{array},$$

and

$$\begin{array}{rcl} h_\lambda\colon V & \to & V \\ u_i & \mapsto & u_i \\ v_j \text{ such that } \overline{i_j}=\overline{1} & \mapsto & w_j \\ v_j \text{ such that } \overline{i_j}=\overline{0} & \mapsto & v_j \\ w_j \text{ such that } \overline{i_j}=\overline{1} & \mapsto & v_j \\ w_j \text{ such that } \overline{i_j}=\overline{0} & \mapsto & w_j \end{array}.$$

Noticing that each autoequivalence must exchange homogeneous components and must preserve orthogonal pairs of elements, if $\psi \in \operatorname{Aut}(\Gamma(U,\delta_U))$, we can define $\sigma \in \operatorname{Sym}_p$ by $\psi(V_{x_i}) = V_{x_{\sigma(i)}}$, $\tau \in \operatorname{Sym}_q$ by $\psi(V_{y_j} \oplus V_{z_j}) = V_{y_{\tau(j)}} \oplus V_{z_{\tau(j)}}$ and we can define $\lambda = (\overline{i_1},...,\overline{i_p}) \in \mathbb{Z}_2$ by defining $\overline{i_k} = \overline{1}$ if $\psi(V_{y_k}) = V_{z_{\tau(k)}}$. In this case, $g_\tau^{-1} \circ h_\lambda \circ f_\sigma^{-1} \circ \psi \in \operatorname{Stab}(\Gamma(U,\delta_U))$. Hence, it follows that we get an isomorphism:

$$\begin{array}{rcl} \operatorname{Sym}_p \times \left(\mathbb{Z}_2^q \rtimes \operatorname{Sym}_q\right) & \to & \operatorname{Aut}(\Gamma(U,\delta_U))/\operatorname{Stab}(\Gamma(U,\delta_U)) \\ (\sigma,\lambda,\tau) & \mapsto & [f_\sigma \circ h_\lambda \circ g_\tau] \end{array}$$

where $[\varphi]$ denotes its class modulo $\operatorname{Stab}(\Gamma(U,\delta_U))$. Hence, we have that $W(\Gamma(U,\delta_U)) \cong \operatorname{Sym}_p \times \left(\mathbb{Z}_2^q \rtimes \operatorname{Sym}_q\right)$.

Example 2.6.6 ([DET20])**.** As an example, we can take $\mathbb{H}$ the algebra of quaternions. Since we work over an algebraically closed field $\mathbb{F}$, the algebra is isomorphic to the algebra of 2×2 matrices where the norm is the determinant. Denote by $-$ the canonical involution. We can define the following trilinear map

$$\begin{aligned} X\colon \mathbb{H}^3 &\longrightarrow \mathbb{H} \\ (x,yz) &\longmapsto X(x,y,z) := x\overline{y}z - z\overline{y}x \end{aligned}$$

We can note that $X(x,x,y) = x\overline{x}y - y\overline{x}x = n(x)y - yn(x) = 0$ and that $X(x,y,y) = x\overline{y}y - y\overline{y}x = xn(y) - n(y)x = 0$ for all $x,y \in \mathbb{H}$. Linearizing this, we get that $X(x,y,z) = -X(y,x,z) = X(y,z,x) = -X(z,y,x)$ for all $x,y,z \in \mathbb{H}$. Hence, taking $x = z$, we get that $X(x,y,x) = 0$. Therefore, X satisfies (2.1.1).

On the other hand, for any $x, y, z \in \mathbb{H}$ and $\lambda \in \mathbb{F}$, we have the equality $n((x+\lambda z)\overline{y}(x+\lambda z)) = n(x+\lambda z)^2 n(y)$. Using that $n(x) = \frac{1}{2}n(x,x)$, for all $x \in \mathbb{H}$, after expanding this equality and taking the terms with λ^2 and we get:

$$n(x\overline{y}z + z\overline{y}x) + n(x\overline{y}x, z\overline{y}z) = (n(x,z)^2 + 2n(x)n(z))n(y) \tag{2.6.8}$$

Expanding $n(x\overline{y}z + z\overline{y}x)$, we get:

$$n(x\overline{y}z + z\overline{y}x) = n(x\overline{y}z) + n(z\overline{y}x) + n(x\overline{y}z, z\overline{y}x) \tag{2.6.9}$$

Finally, we can deduce the following identity:

$$\begin{aligned} n(x\overline{y}x, z\overline{y}z) &= n(x\overline{y}x + y\overline{x}x - y\overline{x}x, z\overline{y}z + y\overline{z}z - y\overline{z}z) \\ &= n(n(x,y)x - n(x)y, n(y,z)z - n(z)y) \\ &= n(x,y)n(y,z)n(z,x) - n(x)n(y,z)^2 - n(z)n(x,y)^2 \\ &+ 2n(x)n(y)n(x) \end{aligned} \tag{2.6.10}$$

Using (2.6.8), (2.6.9) and (2.6.10), we get:

$$\begin{aligned} n(X(x,y,z), X(x,y,z)) &= n(x\overline{y}z - z\overline{y}x, x\overline{y}z - z\overline{y}x) \\ &= 2n(x\overline{y}z) + 2n(z\overline{y}x) - 2n(x\overline{y}z, z\overline{y}x) \\ &= 8n(x)n(y)n(z) - 2n(x\overline{y}z + z\overline{y}x) \text{ (due to (2.6.9))} \\ &= 4n(x)n(y)n(z) + 2n(x\overline{y}z, z\overline{y}x) - 2n(x,z)^2 n(y) \text{ (due to (2.6.8))} \\ &= \begin{vmatrix} n(x,x) & n(x,y) & n(x,z) \\ n(y,x) & n(y,y) & n(y,z) \\ n(z,x) & n(z,y) & n(z,z) \end{vmatrix} \text{ (due to (2.6.10))} \end{aligned}$$

Therefore, X satisfies (2.1.2), which means that X is a 3-fold cross product on a vector space of dimension 4. Due to Corollary 2.6.5 there are three different fine gradings on $(\mathbb{H}, X)$ up to equivalence, with universal groups $\mathbb{Z}^2$, $\mathbb{Z} \times \mathbb{Z}/(4)$ and $\mathbb{Z}/(4) \times (\mathbb{Z}/(2))^2$.

Since we have an inclusion $\mathbf{Aut}(\mathbb{H}) \hookrightarrow \mathbf{Aut}(\mathbb{H}, X)$ any grading on $\mathbb{H}$ induces a grading on $(\mathbb{H}, X)$. However, the converse doesn't hold. Indeed, on $\mathbb{H}$ there are only two fine gradings with universal groups $\mathbb{Z}$ and $(\mathbb{Z}/(2))^2$ as shown in [EK13, Example 2.40].

2.6.3 $n = 7$, $r = 2$f

Due to Theorem 2.3.8, any 2-fold cross product over a 7-dimensional vector space is isomorphic to $(\mathcal{C}_0, X^{\mathcal{C}_0})$. Due to Proposition 2.5.7, there is an isomorphism between $\mathbf{Aut}(\mathcal{C})$ and $\mathbf{Aut}(\mathcal{C}_0, X^{\mathcal{C}_0})$. Therefore, in view of Proposition 1.5.7 for an abelian group G, there is a correspondence between classes of G-gradings up to isomorphism on $\mathcal{C}$ and on $(\mathcal{C}_0, X^{\mathcal{C}_0})$. Moreover, since the isomorphism sends $f \in \mathbf{Aut}(\mathcal{C})$ to $f_{|\mathcal{C}_0}$, the corespondence, sends a grading $\Gamma\colon \mathcal{C} = \bigoplus_{g\in G} \mathcal{C}_g$ to $\theta(\Gamma)\colon \mathcal{C} = \bigoplus_{g\in G} (\mathcal{C}_g \cap \mathcal{C}_0)$.

The classification of G-gradings up to isomorphism on $\mathcal{C}$ is given in Theorem 2.2.11. Concretely, there are two fine gradings up to equivalence with universal groups $\mathbb{Z}^2$ and $(\mathbb{Z}/2)^3$ corresponding respectively to the grading $\Gamma^1_{\mathcal{C}}(\mathbb{Z}^2, ((1, 0), (0, 1), (1, 1)))$ and to the grading $\Gamma^2_{\mathcal{C}}((\mathbb{Z}/2)^3, (\mathbb{Z}/2)^3)$.

Due to the fact that the isomorphism between $\mathbf{Aut}(\mathcal{C})$ and $\mathbf{Aut}(\mathcal{C}_0, X^{\mathcal{C}_0})$ induce an isomorphism between the automorphisms of each grading Γ and $\theta(\Gamma)$ and their stabilizers, the Weyl groups of the previous fine gradings are isomorphic to the Weyl groups of the corresponding fine gradings on $\mathcal{C}$. These groups are computed in [EK13, Theorems 4.17 and 4.19] and they are respectively, the Weyl group of the root system of type G_2, which is the dihedral group of order 12 and the general linear group $\mathrm{GL}(\mathbb{Z}_2^3)$.

2.6.4 $n = 8$, $r = 3$

Again, we assume that $\mathbb{F}$ is an algebraically closed field. Due to Theorem 2.3.8 and Proposition 2.3.10 we have that (V, X) is isomorphic to $(\mathcal{C}, X_1^{\mathcal{C}})$.

We have shown that $\mathbf{Aut}(\mathcal{C}, X_1^{\mathcal{C}}) = \mathbf{Aut}(\mathcal{C}, \{\cdots\})$. Moreover, as shown in [Eld96],

$$\mathrm{Aut}(\mathcal{C}, \{\cdots\}) = \left\{ \prod_{i=1}^{m} L_{x_i} \mid m \geq 0, x_i \in \mathcal{C}_0 \text{ and } \prod_{i=1}^{m} n(x_i) = 1 \right\} \tag{2.6.11}$$

where, as in subsection 2.5.4, L_x is the map defined by $y \in \mathcal{C} \mapsto L_x(y) := xy$.

Denote by $\{e_1, e_2, u_1, u_2, u_3, v_1, v_2, v_3\}$ the Cartan basis defined as in table 2.1.

Proposition 2.6.7. *[DET20] Let $\mathcal{C}$ be a Cayley algebra over an algebraically closed field $\mathbb{F}$.*

(1) The orbit of 1 *under* $\mathrm{Aut}(\mathcal{C}, \{\cdots\})$ *is the set of elements of norm* 1*:*

$$\mathrm{orbit}_{\mathrm{Aut}(\mathcal{C},\{\cdots\})}(1) = \{x \in \mathcal{C} \mid n(x) = 1\}$$

(2) The orbit of e_1 *is the set of nonzero isotropic elements:*

$$\mathrm{orbit}_{\mathrm{Aut}(\mathcal{C},\{\cdots\})}(e_1) = \{x \in \mathcal{C} \mid n(x) = 0, x \neq 0\}$$

(3) The orbit of the pair (e_1, e_2) *under the diagonal action of* $\mathrm{Aut}(\mathcal{C}, \{\cdots\})$ *on* $\mathcal{C} \times \mathcal{C}$ *(i.e.* $\varphi(x, y) = (\varphi(x), \varphi(y))$ *for all* $\varphi \in \mathrm{Aut}(\mathcal{C}, \{\cdots\})$ *and* $x, y \in \mathcal{C}$*) is:*

$$\mathrm{orbit}_{\mathrm{Aut}(\mathcal{C},\{\cdots\})}(e_1, e_2) = \{(x, y) \in \mathcal{C} \mid n(x) = 0 = n(y),\ n(x, y) = 1\}$$

Proof. The proof is in [DET20] but for completeness we write it here. First we notice that all the inclusions "⊆" hold since $\mathbf{Aut}(\mathcal{C}, \{\cdots\})$ is a subgroup of the orthogonal group. Let's prove the other inclusions:

(1) Let $x \in \mathcal{C}$ be an element such that $n(x) = 1$. That means that $1 = x\overline{x} = L_{\overline{x}}L_x(1)$. Since $n(x)n(\overline{x}) = 1$, we have that $L_{\overline{x}}L_x \in \mathbf{Aut}(\mathcal{C}, \{\cdots\})$. Therefore, $x \in \mathrm{orbit}_{\mathrm{Aut}(\mathcal{C},\{\cdots\})}(1)$.

(2) Consider an element $0 \neq x \in \mathcal{C}$ with $n(x) = 0$. Suppose first that $x \in \mathcal{C}_0$. Take $y' \in \mathcal{C}_0$ with $n(x, y') = 1$. Since $n(y' + \lambda x) = \lambda + n(y')$, we have that $y = y' - n(y')x \in \mathcal{C}_0$ satisfies $n(y) = 0$ and $n(x, y) = 1$. Due to the Cayley equation, we have that $x^2 = y^2 = 0$. In this situation, $-xy - yx = x\overline{y} + y\overline{x} = n(x, y)1 = 1$. Moreover, $(xy)(xy) = (xy)(-1 - yx) = -xy$ (due to [Sha66, (3.6)], $(xy)(yx) = x(y^2)x = 0$). Therefore, $e_1' = -xy$ is an idempotent, similarly, $e_2' = -yx$ is another idempotent and as we showed before, $(xy)(yx) = 0$, hence they are orthogonal idempotents. As it is shown on [EK13, Chaper 4], they can be extended to a Cartan basis $e_1', e_2', u_1', u_2', u_2', v_1', v_2', v_3'$ (i.e. satisfying the table 2.1) and so, the morphism φ induced by $\varphi(e_i') = e_i$, $\varphi(u_j') = u_j$ and $\varphi(v_j') = v_j$ is an automorphism of $\mathcal{C}$. Since $\mathrm{Aut}(\mathcal{C}) \subseteq \mathrm{Aut}(\mathcal{C}, \{\cdots\})$ it is an automorphism of $(\mathcal{C}, \{\cdots\})$.

Since $e_2'x = -(yx)x = -yx^2 = 0$, $xe_1' = -x(xy) = -x^2y = 0$, $\varphi(x) \in \{z \in \mathcal{C} \mid ze_1 = 0 = e_2z\} = U := \mathbb{F}u_1 \oplus \mathbb{F}u_2 \oplus \mathbb{F}u_3$. Since in the construction of the Cartan basis in [EK13] we can choose u_1, u_2, u_3 arbitrarily with $n(u_1u_2, u_3) = 1$, we can asume that $\varphi(x) = -u_3$. Now, since $L_{u_2+v_2}L_{u_1+v_1}(e_1) = -u_3$ we have that $\varphi^{-1}L_{u_2+v_2}L_{u_1+v_1}(e_1) = x$. The fact that $n(u_1, v_1) = 1 = n(u_2, v_2)$ implies $n(u_1 + v_1)n(u_2 + v_2) = 1$. Therefore, $L_{u_2+v_2}L_{u_1+v_1} \in \mathrm{Aut}(\mathcal{C}, \{\cdots\})$, which implies that $x \in \mathrm{orbit}_{\mathrm{Aut}(\mathcal{C},\{\cdots\})}(e_1)$.

Now, if $x \notin \mathcal{C}_0$, take $y \in \mathcal{C}_0$, orthogonal to x with $n(y) = 1$, which exists since the field is algebraically closed. $L_y \in \mathrm{Aut}(\mathcal{C}, \{\cdots\})$ and $L_y(x) = yx$, which satisfies $\overline{yx} = -\overline{x}y = -n(x,y)1 + \overline{y}x = -yx$. Hence, $yx \in \mathcal{C}_0$ is such that $n(yx) = n(y)n(x) = 0$ and by the previous argument there is an element $\varphi \in \mathrm{Aut}(\mathcal{C}, \{\cdots\})$ with $\varphi(yx) = ie_1$. Hence $L_y^{-1} \circ \varphi^{-1}(e_1) = x$, which implies $x \in \mathrm{orbit}_{\mathrm{Aut}(\mathcal{C},\{\cdots\})}(e_1)$.

(3) In this case, Let $x, y \in \mathcal{C}$ be such that $n(x) = 0 = n(y)$ and $n(x,y) = 1$. Since, $\mathbb{F}$ is algebraically closed we can take $z \in \mathcal{C}$ to be an element orthogonal to $1, x$ and y such that $n(z) = 1$ and an element t orthogonal to 1 and z with $n(t) = 1$. We denote $a = zt^{-1} = z\overline{t}$ (since $t\overline{t} = n(t)$) and $b = t$. $n(a) = 1 = n(b)$ and $\overline{a} = t\overline{z} = n(t,z)1 - \overline{z}t = -z\overline{t}$ (the last equality because $\overline{z} = -z$ and $\overline{t} = -t$) therefore $a \in \mathcal{C}_0$ and $b \in \mathcal{C}_0$. Thus, $L_aL_b \in \mathrm{Aut}(\mathcal{C}, \{\cdots\})$. Moreover we have that $ab = (z\overline{t})t = z(\overline{t}t) = z$, and so using Proposition 2.2.3, $n(L_aL_b(x), 1) = n(a(bx), 1) = n(x, \overline{b}\overline{a}) = n(x, \overline{z}) = n(x, -z) = 0$ and similarly $n(L_aL_b(y), 1) = 0$. That implies that we can restrict to the case with $x, y \in \mathcal{C}_0$. In the same vein as before, we can show that we can find a Cartan basis $\{e_1', e_2', u_1', u_2', u_3', v_1', v_2', v_3'\}$ such that $xy = -e_2$, $yx = -e_1$, $x = v_1'$ and $y = u_1'$. The morphism ψ induced by $\psi(e_i') = e_i$, $\psi(u_j') = u_j$ and $\psi(v_j') = v_j$ for $i \in \{1,2\}$ and $j \in \{1,2,3\}$, is an automorphism of $\mathbf{Aut}(\mathcal{C}, \{\cdots\})$. Hence, we can asume that $x = v_1$ and $y = u_1$ and finally, since $L_{u_1-v_1}L_{e_1-e_2}(v_1) = e_1$, $L_{u_1-v_1}L_{e_1-e_2}(u_1) = e_2$ and $L_{u_1-v_1}L_{e_1-e_2} \in \mathrm{Aut}(\mathcal{C}, \{\cdots\})$, we have the result.

□

Now we are going to review some gradings on $(\mathcal{C}, \{\cdots\})$, which appear in [DET20], and prove that they are the only gradings up to isomorphism.

Example 2.6.8 (Cartan grading)**.** . We can define a $\mathbb{Z}^3$-grading on $(\mathcal{C}, \{\cdots\})$ called the **Cartan grading** by taking a Cartan basis $\{e_1, e_2, u_1, u_2, u_3, v_1, v_2, v_3\}$ and assigning degrees:

$$\begin{aligned}
\deg(u_1) &= (1,0,0) = -\deg(v_1),\\
\deg(u_2) &= (0,1,0) = -\deg(v_2),\\
\deg(u_3) &= (0,0,1) = -\deg(v_3),\\
\deg(e_2) &= (1,1,1) = -\deg(e_1)
\end{aligned}$$

The grading is denoted by $\Gamma^{\mathcal{C}}_{\mathrm{Cartan}}$. $\Gamma^{\mathcal{C}}_{\mathrm{Cartan}}$ is the weight space decomposition relative to a maximal thorus $\mathbf{T}$ of $\mathbf{Aut}(\mathcal{C}, \{\cdots\})$. Since the homogeneous components have dimension 1, this is a fine grading.

For an abelian group G and a homomorphism $\alpha\colon \mathbb{Z}^3 \to F$, define the grading $\Gamma_1^{\mathcal{C}}(G,\alpha)$ as the coarsening of $\Gamma_{\text{Cartan}}^{\mathcal{C}}$ by α, i.e.,

$$\begin{aligned}
\deg(u_1) &= \alpha(1,0,0) = -\deg(v_1),\\
\deg(u_2) &= \alpha(0,1,0) = -\deg(v_2),\\
\deg(u_3) &= \alpha(0,0,1) = -\deg(v_3),\\
\deg(e_2) &= \alpha(1,1,1) = -\deg(e_1)
\end{aligned}$$

We can calculate the Weyl group of $\Gamma_{\text{Cartan}}^{\mathcal{C}}$ noticing that its universal group is $\mathbb{Z}^3$ then, the image of $\mu_\Gamma\colon \widehat{\mathbb{Z}^3} \to \text{Aut}(\mathcal{C},\{\cdots\})$ is a maximal torus T. Thus, its homogeneous components are the weight spaces with respect to T. Moreover, it corresponds to $\text{Diag}(\Gamma_{\text{Cartan}}^{\mathcal{C}}) = \mathbf{Diag}(\Gamma_{\text{Cartan}}^{\mathcal{C}})(\mathbb{F})$. Since $\text{Aut}(\Gamma_{\text{Cartan}}^{\mathcal{C}})$ is the normalizer of $\text{Diag}(\Gamma_{\text{Cartan}}^{\mathcal{C}})$ and $\text{Stab}(\Gamma_{\text{Cartan}}^{\mathcal{C}})$ is its centralizer, the Weyl group of $\Gamma_{\text{Cartan}}^{\mathcal{C}}$ is isomorphic to the Weyl group of $\text{Aut}(\mathcal{C},\{\cdots\})$ relative to T (see [Hum75, 24.1] for the definition). Due to [Hum75, 27.1], it is isomorphic to the Weyl group of its root system, i.e., it is isomorphic to the Weyl group of the root system Spin(7), which is a group of type B_3 [KMRT, 25.10]. Hence, the Weyl group of $\Gamma_{\text{Cartan}}^{\mathcal{C}}$ is isomorphic to $\mathbb{Z}_2^3 \rtimes \text{Sym}_3$, the signed permutation group of 3 elements [EK13, Apendix B].

Example 2.6.9. For any group G and subgroup H isomorphic to $\mathbb{Z}_2^3$, the grading $\Gamma_{\mathcal{C}}^2(G,H)$ introduced in the section 2.2 is a grading on $(\mathcal{C},\{\cdots\})$ since it is a grading on $\mathcal{C}$ and $\mathbf{Aut}(\mathcal{C})$ is a subgroupscheme of $\mathbf{Aut}(\mathcal{C},\{\cdots\})$.

Example 2.6.10. Let G be an abelian group. Let H be a subgroup isomorphic to $(\mathbb{Z}_2)^4$ and let K a subgroup of H isomorphic to $(\mathbb{Z}_2)^3$. Take an element $h \in H \setminus K$ and consider the shift $\Gamma_{\mathcal{C}}^2(G,K)^{[h]}$, which is a grading of the cross product since $h^2 = e$. Since $H/K \cong \mathbb{Z}_2$, we have that any other $h' \in H \setminus K$ can be written as $h' = hk$. We are going to show that $\Gamma_{\mathcal{C}}^2(G,K)^{[h]} \cong \Gamma_{\mathcal{C}}^2(G,K)^{[h']}$. Indeed $\text{Supp}(\Gamma_{\mathcal{C}}^2(G,K)) = K$, we have that $\text{Supp}(\Gamma_{\mathcal{C}}^2(G,K)^{[h']}) = h'K = hK$. Hence, there is some $k \in \{1,...,7\}$ such that x_k has degree h. Using the automorphism L_{ix_k}, which is an automorphism due to (2.6.11), we obtain an isomorphism from one grading to the other. We have shown that up to isomorphism the grading doesn't depend on the element we choose. Hence, we denote the grading as $\Gamma^{\mathcal{C}}(G,H,K)$.

Theorem 2.6.11. *[DET20] Let $\mathcal{C}$ be the Cayley algebra over an algebraically closed field $\mathbb{F}$, let G be an abelian group and let $\Gamma\colon \mathcal{C} = \bigoplus_{g\in G} \mathcal{C}_g$ be a grading of $(\mathcal{C},\{\cdots\})$. Then Γ is isomorphic to one of the following gradings:*

(1) $\Gamma^{\mathcal{C}}(G,\alpha)$ *for a group homomorphism* $\alpha\colon \mathbb{Z}^3 \to G$

(2) $\Gamma^2_{\mathcal{C}}(G,H)$ *for an elementary* 2*-subgroup* H *of rank* 3*.*

(3) $\Gamma^{\mathcal{C}}(G,H,K)$ *for an elementary* 2*-subgroup* H *of rank* 4 *and a subgroup* K *of* H *of index* 2*.*

Gradings on different items are not isomorphic. Moreover:

- *Two gradings* $\Gamma^{\mathcal{C}}(G,\alpha)$ *and* $\Gamma^{\mathcal{C}}(G,\alpha')$ *are isomorphic if and only if there is an element* ω *in the Weyl group* $W(\Gamma^{\mathcal{C}}_{\text{Cartan}})$ *such that* $\alpha'=\alpha\omega$*.*
- *Two gradings* $\Gamma^2_{\mathcal{C}}(G,H)$ *and* $\Gamma^2_{\mathcal{C}}(G,H')$ *are isomorphic if and only if* $H=H'$*.*
- *Two gradings* $\Gamma^{\mathcal{C}}(G,H,K)$ *and* $\Gamma^{\mathcal{C}}(G,H',K')$ *are isomorphic if and only if* $H=H'$ *and* $K=K'$*.*

Proof. We consider a grading $\Gamma\colon \mathcal{C}=\bigoplus_{g\in G}\mathcal{C}_g$ by an abelian group G of $(\mathcal{C},\{\cdots\})$.

To begin with, we consider the case in which there is a nonzero homogeneous element $x\in\mathcal{C}_g$ such that $n(x)=0$. Due to Theorem 2.5.12, $\mathbf{Aut}(\mathcal{C},\{\cdots\})\subseteq\mathbf{O}^+(\mathcal{C},n)$ hence, $n(\mathcal{C}_{g_1},\mathcal{C}_{g_2})=0$ unless $g_1g_2=e$ (one way to prove it is to show that $\mathbf{O}^+(\mathcal{C},n)=\{\varphi\in\widetilde{\mathbf{O}}(\mathcal{C},n)\mid\det(\varphi)=e\}$) and use the arguments in the subsection 2.6.2). Hence, we can find a homogeneous element $y\in\mathcal{C}$ such that $n(y)=0$ and $n(x,y)=1$. Using Proposition 2.6.7, after applying a suitable automorphism, can assume that there is a Cartan basis $\{e_1,e_2,u_1,u_2,u_3,v_1,v_2,v_3\}$ such that $x=e_1$ and $y=e_2$. Since $(e_1\mathcal{C})e_2=U=\mathbb{F}u_1\oplus\mathbb{F}u_2\oplus\mathbb{F}u_3$ and $(e_2\mathcal{C})e_1=V=\mathbb{F}v_1\oplus\mathbb{F}v_2\oplus\mathbb{F}v_3$ and due to the fact that the linear morphisms $R_{e_2}L_{e_1}$ and $R_{e_1}L_{e_2}$ are graded morphisms of the vector spaces (notice that $L_{e_i}(x)=\{e_i,e_{i+1},x\}$ and $R_{e_i}(x)=\{x,e_{i+1},i\}$ with indices modulo 2), then due to Remark 1.4.9, U and V are graded subspaces. We can take a homogeneous basis $\{u_1',u_2',u_3'\}$ of U such that $n(u_1',u_2'u_3')=1$. Taking $v_i'=\{u_{i+2}',e_1,u_{i+1}\}$ with $i\in\{1,2,3\}$ and indices taken modulo 3, we can see that this implies that we have a dual basis and that this gives a Cartan basis. Hence, the grading is a coarsening of the Cartan grading, which implies that there is a group homomorphism $\alpha\colon\mathbb{Z}^3\to G$ such that Γ is isomorphic to $\Gamma^{\mathcal{C}}(G,\alpha)$.

Otherwise, if all the homogeneous components have no nonzero isotropic elements, they are one dimensional. Indeed, we can prove the statement by contradiction since if x and y are linearly independent homogeneous elements of the same degree such that $n(y)\neq 0$, due to the fact that $n(y)\neq 0$ and $\mathbb{F}$ is algebraically closed, $0=n(x+\lambda y)=n(x)+\lambda n(x,y)+\lambda^2 n(y)$ has a solution, which implies that there is a homogeneous element of zero norm.

Now, the fact that $\{x, x, x\} = n(x)x$ for all $x \in \mathcal{C}$ implies that the support of Γ generates a 2-elementary abelian subgroup. Consider the two following possibilities:

- If the neutral element e of G is in the support, due to Proposition 2.6.7, we can take an element x of norm 1, homogeneous of degree e, and send it to 1 via some automorphism. Hence, we can assume that 1 is homogeneous of degree e. Now, $xy = \{x, 1, y\}$. Hence, this is a grading of the Cayley algebra $\mathcal{C}$. This, due to [EK13, Theorem 4.21] shows that the grading is isomorphic to $\Gamma^2_{\mathcal{C}}(G, H)$ for some subgroup H isomorphic to $\mathbb{Z}_2^3$.

- If the neutral element e of G is not in the support, we can take a homogeneous element of degree g and consider the shift $\Gamma^{[g]}$. Then, we are in the previous case. So there is a 2-elementary abelian subgroup K of G of rank 3 such that $\Gamma^{[g]} = \Gamma^2_{\mathcal{C}}(G, K)$. Since the subgroup H generated by g and K is a 2-elementary abelian subgroup, we get that the grading is isomorphic to $\Gamma^{\mathcal{C}}(G, H, K)$

The gradings on each item are not isomorphic, because in the first one there is an isotropic homogeneous element, in the second one, the neutral element is in the support and there are no isotropic homogeneous elements, and in the third one the neutral element is not in the support and there are no isotropic homogeneous elements.

The isomorphism condition for gradings on the first item follows from [EK13, Proposition 4.22]. And the isomorphism conditions for the gradings in the second and third item follow from the fact that $\operatorname{Supp}\Gamma^2_{\mathcal{C}}(G, H) = H$ and $\operatorname{Supp}\Gamma^{\mathcal{C}}(G, H, K) = H \setminus K$. □

We have the following corollary

Corollary 2.6.12 ([AC20],[DET20])**.** *Up to equivalence, the only fine gradings of* $(\mathcal{C}, \{\cdots\})$ *are* $\Gamma^{\mathcal{C}}_{\text{Cartan}}$ *and* $\Gamma^{\mathcal{C}}_{\text{CD}}$ *with universal groups* $\mathbb{Z}^3$ *and* $(\mathbb{Z}_2)^4$ *respectively.*

In order to calculate the Weyl group of $\Gamma^{\mathcal{C}}_{\text{CD}}$, we notice that any element of the group can be identified with a permutation of the support (which is a coset of $(\mathbb{Z}_2)^4/K$ for some subgroup K isomorphic of $(\mathbb{Z}_2^3)$. Since any representative is an equivalence of the grading, it induces an automorphism of the universal grading group. Since we can identify the group $\mathbb{Z}_2$ with the finite field of two elements $\mathbb{F}_2$, we can embed $W(\Gamma^{\mathcal{C}}_{\text{CD}})$ in $\{\varphi \in \mathrm{GL}((\mathbb{F}_2)^4) \mid \varphi(1 \times \mathbb{F}_2^3) \subseteq 1 \times \mathbb{F}_2^3\}$ which can be identified with the group $\operatorname{Aff}(3, \mathbb{F}_2)$ of invertible affine transformations of $\mathbb{F}^3$. Due to proposition 2.6.7, since $n(iw_k) = 1$ for all $k \in \{0, ..., 7\}$, we know that $W(\Gamma^{\mathcal{C}}_{\mathrm{CD}})$ acts transitively on the support of $\Gamma^{\mathcal{C}}_{\mathrm{CD}}$. Moreover, since $\mathrm{Aut}(\mathcal{C})$ is a subgroup of $\mathrm{Aut}(\mathcal{C}, \{\cdots\})$, due to [EK13, Theorem 4.19], we have that $\mathrm{GL}(\mathbb{F}_2^3)$ is a subgroup of $W(\Gamma^{\mathcal{C}}_{\mathrm{CD}})$. Hence, we get that the Weyl group is $\mathrm{Aff}(3, \mathbb{F}_2)$.

The computations of the Weyl group have been independently computed in [AC20] and in [DET20].

The homogeneous components of the gradings $\Gamma^{\mathcal{C}}(G, H)$ and $\Gamma^{\mathcal{C}}(G, H, K)$ are the subspaces spanned by the elements of a Cayley-Dickson basis. Hence, they are equivalent to the grading $\Gamma^{\mathcal{C}}_{\mathrm{CD}}$ over the group $(\mathbb{Z}_2)^4$ given by:

$$\begin{aligned} \deg(x_0) &= (\bar{1}, \bar{1}, \bar{1}, \bar{1}), \quad & \deg(x_1) &= (\bar{1}, \bar{1}, \bar{0}, \bar{0}), \\ \deg(x_2) &= (\bar{1}, \bar{0}, \bar{1}, \bar{0}), \quad & \deg(x_3) &= (\bar{1}, \bar{0}, \bar{0}, \bar{1}). \end{aligned}$$

Since the automorphism group scheme of $(\mathcal{C}, \{\cdots\})$ is isomorphic to $\mathbf{Spin}(\mathcal{C}_0, -n)$, then, we get the following result

Corollary 2.6.13 ([DET20]). *Let* $\mathbf{Q}$ *be a maximal diagonalizable subgroup scheme of* $\mathbf{Spin}(\mathcal{C}_0, -n)$*. Then, either:*

(a) $\mathbf{Q}$ *is conjugate to* $\mathbf{Diag}(\Gamma^{\mathcal{C}}_{\mathrm{Cartan}})$*, which is a maximal torus, isomorphic to* $\mathbf{G}_m^3$*.*

(b) $\mathbf{Q}$ *is conjugate to* $\mathbf{Diag}(\Gamma^{\mathcal{C}}_{\mathrm{CD}})$*, which is isomorphic to* μ_2^4

Chapter 3

Structurable algebras

From now on, we are going to focus on classifying gradings by abelian groups on a class of nonassociative algebras with involution, called structurable algebras introduced in [Ali78]. Here we are going to introduce them and some of their relevant features. But before we need to set some notation. Let $(\mathcal{A}, -)$ be an algebra with an involution. We define the subspace of hermitian elements $\mathcal{H}(\mathcal{A}, -)$ and the subspace of skew-hermitian elements $\mathcal{S}(\mathcal{A}, -)$ as follows:

$$\mathcal{H}(\mathcal{A}, -) = \{a \in \mathcal{A} \mid \overline{a} = a\}$$
$$\mathcal{S}(\mathcal{A}, -) = \{a \in \mathcal{A} \mid \overline{a} = -a\}$$

We define the subspace

$$\mathcal{K}(\mathcal{A}, -) = \mathrm{Alg}_{\mathbb{F}}(\mathcal{S}(\mathcal{A}, -))$$

to be the subalgebra of $\mathcal{A}$ generated by the skew hermitian elements and

$$\mathcal{M}(\mathcal{A}, -) = \{x \in \mathcal{H}(\mathcal{A}, -) \mid xs + sx = 0 \ \forall s \in \mathcal{S}(\mathcal{A}, -)\}$$

.

We define the **center** of $(\mathcal{A}, -)$ as the subspace

$$\mathcal{Z}(\mathcal{A}, -) = \{z \in \mathcal{A} \mid \overline{z} = z, [z, \mathcal{A}] = [z, \mathcal{A}, \mathcal{A}] = [\mathcal{A}, z, \mathcal{A}] = [\mathcal{A}, \mathcal{A}, z] = 0\}$$

where $[x, y] = xy - yx$ and $[x, y, z] = (xy)z - x(yz)$ for all $x, y, z \in \mathcal{A}$.

When we are working over an algebra $\mathcal{A}$, given an element $x \in \mathcal{A}$ we denote by $L_x, R_x \colon \mathcal{A} \to \mathcal{A}$ the maps defined by $L_x(y) = xy$ and $R_x(y) = yx$.

In Section 3.1, we define Jordan algebras, which are the motivation for structurable algerbas. Additionally, we give a definition in characteristic different from 2. In Section 3.2, we define structurable algebras and give a classification due to Allison and Smirnov. Finally, in Section 3.3 we give the Allison-Kantor construction of a 5-graded Lie algebra from a structurable algebra. Additionally, we explain how the gradings on the structurable algebras are connected with the gradings on the Lie algebras.

3.1 Jordan algebras

3.1.1 Definition and main examples

Definition 3.1.1. An algebra J over a field of characteristic different from 2 is a **Jordan algebra** if for every $x, y \in J$ the following identities are satisfied:

(1) $xy = yx$

(2) $(xy)x^2 = x(yx^2)$

We are going to introduce a few examples of Jordan algebras:

Examples 3.1.2. Let $\mathcal{A}$ be an associative algebra over a field $\mathbb{F}$ of characteristic different form 2. For any two elements $x, y \in \mathcal{A}$, we denote $x \bullet y = \frac{1}{2}(xy + yx)$. The algebra $\mathcal{A}^+ = (\mathcal{A}, \bullet)$ is a Jordan algebra (see [Mcc04, I.2.5]).

Example 3.1.3. Let $(\mathcal{A}, -)$ be an associative algebra with involution. The subspace $\mathcal{H}(\mathcal{A}, -)$ is closed under $\bullet$. Hence, it is a subalgebra of $\mathcal{A}^+$.

Example 3.1.4. Let $(\mathcal{C}, -)$ be a Hurwitz algebra. Denote by $(\mathcal{M}_3(\mathcal{C}), *)$ the algebra of 3×3 matrices over $\mathcal{C}$ with the involution $*$ given by $(a_{i,j})^* = (\overline{a_{j,i}})$. As before, we denote $x \bullet y = \frac{1}{2}(xy + yx)$ for all $x, y \in \mathcal{H}(\mathcal{M}_3(\mathcal{C}), *)$. We denote $\mathcal{H}_3(\mathcal{C}, -) = (\mathcal{H}(\mathcal{M}_3(\mathcal{C}), *), \bullet)$. This is a Jordan algebra (see [Mcc04, I.2.7]). In case $(\mathcal{C}, -)$ is a Cayley algebra, we say that $\mathcal{H}_3(\mathcal{C}, -)$ is an **Albert algebra**.

For elements in these algebras we will use the following notation:

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\iota_1(a) = 2\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \overline{a} \\ 0 & a & 0 \end{pmatrix}, \iota_2(a) = 2\begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ \overline{a} & 0 & 0 \end{pmatrix}, \iota_3(a) = 2\begin{pmatrix} 0 & \overline{a} & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

for all $a \in \mathcal{C}$ ($\iota_1(a) = 2\overline{a}[23]$, $\iota_2(a) = 2\overline{a}[31]$ and $\iota_3(a) = 2\overline{a}[12]$, in [Mcc04, II 4.4] notation).

3.1.2 U-operator

If we have a Jordan algebra J as defined in the subsection 3.1.1, given $x, y \in J$, we can define some operators U_x, $U_{x,y}$ and $V_{x,y}$ by:

$$\begin{aligned}
&U_x = 2L_x^2 - L_{x^2}, \\
&U_{x,y} = \frac{1}{2}(U_{x+y} - U_x - U_y) = (L_x L_y + L_y L_x) - L_{xy} \text{ and}, \\
&V_{x,y}(z) = U_{x,z}(y) = x(zy) + z(xy) - (xz)y = (xy)z + (zy)x - (zx)y.
\end{aligned}$$

Remark 3.1.5. If we want to work over a field of characteristic 2, we have to remove the $\frac{1}{2}$ from the definition, but in our case it will be more convenient to keep this notation.

In case the Jordan algebra has a unit, in a field of characteristic different from 2, we can see that $xy = U_{x,y}(1)$. Hence, we can recover the product from U. In this case, U satisfies the following identities:

$$U_1 = \mathrm{id}_J, \tag{3.1.1}$$
$$V_{x,y}U_x = U_x V_{y,x}, \text{ and} \tag{3.1.2}$$
$$U_{U_x y} = U_x U_y U_x. \tag{3.1.3}$$

Definition 3.1.6. A **unital quadratic Jordan algebra** J **over** $\mathbb{F}$ is an $\mathbb{F}$-vector space J, together with a quadratic map $U\colon x \mapsto U_x$ from J into $\mathrm{End}_{\mathbb{F}}(J)$ and an element $1 \in J$ such that the identities 3.1.1, 3.1.2 and 3.1.3 hold under every scalar extension. We denote the algebra as $\mathfrak{J}(J, U, 1)$.

Remark 3.1.7. Since we work over a field of characteristic different from 2, this definition is equivalent to the one on subsection 3.1.1, in the case where the algebra is unital, by defining a product $xy = U_{x,y}(1)$ (see [Mcc69]).

3.1.3 Jordan algebras of a cubic form

Let J be a vector space over $\mathbb{F}$. A cubic form on J is a map $N\colon J \to \mathbb{F}$ satisfying that $N(x + \lambda y) = N(x) + \lambda \partial_y N\mid_x + \lambda^2 \partial_x N\mid_y + \lambda^3 N(y)$ where $\partial_y N\mid_x$ is a map which is linear in y and quadratic in x.

Given a vector space J, with a cubic from N and a base point c, i.e., an element $c \in J$ such that $N(c) = 1$, we can define a **trace form** $T(x, y) = -\partial_x \partial_y \log N\mid_c = -\partial_x \partial_y N\mid_c + (\partial_x N\mid_c)(\partial_y N\mid_c)$ for all $x, y \in J$. We say that N is **nondegenerate** if T is a nondegenerate bilinear form. In case N is nondegenerate, we can define a quadratic mapping $x \mapsto x^\#$ by $T(x^\#, y) =$

$\partial_y N\mid_x$ for all $x, y \in J$ called the **adjoint operator**. We say that the norm N is **admissible** if the identity $x^{\#\#} = N(x)x$ holds under all scalar extension. We can define a product $x \times y = (x+y)^{\#} - x^{\#} - y^{\#}$ by linearizing the adjoint operator. We can define a quadratic map $U\colon x \to U_x$ form J to $\mathrm{End}_{\mathbb{F}}(J)$ by:

$$U_x(y) = T(x,y)x - x^{\#} \times y.$$

In case N is a nondegenerate admisible cubic form with base point c, $\mathcal{J}(J, U, c)$ is a unital quadratic Jordan algebra over $\mathbb{F}$ which we denote by $\mathcal{J}(N, c)$ and call the **Jordan algebra of the admisible cubic form** N **and base point** c [Mcc69, Theorem 5].

Given a Hurwitz algebra $(\mathcal{C}, -)$, as shown in [Mcc04, II.4.4] we can define an admisible cubic form N and a base point c on $\mathcal{H}_3(\mathcal{C}, -)$ by

$$\begin{aligned} N(x) &= \alpha_1\alpha_2\alpha_3 - 4\sum_{i=1}^{3} \alpha_i n(a_i) + 8n(a_1, \bar{a}_2\bar{a}_3), \text{ and} \\ c &= E_1 + E_2 + E_3. \end{aligned}$$

where $x = \sum_{i=1}^3 \alpha_i E_i + \sum_{i=1}^3 \iota_i(a_i)$. Moreover, we can write the trace the adjoint and the product as:

$$\begin{aligned} T(x,y) &= \sum_{i=1}^{3} \alpha_i\beta_i + 4\sum_{i=1}^{3} n(a_i, b_i), \\ x^{\#} &= \sum_{i=1}^{3} (\alpha_{i+1}\alpha_{i+2} - 4n(a_i))E_i + 2\sum_{i=1}^{3} \iota_i(2\overline{a_{i+1}a_{i+2}} - \alpha_i a_i), \text{ and} \\ x \times y &= \sum_{i=1}^{3} (\alpha_{i+1}\beta_{i+2} + \beta_{i+1}\alpha_{i+2} - 4n(a_i, b_i))E_i + \\ &\qquad 2\sum_{i=1}^{3} \iota_i(2\overline{(a_{i+1}b_{i+2} + b_{i+2}a_{i+1})} - \alpha_i b_i - \beta_i a_i) \end{aligned} .$$

where $x = \sum_{i=1}^3 \alpha_i E_k + \sum_{i=1}^3 \iota_i(a_i)$ and $y = \sum_{i=1}^3 \beta_i E_i + \sum_{i=1}^3 \iota_i(b_i)$ and the indices are taken modulo 3. This is just rewriting the identities in [Mcc04, II.4.4] using our notation.

Any Jordan algebra of an admissible cubic form with a base point c satisfies the equation:

$$x^3 - T(x)x^2 + S(x)x - N(x)c = 0 \tag{3.1.4}$$

For a linear map T defined by $T(x) = T(x, c)$ and $S(x) = T(x^{\#})$ (see [Mcc69]). For the algebra $\mathcal{H}_3(\mathcal{C}, -)$, T and S are:

$$\begin{aligned} T(x) &= \alpha_1 + \alpha_2 + \alpha_3, \text{ and} \\ S(x) &= \frac{1}{2}(T(x)^2 - T(x^2)) = \sum_{i=1}^{3}(\alpha_{i+1}\alpha_{i+2} - 4n(a_i)). \end{aligned}$$

for any $x = \sum_{i=1}^{3} \alpha_i E_i + \sum_{i=1}^{3} \iota_i(a_i)$.

3.2 Structurable algebras

In this section we are going to introduce the concept of structurable algebra, state the classification theorem, and finally, we will review the Kantor-Allison construction of a 5-graded Lie algebra.

3.2.1 Definitions

Let $(\mathcal{A}, -)$ be a finite dimensional algebra with involution and unity over a field $\mathbb{F}$ of characteristic different from 2 or 3. For any $x, y \in \mathcal{A}$, we can define the homomorphisms $V_{x,y}, T_x \in \operatorname{End}_{\mathbb{F}}(\mathcal{A})$ by:

$$\begin{aligned} V_{x,y}(z) &= (x\overline{y})z + (z\overline{y})x - (z\overline{x})y, \text{ and} \\ T_x(z) &= V_{x,1}(z) = xz + zx - z\overline{x}. \end{aligned}$$

Definition 3.2.1. Given a finite dimensional nonassociative algebra with involution $(\mathcal{A}, -)$ over a field $\mathbb{F}$ of characteristic different from 2 and 3, we say that it is a **structurable algebra** if:

$$[T_z, V_{x,y}] = V_{T_z(x),y} - V_{x,T_{\overline{z}}(y)} \tag{3.2.1}$$

for all $x, y, z \in \mathcal{A}$. Where $[A, B] = AB - BA$ for all $A, B \in \operatorname{End}_{\mathbb{F}}(\mathcal{A})$. We say that the structurable algebra is **central** if $\mathcal{Z}(\mathcal{A}, -) = \mathbb{F}1$.

We will often use the following lemma due to Smirnov:

Lemma 3.2.2 ([Smi90b]). *Given a simple structurable algebra $(\mathcal{A}, -)$:*

(i) We have
$$\mathcal{A} = \mathcal{K}(\mathcal{A}, -) \oplus \mathcal{M}(\mathcal{A}, -). \tag{3.2.2}$$

(ii) In $\mathcal{A}$ we have $\mathcal{M}(\mathcal{A}, -)\mathcal{K}(\mathcal{A}, -) + \mathcal{K}(\mathcal{A}, -)\mathcal{M}(\mathcal{A}, -) \subseteq \mathcal{M}(\mathcal{A}, -)$.

(iii) For $e, f \in \mathcal{K}(\mathcal{A}, -)$, $w \in \mathcal{M}(\mathcal{A}, -)$ *and* $x \in \mathcal{A}$*:*

(1) $ew = w\overline{e}$,

(2) $e(fw) = (fe)w$,

(3) $f(wx) = w(\overline{f}x)$.

Proof. [Smi90b, Lemma 3.6] □

For any $A \in \mathrm{End}_{\mathbb{F}}(\mathcal{A})$, we define A^δ, A^ϵ and $\overline{A}$ by $A^\delta = A + R_{\overline{A(1)}}$, $A^\epsilon = A - T_{A(1)+\overline{A}(1)}$ and $\overline{A}(x) = \overline{A(\overline{x})}$.

3.2.2 Examples and description

In this section we will recall some examples of structurable algebras and give the classification of the central simple ones.

Example 3.2.3. Every unital associative algebra with involution is a structurable algebra [Ali78, 8.i]

Example 3.2.4. Let $\mathcal{A}$ be a unital Jordan algebra and let $-$ be the identity map. $(\mathcal{A}, -)$ is structurable [Ali78, 8.ii]. Moreover, if $(\mathcal{A}, \mathrm{id})$ is a structurable algebra, then $\mathcal{A}$ is a unital Jordan algebra [Ali78].

Example 3.2.5. Let $(\mathcal{E}, -)$ be a unital associative algebra with involution. Let $\mathcal{W}$ be a unital associative left $\mathcal{E}$-module and denote the action by $e \circ w$ for all $e \in \mathcal{E}$, $w \in \mathcal{W}$. Let $h \colon \mathcal{W} \times \mathcal{W} \to \mathcal{E}$ be a hermitian form, i.e, it satisfies the following two identities:

$$h(w_1, w_2) = \overline{h(w_2, w_1)} \qquad \text{for all } w_1, w_2 \in \mathcal{W}, \text{ and} \qquad (3.2.3)$$

$$h(e \circ w_1, w_2) = eh(w_1, w_2) \qquad \text{for all } w_1, w_2 \in \mathcal{W} \text{ and } e \in \mathcal{E} \qquad (3.2.4)$$

We denote $\mathcal{A} = \mathcal{E} \oplus \mathcal{W}$ and define a product and an involution by

$$(e_1, w_1)(e_2, w_2) = (e_1e_2 + h(w_2, w_1), e_2 \circ w_1 + \overline{e}_1 \circ w_2), \text{ and}$$
$$\overline{(e_1, w_1)} = (\overline{e}_1, w_1)$$

for all $w_1, w_2 \in \mathcal{W}$ and $e_1, e_2 \in \mathcal{E}$. We call this algebra **the structurable algebra constructed from a hermitian form** h **on a module** W **over a unital associative algebra with involution** and denote it by $S(\mathcal{E}, -, \mathcal{W}, h)$

following the notation in [Rig22]. If we know that $e \in \mathcal{E}$ and $w \in \mathcal{W}$ we will usually denote the element (e, w) as $e + w$.

In this class we can find an example of algebra which is not power associative. Indeed, let $(\mathcal{E}, -) = (\mathcal{M}_2(\mathbb{F}), {}^t)$ where ^{t} is the transposition, let $\mathcal{W} = \mathcal{M}_2(\mathbb{F})$ with the action given by $e \circ w = ew$ for $e \in \mathcal{E}$ and $w \in \mathcal{W}$ and denote by $h\colon \mathcal{W} \times \mathcal{W} \to \mathcal{E}$ the hermitian form given by $h(w_1, w_2) = w_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} w_2^t$. Consider the element $x = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$. Then, $xx^2 = \left(\begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix}\right)$ and $x^2x = \left(\begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix}\right)$.

Example 3.2.6. Let $(\mathcal{C}_1, -)$ and $(\mathcal{C}_2, -)$ be two Hurwitz algebras with standard involutions. Denote $\mathcal{A} = \mathcal{C}_1 \otimes_{\mathbb{F}} \mathcal{C}_2$ and define an involution $-$ by $\overline{x \otimes y} = \overline{x} \otimes \overline{y}$. Then $(\mathcal{A}, -)$ is a structurable algebra [Ali78, 8.iv]

Example 3.2.7. Let J be a Jordan algebra over the field $\mathbb{F}$ constructed from an admisible non-degenerate cubic form N with basepoint c. Denote $T\colon J \times J \to \mathbb{F}$ and $\times\colon J \times J \to J$ as in subsection 3.1.3. Take $0 \neq \theta \in \mathbb{F}$. Denote

$$\mathcal{A} = \left\{ \begin{pmatrix} \alpha & j \\ k & \beta \end{pmatrix} \mid \alpha, \beta \in \mathbb{F}, j, k \in J \right\}$$

and define a product and an involution by:

$$\begin{pmatrix} \alpha_1 & j_1 \\ k_1 & \beta_1 \end{pmatrix} \begin{pmatrix} \alpha_2 & j_2 \\ k_2 & \beta_2 \end{pmatrix} = \begin{pmatrix} \alpha_1\alpha_2 + \theta T(j_1, k_2) & \alpha_1 j_2 + \beta_2 j_1 + \theta k_1 \times k_2 \\ \alpha_2 k_1 + \beta_1 k_2 + j_1 \times j_2 & \beta_1\beta_2 + \theta T(j_2, k_1) \end{pmatrix}$$

and

$$\overline{\begin{pmatrix} \alpha & j \\ k & \beta \end{pmatrix}} = \begin{pmatrix} \beta & j \\ k & \alpha \end{pmatrix}.$$

This is a structurable algebra [Ali79, 7.iii]. We will call this algebra, the **algebra of 2×2 matrices constructed from N, c and θ.**

Example 3.2.8 (The Smirnov algebra)**.** Let $(\mathcal{C}, -)$ be a Cayley-Dickson algebra with the standard involution. Let $\mathcal{S} = \mathcal{S}(\mathcal{C}, -)$. Denote $x \times y = xy - yx$ for every $x, y \in \mathcal{C}$ as in Example 2.3.6 and denote $(x, y) = -b_n(x, y)$ as defined in Example 2.3.6. Let $\mathcal{M}$ be the subspace of $\mathcal{S} \otimes_{\mathbb{F}} \mathcal{S}$ generated by the set $\{s \otimes r - r \otimes s \mid s, r \in \mathcal{S}\}$. We denote $\mathcal{H} = (\mathcal{S} \otimes_{\mathbb{F}} \mathcal{S})/\mathcal{M}$ and write the equivalence class of x as $[x]_{\mathcal{M}}$. We denote $\mathcal{A} = \mathcal{H} \oplus \mathcal{S}$ and define a commutative operation $\circ$ and an anticommutative operation $[\cdot, \cdot]$ by extending linearly:

$$[s_1, s_2] = s_1 \times s_2,\ [s_1, [s_2 \otimes s_3]_{\mathcal{M}}] = [(s_1 \times s_2) \otimes s_3 + s_2 \otimes (s_1 \times s_3)]_{\mathcal{M}},$$
$$[[s_1 \otimes s_2]_{\mathcal{M}}, [s_3 \otimes s_4]_{\mathcal{M}}] = (s_1, s_3)s_2 \times s_4 + (s_1, s_4)s_2 \times s_3 + (s_2, s_3)s_1 \times s_4 + (s_2, s_4)s_1 \times s_3,$$
$$s_1 \circ s_2 = [s_1 \otimes s_2]_{\mathcal{M}},\ s_1 \circ [s_1 \otimes s_2]_{\mathcal{M}} = \frac{1}{2}(s_2, s_3)s_1 + \frac{1}{4}(s_1, s_2)s_3 + \frac{1}{4}(s_1, s_3)s_2,$$

and

$$(s_1 \otimes s_2) \circ (s_3 \otimes s_4) = \left[\frac{1}{4}(s_1 \times s_3) \otimes (s_2 \times s_4) + \frac{1}{4}(s_1 \times s_4) \otimes (s_2 \times s_3) + \frac{1}{2}(s_1, s_2)s_3 \otimes s_4 + \frac{1}{2}(s_3, s_4)s_1 \otimes s_2\right]_{\mathcal{M}}.$$

for all $s_1, s_2, s_3, s_4 \in \mathcal{S}$. We define a product and an involution on $\mathcal{A}$ by:

$$xy = x \circ y + \frac{1}{2}[x, y], \text{ and}$$
$$\overline{h + s} = h - s$$

for all $x, y \in \mathcal{A}$, $h \in \mathcal{H}$ and $s \in \mathcal{S}$. This is a structurable algebra which we denote by $T(\mathcal{C})$ [Smi90a].

There is a description of central simple structurable algebras, which was given in [Ali78] in characteristic 0 with a missing case, and it was completed in [Smi90b] in fields of characteristic different form 2, 3 and 5. There are many restatements of this theorem. We will give the following one which is the same as in [Smi90b] but with the description given in [Ali79] for the algebras on item (4).

Theorem 3.2.9 ([Ali78],[Smi90b])**.** *Let* $(\mathcal{A}, -)$ *be a finite-dimensional central simple structurable algebra over a field* $\mathbb{F}$ *of characteristic different from* 2, 3 *and* 5*. Then,* $(\mathcal{A}, -)$ *is isomorphic to one of the following algebras:*

(1) A central simple associative algebra with involution,

(2) a central simple Jordan algebra with the identity involution,

(3) an algebra constructed from a nondegenerate hermitian form h *on a module* $\mathcal{W}$ *over a unital central simple associative algebra with involution* $(\mathcal{E}, -)$*,*

(4) a twisted form of a tensor product of composition algebras,

(5) *a twisted form of an algebra of* 2×2 *matrices constructed from an admisible nondegenerate cubic from* N *with base point* c *and scalar* $0 \neq \theta \in \mathbb{F}$ *or*

(6) *the algebra* $T(\mathcal{C})$ *for a Cayley-Dickson algebra* $\mathcal{C}$.

This theorem lacks some features in order to be a classification. For example, some of the classes overlap. For instance, if in (3) the unital central simple associative algebra with involution is $(\mathbb{F}, \mathrm{id})$, $\mathcal{W}$ a vector space and h a nondegenerate symmetric bilinear form, we get a Jordan algebra. Another issue is that not all of these descriptions give central simple algebras. For instance, in (3) if we take $(\mathbb{F}, \mathrm{id})$ as the associative algebra, $\mathcal{W} = \mathbb{F}$ and $h(\lambda_1, \lambda_2) = \lambda_1\lambda_2\lambda$ for all $\lambda_1\lambda_2 \in \mathbb{F}$ and a fixed $\lambda \in \mathbb{F}^\times$, we get the algebra $\left(\frac{\mathbb{F}[x]}{(x^2-\lambda)}, \mathrm{id}\right)$, which is not central.

3.3 The Kantor-Allison construction

Probably, the main importance of structurable algebras is the fact that it has a generalization of the Tits-Kantor-Koecher (TKK) construction of Lie algebras from Jordan algebras introduced in [Tit62], [Kan64] and [Koe67]. This construction was introduced in [Ali79] as a modified TKK construction and we will refer to this construction as the Kantor-Allison construction, which is how it is called in [Smi96]. Let's see how this construction works:

Let $(\mathcal{A}, -)$ be a structurable algebra. Define the subspace

$$\mathfrak{Instrl}(\mathcal{A}, -) = T_{\mathcal{A}} \oplus \mathrm{Inder}(\mathcal{A}, -)$$

to be the subspace generated by the endomorphisms of $\mathcal{A}$ of the form T_x for $x \in \mathcal{A}$ and the inner derivations. It is shown in [Ali78, (16)] that

$$\mathfrak{Instrl}(\mathcal{A}, -) = V_{\mathcal{A},\mathcal{A}},$$

i.e., the subspace of $\mathfrak{gl}(\mathcal{A})$ generated by the endomorphisms of the form $V_{x,y}$ with $x, y \in \mathcal{A}$. This is a Lie subalgebra of $\mathfrak{gl}(\mathcal{A})$ due to [Ali78, (12)]. Denote by $N = N(\mathcal{A}, -) = \{(x, s) \mid x \in \mathcal{A}, s \in \mathcal{S}(\mathcal{A}, -)\}$. Denote $\widetilde{N}$ to be a vector space isomorphic to N under an isomorphism $n \mapsto \tilde{n}$ from N to $\widetilde{N}$. For any $A \in \mathfrak{Instrl}$ and $n = (x, s) \in N$, we define $A(x, s) = (Ax, A^\delta s)$ which is well defined due to [Ali78, Corollary 5]. Define the vector space

$$\mathfrak{K}(\mathcal{A}, -) = \widetilde{N} \oplus \mathfrak{Instrl}(\mathcal{A}, -) \oplus N$$

and define a product by:

$$[A,B]=AB-BA,\ [A,n]=An,\ A\tilde{n}=\widetilde{A^{\epsilon}n},$$
$$[(x,r),(y,s)]=(0,x\overline{y}-y\overline{x}),\ [\widetilde{(x,r)},\widetilde{(y,s)}]=(0,\widetilde{x\overline{y}-y\overline{x}}),\ \text{and}$$
$$[(x,r),\widetilde{(y,s)}]=-\widetilde{(sx,0)}+V_{x,y}+L_rL_s+(ry,0)$$

for all $A,B\in\mathfrak{Instrl}(\mathcal{A},-)$, $x,y\in\mathcal{A}$ and $s,r\in\mathcal{S}(\mathcal{A},-)$. Due to [Ali79, Theorem 3], $\mathfrak{K}(\mathcal{A},-)$ is a Lie algebra with a 5-grading

$$\Gamma_{\mathfrak{K}(\mathcal{A},-)}:\mathfrak{K}(\mathcal{A},-)=\mathfrak{K}(\mathcal{A},-)_{-2}\oplus\mathfrak{K}(\mathcal{A},-)_{-1}\oplus\mathfrak{K}(\mathcal{A},-)_{0}\oplus\mathfrak{K}(\mathcal{A},-)_{1}\oplus\mathfrak{K}(\mathcal{A},-)_{2}$$

given by:

$$\mathfrak{K}(\mathcal{A},-)_{-2}=(0,\widetilde{\mathcal{S}(\mathcal{A},-)}),\ \mathfrak{K}(\mathcal{A},-)_{-1}=\widetilde{(\mathcal{A},0)},$$
$$\mathfrak{K}(\mathcal{A},-)_{0}=\mathfrak{Instrl}(\mathcal{A},-),$$
$$\mathfrak{K}(\mathcal{A},-)_{1}=(\mathcal{A},0),\ \mathfrak{K}(\mathcal{A},-)_{2}=(0,\mathcal{S}(\mathcal{A},-)).$$

Definition 3.3.1. Given a structurable algebra $(\mathcal{A},-)$ and a structurable algebra $(\mathcal{A}',-)$, we say that an isomorphisms of vector spaces $\alpha\colon\mathcal{A}\to\mathcal{A}'$ is an **isotopy** if there is another isomorphism $\hat{\alpha}\colon\mathcal{A}\to\mathcal{A}'$ satisfying that $V_{\alpha(x),\hat{\alpha}(y)}\alpha(z)=\alpha(V_{x,y}(z))$ for all $x,y,z\in\mathcal{A}$. We denote by $\mathrm{Str}(\mathcal{A},-)$ the group of isotopies from $\mathcal{A}$ to $\mathcal{A}$ (also called **autotopies**). We can extend this to group schemes by extending scalars, and we denote the affine group scheme of autotopies by $\mathbf{Str}(\mathcal{A},-)$.

In the same vein of [AH81, Theorem 12.3], there is an isomorphism:

$$\mathbf{Str}(\mathcal{A},-)\to\mathbf{Aut}(\mathfrak{K}(\mathcal{A},-),\Gamma_{\mathfrak{K}(\mathcal{A},-)})$$

sending α to $\alpha_{\mathfrak{K}}$ defined by:

$$\alpha_{\mathfrak{K}}(\widetilde{(x,s)}+E+(y,r))=(\widetilde{\hat{\alpha}(x),\hat{\alpha}(s)})+\alpha E\alpha^{-1}+(\alpha(y),\alpha(r)).$$

for every $x,y\in\mathcal{A}\otimes_{\mathbb{F}}R$, $s,r\in\mathcal{S}(\mathcal{A}\otimes_{\mathbb{F}}R,-)$ and $E\in\mathfrak{Instrl}(\mathcal{A}\otimes_{\mathbb{F}}R,-)$. Moreover, it is clear that $\mathbf{Aut}(\mathcal{A},-)$ is a subgroup scheme of $\mathbf{Str}(\mathcal{A},-)$ and that if α is an automorphism, $\hat{\alpha}=\alpha$. Therefore, for any finitely generated abelian group G, a morphism $\theta\colon G^D\to\mathbf{Aut}(\mathcal{A},-)$ induces a morphism $\widetilde{\theta}\colon G^D\to\mathbf{Aut}(\mathfrak{K}(\mathcal{A},-),\Gamma_{\mathfrak{K}(\mathcal{A},-)})$. Hence, any G-grading Γ on $\mathcal{A}$ induces a grading on $(\mathfrak{K}(\mathcal{A},-),\Gamma_{\mathfrak{K}(\mathcal{A},-)})$ given by $\deg(x,0)=\deg(x)=\deg\widetilde{(x,0)}$, $\deg(0,s)=\deg(s)=\deg\widetilde{(0,s)}$ and $\deg V_{x,y}=\deg(x)\deg(y)$ for every homogeneous elements $x,y\in\mathcal{A}$ and $s\in\mathcal{S}(\mathcal{A},-)$.

Chapter 4

Structurable algebras related to a hermitian form

In this chapter, we are going to study structurable algebras related to an hermitian form and their relation with other algebraic structures like associative triple systems of the second kind or associative algebras with a 3-grading and an involution. The characteristic of the ground field will be assumed to be different from 2 and 3. In [Fau94, Example 1.11], it is shown how given a finite dimensional algebra $S(\mathcal{E}, -, \mathcal{W}, h)$, i.e., an algebra constructed from a nondegenerate hermitian form h on a module $\mathcal{W}$ over a unital central simple associative algebra with involution $(\mathcal{E}, -)$, the subspace $\mathcal{W}$ together with a triple product on $\mathcal{W}$ defined by $\{u, v, w\} = h(u, v) \circ w = (uv)w$ is an associative triple system of the second kind (AT2). In the example, given the associative triple system and $(\mathcal{E}, -)$ he can recover $S(\mathcal{E}, -, \mathcal{W}, h)$. In this chapter we are going to modify this idea and use the theory of AT2 in order to give a classification of the gradings on this class of structurable algebras based on a classification of the gradings on a class of 3-graded associative algebras with an involution. Every vector space here will be finite dimensional.

In Section 4.1, we define what an associative triple system of the second kind (AT2) is, and give a construction by Loos of a 3-graded associative algebra with an involution. Additionally, we show that there is an equivalence of categories between the category of central simple AT2 and a full subcategory of the cateogory of 3-graded associative algebras with involution (see Corollary 4.1.21). In Section 4.2, we give a definition of structurable grading, which is a kind of grading which exists if and only if the algebra with involution is a structurable algebra related to a hermitian for, and we show that there is an equivalence of categories between the category of central simple AT2 and the categrory of central simple algebras with involution and a structurable grading (see Proposition 4.2.29). In section 4.3, we show that all the central simple structurable algebras with a hermitian form have only one structurable grading except for the split quartic Cayley algebra (this is shown in Propositions 4.3.3, 4.3.5, 4.3.9 and Lemma 4.3.12).

4.1 Associative triple system

4.1.1 Definition and example

Definition 4.1.1. An **associative triple system of the second kind** (AT2) is a vector space W with a triple product $\{\cdots\}$ satisfying

$$\{\{u, v, x\}, y, z\} = \{u, \{y, x, v\}, z\} = \{u, v, \{x, y, z\}\} \tag{4.1.1}$$

for all $u, v, x, y, z \in W$. In case that the product is known from the context we just write W. We will denote by $AT2$ the category of AT2's.

As shown in Remark 1.1.5, an ideal of an AT2 is a subspace $I \subseteq W$ such that:

$$\{I, W, W\} + \{W, I, W\} + \{W, W, I\} \subseteq I.$$

We say that the AT2 is **simple** if $\{W, W, W\} \neq 0$ and the only ideals are W and 0.

Example 4.1.2. Let $(\mathcal{A}, -)$ be an associative algebra with involution. We denote $\{a, b, c\} = a\bar{b}c$. $(\mathcal{A}, \{\cdots\})$ is an AT2. Moreover, any subspace of $\mathcal{A}$ closed under the triple product is also an AT2 as shown in [Loo72].

4.1.2 Associative envelope

In this subsection $(W, \{\cdots\})$ is an AT2. We are going to study a construction introduced by Loos [Loo72] of an associative algebra with a 3-grading and an involution. We denote $E = \mathrm{End}_{\mathbb{F}}(W)$. For any $x, y \in W$ we define the endomorphisms $L(x, y)$ and $R(x, y)$ as:

$$L(x, y)z = \{x, y, z\}$$
$$R(x, y)z = \{z, y, x\}.$$

for all $z \in W$. We can notice that due to (4.1.1), for every $u, v, x, y, z \in W$ we have the following identities

$$\begin{aligned} L(u,v)L(x,y) &= L(u,L(y,x)v) = L(L(u,v)x,y) \\ R(z,y)R(x,v) &= R(z,R(v,x)y) = R(R(z,y)x,v). \end{aligned} \tag{4.1.2}$$

On $E \oplus E^{op}$ and $E^{op} \oplus E$, we denote by $-$ the involution given by $\overline{(A,B)} = (B,A)$. We define for every $x, y \in W$, an element $\lambda(x,y) \in E \oplus E^{op}$ and an element $\rho(x,y) \in E^{op} \oplus E$ by:

$$\begin{aligned} \lambda(x,y) &= (L(x,y), L(y,x)), \\ \rho(x,y) &= (R(y,x), R(x,y)) \text{ and} \end{aligned}$$

for all $x, y, u, v \in W$. By [Mey72, (4.12) and (4.13)] (4.1.2) imply

$$\begin{aligned} \lambda(x,y)\lambda(u,v) &= \lambda(\{x,y,u\},v) = \lambda(x,\{v,u,y\}) \\ \rho(u,v)\rho(x,y) &= \rho(u,\{v,x,y\}) = \rho(\{x,v,u\},y). \end{aligned} \tag{4.1.3}$$

If we denote by $L_0(W)$ the subspace of $E \oplus E^{op}$ generated by the elements of the form $\lambda(x,y)$ for $x, y \in W$ and by $R_0(W)$ the subspace of $E^{op} \oplus E$ generated by the elements of the form $\rho(x,y)$ for $x, y \in W$, and we denote by e_1 and e_2 the units in $E \oplus E^{op}$ and $E^{op} \oplus E$ respectively, the previous discussion yields the following Lemma.

Lemma 4.1.3 ([Loo72])**.** *$L(W) = \mathbb{F}e_1 + L_0(W)$ (resp. $R(W) = \mathbb{F}e_2 + R_0(W)$) is a subalgebra of $E \oplus E^{op}$ (resp. $E^{op} \oplus E$) stable under the involution. Moreover, $L_0(W)$ (resp. $R_0(W)$) is an ideal of $(L(W), -)$ (resp $(R(W), -)$).*

We can define a left and a right action of $L(W)$ and $R(W)$ over W by:

$$ax = a_1(x) = x\overline{a}, \quad bx = b_2(x) = x\overline{b}$$

where $a = (a_1, a_2) \in L$ and $b = (b_1, b_2) \in R$.

Theorem 4.1.4 ([Loo72])**.** *Let $(W, \{\cdots\})$ be an AT2 and $\overline{W}$ be a copy of W. Denote $\mathcal{A}(W) = L(W) \oplus W \oplus \overline{W} \oplus R(W)$ and denote its elements in matrix form, i.e., write $a + x + y + b$ as $\begin{pmatrix} a & x \\ y & b \end{pmatrix}$ for $a \in L(W)$, $x \in W$, $y \in \overline{W}$ and $b \in R(W)$. With this notation, the following hold:*

(a) With the product:

$$\begin{pmatrix} a & x \\ y & b \end{pmatrix} \begin{pmatrix} a' & x' \\ y' & b' \end{pmatrix} = \begin{pmatrix} aa' + \lambda(x,y') & ax' + xb' \\ ya' + by' & \rho(y,x') + bb' \end{pmatrix},$$

$\mathcal{A}(W)$ *is an associative algebra with unity* $1 = \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix}$ *and* $\mathcal{A}_0(W) = L_0(W) \oplus W \oplus \overline{W} \oplus R_0(W)$ *is an ideal.*

(b) $- : u = \begin{pmatrix} a & x \\ y & b \end{pmatrix} \mapsto \overline{u} = \begin{pmatrix} \overline{a} & y \\ x & \overline{b} \end{pmatrix}$ *is an involution of the algebra* $\mathcal{A}(W)$ *with the product defined in* (a) *and the triple product can be recovered as* $\{u, v, w\} = u\overline{v}w$ *for all* $u, v, w \in W$. *We will denote this algebra with involution as* $(\mathcal{A}(W), -)$.

(c) e_1 *and* e_2 *are othogonal idempotents of* $\mathcal{A}(W)$. *They satisgy* $\overline{e_1} = e_1$ *and* $\overline{e_2} = e_2$ *and they induce a Peirce decomposition:*

$$\mathcal{A}(W)_{1,1} = L(W),\ \mathcal{A}(W)_{1,2} = W,\ \mathcal{A}(W)_{2,1} = \overline{W} \text{ and } \mathcal{A}(W)_{2,2} = R(W)$$

(d) $L(W) \oplus R(W)$ *doesn't contain any nontrivial ideal of* $\mathcal{A}(W)$.

Remark 4.1.5. Whenever the context is clear we will write $\mathcal{A}$ instead of $\mathcal{A}(W)$.

Remark 4.1.6. Let $\widetilde{L}_0(W)$ (resp. $\widetilde{R}_0(W)$) be the subspace of $\mathrm{End}_{\mathbb{F}}(W)$ (resp. $\mathrm{End}_{\mathbb{F}}(W)^{op}$) spanned by the elements of the form $L(u, v)$ with $u, v \in W$ (resp. $R(u, v)$ with $u, v \in W$). $\widetilde{L}_0(W)$ (resp. $\widetilde{R}_0(W)$) is a subalgebra of $\mathrm{End}(W)$ (resp. $\mathrm{End}(W)^{op}$) due to (4.1.2). Denote $\widetilde{L}(W) = \mathbb{F}\mathrm{id} + \widetilde{L}_0(W)$ (resp. $\widetilde{R}(W) = \mathbb{F}\mathrm{id} + \widetilde{R}_0(W)$).

There is a reason why on Theorem 4.1.4, the algebras $L(W)$ and $R(W)$ are used instead of the algebras $\widetilde{L}(W)$ and $\widetilde{R}(W)$. The reason is that the involutions given by $L(x, y) \mapsto \overline{L(x, y)} = L(y, x)$ and $R(x, y) \mapsto \overline{R(x, y)} = R(y, x)$ are not well defined. The following, gives a counterexample:

Let $\mathrm{UT}_n^*(\mathbb{F})$ be the algebra of $n \times n$ strictly upper triangular matrices and $\mathrm{LT}^*(\mathbb{F})$ the algebra of $n \times n$ strictly lower triangular matrices. Denote

$$\overline{(X, Y)} = (Y^t, X^t).$$

$-$ is an involution of the algebra $\mathrm{UT}_n^*(\mathbb{F}) \oplus \mathrm{LT}_n^*(\mathbb{F})$. Denote $W = \mathrm{UT}_4^*(\mathbb{F}) \oplus \mathrm{LT}_4^*(\mathbb{F})$, and denote

$$\{X, Y, Z\} = X\overline{Y}Z$$

for every $X, Y, Z \in W$. Then, as shown in Example 4.1.2, this is an AT2. In this case, the involution of $\widetilde{L}(W)$ is not well defined since

$$\overline{0} = \overline{L(0, 0)} = L(0, 0) = 0,$$

but since for any $(A, B) \in W$,

$$\{(0, E_{3,2}), (E_{1,2}, 0), (A, B)\} = (0, E_{3,2}E_{2,1}B) = (0, E_{3,1}B)$$

and $\mathrm{LT}_4^*(\mathbb{F})$ is spanned by $\{E_{i,j} \mid 1 \leq j < i \leq 4\}$, it follows that:

$$L((0, E_{3,2}), (E_{1,2}, 0)) = 0.$$

However:

$$\overline{L((0, E_{3,2}), (E_{1,2}, 0)))} = L((E_{1,2}, 0), (0, E_{3,2})) \neq 0$$

due to the fact that

$$L((E_{1,2}, 0), (0, E_{3,2}))(E_{3,4}, 0) = (E_{1,4}, 0).$$

Therefore, the involution is not well defined.

We denote by $\Delta(W)$ the $\mathbb{Z}$-grading on $\mathcal{A}(W)$ given by $\mathcal{A}(W)_{-1} = \mathcal{A}(W)_{1,2}$, $\mathcal{A}(W)_0 = \mathcal{A}(W)_{1,1} \oplus \mathcal{A}(W)_{2,2}$, $\mathcal{A}(W)_1 = \mathcal{A}(W)_{2,1}$ and $\mathcal{A}(W)_i = 0$ for all $i \neq -1, 0, 1$. With this grading, for every $i \in \mathbb{Z}$ it happens that $\overline{\mathcal{A}(W)_i} = \mathcal{A}(W)_{-i}$.

We denote by $3\mathit{GrAlgInv}$ the category of Ω-algebras $(\mathcal{A}, -, \Delta)$ where $\mathcal{A}$ is a unital associative algebra, $\Delta\colon \mathcal{A} = \mathcal{A}_{-1} \oplus \mathcal{A}_0 \oplus \mathcal{A}_1$ is a 3-grading and $-$ an involution satisfying $\overline{\mathcal{A}_i} = \mathcal{A}_{-i}$ for all $i \in \mathbb{Z}$. The morphisms are the algebra homomorphisms preserving the involution and the grading.

Example 4.1.7. An example of such kind of algebras is the algebra $(\mathcal{M}(n, m),^t)$ consisting on $(n+m) \times (n+m)$ matrices, with the transpose as the involution and with a grading given by:

$$\mathcal{M}(n,m)_{-1} = \begin{pmatrix} 0 & \mathcal{M}_{n\times m}(\mathbb{F}) \\ 0 & 0 \end{pmatrix}, \ \mathcal{M}(n,m)_0 = \begin{pmatrix} \mathcal{M}_n(\mathbb{F}) & 0 \\ 0 & \mathcal{M}_m(\mathbb{F}) \end{pmatrix},$$
$$\mathcal{M}(n,m)_1 = \begin{pmatrix} 0 & 0 \\ \mathcal{M}_{m\times n}(\mathbb{F}) & 0 \end{pmatrix}.$$

Example 4.1.8. Let $\mathcal{M}_4(\mathbb{F})$ be the algebra of 4×4 matrices. Let Φ_0 and Φ_1 be either:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and let Φ be the block matrix

$$\Phi = \Phi_0 \oplus \Phi_1.$$

We can define an involution on $\mathcal{M}_4(\mathbb{F})$ by:

$$X \mapsto \overline{X} := \Phi X^t \Phi$$

and we can define a 3-grading $\Delta\colon \mathcal{M}_4(\mathbb{F}) = \mathcal{M}_4(\mathbb{F})_{-1} \oplus \mathcal{M}_4(\mathbb{F})_0 \oplus \mathcal{M}_4(\mathbb{F})_1$ by:

$$\mathcal{M}_4(\mathbb{F})_{-1} = \begin{pmatrix} 0 & \mathcal{M}_2(\mathbb{F}) \\ 0 & 0 \end{pmatrix}, \mathcal{M}_4(\mathbb{F})_0 = \begin{pmatrix} \mathcal{M}_2(\mathbb{F}) & 0 \\ 0 & \mathcal{M}_2(\mathbb{F}) \end{pmatrix},$$
$$\mathcal{M}_4(\mathbb{F})_1 = \begin{pmatrix} 0 & 0 \\ \mathcal{M}_2(\mathbb{F}) & 0 \end{pmatrix}.$$

We notice that given four matrices $X, Y, Z, W \in \mathcal{M}_2(\mathbb{F})$, the involution is given by:

$$\overline{\begin{pmatrix} X & Y \\ Z & W \end{pmatrix}} = \begin{pmatrix} \Phi_0 X^t \Phi_0 & \Phi_0 Z^t \Phi_1 \\ \Phi_1 Y^t \Phi_0 & \Phi_1 W^t \Phi_1 \end{pmatrix}. \tag{4.1.4}$$

Thus, $(\mathcal{M}_4(\mathbb{F}), -, \Delta)$ is an element of $3\mathit{GrAlgInv}$.

Example 4.1.9. Let $(\mathcal{D}, \widehat{\ })$ be a simple associative algebra with involution. Let $d_0, d_1 \in \mathcal{D}$ be two invertible hermitian elements with respect to the involution. Let

$$\Phi = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}.$$

Denote by $\mathcal{M}_2(\mathcal{D})$ the algebra of 2×2 matrices over $\mathcal{D}$. We can define an involution on $\mathcal{M}_2(\mathcal{D})$ by:

$$X \mapsto \overline{X} := \Phi^{-1} \widehat{X}^t \Phi$$

and we can define a 3-grading $\Delta\colon \mathcal{M}_2(\mathcal{D}) = \mathcal{M}_2(\mathcal{D})_{-1} \oplus \mathcal{M}_2(\mathcal{D})_0 \oplus \mathcal{M}_2(\mathcal{D})_1$ by:

$$\mathcal{M}_2(\mathcal{D})_{-1} = \begin{pmatrix} 0 & \mathcal{D} \\ 0 & 0 \end{pmatrix}, \mathcal{M}_2(\mathcal{D})_0 = \begin{pmatrix} \mathcal{D} & 0 \\ 0 & \mathcal{D} \end{pmatrix},$$
$$\mathcal{M}_4(\mathcal{D})_1 = \begin{pmatrix} 0 & 0 \\ \mathcal{D} & 0 \end{pmatrix}.$$

Given four elements $a, b, c, d \in \mathcal{D}$, the involution is given by:

$$\overline{\begin{pmatrix} X & Y \\ Z & W \end{pmatrix}} = \begin{pmatrix} d_0^{-1} \widehat{a} d_0 & d_0^{-1} \widehat{c} d_1 \\ d_1^{-1} \widehat{b} d_0 & d_1^{-1} \widehat{d} d_1 \end{pmatrix}. \tag{4.1.5}$$

Thus, $(\mathcal{M}_2(\mathcal{D}), -, \Delta)$ is an element of $3\mathit{GrAlgInv}$.

A classification can be found in [Loo72].

Remark 4.1.10. In case the context is clear, we are going to write $\mathcal{A}$ instead of $(\mathcal{A}, -, \Delta)$. For example, in the case of $(\mathcal{A}(W), -, \Delta(W))$, we will sometimes do it.

We can define a functor $\mathbf{W}\colon 3\mathit{GrAlgInv} \to AT2$ in the following way, for an object $(\mathcal{A}, -, \Delta)$ of $3\mathit{GrAlgInv}$, we define $\mathbf{W}(\mathcal{A}, -, \Delta)$ (or just $\mathbf{W}(\mathcal{A})$ when the context is clear) to be the AT2 whose underlying vector space is $\mathcal{A}_{-1}$ and the triple product is given by $\{x, y, z\} = x\overline{y}z$ for all $x, y, z \in \mathcal{A}_{-1}$. For a morphism $\varphi\colon \mathcal{A} \to \mathcal{B}$ between two elements of $3\mathit{GrAlgInv}$ we define $\mathbf{W}(\varphi)$ to be the restriction of φ to $\mathbf{W}(\mathcal{A})$.

Remark 4.1.11. Using theorem 4.1.4, it is easy to check that $\mathbf{W}(\mathcal{A}(W)) = W$ for any AT2 W.

We will try to find a subcategory of $3\mathit{GrAlgInv}$ in which we can extend a morphism from the subspace W to the whole algebra $\mathcal{A}(W)$ in order to define a functor sending an AT2 W to $\mathcal{A}(W)$. We shall make use of the following theorem.

Theorem 4.1.12 ([Loo72]). *An AT2 W is simple if and only if $(\mathcal{A}(W), -)$ is a simple algebra with involution. In particular, in this case, $\mathcal{A}_0(\mathcal{W}) = \mathcal{A}(\mathcal{W})$.*

Lemma 4.1.13. *If $(\mathcal{A}, -)$ is a simple algebra with involution and Δ a 3-grading $\Delta\colon \mathcal{A} = \mathcal{A}_{-1} \oplus \mathcal{A}_0 \oplus \mathcal{A}_1$ such that $(\mathcal{A}, -, \Delta)$ is an object of $3\mathit{GrAlgInv}$, then either $\mathcal{A}_1 = \mathcal{A}_{-1} = 0$ or $\mathcal{A}_0 = \mathcal{A}_1\mathcal{A}_{-1} \oplus \mathcal{A}_{-1}\mathcal{A}_1$.*

Proof. Assume that $\mathcal{A}_1 \neq 0$ (which is equivalent to $\mathcal{A}_{-1} \neq 0$ by applying the involution). Consider $I = \mathcal{A}_{-1} \oplus (\mathcal{A}_1\mathcal{A}_{-1} + \mathcal{A}_{-1}\mathcal{A}_1) \oplus \mathcal{A}_1$. This is an ideal because if we multiply a homogeneous element on I by a homogeneous element on $\mathcal{A}$, by counting degrees we can see that the product is in I. Moreover, it is clear that $\overline{I} = I$. Since the ideal is not zero, and $(\mathcal{A}, -)$ is simple, $I = \mathcal{A}$. Hence, $\mathcal{A}_0 = \mathcal{A}_1\mathcal{A}_{-1} + \mathcal{A}_{-1}\mathcal{A}_1$. We still have to show that this is a direct sum.

Let $J = (\mathcal{A}_1\mathcal{A}_{-1}) \cap (\mathcal{A}_{-1}\mathcal{A}_1)$. If $a \in \mathcal{A}_1$, $aJ + Ja \subseteq (\mathcal{A}_1\mathcal{A}_1)\mathcal{A}_{-1} + \mathcal{A}_{-1}(\mathcal{A}_1\mathcal{A}_1) = 0$. Similarly, if $a \in \mathcal{A}_{-1}$. And if $a \in \mathcal{A}_0$, it follows that $aJ + Ja \subseteq [(\mathcal{A}_0\mathcal{A}_1)\mathcal{A}_{-1} \cap (\mathcal{A}_0\mathcal{A}_{-1})\mathcal{A}_1] + [\mathcal{A}_1(\mathcal{A}_{-1}\mathcal{A}_0) \cap \mathcal{A}_{-1}(\mathcal{A}_1\mathcal{A}_0)] \subseteq J$. Also, it is easy to see that $\overline{J} = J$. Thus, J is an ideal of $(\mathcal{A}, -)$. Hence, since J is not $\mathcal{A}$, due to the fact that $\mathcal{A}_1 \cap J = 0$, the only possibility is $J = 0$. Hence $\mathcal{A}_0 = \mathcal{A}_1\mathcal{A}_{-1} \oplus \mathcal{A}_{-1}\mathcal{A}_1$. □

Lemma 4.1.14. *If $(\mathcal{A}, -)$ is a simple associative algebra with involution and $\Delta\colon \mathcal{A} = \mathcal{A}_{-1} \oplus \mathcal{A}_0 \oplus \mathcal{A}_1$ is a 3-grading such that $(\mathcal{A}, -, \Delta)$ is an object of $3\mathit{GrAlgInv}$, then $W = \mathbf{W}(\mathcal{A})$ is simple or 0.*

Proof. If $\mathcal{A}_{-1} = 0$, $W = 0$. Hence, we will focus on the case $\mathcal{A}_{-1} \neq 0$. First, we have to show that $\{W, W, W\} \neq 0$. We can argue by contradiction. Assume that $\{W, W, W\} = 0$, by definition that implies that $\mathcal{A}_{-1}\mathcal{A}_1\mathcal{A}_{-1} = \mathcal{A}_{-1}\overline{\mathcal{A}_{-1}}\mathcal{A}_{-1} = 0$. Applying the involution, it also implies that $\mathcal{A}_1\mathcal{A}_{-1}\mathcal{A}_1 = 0$. This together with the fact that $\mathcal{A}_1^2 = 0 = \mathcal{A}_{-1}^2$ implies that $\mathcal{A}_0 = \mathcal{A}_1\mathcal{A}_{-1} \oplus \mathcal{A}_{-1}\mathcal{A}_1$ is an ideal of the algebra $\mathcal{A}$. Since $\overline{\mathcal{A}_0} = \mathcal{A}_0$, it is an ideal of $(\mathcal{A}, -)$ which is different from $\mathcal{A}$. Hence, the simplicity of $(\mathcal{A}, -)$ implies that $\mathcal{A}_0 = 0$. Therefore, $\mathcal{A}^2 = 0$ and this is a contradiction with the fact that $(\mathcal{A}, -)$ is simple.

The only thing left to prove is that the only nonzero ideal is W. Let I be an ideal of W. We claim that $J = I \oplus (I\mathcal{A}_1 + \mathcal{A}_1 I + \overline{I}\mathcal{A}_{-1} + \mathcal{A}_{-1}\overline{I}) \oplus \overline{I}$ is an ideal of $(\mathcal{A}, -)$. Indeed, it is stable under the involution. Moreover, since I is an ideal of W, the inclusion $I\mathcal{A}_1\mathcal{A}_{-1} + \mathcal{A}_{-1}\overline{I}\mathcal{A}_{-1} + \mathcal{A}_{-1}\mathcal{A}_1 I \subseteq I$ is satisfied. From this inclusion and the fact that $\mathcal{A}_1^2 = \mathcal{A}_{-1}^2 = 0$, it follows that $\mathcal{A}_i J + J\mathcal{A}_i \subseteq J$ for $i = 1$ and $i = -1$, and using lemma 4.1.13 it follows that $\mathcal{A}_0 J + J\mathcal{A}_0 \subseteq J$. Therefore, J is an ideal. Since J is nonzero and $(\mathcal{A}, -)$ is simple, $J = \mathcal{A}$ and $\mathcal{A}_{-1} = J \cap \mathcal{A}_{-1} = I$. Therefore $I = W$. □

Proposition 4.1.15. *If $(\mathcal{A}, -)$ is a simple associative algebra with involution with a 3-grading $\Delta\colon \mathcal{A} = \mathcal{A}_{-1} \oplus \mathcal{A}_0 \oplus \mathcal{A}_1$ such that $(\mathcal{A}, -, \Delta)$ is an object of 3GrAlgInv and $\mathcal{A}_{-1} \neq 0$. Then, $(\mathcal{A}, -, \Delta) \cong \mathcal{A}(\mathbf{W}(\mathcal{A}), -, \Delta(\mathbf{W}(\mathcal{A})))$.*

Proof. We need to define a homomorphism $\Psi\colon \mathcal{A} \to \mathcal{A}(\mathbf{W}(\mathcal{A}))$. We are going to define it as the homomorphism induced by $\Psi(a) = a$ for all $a \in \mathcal{A}_{-1}$, $\Psi(b) = \overline{\Psi(\overline{b})}$ for all $b \in \mathcal{A}_1$, and $\Psi(x) = \sum_{i=1}^k \Psi(a_i)\Psi(b_i)$ if $x = \sum_{i=1}^k a_i b_i$ and $a_i \in \mathcal{A}_{-1}$ and $b_i \in \mathcal{A}_1$ for all $i \in \{1, ..., k\}$ or viceversa.

We have to prove that it is well defined. In order to do so, we have to show that if $0 = \sum_{i=1}^k a_i b_i$ with $a_i \in \mathcal{A}_{-1}$ and $b_i \in \mathcal{A}_1$ for all $i \in \{1, ..., k\}$ or viceversa, then $\sum_{i=1}^k \Psi(a_i)\Psi(b_i) = 0$. Theorem 4.1.4 shows that $\sum_{i=1}^k \Psi(a_i)\Psi(b_i) = (f_1, f_2) \in \mathrm{End}_{\mathbb{F}}(\Psi(\mathcal{A}_{-1})) \oplus \mathrm{End}_{\mathbb{F}}(\Psi(\mathcal{A}_{-1}))^{op}$. For $a \in \mathcal{A}_{-1}$ we have that in case $a_i \in \mathcal{A}_{-1}$ and $b_i \in \mathcal{A}_1$ for all $i \in \{1, ..., k\}$:

$$f_1(a) = (f_1, f_2)a = \sum_{i=1}^k \Psi(a_i)\overline{\Psi(\overline{b_i})}\Psi(a) = \sum_{i=1}^k a_i b_i a = 0.$$

This shows that $f_1 = 0$. Similarly we show that $f_2 = 0$, taking $a \in \mathcal{A}_1$ and using the fact that the action is defined as $a(f_1, f_2) = \overline{(f_1, f_2)}a = (f_2, f_1)a$. In the case with $a_i \in \mathcal{A}_{-1}$ and $b_i \in \mathcal{A}_1$ for all $i \in \{1, ..., k\}$, the same argument works.

Now, we need to prove that this is an algebra homomorphism. In order to do so, we have to check that $\Psi(a)\Psi(b) = \Psi(ab)$ for any homogeneous elements $a, b \in \mathcal{A}$. If $a \in \mathcal{A}_i$ and $b \in \mathcal{A}_{-i}$ for $i \in \{-1, 1\}$, the definition of Ψ gives the identity. If a or b have degree 0, by Lemma 4.1.13, they are a sum of elements from $\mathcal{A}_1\mathcal{A}_{-1}$ and $\mathcal{A}_{-1}\mathcal{A}_1$. Hence, we can restrict the proof to the case in which they are a product of an element from $\mathcal{A}_i$ and an element from $\mathcal{A}_{-i}$ with $i \in \{-1, 1\}$. Therefore, we only need to check that $\Psi(a)\Psi(b)\Psi(c) = \Psi(abc)$ and that $\Psi(a)\Psi(b)\Psi(c)\Psi(d) = \Psi(abcd)$ for $a, b, c, d \in \mathcal{A}_1 \cup \mathcal{A}_{-1}$. The first is just a direct calculation and the second follows from the first since $\Psi(a)\Psi(b)\Psi(c)\Psi(d) = \Psi(abc)\Psi(d) = \Psi(abcd)$.

We have shown that Ψ is a homomorphism of algebras and now we have to show that it is one of algebras with involution. If $a \in \mathcal{A}_1$, form the definition, we have that $\Psi(a) = \overline{\Psi(\overline{a})}$. Hence, $\Psi(\overline{a}) = \overline{\overline{\Psi(\overline{a})}} = \overline{\Psi(a)}$. If $a \in \mathcal{A}_{-1}$, we have $\Psi(\overline{a}) = \overline{\Psi(\overline{\overline{a}})} = \overline{\Psi(a)}$. If $a \in \mathcal{A}_0$, due to Lemma 4.1.13 and since the involution is linear, without loss of generality, we can assume that a is a product of an element from $\mathcal{A}_i$ and an element from $\mathcal{A}_{-i}$ with $i \in \{-1, 1\}$ and it follows that Ψ commutes with the involution.

Since $\mathcal{A}(\mathbf{W}(\mathcal{A}))$ is simple, Theorem 4.1.12, implies that it is generated as an algebra by $\mathbf{W}(\mathcal{A})$ and $\overline{\mathbf{W}(\mathcal{A})}$ as an algebra. Thus, it is generated as an algebra with involution by $\mathbf{W}(\mathcal{A})$. Therefore, since $\mathbf{W}(\mathcal{A}) = \Psi(\mathcal{A}_{-1})$, it follows that Ψ is surjective.

Since $\ker \Psi$ is an ideal of $(\mathcal{A}, -)$, which is a simple algebra with involution, and $\ker \Psi \cap \mathcal{A}_{-1} = 0$, then $\ker \Psi = 0$. Therefore, Ψ is injective. □

The previous proposition together with theorem 4.1.12 imply that for any simple algebra with involution $(\mathcal{A}, -)$ and a 3-grading $\Delta\colon \mathcal{A} = \mathcal{A}_{-1} \oplus \mathcal{A}_0 \oplus \mathcal{A}_1$ with $\mathcal{A}_{-1} \neq 0$ such that $(\mathcal{A}, -, \Delta)$ is an object of $3\mathit{GrAlgInv}$, $(\mathcal{A}, -, \Delta)$ is isomorphic to $\mathcal{A}(W)$ for a simple AT2 W. We denote by $3\mathit{SGrAlgInv}$ the full subcategory of $3\mathit{GrAlgInv}$ whose objects are the simple algebras with involution $(\mathcal{A}, -)$ with a 3-grading $\Delta\colon \mathcal{A} = \mathcal{A}_{-1} \oplus \mathcal{A}_0 \oplus \mathcal{A}_1$ such that $\mathcal{A}_{-1} \neq 0$ we denote by $AT2^*$ the full subcategory of $AT2$ consisting on those $AT2$ W such that $L_0(W) = L(W)$ and we denote by $SAT2$, the full subcategory of $AT2$ whose objects are the simple AT2's.

We can define a functor

$$\mathbf{A}\colon SAT2 \to 3\mathit{SGrAlgInv} \tag{4.1.6}$$

sending a simple AT2 W to $\mathcal{A}(W)$ and a morphism $\Psi\colon W_1 \to W_2$ to the morphism $\mathbf{A}(\Psi)\colon \mathcal{A}(W_1) \to \mathcal{A}(W_2)$ defined as the linear extension of the map sending $\mathbf{A}(\Psi)(a) = \Psi(a)$ if $a \in \mathcal{A}(W_1)_{-1}$, $\mathbf{A}(\Psi)(a) = \overline{\Psi(\overline{a})}$ if $a \in \mathcal{A}(W_1)_1$, and $\mathbf{A}(\Psi(a)) = \sum_{i=1}^{k} \Psi(a_i)\Psi(b_i)$ if $a = \sum_{i=1}^{k} a_i b_i$ with $a_i \in \mathcal{A}_1$ and $b_i \in \mathcal{A}_{-1}$ for all $i \in \{1, ..., k\}$ or viceversa.

In order to show that $\mathbf{A}$ is well defined, we will need an intermediate lemma. To state it, we need a definition.

Definition 4.1.16. Let W_1 and W_2 be two simple $AT2$. Let $\Psi\colon W_1 \to W_2$ be a morphism of $AT2$. We denote by $L(W_2)_{|\Psi(W_1)}$ ($R(W_2)_{|\Psi(W_1)}$) the vector subspace of $L(W_2)$ spanned by the elements of the form $\lambda(\Psi(u), \Psi(v))$ ($\rho(\Psi(u), \Psi(v))$) with $u, v \in W_1$. $L(W_2)_{|\Psi(W_1)}$ ($R(W_2)_{|\Psi(W_1)}$) is a subalgebra with involution of $L(W_2)$ ($R(W_2)$) due to (4.1.3).

Lemma 4.1.17. *Let W_1 and W_2 be two objects of $AT2^*$ and $\Psi\colon W_1 \to W_2$ a monomorphism. Then, it follows that $(L(W_2)_{|\Psi(W_1)}, -) \cong (L(\Psi(W_1)), -)$ and $(R(W_2)_{|\Psi(W_1)}, -) \cong (R(\Psi(W_1)), -)$. In particular, the result holds if W_1 and W_2 are simple and Ψ is nonzero.*

Proof. We are just going to prove that $(L(W_2)_{|\Psi(W_1)}, -) \cong (L(\Psi(W_1)), -)$. The other isomorphism is analogous.

In case Ψ is a monomorphism, $\Psi(W_1)$ is an $AT2$ isomorphic to W_1. Therefore $L_0(\Psi(W_1)) = L(\Psi(W_1))$.

Define $\varphi\colon L(W_2)_{|\Psi(W_1)} \to L(\Psi(W_1))$ by $\varphi(f, g) = (f_{|\Psi(W_1)}, g_{|\Psi(W_1)})$. This is a well defined vector space homomorphism since $L(\Psi(u), \Psi(v))_{|\Psi(W_1)}$ is the same as $L(\Psi(u), \Psi(v))$ in $\mathrm{End}_{\mathbb{F}}(\Psi(W_1))$, so the image falls in $L(\Psi(W_1))$. It is clearly a morphism of algebras, and it preserves the exchange involution. By definition of $L(W_2)_{|\Psi(W_1)}$ and the fact that $L_0(\Psi(W_1)) = L(\Psi(W_1))$ it is clearly surjective.

In order to show that it is injective, since $L(W_1) = L_0(W_1)$, there are $u_1, v_1, ..., u_k, v_k \in W_1$ are such that $\sum_{i=1}^k \lambda(u_i, v_i) = (\mathrm{id}_{W_1}, \mathrm{id}_{W_1})$. Since Ψ is a homomorphism of AT2, $\varphi(\sum_{i=1}^k \lambda(\Psi(u_i), \Psi(v_i))) = (\mathrm{id}_{\Psi(W_1)}, \mathrm{id}_{\Psi(W_1)})$, then (4.1.3) implies that $\sum_{i=1}^k \lambda(\Psi(u_i), \Psi(v_i))$ is the unity in $L(W_2)_{|\Psi(W_1)}$. Now, if $x \in L(W_2)_{|\Psi(W_1)}$ is such that $\varphi(x) = 0$, it implies that there are $x_1, y_1, ..., x_l, y_l \in W_1$ are such that $x = \sum_{j=1}^l \lambda(\Psi(x_j), \Psi(y_j))$. This implies that the equality $\varphi(\sum_{i=1}^k \lambda(\Psi(u_i), \Psi(v_i)) + \sum_{j=1}^l \lambda(\Psi(x_j), \Psi(y_j))) = (\mathrm{id}_{\Psi(W_1)}, \mathrm{id}_{\Psi(W_1)})$ holds. Therefore, due to (4.1.3), $\sum_{i=1}^k \lambda(\Psi(u_i), \Psi(v_i)) + \sum_{j=1}^l \lambda(\Psi(x_j), \Psi(y_j))$ is the unity too, and this implies that $x = 0$. From here, it follows that φ is injective. □

Proposition 4.1.18. **A** *is a well defined functor.*

Proof. To begin with, we need to assume that $\Psi\colon W_1 \to W_2$ is a morphism between two simple AT2's. We want to show that $\mathbf{A}(\Psi)$ is a well defined vector space homomorphism. In order to do so, due to the fact that $\mathbf{A}(W)$ has the vector space decomposition $\mathbf{A}(W) = L(W) \oplus W \oplus \overline{W} \oplus R(W)$, we just need to show that if we have $x_1, ..., x_k, y_1, ..., y_k \in W$, then $\sum_{i=1}^k \lambda(x_i, y_i) = 0$ implies $\sum_{i=1}^k \lambda(\Psi(x_i), \Psi(y_i)) = 0$, and if we have $\sum_{i=1}^k \rho(y_i, x_i) = 0$, then $\sum_{i=1}^k \rho(\Psi(y_i), \Psi(x_i)) = 0$. We are just going to prove that $\sum_{i=1}^k \lambda(x_i, y_i) = 0$

implies $\sum_{i=1}^{k} \lambda(\Psi(x_i), \Psi(y_i)) = 0$ and the case with ρ is analogous. Since W_1 is simple, we can assume that Ψ is a monomorphism (otherwise Ψ is 0 and it is trivial).

Due to Lemma 4.1.17, if φ is the isomorphism defined in its proof, we just need to show that if $\sum_{i=1}^{k} \lambda(x_i, y_i) = 0$, then $\varphi(\sum_{i=1}^{k} \lambda(\Psi(x_i), \Psi(y_i))) = (\sum_{i=1}^{k} L(\Psi(x_i), \Psi(y_i)), \sum_{i=1}^{k} L(\Psi(y_i), \Psi(x_i)) = 0$. Thus, we just need to show that each coordinate is 0. Take $\Psi(u) \in \Psi(W_1)$. Then, the fact that Ψ is a morphism of $AT2$ implies that $\sum_{i=1}^{k} L(\Psi(x_i), \Psi(y_i))\Psi(u) = \Psi(\sum_{i=1}^{k} L(x_i, y_i)u) = \Psi(0) = 0$. Therefore, $\sum_{i=1}^{k} L(\Psi(x_i), \Psi(y_i)) = 0$.

The proof that $\mathbf{A}(\Psi)$ preserves the product and the involution follows from a calculation similar to the ones in Proposition 4.1.15.

To show that $\mathbf{A}$ is a functor we just need to show that $\mathbf{A}(\mathrm{id}_W) = \mathrm{id}_{\mathcal{A}(W)}$ and that $\mathbf{A}(\Psi \circ \Phi) = \mathbf{A}(\Psi) \circ \mathbf{A}(\Phi)$ but this is direct from the definition. □

Remark 4.1.19. If $\Psi \colon W_1 \to W_2$ is a monomorphism between two objects of $AT2^*$, the same proof as in Proposition 4.1.18 works to show that it makes sense to define $\mathbf{A}(\Psi) \colon \mathcal{A}(W_1) \to \mathcal{A}(W_2)$.

Proposition 4.1.20. $\mathbf{A}$ *is an equivalence from* $SAT2$ *to* $3SGrAlgInv$*. Its inverse is the restriction of* $\mathbf{W}$*.*

Proof. The fact that $(\mathcal{A}(W), -, \Delta(W))$ is simple for any simple AT2 follows from Theorem 4.1.12. It is clear that $\mathbf{W} \circ \mathbf{A} = \mathrm{id}$ due to Remark 4.1.11.

In order to show that $\mathbf{A} \circ \mathbf{W} \cong \mathrm{id}$, we denote as $\Psi_{\mathcal{A}}$ the isomorphism from $\mathcal{A}$ to $\mathcal{A}(\mathbf{W}(\mathcal{A}))$ defined in proposition 4.1.15 and we will show that $\{\Psi_{\mathcal{A}}\}$ is a natural transformation. In order to do so, we have to show that for every homomorphism of graded algebras with involution $f \colon \mathcal{A} \to \mathcal{B}$, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\Psi_{\mathcal{A}}} & \mathcal{A}(\mathbf{W}(\mathcal{A})) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathbf{A}(\mathbf{W}(f))} \\ \mathcal{B} & \xrightarrow{\Psi_{\mathcal{B}}} & \mathcal{A}(\mathbf{W}(\mathcal{B})). \end{array}$$

The diagram commutes because the algebra $\mathcal{A}$ is generated as an algebra with involution by $\mathcal{A}_{-1}$ and the restriction of $\mathbf{A}(\mathcal{W}(f)) \circ \Psi_{\mathcal{A}}$ to $\mathcal{A}_{-1}$ is by definition the same as the restriction of $\Psi_{\mathcal{B}} \circ f$. □

We denote by $3CSGrAlgInv$ the full subcategory of $3SGrAlgInv$ consisting of the objects $(\mathcal{A}, -, \Delta)$ such that $(\mathcal{A}, -)$ is central simple and we denote by $CSAT2$ the full subcategory of $SAT2$ consisting of the central simple AT2.

Corollary 4.1.21. **A** *induces an equivalence from CSAT2 to 3CSGrAlgInv. Its inverse is the restriction of* **W** *to 3CSGrAlgInv .*

Proof. This follows from Proposition 1.1.13, Proposition 4.1.20 and the fact that $\mathcal{A}(W \otimes_{\mathbb{F}} \mathbb{K}) \cong \mathcal{A}(W) \otimes_{\mathbb{F}} \mathbb{K}$. □

The following Lemma will be needed later on for a proof.

Lemma 4.1.22. *Let $(\mathcal{A}, -)$ be an associative algebra with involution and e an idempotent such that $\overline{e} = e$. Denote $\mathcal{A}_{1,1} = e\mathcal{A}e$. Then:*

(1) $\overline{\mathcal{A}_{1,1}} = \mathcal{A}_{1,1}$.

(2) If $(\mathcal{A}, -)$ is central simple, $(\mathcal{A}_{1,1}, -)$ is central simple.

Proof. (1) follows from the fact that $\overline{e} = e$ since $\overline{e\mathcal{A}e} = \overline{e}\overline{\mathcal{A}}\overline{e} = e\mathcal{A}e$.

In order to prove (2), as $(\mathcal{A}, -)$ is central simple if and only if $(\mathcal{A} \otimes \overline{\mathbb{F}}, -)$ is simple, where $\overline{\mathbb{F}}$ is an algebraic closure of $\mathbb{F}$, we may assume that $\mathbb{F}$ is algebraically closed and just prove that $(\mathcal{A}_{1,1}, -)$ is simple (here we are using the notation $-$ for the involution on $\mathcal{A}$ and its restriction to $\mathcal{A}_{1,1}$). Then, up to isomorphism there are two possibilities: either $\mathcal{A}$ is simple, or there is a simple algebra $\mathcal{B}$ such that $(\mathcal{A}, -) \cong (\mathcal{B} \oplus \mathcal{B}^{op}, \mathrm{ex})$.

In case $\mathcal{A}$ is simple, then there is an $\mathbb{F}$-vector space V such that $\mathcal{A} = \mathrm{End}_{\mathbb{F}}(V)$. Let $e \in \mathrm{End}_{\mathbb{F}}(V)$ be an idempotent. Then, $\mathrm{im}\,(e) = \{v \in V \mid e(v) = v\}$ and $V = \mathrm{im}\,(e) \oplus \ker(e)$. The morphism $\Psi\colon e\,\mathrm{End}_{\mathbb{F}}(V)e \to \mathrm{End}_{\mathbb{F}}(\mathrm{im}\,(e))$ given by $\Psi(f) = f_{|\mathrm{im}\,(e)}$ is an isomorphism. Indeed, it is clearly injective and it is surjective because given a morphism $f \in \mathrm{End}_{\mathbb{F}}(\mathrm{im}\,(e))$, the morphism g defined by $g(v) = f(v)$ if $v \in \mathrm{im}\,(e)$ and $g(v) = 0$ if $v \in \ker(e)$ satisfies that $\Psi(g) = f$. Therefore, $e\,\mathrm{End}_{\mathbb{F}}(V)e \cong \mathrm{End}_{\mathbb{F}}(\mathrm{im}\,(e))$ is simple and so $(\mathcal{A}_{1,1}, -)$ is simple.

In case $\mathcal{A}$ is not simple, there is a simple algebra $\mathcal{B}$ such that $(\mathcal{A}, -) \cong (\mathcal{B} \oplus \mathcal{B}^{op}, \mathrm{ex})$. Moreover, any idempotent on $\mathcal{B} \oplus \mathcal{B}^{op}$ is of the form (e_1, e_2) with e_1 and e_2 idempotents of $\mathcal{B}$. Concretely, the idempotent which induces the Peirce decomposition is fixed under the involution. Thus, the fact that $\mathrm{ex}(e_1, e_2) = (e_2, e_1)$ implies that $e_1 = e_2$. Hence, $(\mathcal{A}_{1,1} -) \cong (f\mathcal{B}f \oplus f\mathcal{B}^{op}f, \mathrm{ex})$ for some idempotent f, which is a simple associative algebra with involution since $f\mathcal{B}f$ is simple and $f\mathcal{B}^{op}f = (f\mathcal{B}f)^{op}$. Thus, $(\mathcal{A}_{1,1}, -)$ is simple. □

4.2 Some functors

4.2.1 Definition and basic properties

Let $(\mathcal{E}, -)$ be a unital central simple associative algebra with involution. Let $\mathcal{W}$ be a unital associative left $\mathcal{E}$-module and denote the action by $e \circ w$ for all $e \in \mathcal{E}$, $w \in \mathcal{W}$. Let $h\colon \mathcal{W} \times \mathcal{W} \to \mathcal{E}$ be a nondegenerate hermitian form. We denote $(\mathcal{A}, -) = S(\mathcal{E}, -, \mathcal{W}, h)$. From the product defined in Example 3.2.5 we notice that there is a $\mathbb{Z}_2$-grading $\Gamma\colon \mathcal{A} = \mathcal{A}_{\bar{0}} \oplus \mathcal{A}_{\bar{1}}$ given by $\mathcal{A}_{\bar{0}} = \mathcal{E}$ and $\mathcal{A}_{\bar{1}} = \mathcal{W}$ which satisfies:

(1) $\mathcal{A}_{\bar{0}}$ is an associative subalgebra of $\mathcal{A}$ stable under the involution,

(2) $\mathcal{A}_{\bar{1}} \subseteq \mathcal{H}(\mathcal{A}, -)$,

(3) $e(fw) = (fe)w$ for all $e, f \in \mathcal{A}_{\bar{0}}$ and $w \in \mathcal{A}_{\bar{1}}$, and

(4) $f(wv) = w(\bar{f}v)$ for all $f \in \mathcal{A}_{\bar{0}}$ and $w, v \in \mathcal{A}_{\bar{1}}$.

Definition 4.2.1. Given a unital nonassociative algebra with involution, $(\mathcal{A}, -)$ and a $\mathbb{Z}_2$-grading $\Gamma\colon \mathcal{A} = \mathcal{A}_{\bar{0}} \oplus \mathcal{A}_{\bar{1}}$, we say that Γ is a **structurable grading** if $\mathcal{A}_{\bar{1}} \neq 0$ and it satisfies the conditions $(1) - (4)$.

Remark 4.2.2. Sometimes, we will say that $\Gamma\colon \mathcal{A} = \mathcal{E} \oplus \mathcal{W}$ is a structurable grading without specifying the degrees. That will mean that $\mathcal{A}_{\bar{0}} = \mathcal{E}$ and that $\mathcal{A}_{\bar{1}} = \mathcal{W}$.

Remark 4.2.3. Given an algebra with a structurable grading $\mathcal{A} = \mathcal{E} \oplus \mathcal{W}$, for all $f \in \mathcal{E}$ and for all $w \in \mathcal{W}$, $fw \in \mathcal{W}$ (because of the degrees). Since property (2) says that $\mathcal{W} \subseteq \mathcal{H}(\mathcal{A}, -)$, then $fw = \overline{fw} = w\bar{f}$.

Remark 4.2.4. Let $(\mathcal{A}, -)$ be a nonassociative algebra with involution and $\Gamma\colon \mathcal{A} = \mathcal{E} \oplus \mathcal{W}$ a structurable grading. Consider the bilinear map:

$$\begin{array}{rccc} h\colon & \mathcal{W} \times \mathcal{W} & \to & \mathcal{E} \\ & (w, v) & \mapsto & vw. \end{array}$$

Consider too the action $\circ\colon \mathcal{E} \times \mathcal{W} \to \mathcal{W}$ given by $f \circ w = \bar{f}w$. Due to property (1), the algebra $(\mathcal{E}, -)$ is an associative algebra with involution. Due to property (3), $\mathcal{W}$ is a unital associative module with this action. Due to property (4), $h(f \circ v, w) = fh(v, w)$ for all $v, w \in \mathcal{W}$ and $f \in \mathcal{E}$. Due to (1) and (2), $h(v, w) = \overline{h(w, v)}$. Hence, h is an hermitian form. By definition, we can write the product as

$$(e + v)(f + w) = (ef + h(w, v)) + (\bar{e} \circ v + f \circ w)$$

for all $e, f \in \mathcal{E}$ and $v, w \in \mathcal{W}$. And due to property (2) we have

$$\overline{e+v} = \overline{e} + v$$

for all $e \in \mathcal{E}$ and $v \in \mathcal{W}$. Hence, $(\mathcal{A}, -)$ is isomorphic to the structurable algebra related to a hermitan form $S(\mathcal{E}, -, \mathcal{W}, h)$

Definition 4.2.5. Let $(\mathcal{A}, -)$ be a nonassociative algebra with a unit and $\Gamma \colon \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$ a G-grading. We say that $(\mathcal{A}, -, \Gamma)$ is **graded-central**, as an algebra with involution, if $\mathcal{Z}(\mathcal{A}, -)_e = \mathbb{F}1$. We say that $(\mathcal{A}, -, \Gamma)$ is **graded-central-simple**, as an algebra with involution, if it is graded-central and graded-simple (i.e. the only graded ideals are $\mathcal{A}$ and 0).

Remark 4.2.6. Due to [AC20, Lemma 3.8], the center of a G-graded algebra, $\mathcal{Z}(\mathcal{A})$, is a graded subspace and the space of hermitian elements, $\mathcal{H}(\mathcal{A}, -)$ is also a graded subspace. Hence, $\mathcal{Z}(\mathcal{A}, -) = \mathcal{Z}(\mathcal{A}) \cap \mathcal{H}(\mathcal{A}, -)$ is graded. Thus, the definition above makes sense.

Lemma 4.2.7. *Let $(\mathcal{A}, -)$ be a unital nonassociative algebra and Γ a G-grading. $(\mathcal{A}, -, \Gamma)$ is graded-central if and only if it is central as an Ω-algebra. Moreover, $(\mathcal{A}, -, \Gamma)$ is graded-central-simple if and only if it is central simple as an Ω-algebra.*

Proof. As in Lemma 1.1.11, we can show that if $\mathcal{C}(\mathcal{A}, -, \Gamma) = \mathbb{F}\mathrm{id}$, then $\mathcal{Z}(\mathcal{A}, -)_e = \mathbb{F}1$. Assume now that $\mathcal{Z}(\mathcal{A}, -)_e = \mathbb{F}1$. As in Lemma 1.1.11, we can show that if $f \in \mathcal{C}(\mathcal{A}, -, \Gamma)$, then $f = L_{f(1)}$ and $f(1) \in \mathcal{Z}(\mathcal{A}, -)$. We just have to prove that $\deg(f(1)) = e$. In order to do so, with the notation of Remark 1.4.2, we have that for every $g \in G$, $\pi_g(f(1)) = f(\pi_g(1)) = f(1)\delta_{g,e}$ where $\delta_{g,e} = 1$ if $g = e$ and $\delta_{g,e} = 0$ otherwise. Hence, $\deg f(1) = e$. That implies that $\mathcal{C}(\mathcal{A}, -, \Gamma) = \{L_x \mid x \in \mathcal{Z}(\mathcal{A}, -)_e\} = \mathbb{F}\mathrm{id}$.

The second part is clear since a graded ideal of $(\mathcal{A}, -)$ is the same as an ideal of $(\mathcal{A}, -, \Gamma)$. □

Our purpose for this subsection is to give a description of the graded-central-simple nonassociative algebras with involution and a structurable grading $(\mathcal{A}, -, \Gamma)$.

Lemma 4.2.8. *Let $(\mathcal{A}, -)$ be a structurable algebra and let $\Gamma \colon \mathcal{E} \oplus \mathcal{W}$ be a stucturable grading. The following are (graded) ideals:*

(a) $J_1 = I \oplus I\mathcal{W}$ *where I is an ideal of $(\mathcal{E}, -)$.*

(b) $J_2 = \mathcal{W} \oplus \mathcal{W}\mathcal{W}$.

Proof. (a) Clearly, this is a graded subspace. Given $e \in I$ and $w \in \mathcal{W}$, we have that $\overline{e} \in I$ (because I is an ideal of an algebra with involution), and $\overline{ew} = ew$ (since $ew \in \mathcal{W} \subseteq \mathcal{H}(\mathcal{A}, -)$). This shows that $\overline{J_1} \subseteq J_1$.

We only have left to prove that this is an ideal of $\mathcal{A}$ as an algebra. First, for any $e \in \mathcal{E}$, $f \in I$ and $w \in \mathcal{W}$, $ef \in I$, and $e(fw) = \overline{e} \circ (\overline{f} \circ w) = \overline{fe} \circ w = (fe)w \in I\mathcal{W}$. Hence $\mathcal{E}J_1 \subseteq J_1$. Given $v \in \mathcal{W}$, we have that $vf = \overline{f}v \in I\mathcal{W}$ and $v(fw) = \overline{f}(vw) \in I\mathcal{E} \subseteq I$. Hence, $\mathcal{W}J_1 \subseteq J_1$. We have shown that $\mathcal{A}J_1 \subseteq J_1$. Finally, applying the involution we get $J_1\mathcal{A} = \overline{\mathcal{A}J_1} \subseteq \overline{J_1} = J_1$.

(b) Since $\overline{\mathcal{W}} = \mathcal{W}$, we get that $\overline{J_2} = J_2$. For $e \in \mathcal{E}$ and $v, w \in \mathcal{W}$, we have that $ew \in \mathcal{W}$, and that $e(vw) = v(\overline{e}w) \in \mathcal{W}\mathcal{W}$. Hence $\mathcal{E}J_2 \subseteq J_2$. For $u \in \mathcal{W}$ we have that $uw \in \mathcal{W}\mathcal{W}$ and that $u(vw) \in \mathcal{W}\mathcal{E} \subseteq \mathcal{W}$. Hence, $\mathcal{A}J_2 \subseteq J_2$. As in (a), applying the involution we get that $J_2\mathcal{A} \subseteq J_2$. □

Corollary 4.2.9. *Let $(\mathcal{A}, -, \Gamma)$ be a graded-simple non associative algebra with involution and a structurable grading $\Gamma\colon \mathcal{A} = \mathcal{E} \oplus \mathcal{W}$ with $\mathcal{W} \neq 0$. Then, $\mathcal{W}\mathcal{W} = \mathcal{E}$.*

Proof. Due to Lemma 4.2.8, $\mathcal{W} \oplus \mathcal{W}\mathcal{W} = \mathcal{A}$. Since $\mathcal{W}\mathcal{W} \subseteq \mathcal{E}$, the corollary follows. □

Remark 4.2.10. We should notice that if $(\mathcal{A}, -)$ is simple, then, $(\mathcal{A}, -, \Gamma)$ is graded-simple.

Lemma 4.2.11. *Let $(\mathcal{A}, -, \Gamma)$ be a nonassociative algebra and $\Gamma\colon \mathcal{A} = \mathcal{E} \oplus \mathcal{W}$ a non trivial structurable grading (i.e. $\mathcal{W} \neq 0$). Let $h\colon \mathcal{W} \times \mathcal{W} \to \mathcal{E}$ be the hermitian form defined by $h(w, v) = vw$. $(\mathcal{A}, -, \Gamma)$ is graded-central-simple if and only if $(\mathcal{E}, -)$ is central simple and h is nondegenerate.*

Proof. Assume that $(\mathcal{A}, -, \Gamma)$ is graded-central-simple. By contradiction, if h is degenerate, there is an element $w \in \mathcal{W}$ such that $h(w, \mathcal{W}) = 0$. In that case, $\mathcal{E}w$ is an ideal of $(\mathcal{A}, -, \Gamma)$. It is not trivial since $w = 1w \in \mathcal{E}w$. It is not $\mathcal{A}$ since $\mathcal{E}w \subseteq \mathcal{W}$. Hence, we get a contradiction. Assume now that $(\mathcal{E}, -)$ is not simple. Let I be a nontrivial proper ideal of $(\mathcal{E}, -)$. Due to Lemma 4.2.8, $I \oplus IW$ is a nontrivial proper ideal of $(\mathcal{A}, -, \Gamma)$. Thus, we arrive to a contradiction. Finally, for $e \in \mathcal{Z}(\mathcal{E}, -)$ and for any $w \in \mathcal{W}$, $ew = w\overline{e} = we$. Moreover, $e(fw) = (ef)w = (fe)w$ and $e(wf) = e(\overline{fw}) = (e\overline{f})w = \overline{f}(ew) = (ew)f$ for all $f \in \mathcal{E}$ and $w \in W$, $e(wv) = (wv)e = \overline{e(vw)} = \overline{v(ew)} = (ew)v$. Hence $[e, \mathcal{A}, \mathcal{A}] = 0$. Similarly, $[\mathcal{A}, e, \mathcal{A}] = 0$ and $[\mathcal{A}, \mathcal{A}, e] = 0$. Therefore, $\mathcal{Z}(\mathcal{E}, -) \subseteq \mathcal{Z}(\mathcal{A}, -) = \mathbb{F}1$. Hence $(\mathcal{E}, -)$ is central.

If $(\mathcal{E}, -)$ is central simple and h is nondegenerate, since $\mathcal{Z}(\mathcal{A}, -)_{\bar{0}} \subseteq \mathcal{Z}(\mathcal{E}, -) = \mathbb{F}1$, $(\mathcal{A}, -, \Gamma)$ is central. If I is a nontrivial ideal of $(\mathcal{A}, -, \Gamma)$, since it is a graded ideal, either $I_{\bar{0}}$ or $I_{\bar{1}}$ are nonzero. If $I_{\bar{1}} \neq 0$, the fact that h is nondegenerate implies $0 \neq h(I, \mathcal{W}) \subseteq I_{\bar{0}}$. Therefore, $I_{\bar{0}}$ is not zero. $I_{\bar{0}}$ is a nontrivial ideal of $(\mathcal{E}, -)$ and since it is simple, $I_{\bar{0}} = \mathcal{E}$. Thus, $1 \in I$ and that implies that $I = \mathcal{A}$. Hence, $(\mathcal{A}, -, \Gamma)$ is simple. □

Lemma 4.2.12. *Let $(\mathcal{A}, -, \Gamma)$ be an algebra with involution with a structurable grading $\Gamma\colon \mathcal{A} = \mathcal{E} \oplus \mathcal{W}$ is a structurable grading. Then, $(\mathcal{A}, -, \Gamma)$ is graded-central-simple if and only if $(\mathcal{A}, -)$ is central simple or it is isomorphic to $\left(\frac{\mathbb{F}[X]}{(X^2-\lambda)}, \mathrm{id}, \Gamma\right)$ with $\Gamma\colon \mathbb{F}[x] = \mathbb{F}1 \oplus \mathbb{F}x$ for $x = X + (X^2 - \lambda)$ and $\lambda \in \mathbb{F}^\times$.*

Proof. One direction is clear. Assume that $(\mathcal{A}, -, \Gamma)$ is graded-central-simple. From Lemma 4.2.11, h is nondegenerate and $(\mathcal{E}, -)$ is central simple. We can divide the proof into two cases:

(a) If $- = \mathrm{id}$ on $\mathcal{A}$, $\mathcal{E}$ is commutative and $\mathcal{H}(\mathcal{E}, -) = \mathcal{E}$. Since additionally it is associative, $\mathcal{Z}(\mathcal{E}, -) = \mathcal{E}$. Thus, the fact that $(\mathcal{E}, -)$ is central implies that $\mathcal{E} = \mathbb{F}$. Let I be a nonzero ideal of $(\mathcal{A}, -)$. In case there is an element $0 \neq w \in \mathcal{W} \cap I$, the fact that h is nondegenerate implies that there is an element $v \in \mathcal{W}$ such that $0 \neq h(w, v) \in \mathbb{F}1 \cap I$. Therefore, $1 \in I$, which implies that $I = \mathcal{A}$. Otherwise, there is an element $w \in \mathcal{W}$ such that $1 + w \in I$. If $w^2 \neq 1$, we have that $(1 + w)(1 - w) = 1 - w^2 \in I \cap \mathbb{F}^\times 1$, which implies that $1 \in I$ and so, $I = \mathcal{A}$. If $w^2 = 1$ and $\dim \mathcal{W} > 1$, we can take an element $v \in \mathcal{W}$ such that $h(w, v) = 0$ and $(1 + w)v = v \in I \cap \mathcal{W}$ so we are in the previous case. Therefore, $(\mathcal{A}, -)$ is simple. In this case, if $0 \neq x \in \mathcal{A}$, there is $\lambda \in \mathbb{F}$ and $0 \neq w \in \mathcal{W}$ such that $x = \lambda 1 + w$. Since $\dim \mathcal{W} > 1$ there are $u, v \in \mathcal{W}$ such that $h(u, w) \neq 0$ and $h(v, w) = 0$. Hence $[u, (\lambda 1 + w), v] = [u, \lambda 1, w] + (uw)v - u(wv) = h(w, u)v \neq 0$. Therefore, $\mathcal{Z}(\mathcal{A}, -) = \mathbb{F}1$ and $(\mathcal{A}, -)$ is central. In case $\dim(\mathcal{W}) = 1$, the fact that h is nondegenerate implies that for any $0 \neq w \in \mathcal{W}$, $w^2 = h(w, w) = \lambda 1 \in \mathbb{F}^\times 1$. This implies that $(\mathcal{A}, -, \Gamma) \cong \left(\frac{\mathbb{F}[x]}{(x^2-\lambda)}, \mathrm{id}, \Gamma\right)$ with $\Gamma\colon \mathbb{F} = \mathbb{F}1 \oplus \mathbb{F}x$, which is not central and is not even simple in case λ is a square in $\mathbb{F}$.

(b) Assume now that $- \neq \mathrm{id}$. Let J be an ideal of $(\mathcal{A}, -)$ and $0 \neq x \in J$. Take $e \in \mathcal{E}$ and $w \in \mathcal{W}$ such that $x = e + w$. Since $(\mathcal{E}, -)$ is simple and $\mathcal{E}e\mathcal{E} + \mathcal{E}\bar{e}\mathcal{E}$ is an ideal, we have that $\mathcal{E}e\mathcal{E} + \mathcal{E}\bar{e}\mathcal{E} = \mathcal{E}$ therefore, for any $s \in \mathcal{E}$ there are $e_1, ..., e_n, e_{n+1}, ..., e_m, f_1, ..., f_n, f_{n+1}, ..., f_m \in \mathcal{E}$ such that

$$s = \sum_{i=1}^{n} e_i e f_i + \sum_{i=n+1}^{m} e_i \bar{e} f_i$$

. Assume that $0 \neq s$ is skew hermitian with respect to the involution. Hence, there is $v \in \mathcal{W}$ such that

$$0 \neq \sum_{i=1}^{n} e_i x f_i + \sum_{i=n+1}^{m} e_i \overline{x} f_i = s + v \in J$$

. Thus $s = \frac{1}{2}((s+v) - (\overline{s+v})) \in J \cap \mathcal{E}$, so $J \cap \mathcal{E} \neq 0$. Therefore, $J \cap \mathcal{E}$ is nonzero. Since it is an ideal of $(\mathcal{E}, -)$ which is simple, it follows that $J \cap \mathcal{E} = \mathcal{E}$. Thus, $1 \in J$, implying that $J = \mathcal{A}$ and that $(\mathcal{A}, -)$ is simple.

We are going to prove that $(\mathcal{A}, -)$ is central by contradiction. Assume it is not. In this case, since $\mathcal{Z}(\mathcal{A}, -)_{\overline{0}} = \mathbb{F}1$ then, the fact that $\mathcal{Z}(\mathcal{A}, -)$ is graded implies that there should be an element $0 \neq w \in \mathcal{Z}(\mathcal{A}, -)_{\overline{1}}$. Take $0 \neq s \in \mathcal{S}(\mathcal{A}, -)$. Then, $sw = ws = \overline{s}w = -sw$. Therefore, $sw = 0$. However, since $(\mathcal{A}, -)$ is simple, $\mathcal{Z}(\mathcal{A}, -)$ is a field. Thus, w is invertible, and that implies that $s = 0$, which is a contradiction. □

We finish by stating the following corollary

Corollary 4.2.13. *Let $(\mathcal{A}, -, \Gamma)$ be a nonassociative algebra with $\Gamma\colon \mathcal{E} \oplus \mathcal{W}$ a structurable grading with $\mathcal{W} \neq 0$. Then, if $(\mathcal{A}, -, \Gamma)$ is graded-central-simple, $\mathcal{W}\mathcal{W} = \mathcal{E}$.*

Proof. If $(\mathcal{A}, -)$ is central simple, this result follows from Corollary 4.2.9 and if $(\mathcal{A}, -, \Gamma) \cong \left(\frac{\mathbb{F}[x]}{(x^2-\lambda)}, \mathrm{id}, \Gamma\right)$, it clearly follows. □

4.2.2 Associative triple systems and structurable algebras

We have studied a construction of an associative algebra with an involution and a 3-grading from an AT2. Previously we have defined the concept of structurable grading, which is something that characterizes structurable algebras related to an hermitian form. Our purpose now is to give a construction of an algebra with a structurable grading from an AT2 and to study it.

Definition 4.2.14. Let $(\mathcal{A}, -, \Gamma)$ be a structurable algebra with a structurable grading $\Gamma\colon \mathcal{A} = \mathcal{E} \oplus \mathcal{W}$, we define $\mathfrak{W}(\mathcal{A}, -, \Gamma)$ as the Ω-algebra $(\mathcal{W}, \{\cdots\}_\Gamma)$ where $\{\cdots\}_\Gamma$ is the operator of arity 3 given by

$$\{u, v, w\}_\Gamma = h(u, v) \circ w \tag{4.2.1}$$

for all u, v, w.

Remark 4.2.15. Notice that for a structurable algebra $(\mathcal{A}, -, \Gamma)$ with a structurable grading $\Gamma\colon \mathcal{A} = \mathcal{E} \oplus \mathcal{W}$, we have that $\{u, v, w\}_\Gamma = (uv)w$.

The following result appears in [Fau94]. However, since the proof is not given, we write it here.

Lemma 4.2.16 ([Fau94])**.** *Let $(\mathcal{A}, -)$ be a structurable algebra and $\Gamma\colon \mathcal{A} = \mathcal{E} \oplus \mathcal{W}$ a structurable grading. Then $\mathfrak{W}(\mathcal{A}, -, \Gamma)$ is an associative triple system of the second kind.*

Proof. For all $u, v, w, x, y, z \in \mathcal{W}$ we have:

$$\begin{aligned} \{\{u, v, x\}, y, z\} &= h(h(u, v) \circ x, y) \circ z \\ &= (h(u, v)h(x, y)) \circ z \\ &= h(u, v) \circ (h(x, y) \circ z) \\ &= \{u, v, \{x, y, z\}\} \end{aligned}$$

and

$$\begin{aligned} \{u, v, \{x, y, z\}\} &= (h(u, v)h(x, y)) \circ z \\ &= h(u, \overline{h(x, y)} \circ v) \circ z \\ &= h(u, h(y, x) \circ v) \circ z \\ &= \{u, \{y, x, v\}, z\}. \end{aligned}$$

□

In case we have two algebras with involution and a structurable grading, $(\mathcal{A}, -, \Gamma_\mathcal{A})$ and $(\mathcal{B}, -, \Gamma_\mathcal{B})$, and a morphism $f\colon (\mathcal{A}, -, \Gamma_\mathcal{A}) \to (\mathcal{B}, -, \Gamma_\mathcal{B})$, its restriction induces a morphism $\mathfrak{W}(f)\colon \mathfrak{W}(\mathcal{A}, -, \Gamma_\mathcal{A}) \to \mathfrak{W}(\mathcal{B}, -, \Gamma_\mathcal{B})$. Hence, $\mathfrak{W}$ defines a functor from the category of nonassociative algebras with involution and a structurable grading which we denote by *AlgStrGr* to $AT2$.

From this construction we can also define some ideals.

Lemma 4.2.17. *Let $(\mathcal{A}, -)$ be an algebra with involution and $\Gamma\colon \mathcal{A} = \mathcal{E} \oplus \mathcal{W}$ a structurable grading. If I is an ideal of $(\mathcal{A}, -, \Gamma)$, $J_1 = I \cap \mathcal{W}$ is an ideal of $\mathfrak{W}(\mathcal{A}, -, \Gamma)$. Moreover, if $\mathcal{E} = \mathcal{W}\mathcal{W}$ and I is an ideal of $\mathfrak{W}(\mathcal{A}, -, \Gamma)$, then, $J_2 = I \oplus (\mathcal{W}I + I\mathcal{W})$ is an ideal of $(\mathcal{A}, -, \Gamma)$. In particular, if $(\mathcal{A}, -, \Gamma)$ is graded-simple, then $I = 0$ or $I = \mathcal{W}$, so $\mathfrak{W}(\mathcal{A}, -, \Gamma)$ is simple.*

Proof. The fact that if I is an ideal of $(\mathcal{A}, -, \Gamma)$, $J_1 = I \cap \mathcal{W}$ is an ideal of $\mathfrak{W}(\mathcal{A}, -, \Gamma)$ is easy to prove from the definition of $\{\cdots\}_\Gamma$.

If I is an ideal of $\mathfrak{W}(\mathcal{A}, -, \Gamma)$, since the elements of I are hermitian, we have that $\overline{J_2} = \overline{I} \oplus (\overline{\mathcal{W}I} + \overline{I\mathcal{W}}) = I \oplus (I\mathcal{W} + \mathcal{W}I) = J_2$. J_2 is graded, since $I \subseteq \mathcal{W}$ and $(I\mathcal{W} + \mathcal{W}I) \subseteq \mathcal{E}$. Since J_2 is stable under the involution, we just need to show that $\mathcal{A}J_2 \subseteq J_2$ since $J_2\mathcal{A} = \overline{\mathcal{A}J_2}$ and applying the involution to the previous inclusion, we would get $J_2\mathcal{A} \subseteq J_2$. Since we have a grading, we just need to show that $\mathcal{E}J_2 \subseteq J_2$ and $\mathcal{W}J_2 \subseteq J_2$. We have:

$$\begin{aligned}\mathcal{E}J_2 &= \mathcal{E}I + \mathcal{E}(\mathcal{W}I) + \mathcal{E}(I\mathcal{W}) \\ &= (\mathcal{W}\mathcal{W})I + \mathcal{W}(\mathcal{E}I) + I(\mathcal{E}\mathcal{W}) \\ &\subseteq \{\mathcal{W}, \mathcal{W}, I\}_\Gamma + \mathcal{W}((\mathcal{W}\mathcal{W})I) + I\mathcal{W} \\ &= \{\mathcal{W}, \mathcal{W}, I\}_\Gamma + \mathcal{W}\{\mathcal{W}, \mathcal{W}, I\}_\Gamma + I\mathcal{W} \\ &\subseteq I + \mathcal{W}I + I\mathcal{W},\end{aligned}$$

which implies that $\mathcal{E}J_2 \subseteq J_2$. Moreover, since $u(vw) = \overline{(vw)}u = (wv)u = \{w, v, u\}_\Gamma$ we have that:

$$\begin{aligned}\mathcal{W}J_2 &= \mathcal{W}I + \mathcal{W}(\mathcal{W}I) + (\mathcal{W}I)W \\ &= \mathcal{W}I + \{I, \mathcal{W}, \mathcal{W}\}_\Gamma + \{\mathcal{W}, I, \mathcal{W}\}_\Gamma \\ &\subseteq \mathcal{W}I + I \\ &\subseteq J_2.\end{aligned}$$

In case $(\mathcal{A}, -\Gamma)$ is graded-simple, Corollary 4.2.9 implies that $\mathcal{W}\mathcal{W} = \mathcal{E}$. Thus, the statement follows. □

Remark 4.2.18. Again, we should recall that in case $(\mathcal{A}, -)$ is simple, $(\mathcal{A}, -, \Gamma)$ is graded simple, so Lemma 4.2.17 holds.

Remark 4.2.19. In Lemma 4.2.17, J_2 is the minimal ideal of $(\mathcal{A}, -)$ containing I.

Lemma 4.2.20. *Given two algebras with involution and a structurable grading $(\mathcal{A}, -, \Gamma_\mathcal{A})$ where $\Gamma_\mathcal{A} \colon \mathcal{A} = \mathcal{E}_\mathcal{A} \oplus \mathcal{W}_\mathcal{B}$ and $(\mathcal{B}, -, \Gamma_\mathcal{B})$ where $\Gamma_\mathcal{B} \colon \mathcal{B} = \mathcal{E}_\mathcal{B} \oplus \mathcal{W}_\mathcal{B}$, and given a homomorphism $f \colon (\mathcal{A}, -, \Gamma_\mathcal{A}) \to (\mathcal{B}, -, \Gamma_\mathcal{B})$, the following assertions hold:*

(a) If $(\mathcal{A}, -, \Gamma_\mathcal{A})$ is graded-simple, f is injective if and only if $\mathfrak{W}(f)$ is injective.

(b) If $(\mathcal{B}, -, \Gamma_\mathcal{B})$ is graded-simple, f is surjective if and only if $\mathfrak{W}(f)$ is surjective.

Proof.

(a) From the definition of $\mathfrak{W}(f)$, it follows that $\ker(\mathfrak{W}(f)) = \ker(f) \cap \mathcal{W}_{\mathcal{A}}$. Therefore, if f is injective, so is $\mathfrak{W}(f)$. Aditionally, if $\mathfrak{W}(f)$ is injective, $\ker(f)$ is an ideal of $(\mathcal{A}, -, \Gamma_{\mathcal{A}})$ which is not $\mathcal{A}$ since $\ker(f) \cap \mathcal{W}_{\mathcal{A}} = 0$. Thus, the fact that $(\mathcal{A}, -, \Gamma_{\mathcal{A}})$ is graded-simple implies that $\ker(f) = 0$.

(b) This part of the lemma follows from the fact that since $(\mathcal{B}, -, \Gamma_{\mathcal{B}})$ is graded-simple, $\mathcal{E}_B = \mathcal{W}_B^2$. □

We would like to construct a functor from some full subcategory of $AT2$ to *AlgStrGr*. In order to do so, given an AT2 W and $v, w \in W$, we will denote $L(x, y)$ and λ and $L(W)$ as in 4.1.2 and $-$ the exchange involution in $L(W)$ i.e., $\overline{(x, y)} = (y, x)$. We define $(\mathfrak{A}(W), -, \Gamma_{\mathfrak{A}}(W))$ as the structurable algebra $S(L(W), -, W, h)$ where W is an $L(W)$ module with the action $(a_1, a_2) \circ w = a_1(w)$ for all $a = (a_1, a_2) \in L(W)$ and $w \in W$, $h \colon W \times W \to L(W)$ is defined as $h(w_1, w_2) = \lambda(w_1, w_2)$ and where $\Gamma_{\mathfrak{A}}(W) \colon \mathfrak{A}(W) = L(W) \oplus W$ is the structurable grading.

Remark 4.2.21. Let $(\mathcal{A}, -)$ be a simple associative algebra with involution and let $\Delta \colon \mathcal{A} = \mathcal{A}_{-1} \oplus \mathcal{A}_0 \oplus \mathcal{A}_1$ be a 3-grading of $\mathcal{A}$ such that $\varphi(\mathcal{A}_i) = \mathcal{A}_{-i}$. Denote $W = \mathbf{W}(\mathcal{A}, -, \Delta)$. Due to Proposition 4.1.15, it follows that $(L(W), -) \cong (\mathcal{A}_{-1}\mathcal{A}_1, -)$ where the involution in $\mathcal{A}_{-1}\mathcal{A}_1$ is given by the restriction of the involution in $\mathcal{A}$ and as shown in the proof, the isomorphism is given by:

$$\sum_{i=1}^{k} \lambda(v_i, w_i) \mapsto \sum_{i=1}^{k} v_i \overline{w}_i$$

for all $v_1, ..., v_k, w_1, ..., w_k \in \mathcal{A}_{-1}$. Then, it follows that $(\mathfrak{A}(W), -, \Gamma(W)$ is isomorphic to $S(\mathcal{A}_{-1}\mathcal{A}_1, -, \mathcal{A}_1, h)$ where:

- $\mathcal{A}_{-1}$ is a left-module of $\mathcal{A}_{-1}\mathcal{A}_1$ with the action consisting in multiplying on the left in $\mathcal{A}$, and
- $h(v, w) = v\overline{w}$

Example 4.2.22. Let $(\mathcal{M}_4(\mathbb{F}), -, \Delta)$ be the algebra from Example 4.1.8. Let $W = \mathbf{W}(\mathcal{M}_4(\mathbb{F}), -, \Delta)$. Then, here we can identify $(\mathfrak{A}(W), -, \Gamma(W))$ with $S(\mathcal{M}_2(\mathbb{F}), -, \mathcal{M}_2(\mathbb{F}), h)$ where:

- $\overline{X} = \Phi_0 X^t \Phi_0$ for all $X \in \mathcal{M}_2(\mathbb{F})$ (due to (4.1.4)),
- $\mathcal{M}_2(\mathbb{F})$ is a left-module of $\mathcal{M}_2(\mathbb{F})$ with the action consisting in multiplying on the left, and

- $h(X,Y) = X\Phi_1 Y^t \Phi_0$ for all $X, Y \in \mathcal{M}_2(\mathbb{F})$ (due to (4.1.4)).

Therefore, we can identify $(\mathfrak{A}(W), -, \Gamma(W))$ with $\mathcal{M}_2(\mathbb{F}) \times \mathcal{M}_2(\mathbb{F})$ by identifying for every $Y, X \in \mathcal{M}_2(\mathbb{F})$, the matrix:

$$\begin{pmatrix} X & Y \\ 0 & 0 \end{pmatrix}$$

with the elemet (X,Y) and we have:

- $(X,Y)(Z,W) = (XZ + W\Phi_1 Y^t \Phi_0, ZY + \Phi_0 X^t \Phi_0 W)$ for all $(X,Y), (Z,W) \in \mathcal{M}_2(\mathbb{F}) \times \mathcal{M}_2(\mathbb{F})$
- $\overline{(X,Y)} = (\Phi_0 X^t \Phi_0, Y)$ for all $(X,Y) \in \mathcal{M}_2(\mathbb{F}) \times \mathcal{M}_2(\mathbb{F})$
- The structurable grading is $\Gamma(W)\colon \mathcal{M}_2(\mathbb{F}) \times \mathcal{M}_2(\mathbb{F}) = \mathcal{E} \oplus \mathcal{W}$ where $\mathcal{E} = \mathcal{M}_2(\mathbb{F}) \times 0$ and $\mathcal{W} = 0 \times \mathcal{M}_2(\mathbb{F})$.

In view of this we can check that we have the following subspaces:

(1) If $\Phi_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

  ⋆ $\mathcal{S}(\mathfrak{A}(W), -) = \mathbb{F}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 0\right)$

  ⋆ $\mathcal{H}(\mathfrak{A}(W), -) = \left\{\left(\begin{pmatrix} \lambda & \beta \\ \beta & \gamma \end{pmatrix}, X\right) \mid \lambda, \beta, \gamma \in \mathbb{F},\ X \in \mathcal{M}_2(\mathbb{F})\right\}$

  ⋆ $\mathcal{K}(\mathfrak{A}(W), -) = \mathbb{F}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, 0\right) \oplus \mathbb{F}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 0\right)$

  ⋆ $\mathcal{M}(\mathfrak{A}(W), -) = \left\{\left(\begin{pmatrix} \lambda & \beta \\ \beta & -\lambda \end{pmatrix}, X\right) \mid \lambda, \beta \in \mathbb{F},\ X \in \mathcal{M}_2(\mathbb{F})\right\}$

(2) If $\Phi_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

  ⋆ $\mathcal{S}(\mathfrak{A}(W), -) = \mathbb{F}\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, 0\right)$

  ⋆ $\mathcal{H}(\mathfrak{A}(W), -) = \left\{\left(\begin{pmatrix} \lambda & \beta \\ \gamma & \lambda \end{pmatrix}, X\right) \mid \lambda, \beta, \gamma \in \mathbb{F},\ X \in \mathcal{M}_2(\mathbb{F})\right\}$

  ⋆ $\mathcal{K}(\mathfrak{A}(W), -) = \mathbb{F}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, 0\right) \oplus \mathbb{F}\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, 0\right)$

$$\star\ \mathcal{M}(\mathfrak{A}(W), -) = \left\{ \left(\begin{pmatrix} 0 & \lambda \\ \beta & 0 \end{pmatrix}, X \right) \mid \lambda, \beta \in \mathbb{F},\ X \in \mathcal{M}_2(\mathbb{F}) \right\}$$

Example 4.2.23. Let $(\mathcal{M}_2(\mathcal{D}), -, \Delta)$ be the algebra from Example 4.1.9. Let $W = \mathbf{W}(\mathcal{M}_2(\mathcal{D}), -, \Delta)$. Then, here we can identify $(\mathfrak{A}(W), -, \Gamma(W))$ with $S(\mathcal{D}, -, \mathcal{D}, h)$ where:

- $\overline{a} = d_0^{-1}\widehat{a}\Phi_0$ for all $a \in \mathcal{D}$. This is due to (4.1.5),
- $\mathcal{D}$ is a left-module of $\mathcal{D}$ with the action consisting in multiplying on the left , and
- $h(a, b) = ad_1^{-1}\widehat{b}d_0$ for all $a, b \in \mathcal{M}_2(\mathbb{F})$. This is due to (4.1.5).

Therefore, we can identify $(\mathfrak{A}(W), -, \Gamma(W))$ with $\mathcal{M}_2(\mathbb{F}) \times \mathcal{M}_2(\mathbb{F})$ by identifying for every $a, b \in \mathcal{M}_2(\mathbb{F})$, the matrix:

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$$

with the element (a, b) and we have:

- $(a, b)(c, d) = (ac + dd_1^{-1}\widehat{b}d_0, cb + d_0^{-1}\widehat{a}d_0 d)$ for all $(a, b), (c, d) \in \mathcal{D} \times \mathcal{D}$
- $\overline{(a, b)} = (d_0^{-1}\widehat{a}d_0, b)$ for all $(a, b) \in \mathcal{D} \times \mathcal{D}$
- The structurable grading is $\Gamma(W) \colon \mathcal{D} \times \mathcal{D} = \mathcal{E} \oplus \mathcal{W}$ where $\mathcal{E} = \mathcal{D} \times 0$ and $\mathcal{W} = 0 \times \mathcal{D}$.

In case $(\mathcal{D}, \widehat{\ }) = (\mathcal{M}_2(\mathbb{F}), {}^t)$, recall the $\mathbb{Z}_2^2$-grading on $\mathcal{M}_2(\mathbb{F})$ given by:

$$\deg(z_1) = a_1, \quad \deg(z_2) = a_2, \quad \deg(z_3) = a_3$$

where:

$$z_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad z_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad z_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and:

$$a_1 = (\overline{1}, \overline{0}), \quad a_2 = (\overline{0}, \overline{1}), \quad a_3 = (\overline{1}, \overline{1})$$

Denote $z_0 = \mathrm{id}$. We can choose $d_0 = z_0$ or $d_0 = z_2$ and we have the following subspaces:

(1) Case $d_0 = z_0$:

- ⋆ $\mathcal{S}(\mathfrak{A}(W), -) = \mathbb{F}(z_3, 0)$,
- ⋆ $\mathcal{H}(\mathfrak{A}(W), -) = \{(\lambda z_0 + \beta z_1 + \gamma z_2, d) \mid \lambda, \beta, \gamma \in \mathbb{F},\ d \in \mathcal{D}\}$,
- ⋆ $\mathcal{K}(\mathfrak{A}(W), -) = \mathbb{F}(z_0, 0) \oplus \mathbb{F}(z_3, 0)$ and
- ⋆ $\mathcal{M}(\mathfrak{A}(W), -) = \{(\lambda z_1 + \beta z_2, d) \mid \lambda, \beta \in \mathbb{F},\ d \in \mathcal{D}\}$.

(2) Case $d_0 = z_2$:

- ⋆ $\mathcal{S}(\mathfrak{A}(W), -) = \mathbb{F}(z_1, 0)$,
- ⋆ $\mathcal{H}(\mathfrak{A}(W), -) = \{(\lambda z_0 + \beta z_2 + \gamma z_3, d) \mid \lambda, \beta, \gamma \in \mathbb{F},\ d \in \mathcal{D}\}$,
- ⋆ $\mathcal{K}(\mathfrak{A}(W), -) = \mathbb{F}(z_0, 0) \oplus \mathbb{F}(z_1, 0)$ and
- ⋆ $\mathcal{M}(\mathfrak{A}(W), -) = \{(\lambda z_2 + \beta z_3, d) \mid \lambda, \beta \in \mathbb{F},\ d \in \mathcal{D}\}$.

Proposition 4.2.24. *An AT2 $(W, \{\cdots\})$ is central simple if and only if the algebra $(\mathfrak{A}(W), -, \Gamma_{\mathfrak{A}}(W))$ is graded-central-simple.*

Proof. Assume that $(W, \{\cdots\})$ is central simple. Therefore, for every field extension $\mathbb{K}/\mathbb{F}$, $(W \otimes_{\mathbb{F}} \mathbb{K}, \{\cdots\}_{\mathbb{K}})$ is simple. Using theorem 4.1.12, it follows that $(W \otimes_{\mathbb{F}} \mathbb{K}, \{\cdots\}_{\mathbb{K}})$ is simple if and only if $(\mathcal{A}(W) \otimes_{\mathbb{F}} \mathbb{K}, -)$ is simple, which implies that $(\mathcal{A}(W), -)$ is central simple. Due to Lemma 4.1.22, it follows that $(L(W), -)$ is central simple.

We just have to show that $h\colon W \times W \to L(W)$ is nondegenerate. Without loss of generality we can assume that $\mathbb{F}$ is algebraically closed. Due to [Loo72] $(W, \{\cdots\})$ is isomorphic to the set of $p \times q$ matrices over an associative composition algebra with the product given by $\{x, y, z\} = x\overline{y}^t z$ where $-$ is an involution on the composition algebra. Hence, it is trivial that h is nondegenerate. Therefore, $(\mathfrak{A}(W), -, \Gamma_{\mathfrak{A}}(W))$ is graded-central-simple.

In case $(\mathfrak{A}(W), -, \Gamma_{\mathfrak{A}}(W))$ is graded-central-simple, $(L(W), -)$ is central simple and the hermitian form $h\colon W \times W \to L(W)$ is nondegenerate. Since this is a property preserved by extension of scalars, in view of Proposition 1.1.13, we just need to show that this implies that $(W, \{\cdots\})$ is simple.

Let I be a nonzero ideal of $(W, \{\cdots\})$. Then, let $J = \lambda(I, W) + \lambda(W, I)$. Since $L(W)J + JL(W) = \lambda(W, W)J + J\lambda(W, W) \subseteq \lambda(I, W) + \lambda(W, I)$ due to (4.1.3), J is an ideal of $(L(W), -)$. The fact that h is nondegenerate, implies that $J \neq 0$. Since $(L(W), -)$ is simple, $J = L(W)$. Thus, since $(\mathrm{id}, \mathrm{id}) \in L(W)$, $W = L(W)W = \{I, W, W\} + \{W, I, W\} \subseteq I$, we have that $(W, \{\cdots\})$ is simple.

□

Remark 4.2.25. If $(W, \{\cdots\})$ is an AT2, $\mathfrak{W}(\mathfrak{A}(W), -, \Gamma_{\mathfrak{A}}(W)) = (W, \{\cdots\})$

We denote by *CSAlgStrGr* to the full subcategory of *AlgStrGr* consisting of the graded-central-simple algebras with a structurable grading.

Proposition 4.2.26. *Let $(\mathcal{A}, -, \Gamma)$ be an algebra with a structurable grading $\Gamma\colon \mathcal{A} = \mathcal{E} \oplus \mathcal{W}$. Assume that $(\mathcal{A}, -, \Gamma)$ is graded-central-simple. Then, $(W, \{\cdots\}) = \mathfrak{W}(\mathcal{A}, -, \Gamma)$ is central simple. Moreover:*

$$(\mathcal{A}, -, \Gamma) \cong (\mathfrak{A}(W), -, \Gamma_{\mathfrak{A}}(W)).$$

Proof. For the proof, we will denote just by λ the hermitian form in $\mathfrak{A}(\mathcal{W})$.

In order to prove the first part, since $(\mathcal{A}, -, \Gamma)$ is graded central simple if and only if it $(\mathcal{A} \otimes_{\mathbb{F}} \mathbb{K}, -_{\mathbb{K}}, \Gamma_{\mathbb{K}})$ is simple for every field extension $\mathbb{K}/\mathbb{F}$, in view of Proposition 1.1.13, we only need to show that $\mathfrak{W}(\mathcal{A}, -, \Gamma)$ is simple, but this follows from Lemma 4.2.17

In order to prove the second part, we need to define a vector space homomorphism:

$$\psi\colon \mathcal{A} \to \mathfrak{A}(\mathcal{W})$$

by $\psi(w) = w$ for all $w \in \mathcal{W}$ and $\psi(\sum_{i=1}^k w_i v_i) = \sum_{i=1}^k \lambda(v_i, w_i)$ for all $w_1, ..., w_k, v_1, ..., v_k \in \mathcal{W}$. Since $\mathcal{E} = \mathcal{W}\mathcal{W}$ due to Corollary 4.2.9, in order to show that it is well defined we just need to show that if $\sum_{i=1}^k w_i v_i = 0$, then $\sum_{i=1}^k \lambda(v_i, w_i) = 0$. Let $(x_1, x_2) = \sum_{i=1}^k \lambda(v_i, w_i)$. Then for all $w \in \mathcal{W}$:

$$\begin{aligned} x_1(w) = \sum_{i=1}^k \lambda(v_i, w_i) \circ w = \sum_{i=1}^k \{v_i, w_i, w\} &= \sum_{i=1}^k h(v_i, w_i) \circ w \\ &= \left(\sum_{i=1}^k w_i v_i\right) \circ w = 0. \end{aligned}$$

Therefore, $x_1 = 0$ and similarly $x_2 = 0$. In order to prove that ψ is a homomorphism of algebras we just need to prove that it preserves the product and the involution for homogeneous elements. Using the fact that $\mathcal{E} = \mathcal{W}\mathcal{W}$, given $w, w_1, w_2, w_3, w_4 \in \mathcal{W}$, the following identities are enough:

- $\psi(\overline{w}) = \psi(w) = w = \overline{w} = \overline{\psi(w)}$,
- $\psi(\overline{w_1 w_2}) = \psi(w_2 w_1) = \lambda(w_1, w_2) = \overline{\lambda(w_2, w_1)} = \overline{\psi(w_1 w_2)}$,
- $\psi(w_1 w_2) = \lambda(w_2, w_1) = w_1 w_2 = \psi(w_1)\psi(w_2)$,
- $\psi((w_1 w_2) w_3) = \psi(h(w_1, w_2) \circ w_3) = h(w_1, w_2) \circ w_3 = \{w_1, w_2, w_3\} = \lambda(w_1, w_2) \circ w_3 = \lambda(w_2, w_1) w_3 = \psi(w_1 w_2) w_3$,
- $\psi((w_1 w_2)(w_3 w_4)) = \psi(w_3((w_2 w_1) w_4)) = \psi(w_3)\psi((w_2 w_1) w_4) = \psi(w_3)((\psi(w_2)\psi(w_1))\psi(w_4)) = (\psi(w_1)\psi(w_2))(\psi(w_3)\psi(w_4))$,

where $(w_1w_2)(w_3w_4) = w_3((w_2w_1)w_4)$ due to (4.1.2). The fact that $L(\mathcal{W}) = \mathcal{W}\mathcal{W}$ implies that ψ is surjective. Since ψ is a graded homomorphism, $\ker\psi$ is a graded ideal. Since $\ker\psi \cap \mathcal{W} = 0$ and $\mathfrak{A}(\mathcal{W})$ is graded-simple, $\ker\psi = 0$. Hence, ψ is injective. □

Remark 4.2.27. The previous proposition implies that for an object $(\mathcal{A}, -, \Gamma)$ of *CSAlgStrGr* with $\Gamma\colon \mathcal{A} = \mathcal{E} \oplus \mathcal{W}$, $L(\mathfrak{W}(\mathcal{A}, -, \Gamma)) \cong \mathcal{E}$.

We can define a functor $\mathfrak{A}\colon CSAT2 \to CSAlgStrGr$ given by $\mathfrak{A}(W) = (\mathfrak{A}(W), -, \Gamma_{\mathfrak{A}}(W))$ for every central-simple AT2 W and for any morphism $f\colon W \to V$, we define $\mathfrak{A}(f)$ by $\mathfrak{A}(f)(w) = f(w)$ and $\mathfrak{A}(f)(\sum_{i=1}^k h(v_i, w_i)) = \sum_{i=1}^k h(f(v_i), f(w_i))$ for every $w, v_1, w_1, ..., v_k, w_k \in W$.

Proposition 4.2.28. $\mathfrak{A}$ *is a well defined functor.*

Proof. The only thing that we have to show is that given a morphism $f\colon W \to V$ of $CSAT2$, then, $\mathfrak{A}(f)$ is a well defined morphism of *CSAlgStrGr*.

Let W and V be elements of $CSAT2$ and $f\colon W \to V$ a nonzero morphism. Since W and V are simple, f is injective. Thus, $f(W)$ is a subtriple system of V isomorphic to W. The homomorphism $\varphi\colon L(V)_{|f(W)} \to L(f(W))$ given by restricting each endomorphism to $f(W)$ is a well defined isomorphism of algebras with involution due to Lemma 4.1.17.

The homomorphism f^* from $L(W)$ to $L(f(W))$ given by $f^*\lambda(u,v) = \lambda(f(u), f(v))$ is well defined due to the fact that if $v_1, w_1, ..., v_k, w_k \in W$, are elements which satisfy that $\sum_{i=1}^k \lambda(v_i, w_i) = 0$, then for all $u \in W$,

$$\sum_{i=1}^k \lambda(f(v_i), f(w_i))f(u) = f(\sum_{i=1}^k \lambda(v_i, w_i)u) = 0$$

, and again, due to (4.1.3) and the fact that f is a morphism of $AT2$, it is a morphism of algebras with involution.

Finally, $\mathfrak{A}(f)$ is well defined due to the fact that if $v_1, w_1, ..., v_k, w_k \in W$, are such that $\sum_{i=1}^k h(v_i, w_i) = 0$ in $\mathfrak{A}(W)$, then, the fact that f^* and φ are monomorphisms implies that

$$\sum_{i=1}^k h(f(v_i), f(w_i)) = \sum_{i=1}^k \lambda(f(v_i), f(w_i)) = 0$$

. This implies that $\mathfrak{A}$ is well defined.

□

Proposition 4.2.29. $\mathfrak{A} \circ \mathfrak{W} \cong \mathrm{id}$ *and* $\mathfrak{W} \circ \mathfrak{A} \cong \mathrm{id}$, *i.e.,* $\mathfrak{A}$ *and* $\mathfrak{W}$ *are equivalences of categories between* $CSAT2$ *and* *CSAlgStrGr*.

Proof. In order to show that $\mathfrak{A} \circ \mathfrak{W} \cong \mathrm{id}$, we define $\psi_{(\mathcal{A}, -, \Gamma)}$ as the isomorphism defined in Proposition 4.2.26. This is a natural isomorphism if and only if for every graded-central-simple $(\mathcal{A}, -, \Gamma_{\mathcal{A}})$ and $(\mathcal{B}, -, \Gamma_{\mathcal{B}})$, and every homomorphism $f\colon \mathcal{A} \to \mathcal{B}$ of graded algebras with involution, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\psi_{\mathcal{A}}} & \mathfrak{A}(\mathfrak{W}(\mathcal{A}, -, \Gamma_{\mathcal{A}})) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathfrak{A}(\mathfrak{W}(f))} \\ \mathcal{B} & \xrightarrow{\psi_{\mathcal{B}}} & \mathfrak{A}(\mathfrak{W}(\mathcal{B}, -, \Gamma_{\mathcal{B}})). \end{array}$$

Due to Corollary 4.2.9, we just need to show that it commutes for element of the space $\mathcal{W}$ given by the structurable grading $\Gamma_{\mathcal{A}}\colon \mathcal{A} = \mathcal{E} \oplus \mathcal{W}$, but this is a straightforward calculation.

Finally, it is clear that $\mathfrak{W} \circ \mathfrak{A} = \mathrm{id}$.

□

4.3 Structurable gradings

Given an algebra with involution and a structurable grading, $(\mathcal{A}, -, \Gamma)$, where $\Gamma\colon \mathcal{A} = \mathcal{E} \oplus \mathcal{W}$, for any $\psi \in \mathrm{Aut}(\mathcal{A}, -)$, since any homomorphism of algebras with involution preserve the properties (1) − (4) of the definition of structurable grading, then, the grading $\Gamma^{\psi}\colon \mathcal{A} = \psi(\mathcal{E}) \oplus \psi(\mathcal{W})$ is a structurable grading. If the algebra with involution $(\mathcal{A}, -)$ admits a unique structurable grading, then $\mathrm{Aut}(\mathcal{A}, -) \cong \mathrm{Aut}(\mathcal{A}, -, \Gamma)$. Our purpose in this section, will be to classify the structurable gradings on structurable algebras related to an hermitian form.

4.3.1 Uniqueness property

Definition 4.3.1. Let $(\mathcal{A}, -)$ be a structurable algebra related to an hermitian form. We say that it has the uniqueness property if there is a unique grading Γ such that $(\mathcal{A}, -, \Gamma)$ is an object of *CSAlgStrGr*.

Proposition 4.3.2. *Let $(\mathcal{A}, -, \Gamma)$ be an object of CSAlgStrGr with $\Gamma\colon \mathcal{A} = \mathcal{E} \oplus \mathcal{W}$. If $\mathcal{E} = \mathcal{K}(\mathcal{A}, -)$, i.e., the algebra generated by the skew symmetric elements, then, $\mathcal{W} = \mathcal{M}(\mathcal{A}, -)$ and $(\mathcal{A}, -)$ has the uniqueness property.*

Proof. $\mathcal{W} \subseteq \mathcal{M}(\mathcal{A}, -)$ due to Remark 4.2.3 and $\mathcal{W} = \mathcal{M}(\mathcal{A}, -)$ because due to (3.2.2), their dimensions are the same.

Let $\widetilde{\Gamma}\colon \mathcal{A} = \widetilde{\mathcal{E}} \oplus \widetilde{\mathcal{W}}$ be a structurable grading such that $(\mathcal{A}, -, \widetilde{\Gamma})$ is graded-central-simple. Then, due to properties (1) and (2) of structurable gradings, $\mathcal{S}(\mathcal{A}, -) \subseteq \widetilde{\mathcal{E}}$. Therefore, $\mathcal{E} \subseteq \widetilde{\mathcal{E}}$. Due to Remark 4.2.3, we have that $\mathcal{W} \subseteq \widetilde{\mathcal{W}}$. But $\mathcal{A} = \mathcal{E} \oplus \mathcal{W} = \widetilde{\mathcal{E}} \oplus \widetilde{\mathcal{W}}$. Therefore, $\mathcal{E} \subseteq \widetilde{\mathcal{E}}$ and $\mathcal{W} \subseteq \widetilde{\mathcal{W}}$ imply that $\mathcal{E} = \widetilde{\mathcal{E}}$ and $\mathcal{W} = \widetilde{\mathcal{W}}$. $\square$

We just have left the case in which $\mathcal{E} \neq \mathcal{K}(\mathcal{A}, -)$. Since $\mathcal{S}(\mathcal{A}, -) \subseteq \mathcal{E}$, that is equivalent to $\mathcal{E} \neq \mathcal{K}(\mathcal{E}, -)$. Hence, we can make use of the following proposition:

Proposition 4.3.3. *Let $(\mathcal{E}, -)$ be a central simple associative algebra with involution which is not generated by its skew-symmetric elements. Then, $(\mathcal{E} \otimes_{\mathbb{F}} \overline{\mathbb{F}}, -) \cong (\mathcal{M}_2(\mathbb{F}), {}^t)$ or $(\mathcal{E} \otimes_{\mathbb{F}} \overline{\mathbb{F}}, -) \cong (\overline{\mathbb{F}}, \mathrm{id})$.*

Proof. Let $(\mathcal{E}, -)$ be a central simple associative algebra with involution. Then, $(\mathcal{E} \otimes_{\mathbb{F}} \overline{\mathbb{F}}, -)$ is a central simple algebra with involution over $\overline{\mathbb{F}}$. Therefore, it is enough to assume that the field is algebraically closed. We can divide the proof into two parts:

- If $\mathcal{E}$ is not simple, there is a simple associative algebra $\mathcal{A}$ such that $(\mathcal{E}, -) \cong (\mathcal{A} \oplus \mathcal{A}^{op}, \mathrm{ex})$, where $\mathrm{ex}(x, y) = (y, x)$ is the exchange involution. Since
$$\mathcal{S}(\mathcal{A} \oplus \mathcal{A}^{op}, \mathrm{ex}) = \{(x, -x) \mid x \in \mathcal{A}\}$$
and
$$\mathcal{H}(\mathcal{A} \oplus \mathcal{A}^{op}, \mathrm{ex}) = \{(x, x) \mid x \in \mathcal{A}\} = (1, -1)\mathcal{S}(\mathcal{A} \oplus \mathcal{A}^{op}, \mathrm{ex}),$$
we have shown that $(\mathcal{E}, -) = \mathcal{K}(\mathcal{E}, -)$.

- If $\mathcal{E}$ is simple, since we are assuming that $\mathbb{F}$ is algebraically closed, it is central simple. In view of [KMRT, Theorem 1.1], $\mathcal{E} \cong \mathcal{M}_n(\mathbb{F})$ for some $n \in \mathbb{N}$. Thus, $\mathcal{Z}(\mathcal{E}) = \mathbb{F}$. Due to [HER68, Theorem 2.1.10], if $\dim_{\mathbb{F}} \mathcal{E} > 4$, $\mathcal{K}(\mathcal{E}, -) = \mathcal{E}$. Assume that $\dim_{\mathbb{F}} \mathcal{E} \leq 4$. This implies that $n = 1$ or $n = 2$. If $n = 1$, $(\mathcal{E}, -) \cong (\mathbb{F}, \mathrm{id})$. If $n = 2$, [KMRT, Proposition 2.1] implies that one of the following cases holds:

  (a) $(\mathcal{E}, -) \cong (\mathcal{M}_2(\mathbb{F}), {}^t)$ or

  (b) $(\mathcal{E}, -) \cong (\mathcal{M}_2(\mathbb{F}), -)$, where $\overline{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} A^t \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

  In case (a), using [KMRT, Proposition 2.6], $\mathcal{S}(\mathcal{E}, -)$ has dimension 1. Thus, it is straightforward to show that $\mathcal{S}(\mathcal{E}, -) = \mathbb{F}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Hence, the fact that $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, implies that $\mathcal{K}(\mathcal{E}, -) \neq \mathcal{E}$. In case

(b), [KMRT, Proposition 2.6] implies that $\mathcal{S}(\mathcal{E}, -)$ has dimension 3. Hence, by showing that the elements $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ are skew-hermitian, it follows that $\mathcal{S}(\mathcal{E}, -) = \{x \in \mathcal{M}_2(\mathbb{F}) \mid \mathrm{tr}(x) = 0\}$. The fact that $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ implies that $\mathcal{K}(\mathcal{E}, -) = \mathcal{E}$.

□

Remark 4.3.4. [KMRT, Proposition 2.6] implies that $(\mathcal{M}_2(\mathbb{F}), {}^t)$ is isomorphic to $(\mathcal{M}_2(\mathbb{F}), \widehat{\ })$ where $\widehat{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} = \begin{pmatrix} d & b \\ c & a \end{pmatrix}$ due to the fact that the dimension of the space of skew-symmetric elements is the same. We will denote $E_{i,j}$ to the matrix with 1 in the coordinate (i, j) and 0 in the rest.

We will start by classifying the structurable gradings in the case on the structurable algebras related to an hermitian form, of the form $S(\mathbb{F}, \mathrm{id}, \mathcal{W}, h)$.

Proposition 4.3.5. *If $(\mathcal{A}, -, \Gamma)$ is a graded-central-simple algebra with a structurable grading $\Gamma\colon \mathcal{A} = \mathcal{E} \oplus \mathcal{W}$ where $\mathcal{E} = \mathbb{F}$ and $- = \mathrm{id}$, then it has the uniqueness property.*

Proof. Assume that $\widetilde{\Gamma}\colon \mathcal{A} = \widetilde{\mathcal{E}} \oplus \widetilde{\mathcal{W}}$ is a structurable grading such that $(\mathcal{A}, -, \widetilde{\Gamma})$ is graded-central-simple. The fact that $(\mathcal{E}, -) \cong (\mathbb{F}, \mathrm{id})$ implies that $-$ is the identity involution on $\mathcal{A}$. Since any associative algebra with the identity involution is commutative, it has to be its own center. Since $(\mathcal{A}, -, \widetilde{\Gamma})$ is graded-central-simple, $(\widetilde{\mathcal{E}}, -)$ is central-simple, which implies that its center is the field. Thus, $\widetilde{\mathcal{E}} = \mathbb{F}1$. Now, $\{x \in \mathcal{A} \setminus \mathbb{F}1 \mid x^2 \in \mathbb{F}1\} = \mathcal{W}$ due to the fact that if $0 \neq x \in \mathcal{A} \setminus \mathbb{F}1$, there is $0 \neq w \in \mathcal{W}$ and $\lambda \in \mathbb{F}$ such that $x = \lambda 1 + w$, and $x^2 = \lambda^2 + w^2 + 2\lambda w$. So $x^2 \in \mathbb{F}1$ if and only if $\lambda = 0$. Finally, $\widetilde{\mathcal{W}} = \mathcal{W}$ due to the fact that $\widetilde{\mathcal{W}} \subseteq \{x \in \mathcal{A} \setminus \mathbb{F}1 \mid x^2 \in \mathbb{F}1\} = \mathcal{W}$ and they have the same dimension. □

Now, we will approach the case in which $(\mathcal{E} \otimes_{\mathbb{F}} \overline{\mathbb{F}}, -) \cong (\mathcal{M}_2(\overline{\mathbb{F}}), \widehat{\ })$.

Lemma 4.3.6. *Let $(\mathcal{A}, -, \Gamma)$ be a graded-central-simple algebra with involution and a structurable grading $\Gamma\colon \mathcal{A} = \mathcal{E} \oplus \mathcal{W}$ such that $(\mathcal{E} \otimes_{\mathbb{F}} \overline{\mathbb{F}}, -) \cong (\mathcal{M}_2(\overline{\mathbb{F}}), \widehat{\ })$. Then, $\dim_{\mathbb{F}} \mathcal{W}$ is even.*

Proof. Under this assumptions, $\mathcal{W} \otimes_{\mathbb{F}} \overline{\mathbb{F}}$ is a left unital associative module for $\mathcal{M}_2(\overline{\mathbb{F}})$. Thus, its dimension is even. □

Lemma 4.3.7. *Let $(\mathcal{A}, -, \Gamma)$ be a graded-central-simple algebra with involution and a structurable grading $\Gamma\colon \mathcal{A} = \mathcal{E} \oplus \mathcal{W}$ such that $(\mathcal{E} \otimes_{\mathbb{F}} \overline{\mathbb{F}}, -) \cong (\mathcal{M}_2(\overline{\mathbb{F}}), \widehat{\ })$. Then:*

(1) If $m \in \mathcal{M}(\mathcal{A}, -)$ *is such that* $0 \neq m^2 \in \mathbb{F}1$*, then either* $m \in \mathcal{E}$ *or* $m \in \mathcal{W}$*.*

(2) If $\widetilde{\Gamma}\colon \mathcal{A} = \widetilde{\mathcal{E}} \oplus \widetilde{\mathcal{W}}$ *is a structurable grading such that* $(\mathcal{A}, -, \widetilde{\Gamma})$ *is graded-central-simple, then, either* $\widetilde{\mathcal{E}} = \mathcal{E}$ *or* $\widetilde{\mathcal{E}} \cap \mathcal{M}(\mathcal{A}, -) \subseteq \mathcal{W}$*.*

(3) If $\widetilde{\Gamma}\colon \mathcal{A} = \widetilde{\mathcal{E}} \oplus \widetilde{\mathcal{W}}$ *is a structurable grading such that* $(\mathcal{A}, -, \widetilde{\Gamma})$ *is graded-central-simple, and* $\mathcal{E} \neq \widetilde{\mathcal{E}}$*, then* $\mathcal{E} \cap \widetilde{\mathcal{E}} = \mathcal{K}(\mathcal{A}, -)$*.*

Proof. Let $\mathcal{M} = \mathcal{M}(\mathcal{A}, -)$ and $\mathcal{K} = \mathcal{K}(\mathcal{A}, -)$. Without loss of generality, we assume that we work over an algebraically closed field. Let $m \in \mathcal{M}$ be such that $0 \neq m^2 \in \mathbb{F}1$. Using that Γ is a grading, we can take $e \in \mathcal{E}$ and $w \in \mathcal{W}$ such that $m = e + w$. We can identify $(\mathcal{E}, -)$ with $(\mathcal{M}_2(\mathbb{F}), \widehat{\ })$ as in Remark 4.3.4. Then, we get that $\mathcal{M} \cap \mathcal{E}$ is the subspace generated by $E_{1,2}$ and $E_{2,1}$. Moreover, we can see that $\mathcal{K}$ is the subspace generated by $s = E_{1,1} - E_{2,2}$ and 1. Hence, $E_{1,2}^2, E_{2,1}^2, E_{1,2}E_{2,1}, E_{2,1}E_{1,2} \in \mathcal{K}$, implying that

$$(\mathcal{E} \cap \mathcal{M})^2 \subseteq \mathcal{K}. \tag{4.3.1}$$

Since $ew \in \mathcal{W}$, $ew \in \mathcal{M}$. Since $w \in \mathcal{W} \subseteq \mathcal{M}$ and $m \in \mathcal{M}$, $e \in \mathcal{M}$. Therefore, $e, w, ew \in \mathcal{M}$. Thus, $ew = \overline{ew} = \bar{w}\bar{e} = we$, implying that $m^2 = e^2 + w^2 + 2ew$. Since $0 \neq m^2 \in \mathbb{F}1$, $e^2 \in \mathcal{K} \cap \mathcal{H}(\mathcal{A}, -)$ (due to (4.3.1)), $w^2 \in \mathcal{E}$ and $ew \in \mathcal{W}$, it follows that $ew = 0$. Letting $e^2 = \lambda 1$ with $\lambda \in \mathbb{F}$, we get that $0 = e(ew) = e^2 w = \lambda w$. Thus, either $\lambda = 0$ or $w = 0$. If $w = 0$, $m \in \mathcal{E}$. In case $w \neq 0$, since $e^2 = 0$, we have that $0 \neq m^2 = w^2 \in \mathbb{F}1$. Letting $w^2 = \beta 1$ with $\beta \in \mathbb{F}^\times$, we get that $0 = w(ew) = ew^2 = \beta e$. Thus, $e = 0$ and $m \in \mathcal{W}$, so (1) is proved.

In order to prove (2), we use that $(\widetilde{\mathcal{E}}, -) \cong (\mathcal{M}_2(\mathbb{F}), \widehat{\ })$. Using this identification and letting $w = E_{1,2} + E_{2,1}$, we have that $\widetilde{\mathcal{E}} = \mathcal{K} \oplus \mathcal{K}w$. Therefore, if $w \in \mathcal{E}$, $\widetilde{\mathcal{E}} = \mathcal{E}$. Otherwise, since $w^2 = 1$, using (1) we get $w \in \mathcal{W}$. hence $\widetilde{\mathcal{E}} \cap \mathcal{M} = \mathcal{K}w \subseteq \mathcal{W}$.

Due to the fact that $\mathcal{S}(\mathcal{A}, -) \subseteq \mathcal{E}$ and $\mathcal{S}(\mathcal{A}, -) \subseteq \widetilde{\mathcal{E}}$, it follows that $\mathcal{K} \subseteq \mathcal{E} \cap \widetilde{\mathcal{E}}$. The fact that associative algebras with involution are structurable algebras imply that equation (3.2.2) holds, i.e. $\mathcal{E} = \mathcal{K} \oplus (\mathcal{E} \cap \mathcal{M})$. Using (2) we get (3). □

Lemma 4.3.8. *Let* $(\mathcal{A}, -, \Gamma)$ *be a graded-central-simple with a structurable grading* $\Gamma\colon \mathcal{E} \oplus \mathcal{W}$ *such that* $- \neq \mathrm{id}$*. Then,* $\mathcal{W} = \{h \in \mathcal{H}(\mathcal{A}, -) \mid \frac{1}{2}(he - \bar{e}h) = 0\, \forall e \in \mathcal{E}\}$*.*

Proof. Without loss of generality we can assume that $\mathbb{F}$ is algebraically closed. The fact that $\mathcal{W} \subseteq \mathcal{H}(\mathcal{A}, -)$ and $\mathcal{E}\mathcal{W} \subseteq \mathcal{W}$ implies that $ew = \overline{ew} = w\bar{e}$.

Therefore, $\mathcal{W} \subseteq \{h \in \mathcal{H}(\mathcal{A}, -) \mid \frac{1}{2}(he - \bar{e}h) = 0 \,\forall e \in \mathcal{E}\}$. In case $\mathcal{E} = \mathcal{K}$, the result follows since $\mathcal{W} \subseteq \mathcal{M}(\mathcal{A}, -)$ and the dimensions are the same. Otherwise, $(\mathcal{E}, -) \cong (\mathcal{M}_2(\mathbb{F}), \widehat{\ })$. Let $A = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \in \{h \in \mathcal{H}(\mathcal{A}, -) \mid \frac{1}{2}(he - \bar{e}h) = 0 \,\forall e \in \mathcal{E}\} \cap \mathcal{E}$. Let $B = E_{1,1} - E_{2,2}$. Since $AB - \widehat{B}A = 0$, it follows that $a_{1,1} = a_{2,2} = 0$, since $E_{1,2}A - AE_{1,2} = 0$, it follows that $a_{2,1} = 0$, and since $E_{2,1}A - AE_{2,1} = 0$, it follows that $a_{1,2} = 0$. Therefore, $A = 0$ and that implies that $\{h \in \mathcal{H}(\mathcal{A}, -) \mid \frac{1}{2}(he - \bar{e}h) = 0 \,\forall e \in \mathcal{E}\} \subseteq \mathcal{W}$. □

Proposition 4.3.9. *Let $(\mathcal{A}, -, \Gamma)$ be a graded-central-simple algebra with involution and a structurable grading $\Gamma\colon \mathcal{A} = \mathcal{E} \oplus \mathcal{W}$ such that $(\mathcal{E}, -) \cong (\mathcal{M}_2(\mathbb{F}), \widehat{\ })$. If $\dim_{\mathbb{F}} \mathcal{W} \neq 4$, then $(\mathcal{A}, -)$ has the uniqueness property.*

Proof. Without loss of generality, we assume that $\mathbb{F}$ is algebraically closed. Suppose that $\widetilde{\Gamma}\colon \mathcal{A} = \widetilde{\mathcal{E}} \oplus \widetilde{\mathcal{W}}$ is a structurable grading such that $(\mathcal{A}, -, \Gamma)$ is graded-central-simple.

Proposition 4.3.2 and Proposition 4.3.5 imply that $(\mathcal{E}, -) \cong (\widetilde{\mathcal{E}}, -) \cong (\mathcal{M}_2(\mathbb{F}), \widehat{\ })$. Therefore, $\dim_{\mathbb{F}} \mathcal{E} = \dim_{\mathbb{F}} \widetilde{\mathcal{E}} = 4$ and $\dim_{\mathbb{F}}(\mathcal{W}) = \dim_{\mathbb{F}}(\widetilde{\mathcal{W}}) = k$.

Denote $\mathcal{E}_{1,2} = \mathcal{E} \cap \widetilde{\mathcal{E}}$ and $\mathcal{W}_{1,2} = \mathcal{W} \cap \widetilde{\mathcal{W}}$ and let $s \in \mathcal{S}(\mathcal{A}, -)$ be such that $s^2 = 1$. Such s exists since identifying $(\mathcal{E}, -) \cong (\mathcal{M}_2(\mathbb{F}), \widehat{\ })$ we can choose $s = E_{1,1} - E_{2,2}$.

We will prove by contradiction that $\mathcal{E} = \widetilde{\mathcal{E}}$. Thus, assume $\mathcal{E} \neq \widetilde{\mathcal{E}}$. Then, Lemma 4.3.7 (3) implies that $\mathcal{E}_{1,2} = \mathcal{K}(\mathcal{A}, -)$. Additionally, since $\mathcal{M}(\mathcal{A}, -) = (\mathcal{E} \cap \mathcal{M}(\mathcal{A}, -)) \oplus \mathcal{W}$, due to Lemma 4.3.7 (2) it follows that $\mathcal{M}(\mathcal{A}, -) = \mathcal{W} + \widetilde{\mathcal{W}}$. Since $k + 2 = \dim_{\mathbb{F}} \mathcal{M}(\mathcal{A}, -) = \dim_{\mathbb{F}} \mathcal{W} + \dim_{\mathbb{F}} \widetilde{\mathcal{W}} - \dim_{\mathbb{F}} \mathcal{W}_{1,2} = 2k - \dim_{\mathbb{F}} \mathcal{W}_{1,2}$, it follows, that $\dim_{\mathbb{F}} \mathcal{W}_{1,2} = k - 2$. We divide the proof into two cases:

case 1: $k \geq 6$.

Since we can identify $(\mathcal{E}, -)$ with $(\mathcal{M}_2(\mathbb{F}), \widehat{\ })$, we can take $x = E_{1,2} + E_{2,1}$. In this situation, $x^2 = 1$. For that reason, the map $\mathcal{W} \to \mathcal{W}$ given by $w \mapsto xw$ is an isomorphism. Hence, $\dim_{\mathbb{F}} \mathcal{W}_{1,2} = \dim_{\mathbb{F}} x\mathcal{W}_{1,2} = k - 2$. Since $x \in (\mathcal{E} \cap \mathcal{M}(\mathcal{A}, -)) \subseteq \widetilde{\mathcal{W}}$, $x\mathcal{W}_{1,2} \subseteq (\mathcal{E}\mathcal{W}) \cap (\widetilde{\mathcal{W}}\widetilde{\mathcal{W}}) = \mathcal{W} \cap \widetilde{\mathcal{E}}$. The fact that $\mathcal{W} \cap \widetilde{\mathcal{E}} = \widetilde{\mathcal{E}} \cap \mathcal{M}(\mathcal{A}, -)$, which has dimension 2, is a contradiction with $k > 4$.

case 2: $k = 2$.

In this case, Lemma 4.3.7 and the fact that $\dim_{\mathbb{F}}(\mathcal{W} \cap \widetilde{\mathcal{E}}) = 2$, we get that $\mathcal{W} = \widetilde{\mathcal{E}} \cap \mathcal{M}(\mathcal{A}, -)$. We identify $(\widetilde{\mathcal{E}}, -)$ with $(\mathcal{M}_2(\mathbb{F}), \widehat{\ })$ and take $x = E_{1,2} + E_{2,1} \in \widetilde{\mathcal{E}} \cap \mathcal{M}(\mathcal{A}, -)$. Then, for any $y \in \mathcal{E}$ we have that

$$h(yx, x) = \bar{y}h(x, x) = \bar{y}x^2 = \bar{y}1 = \bar{y}.$$

Therefore, the map $\mathcal{E} \to \mathcal{W}$ given by $y \mapsto yx$ is an injective homomorphism of vector spaces. Thus, $\dim_{\mathbb{F}} \mathcal{W} \geq \dim_{\mathbb{F}} \mathcal{E} = 4$, which is a contradiction. □

We are left with the case in which $(\mathcal{E} \otimes_{\mathbb{F}} \overline{\mathbb{F}}, -) \cong (\mathcal{M}_2(\overline{\mathbb{F}}), \widehat{\ })$ and $\mathcal{W}$ has dimension 4.

4.3.2 Split quartic Cayley algebra

Let $(\mathcal{A}, -, \Gamma)$ be a graded-central-simple algebra with involution and a structurable grading $\Gamma \colon \mathcal{A} = \mathcal{E} \oplus \mathcal{W}$. We have shown that $(\mathcal{A}, -)$ has the uniqueness property unless we have $(\mathcal{E} \otimes_{\mathbb{F}} \overline{\mathbb{F}}, -) \cong (\mathcal{M}_2(\overline{\mathbb{F}}), \widehat{\ })$ and $\mathcal{W}$ has dimension 4. In this case we will solve the problem over an algebraically closed field and we will give some examples of what we can expect over arbitrary fields. Thus, unless otherwise stated, our algebras will be over an algebraically closed field $\mathbb{F}$. In this section, we will denote $\mathcal{K} = \mathcal{K}(\mathcal{A}, -)$ and $\mathcal{M} = \mathcal{M}(\mathcal{A}, -)$.

Remark 4.3.10. The algebras on Example 4.2.22 and on Example 4.2.23 with $(\mathcal{D}, \widehat{\ }) \cong (\mathcal{M}_2(\overline{\mathbb{F}}), {}^t)$ satisfying that $(\mathcal{E} \otimes_{\mathbb{F}} \overline{\mathbb{F}}, -) \cong (\mathcal{M}_2(\overline{\mathbb{F}}), \widehat{\ })$ and $\mathcal{W}$ has dimension 4. Thus, they fall into the class of algebras we are going to study in this section.

Lemma 4.3.11. *Let $(\mathcal{A}, -, \Gamma)$ be a graded-central-simple algebra over an algebraically closed field $\mathbb{F}$ with involution and a structurable grading $\Gamma \colon \mathcal{A} = \mathcal{E} \oplus \mathcal{W}$ such that $(\mathcal{E}, -) \cong (\mathcal{M}_2(\mathbb{F}), \widehat{\ })$ and $\mathcal{W}$ has dimension* 4*. Then, there is an element $w \in \mathcal{W}$ such that $w^2 = 1$.*

Proof. Denote $\mathcal{W} = \mathbf{W}(\mathcal{A}, -, \Gamma)$. Then, Remark 4.2.27 shows that $(\mathcal{E}, -) \cong (L(\mathcal{W}), -)$. Therefore, we just need to prove that there is an element $w \in \mathcal{W}$ such that $\{w, w, \cdot\} = \mathrm{id}$. Using theorem 4.1.4, this is equivalent to proving (using the same notation as in the theorem) the existence of an element $w \in \mathcal{A}(\mathcal{W})_{1,2}$ such that $w\overline{w} = e_1$, where e_1 is the idempotent which gives the Peirce decomposition as in theorem 4.1.4 .

Since we work over an algebraically closed field, there is a vector space V such that either $\mathcal{A}(\mathcal{W}) \cong \mathrm{End}_{\mathbb{F}}(V)$ or $\mathcal{A}(\mathcal{W}) \cong \mathrm{End}_{\mathbb{F}}(V) \oplus \mathrm{End}_{\mathbb{F}}(V)^{op}$. Since $\mathcal{E}$ is simple and it is isomorphic to $L(\mathcal{W})$, which is isomorphic to $\mathcal{A}(\mathcal{W})_{1,1}$, then $\mathcal{A}(\mathcal{W})$ is isomorphic to $\mathrm{End}_{\mathbb{F}}(V)$. Moreover, the restriction of the involution to $\mathcal{A}(\mathcal{W})_{1,1}$ is orthogonal, which implies that the involution of $\mathcal{A}(\mathcal{W})$ is orthogonal. Thus, if $-$ is the involution of $\mathcal{A}(\mathcal{W})$ (after the identification with $\mathrm{End}_{\mathbb{F}}(V)$), there is a symmetric bilinear form $b \colon V \times V \to \mathbb{F}$ such that $b(\overline{r}v, w) = b(v, rw)$ for all $r \in \mathrm{End}_{\mathbb{F}}(V)$ and $v, w \in V$. We identify the idempotent e_1 with the corresponding endomorphism. We denote $V^1 = e_1 V$ and $V^0 = \{v \in V \mid e_1 v = 0\} = \ker L_{e_1}$. Since for any vector $v \in V$, $v = (1 - e_1)v + e_1 v$, the fact that $e_1(1 - e_1) = 0$, implies that $V = V^0 + V^1$. The fact that e_1 is an idempotent implies that $V = V^0 \oplus V^1$. Since $\overline{e_1} = e_1$, we have that $b(e_1 v, w) = b(v, e_1 w)$. Therefore, $b(V^1, V^0) = b(V^0, V^1) = 0$.

Therefore, we can take an orthonormal basis $\{v_1^1, ..., v_n^1\}$ of V^1 and extend it with an orthonormal basis $\{v_1^0, ..., v_m^0\}$ of V^1 obtaining an orthonormal basis of V. Using this basis, we deduce that $(\mathcal{A}(\mathcal{W}), -) \cong (\mathcal{M}_{n+m}(\mathbb{F}), {}^t)$ and that with this identification $e_1 = E_{1,1} + ... + E_{n,n}$. This implies that with this identification, $\mathcal{A}_{1,1} = \mathcal{M}_n(\mathbb{F})$ and $\mathcal{A}_{1,2} = \mathcal{M}_{n,m}(\mathbb{F})$. Since $\dim_{\mathbb{F}}(\mathcal{E}) = 4$, $n = 2$ and since $\dim_{\mathbb{F}} \mathcal{W} = 4$, $m = 2$. Taking $w = E_{1,3} + E_{2,4}$, we get that $w\overline{w} = e_1$. □

Choose an element $w \in \mathcal{W}$ such that $w^2 = 1$. Then $\mathcal{E} = \mathcal{E}h(w, w) = h(\mathcal{E}w, w)$. Thus, $\dim_{\mathbb{F}}(\mathcal{E}w) = 4$, which implies that $\mathcal{W} = \mathcal{E}w$. identifying $\mathcal{E}$ with $(\mathcal{M}_2(\mathbb{F}), \widehat{\ })$, we can take $x_1 = E_{1,2} + E_{2,1}$, $x_2 = w$ and $x_3 = x_1x_2$. Since $\mathcal{E} = \mathcal{K} \oplus \mathcal{K}x_1$, we get the following decomposition:

$$\mathcal{A} = \mathcal{K} \oplus \mathcal{K}x_1 \oplus \mathcal{K}x_2 \oplus \mathcal{K}x_3 \tag{4.3.2}$$

and

$$\mathcal{M} = \mathcal{K}x_1 \oplus \mathcal{K}x_2 \oplus \mathcal{K}x_3 \tag{4.3.3}$$

Lemma 4.3.12. *With the notation of* (4.3.2), *for every* $e, f \in \mathcal{K}$ *we have* $(ex_i)(fx_j) = (\overline{fe})x_k$, $(ex_i)(fx_i) = \overline{f}e$, $e(fx_i) = (fe)x_i$ *and* $(fx_i)e = (f\overline{e})x_i$ *for* $\{i, j, k\} = \{1, 2, 3\}$.

Proof. Assume first that $e, f = 1$. Then, the fact that $\overline{x_1} = x_1$, implies that $x_2x_3 = x_2(x_1x_2) = h(x_1x_2, x_2) = x_1h(x_2, x_2) = x_1$. Additionally, $x_1x_3 = x_1(x_1x_2) = x_1 \circ (x_1 \circ x_2) = (x_1^2) \circ x_2 = x_2$. If we have $\{i, j, k\} = \{1, 2, 3\}$ such that, $x_ix_j = x_k$, applying the involution, we get $x_jx_i = x_k$. Finally, $x_3^2 = h(x_1x_2, x_1x_2) = x_1h(x_2, x_2)x_1 = x_1x_1 = 1$.

Now, for arbitrary $e, f \in \mathcal{K}$, we have that $(ex_i)(fx_j) = \overline{f}((ex_i)x_j)$ due to Lemma 3.2.2 and $(ex_i)x_j = \overline{\overline{x_j}(\overline{ex_i})} = \overline{x_j(ex_i)} = \overline{\overline{e}(x_jx_i)} = (x_ix_j)e$. Thus, $(ex_i)(fx_j) = \overline{f}((x_ix_j)e)$. In case $\{i, j, k\} = \{1, 2, 3\}$, we get $(ex_i)(fx_j) = \overline{f}(x_ke) = \overline{f}(\overline{e}x_k) = (\overline{fe})x_k$. And finally, $(ex_i)(fx_i) = \overline{f}(x_i^2e) = \overline{f}e$.

$e(fx_i) = (fe)x_i$ follows from Lemma 3.2.2 and using that we have:

$$(fx_i)e = \overline{\overline{(fx_i)e}} = \overline{\overline{e}(x_i\overline{f})} = \overline{\overline{e}(fx_i)} = \overline{(f\overline{e})x_i} = x_i\overline{(f\overline{e})} = (f\overline{e})x_i$$

□

Corollary 4.3.13. *With the previous notation, given* $\sigma \in \mathrm{S}_3$ *(the symmetric group of* 3 *elements), and an unitary commutative associative algebra* R*, the map* $\varphi_R^\sigma \colon \mathcal{A} \otimes_{\mathbb{F}} R \to \mathcal{A} \otimes_{\mathbb{F}} R$ *given by* $\varphi_R^\sigma(e_0 + e_1x_1 + e_2x_2 + e_3x_3) = e_0 + e_1x_{\sigma(1)} + e_2x_{\sigma(2)} + e_3x_{\sigma(3)}$ *for every* $e_0, e_1, e_2, e_3 \in \mathcal{K} \otimes_{\mathbb{F}} R$ *is in* $\mathbf{Aut}(\mathcal{A}, -)(R)$.

Corollary 4.3.14. *With the previous notation, let* $\{i, j, k\} = \{1, 2, 3\}$*. Denote* $\mathcal{E}_i = \mathcal{K} \oplus \mathcal{K}x_i$ *and* $\mathcal{W}_i = \mathcal{K}x_j \oplus \mathcal{K}x_k$*. Then, the grading* $\Gamma_i \colon \mathcal{A} = \mathcal{E}_i \oplus \mathcal{W}_i$ *is a structurable grading and* $(\mathcal{A}, -, \Gamma_i)$ *is graded-central-simple. Moreover, those are the only structurable gradings satisfying this condition.*

Proof. Since Γ_1 is the structurable grading used to obtain (4.3.2), for any $i \in \{1, 2, 3\}$ taking $\sigma \in \mathrm{S}_3$ such that $\sigma(1) = i$, $\varphi_\sigma(\mathcal{E}_1) = \mathcal{E}_i$ and $\varphi_\sigma(W_1) = W_i$. Thus applying φ_σ we get that Γ_i is a structurable grading.

Due to Lemma 4.3.8, we only need to prove that for a structurable grading $\Gamma \colon \mathcal{A} = \mathcal{E} \oplus W$, then $\mathcal{E} = \mathcal{E}_i$ for some $i \in \{1, 2, 3\}$. We will prove it by contradiction. Hence, assume that $\mathcal{E} \neq \mathcal{E}_i$ for any $i \in \{1, 2, 3\}$. Using Lemma 4.3.7, we have that $\mathcal{E} \cap \mathcal{M} \subseteq \mathcal{K}x_j \oplus \mathcal{K}x_k$ for all $\{i, j, k\} = \{1, 2, 3\}$. Therefore, this would imply $\mathcal{E} = \mathcal{K}$ and $\mathcal{W} = \mathcal{M}$. However, this is a contradiction with Proposition 4.3.2 and the fact that there are three different structurable gradings. □

Remark 4.3.15. Notice that since $(\mathcal{M}_2(\mathbb{F}), \widehat{\ })$ is a quaternion algebra, if $\tilde{x}_1$ is an hermitian element in $\mathcal{E} \cap \mathcal{M}$ such that $\tilde{x}_1^2 = 1$, then there is an automorphism sending $E_{1,1} - E_{2,2}$ to itself and $x_1 \mapsto \tilde{x}_1$. Moreover, this automorphism preserves the involution. Now, due to [Rig22, (8.2.2)], this automorphism extends to an automorphism of $\mathcal{A}$ preserving $\mathcal{E}$ and $\mathcal{W}$. Thus, up to isomorphism, we could have taken x_1 to be any element in $\mathcal{E} \cap \mathcal{M}$ such that $x_1^2 = 1$, x_2 to be any element in $\mathcal{W}$ such that $x_2^2 = 1$ and $x_3 = x_1 x_2$.

Example 4.3.16. Recall the structurable algebras from Example 4.2.22. Due to Lemma 4.3.12 they are isomorphic. We are going to use Remark 4.3.15 in order to give choices of x_1, x_2 and x_3 in each of the models given in the example depending on the choice of Φ_0 and Φ_1. It will be enough to take x_1 to be an element of $\mathcal{E} \cap \mathcal{M}$ whose square is 1, x_2 an element of W whose square is 1 and $x_3 = x_1 x_2$. Since in the example the product is defined for elements of $\mathcal{M}_2(\mathbb{F}) \times \mathcal{M}_2(\mathbb{F})$, to calculate that this works is straightforward.

(1) Case $\Phi_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \Phi_1$. In this case, we can take:

- ⋆ $x_1 = \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, 0 \right)$
- ⋆ $x_2 = \left(0, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right)$
- ⋆ $x_3 = \left(0, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right)$

(2) Case $\Phi_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\Phi_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

- $\star$ $x_1 = \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, 0\right)$
- $\star$ $x_2 = \left(0, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\right)$
- $\star$ $x_3 = \left(0, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}\right)$

(3) Case $\Phi_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\Phi_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

- $\star$ $x_1 = \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, 0\right)$
- $\star$ $x_2 = \left(0, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}\right)$
- $\star$ $x_3 = \left(0, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}\right)$

(4) Case $\Phi_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\Phi_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

- $\star$ $x_1 = \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, 0\right)$
- $\star$ $x_2 = \left(0, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$
- $\star$ $x_3 = \left(0, \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$

Example 4.3.17. Recall the structurable algebras from Example 4.2.23 with $(\mathcal{D}, \widehat{\ }) = (\mathcal{M}_2(\mathbb{F}), {}^t)$. In this case, in the same vein as in Example 4.3.16, we will find different choices of x_1, x_2 and x_3 for two different choices of d_1 and d_2.

(1) Case $d_0 = z_0 = d_1$ or $d_0 = z_2 = d_1$:

- $\star$ $x_1 = (z_2, 0)$,
- $\star$ $x_2 = (0, z_0)$,

- $\star$ $x_3 = (0, z_2)$.

(2) Case $d_0 = z_0$, $d_1 = z_i$ with $i \neq 0$: take $\{i, j, k\} = \{1, 2, 3\}$ and ζ such that $z_j z_i z_k = \zeta^2 z_0$, then:

- $\star$ $x_1 = (z_2, 0)$,
- $\star$ $x_2 = (0, \zeta(z_j + z_k))$,
- $\star$ $x_3 = (0, \zeta(z_{j+2} + z_{k+2})$.

This algebra, which we have shown that doesn't have the uniqueness property, is a known algebra called the **split quartic Cayley algebra**, which appears for example in [Ali91], in order to give constructions of Lie algebras of type D_4. We will show how it is constructed. In order to do so, we need to recall a modified Cayley-Dickson process, which is introduced in [AF84], starting with the algebra $\mathcal{B} = \mathbb{F} \oplus \mathbb{F} \oplus \mathbb{F} \oplus \mathbb{F}$. Take $\mu \in \mathbb{F}^\times$ and denote by t the trace of $\mathcal{B}$, i.e., $t(a, b, c, d) = \frac{1}{4}(a + b + c + d)$, and by $1_\mathcal{B}$ the unit of $\mathcal{B}$. Define $b^\theta = -b + \frac{1}{2}t(b)1_\mathcal{B}$ for every $b \in \mathcal{B}$. Let $\mathcal{A} = \mathcal{B} \oplus \mathcal{B}$. We can define a product $\cdot$ and an involution $-$ of $\mathcal{A}$ by:

$$(b_1, b_2) \cdot (b_3, b_4) = (b_1 b_3 + \mu(b_2 b_4^\theta)^\theta, b_1^\theta b_4 + (b_2^\theta b_3^\theta)^\theta), \text{ and}$$
$$\overline{(b_1, b_2)} = (b_1, -b_2^\theta).$$

We can denote $s = (0, 1_\mathcal{B})$ and we denote any element $(b_1, b_2) \in \mathcal{B} \oplus \mathcal{B}$ as $b_1 + sb_2$. This algebra with involution is denoted by $\mathfrak{CD}(\mathcal{B}, \mu)$. Notice that the morphism $b_1 + sb_2 \mapsto b_1 + \sqrt{\mu} sb_2$ is an isomorphism from $\mathfrak{CD}(\mathcal{B}, \mu)$ to $\mathfrak{CD}(\mathcal{B}, 1)$. This is what we call the **split quartic Cayley algebra**. If we denote $\mathcal{K} = \mathbb{F}1 \oplus \mathbb{F}s$ and we denote $x_1 = (1, 1, -1, -1)$, $x_2 = (1, -1, 1, -1)$ and $x_3 = (1, -1, -1, 1)$, we obtain an isomorphism from $\mathfrak{CD}(\mathcal{B}, 1)$ to $(\mathcal{A}, -)$.

We will give now some examples of what can happen over arbitrary fields. In this case, any twisted form of an algebra with the uniqueness property over an algebraically closed field will have the uniqueness property, since any structurable grading, after extension of scalars is a structurable grading. Thus, we will only be interested in twisted forms of the split quartic Cayley algebras (or just quartic Cayley algebras as called in [Ali90]). The following examples shed some light on the cases that might happen.

Example 4.3.18. Consider the $\mathbb{R}$-algebra with involution $(\mathcal{E}, -) = (\mathcal{M}_2(\mathbb{R}), {}^t)$ and we denote $W = \mathcal{M}_2(\mathbb{R})$. We define the action $\circ \colon \mathcal{E} \times W \to W$ by $X \circ Y = XY$, i.e., we endow $\mathcal{M}_2(\mathbb{R})$ with a structure of left-unitary associative module. Consider now the hermitian form $h \colon W \times W \to \mathcal{E}$ by

$$h(X, Y) = X \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} Y^t.$$

This is a hermitian form due to the fact that

$$h(X \circ Y, Z) = (XY)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} Z^t = X\left(Y\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} Z^t\right)$$

and

$$\overline{h(X,Y)} = \left(X\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} Y^t\right)^t = Y\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} X^t = h(Y,X).$$

Let $(\mathcal{A}, -) = S(\mathcal{E}, -, W, h)$. This is a quartic Cayley algebra. Denote $\mathcal{K} = \mathcal{K}(\mathcal{A}, -)$ and $\mathcal{M} = \mathcal{M}(\mathcal{A}, -)$.

We will prove that $(\mathcal{A}, -)$ has the uniqueness property by contradiction. Assume that it doesn't have the uniqueness property and let $\widetilde{\Gamma}\colon \mathcal{A} = \widetilde{\mathcal{E}} \oplus \widetilde{W}$ be a structurable grading such that $(\mathcal{A}, -, \widetilde{\Gamma})$ is a graded-central-simple algebra. Due to Proposition 4.3.3 and the fact that the involution is not the identity, $(\widetilde{\mathcal{E}} \otimes_{\mathbb{R}} \mathbb{C}, -) \cong (\mathcal{M}_2(\mathbb{C}), \widehat{\ })$. Due to (3.2.2), it follows that $\widetilde{\mathcal{E}} = \mathcal{K} \oplus (\mathcal{M} \cap \widetilde{\mathcal{E}})$. Due to Lemmma (4.3.7), it follows that $(\mathcal{M} \cap \widetilde{\mathcal{E}}) \subseteq \mathcal{W}$. Thus, $\mathcal{K} \oplus (\mathcal{M} \cap \widetilde{\mathcal{E}})$ is a $\mathbb{Z}_2$-grading of $\widetilde{\mathcal{E}}$. Since $\widetilde{\mathcal{E}}$ is a quaternionic algebra, this implies that there is an element $m \in \mathcal{M}(\mathcal{A}, -)$ such that $m^2 \in \mathbb{R}^\times 1$, and either $\mathcal{E} = \widetilde{\mathcal{E}}$, which due to Lemma 4.3.8 implies that $\Gamma = \widetilde{\Gamma}$, or $m \in W$, which would imply that there is an element $X \in \mathcal{M}_2(\mathbb{R})$ such that

$$X\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} X^t = \lambda \mathrm{Id}$$

for some $\lambda \in \mathbb{R}^\times$ which is a contradiction due to the fact that $\lambda\mathrm{Id}$ and $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ have different signature.

Example 4.3.19. For this example, we denote by σ the nontrivial automorphism of $\mathbb{Q}[i]$ and consider the $\mathbb{Q}$-algebra $\mathcal{E} = \mathfrak{CD}(\mathbb{Q}[i], \sigma, 2) = \mathbb{Q}[i] \oplus w\mathbb{Q}[i]$ with $w^2 = 2$ and with the involution $-$ given by

$$\overline{\lambda_1 + w\lambda_2} = \sigma(\lambda_1) + w\lambda_2$$

for every $\lambda_1, \lambda_2 \in \mathbb{Q}[i]$, i.e, the composition of the standard involution and the automorphism given by $\lambda_1 + w\lambda_2 \mapsto \lambda_1 - w\lambda_2$. We consider $W = \mathcal{E}$ and the action $\circ\colon \mathcal{E} \times W \to W$ given by left multiplication, which endows W with a structure of unitary associative left module. Finally, we define the hermitian form $h(v, w) = v3\overline{w}$. We denote $(\mathcal{A}, -) = S(\mathcal{E}, -, W, h)$. Hence $\mathcal{A} = \mathcal{E} \oplus W$. We denote $x_1 = (w, 0), x_2 = (0, 1), x_3 = (0, w)$. Here $\mathcal{K} = \mathcal{K}(\mathcal{A}, -) = \mathbb{Q}1 \oplus \mathbb{Q}(i, 0)$. We can identify it with $\mathbb{Q}[i]$. With this notation we have that:

$$\mathcal{A} = \mathcal{K}1 \oplus \mathcal{K}x_1 \oplus \mathcal{K}x_2 \oplus \mathcal{K}x_3.$$

We have three structurable gradings denoted $\Gamma_i\colon \mathcal{A} = \mathcal{E}_i \oplus W_i$ for $i = 1, 2, 3$ where if $\{i, j, k\} = \{1, 2, 3\}$, $\mathcal{E}_i = \mathcal{K}1 \oplus \mathcal{K}x_i$ and $W = \mathcal{K}x_j \oplus \mathcal{K}x_k$ and they satisfy that $(\mathcal{A}, -\ , \Gamma_i)$ is graded-central-simple. Clearly, after extending scalars these are isomorphic to the split Cayley algebra and the three structurable gradings. However, in this case, the gradings are not isomorphic since $\mathcal{E}_1 \cong \mathfrak{CD}(\mathbb{Q}[i], \sigma, 2), \mathcal{E}_2 \cong \mathfrak{CD}(\mathbb{Q}[i], \sigma, 3)$ and $\mathcal{E}_3 \cong \mathfrak{CD}(\mathbb{Q}[i], \sigma, 6)$.

Chapter 5

Automorphisms and gradings on structurable algebras

In this chapter, we are going to classify the gradings on central simple structurable algebras related to a hermitian form by an abelian group up to isomorphism. In order to find the classification, we are going to rely in the equivalences of categories and the uniqueness property shown in the previous Chapter in order to find an isomorphism the automorphism group schemes of the structurable algebras with the uniqueness property and the automorphism group schemes of some associative algebras with some 3-grading and an involution.

In Section 5.1 we show that there is an isomorphism betweem the automorphism group scheme of the algebras with involution with only one structurable grading and the corresponding 3-graded associative algebra with an involution (see Theorem 5.1.7). Moreover, we give a description of the automorphism group scheme of the split quartic Cayley algebra in term of known groups (see Theorem 5.1.13). Finally, in Section 5.2 we classify the gradings on the associative setting, and use this and the fact that we have an isomorphism group scheme between the automorphisms of certain graded associative algebras with involution and certain algebras with involution and a structurabl gradings in order to classify up to isomorphism the gradings on the central simple structurable algebras related to an hermitian form. There are two main complications with this approach: first, not all the gradings on structurable algebras are gradings on structurable algebras with a structurable grading, and then, some isomorphic gradings of the structurable algebra, are not isomorphic as gradings of the structurable algebra with the structurable grading. For that reason, before giving the complete classification, we classify these gradings up to isomorphism, and then use the results on the associative algebras in order to give a complete classification which is divided into the Theorems 5.2.63 and 5.2.65.

5.1 Automorphism group schemes

Our purpose in this section is to give a description of the automorphism group scheme of the structurable algebra. In order to do so, we will use the concepts we have been developing in the previous sections. Our main results in this section are Theorem 5.1.7 and Theorem 5.1.13.

5.1.1 Associative triple systems

Recall that we have a functor $\mathbf{A}$, which is an equivalence of categories from *SAT*2 to 3*SGrAlgInv*. Hence for a simple AT2 $(\mathcal{W}, \{\cdots\})$, $\mathrm{Aut}(\mathcal{W}, \{\cdots\}) \cong \mathrm{Aut}(\mathcal{A}(\mathcal{W}), -, \Delta(\mathcal{W}))$. We are going to show that this isomorphism can be extended to affine group schemes in Proposition 5.1.1 and use it in Corollary 5.1.3 to prove that $\mathbf{Aut}(\mathcal{W}, \{\cdots\})$ is smooth.

Proposition 5.1.1. *Let $(\mathcal{W}, \{\cdots\})$ be a simple AT2. The morphism of affine group schemes:*

$$\theta\colon \mathbf{Aut}(\mathcal{A}(\mathcal{W}), -, \Delta(\mathcal{W})) \to \mathbf{Aut}(\mathcal{W}, \{\cdots\})$$

given by $\theta_R(\psi) = \psi_{|\mathcal{W}\otimes_{\mathbb{F}} R}$ is an isomorphism.

Proof. Due to Theorem 4.1.12 $(\mathcal{A}(\mathcal{W}), -)$ is simple. Moreover, due to Lemma 4.1.13, $\mathcal{A}(\mathcal{W})$ is generated as an algebra with involution by $\mathcal{W}$. Thus, $\mathcal{A}(\mathcal{W})\otimes_{\mathbb{F}} R$ is generated as an algebra with involution by $\mathcal{W} \otimes_{\mathbb{F}} R$.

Given $\psi \in \mathbf{Aut}(\mathcal{W}, \{\cdots\})(R)$, $\mathbf{A}(\psi)$ is well defined as shown in Remark 4.1.19. $\mathbf{A}(\psi)$ is an automorphism due the fact that its inverse is $\mathbf{A}(\psi^{-1})$. We define $\theta^{-1}\colon \mathbf{Aut}(\mathcal{W}, \{\cdots\}) \to \mathbf{Aut}(\mathcal{A}(\mathcal{W}), -, \Delta(\mathcal{W}))$ as $\theta_R^{-1}(\psi) = \mathbf{A}(\psi)$. By definition, we can see that $\theta_R \circ \theta_R^{-1} = \mathrm{id}$. On the other hand we have that $\theta_R \circ (\theta_R^{-1} \circ \theta_R) = (\theta_R \circ \theta_R^{-1}) \circ \theta_R = \theta_R$. Since θ_R is injective, $\theta_R^{-1} \circ \theta_R = \mathrm{id}$. Thus, θ is an isomorphism with inverse θ^{-1}. □

In order to finish this section, we are going to prove that $\mathbf{Aut}(\mathcal{W}, \{\cdots\})$ is smooth for any central simple AT2 $(\mathcal{W}, \{\cdots\})$. Using Proposition 5.1.1 and Proposition 4.1.20 we prove that for any object $(\mathcal{A}, -, \Delta)$ of 3*SGrAlgInv*, $\mathbf{Aut}(\mathcal{A}, -, \Delta)$ is smooth. In order to do so, let $(\mathcal{A}, -)$ be a central simple associative algebra. We denote the center of $\mathcal{A}$ by $\mathcal{Z}$. We define the affine group schemes $\mathbf{Aut}_{\mathcal{Z}}(\mathcal{A}), \mathbf{Aut}_{\mathcal{Z}}(\mathcal{A}, -, \Delta), \mathbf{Sim}(\mathcal{A}, -), \mathbf{Sim}_0(\mathcal{A}, -)$ and $\mathbf{Iso}(\mathcal{A}, -)$ as the group schemes whose R-points are:

$$\begin{aligned}
&\mathbf{Aut}_{\mathcal{Z}}(\mathcal{A})(R) = \{\psi \in \mathbf{Aut}(\mathcal{A})(R) \mid \psi_{|\mathcal{Z}} = \mathrm{id}\},\\
&\mathbf{Aut}_{\mathcal{Z}}(\mathcal{A}, -)(R) = \mathbf{Aut}_{\mathcal{Z}}(\mathcal{A})(R) \cap \mathbf{Aut}(\mathcal{A}, -)(R),\\
&\mathbf{Aut}_{\mathcal{Z}}(\mathcal{A}, -, \Delta)(R) = \mathbf{Aut}_{\mathcal{Z}}(\mathcal{A})(R) \cap \mathbf{Aut}(\mathcal{A}, -, \Delta)(R),\\
&\mathbf{Sim}(\mathcal{A}, -)(R) = \{u \in (\mathcal{A} \otimes_{\mathbb{F}} R)^{\times} \mid u\overline{u} \in (\mathcal{Z} \otimes_{\mathbb{F}} R)^{\times}\},\\
&\mathbf{Sim}_0(\mathcal{A}, -)(R) = \{u \in (\mathcal{A}_0 \otimes_{\mathbb{F}} R)^{\times} \mid u\overline{u} \in (\mathcal{Z} \otimes_{\mathbb{F}} R)^{\times}\}, \text{ and}\\
&\mathbf{Iso}(\mathcal{A}, -)(R) = \{u \in (\mathcal{A} \otimes_{\mathbb{F}} R)^{\times} \mid u\overline{u} = 1\}.
\end{aligned}$$

We can now prove the following proposition.

Proposition 5.1.2. *Let* $(\mathcal{A}, -, \Delta)$ *be an object of* 3*CSGrAlgInv. Then* $\mathbf{Aut}(\mathcal{A}, -, \Delta)$ *is smooth.*

Proof. Denote by e_1 and e_2 two idempotents such that $e_2 = 1 - e_1$, $\overline{e_i} = e_i$ for $i \in \{1, 2\}$, $\mathcal{A}_{-1} = \mathcal{A}_{1,2}$, $\mathcal{A}_0 = \mathcal{A}_{1,1} \oplus \mathcal{A}_{2,2}$ and $\mathcal{A}_1 = \mathcal{A}_{2,1}$, where $\mathcal{A}_{i,j} = e_i \mathcal{A} e_j$ for $i, j \in \{1, 2\}$. Such e_1 exists since due to Proposition 4.1.20, there is a simple AT2 $\mathcal{W}$ such that $(\mathcal{A}, -, \Delta) \cong (\mathcal{A}(\mathcal{W}), -, \Delta(\mathcal{W}))$ and we can take as e_1 the unit in $L(\mathcal{W})$.

We will also denote as $-$, the linear extension of $-$ to $\mathcal{A} \otimes_{\mathbb{F}} R$ and its restriction to other subspaces. Denote by $\mathcal{Z}$ the center of $\mathcal{A}$. $\mathcal{Z}$ is a graded subspace of $\mathcal{A}$ [AC20, Lemma 3.8]. Call the induced grading $\Delta_{\mathcal{Z}}$. Since $\Delta_{\mathcal{Z}}$ is a 3-grading on a separable algebra of dimension 1 or 2, it is trivial. Thus, $\mathbf{Aut}(\mathcal{Z}, -, \Delta_{\mathcal{Z}}) = \mathbf{Aut}(\mathcal{Z}, -)$. Due to [KMRT, 22.10] we have the following exact sequence:

$$\mathbf{1} \to \mathbf{Aut}_{\mathcal{Z}}(\mathcal{A}, -, \Delta) \to \mathbf{Aut}(\mathcal{A}, -, \Delta) \to \mathbf{Aut}(\mathcal{Z}, -) \to \mathbf{1}$$

Where the first morphism is given by the inclusion and the second one is given by the restriction to $\mathcal{Z}$.

Thus, since the characteristic of $\mathbb{F}$ is different from 2, and $(\mathcal{Z}, -)$ is a central simple algebra with involution, $\mathbf{Aut}(\mathcal{Z}, -)$ is smooth (see [KMRT, 23]). Using [KMRT, 22.12] and the previous exact sequence, we just need to prove that $\mathbf{Aut}_{\mathcal{Z}}(\mathcal{A}, -, \Delta)$ is smooth.

For an element $u \in (\mathcal{A} \otimes_{\mathbb{F}} R)^{\times}$, we define

$$\mathrm{Int}_R(u) \in \mathbf{Aut}_{\mathcal{Z}}(R)$$

as $\mathrm{Int}_R(u)(a) = uau^{-1}$. As it is shown in [KMRT, 23], the morphism $\mathrm{Int}\colon \mathbf{GL}_1(\mathcal{A}) \to \mathbf{Aut}_{\mathcal{Z}}(\mathcal{A})$ is surjective. Moreover, the inverse image of $\mathbf{Aut}_{\mathcal{Z}}(\mathcal{A}, -)$ by Int is $\mathbf{Sim}(\mathcal{A}, -)$. We will show that the inverse image of $\mathbf{Aut}_{\mathcal{Z}}(\mathcal{A}, -, \Delta)$ is $\mathbf{Sim}_0(\mathcal{A}, -, \Delta)$. Indeed, if we take an element $u \in$

$(\mathcal{A} \otimes_{\mathbb{F}} R)^\times$ satisfying $u\overline{u} \in \mathcal{Z}^\times$ and such that $\mathrm{Int}_R(u) \in \mathbf{Aut}_{\mathcal{Z}}(\mathcal{A}, -, \Delta)(R)$. Since $\mathrm{Int}_R(u)$ preserves the grading, it preserves $(\mathcal{A}_{-1} \otimes R)(\mathcal{A}_1 \otimes R)$. Since e_1 is the unit of $(\mathcal{A}_{-1} \otimes R)(\mathcal{A}_1 \otimes R)$ (due to the fact that for a simple AT2 $\mathcal{W}$, Theorem 4.1.12 implies $L(W) = \mathcal{A}_{-1}(W)\mathcal{A}_1(W)$), then $\mathrm{Int}_R(u)(e_i) = e_i$. Thus $ue_i = e_iu$ for $i \in \{1, 2\}$. Since $e_iue_i = e_i^2u = e_iu$ for $i \in \{1, 2\}$, we have that $u = e_1u + e_2u = e_1ue_1 + e_2ue_2 = u_1 + u_2$. Since $u_1 + u_2 \in \mathcal{A}_0$, it follows that the inverse image of $\mathbf{Aut}_{\mathcal{Z}}(\mathcal{A}, -, \Delta)$ by Int is $\mathbf{Sim}_0(\mathcal{A}, -, \Delta)$.

Due to [KMRT, 22.4], the restriction of Int to $\mathbf{Sim}_0(\mathcal{A}, -, \Delta)$ gives a surjective morphism $\mathbf{Sim}_0(\mathcal{A}, -, \Delta) \to \mathbf{Aut}_{\mathcal{Z}}(\mathcal{A}, -, \Delta)$ and also, $\mathbf{Aut}_{\mathcal{Z}}(\mathcal{A}, -, \Delta)$ is smooth if $\mathbf{Sim}_0(\mathcal{A}, -, \Delta)$ is smooth. Thus, we just need to prove that $\mathbf{Sim}_0(\mathcal{A}, -, \Delta)$ is smooth.

Since $\mathbb{F}1$ is the subalgebra of $\mathcal{Z}$ which consists on the invariant elements under $-$, we have the following exact sequence [KMRT, 23.4]:

$$\mathbf{1} \to \mathbf{Iso}(\mathcal{A}_0, -) \to \mathbf{Sim}_0(\mathcal{A}, -, \Delta) \xrightarrow{\mu} \mathbf{G}_{\mathrm{m}} \to \mathbf{1}$$

where $\mu_R(u) = u\overline{u}$. Due to the fact that $\mathbf{GL}_1$ is smooth as shown in [KMRT, 22.11], if $\mathbf{Iso}(\mathcal{A}_0, -)$ is smooth, [KMRT, 22.10] would imply that $\mathbf{Sim}_0(\mathcal{A}, -, \Delta)$ is smooth. Therefore, we just have to prove that $\mathbf{Iso}(\mathcal{A}_0, -)$ is smooth.

Finally, given $u, v \in (\mathcal{A}_0 \otimes_{\mathbb{F}} R)^\times$, let $u_i = e_iue_i$ and $v_i = e_ive_i$ as before for each $i \in \{1, 2\}$. Then, since $u_1v_2 = u_2v_1 = 0$, it follows that $uv = 1$ if and only if $u_iv_i = e_i$ for $i \in \{1, 2\}$. Therefore, we have an isomorphism

$$\theta\colon \mathbf{Iso}(\mathcal{A}_{1,1}, -) \times \mathbf{Iso}(\mathcal{A}_{2,2}, -) \to \mathbf{Iso}(\mathcal{A}_0, -)$$

given by $\theta_R(u_1, u_2) = u_1 + u_2$. Since Lemma 4.1.22 implies that $(\mathcal{A}_{i,i}, -)$ is central simple for $i \in \{1, 2\}$, due to [KMRT, 23], $\mathbf{Iso}(\mathcal{A}_{i,i}, -)$ is smooth for $i \in \{1, 2\}$. Thus, [KMRT, 21.10] implies that $\mathbf{Iso}(\mathcal{A}_0, -)$ is smooth. □

Corollary 5.1.3. *Let $(\mathcal{W}, \{\cdots\})$ be a central simple AT2. Then the automorphism group scheme $\mathbf{Aut}(\mathcal{W}, \{\cdots\})$ is smooth.*

5.1.2 Structurable algebras with the uniqueness property

In this subsection we will give a description of the affine group scheme $\mathbf{Aut}(\mathcal{A}, -, \Gamma)$ for graded-central-simple structurable algebras $(\mathcal{A}, -, \Gamma)$ in terms of AT2. We will show in Proposition 5.1.6 that in case $(\mathcal{A}, -)$ has the uniqueness property,

$$\mathbf{Aut}(\mathcal{A}, -, \Gamma) \cong \mathbf{Aut}(\mathcal{A}, -).$$

The goal will be to prove Theorem 5.1.7

We will begin recalling that for a graded-central-simple structurable algebra $(\mathcal{A}, -, \Gamma)$ with a structurable grading $\Gamma\colon \mathcal{A} = \mathcal{E} \oplus \mathcal{W}$, due to Corollary 4.2.9, $\mathcal{E} = \mathcal{W}\mathcal{W}$. Thus, given a unital commutative associative algebra R, and $\varphi \in \mathbf{Aut}(\mathcal{A}, -)(R)$, $\varphi(W \otimes_{\mathbb{F}} R) = W \otimes_{\mathbb{F}} R$, implies $\varphi(\mathcal{E} \otimes_{\mathbb{F}} R) = \mathcal{E} \otimes_{\mathbb{F}} R$.

Theorem 5.1.4. *Let $(\mathcal{W}, \{\cdots\})$ be a central simple associative triple system of the second kind. Then the morphism of affine group schemes:*

$$\theta\colon \mathbf{Aut}(\mathfrak{A}(\mathcal{W}), -, \Gamma_{\mathfrak{A}}(\mathcal{W})) \to \mathbf{Aut}(\mathcal{W}, \{\cdots\})$$

given by $\theta_R(\varphi) = \varphi_{|\mathcal{W}\otimes_{\mathbb{F}} R}$ for every algebra R, is an isomorphism.

Proof. Define $\theta_R^{-1}(\varphi)$ to be the morphism satisfying:

(1) $\theta_R^{-1}(\varphi)(w) = \varphi(w)$ for all $w \in \mathcal{W}$ and

(2) $\theta_R^{-1}(\varphi)(x) = \sum_{i=1}^{k} \varphi(v_i \otimes 1)\varphi(w_i \otimes 1)(1 \otimes r_i)$ if

$$x = \sum_{i=1}^{k} v_i w_i \otimes r_i \in L(\mathcal{W}) \otimes R$$

for any $v_1, w_1, ..., v_k, w_k \in \mathcal{W}$ and $r_1, ..., r_k \in R$.

Since $(\mathcal{W}, \{\cdots\})$ is simple, it implies that $(\mathfrak{A}(\mathcal{W}), -, \Gamma_{\mathfrak{A}}(\mathcal{W}))$ is graded-simple. $(\mathcal{W} \otimes_{\mathbb{F}} R)(\mathcal{W} \otimes_{\mathbb{F}} R) = \mathcal{E} \otimes_{\mathbb{F}} R$. Thus, in order to show that it is well defined we just need to prove that if $\sum_{i=1}^{k} v_i w_i \otimes r_i = 0$, then $\sum_{i=1}^{k} \varphi(v_i \otimes 1)\varphi(w_i \otimes 1)(1 \otimes r_i) = 0$ but this can be shown in the same way as we did to show that the functor $\mathfrak{A}$ was well defined. Similarly, we show that it is an algebra homomorphism.

Due to the fact that $\mathcal{W} \otimes R$ generates the whole algebra, it is clear that $\theta^{-1} \circ \theta = \mathrm{id}$ and that $\theta \circ \theta^{-1} = \mathrm{id}$. □

Now we will prove that if $(\mathcal{A}, -, \Gamma)$ is an object of *CSAlgStrGr* such that $(\mathcal{A}, -)$ has the uniqueness property, then $\mathbf{Aut}(\mathcal{A}, -) = \mathbf{Aut}(\mathcal{A}, -, \Gamma)$.

Lemma 5.1.5. *Let $(\mathcal{A}, -, \Gamma)$ be an object of CSAlgStrGr where $\Gamma\colon \mathcal{A} = \mathcal{E} \oplus \mathcal{W}$ is the structurable grading. Then,* $\mathrm{Der}(\mathcal{A}, -) = \mathrm{Der}(\mathcal{A}, -, \Gamma)$.

Proof. Clearly $\mathrm{Der}(\mathcal{A}, -) \supseteq \mathrm{Der}(\mathcal{A}, -, \Gamma)$. Hence, we just need to show that $\mathrm{Der}(\mathcal{A}, -) \subseteq \mathrm{Der}(\mathcal{A}, -, \Gamma)$. Let $d \in \mathrm{Der}(\mathcal{A}, -)$.

In case $\mathcal{E} = \mathcal{K}(\mathcal{A}, -)$, then since $d(\mathcal{S}(\mathcal{A}, -)) \subseteq \mathcal{S}(\mathcal{A}, -)$, it is clear that $d(\mathcal{E}) \subseteq \mathcal{E}$. Moreover, if $w \in \mathcal{W}$ and $e \in \mathcal{E}$, we have that

$$\begin{aligned} ed(w) = ed(w) + d(e)w - d(e)w = d(ew) - d(e)w = d(w\bar{e}) - w\overline{d(e)} \\ = d(w)\bar{e} + wd(\bar{e}) - wd(\bar{e}) = d(w)\bar{e}. \end{aligned}$$

Since $d(\mathcal{H}(\mathcal{A}, -)) \subseteq d(\mathcal{H}(\mathcal{A}, -))$, it follows that $d(w) \in \mathcal{M}(\mathcal{A}, -)$. Since in this case, $\mathcal{W} = \mathcal{M}(\mathcal{A}, -)$ due to Lemma 4.3.8, the result follows.

In case $\mathcal{E} \neq \mathcal{K}(\mathcal{A}, -)$, since after extending scalars, the map induced by d is also a derivation, we can assume that we work over an algebraically closed field. Denote:

$$\begin{aligned} \mathrm{Der}_{\bar{0}} &= \{d \in \mathrm{Der}(\mathcal{A}, -) \mid d(\mathcal{E}) \subseteq \mathcal{E}, d(\mathcal{W}) \subseteq \mathcal{W}\}, \text{ and} \\ \mathrm{Der}_{\bar{1}} &= \{d \in \mathrm{Der}(\mathcal{A}, -) \mid d(\mathcal{E}) \subseteq \mathcal{W}, d(\mathcal{W}) \subseteq \mathcal{E}\} \end{aligned}$$

We have that $\mathrm{Der}(\mathcal{A}, -) = \mathrm{Der}_{\bar{0}}(\mathcal{A}, -) \oplus \mathrm{Der}_{\bar{1}}(\mathcal{A}, -)$. Hence, if we prove that $\mathrm{Der}_{\bar{1}}(\mathcal{A}, -) = 0$, the statement follows. Due to Proposition 4.3.3, we only need to consider two cases:

(1) In case $\mathcal{E} = \mathbb{F}1$, for $d \in \mathrm{Der}_{\bar{1}}(\mathcal{A}, -)$, we have that $d(1) = d(1^2) = 2d(1)$, which implies that $d(1) = 0$. For an element $w \in \mathcal{W}$, since $d(w) \in \mathbb{F}1$, we have that $d(w) = \lambda 1$. Since $w^2 \in \mathbb{F}1$, we have that $0 = d(w^2) = wd(w) + d(w)w = 2\lambda w$. Thus, $\lambda = 0$ and it follows that $d = 0$.

(2) In case $(\mathcal{E}, -) = (\mathcal{M}_2(\mathbb{F}), \widehat{\ })$, we can identify those two algebras. Take $d \in \mathrm{Der}_{\bar{1}}(\mathcal{A}, -)$ and denote $s = E_{1,1} - E_{2,2}$. Clearly $s \in \mathcal{S}(\mathcal{A}, -)$. Moreover, since $d(\mathcal{S}(\mathcal{A}, -)) \subseteq \mathcal{S}(\mathcal{A}, -)$ and $\mathcal{S}(\mathcal{A}, -) = \mathbb{F}s$, $d(s) \in \mathcal{S}(\mathcal{A}, -) \cap \mathcal{W} = (0)$. Due to the fact that $\mathcal{K}(\mathcal{A}, -) = \mathbb{F}1 \oplus \mathbb{F}s$, it follows, $d(\mathcal{K}(\mathcal{A}, -)) = 0$. Denote $x = E_{1,2} + E_{2,1}$. In this case, $x \in \mathcal{E} \cap \mathcal{M}(\mathcal{A}, -)$. Because $d(x) \in \mathcal{W}$, we have that $xd(x) = \overline{xd(x)} = \overline{d(x)}\bar{x} = d(x)x$. Since $x^2 = 1$, it follows that

$$0 = d(x^2) = xd(x) + d(x)x = 2xd(x).$$

Thus,

$$0 = \frac{1}{2}x(2xd(x)) = x^2 d(x) = d(x).$$

Moreover, as it happens in $(\mathcal{M}_2(\mathbb{F}), \widehat{\ })$, $\mathcal{E} = \mathcal{K}(\mathcal{A}, -) \oplus \mathcal{K}(\mathcal{A}, -)x$. Therefore, $d(\mathcal{E}) = 0$.

Finally, given $w \in \mathcal{W}$ and $e \in \mathcal{E}$, we have that

$$ed(w) = d(ew) - d(e)w = d(ew) = d(w\bar{e}) = d(w)\bar{e},$$

But if $x \in \mathcal{E}$ and $ex = x\bar{e}$ for all $e \in \mathcal{E}$, this implies that $x = 0$. Thus, $d(w) = 0$. Therefore, we have shown that $d(\mathcal{W}) = 0$, which implies that $d = 0$.

□

We can use this Lemma in order to prove the equality $\mathbf{Aut}(\mathcal{A}, -, \Gamma) = \mathbf{Aut}(\mathcal{A}, -)$ in the following way:

Proposition 5.1.6. *Let $(\mathcal{A}, -, \Gamma)$ be an object of CSAlgStrGr such that $(\mathcal{A} \otimes_{\mathbb{F}} \overline{\mathbb{F}}, -)$ has the uniqueness property. Let $\Gamma\colon \mathcal{A} = \mathcal{E} \oplus W$ be the structurable grading. Then:*

$$\mathbf{Aut}(\mathcal{A}, -, \Gamma) = \mathbf{Aut}(\mathcal{A}, -)$$

Proof. Denote by $\theta\colon \mathbf{Aut}(\mathcal{A}, -, \Gamma) \to \mathbf{Aut}(\mathcal{A}, -)$, the natural embedding, i.e., the morphism given by $\theta_R(\varphi) = \varphi$. We will prove that it is an isomorphism.

Since $(\mathcal{A} \otimes_{\mathbb{F}} \overline{\mathbb{F}}, -)$ has the uniqueness property, and for any automorphism $\varphi \in \mathbf{Aut}(\mathcal{A}, -)(\overline{\mathbb{F}})$, $\Gamma^{\varphi}\colon \mathcal{A} \otimes_{\mathbb{F}} \overline{\mathbb{F}} = \varphi(\mathcal{E} \otimes_{\mathbb{F}} \overline{\mathbb{F}}) \oplus \varphi(W \otimes_{\mathbb{F}} \overline{\mathbb{F}})$ is a structurable grading, it follows, that $\Gamma^{\varphi} = \Gamma$. Hence, if $\varphi \in \mathbf{Aut}(\mathcal{A}, -)(\overline{\mathbb{F}})$, then $\varphi \in \mathbf{Aut}(\mathcal{A}, -, \Gamma)(\overline{\mathbb{F}})$, which implies that $\theta_{\overline{\mathbb{F}}}$ is an isomorphism.

$$d(\theta)\colon \operatorname{Der}(\mathcal{A}, -, \Gamma) \to \operatorname{Der}(\mathcal{A}, -)$$

is injective since $\theta_{\mathbb{F}[\tau]}$ is a monomorphism, hence, it is an isomorphism because due to Lemma 5.1.5, $\operatorname{Der}(\mathcal{A}, -, \Gamma) = \operatorname{Der}(\mathcal{A}, -)$. Finally, because of Theorem 5.1.4, Proposition 5.1.1 and Corolary 5.1.3, we have that $\mathbf{Aut}(\mathcal{A}, -, \Gamma)$ is smooth. Therefore, [EK13, Theorem A.50] implies that θ is an isomorphism. □

As a corollary of this, we get the following theorem.

Theorem 5.1.7. *Let $(\mathcal{A}, -, \Gamma)$ be an object of CSAlgStrGr such that $(\mathcal{A} \otimes_{\mathbb{F}} \overline{\mathbb{F}}, -)$ has the uniqueness property. Denote $\mathcal{W} = \mathfrak{W}(\mathcal{A}, -, \Gamma)$. Then:*

$$\mathbf{Aut}(\mathcal{A}, -) \cong \mathbf{Aut}(\mathcal{A}(\mathcal{W}), -, \Delta(\mathcal{W}))$$

via the morphism $\theta_2^{-1} \circ \theta_1$ where

$$\theta_1\colon \mathbf{Aut}(\mathcal{A}, -) \to \mathbf{Aut}(\mathfrak{W}(\mathcal{A}, -, \Gamma))$$

and

$$\theta_2\colon \mathbf{Aut}(\mathcal{A}(\mathcal{W}), -, \Delta(\mathcal{W})) \to \mathbf{Aut}(\mathcal{W})$$

are defined by ${\theta_1}_R(\psi) = \psi_{|\mathcal{W} \otimes_{\mathbb{F}} R}$ for every $\psi \in \mathbf{Aut}(\mathcal{A}, -, \Gamma)(R)$ and ${\theta_2}_R(\psi) = \psi_{|\mathcal{W} \otimes_{\mathbb{F}} R}$ for every $\psi \in \mathbf{Aut}(\mathcal{A}(\mathcal{W}), -, \Delta(\mathcal{W}))(R)$

Proof. This follows from Proposition 5.1.6, Theorem 5.1.4, Proposition 5.1.1 and Proposition 4.2.26 □

5.1.3 Automorphisms on the split quartic Cayley algebra

When we work over an algebraically closed field, the only algebra which doesn't have the uniqueness property, is the split quartic Cayley algebra, which we will denote until the end of the section by $(\mathcal{A}, -)$. Our goal will be to prove Theorem 5.1.13, which gives a description of its automorphism group scheme.

If we use the notation from (4.3.2) we can define a $\mathbb{Z}_2^2$-grading by:

$$\mathcal{A}_{(\bar{0},\bar{0})} = \mathcal{K},\ \mathcal{A}_{(\bar{0},\bar{1})} = \mathcal{K}x_1,\ \mathcal{A}_{(\bar{1},\bar{0})} = \mathcal{K}x_2,\ \mathcal{A}_{(\bar{1},\bar{1})} = \mathcal{K}x_3$$

We call this grading the **standard quartic grading** and denote it by Γ_{SQ}. Recall that for a finite abelian group G, the constant affine group scheme is the affine group scheme whose R points are $\mathrm{Hom}_{\mathrm{Alg}_{\mathbb{F}}}(\mathrm{Maps}(G, \mathbb{F}), R)$ and that $\mathrm{Maps}(G, \mathbb{F})$ is spanned as a vector space by the maps e_σ for each $\sigma \in G$, where $e_\sigma(\sigma) = 1$ and $e_\sigma(\tau) = 0$ if $\tau \neq \sigma$.

Lemma 5.1.8. *Let $(\mathcal{A}, -)$ be the split quartic Cayley algebra. There is a morphism:*

$$\theta\colon \mathbf{Aut}(\mathcal{A}, -, \Gamma_{\mathrm{SQ}}) \rtimes \mathrm{Sym}_3 \to \mathbf{Aut}(\mathcal{A}, -)$$

Defined as:

$$\theta_R(\psi, f)(y_0+y_1x_1+y_2x_2+y_3x_3) = \psi\left(y_0 + \sum_{\sigma\in\mathrm{Sym}_3}\sum_{i=1}^{3} y_i(x_{\sigma(i)}\otimes 1)\,(1\otimes f(e_\sigma))\right)$$

for every $y_0, y_1, y_2, y_3 \in \mathcal{K}\otimes_{\mathbb{F}} R$ and $f \in \mathrm{Sym}_3(R)$.

Proof. To begin with, we notice that the elements e_σ for $\sigma \in \mathrm{Sym}_3$ are orthogonal idempotents. Moreover, $1 = \sum_{\sigma\in\mathrm{Sym}_3} e_\sigma$. Therefore, their images under f are also orthogonal idempotents of R whose sum is 1. Since for every $\psi \in \mathbf{Aut}(\mathcal{A}, -, \Gamma_{\mathrm{SQ}})(R)$ and $f \in \mathrm{Sym}_3(R)$ we have that $\theta_R(\psi, f) = \psi \circ \theta_R(\mathrm{id}, f)$, in order to prove that $\theta_R(\psi, f) \in \mathbf{Aut}(\mathcal{A}, -)$, we just need to show that $\theta_R(\mathrm{id}, f) \in \mathbf{Aut}(\mathcal{A}, -)$.

$\theta_R(\mathrm{id}, f)$ clearly preserves the involution. Moreover, $\theta_R(\mathrm{id}, f)_{|\mathcal{K}\otimes R} = \mathrm{id}$ and for any $i \in \{1, 2, 3\}$ and $y \in \mathcal{K}\otimes R$, we have that

$$\theta_R(\mathrm{id}, f)(yx_i) = y\theta_R(\mathrm{id}, f)(x_i) = \theta_R(\mathrm{id}, f)(y)\theta_R(\mathrm{id}, f)(x_i).$$

Hence, we just need to prove that

$$\theta_R(\mathrm{id}, f)(x_i x_j) = \theta_R(\mathrm{id}, f)(x_i)\theta_R(\mathrm{id}, f)(x_j)$$

for every $i, j \in \{1, 2, 3\}$. In case $i = j$ we have:

$$\begin{aligned}
(\theta_R(\mathrm{id}, f)(x_i))^2 &= \left(\sum_{\sigma \in \mathrm{Sym}_3} x_{\sigma(i)} \otimes f(e_\sigma) \right)^2 \\
&= \sum_{\sigma, \tau \in \mathrm{Sym}_3} x_{\sigma(i)} x_{\tau(i)} \otimes f(e_\sigma) f(e_\tau) \\
&= \sum_{\sigma \in \mathrm{Sym}_3} x_{\sigma(i)} x_{\sigma(i)} \otimes f(e_\sigma) \\
&= \sum_{\sigma \in \mathrm{Sym}_3} 1 \otimes f(e_\sigma) \\
&= 1 \\
&= \theta_R(\mathrm{id}, f)(x_i^2).
\end{aligned}$$

Finally, in case $i \neq j$, let $\{i, j, k\} = \{1, 2, 3\}$:

$$\begin{aligned}
\theta_R(\mathrm{id}, f)(x_i)\theta_R(\mathrm{id}, f)(x_j) &= \left(\sum_{\sigma \in \mathrm{Sym}_3} x_{\sigma(i)} \otimes f(e_\sigma) \right) \left(\sum_{\sigma \in \mathrm{Sym}_3} x_{\sigma(j)} \otimes f(e_\sigma) \right) \\
&= \sum_{\sigma, \tau \in \mathrm{Sym}_3} x_{\sigma(i)} x_{\tau(j)} \otimes f(e_\sigma) f(e_\tau) \\
&= \sum_{\sigma \in \mathrm{Sym}_3} x_{\sigma(i)} x_{\sigma(j)} \otimes f(e_\sigma) \\
&= \sum_{\sigma \in \mathrm{Sym}_3} x_{\sigma(k)} \otimes f(e_\sigma) \\
&= \theta_R(\mathrm{id}, f)(x_k) \\
&= \theta_R(\mathrm{id}, f)(x_i x_j)
\end{aligned}$$

□

Lemma 5.1.9. *Let* $(\mathcal{A}, -)$ *be the split quartic Cayley algebra over an algebraically closed field* $\mathbb{F}$*. Using the notation from* (4.3.2) *and the grading* Γ_1 *from Corollary 4.3.14. Let* C_2 *be the subgroup of* Sym_3 *generated by the transposition* $(2, 3)$*. Then:*

$$\mathbf{Aut}(\mathcal{A}, -, \Gamma_{\mathrm{SQ}}) \rtimes \mathrm{C}_2 \cong \mathbf{Aut}(\mathcal{A}, -, \Gamma_1)$$

Proof. The restriction of the morphism θ on Lemma 5.1.8 to the affine group scheme $\mathbf{Aut}(\mathcal{A}, -, \Gamma_{\mathrm{SQ}}) \rtimes \mathrm{C}_2$, factors through $\mathbf{Aut}(\mathcal{A}, -, \Gamma_1)$. Denote the morphism defined by the restriction of θ as $\tilde{\theta}\colon \mathbf{Aut}(\mathcal{A}, -, \Gamma_{\mathrm{SQ}}) \rtimes \mathrm{C}_2 \to \mathbf{Aut}(\mathcal{A}, -, \Gamma_1)$. Due to Theorem 5.1.4 and Corollary 5.1.3, it follows that $\mathbf{Aut}(\mathcal{A}, -, \Gamma_1)$ is smooth. Hence, due to [EK13, Theorem A.50], we just need to prove that $\tilde{\theta}_{\mathbb{F}}$ is an isomorphism and that $d\tilde{\theta}$ is also an isomorphism.

If $f \in \mathrm{Hom}_{\mathrm{Alg}_{\mathbb{F}}}(\mathrm{Maps}(\mathrm{C}_2, \mathbb{F}), \mathbb{F})$, since $e_{\mathrm{id}}^2 = e_{\mathrm{id}}$, $e_{(2,3)}^2 = e_{(2,3)}$ and $e_{\mathrm{id}} + e_{(2,3)} = 1$, then either $f(e_{\mathrm{id}}) = 1$ and $f(e_{(2,3)}) = 0$ or $f(e_{\mathrm{id}}) = 0$ and $f(e_{(2,3)}) = 1$ ($\mathrm{C}_2(\mathbb{F})$ is the cyclic group of order 2). In the first case $\theta(\mathrm{id}, f) = \varphi_{\mathbb{F}}^{\mathrm{id}} = \mathrm{id}$ and in the second case $\theta(\mathrm{id}, f) = \varphi_{\mathbb{F}}^{(2,3)}$ (φ_R^σ is defined as in Corollary 4.3.13).

Due to Corollary 4.3.14, it follows that if $\psi \in \mathrm{Aut}(\mathcal{A}, -, \Gamma_1)$, there is some $\sigma \in \mathrm{C}_2$ such that $\psi(\mathcal{K}x_i) \subseteq \mathcal{K}x_{\sigma(i)}$ for all $i \in \{1, 2, 3\}$. In this case, $\psi = (\psi\varphi_{\mathbb{F}}^\sigma)\varphi_{\mathbb{F}}^\sigma$. So, due to the fact that $(\psi\varphi_{\mathbb{F}}^\sigma) \in \mathrm{Aut}(\mathcal{A}, -, \Gamma_{\mathrm{SQ}})$, if $f(\sigma) = 1$ and $f(\tau) = 0$ for $\tau \neq \sigma$, we have that $\psi = \tilde{\theta}_{\mathbb{F}}(\psi\varphi_{\mathbb{F}}^\sigma, f)$. Thus, $\tilde{\theta}_{\mathbb{F}}$ is surjective. In order to prove that $\tilde{\theta}_{\mathbb{F}}$ is a monomorphism, assume that $\tilde{\theta}_{\mathbb{F}}(\psi, f) = \mathrm{id}$. This implies that $\psi^{-1} = \psi^{-1}\tilde{\theta}_{\mathbb{F}}(\psi, f) = \tilde{\theta}(\mathrm{id}, f)$. Thus, $\tilde{\theta}_{\mathbb{F}}(\mathrm{id}, f) \neq \varphi_{\mathbb{F}}^{(1,2)}$ and that, as we saw before, implies that $\tilde{\theta}_{\mathbb{F}}(\mathrm{id}, f) = \mathrm{id}$, and so, $\psi = \mathrm{id}$. Therefore, $\tilde{\theta}_{\mathbb{F}}$ is injective. Thus, it is an isomorphism.

By Lemma 5.1.5, any derivation $d \in \mathrm{Der}(\mathcal{A}, -)$, preserves Γ_1 and Γ_2. Hence $d(\mathcal{K}) \subseteq \mathcal{K}$ and $d(\mathcal{K}x_i) \subseteq \mathcal{K}x_i$ for all $i \in \{1, 2, 3\}$. Thus,

$$Der(\mathcal{A}, -) = \mathrm{Der}(\mathcal{A}, -, \Gamma_1) = \mathrm{Der}(\mathcal{A}, -, \Gamma_{\mathrm{SQ}}).$$

Now, on the one hand,

$$\mathrm{Lie}(\mathbf{Aut}(\mathcal{A}, -, \Gamma_1)) = \mathrm{Der}(\mathcal{A}, -, \Gamma_1) = \mathrm{Der}(\mathcal{A}, -, \Gamma_{\mathrm{SQ}})$$

and if we consider the projection $\pi\colon \mathbf{Aut}(\mathcal{A}, -, \Gamma_{\mathrm{SQ}}) \rtimes \mathrm{C}_2 \to \mathrm{C}_2$, we have

$$\mathrm{Lie}(\ker d\pi) = \mathrm{Lie}(\mathbf{Aut}(\mathcal{A}, -, \Gamma_{\mathrm{SQ}})) = \mathrm{Der}(\mathcal{A}, -, \Gamma_{\mathrm{SQ}})$$

where the first equality is proved in [KMRT, (21.4)] but since $\mathrm{Lie}(\mathrm{C}_2) = 0$, then $d\pi = 0$ and $\ker d\pi = \mathbf{Aut}(\mathcal{A}, -, \Gamma_{\mathrm{SQ}}) \rtimes \mathrm{C}_2$. Therefore, $d\tilde{\theta}$ is an isomorphism. □

Corollary 5.1.10. $\mathbf{Aut}(\mathcal{A}, -, \Gamma_{\mathrm{SQ}})$ *is smooth.*

Proof. Since $\mathbf{Aut}(\mathcal{A}, -, \Gamma_1)$ is smooth and it is isomorphic to $\mathbf{Aut}(\mathcal{A}, -, \Gamma_{\mathrm{SQ}}) \rtimes \mathrm{C}_2$, thus there is an isomorphism of algebras

$$\mathbb{F}[\mathbf{Aut}(\mathcal{A}, -, \Gamma_{\mathrm{SQ}}) \rtimes \mathrm{C}_2] \cong \mathbb{F}[\mathbf{Aut}(\mathcal{A}, -, \Gamma_{\mathrm{SQ}})] \otimes_{\mathbb{F}} \mathbb{F}[\mathrm{C}_2]$$

which implies that $\mathbb{F}[\mathbf{Aut}(\mathcal{A}, -, \Gamma_{\mathrm{SQ}})] \otimes_{\mathbb{F}} \mathbb{F}[\mathrm{C}_2]$ is reduced. Hence, $\mathbb{F}[\mathbf{Aut}(\mathcal{A}, -, \Gamma_{\mathrm{SQ}})]$, is reduced. Therefore, since $\mathbb{F}$ is algebraically closed, it follows that $\mathbf{Aut}(\mathcal{A}, -, \Gamma_{\mathrm{SQ}})$ is smooth [KMRT, 21.9]). □

Proposition 5.1.11. *Let $(\mathcal{A}, -)$ be the split quartic Cayley algebra over an algebraically closed field $\mathbb{F}$. Then:*

$$\mathbf{Aut}(\mathcal{A}, -, \Gamma_{\mathrm{SQ}}) \rtimes \mathrm{Sym}_3 \cong \mathbf{Aut}(\mathcal{A}, -)$$

Proof. We are going to prove that the morphism θ defined in Lemma 5.1.8, is an isomorphism. Since $\mathbf{Aut}(\mathcal{A}, -, \Gamma_{\mathrm{SQ}})$ is smooth due to Corollary 5.1.10 and Sym_3 is smooth [KMRT, 21.11], using the exact sequence:

$$1 \to \mathbf{Aut}(\mathcal{A}, -, \Gamma_{\mathrm{SQ}}) \to \mathbf{Aut}(\mathcal{A}, -, \Gamma_{\mathrm{SQ}}) \rtimes \mathrm{Sym}_3 \to \mathrm{Sym}_3 \to 1$$

induced by the inclusion of $\mathbf{Aut}(\mathcal{A}, -, \Gamma_{\mathrm{SQ}})$ and the projection on Sym_3, we can prove that $\mathbf{Aut}(\mathcal{A}, -, \Gamma_{\mathrm{SQ}}) \rtimes \mathrm{Sym}_3$ is smooth [KMRT, 22.12]. Due to [EK13, Theorem A.50], it follows that θ is an isomorphism if and only if $\theta_{\mathbb{F}}$ is an isomorphism and $d\theta$ is an isomorphism.

Use the notation of (4.3.2), as in the proof of Lemma 5.1.9, we have that $\mathrm{Der}(\mathcal{A}, -) = \mathrm{Der}(\mathcal{A}, -, \Gamma_{\mathrm{SQ}})$. Thus,

$$\mathrm{Lie}(\mathbf{Aut}(\mathcal{A}, -)) \cong \mathrm{Lie}(\mathbf{Aut}(\mathcal{A}, -, \Gamma_{\mathrm{SQ}})).$$

Moreover, since as in the proof of Lemma 5.1.9, we have

$$\mathrm{Lie}(\mathbf{Aut}(\mathcal{A}, -, \Gamma_{\mathrm{SQ}}) \rtimes \mathrm{Sym}_3) = \mathrm{Lie}(\mathbf{Aut}(\mathcal{A}, -, \Gamma_{\mathrm{SQ}}),$$

then it follows that $d\theta$ is an isomorphism.

Since for every $\sigma, \tau \in \mathrm{Sym}_3$ we have that $e_\sigma^2 = e_\sigma$, $e_\sigma e_\tau = 0$ if $\sigma \neq \tau$ and $\sum_{\sigma \in \mathrm{Sym}_3} e_\sigma = 1$, the same relations will hold after applying an element $f \in \mathrm{Hom}_{\mathrm{Alg}_{\mathbb{F}}}(\mathrm{Maps}(\mathrm{Sym}_3, \mathbb{F}), \mathbb{F})$. Therefore, there is some $\sigma \in \mathrm{Sym}_3$ such that $f(e_\sigma) = 1$ and $f(e_\tau) = 0$ for every $\sigma \neq \tau \in \mathrm{Sym}_3$. We will denote such f as f_σ. In order to show that $\theta_{\mathbb{F}}$ is an isomorphism, note that $\theta_{\mathbb{F}}(\psi, f_\sigma) = \psi\theta(\mathrm{id}, f_\sigma) = \psi\varphi_{\mathbb{F}}^\sigma$. In order to show that $\theta_{\mathbb{F}}$ is injective, we notice that if $\theta(\psi, f_\sigma) = \mathrm{id}$, then, since $\psi, \psi\varphi_{\mathbb{F}}^\sigma \in \mathrm{Aut}(\mathcal{A}, -, \Gamma_{\mathrm{SQ}}))$, it implies that $\sigma = \mathrm{id}$. Thus, $\varphi_{\mathbb{F}}^\sigma = \mathrm{id}$ and this implies that $\psi = \mathrm{id}$. Therefore, $\theta_{\mathbb{F}}$ is injective. In order to prove that it is surjective, we use that due to corollary 4.3.14, for every $i \in \{1, 2, 3\}$ there is j such that $\psi(\mathcal{E}_i) = \mathcal{E}_j$. Moreover, since $\mathcal{K}x_i = \mathcal{E}_i \cap \mathcal{M}(\mathcal{A}, -)$, then there is a permutation $\sigma \in \mathrm{Sym}_3$ such that $\psi(\mathcal{K}x_i) = \mathcal{K}x_{\sigma(i)}$ for all $i \in \{1, 2, 3\}$. Thus, $\psi\varphi_{\mathbb{F}}^{\sigma^{-1}} \in \mathrm{Aut}(\mathcal{A}, -, \Gamma_{\mathrm{SQ}})$. Hence, $\psi = \theta_{\mathbb{F}}(\psi\varphi_{\mathbb{F}}^{\sigma^{-1}}, f_\sigma)$. □

Given an associative algebra with involution and unity $(\mathcal{E}, -)$, we defined the automorphism group scheme $\mathbf{Iso}(\mathcal{E}, -)$ as the affine group scheme whose R points are

$$\mathbf{Iso}(\mathcal{E}, -)(R) = \{x \in (\mathcal{E} \otimes R)^\times \mid x\overline{x} = 1\}.$$

In case $(\mathcal{E},\ ) = (\mathbb{F}\oplus\mathbb{F}, \mathrm{ex})$ we identify $(1,1)\otimes a + (1,-1)\otimes ib$ with $(a,b)\in R\times R$, we get that $x\overline{x} = a^2+b^2$. On the other hand, for $x=(a,b)\in(\mathcal{E}\otimes R)^\times \cong (R\oplus R)^\times$, we get $x\overline{x} = (ab,ab)$, thus, we have an isomorphism of affine group schemes´

$$\mathbf{G}_{\mathrm{m}} \cong \mathbf{Iso}(\mathcal{E},\,-\,) \tag{5.1.1}$$

given by the morphism sending $a\to(a,a^{-1})$.

Lemma 5.1.12. *Let $(\mathcal{A},\,-\,)$ be the split quartic Cayley algebra over an algebraically closed field $\mathbb{F}$. Then:*

$$\mathbf{Aut}(\mathcal{A},\,-\,,\Gamma_{\mathrm{SQ}}) \cong (\mathbf{Iso}(\mathbb{F}\oplus\mathbb{F},\mathrm{ex})\times\mathbf{Iso}(\mathbb{F}\oplus\mathbb{F},\mathrm{ex}))\rtimes\mathbf{Aut}(\mathbb{F}\oplus\mathbb{F})$$

where

$$((1,1),\psi)((r_1,r_2),\mathrm{id}) = ((\psi(r_1),\psi(r_2)),id)((1,1),\psi)$$

for every algebra R, $r_1,r_2\in\mathbf{Iso}(\mathbb{F}\oplus\mathbb{F},\mathrm{ex})(R)$ and $\psi\in\mathbf{Aut}(\mathbb{F}\oplus\mathbb{F})(R)$.

Proof. Use the notation from (4.3.2). Define the morphism

$$\theta\colon (\mathbf{Iso}(\mathbb{F}\oplus\mathbb{F},\mathrm{ex})\times\mathbf{Iso}(\mathbb{F}\oplus\mathbb{F},\mathrm{ex}))\rtimes\mathbf{Aut}(\mathbb{F}\oplus\mathbb{F})\to\mathbf{Aut}(\mathcal{A},\,-\,,\Gamma_{\mathrm{SQ}})$$

as

$$\begin{aligned}\theta_R(r_1,r_2,\psi)&(y_0+y_1x_1+y_2x_2+y_3x_3)\\ &=\psi(y_0)+\psi(y_1)(r_1x_1)+\psi(y_2)(r_2x_2)+\psi(y_3)(\overline{r_1r_2}x_3).\end{aligned} \tag{5.1.2}$$

Using Lemma 4.3.12 we can check that this is an homomorphism of the algebra $(\mathcal{A},\,-\,)$. Fix an algebra R. Denote $r_3=\overline{r_1r_2}$. Since $r_i(\overline{r_i}x_i)=x_i$, we can check that

$$\theta_R(r_1,r_2,\psi)\circ\theta_R(\psi^{-1}(\overline{r_1}),\psi^{-1}(\overline{r_2}),\psi^{-1})$$

and

$$\theta_R(\psi^{-1}(\overline{r_1}),\psi^{-1}(\overline{r_2}),\psi^{-1})\circ\theta_R(r_1,r_2,\psi)$$

are the identity.

Since

$$\overline{x_i\theta_R(r_1,r_2,\psi)(x_i)} = \overline{x_i(r_ix_i)} = r_i$$

and for every $y\in\mathcal{K}\otimes R$, $\psi(y)=\theta_R(r_1,r_2,\psi)(y)$, it follows that θ_R is injective. Moreover, if for some algebra R, $\widetilde{\psi}\in\mathbf{Aut}(\mathcal{A},\,-\,,\Gamma_{\mathrm{SQ}})$, then, for every $i\in\{1,2,3\}$ there is an element $r_i\in\mathcal{K}\otimes_{\mathbb{F}}R$ such that $\widetilde{\psi}(x_i)=r_ix_i$. Since $x_i^2=1$, applying $\widetilde{\psi}$, we get that $\overline{r_i}r_i=1$. Since $x_1x_2=x_3$, applying $\widetilde{\psi}$, we get that $r_3 = \overline{r_1 r_2}$. If we denote $\psi = \widetilde{\psi}_{|\mathcal{K}\otimes_{\mathbb{F}}R}$, we have that $\widetilde{\psi} = \theta_R(r_1, r_2, \psi)$. Therefore, θ_R is surjective and this implies that θ is an isomorphism.

Given an algebra R, $r_1, r_2 \in \mathbf{Iso}(\mathbb{F}\oplus\mathbb{F}, \mathrm{ex})(R)$ and $\psi \in \mathbf{Aut}(\mathbb{F}\oplus\mathbb{F})(R)$, in order to show that

$$((1,1),\psi)((r_1,r_2),\mathrm{id}) = ((\psi(r_1),\psi(r_2)),\psi),$$

we just have to apply θ_R and compute on x_i for $i \in \{1,2,3\}$ and on $r1$ for $r \in (\mathbb{F}\oplus\mathbb{F})\otimes_{\mathbb{F}} R$.

□

As a corollary of the previous lemma, we get the following theorem.

Theorem 5.1.13. *Let $(\mathcal{A}, -)$ be the split quartic Cayley algebra over an algebraically closed field $\mathbb{F}$.*

$$\mathbf{Aut}(\mathcal{A}, -) \cong ((\mathbf{Iso}(\mathbb{F}\oplus\mathbb{F},\mathrm{ex}) \times \mathbf{Iso}(\mathbb{F}\oplus\mathbb{F},\mathrm{ex})) \rtimes \mathrm{C}_2) \rtimes \mathrm{Sym}_3$$

where for every algebra R, $r_1, r_2 \in \mathbf{Iso}(\mathbb{F}\oplus\mathbb{F},\mathrm{ex})(R)$, $\psi \in \mathbf{Aut}(\mathbb{F}\oplus\mathbb{F})(R)$ and $f \in \mathrm{Sym}_3(R)$,

$$(((1,1),\mathrm{id}),f)(((r_1,r_2),\psi),1) = (((s_1,s_2),\psi),f)$$

where for each $j \in \{1,2\}$, we denote:

$$s_j = \sum_{i=1}^{3} r_i \sum_{\substack{\sigma\in\mathrm{Sym}_3\\ \sigma(i)=j}} (1\otimes f(e_\sigma))$$

where $r_3 = \overline{r_1 r_2}$.

Proof. In order to prove the first part of the theorem we only need to notice that $\mathbf{Aut}(\mathbb{F}\oplus\mathbb{F}) \cong \mathrm{C}_2$ and use proposition 5.1.11.

In order to prove the commutation rule, we take θ_1 as the morphism defined in Lemma 5.1.8 and $\theta_2 = \theta \times \mathrm{id}$ where θ is the morphism as defined in Lemma 5.1.12. In order to prove that the commutation rule works, take an algebra R, $r_1, r_2 \in \mathbf{Iso}(\mathbb{F}\oplus\mathbb{F},\mathrm{ex})(R)$, $\psi \in \mathbf{Aut}(\mathbb{F}\oplus\mathbb{F})(R)$ and $f \in \mathrm{Sym}_3(R)$. Assume that there are s_1, s_2 and $\varphi \in \mathbf{Aut}(\mathbb{F}\oplus\mathbb{F})(R)$ such that

$$(((1,1),\mathrm{id}),f)(((r_1,r_2),\psi),1) = (((s_1,s_2),\psi),f).$$

Then, on one hand, the equality

$$(\theta_1\circ\theta_2)(((1,1),\mathrm{id}),f)\circ(\theta_1\circ\theta_2(((r_1,r_2),\psi),1) = (\theta_1\circ\theta_2)(((s_1,s_2),\psi),f)$$

applied on elements of $\mathcal{K}$ implies $\psi = \varphi$. Then, defining the morphisms $\Psi_1 = (\theta_1 \circ \theta_2)(((1,1),\mathrm{id}), f)$ and $\Psi_2 = \theta_1 \circ \theta_2(((r_1,r_2),\psi),1)$, we have:

$$\begin{aligned}\Psi_1 \circ \Psi_2(x_i \otimes 1)) &= \Psi_1(r_i(x_i \otimes 1)) \\ &= r_i \sum_{\sigma \in \mathrm{Sym}_3} (x_{\sigma(i)} \otimes 1)(1 \otimes f(e_\sigma))\end{aligned}$$

and:

$$\begin{aligned}(\theta_1 \circ \theta_2)(((s_1,s_2),\psi),f)(x_i \otimes 1) &= \sum_{\sigma \in \mathrm{Sym}_3} s_{\sigma(i)}(x_{\sigma(i)} \otimes 1)(1 \otimes f(e_\sigma)) \\ &= \sum_{j=1}^{3} \left(s_j(x_j \otimes 1) \sum_{\substack{\sigma \in \mathrm{Sym}_3 \\ \sigma(i)=j}} (1 \otimes f(e_\sigma)) \right)\end{aligned}$$

which implies that $r_i \sum_{\substack{\sigma \in \mathrm{Sym}_3 \\ \sigma(i)=j}} (1 \otimes f(e_\sigma)) = s_j \sum_{\substack{\sigma \in \mathrm{Sym}_3 \\ \sigma(i)=j}} (1 \otimes f(e_\sigma))$ for every $i,j \in \{1,2,3\}$. Thus:

$$s_j = \sum_{i=1}^{3} s_j \sum_{\substack{\sigma \in \mathrm{Sym}_3 \\ \sigma(i)=j}} (1 \otimes f(e_\sigma)) = \sum_{i=1}^{3} r_i \sum_{\substack{\sigma \in \mathrm{Sym}_3 \\ \sigma(i)=j}} (1 \otimes f(e_\sigma)).$$

□

Remark 5.1.14. Let $(\mathcal{A}, -)$ be the split quartic Cayley algebra over an algebraically closed field $\mathbb{F}$. Then:

$$\mathbf{Aut}(\mathcal{A}, -) \cong (\mathbf{G}_{\mathrm{m}} \times \mathbf{G}_{\mathrm{m}}) \rtimes (\mathrm{C}_2 \times \mathrm{Sym}_3) \tag{5.1.3}$$

where if we identify $\mathbf{G}_{\mathrm{m}} \times \mathbf{G}_{\mathrm{m}}$ with $\{(a,b,c) \in \mathbf{G}_{\mathrm{m}}^3 \mid abc = 1\}$ via the morphism sending (a,b) to $(a,b,(ab)^{-1})$, Sym_3 acts by permutations and C_2 acts by $(a,b,c) \to (a^{-1},b^{-1},c^{-1})$. This is due to the fact that we have the isomorphism (5.1.1) and the fact that C_2 commutes with Sym_3

5.2 Gradings

Our purpose in this section is to prove Theorems 5.1.7 and 5.1.13, which give a complete classification up to isomorphism of gradings on central simple structurable algebras related to an hermitian form.

5.2.1 Associative algebras

Here G will be a finitely generated abelian group. Our purpose is to classify gradings on the objects of 3*CSGrAlgInv* up to isomorphism, i.e, we have to classify up to isomorphism the Ω-algebras $(\mathcal{A}, \varphi, \Delta_1, \Delta_2)$ satisfying:

(Q1) $(\mathcal{A}, \varphi)$ is a simple associative algebra with involution,

(Q2) Δ_1 is a $\mathbb{Z}$-grading of $\mathcal{A}$,

(Q3) $\mathrm{Supp}(\Delta_1) = \{-1, 0, 1\}$ and $\varphi(\mathcal{A}_i) = \mathcal{A}_{-i}$ for all $i \in \mathbb{Z}$, and

(Q4) Δ_2 is a G-grading of $(\mathcal{A}, \varphi, \Delta_1)$.

If we denote the homogeneous component by $\Delta_1 \colon \mathcal{A} = \bigoplus_{i\in\mathbb{Z}} \mathcal{A}_i$ and $\Delta_2 \colon \mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$, we can define a $(\mathbb{Z}\times G)$-grading $\Delta \colon \mathcal{A} = \bigoplus_{(i,g)\in\mathbb{Z}\times G} \mathcal{A}_{(i,g)}$ where $\mathcal{A}_{(i,g)} = \mathcal{A}_g \cap \mathcal{A}_i$. Denote $\pi_1 \colon \mathbb{Z} \times G \to \mathbb{Z}$ and $\pi_2 \colon \mathbb{Z} \times G \to G$ to be the group homomorphisms defined by $\pi_1(i, g) = i$ and $\pi_2(i, g) = g$. Then, $(\mathcal{A}, \varphi, \Delta)$ satisfies:

(T1) $(\mathcal{A}, \varphi)$ is a simple associative algebra with involution,

(T2) Δ is a $(\mathbb{Z} \times G)$-grading of $\mathcal{A}$,

(T3) $\mathrm{Supp}\ {}^{\pi_1}\Delta = \{-1, 0, 1\}$, and

(T4) $\varphi(\mathcal{A}_{(i,g)}) = \mathcal{A}_{(-i,g)}$ for all $(i, g) \in \mathbb{Z} \times G$.

Moreover if we have an Ω-algebra $(\mathcal{A}, \varphi, \Delta)$ satisfying $(T1) - (T4)$, then the algebra $(\mathcal{A}, \varphi, {}^{\pi_1}\Delta, {}^{\pi_2}\Delta)$ satisfies $(Q1) - (Q4)$. Additionally, for a pair of algebras $(\mathcal{A}_1, \varphi_1, \Delta_1)$ and $(\mathcal{A}_2, \varphi_2, \Delta_2)$, we have that

$$\mathrm{Hom}_{\mathbb{F}}((\mathcal{A}_1, \varphi_1, \Delta_1), (\mathcal{A}_2, \varphi_2, \Delta_2)) = \mathrm{Hom}_{\mathbb{F}}((\mathcal{A}_1, \varphi_1, {}^{\pi_1}\Delta_1, {}^{\pi_2}\Delta_1), (\mathcal{A}_2, \varphi_2, {}^{\pi_1}\Delta_2, {}^{\pi_2}\Delta_2)).$$

Therefore, the problem reduces to classify the algebras $(\mathcal{A}, \varphi, \Delta)$ satisfying $(T1) - (T4)$.

Let $(\mathcal{A}, \varphi, \Delta)$ be such algebra. We can reduce the problem into two cases, i.e., the case where $(\mathcal{A}, \Delta)$ is not graded-simple and the case where it is.

Case 1: $(\mathcal{A}, \Delta)$ is not graded simple

We will begin with this case introducing a definition:

Definition 5.2.1. Let I be an associative algebra. Let $\Delta\colon I = \bigoplus_{g\in G} I_g$ be a $(\mathbb{Z}\times G)$-grading of I. Denote by $\mathrm{ex}\colon I\oplus I^{op}\to I\oplus I^{op}$ the involution given by $\mathrm{ex}(x, y) = (y, x)$. We denote by Δ^{ex}, the $(\mathbb{Z}\times G)$-grading on $(I\oplus I^{op}, \mathrm{ex})$ defined by $\deg(x, 0) = (i, g)$ and $\deg(0, x) = (-i, g)$ whenever $\deg(x) = (i, g)$ on Δ.

Lemma 5.2.2. *Let $(\mathcal{A}, \varphi, \Delta)$ be an Ω-algebra satisfying $(T1) - (T4)$ such that $(\mathcal{A}, \Delta)$ is not graded simple. Then, there is a simple algebra I and a $(\mathbb{Z}\times G)$-grading Δ_I of I with* $\mathrm{Supp}\,^{\pi_1}\Delta_I = \{-1, 0, 1\}$ *such that* $(\mathcal{A}, \varphi, \Delta)\cong (I\oplus I^{op}, \mathrm{ex}, \Delta_I^{\mathrm{ex}})$.

Proof. The fact that $(\mathcal{A}, \Delta)$ is not graded simple, implies that there is a nontrivial proper graded ideal I of $(\mathcal{A}, \Delta)$. The ideal $J_1 = I + \varphi(I)$ is a nontrivial ideal of $(\mathcal{A}, \varphi)$. Thus, the fact that $(\mathcal{A}, \varphi)$ is simple implies that $I + \varphi(I) = \mathcal{A}$. Moreover, $J_2 = I\cap\varphi(I)$ is also an ideal of $(\mathcal{A}, \varphi)$. Since J_2 is a subspace of I, it is not $\mathcal{A}$. Therefore, since $(\mathcal{A}, \varphi)$ is simple, $J_2 = 0$. Hence, $\mathcal{A} = I\oplus\varphi(I)$. Denote by $\Delta_I\colon I = \bigoplus_{g\in G} I_g$ the $(\mathbb{Z}\times G)$-grading of I given by restriction, i.e., $I_g = \mathcal{A}_g\cap I$ for all $g\in G$. Then, there is a graded isomorphism $(\mathcal{A}, \varphi, \Delta)\to(I\oplus I^{op}, \mathrm{ex}, \Delta_I^{\mathrm{ex}})$ given by $x + \varphi(y)\mapsto (x, y)$ for every $x, y\in I$. □

The previous Lemma implies that we just need to classify up to isomorphism, the graded algebras of the form $(I\oplus I^{op}, \mathrm{ex}, \Delta_I^{\mathrm{ex}})$ where I is a simple associative algebra.

Definition 5.2.3. (1) Let G be an abelian group, and let $\mathcal{D}$ be a G-graded-division algebra with support T, let k be a positive integer and let $\gamma = (g_1, ..., g_k)\in G^k$. We denote by $\Delta(G, D, \gamma)$, the G-grading on $\mathcal{M}_n(\mathcal{D})$ given by $\deg(dE_{i,j}) = g_i(\deg d)g_j^{-1}$ for every homogeneous $d\in\mathcal{D}$.

(2) Let G be a group, $\mathcal{D}$ a $(\mathbb{Z}\times G)$-graded division algebra with support T (notice that since T is a finite group, $T\subseteq\{0\}\times G$), let $g_1, ..., g_m$ be elements of the group G and let k_0, k_1 be possitive integers such that $k_0 + k_1 = m$. Denote $\gamma_0 = (g_1, ..., g_{k_0})$ and $\gamma_1 = (g_{k_0+1}, ..., g_{k_0+k_1})$. We denote by $\Delta(G, \mathcal{D}, \gamma_0, \gamma_1)$, the $\mathbb{Z}\times G$-grading on $\mathcal{M}_m(\mathcal{D})$ given by $\deg(dE_{i,j}) = (\delta_i - \delta_j, g_i t g_j^{-1})$, for every homogeneous $d\in\mathcal{D}$ of degree $(0, t)$ and where $\delta_k = 0$ if $k\le k_0$ and $\delta_k = 1$ otherwise. We denote $\mathcal{M}(G, \mathcal{D}, \gamma_0, \gamma_1) = (\mathcal{M}_m(\mathcal{D}), \Delta(G, \mathcal{D}, \gamma_0, \gamma_1))$.

(3) With the previous notation, we denote $\mathcal{M}(G,\mathcal{D},\gamma_0,\gamma_1)^{\text{ex}} = (\mathcal{M}_m(\mathcal{D}) \oplus \mathcal{M}_m(\mathcal{D})^{op}, \text{ex}, \Delta(G,\mathcal{D},\gamma_0,\gamma_1)^{\text{ex}})$

(4) Let $\gamma = (g_1,...,g_n) \in G^n$ be an n-tuple consisting on n elements of G which are different modulo T. Let $\kappa = (\kappa_1,...,\kappa_n) \in \mathbb{Z}^n_{>0}$ and let $m = \kappa_1 + ... + \kappa_n$. We denote $\boldsymbol{\gamma}(\kappa,\gamma) = (h_1,...,h_m) \in G^m$ to be a m-tuple of elements of G such that $h_i = g_j$ if and only if $\kappa_1 + ... + \kappa_{j-1} < i \le \kappa_1 + ... + \kappa_j$ for every $i \in \{1,...,m\}$ and $j \in \{1,...,n\}$.

(5) for $\gamma_0 = (g_1,...,g_n) \in G^n$, $\gamma_1 = (g'_1,...,g'_m) \in G^m$ an n-tuple and an m-tuple consisting of distinct elements of G módulo T, and $\kappa_0 \in \mathbb{Z}^n_{>0}$, $\kappa_1 \in \mathbb{Z}^m_{>0}$, we denote

$$\begin{aligned}
&\Delta(G,\mathcal{D},\kappa_1,\gamma_1) = \Delta(G,\mathcal{D},\boldsymbol{\gamma}(\kappa_1,\gamma_1)),\\
&\Delta(G,\mathcal{D},\kappa_0,\kappa_1,\gamma_0,\gamma_1) = \Delta(G,\mathcal{D},\boldsymbol{\gamma}(\kappa_0,\gamma_0),\boldsymbol{\gamma}(\kappa_1,\gamma_1)),\\
&\mathcal{M}(G,\mathcal{D},\kappa_0,\kappa_1,\gamma_0,\gamma_1) = \mathcal{M}(G,\mathcal{D},\boldsymbol{\gamma}(\kappa_0,\gamma_0),\boldsymbol{\gamma}(\kappa_1,\gamma_1)) \text{ and}\\
&\mathcal{M}(G,\mathcal{D},\kappa_0,\kappa_1,\gamma_0,\gamma_1)^{\text{ex}} = \mathcal{M}(G,\mathcal{D},\boldsymbol{\gamma}(\kappa_0,\gamma_0),\boldsymbol{\gamma}(\kappa_1,\gamma_1))^{\text{ex}}.
\end{aligned}$$

For an n-tuple $\gamma = (g_1,...,g_n) \in G^n$ we denote by $\Xi(\gamma)$ the multiset $\{g_1T,...,g_nT\}$ with multiplicity for each g_iT, the number of entries of g_i modulo T in γ. In case we have an n-tuple $\gamma = (g_1,...,g_n) \in G^k$ consisting on distinct elements and $\kappa \in \mathbb{Z}^n_{\ge 0}$, we denote $\Xi(\kappa,\gamma) = \Xi(\boldsymbol{\gamma}(\kappa,\gamma))$.

Remark 5.2.4. For $\gamma_0 = (g_1,...,g_n) \in G^n$, $\gamma_1 = (g_{n+1},...,g_{n+m}) \in G^m$ $\mathcal{M}(G,\mathcal{D},\gamma_0,\gamma_1)$ is a graded algebra isomorphic to $\text{End}_{\mathcal{D}}(V)$ where V has a grading given by a homogeneous basis $v_1,...,v_m$, where the degree of v_i is (δ_i, g_i) with $\delta_i = 0$ if $i \in \{1,...,n\}$ and $\delta_i = 1$ if $i \in \{n+1,...,m\}$.

Lemma 5.2.5. *Let $\mathcal{A}$ be a simple associative algebra and let Δ be a $(\mathbb{Z}\times G)$-grading with support contained in $\{-1,0,1\} \times G$. Then, $(\mathcal{A},\Delta)$ is isomorphic to $\mathcal{M}(G,\mathcal{D},\gamma_0,\gamma_1)$ for some $(\mathbb{Z}\times G)$-graded-division simple algebra $\mathcal{D}$, $\gamma_0 = (g_1,...,g_{k_0}) \in G^{k_0}$ and $\gamma_1 = (g_{k_0+1},...,g_{k_0+k_1}) \in G^{k_1}$ and with $k_0 \ne 0$. Moreover, the graded algebras $\mathcal{M}(G,\mathcal{D},\gamma_0,\gamma_1)$ and $\mathcal{M}(G,\mathcal{D}',\gamma'_0,\gamma'_1)$ are isomorphic if and only if $\mathcal{D}$ is isomorphic to $\mathcal{D}'$ as graded algebras and there is an element $g \in G$ such that $\Xi(\gamma_i) = g\Xi(\gamma'_i)$ for $i \in \{0,1\}$.*

Proof. In [HSK19] something similar has been done but in the case of algebraically closed fields.

Due to [EK13, Theorem 2.10.], $(\mathcal{A},\Delta)$ is isomorphic to $\text{End}_{\mathcal{D}}(V)$ for some graded division algebra $\mathcal{D}$ and some graded right $\mathcal{D}$-module V. Since $T = \text{Supp}(\mathcal{D}) \subseteq \{-1,0,1\} \times G$ is a finite group, then $T \subseteq \{0\} \times G$. Moreover, since $\text{Supp}(\Delta) \subseteq \{-1,0,1\} \times G$, there is an integer i such that $\text{Supp}\, V \subseteq$

$\{i, i+1\} \times G$. Since we can do a shift to the grading of V (i.e., multiplying each the degree of each group element by an element $(j, g) \in \mathbb{Z} \times G$), we can assume that $i = 0$. Moreover, we can assume that there is at least one element of V with degree on $\{0\} \times G$.

Taking an homogeneous basis $v_1, ..., v_{k_0+k_1}$ of V of degrees $\deg(v_i) = (\delta_i, g_i)$, with $\delta_i = 0$ if $i \leq k_0$ and $\delta_i = 1$ otherwise, we see that $(\mathcal{A}, \Delta)$ is isomorphic to $\mathcal{M}(G, \mathcal{D}, \gamma_0, \gamma_1)$ where $\gamma_0 = (g_1, ..., g_{k_0}) \in G^{k_0}$ and $\gamma_1 = (g_{k_0+1}, ..., g_{k_0+k_1}) \in G^{k_1}$ and $k_0 \neq 0$. Since $\mathcal{A}$ is simple, $\mathcal{D}$ is simple.

The second part directly follows from [EK13, Corollary 2.12]. □

Corollary 5.2.6. *Let $\mathcal{A}$ be a simple associative algebra and let Δ be a $(\mathbb{Z} \times G)$-grading with support in $\{-1, 0, 1\} \times G$. Then, $(\mathcal{A}, \Delta)$ is isomorphic to the graded algebra $\mathcal{M}(G, \mathcal{D}, \kappa_0, \kappa_1, \gamma_0, \gamma_1)$ for some $(\mathbb{Z} \times G)$-graded-division simple algebra $\mathcal{D}$, $\gamma_0 = (g_1, ..., g_{k_0}) \in G^{k_0}$ and $\gamma_1 = (g_{k_0+1}, ..., g_{k_0+k_1}) \in G^{k_1}$ consisting on distinct elements modulo $T = \operatorname{Supp} \mathcal{D}$, $\kappa_0 \in \mathbb{Z}_{>0}^{k_0}$, $\kappa_1 \in \mathbb{Z}_{>0}^{k_1}$ and with $k_0 \neq 0$. Moreover, the graded algebras $\mathcal{M}(G, \mathcal{D}, \kappa_0, \kappa_1, \gamma_0, \gamma_1)$ and $\mathcal{M}(G, \mathcal{D}', \kappa_0', \kappa_1', \gamma_0', \gamma_1')$ are isomorphic if and only if $\mathcal{D}$ is isomorphic to $\mathcal{D}'$ as graded algebras and there is an element $g \in G$ such that $\Xi(\kappa_i, \gamma_i) = g\Xi(\kappa_i', \gamma_i')$.*

Remark 5.2.7. In Lemma 5.2.5 and Corollary 5.2.6, if the algebra satisfies $(T3)$, $k_1 \neq 0$.

Definition 5.2.8. Let I be an algebra and $\Delta \colon I = \bigoplus_{(i,g) \in \mathbb{Z} \times G} I_{(i,g)}$. We define the $(\mathbb{Z} \times G)$-grading $\Delta^{op} \colon I^{op} = \bigoplus_{(i,g) \in \mathbb{Z} \times G} I^{op}_{(i,g)}$ on the algebra I^{op} by $I^{op}_{(i,g)} = I_{(-i,g)}$ for every $(i, g) \in \mathbb{Z} \times G$.

Lemma 5.2.9. *Let I and J be simple associative algebras. Let Δ_I and Δ_J be $(\mathbb{Z} \times G)$-gradings of I and J respectively. Then, $(I \oplus I^{op}, \mathrm{ex}, \Delta_I^{\mathrm{ex}})$ is isomorphic to $(J \oplus J^{op}, \mathrm{ex}, \Delta_J^{\mathrm{ex}})$ if and only if (I, Δ_I) is isomorphic to either (J, Δ_J) or (J^{op}, Δ_J^{op}).*

Proof. Clearly, if (I, Δ_I) is isomorphic to either (J, Δ_J) or (J^{op}, Δ_J^{op}), then, $(I \oplus I^{op}, \mathrm{ex}, \Delta_I^{\mathrm{ex}})$ is isomorphic to $(J \oplus J^{op}, \mathrm{ex}, \Delta_J^{\mathrm{ex}})$.

In order to prove the converse, assume that $\psi \colon I \oplus I^{op} \to J \oplus J^{op}$ is an isomorphism preserving the involutions and the gradings. In this situation, $\psi(I)$ is a proper nontrivial ideal of $J \oplus J^{op}$. Therefore, $\psi(I)$ is either J or J^{op}. In the first case (I, Δ_I) is isomorphic to (J, Δ_J) and in the second case, it is isomorphic to (J^{op}, Δ_J^{op}) via $\Psi_{|I}$. □

Remark 5.2.10. If we have a group G and an m-tuple $\gamma = (g_1, ..., g_m) \in G^m$, we denote $\gamma^{-1} = (g_1^{-1}, ..., g_m^{-1})$. Also, if $\mathcal{M}(G, \mathcal{D}, \gamma_0, \gamma_1) = (\mathcal{M}_m(\mathcal{D}), \Delta)$, we denote $\mathcal{M}(G, \mathcal{D}, \gamma_0, \gamma_1)^{op} = (\mathcal{M}_m(\mathcal{D})^{op}, \Delta^{op})$.

Lemma 5.2.11. *Let $\mathcal{D}$ be a $\mathbb{Z}\times G$-graded-division algebra. Let $\gamma_0 = (g_1, ..., g_{k_0})$ and $\gamma_1 = (g_{k_0+1}, ..., g_{k_0+k_1})$. Then $\mathcal{M}(G, \mathcal{D}, \gamma_0, \gamma_1)^{op}$, as defined in the previous paragraph, is isomorphic to the graded algebra $\mathcal{M}(G, \mathcal{D}^{op}, \gamma_0^{-1}, \gamma_1^{-1})$. Concretely, if γ_0 and γ_1 consist on distinct elements modulo $T = \operatorname{Supp}\mathcal{D}$ and we have $\kappa_0 \in \mathbb{Z}_{>0}^{k_0}$ and $\kappa_1 \in \mathbb{Z}_{>0}^{k_1}$, we have that $\mathcal{M}(G, \mathcal{D}, \kappa_0, \kappa_1, \gamma_0, \gamma_1)^{op}$ is isomorphic to $\mathcal{M}(G, \mathcal{D}^{op}, \kappa_0, \kappa_1, \gamma_0^{-1}, \gamma_1^{-1})$.*

Proof. Let

$$\gamma = ((\delta_1, g_1), ..., (\delta_{k_0+k_1}, g_{k_0+k_1})).$$

With this notation, $\mathcal{M}(G, \mathcal{D}, \gamma_0, \gamma_1)$ is the same as $(\mathcal{M}_m(\mathcal{D}), \Delta(G, \mathcal{D}, \gamma))$, i.e, for any homogeneous $d \in \mathcal{D}$ of degree t, the degree of $dE_{i,j}$ is $((\delta_i - \delta_j), g_i t g_j^{-1})$. Denote

$$\hat{\gamma} = ((-\delta_1, g_1), ..., (-\delta_{k_0+k_1}, g_{k_0+k_1})).$$

Since G is an abelian group, the grading, $\Delta(G, \mathcal{D}, \gamma)$ is both a grading on $\mathcal{M}_m(\mathcal{D})$ and on $\mathcal{M}_m(\mathcal{D})^{op}$. Thus, $(\mathcal{M}_m(\mathcal{D})^{op}, \Delta(G, \mathcal{D}, \hat{\gamma}))$ is well defined, and the fact that for any homogeneous $d \in \mathcal{D}$ of degree t, the degree of $dE_{i,j}$ is $((-\delta_i + \delta_j), g_i t g_j^{-1})$, implies that $\mathcal{M}(G, \mathcal{D}, \gamma_0, \gamma_1)^{op}$ is the same as $(\mathcal{M}_m(\mathcal{D})^{op}, \Delta(G, \mathcal{D}, \hat{\gamma}))$.

Consider:

$$\psi \colon \mathcal{M}_m(\mathcal{D})^{op} \to \mathcal{M}_m(\mathcal{D}^{op})$$
$$dE_{i,j} \mapsto dE_{j,i}.$$

This is clearly an isomorphism of algebras. We just need to check that the grading on $\mathcal{M}_m(\mathcal{D}^{op})$ induced by ψ (i.e., the one given by $\deg(\psi x) = \deg(x)$) is $\Delta(G, \mathcal{D}^{op}, \hat{\gamma}^{-1})$. This is true due to the fact that G is commutative and due to the fact that for any homogeneous element $d \in \mathcal{D}_t$ for some $t \in G$,

$$\deg dE_{i,j} = (-\delta_i - (-\delta_j), g_i t g_j^{-1})$$

in $\Delta(G, \mathcal{D}, \hat{\gamma})$ and $\deg(dE_{j,i}) = (\delta_j - \delta_i, g_j^{-1} t (g_i^{-1})^{-1})$ in $\Delta(G, \mathcal{D}^{op}, \hat{\gamma}^{-1})$. Thus,

$$(\mathcal{M}_m(\mathcal{D})^{op}, \Delta(G, \mathcal{D}, \hat{\gamma})) \cong (\mathcal{M}_m(\mathcal{D}^{op}), \Delta(G, \mathcal{D}^{op}, \hat{\gamma}^{-1}))$$

but since we have $\hat{\gamma}^{-1} = ((\delta_1, g_1^{-1}), ..., (\delta_{k_0+k_1}, g_{k_0+k_1}^{-1}))$, this is isomorphic to the algebra $\mathcal{M}(G, \mathcal{D}^{op}, \gamma_0^{-1}, \gamma_1^{-1})$. □

We can summarize all these results in the following theorem:

Theorem 5.2.12. *Let $(\mathcal{A}, \varphi, \Delta)$ be an Ω-algebra satisfying $(T1) - (T4)$ and such that $(\mathcal{A}, \Delta)$ is not graded simple. Then, there is a $(\mathbb{Z} \times G)$-graded-division simple algebra $\mathcal{D}$, $k_0, k_1 > 0$, $\gamma_0 \in G^{k_0}$ and $\gamma_1 \in G^{k_1}$ consisting on* *distinct elements modulo* $T = \operatorname{Supp}\mathcal{D}$ *and* $\kappa_0 \in \mathbb{Z}_{>0}^{k_0}$, *and* $\kappa_1 \in \mathbb{Z}_{>0}^{k_1}$ *such that* $(\mathcal{A}, \varphi, \Gamma)$ *is isomorphic to* $\mathcal{M}(G, \mathcal{D}, \kappa_0, \kappa_1, \gamma_0, \gamma_1)^{ex}$.

Moreover, $\mathcal{M}(G, \mathcal{D}, \kappa_0, \kappa_1, \gamma_0, \gamma_1)^{ex}$ *is isomorphic to* $\mathcal{M}(G, \mathcal{D}', \kappa_0', \kappa_1', \gamma_0', \gamma_1')^{ex}$ *if either:*

(1) $\mathcal{D}$ *is isomorphic to* $\mathcal{D}'$ *as graded algebras and there is* $g \in G$ *such that* $\Xi(\kappa_0, \gamma_0) = g\Xi(\kappa_0', \gamma_0')$ *and* $\Xi(\kappa_1, \gamma_1) = g\Xi(\kappa_1', \gamma_1')$.

(2) $\mathcal{D}$ *is isomorphic to* $\mathcal{D}'^{op}$ *as graded algebras and there is* $g \in G$ *such that* $\Xi(\kappa_0, \gamma_0) = g\Xi(\kappa_0', \gamma_0'^{-1})$ *and* $\Xi(\kappa_1, \gamma_1) = g\Xi(\kappa_1', \gamma_1'^{-1})$.

Proof. This is the result of combining Lemmas 5.2.2, 5.2.5, 5.2.9 and 5.2.11 and Remark 5.2.7. □

Case 2: $(\mathcal{A}, \Delta)$ is graded simple

In this case, due to [EK13, Theorem 2.6], there is a $(\mathbb{Z} \times G)$-graded-division algebra $\mathcal{D}$ and a $(\mathbb{Z}\times G)$-graded right $\mathcal{D}$-module V such that $(\mathcal{A}, \Delta)$ is isomorphic to $\operatorname{End}_{\mathcal{D}}(V)$. As in the previous case, we can assume that the support of V is contained in $\{0, 1\} \times G$.

Remark 5.2.13. Given an abelian group G, two G-graded-division algebras $\mathcal{D}$ and $\mathcal{D}'$, a G-graded right $\mathcal{D}$-module V, a G-graded right $\mathcal{D}'$-module V' and $g \in G$, given an isomorphism of graded algebras $\psi_0 \colon \mathcal{D} \to \mathcal{D}'$ and an isomorphism of graded modules $\psi_1 \colon V \to V'^{[g]}$ such that $\psi_1(wd) = \psi_1(w)\psi_0(d)$ for all $w \in V$ and $d \in \mathcal{D}$, there is a unique homomorphism of graded algebras $\psi\colon \operatorname{End}_{\mathcal{D}}(V) \to \operatorname{End}_{\mathcal{D}'}(V')$ such that $\psi_1(rw) = \psi(r)\psi_1(w)$ for all $r \in \operatorname{End}_{\mathcal{D}}(V)$ and $w \in V$. Moreover, all isomorphisms between these two graded algebras can be determined by a choice of a pair (ψ_0, ψ_1) [EK13, Theorem 2.10]. In this situation, we denote $\psi = (\psi_0, \psi_1)$.

Remark 5.2.14. Sometimes, if we have a group G and G-graded vector space (V, Γ), we just denote it by V, and if we have another group H and a group homomorphism $\alpha\colon G \to H$, we denote ${}^{\alpha}V = (V, {}^{\alpha}\Gamma)$.

Denote by $\tau\colon \mathbb{Z} \times G \to \mathbb{Z}_2 \times G$ the morphism given by $\tau(i, g) = (\bar{i}, g)$, where $\bar{i}$ denotes the class of i modulo 2, then ${}^{\tau}\operatorname{End}_{\mathcal{D}}(V)$ is the same as $\operatorname{End}_{{}^{\tau}\mathcal{D}}({}^{\tau}V)$. Since $T = \operatorname{Supp}\mathcal{D}$ is a finite group, it is contained in $\{0\} \times G$. Hence, it can be identified with a subgroup of G in a natural way. By doing so, we can just write $\mathcal{D}$ instead of ${}^{\tau}\mathcal{D}$.

In case φ is an involution of the algebra $\operatorname{End}_{\mathcal{D}}(V)$ such that $\deg\varphi(x) = (i, g)$ whenever $\deg(x) = (-i, g)$, then φ is an involution of the graded algebra ${}^{\tau}\operatorname{End}_{\mathcal{D}}(V)$.

Definition 5.2.15. Given a group G, a G-graded division algebra $\mathcal{D}$, a G-graded right $\mathcal{D}$ module V, an involution φ_0 of the graded algebra $\mathcal{D}$ and a nondegenerate φ_0-sesquilinear form $B\colon V \times V \to \mathcal{D}$, we denote by (φ_0, B) the involution φ of $\mathrm{End}_{\mathcal{D}}(V)$ which satisfies $B(rv, w) = B(v, \varphi(r)w)$ for every $r \in \mathrm{End}_{\mathcal{D}}(V)$, and $v, w \in V$.

Definition 5.2.16. Let $\mathcal{D}$ be a G-graded-division algebra and V a G-graded right $\mathcal{D}$-module. Let φ_0 be an involution of $\mathcal{D}$ and $B\colon V \times V \to \mathcal{D}$ a nondegenerate hermitian or skew-hermitian φ_0-sesquilinear form. We denote by $\mathrm{End}(G, \mathcal{D}, V, \varphi_0, B)$ the Ω-algebra $(\mathrm{End}_{\mathcal{D}}(V), (\varphi_0, B), \Delta)$, where Δ is the grading on $\mathrm{End}_{\mathcal{D}}(V)$ induced by the gradings on $\mathcal{D}$ and V.

Proposition 5.2.17. *Let $(\mathcal{A}, \varphi, \Delta)$ be a triple satisfying $(T1)-(T4)$ such that $(\mathcal{A}, \Delta)$ is graded simple. Then, there exist a $(\mathbb{Z} \times G)$-graded-division algebra $\mathcal{D}$, a graded right $\mathcal{D}$-module V with support in $\{0, 1\} \times G$, an involution φ_0 of the graded algebra $\mathcal{D}$, such that $(\mathcal{D}, \varphi_0)$ is a simple algebra with involution and a nondegenerate hermitian or skew-hermitian φ_0-sesquilinear form $B\colon V \times V \to \mathcal{D}$ such that the induced form $B\colon {}^{\tau}V \times {}^{\tau}V \to \mathcal{D}$ is graded of degree $(\overline{0}, g)$ for some $g \in G$, such that $(\mathcal{A}, \varphi, \Delta)$ is isomorphic to $\mathrm{End}(\mathbb{Z} \times G, \mathcal{D}, V, \varphi_0, B)$. Any such other pair (φ_0', B') inducing the same involution on $\mathrm{End}_{\mathcal{D}}(V)$ is of the form $(\mathrm{Int}(d) \circ \varphi_0, dB)$ where d is a homogeneous element such that $\varphi_0(d) = \pm d$ and $\mathrm{Int}(d)(x) = dxd^{-1}$ for all $x \in \mathcal{D}$.*

In addition, for every 5-tuple $(G, \mathcal{D}, V, \varphi_0, B)$ consisting on a group G, a $(\mathbb{Z} \times G)$-graded-division algebra $\mathcal{D}$, a graded right $\mathcal{D}$-module V such that $\mathrm{Supp}\,{}^{\pi_1}V = \{0, 1\} \times G$, an involution φ_0 of the graded algebra $\mathcal{D}$, such that $(\mathcal{D}, \varphi_0)$ is a simple algebra with involution and a nondegenerate hermitian or skew-hermitian φ_0-sesquilinear form $B\colon V \times V \to \mathcal{D}$ such that the induced form $B\colon {}^{\tau}V \times {}^{\tau}V \to \mathcal{D}$ is graded of degree $(\overline{0}, g)$ for some $g \in G$, the graded algebra with involution $\mathrm{End}(\mathbb{Z} \times G, \mathcal{D}, V, \varphi_0, B)$ satisfies $(T1) - (T4)$.

Proof. As it was mentioned before, $(\mathcal{A}, \Delta)$ is isomorphic to $\mathrm{End}_{\mathcal{D}}(V)$ for some $(\mathbb{Z} \times G)$-graded-division algebra and a $(\mathbb{Z} \times G)$-graded right $\mathcal{D}$-module V with support in $\{0, 1\}$. $(T3)$ implies $\mathrm{Supp}\,{}^{\pi_1}V = \{0, 1\}$. Using [EKR22, Theorem 3.1.] and the fact that φ is an involution of ${}^{\tau}\,\mathrm{End}_{\mathcal{D}}(V) = \mathrm{End}_{\mathcal{D}}({}^{\tau}V)$, we can show that there is an involution φ_0 of the graded algebra $\mathcal{D}$ and a graded nondegenerated hermitian or skew-hermitian φ_0-sesquilinear form $B\colon {}^{\tau}V \times {}^{\tau} V \to \mathcal{D}$ such that $B(rv, w) = B(v, \varphi(r)w)$ for all $r \in \mathrm{End}_{\mathcal{D}}(V)$ and $v, w \in V$.

Since B is graded, there is an element $g \in G$ and $\bar{i} \in \mathbb{Z}_2$ such that B has degree $(\bar{i}, g)$. In order to prove that the degree of B is $(\overline{0}, g)$, we will assume that $\bar{i} = \overline{1}$ and we will arrive to a contradiction.

Take an element $h_1 \in G$ such that ${}^{\tau}V_{(\overline{0},h_1)} \neq 0$ and take $v \in V_{(\overline{0},h_1)}$. Clearly $v \in V_{(0,h_1)}$. Since B is nondegenerate, $\operatorname{Supp} \mathcal{D} \subseteq \{\overline{0}\} \times G$ and it has degree $(\overline{1}, g)$, then there is $h_2 \in G$ and $w \in V_{(\overline{1},h_2)}$ such that $B(v, w) \neq 0$. Since $\operatorname{Supp} V \subseteq \{0, 1\} \times G$, $w \in V_{(1,h_2)}$. Let $v, w, u_1, ..., u_k$ be a homogeneous basis of V as a $\mathcal{D}$ module and let $r \in \operatorname{End}_{\mathcal{D}}(V)$ be the morphism such that $r(w) = v$, $r(v) = 0$ and $r(u_i) = 0$ for all $i \in \{1, ..., k\}$. Then r is a homomorphism of degree $(-1, h_2^{-1}h_1)$. Hence, $\varphi(r)$ is a homomorphism of degree $(1, h_2^{-1}h_1)$, which implies that $\varphi(r)(w) = 0$. Thus:

$$0 \neq B(v, w) = B(r(w), w) = B(w, \varphi(r)(w)) = 0$$

which leads to a contradiction.

If we have another pair (φ_0', B') inducing the same involution on $\operatorname{End}_{\mathcal{D}}(V)$, due to [EKR22, Theorem 3.1] we get that there is a homogeneous element $d \in \mathcal{D}$ such that $\varphi_0(d) = \pm d$ and such that $(\varphi_0', B') = (\operatorname{Int}(d) \circ \varphi_0, dB)$.

In order to prove the second part of the proposition, we take $\mathcal{D}$, φ_0, V and B as in the statement. The fact that $\operatorname{End}_{\mathcal{D}}(V)$ is a simple algebra with involution implies that $(\operatorname{End}_{\mathcal{D}}(V), \varphi)$ is a simple associative algebra with involution, the fact that $\mathcal{D}$ and V are $(\mathbb{Z} \times G)$-graded implies $(T2)$ and the fact that $\operatorname{Supp} \mathcal{D} \subseteq \{0\} \times G$ and $\operatorname{Supp} {}^{\pi_1}V = \{0, 1\}$ implies $(T3)$. Therefore, we just need to prove $(T4)$.

We denote by φ the involution (φ_0, B). We need to prove that $\deg(\varphi(r)) = (i, h)$ whenever $\deg(r) = (-i, h)$. Due to the fact that φ is an involution of ${}^{\tau}\operatorname{End}_{\mathcal{D}}(V)$, it implies that $\varphi(\operatorname{End}_{\mathcal{D}}(V)_{(0,h)}) = \operatorname{End}_{\mathcal{D}}(V)_{(0,h)}$ for all $h \in G$. Now, if $r \in \operatorname{End}_{\mathcal{D}}(V)_{(1,h)}$ for some $h \in G$, due to the fact that φ preserves the grading in ${}^{\tau}\operatorname{End}_{\mathcal{D}}(V)$, there is $r_1 \in \operatorname{End}_{\mathcal{D}}(V)_{(1,h)}$ and $r_{-1} \in \operatorname{End}_{\mathcal{D}}(V)_{(-1,h)}$ such that $\varphi(r) = r_1 + r_{-1}$. We will prove that $r_1 = 0$ by contradiction.

Assume that $r_1 \neq 0$. Then, there is $v \in V_{(0,h_1)}$ for some $h_1 \in G$ such that $0 \neq w = r_1(v) \in V$. Moreover $r_{-1}(v) = 0$. Therefore, $\varphi(r)(v) = w$. Since B is a graded form of degree $(\overline{0}, g)$ of ${}^{\tau}V$, there is $h_2 \in G$ and $u \in V_{(1,h_2)}$ such that $B(u, w) \neq 0$. Using the grading, since r has degree $(1, h)$, then $r(u) = 0$. Thus:

$$0 \neq B(u, w) = B(u, \varphi(r)(v)) = B(ru, v) = 0$$

which leads to a contradiction with the fact that $r_1 \neq 0$. Therefore, $\deg(\varphi(r)) = (-1, h)$ whenever $\deg(r) = (1, h)$. In case $\deg(r) = (-1, h)$ for some $h \in G$, the proof is analogous. □

5.2.2 Associative algebras: the algebraically closed setting

In this section we will assume that $\mathbb{F}$ is an algebraically closed field and G a commutative group. Our first main results will be the classification up to isomrphisms of Graded-division algebras with involution whose underlying algebra with involution is simple. This is obtained in Proposition 5.2.25 in the case where the underlying algebra is simple and in Theorem 5.2.31 in case the underlying algebra is not simple. Then, we will use it in order to find a classification up to isomorphism of algebras satisfying the properties $(T1)-(T4)$ from Section 5.2.1 (see Theorem 5.2.37).

Graded-division algebras

We will start by classifying the G-graded-division simple algebras with involution and the G-graded-division simple algebras. In order to do so, we will recall the classification of G-graded-division simple algebras.

Let T be a finite subgroup of G such that char($\mathbb{F}$) doesn't divide its order and let $\beta\colon T\times T\to\mathbb{F}$ be a nondegenerate alternating bicharacter on T, i.e., a map which is multiplicative in each variable and such that $\beta(t,t)=1$. Since T is alternating and nondegenerate, we can decompose $T=H_1'\times H_1''\times ...\times H_r'\times H_r''$ for some $r\in\mathbb{Z}_{>0}$ in such way that $H_i'\times H_i''$ and $H_j'\times H_j''$ are orthogonal if $i\neq j$, H_i' and H_i'' are in duality by β for every $i\in\{1,...,r\}$ and they are cyclic groups of order l_i. Choose generators a_i,b_i of H_i' and H_i'', denote $\epsilon_i=\beta(a_i,b_i)$. Then we can endow $\mathcal{M}_{l_1}(\mathbb{F})\otimes_{\mathbb{F}}...\otimes_{\mathbb{F}}\mathcal{M}_{l_r}(\mathbb{F})$ with a G-grading given by:

$$\deg(I\otimes...\otimes I\otimes X_i\otimes I\otimes...\otimes I)=a_i$$

and

$$\deg(I\otimes...\otimes I\otimes Y_i\otimes I\otimes...\otimes I)=b_i$$

where:

$$X_i=\sum_{k=1}^{l_i}\epsilon_i^{k-1}E_{k,k},\ \text{ and }\ Y_i=\sum_{k=1}^{l_i}E_{k,k+1}$$

with the indices taken modulo l_i. Note that $Y_iX_i=\epsilon_iX_iY_i$. This is a grading on $\mathcal{M}_{l_1}(\mathbb{F})\otimes_{\mathbb{F}}...\otimes_{\mathbb{F}}\mathcal{M}_{l_r}(\mathbb{F})$ which makes it a graded-division algebra. This grading induces naturally a grading on $\mathcal{M}_{l_1\cdots l_r}(\mathbb{F})$ via the Kronecker product.

Definition 5.2.18. We call the previous graded division algebra a **standard realization** of T and β.

Remark 5.2.19. We will usually denote $X_{a_i} = I \otimes ... \otimes I \otimes X_i \otimes I \otimes ... \otimes I$, $X_{b_i} = I \otimes ... \otimes I \otimes Y_i \otimes I \otimes ... \otimes I$ and $X_{(a_1^{i_1}, b_1^{j_1}, ..., a_r^{i_r}, b_r^{j_r})} = X_{a_1}^{i_1} X_{b_1}^{b_1} \cdots X_{a_r}^{i_r} X_{b_r}^{b_r}$ for every $i \in \{1, ..., r\}$ and $i_1, j_1, ..., i_r, j_r \in \mathbb{Z}$. We should note that $X_{a_i}^{l_i} = X_{b_i}^{l_i} = 1$ for each $i \in \{1, ..., r\}$. Finally, we will note that for $t_1, t_2 \in T$, we have that $X_{t_1} X_{t_2} = \beta(t_1, t_2) X_{t_2} X_{t_1}$.

The standard realizations of T and β are isomorphic. Therefore, we will denote a realization as $\mathcal{D}(T, \beta)$, which is a notation well defined up to isomorphism.

This discussion is in [EK13, Chapter 2]. The following Theorem classifies all the finite dimensional G-graded-division simple associative algebras.

Theorem 5.2.20 ([EK13])**.** *Let T be a finite abelian group and let $\mathbb{F}$ be an algebraically closed field. There exist a grading on the matrix algebra $\mathcal{M}_n(\mathbb{F})$ with support T making $\mathcal{M}_n(\mathbb{F})$ a graded-division algebra if and only if* $\operatorname{char}(\mathbb{F})$ *does not divide n and $T \cong \mathbb{Z}_{l_1}^2 \times ... \times \mathbb{Z}_{l_r}^2$ with $l_1 \cdots l_r = n$. The isomorphism classes of such gradings are in one to one correspondence with the isomorphism classes of nondegenerate alternating bicharacters $\beta \colon T \times T \to \mathbb{F}$.*

The following result will be interesting:

Lemma 5.2.21. *Let $T \cong \mathbb{Z}_{l_1}^2 \times ... \times \mathbb{Z}_{l_r}^2$ be a subgroup of $\mathbb{Z} \times G$ with* $\operatorname{char}(\mathbb{F})$ *not dividing $l_1 \cdots l_r = n$, and a nondegenerate alternating bicharacter $\beta \colon T \times T \to \mathbb{F}$, the $(\mathbb{Z} \times G)$-graded algebra $\mathcal{D}(T, \beta)$ satisfies $\mathcal{D}(T, \beta)^{op} \cong \mathcal{D}(T, \beta \circ \operatorname{ex})$ where* $\operatorname{ex} \colon T \times T \to T \times T$ *is the map given by* $\operatorname{ex}(t_1, t_2) = (t_2, t_1)$.

Proof. This is due to the fact that $T \subseteq \{0\} \times G$ and the fact that due to Remark 5.2.19, $X_{t_1} X_{t_2} = \beta(t_1, t_2) X_{t_2} X_{t_1}$ for every $t_1, t_2 \in T$. □

Remark 5.2.22. Given a nondegenerate alternating bicharacter $\beta \colon \mathbb{Z}_2^n \times \mathbb{Z}_2^n \to \mathbb{F}$, the fact that for every $u, v \in \mathbb{Z}_2^n$, $\beta(u, v) = \beta(v, u)^{-1}$ and the fact that its image is in $\{\pm 1\}$, implies that $\beta = \beta \circ \operatorname{ex}$.

Now, we will start by classifying up to isomorphism the G-graded-simple graded-division simple algebras with involution. We will start with the case where the algebra is simple.

Lemma 5.2.23. *Let $(\mathcal{D}, \varphi_0)$ be a G-graded-simple graded-division simple algebras with involution. Then, $T =$* $\operatorname{Supp} \mathcal{D}$ *is an elementary abelian 2-group. Moreover, every homogeneous component is one dimensional.*

Proof. $\mathcal{D}_e$ is a division algebra. Therefore, the fact that $\mathbb{F}$ is algebraically closed, implies that $\mathcal{D}_e$ is one dimensional. Hence, the support is an elementary abelian 2 group due to [EKR22, Lemma 3.5.]. Moreover the arguments at the beginning of [EK13, 2.2.], show that each homogeneous component of $\mathcal{D}$ has dimension 1. □

Definition 5.2.24. Let T be an elementay abelian 2-subgroup of $\mathbb{Z} \times G$, $\beta\colon T \times T \to \mathbb{F}$ a nondegenerate alternating bicharacter and $\tau\colon T \to \mathbb{F}$ a quadratic form on T whose polar form is β, i.e., such that $\beta(t_1, t_2) = \tau(t_1 t_2)\tau(t_1)\tau(t_2)$. We denote by $\mathcal{D}_{\text{inv}}(T, \beta, \tau)$ the graded algebra $\mathcal{D}(T, \beta)$ with the involution given by

$$d \mapsto \tau(t)d$$

for all $t \in T$ and $d \in \mathcal{D}_t$, which is an involution as shown in [EK13, (2.13)].

On the algebra $\mathcal{D}(T, \beta)$, the transpose is an involution of the graded algebra (see [EK13, Proposition 2.51]). We denote it as $\mathcal{D}_{\text{inv}}(T, \beta)$ and denote also by β the quadratic from for which the involution is given by

$$d \mapsto \beta(t)d$$

for every $t \in T$ and $d \in \mathcal{D}_t$.

Proposition 5.2.25. *T be an abelian group and $\beta\colon T \times T \to \mathbb{F}$ a nondegenerate alternating bicharacter. Then, $\mathcal{D}(T, \beta)$ admits an involution if and only if T is an elementary 2-group. Then, this algebra with involution is of the form $\mathcal{D}_{\text{inv}}(T, \beta, \tau)$ for a quadratic form $\tau\colon T \to \mathbb{F}$. Any involution of $\mathcal{D}(T, \beta)$ is of the form*

$$X \mapsto X_t^{-1} X^t X_t$$

for a uniquelly determined $t \in T$. Moreover, $\mathcal{D}_{\text{inv}}(T, \beta, \tau_1) \cong \mathcal{D}_{\text{inv}}(T, \beta, \tau_2)$ if and only if $\tau_1 = \tau_2$

Proof. The first part is due to [EK13, Proposition 2.51]. Now, it is clear that if $\tau_1 = \tau_2$, then $\mathcal{D}_{\text{inv}}(T, \beta, \tau_1) \cong \mathcal{D}_{\text{inv}}(T, \beta, \tau_2)$. Hence, denote by $\varphi_0^{(1)}$ and $\varphi_0^{(2)}$ the involutions of $\mathcal{D}_{\text{inv}}(T, \beta, \tau_1)$ and $\mathcal{D}_{\text{inv}}(T, \beta, \tau_2)$ respectively and assume that $\mathcal{D}_{\text{inv}}(T, \beta, \tau_1) \cong \mathcal{D}_{\text{inv}}(T, \beta, \tau_2)$. Then, there is an automorphism φ of $\mathcal{D}(T, \beta)$ such that

$$\varphi \circ \varphi_0^{(1)} = \varphi_0^{(2)} \circ \varphi. \tag{5.2.1}$$

Moreover, since the dimension of each homogeneous component of $\mathcal{D}(T, \beta)$ is one, there is a map $\lambda\colon T \to \mathbb{F}$ such that $\varphi(d) = \lambda(s)d$ for all $t \in T$ and d homogeneous of degree t. Hence, applying equation (5.2.1) to an element d of degree t, we get that $\tau_1(t) = \tau_2(t)$.

□

Definition 5.2.26. Let $(\mathcal{A}, \varphi, \Delta)$ be a G-graded algebra with involution and $t \in G \setminus \operatorname{Supp} \Delta$ an order 2 element. We denote by $(\mathcal{A}, \varphi, \Delta)_t^{\text{ex}}$, the G-graded algebra $(\mathcal{A} \oplus \mathcal{A}^{op}, \mathrm{ex}, \Delta_t^{ex})$, where Δ_t^{ex} is a G-grading of algebras given by $\deg(x, \varphi(x)) = \deg(x)$ and $\deg(x, -\varphi(x)) = \deg(x)t$ for all homogeneous $x \in \mathcal{A}$.

The previous grading is well defined as shown in [BG08].

Definition 5.2.27. Let T be an elementary abelian 2-group and T' be a subgroup of index 2, let $t \in T \backslash T'$ be an order 2 element and let $\beta\colon T' \times T' \to \mathbb{F}$ be an alternating bicharacter. We denote by $\beta^{[t]}\colon T \times T \to \mathbb{F}$, the bicharacter given by

$$\beta^{[t]}(ut_1^k, vt^{k_2}) = \beta(u, v)$$

for every $u, v \in T$ and $k_1, k_2 \in \{0, 1\}$. Moreover, if we denote by τ a quadratic form whose polar form is β, then, we denote by $\tau^{[t]}\colon T \to \mathbb{F}$ the map given by

$$\tau^{[t]}(ut^k) = \tau(u)(-1)^k$$

for every $k \in \{0, 1\}$.

Remark 5.2.28. If T is an elementary abelian 2-subgroup of G, $t \in G \setminus T$ an order 2 element, $\beta\colon T \times T \to \mathbb{F}$ an alternating bicharacter and $\tau\colon T \to \mathbb{F}$ a quadratic form whose polar form is β, then $\tau^{[t]}$ is a quadratic from whose polar form is $\beta^{[t]}$ due to the fact that

$$\begin{aligned}\beta^{[t]}(ut^{k_1}, vt^{k_2}) &= \tau(uv)\tau(u)\tau(v)\\ &= \tau(uv)(-1)^{k_1+k_2}\tau(u)(-1)^{k_1}\tau(v)(-1)^{k_2}\\ &= \tau^{[t]}(uv)\tau^{[t]}(u)\tau^{[t]}(v).\end{aligned}$$

Remark 5.2.29. Let T be an elementary abelian 2-group and T' be a subgroup of index 2, let $t \in T \setminus T'$ be an order 2 element and let $\beta\colon T' \times T' \to \mathbb{F}$ be an alternating bicharacter and $\tau\colon T \to \mathbb{F}$ a quadratic form whose polar form is β. The involution on $\mathcal{D}_{\mathrm{inv}}(T', \beta, \tau)_t^{\mathrm{ex}}$ is given by:

$$d \mapsto \tau^{[t]}(s)d$$

for every $s \in T$ and every homogeneous element d with degree s. Concretely, the involution on $\mathcal{D}_{\mathrm{inv}}(T', \beta)_t^{\mathrm{ex}}$ is given by $(X, Y) \mapsto (Y^t, X^t)$.

Proposition 5.2.30. *Let T be an elementary abelian 2-group or rank $2m+1$ for some m, let $t \in T$ be an order 2 element and let T_1 and T_2 be two elementary 2-subgroups of index 2 not containing t. Let $\beta_1\colon T_1 \times T_1 \to \mathbb{F}$ and $\beta_2\colon T_2 \times T_2 \to \mathbb{F}$ be two nondegenerate alternating bicharacters which are polar forms of two quadratic forms τ_1, τ_2 respectively, such that $\beta_1^{[t]} = \beta_2^{[t]}$ and $\tau_1^{[t]} = \tau_2^{[t]}$. Then, $\mathcal{D}_{\mathrm{inv}}(T_1, \beta_1, \tau_1)_t^{\mathrm{ex}} \cong \mathcal{D}_{\mathrm{inv}}(T_2, \beta_2, \tau_2)_t^{\mathrm{ex}}$.*

Proof. Denote the underlying algebra with involution in $\mathcal{D}_{\mathrm{inv}}(T_1, \beta_1, \tau_1)$ as $(\mathcal{D}, \varphi_0)$. Write

$$\mathcal{D}_{\mathrm{inv}}(T_1, \beta_1, \tau_1)_{T_2} = \bigoplus_{h \in T_2} (\mathcal{D}_{\mathrm{inv}}(T_1, \beta_1 \tau_1)^{\mathrm{ex}}_t)_h \, .$$

Then $\mathcal{D}_{\mathrm{inv}}(T_1, \beta_1, \tau_1)_{T_2}$ with the exchange involution is a G-graded-division algebra. We claim that the projection $\psi\colon \mathcal{D}_{\mathrm{inv}}(T_1, \beta_1, \tau_1)_{T_2} \to \mathcal{D}$ given by $(x, y) \mapsto x$ is an algebra isomorphism. In order to show that it is surjective, we just need to notice that since $T = T_1\langle t\rangle$, if we have a homogeneous element $x \in \mathcal{D}_{\mathrm{inv}}(T_1, \beta_1, \tau_1)$, either $(x, \varphi(x)) \in \mathcal{D}_{\mathrm{inv}}(T_1, \beta_1, \tau_1)_{T_2}$ or $(x, -\varphi(x)) \in \mathcal{D}_{\mathrm{inv}}(T_1, \beta_1, \tau_1)_{T_2}$. In order to show that it is an isomorphism we just need to notice that due to 5.2.23, $\dim \mathcal{D}_{\mathrm{inv}}(T_1, \beta_1, \tau_1)_{T_2} = |T_2| = |T_1| = \dim_{\mathbb{F}}(\mathcal{D})$ where for a finite group H, we denote by $|H|$ the cardinal of H. Then ψ induces a division grading on $\mathcal{D}$ by the group T_2, which we denote by Δ_{T_2} and an involution $\bar{\varphi}_0$ in such way that ψ is a homomorphism of graded algebras with involution.

We take a homogeneous basis $\{Y_h\}_{h \in T_2}$ of $(\mathcal{D}, \bar{\varphi}_0, \Delta_0)$ in such way that

$$Y_h = \psi(X_{ht^k}, (-1)^k \varphi_0(X_{ht^k})) = X_{ht^k},$$

where $k \in \{0, 1\}$ is such that $ht^k \in T_1$ (which exists and is unique since T_2 and T_1 have index 2 over T). Clearly $\{Y_h\}_{h \in T_2}$ is a basis as in 5.2.19. We will show that this algebra with involution is $\mathcal{D}_{\mathrm{inv}}(T_2, \beta_2, \tau_2)$. Hence, we need to show that $Y_{h_1} Y_{h_2} = \beta_2(h_1, h_2) Y_{h_2} Y_{h_1}$ and we have to show that $\bar{\varphi}_0(Y_h) = \tau_2(h) Y_h$ for all $h_1, h_2, h \in H$.

Given $h_1, h_2 \in T$, we have that there are k_1, k_2 such that $h_1 t^{k_1}, h_2 t^{k_2} \in T$. Then:

$$\begin{aligned} Y_{h_1} Y_{h_2} &= \psi(X_{h_1 t^{k_1}}, (-1)^k \varphi_0(X_{h_1 t^{k_1}}))\psi(X_{h_2 t^{k_2}}, (-1)^k \varphi_0(X_{h_2 t^{k_2}})) \\ &= \beta_1(h_1 t^{k_1}, h_2 t^{k_2}) \psi(X_{h_2 t^{k_2}}, (-1)^k_2 \varphi_0(X_{h_2 t^{k_2}}))\psi(X_{h_1 t^{k_1}}, (-1)^k \varphi_0(X_{h_1 t^{k_1}})) \\ &= \beta_1(h_1 t^{k_1}, h_2 t^{k_2}) Y_{h_2} Y_{h_1} \end{aligned}$$

but

$$\beta_1(h_1 t^{k_1}, h_2 t^{k_2}) = \beta_1^{[t]}(h_1 t^{k_1}, h_2 t^{k_2}) = \beta_2^{[t]}(h_1 t^{k_1}, h_2 t^{k_2}) = \beta_2(h_1, h_2).$$

Similarly, choosing $k \in \{0, 1\}$ such that $ht^k \in T_1$, since $(X_{ht^k}, (-1)^k \varphi_0(X_{ht^k}))$ has degree h, using Remark 5.2.29 and the fact that $(X_{ht^k}, (-1)^k \varphi_0(X_{ht^k})$ has degree h, it follows that:

$$\begin{aligned} \bar{\varphi}_0(Y_h) &= \bar{\varphi}_0(\psi(X_{ht^k}, (-1)^k \varphi_0(X_{ht^k})) \\ &= \psi(\mathrm{ex}(X_{ht^k}, (-1)^k \varphi_0(X_{ht^k})) \\ &= \tau_1(h) \psi(X_{ht^k}, (-1)^k \varphi_0(X_{ht^k}) \\ &= \tau_1(h) Y_h. \end{aligned}$$

Thus, since

$$\tau_1(h) = \tau_1^{[t]}(ht^k)(-1)^k = \tau_2^{[t]}(ht^k)(-1)^k = \tau_2(h)(-1)^k(-1)^k = \tau_2(h),$$

we have shown that $\beta = \beta_2$ and $\tau = \tau_2$. Hence, $(\mathcal{D}, \bar{\varphi}_0, \Delta_{T_2}) = \mathcal{D}_{\text{inv}}(T_2, \beta_2, \tau_2)$.

Finally, the fact that ψ is a homogeneous isomorphism, and the fact that $(1, -1)$ has degree t in $\mathcal{D}_{\text{inv}}(T_1, \beta_1, \tau_1)_t^{ex}$ implies that if x is homogeneous of degree h in $\mathcal{D}_{\text{inv}}(T_2, \beta_2, \tau_2)$, then $(x, \bar{\varphi}_0(x))$ has degree h and $(x, -\bar{\varphi}_0(x))$ has degree ht. Therefore, $\mathcal{D}_{\text{inv}}(T_1, \beta_1, \tau_1)_t^{ex} = \mathcal{D}_{\text{inv}}(T_2, \beta_2, \tau_2)_t^{ex}$.

□

Theorem 5.2.31. *Let $T \cong \mathbb{Z}_2^{2m}$ be a subgroup of G and let $t \in G \setminus T$ be an order 2 element, let $\beta\colon T \times T \to \mathbb{F}$ be a nondegenerate alternating bicharacter and $\tau\colon T \to \mathbb{F}$ a quadratic form whose polar form is β. Then $\mathcal{D}_{\text{inv}}(T, \beta, \tau)_t^{\text{ex}}$ is a graded-division simple algebra with involution such that the underlying algebra is not simple. Moreover, given a G-graded-division simple algebra with involution $(\mathcal{D}, \varphi_0, \Delta)$ with $\mathcal{D}$ not simple, there is a subgroup $T \cong \mathbb{Z}_2^{2m}$ of G, an order 2 element $t \in G \setminus T$ and a nondegenerate alternating bicharacter $\beta\colon T \times T \to \mathbb{F}$ such that $(\mathcal{D}, \varphi_0, \Delta) \cong \mathcal{D}_{\text{inv}}(T, \beta, \tau)_t^{\text{ex}}$.*

Two such algebras, $\mathcal{D}_{\text{inv}}(T, \beta, \tau)_t^{\text{ex}}$ and $\mathcal{D}_{\text{inv}}(T', \beta', \tau')_{t'}^{\text{ex}}$, are isomorphic if and only if $T\langle t\rangle = T'\langle t\rangle$, $t = t'$, $\beta^{[t]} = \beta'^{[t']}$ and $\tau^{[t]} = \tau'^{[t]}$.

Proof. Denote by φ_0 the involution in $\mathcal{D}_{\text{inv}}(T, \beta, \tau)$. Since every homogeneous component of $\mathcal{D}_{\text{inv}}(T, \beta, \tau)$ has dimension 1, every homogeneous component of $\mathcal{D}_{\text{inv}}(T, \beta, \tau)_t^{\text{ex}}$ has dimension 1. Therefore, (x, y) is homogeneous if and only if x is homogeneous in $\mathcal{D}_{\text{inv}}(T, \beta, \tau)$ and $y = \pm\varphi_0(x)$. Hence, x and y are invertible. Therefore, (x, y) is invertible, which implies that $\mathcal{D}_{\text{inv}}(T, \beta, \tau)_t^{\text{ex}}$ is a graded-division algebra. Moreover, since the underlying algebra of $\mathcal{D}_{\text{inv}}(T, \beta, \tau)$ is simple, the underlying algebra with involution of $\mathcal{D}_{\text{inv}}(T, \beta, \tau)_t^{\text{ex}}$, is simple.

Assume that $(\mathcal{D}, \varphi_0, \Delta)$ is a G-graded-division simple algebra with involution such that $\mathcal{D}$ is not simple. Then, there is an ideal I such that $(\mathcal{D}, \varphi_0) \cong (I \oplus I^{op}, \text{ex})$. Thus, we are going to identify $(\mathcal{D}, \varphi_0)$ and $(I \oplus I^{op}, \text{ex})$. Since $\mathcal{D}$ is a graded division algebra, the support Δ is a group. Therefore, we are going to assume that the grading group T is the support. The center of $\mathcal{D}$, $\mathcal{Z}(\mathcal{D}) = \mathbb{F}(1, 1) \oplus \mathbb{F}(1, -1)$ is a graded subspace. Thus, $\deg(1, 1) = e$ and $\deg(1, -1) = t$ for some $t \in T$. Taking the coarsening by the group $\overline{T} = T/\langle t\rangle$ given by :

$$\overline{\Delta}\colon \mathcal{D} = \bigoplus_{\bar{t}' \in \overline{T}} \mathcal{D}_{\bar{t}'}$$

where $\mathcal{D}_{\bar{t}'} = \mathcal{D}_{t'} \oplus \mathcal{D}_{t't}$. Since the support of Δ is a 2-group due to Lemma 5.2.23, we can identify $\overline{T}$ with a subgroup T' of T in such way that $T \cong$

$T' \times \langle t \rangle$, and in this coarsening, the homogeneous components have dimension 2. Now, in this coarsening $\mathcal{Z}(\mathcal{D})_e = \mathcal{Z}(\mathcal{D})$. Thus, if we denote $e_1 = (1,0)$ and $e_2 = (0,1)$, the subspaces $e_1\mathcal{D} = I$ and $e_2\mathcal{D} = \varphi_0(I)$ are graded, and for every $x \in I_{t'}$, $\varphi_0(x) \in \varphi(I)_{t'}$. Thus, in I, every homogeneous compontent has dimension 1, we denote the grading on I by Δ_I. Therefore, for every $x \in I_{t'}$, there is a unique $y \in I$ such that $x + \varphi(y) \in \mathcal{D}_{t'}$ and multiplying by $(1,-1)$, it also satisfies that $x - \varphi(y) \in \mathcal{D}_{t't}$. We can define an involution $\bar{\varphi}_0$ in I by sending x to y. Identifying $\mathcal{D}$ with $I \oplus I^{op}$, Δ is given by $\deg(x, \bar{\varphi}_0(x)) = \deg(x)$ and $\deg(x, -\bar{\varphi}_0(x)) = \deg(x)t$ for every homogeneous element $x \in I$. We are going to show that (I, Δ_I) is a graded-division algebra. Indeed, if x is a homogeneous element of (I, Δ_I), $(x, \bar{\varphi}_0(x))$ is a homogeneous element of $(\mathcal{D}, \Delta)$, which is a graded-division algebra, and this implies that x is invertible, from which follows that (I, Δ_I) is a graded division algebra. Hence, $(I, \bar{\varphi}_0, \Delta_I) \cong \mathcal{D}_{\text{inv}}(T', \beta, \tau)$ for some nondegenerate alternating bicharacter β and a quadratic form τ whose polar form is β. Therefore, $(\mathcal{D}, \varphi_0, \Delta)$ is isomorphic to $\mathcal{D}_{\text{inv}}(T, \beta, \tau)_t^{ex}$.

For the isomorphism problem, if $T\langle t \rangle = T'\langle t \rangle$, $t = t'$, $\beta^{[t]} = \beta'^{[t']}$ and $\tau^{[t]} = \tau'^{[t]}$, due to proposition 5.2.30, the algebras are isomorphic. Conversely, given two such algebras $\mathcal{D}_{\text{inv}}(T, \beta, \tau)_t^{\text{ex}}$ and $\mathcal{D}_{\text{inv}}(T', \beta', \tau')_{t'}^{\text{ex}}$, if they are isomorphic, since

$$\{1, t\} = \operatorname{Supp} \mathcal{Z}(\mathcal{D}_{\text{inv}}(T, \beta, \tau)_t^{\text{ex}}) = \operatorname{Supp} \mathcal{Z}(\mathcal{D}_{\text{inv}}(T', \beta', \tau')_{t'}^{\text{ex}}) = \{1, t'\},$$

it follows that $t = t'$. Moreover,

$$T\langle t \rangle = \operatorname{Supp} \mathcal{D}_{\text{inv}}(T, \beta, \tau)_t^{\text{ex}} = \operatorname{Supp} \mathcal{D}_{\text{inv}}(T', \beta', \tau')_{t'}^{\text{ex}} = T'\langle t \rangle.$$

Finally, given $h_1, h_2 \in T$, $X_{h_1}, X_{h_2} \in \mathcal{D}_{\text{inv}}(T, \beta, \tau)$ homogeneous of degree h and $k_1, k_2 \in \{0, 1\}$, if φ_0 is the involution in $\mathcal{D}_{\text{inv}}(T, \beta, \tau)$ the elements $(X_{h_1}, (-1)^{k_1}\beta(h_1)X_{h_1})$ and $(X_{h_2}, (-1)^{k_2}\beta(h_2)X_{h_2})$ have respective degrees ht^{k_1} and ht^{k_2}. The identity:

$$\begin{aligned}(X_{h_1}, (-1)^{k_1}\varphi_0(X_{h_1}))&(X_{h_2}, (-1)^{k_2}\varphi_0(X_{h_2})) \\ &= \beta(h_1, h_2)(X_{h_2}, (-1)^{k_2}\varphi_0(X_{h_2}))(X_{h_1}, (-1)^{k_1}\varphi_0(X_{h_1}))\end{aligned}$$

holds due to the fact that, as shown in Remark 5.2.22, $\beta(h_1, h_2) = \beta(h_2, h_1)$. The fact that the homogeneous components have dimension 1, implies that for any elements x, y of degree $h_1, h_2 \in \mathcal{D}_{\text{inv}}(T, \beta, \tau)_t^{\text{ex}}$, $xy = \beta^{[t]}(h_1, h_2)yx$. Therefore, it follows that $\beta^{[t]} = \beta'^{[t']}$. Finally, Remark 5.2.29 implies that $\tau^{[t]} = \tau'^{[t]}$.

□

Remark 5.2.32. In a standard realization of a 2-elementary abelian subgroup T of a group G and a nondegenerate alternate bicharacter β, for each $s \in T$ we denote X_s as usual. For an order 2 element $t \in G \setminus T$, if we denote by φ_0 the involution of $\mathcal{D}_{\mathrm{inv}}(T, \beta)$, we denote $Y_{st^k} = (X_s, (-1)^k \varphi_0(X_s))$ for every $s \in T$ and $k \in \{0, 1\}$. Taking a quadratic form τ whose polar form is β such that $\varphi_0(s) = \tau(s) X_s$ for each $s \in T$, clearly, for each $s \in T\langle t\rangle$, $\overline{Y_s} = \beta^{[t]}(s) Y_s$.

Definition 5.2.33. Let $T \cong \mathbb{Z}_2^{2m}$ be a subgroup of G and let $t \in G \setminus T$ be an order 2 element, let $\beta\colon T \times T \to \mathbb{F}$ be a nondegenerate alternating bicharacter and $\tau\colon T \to \mathbb{F}$ a quadratic form whose polar form is β. Denote $(\mathcal{D}, \varphi, \Delta) = \mathcal{D}_{\mathrm{inv}}(T, \beta, \tau)^{\mathrm{ex}}_t$. For an homogeneous element d, we denote $\mathcal{D}_{\mathrm{inv}}(T, \beta, \tau, d)^{\mathrm{ex}}_t = (\mathcal{D}, \mathrm{Int}(d) \circ \varphi, \Delta)$ where $\mathrm{Int}(d)(x) = dxd^{-1}$ for all $x \in \mathcal{D}$.

Proposition 5.2.34. *Let T_1, T_2 be two elementary 2-subgroups of G, let $t_i \in G \setminus T_i$ for $i \in \{1, 2\}$ be two order 2 elements, let $\beta_1\colon T_1 \times T_1 \to \mathbb{F}$ and $\beta_2\colon T_2 \times T_2 \to \mathbb{F}$ be two nondegenerate alternating bicharacter and τ_1, τ_2 two quadratic forms whose polar forms are β_1 and β_2. Then, there exists a homogeneous element $d \in \mathcal{D}_{\mathrm{inv}}(T_1, \beta_1, \tau_1)$ such that $\mathcal{D}_{\mathrm{inv}}(T_1, \beta_1, \tau_1, d)^{\mathrm{ex}}_{t_1} \cong \mathcal{D}_{\mathrm{inv}}(T_2, \beta_2, \tau_2)^{\mathrm{ex}}_{t_2}$ if and only if $T_1\langle t_1\rangle = T_2\langle t_2\rangle$, $t_1 = t_2$, $\beta_1^{[t_1]} = \beta_2^{[t_2]}$.*

Proof. The proof that if there exist a homogeneous element $d \in \mathcal{D}_{\mathrm{inv}}(T_1, \beta_1, \tau_1)$ such that $\mathcal{D}_{\mathrm{inv}}(T_1, \beta_1, \tau_1, d)^{\mathrm{ex}}_{t_1} \cong \mathcal{D}_{\mathrm{inv}}(T_2, \beta_2, \tau_2)^{\mathrm{ex}}_{t_2}$ then $T_1\langle t_1\rangle = T_2\langle t_2\rangle$, $t_1 = t_2$, $\beta_1^{[t_1]} = \beta_2^{[t_2]}$ follows in the same way as in Theorem 5.2.31.

Now, in case $T_1\langle t_1\rangle = T_2\langle t_2\rangle$, $t_1 = t_2$, $\beta_1^{[t_1]} = \beta_2^{[t_2]}$, denote $\tau'\colon T_1 \to \mathbb{F}$ the map given by $\tau'(s) = \tau_2^{[t_2]}(s)$ for all $s \in T_1$. Then, due to Theorem 5.2.31 it follows that $\mathcal{D}_{\mathrm{inv}}(T_2, \beta_2, \tau_2)^{\mathrm{ex}}_{t_2} \cong \mathcal{D}_{\mathrm{inv}}(T_1, \beta_1, \tau')^{\mathrm{ex}}_{t_1}$.

Denote $(\mathcal{D}, \varphi_1, \Delta_1) = \mathcal{D}_{\mathrm{inv}}(T_1, \beta_1, \tau_1)^{\mathrm{ex}}_{t_1}$ and $(\mathcal{D}, \varphi_2, \Delta_2) = \mathcal{D}_{\mathrm{inv}}(T_1, \beta_1, \tau')^{\mathrm{ex}}_{t_1}$. There is a homomorphism of graded-algebras:

$$\Psi\colon (\mathcal{D}, \Delta_2) \mapsto (\mathcal{D}, \Delta_1)$$

induced by $(d, 0) \mapsto (d, 0)$ and $(0, d) \mapsto (0, \tau_1(t)\tau'(t)d)$ for every homogeneous $d \in \mathcal{D}(T_1, \beta_1)$ of degree t. This is a well defined morphism of algebras due to the fact that in the first component it is the identity and in the second it is the composition of two involutions. It preserves the gradings since it sends $(d, \tau'(t)d)$ to $(d, \tau_1(t)d)$ and $(d, -\tau'(t)d)$ to $(d, -\tau_1(t)d)$ for every homogeneous $d \in \mathcal{D}(T_1, \beta_1)$ of degree t. Therefore, $(\mathcal{D}, \varphi_2, \Delta_2) \cong (\mathcal{D}, \Psi \circ \varphi_2 \circ \Psi^{-1}, \Delta_1)$. Finally, the involution $\Psi \circ \varphi_2 \circ \Psi^{-1}$ preserves the grading and is given by:

$$\Psi \circ \varphi_2 \circ \Psi^{-1}(d, \pm\tau_1(t)d) = \tau'(t)(d, \pm\tau_1(t)d)$$

for every $d \in \mathcal{D}(T_1, \beta_1)$ of degree t. Due to Proposition 5.2.25, there is $s \in T$ and an element $d_s \in \mathcal{D}(T_1, \beta_1)$ of degree s such that for all $t \in T$ and

$d \in \mathcal{D}(T_1, \beta_1)$ of degree t, $\tau'(t)d = \tau_1(t)d_s d d_s^{-1}$. Thus, there is $s \in T_1$ such that $\varphi_1 = \text{Int}(X_s) \circ (\Psi \circ \varphi_2 \circ \Psi^{-1})$, which implies that $(\mathcal{D}, \Psi \circ \varphi_2 \circ \Psi^{-1}, \Delta_1) \cong \mathcal{D}_{\text{inv}}(T_1, \beta_1, \tau_1, X_s)^{\text{ex}}_{t_1}$ □

Associative algebras

Let $\mathcal{D}$ be a G-graded-division algebra with support T and V a G-graded right $\mathcal{D}$-module. Let φ_0 be an involution of $\mathcal{D}$ and $B\colon V \times V \to \mathcal{D}$ a nondegenerate hermitian or skew-hermitian φ_0-sesquilinear form such that $B\colon {}^{\tau}V \times {}^{\tau}V \to \mathcal{D}$ is homogeneous of degree $(\overline{0}, g)$. Due to Proposition 5.2.34 and Proposition 5.2.17, we can choose an adequate homogeneous element $d \in \mathcal{D}$ and changing φ_0 by $\text{Ind}(d) \circ \varphi_0$ and B by dB, we can assume that $\mathcal{D}$ is either of the form $\mathcal{D}_{\text{inv}}(T, \beta)$ or of the form $\mathcal{D}_{\text{inv}}(T, \beta)^{\text{ex}}_t$. We write $\text{Sgn}(B) = 1$ if it is hermitian and $\text{Sgn}(B) = -1$ if it is skew-hermitian. We can decompose V into isotypic components as:

$$\begin{aligned} V = V_1 \oplus \ldots \oplus V_{m_0} \oplus (V'_{m_0+1} \oplus V''_{m_0+1}) \oplus \ldots \oplus (V'_{k_0} \oplus V''_{k_0}) \\ \oplus W_1 \oplus \ldots \oplus W_{m_1} \oplus (W'_{m_1+1} \oplus W''_{m_1+1}) \oplus \ldots \oplus (W'_{k_1} \oplus W''_{k_1}) \end{aligned} \quad (5.2.2)$$

Such that, each of the isotypic components V_i, V'_i and V''_i are isomorphic to the sum of all the submodules of $\mathcal{V}$ isomorphic to $\mathcal{D}^{[(0,g)]}$ for a fixed $g \in G$, and each of the isotypic components W_i, W'_i and W''_i are isomorphic to the sum of all the submodules of $\mathcal{V}$ isomorphic to $\mathcal{D}^{[(1,g)]}$ for a fixed $g \in G$ in such way that each V_i and W_i are self-dual by B and each pair V'_i, V''_i or W'_i and W''_i are in duality. Thus, $\dim_{\mathcal{D}}(V'_i) = \dim_{\mathcal{D}}(V''_i)$ for every $i \in \{m_0+1, \ldots, k_0\}$ and $\dim_{\mathcal{D}}(W'_j) = \dim_{\mathcal{D}}(W''_j)$ for every $j \in \{m_1+1, \ldots, k_1\}$. We can order each $V_1, \ldots, V_{m_0}$ such that the first l_0 have odd dimension and the remaining even dimension.

We write:

$$\kappa_0 = (q_1^{(0)}, \ldots, q_{l_0}^{(0)}, 2q_{l_0+1}^{(0)}, \ldots, 2q_{m_0}^{(0)}, q_{m_0+1}^{(0)}, q_{m_0+1}^{(0)}, \ldots, q_{k_0}^{(0)}, q_{k_0}^{(0)}) \in \mathbb{Z}_{>0}^{k_0} \quad (5.2.3)$$

to be the vector with the dimensions of each V_i, V'_i and V''_i and

$$\kappa_1 = (q_1^{(1)}, \ldots, q_{l_1}^{(1)}, 2q_{l_1+1}^{(1)}, \ldots, 2q_{m_1}^{(1)}, q_{m_1+1}^{(1)}, q_{m_1+1}^{(1)}, \ldots, q_{k_1}^{(1)}, q_{k_1}^{(1)}) \in \mathbb{Z}_{>0}^{k_1} \quad (5.2.4)$$

to be the vector with the dimensions of each W_i, W'_i and W''_i. Here $q_1^{(0)}, \ldots, q_{l_0}^{(0)}$, and $q_1^{(1)}, \ldots, q_{l_1}^{(1)}$ are odd numbers, and write:

$$\gamma_0 = (g_1^{(0)}, ..., g_{l_0}^{(0)}, g_{l_0+1}^{(0)}, ..., g_{m_0}^{(0)}, g_{m_0+1}'^{(0)}, g_{m_0+1}''^{(0)}, ..., g_{k_0}'^{(0)}, g_{k_0}''^{(0)}) \in \mathbb{Z}_{>0}^{k_0} \quad (5.2.5)$$

and

$$\gamma_1 = (g_1^{(1)}, ..., g_{l_1}^{(1)}, g_{l_1+1}^{(1)}, ..., g_{m_0}^{(1)}, g_{m_1+1}'^{(1)}, g_{m_1+1}''^{(1)}, ..., g_{k_1}'^{(1)}, g_{k_1}''^{(1)}) \in \mathbb{Z}_{>0}^{k_1} \quad (5.2.6)$$

in such way that $\operatorname{Supp} V_i = g_i^{(0)}T$, $\operatorname{Supp} V_i' = g_i'^{(0)}T$, $\operatorname{Supp} V_i'' = g_i''^{(0)}T$, $\operatorname{Supp} W_i = g_i^{(1)}T$, $\operatorname{Supp} W_i' = g_i'^{(1)}T$ and $\operatorname{Supp} W_i'' = g_i''^{(1)}$ for each possible choice of i. The fact that for $h_1, h_2 \in G$, $i, j \in \{0, 1\}$, $v \in V_{(i,h_1)}$ and $w \in V_{(j,h_2)}$, $B(v, w) = 0$ unless $i = j$ and $h_1h_2g \in T$, implies that there are $t_1^{(0)}, ..., t_{k_0}^{(0)}, t_1^{(1)}, ..., t_{k_1}^{(1)} \in T$, such that:

$$\begin{aligned}(g_1^{(0)})^2 t_1^{(0)} &= ... = (g_{m_0}^{(0)})^2 t_{m_0}^{(0)} = g_{m_0+1}'^{(0)} g_{m_0+1}''^{(0)} t_{m_0+1}^{(0)} = ... = g_{k_0}'^{(0)} g_{k_0}''^{(0)} t_{k_0}^{(0)} \\ = (g_1^{(1)})^2 t_1^{(1)} &= ... = (g_{m_1}^{(1)})^2 t_{m_1}^{(1)} = g_{m_1+1}'^{(1)} g_{m_1+1}''^{(1)} t_{m_1+1}^{(1)} = ... = g_{k_1}'^{(1)} g_{k_1}''^{(1)} t_{k_1}^{(1)} = g^{-1}.\end{aligned}$$

Taking $g_i''^{(0)} t_i^{(0)}$ instead of $g_i''^{(0)}$ for every $i \in \{m_0 + 1, ..., k_0\}$ and $g_i''^{(1)} t_i^{(1)}$ instead of $g_i''^{(1)}$, we can assume that γ_0 and γ_1 satisfy:

$$\begin{aligned}(g_1^{(0)})^2 t_1^{(0)} &= ... = (g_{m_0}^{(0)})^2 t_{m_0}^{(0)} = g_{m_0+1}'^{(0)} g_{m_0+1}''^{(0)} = ... = g_{k_0}'^{(0)} g_{k_0}''^{(0)} \\ = (g_1^{(1)})^2 t_1^{(1)} &= ... = (g_{m_1}^{(1)})^2 t_{m_1}^{(1)} = g_{m_1+1}'^{(1)} g_{m_1+1}''^{(1)} = ... = g_{k_1}'^{(1)} g_{k_1}''^{(1)} = g^{-1}. \quad (5.2.7)\end{aligned}$$

Notice that the elements

$$t_1^{(0)}, ..., t_{m_0}^{(0)}, t_1^{(1)}, ..., t_{m_1}^{(1)}$$

and

$$g_{m_0+1}''^{(0)}, ..., g_{k_0}''^{(0)}, g_{m_1+1}''^{(1)}, ..., g_{k_1}''^{(1)},$$

are determined by

$$g_1^{(0)}, ..., g_{m_0}^{(0)}, g_{m_0+1}'^{(0)}, ..., g_{k_0}'^{(0)}, g_1^{(1)}, ..., g_{m_1}^{(1)}, g_{m_1+1}'^{(1)}, ..., g_{k_1}'^{(1)},$$

and g.

Definition 5.2.35. Let T' be an elementary 2-subgroup of $\mathbb{Z} \times G$ and $\beta \colon T' \times T' \to \mathbb{F}$ a nondegenerate alternating bicharacter. Let κ_0, κ_1, γ_0, γ_1 and $g \in G$ be as in (5.2.3), (5.2.4), (5.2.5) and (5.2.6) satisfying (5.2.7) and let

$\delta = \pm 1$. We denote $T = T'$ and define $\mathcal{M}(G, T, \beta, \kappa_0, \kappa_1, \gamma_0, \gamma_1, \delta, g)$ as the graded algebra $\mathcal{M}(G, \mathcal{D}_{\text{inv}}(T, \beta), \kappa_0, \kappa_1, \gamma_0, \gamma_1)$ with the involution given by $\varphi(X) = \Phi^{-1} X^t \Phi$ where Φ is the matrix given in the following block-diagonal form:

$$\begin{aligned}\Phi = \sum_{i=1}^{l_0} \mathrm{I}_{q_i^{(0)}} \otimes X_{t_i^{(0)}} \oplus \sum_{i=l_0+1}^{m_0} \mathrm{S}^{(0)}{}_i \otimes X_{t_i^{(0)}} \oplus \sum_{m_0+1}^{k_0} \begin{pmatrix} 0 & I_{q_i^{(0)}} \\ \delta I_{q_i^{(0)}} & 0 \end{pmatrix} \\ \oplus \sum_{i=1}^{l_1} \mathrm{I}_{q_i^{(1)}} \otimes X_{t_i^{(1)}} \oplus \sum_{i=l_1+1}^{m_1} \mathrm{S}^{(1)}{}_i \otimes X_{t_i^{(1)}} \oplus \sum_{m_1+1}^{k_1} \begin{pmatrix} 0 & I_{q_i^{(1)}} \\ \delta I_{q_i^{(1)}} & 0 \end{pmatrix} \end{aligned} \tag{5.2.8}$$

where for $i \in \{l_0 + 1, ..., m_0\}$, $S_i^{(0)} = \mathrm{I}_{2q_i^{(0)}}$ or $S_i^{(0)} = \begin{pmatrix} 0 & \mathrm{I}_{q_i^{(0)}} \\ -\mathrm{I}_{q_i^{(0)}} & 0 \end{pmatrix}$, we write $\mathrm{sgn}(S_i^{(0)}) = 1$ in the first case and $\mathrm{sgn}(S_i^{(0)}) = -1$ in the second case, and similarly, for $i \in \{l_1 + 1, ..., m_1\}$, $S_i^{(1)} = \mathrm{I}_{2q_i^{(1)}}$ or $S_i^{(1)} = \begin{pmatrix} 0 & \mathrm{I}_{q_i^{(1)}} \\ -\mathrm{I}_{q_i^{(1)}} & 0 \end{pmatrix}$, and we write $\mathrm{sgn}(S_i^{(1)}) = 1$ in the first case and $\mathrm{sgn}(S_i^{(1)}) = -1$ in the second case. All this subject to:

$$\begin{aligned}\delta = \beta(t_1^{(0)}) = ... = \beta(t_{l_0}^{(0)}) = \mathrm{sgn}(S_{l_0+1}^{(0)})\beta(t_{l_0+1}^{(0)}) = ... = \mathrm{sgn}(S_{m_0}^{(0)})\beta(t_{m_0}^{(0)}) \\ = \beta(t_1^{(1)}) = ... = \beta(t_{l_1}^{(1)}) = \mathrm{sgn}(S_{l_1+1}^{(1)})\beta(t_{l_1+1}^{(1)}) = ... = \mathrm{sgn}(S_{m_1}^{(1)})\beta(t_{m_1}^{(1)})\end{aligned} \tag{5.2.9}$$

(recall that β is the quadratic form defined as in Definition 5.2.24). With the same notation, for an order 2 element $t \in G \setminus T'$, we denote $T = T'\langle t\rangle$ and define $\mathcal{M}(G, T', t, \beta, \kappa_0, \kappa_1, \gamma_0, \gamma_1, \delta, g)$ to be $\mathcal{M}(G, \mathcal{D}_{\text{inv}}(T', \beta)_t^{\text{ex}}, \kappa_0, \kappa_1, \gamma_0, \gamma_1)$ with the involution given by the expression $\varphi(X) = \Phi^{-1}\mathrm{ex}(X)^t\Phi$ where ex is applied componentwise and Φ is the matrix given in the following block-diagonal form:

$$\begin{aligned}\Phi = \sum_{i=1}^{l_0} \mathrm{I}_{q_i^{(0)}} \otimes Y_{t_i^{(0)}} \oplus \sum_{i=l_0+1}^{m_0} \mathrm{S}^{(0)}{}_i \otimes Y_{t_i^{(0)}} \oplus \sum_{m_0+1}^{k_0} \begin{pmatrix} 0 & I_{q_i^{(0)}} \\ \delta I_{q_i^{(0)}} & 0 \end{pmatrix} \\ \oplus \sum_{i=1}^{l_1} \mathrm{I}_{q_i^{(1)}} \otimes Y_{t_i^{(1)}} \oplus \sum_{i=l_1+1}^{m_1} \mathrm{S}^{(1)}{}_i \otimes Y_{t_i^{(1)}} \oplus \sum_{m_1+1}^{k_1} \begin{pmatrix} 0 & I_{q_i^{(1)}} \\ \delta I_{q_i^{(1)}} & 0 \end{pmatrix} \end{aligned} \tag{5.2.10}$$

Where for $i \in \{l_0+1, ..., m_0\}$, $S_i^{(0)} = \mathrm{I}_{2q_i^{(0)}}$ or $S_i^{(0)} = \begin{pmatrix} 0 & \mathrm{I}_{q_i^{(0)}} \\ -\mathrm{I}_{q_i^{(0)}} & 0 \end{pmatrix}$, we write $\mathrm{sgn}(S_i^{(0)}) = 1$ in the first case and $\mathrm{sgn}(S_i^{(0)}) = -1$ in the second case, and similarly, for $i \in \{l_1+1, ..., m_1\}$, $S_i^{(1)} = \mathrm{I}_{2q_i^{(1)}}$ or $S_i^{(1)} = \begin{pmatrix} 0 & \mathrm{I}_{q_i^{(1)}} \\ -\mathrm{I}_{q_i^{(1)}} & 0 \end{pmatrix}$, and we write $\mathrm{sgn}(S_i^{(1)}) = 1$ in the first case and $\mathrm{sgn}(S_i^{(1)}) = -1$ in the second case. All this subject to:

$$\begin{aligned} \delta &= \beta^{[t]}(t_1^{(0)}) = ... = \beta^{[t]}(t_{l_0}^{(0)}) = \mathrm{sgn}(S_{l_0+1}^{(0)})\beta^{[t]}(t_{l_0+1}^{(0)}) = ... = \mathrm{sgn}(S_{m_0}^{(0)})\beta^{[t]}(t_{m_0}^{(0)}) \\ &= \beta^{[t]}(t_1^{(1)}) = ... = \beta^{[t]}(t_{l_1}^{(1)}) = \mathrm{sgn}(S_{l_1+1}^{(1)})\beta^{[t]}(t_{l_1+1}^{(1)}) = ... = \mathrm{sgn}(S_{m_1}^{(1)})\beta^{[t]}(t_{m_1}^{(1)}). \end{aligned} \tag{5.2.11}$$

Proposition 5.2.36. *Over an algebraically closed field $\mathbb{F}$, with the notation of this section:*

(1) In case $(\mathcal{D}, \varphi_0)$ is isomorphic to $\mathcal{D}_{\mathrm{inv}}(T, \beta)$, the graded algebra with involution $\mathrm{End}(G, \mathcal{D}, V, \varphi_0, B)$ is isomorphic to $\mathcal{M}(G, T', \beta, \kappa_0, \kappa_1, \gamma_0, \gamma_1, \delta, g)$ with κ_0, κ_1, γ_0, γ_1 and $g \in G$ as in (5.2.3), (5.2.4), (5.2.5) and (5.2.6) and $\delta = \mathrm{sgn}(B)$.

(2) In case $(\mathcal{D}, \varphi_0)$ is isomorphic to $\mathcal{D}_{\mathrm{inv}}(T', \beta)_t^{\mathrm{ex}}$, the graded algebra with involution $\mathrm{End}(G, \mathcal{D}, V, \varphi_0, B)$ is isomorphic to $\mathcal{M}(G, T', t, \beta, \kappa_0, \kappa_1, \gamma_0, \gamma_1, \delta, g)$ with κ_0, κ_1, γ_0, γ_1 and $g \in G$ as in (5.2.3), (5.2.4), (5.2.5) and (5.2.6) and $\delta = \mathrm{sgn}(B)$. Moreover, with the same choice of $\mathcal{D}, V$ and φ_0, we can choose B so that $\mathrm{sgn}(B) = 1$.

Proof. We will abuse of the notation and denote $\beta^{[t]}$ just by β, in case $(\mathcal{D}, \varphi_0)$ is isomorphic to $\mathcal{D}_{\mathrm{inv}}(T', \beta)_t^{\mathrm{ex}}$, in order to work with (1) and (2) at the same time. Also, for every $s \in T$, we will denote $Z_s = X_s$ in case $(\mathcal{D}, \varphi_0)$ is isomorphic to $\mathcal{D}_{\mathrm{inv}}(T, \beta)$, and $Z_s = Y_s$ in case it is isomorphic to $\mathcal{D}_{\mathrm{inv}}(T', \beta)_t^{\mathrm{ex}}$. As shown, in [EK13, (2.19)], after taking a basis $\{v_1, ..., v_r\}$ of V, the involution φ of $\mathrm{End}(G, \mathcal{D}, V, \varphi_0, B)$, can be written in matrix form as $\varphi(X) = \Phi^{-1}\varphi_0(X)^t\Phi$, where the coordinate i, j of Φ is $B(v_i, v_j)$. Thus, in order to prove the proposition, we just need to find the right homogeneous basis.

For $i \in \{1, ..., m_0\}$ and $j \in \{1, ..., m_1\}$, denote in each case $\overline{V}_i = V_{g_i^{(0)}}$ and $\overline{W}_j = V_{g_j^{(1)}}$. Thus, we can identify V_i with $\overline{V_i} \otimes \mathcal{D}$ and W_j with $\overline{W}_j \otimes \mathcal{D}$. Then, we have:

$$B(u,v) = B_i^{(0)}(u,v)Z_{t_i^{(0)}} \text{ for all } u,v \in \overline{V_i}$$
$$B(u,v) = B_j^{(1)}(u,v)Z_{t_i^{(1)}} \text{ for all } u,v \in \overline{W_j}.$$

Where $B_i^{(0)} \colon \overline{V_i} \times \overline{V_i} \to \mathbb{F}$ and $B_i^{(1)} \colon \overline{W_i} \times \overline{W_i} \to \mathbb{F}$ are nondegenerate bilinear forms. The fact that B is hermitian or skew-hermitian implies that $B_i^{(0)}$ is symmetric if $\operatorname{sgn}(B) = \beta(t_i^{(0)})$, in which case, we can take a basis such that the matrix of $B_i^{(0)}$ with respect to this basis is $\mathrm{I}_{\dim_{\mathbb{F}}(\overline{V}_i)}$, or it implies that $B_i^{(0)}$ is skew-symmetric if $\operatorname{sgn}(B) = -\beta(t_i^{(0)})$, in which case, the dimension of $\overline{V}_i$ is even, so we denote $2k_i = \dim_{\mathbb{F}}(\overline{V}_i)$ we can take a basis of $\overline{V_i}$ such that the matrix of $B_i^{(0)}$ is $\begin{pmatrix} 0 & \mathrm{I}_{k_i} \\ -\mathrm{I}_{k_i} & 0 \end{pmatrix}$. Similarly, we can do with each $B_j^{(1)}$.

Now, for each $i \in \{m_0+1, ..., k_0\}$ and $j \in \{m_1+1, ..., k_1\}$, we denote $\overline{V}'_i = V_{g_i'^{(0)}}$, $\overline{V}''_i = V_{g_i''^{(0)}}$, $\overline{W}'_j = V_{g_j'^{(1)}}$ and $\overline{W}''_j = V_{g_j''^{(1)}}$. Hence, we can identify, V'_i with $\overline{V}'_i \otimes \mathcal{D}$, V''_i with $\overline{V}''_i \otimes \mathcal{D}$, w'_j with $\overline{W}'_j \otimes \mathcal{D}$ and V''_i with $\overline{W}''_i \otimes \mathcal{D}$. Then, in a similar way as before, we have that there are nondegenerate pairings $B_i^{(0)} \colon \overline{V}'_i \times \overline{V}''_i \to \mathbb{F}$ and $B_i^{(1)} \colon \overline{W}'_j \times \overline{W}''_j \to \mathbb{F}$ such that:

$$B(u,v) = B_i^{(0)}(u,v)Z_{t_i^{(0)}} \text{ for all } u \in \overline{V}_1 \text{ and } v \in \overline{V}''_i$$
$$B(u,v) = B_j^{(1)}(u,v)Z_{t_i^{(1)}} \text{ for all } u \in \overline{W}'_j \text{ and } v \in \overline{W}''_j.$$

Thus, we can take a basis of $\overline{V}'_i$ and a dual basis of $\overline{V}''_i$ and similarly, take a basis for $\overline{W}'_i$ and a dual basis of $\overline{W}''_i$. Ordering the basis of each subspace as in (5.2.2), we can write the algebra in matrix form and get the result.

Finally, if $(\mathcal{D}, \varphi_0)$ is isomorphic to $\mathcal{D}_{\mathrm{inv}}(T', \beta)_t^{\mathrm{ex}}$ and $\operatorname{sgn}(B) = -1$, due to Proposition 5.2.17

$$\operatorname{End}(G, \mathcal{D}, V, \varphi_0, B) = \operatorname{End}(G, \mathcal{D}, V, \operatorname{Int}((1,-1)) \circ \varphi_0, (1,-1)B)$$

but $\operatorname{Int}(1,-1) \circ \varphi_0 = \varphi_0$ and $\operatorname{sgn}((1,-1)B) = 1$. □

Theorem 5.2.37. *Let $(\mathcal{A}, \varphi, \Gamma)$ be an Ω-algebra satisfying $(T1)-(T4)$ over an algebraically closed field $\mathbb{F}$. Then, one of the following holds:*

(1) There is an elementary 2-subgroup T of G, a nondegenerate alternating bicharacter $\beta \colon T \times T \to \mathbb{F}$, $k_0, k_1 > 0$, $\gamma_0 \in G^{k_0}$ and $\gamma_1 \in G^{k_1}$ consisting on distinct elements modulo T, $\kappa_0 \in \mathbb{Z}_{>0}^{k_0}$ and $\kappa_1 \in \mathbb{Z}_{>0}^{k_1}$ such that $(\mathcal{A}, \varphi, \Gamma) \cong \mathcal{M}(G, \mathcal{D}(T, \beta), \kappa_0, \kappa_1, \gamma_0, \gamma_1)^{\mathrm{ex}}$.

(2) There are, an elementary 2*-subgroup* T *of* G*, a nondegenerate alternating bicharacter* $\beta\colon T\times T\to\mathbb{F}$*,* κ_0*,* κ_1*,* γ_0*,* γ_1 *and* $g\in G$ *as in* (5.2.3)*,* (5.2.4)*,* (5.2.5) *and* (5.2.6) *and* $\delta=\pm1$ *such that we have the isomorphism* $(\mathcal{A},\varphi,\Gamma)\cong\mathcal{M}(G,T,\beta,\kappa_0,\kappa_1,\gamma_0,\gamma_1,\delta,g)$*.*

(3) There are, an elementary 2*-subgroup* T *of* G*, a nondegenerate alternating bicharacter* $\beta\colon T\times T\to\mathbb{F}$*, an order* 2 *element* $t\in G\setminus T$*,* κ_0*,* κ_1*,* γ_0*,* γ_1 *and* $g\in G$ *as in* (5.2.3)*,* (5.2.4)*,* (5.2.5) *and* (5.2.6) *such that* $(\mathcal{A},\varphi,\Gamma)\cong\mathcal{M}(G,T,\beta,t,\kappa_0,\kappa_1,\gamma_0,\gamma_1,1,g)$*.*

Moreover, the algebras on each item are non isomorphic and:

(i) $\mathcal{M}(G,\mathcal{D}(T,\beta),\kappa_0,\kappa_1,\gamma_0,\gamma_1)^{\mathrm{ex}}\cong\mathcal{M}(G,\mathcal{D}(T',\beta'),\kappa_0',\kappa_1',\gamma_0',\gamma_1')^{\mathrm{ex}}$ *if and only if either:*

- *(a)* $T=T'$*,* $\beta=\beta'$ *and there is* $g\in G$ *such that* $\Xi(\kappa_0,\gamma_0)=g\Xi(\kappa_0',\gamma_0')$ *and* $\Xi(\kappa_1,\gamma_1)=g\Xi(\kappa_1',\gamma_1')$ *or*
- *(b)* $T=T'$*,* $\beta=\beta'$ *and there is* $g\in G$ *such that* $\Xi(\kappa_0,\gamma_0)=g\Xi(\kappa_0',\gamma_0'^{-1})$ *and* $\Xi(\kappa_1,\gamma_1)=g\Xi(\kappa_1'^{-1},\gamma_1')$*.*

(ii) $\mathcal{M}(G,T,\beta,\kappa_0,\kappa_1,\gamma_0,\gamma_1,\delta,g)\cong\mathcal{M}(G,T',\beta',\kappa_0',\kappa_1',\gamma_0',\gamma_1',\delta',g')$ *if and only if* $T=T'$*,* $\beta=\beta'$*,* $\delta=\delta'$ *and there is* $g''\in G$ *such that* $\Xi(\kappa_0,\gamma_0)=g''\Xi(\kappa_0',\gamma_0')$*,* $\Xi(\kappa_1,\gamma_1)=g''\Xi(\kappa_1',\gamma_1')$ *and* $g=g'g''^{-1}$*.*

(iii) $\mathcal{M}(G,T,\beta,t,\kappa_0,\kappa_1,\gamma_0,\gamma_1,1,g)\cong\mathcal{M}(G,T',\beta',t',\kappa_0',\kappa_1',\gamma_0',\gamma_1',1,g')$ *if and only if* $T\langle t\rangle=T'\langle t'\rangle$*,* $t=t'$*,* $\beta^{[t]}=\beta^{[t']}$ *and there is* $g''\in G$ *such that* $\Xi(\kappa_0,\gamma_0)=g''\Xi(\kappa_0',\gamma_0')$*,* $\Xi(\kappa_1,\gamma_1)=g''\Xi(\kappa_1',\gamma_1')$ *and* $g=g'g''^{-1}$*.*

Proof. Let's prove the first part. In order to do so, assume that $(\mathcal{A},\varphi,\Gamma)$ is an Ω-algebra satisfying $(T1)-(T4)$ over an algebraically closed field $\mathbb{F}$. In case $(\mathcal{A},\varphi,\Gamma)$ is not graded simple, Theorems 5.2.12 and 5.2.20 imply (1). In case $\mathcal{A}$ is simple, due to Proposition 5.2.36, we get (2) and (3).

In order to prove the second part, the algebras on each item are not isomorphic since in (1), $(\mathcal{A},\Gamma)$ is not graded-simple, in (2), $\mathcal{A}$ is simple, hence $(\mathcal{A},\Gamma)$ is graded-simple and in (3) $(\mathcal{A},\Gamma)$ is graded simple but $\mathcal{A}$ is not simple. Hence, let's prove item by item:

(i) This is a consequence again of Theorems 5.2.12, 5.2.20 together with Lemma 5.2.21 and Remark 5.2.22.

(ii) Let $\mathcal{D}$, V, φ_0 and B be such that the isomorphism $\mathrm{End}(G,\mathcal{D},V,\varphi_0,B)\cong\mathcal{M}(G,T,\beta,\kappa_0,\kappa_1,\gamma_0,\gamma_1,\delta,g)$ holds. If $T=T'$, $\beta=\beta'$, $\delta=\delta'$ and there is $g''\in G$ such that $\Xi(\kappa_0,\gamma_0)=g\Xi(\kappa_0',\gamma_0')$, $\Xi(\kappa_1,\gamma_1)=g\Xi(\kappa_1',\gamma_1')$

and $g = g'g''^{-1}$. Then, by the construction, shown before, it follows $\operatorname{End}(G, \mathcal{D}, V^{[(0,g'')]}, \varphi_0, B) \cong \mathcal{M}(G, T', \beta', \kappa_0', \kappa_1', \gamma_0', \gamma_1', \delta', g')$ and it also follows , $\operatorname{End}(G, \mathcal{D}, V, \varphi_0, B) \cong \operatorname{End}(G, \mathcal{D}, V^{[(0,g'')]}, \varphi_0, B)$ via the isomorphism induced by $(\mathrm{id}_{\mathcal{D}}, \mathrm{id}_V)$. The converse is just a consequence of [EK13, Theorem 2.64] since here we are considering a subgroup of the automorphisms in the theorem mentioned before.

(iii) If $T\langle t\rangle = T'\langle t'\rangle$, $t = t'$, $\beta^{[t]} = \beta^{[t']}$ and there is $g'' \in G$ such that $\Xi(\kappa_0, \gamma_0) = g\Xi(\kappa_0', \gamma_0')$, $\Xi(\kappa_1, \gamma_1) = g\Xi(\kappa_1', \gamma_1')$ and $g = g'g''^{-1}$, the prove is as before.

For the converse, Let $\mathcal{D}$, V, φ_0 and B be such that $\operatorname{End}(G, \mathcal{D}, V, \varphi_0, B) \cong \mathcal{M}(G, T, \beta, \kappa_0, \kappa_1, \gamma_0, \gamma_1, 1, g)$. Due to [EK13, Theorem 2.10], the pair $(\mathcal{D}, V)$ is determined up to isomorphism by the grading on the Ω-algebra $\mathcal{M}(G, T, \beta, \kappa_0, \kappa_1, \gamma_0, \gamma_1, 1, g)$ and a shift on the grading of V by an element $(0, g'') \in \mathbb{Z} \times G$. Due to Proposition 5.2.36 the sesquilinear form B is determided up to one homogeneous element of the center of the algebra $\operatorname{End}_{\mathcal{D}}(V)$ by the involution of $\operatorname{End}(G, \mathcal{D}, V, \varphi_0, B)$ and the choice of φ_0. The restriction $\operatorname{sgn}(B) = 1$, implies that it is uniquely determined by scalar. Therefore, the decomposition (5.2.2) is determined up to a permutation of $V_1, ..., V_{m_0}$, a permutation of $V_{m_0+1}', V_{m_0+1}'', ..., V_{k_0}', V_{k_0}''$ preserving the pairing of the V_i' and V_i'', a permutation of $W_1, ..., W_{m_1}$, and a permutation of $W_{m_1+1}', W_{m_1+1}'', \dots$ $, W_{k_1}', W_{k_1}''$ preserving the pairing of the V_i' and V_i''. If we shift the grading on V by $(0, g'')$, the degree of ${}^{[\tau]}B$ changes to $(\overline{0}, gg''^{-1})$. Thus, the elements the elements $t_1^{(0)}, ..., t_{m_0}^{(0)}, t_1^{(1)}, ..., t_{m_1}^{(1)}$ are uniquely determined by B since the equation (5.2.7) is invariant under these shifts and any changes of the elements $g_1^{(0)}, ..., g_{m_0}^{(0)}, g_1^{(1)}, ..., g_{m_1}^{(1)}$ by other elements on their cosets (since T is an elementary 2-group). Therefore, this argument together with Theorem 5.2.31 implies that in case $\mathcal{M}(G, T, \beta, t, \kappa_0, \kappa_1, \gamma_0, \gamma_1, 1, g') \cong \mathcal{M}(G, T', \beta', t', \kappa_0', \kappa_1', \gamma_0', \gamma_1', 1, g')$, then $T\langle t\rangle = T'\langle t'\rangle$, $t = t'$, $\beta^{[t]} = \beta^{[t']}$ and there is $g'' \in G$ such that $\Xi(\kappa_0, \gamma_0) = g\Xi(\kappa_0', \gamma_0')$, $\Xi(\kappa_1, \gamma_1) = g\Xi(\kappa_1', \gamma_1')$ and $g = g'g''^{-1}$.

□

Definition 5.2.38. Denote by π_1 and π_2 the projection of $\mathbb{Z} \times G$ on $\mathbb{Z}$ and on G respectively. Let $(\mathcal{A}, \varphi, \Delta)$ be an Ω-algebra satisfying $(T1) - (T4)$. Denote ${}^{\pi_1}\Delta\colon \mathcal{A} = \mathcal{A}_{-1} \oplus \mathcal{A}_0 \oplus \mathcal{A}_1$ and $W = \mathbf{W}(\mathcal{A}, \varphi, {}^{\pi_1}\Delta)$. Recall that as shown in Remark 4.2.21 that $\mathfrak{A}(W) = \mathcal{A}_{-1}\mathcal{A}_1 \oplus \mathcal{A}_1$ Then, due to Proposition 5.1.1 and Theorem 5.1.4, the grading ${}^{\pi_2}\Delta$ induces a grading on $\mathfrak{A}(W)$ given by $\deg(a) = g$ for every $a \in \mathcal{A}_{(1,g)} \oplus (\mathcal{A}_{(0,g)} \cap (\mathcal{A}_{-1}\mathcal{A}_1))$. We denote this graded algebra $\mathbf{GrA}(\mathcal{A}, \varphi, \Delta)$.

Examples 5.2.39. We are going to study the grading on the graded algebra $\mathbf{GrA}(\mathcal{M}(G,T,\beta,\kappa_0,\kappa_1,\gamma_0,\gamma_1,\delta,g))$ for different choices of $G,T,\beta,\kappa_0,\kappa_1,\gamma_0,\gamma_1,\delta$ and g.

(1) If $T=\{e\}$, $\kappa_0(1,1)$, $\kappa_1=(2)$, $\gamma_0=(g_1^{(0)},g_2^{(0)})$, $\gamma_1=(g^{(1)})$, $\delta=1$ and the relations are:

$$(g_1^{(0)})^2=(g_2^{(0)})^2=(g^{(1)})^2=g,$$

this is the model of algebra given in Example 4.2.21 with $\Phi_0=\begin{pmatrix}1&0\\0&1\end{pmatrix}=\Phi_1$. In this case, on $\mathcal{W}$ there are two graded subspaces which are the subspace:

$$\mathcal{W}_{g_1^{(0)}(g^{(1)})^{-1}}=\left\{\left(0,\begin{pmatrix}\lambda&\beta\\0&0\end{pmatrix}\right)\mid\lambda,\beta\in\mathbb{F}\right\}$$

whose elements have degree $g_1^{(0)}(g^{(1)})$ and the subspace:

$$\mathcal{W}_{g_2^{(0)}(g^{(1)})^{-1}}=\left\{\left(0,\begin{pmatrix}\lambda&\beta\\0&0\end{pmatrix}\right)\mid\lambda,\beta\in\mathbb{F}\right\}\tag{5.2.12}$$

whose elements have degree $g_2^{(0)}(g^{(1)})$. Moreover, since $g_1^{(0)}\neq g_2^{(0)}$ it follows that their degrees are different. Notice that with the choice of x_1,x_2 and x_3 choosen in Example 4.3.16, in this case, choosing the skew symmetric element

$$s=\left(\begin{pmatrix}0&1\\-1&0\end{pmatrix},0\right),\tag{5.2.13}$$

the elements x_1, sx_1, (x_2+x_3), $s(x_2+x_3)$, (x_2-x_3) and $s(x_2-x_3)$ are all homogeneous and they span $\mathcal{M}$.

(2) If $T=\{e\}$, $\kappa_0=(1,1)$, $\kappa_1=(1,1)$, $\gamma_0=(g_1^{(0)},g_2^{(0)})$, $\gamma_1=(g'^{(1)},g''^{(1)})$, $\delta=1$ and the relations are:

$$(g_1^{(0)})^2=(g_2^{(0)})^2=g'^{(1)}g''^{(1)}=g,$$

this is the model of algebra given in Example 4.2.21 with $\Phi_0=\begin{pmatrix}1&0\\0&1\end{pmatrix}$ and $\Phi_1=\begin{pmatrix}0&1\\1&0\end{pmatrix}$. In this case, the subspace of $\mathcal{W}$:

$$\mathcal{W}_1=\left\{\left(0,\begin{pmatrix}\lambda&0\\0&\beta\end{pmatrix}\right)\mid\lambda,\beta\in\mathbb{F}\right\}\tag{5.2.14}$$

is a graded subspace whose elements have degrees $g_1^{(0)}(g'^{(1)})^{-1}$ or $g_2^{(0)}(g''^{(1)})^{-1}$, and the subspace

$$\mathcal{W}_2 = \left\{ \left(0, \begin{pmatrix} 0 & \lambda \\ \beta & 0 \end{pmatrix}\right) \mid \lambda, \beta \in \mathbb{F} \right\} \tag{5.2.15}$$

is a graded subspace whose elements have degrees $g_1^{(0)}(g''^{(1)})^{-1}$ or $g_2^{(0)}(g'^{(1)})^{-1}$. Since $g_1^{(0)} \neq g_2^{(0)}$ and $g'^{(1)} \neq g''^{(1)}$, the degrees of the elements in $\mathcal{W}_1$ are different to the degrees of the elements in $\mathcal{W}_2$. Thus, every homogeneous element is is $\mathcal{W}_1$ or $\mathcal{W}_2$.

We can choose the element:

$$s = \left(\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, 0 \right)$$

to be a choice of skew-symmetric element such that $s^2 = 1$ and choose x_1, x_2, x_3 as in Example 4.3.16. We denote $e_\sigma = \frac{1}{2}(1 + (\sigma s))$ for $\sigma = \pm$.

We can check that the elements x_1, sx_1, $e_+x_2 \pm e_-x_3$ and $e_-x_2 \pm e_+x_3$ are homogeneous.

(3) If $T = \{e\}$, $\kappa_0 = (1, 1)$, $\kappa_1 = (1, 1)$, $\gamma_0 = (g'^{(0)}, g''^{(0)})$, $\gamma_1 = (g_1^{(1)}, g_2^{(1)})$, $\delta = 1$ and the relations are:

$$g'^{(0)}g''^{(0)} = (g_1^{(1)})^2 = (g_2^{(1)})^2 = g,$$

this is the model of algebra given in Example 4.2.21 with $\Phi_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\Phi_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. In this model, we can choose:

$$s = \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, 0 \right)$$

as a skew-symmetric element whose square is 1. This element has degree e. Therefore, if we denote $e_\sigma = \frac{1}{2}(1 + \sigma s)$ with $\sigma = \pm$, we have that:

$$e_+ = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, 0 \right), \quad \text{and } e_- = \left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, 0 \right),$$

and the subspaces $\mathcal{W}_\sigma = e_\sigma \mathcal{W}$ are graded subspaces. More concretely, these subspaces are the following:

$$\mathcal{W}_+ = \left\{ \left(0, \begin{pmatrix} 0 & 0 \\ \lambda & \beta \end{pmatrix} \right) \mid \lambda, \beta \in \mathbb{F} \right\}$$

and

$$\mathcal{W}_- = \left\{ \left(0, \begin{pmatrix} \lambda & \beta \\ 0 & 0 \end{pmatrix} \right) \mid \lambda, \beta \in \mathbb{F} \right\}.$$

Now, since $g_1^{(1)} \neq g_2^{(1)}$ we have that the degree of $(0, E_{1,1})$ is $g'^{(0)}(g_1^{(1)})^{-1}$ which is different to the degree of $(0, E_{1,2})$ which is $g'^{(0)}(g_2^{(1)})^{-1}$ and we have that the degree of $(0, E_{2,1})$ is $g''^{(0)}(g_1^{(1)})^{-1}$ different to the degree of $(0, E_{2,2})$ which is $g''^{(0)}(g_2^{(1)})^{-1}$.

With the choice of x_1, x_2 and x_3 from Example 4.3.16 we have that $e_+x_1, e_-x_1, e_+(x_2 + x_3), e_-(x_2 + x_3), e_+(x_2 - x_3)$ and $e_-(x_2 - x_3)$ are homogeneous.

(4) If $T = \{e\}$, $\kappa_0 = (1, 1)$, $\kappa_1 = (1, 1)$, $\gamma_0 = (g'^{(0)}, g''^{(0)})$, $\gamma_1 = (g_1^{(1)}, g_2^{(1)})$, $\delta = 1$ and the relations are:

$$g'^{(0)} g''^{(0)} = (g_1^{(1)})^2 = (g_2^{(1)})^2 = g,$$

this is the model of algebra given in Example 4.2.21 with $\Phi_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \Phi_1$. In this model, as before, we can choose:

$$s = \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, 0 \right)$$

as a skew-symmetric element whose square is 1. This element has degree e and if we denote $e_\sigma = \frac{1}{2}(1 + \sigma s)$ with $\sigma = \pm$, we have that:

$$e_+ = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, 0 \right), \quad \text{and } e_- = \left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, 0 \right),$$

and the subspaces $\mathcal{W}_\sigma = e_\sigma \mathcal{W}$ are graded subspaces and they are the following:

$$\mathcal{W}_+ = \left\{ \left(0, \begin{pmatrix} 0 & 0 \\ \lambda & \beta \end{pmatrix} \right) \mid \lambda, \beta \in \mathbb{F} \right\}$$

and

$$\mathcal{W}_- = \left\{ \left(0, \begin{pmatrix} \lambda & \beta \\ 0 & 0 \end{pmatrix} \right) \mid \lambda, \beta \in \mathbb{F} \right\}.$$

(5) If $T = \{e\}$, $\kappa_0 = (2)$, $\kappa_1 = (1,1)$, $\gamma_0 = (g^{(0)})$, $\gamma_1 = (g_1^{(1)}, g_2^{(1)})$, $\delta = 1$ and the relations are:

$$(g^{(0)})^2 = (g_1^{(1)})^2 = (g_2^{(1)})^2 = g,$$

this is the model of algebra given in Example 4.2.21 with $\Phi_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \Phi_1$. In this case with the choice of x_1, x_2, x_3 from Example 4.3.16 and choosing s to be the element

$$\left(\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, 0 \right),$$

which is a skew-symmetric element whose square is 1, we can check that $x_1, sx_1, (x_2 + x_3), (x_2 - x_3), s(x_2 + x_3)$ and $s(x_2 - x_3)$ are homogeneous elements .

(6) If $T = \{e\}$, $\kappa_0 = (1,1)$, $\kappa_1 = (1,1)$, $\gamma_0 = (g_1^{(0)}, g_2^{(0)})$, $\gamma_1 = (g_1^{(1)}, g_2^{(1)})$, $\delta = 1$ and the relations are:

$$(g_1^{(0)})^2 = (g_2^{(0)})^2 = (g_1^{(1)})^2 = (g_2^{(1)})^2 = g,$$

this is the model of algebra given in Example 4.2.21 with $\Phi_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \Phi_1$. With this model we have that

$$\mathrm{Supp}(\mathcal{W}) = \{g_1^{(0)}(g_1^{(1)})^{-1}, g_1^{(0)}(g_2^{(1)})^{-1}, g_2^{(0)}(g_1^{(1)})^{-1}, g_2^{(0)}(g_2^{(1)})^{-1}\}.$$

Since $g_1^{(0)} \neq g_2^{(0)}$ and $g_1^{(1)} \neq g_2^{(1)}$, it follows that this set has cardinal 4 unless either:

$$g_1^{(0)}(g_1^{(1)})^{-1} = g_2^{(0)}(g_2^{(1)})^{-1} \tag{5.2.16}$$

or

$$g_1^{(0)}(g_2^{(1)})^{-1} = g_1^{(0)}(g_2^{(1)})^{-1} \tag{5.2.17}$$

but (5.2.16) and (5.2.17) are equivalent since $g_2^{(0)}(g_1^{(0)})^{-1} = g_1^{(0)}(g_2^{(0)})^{-1}$.

(7) If $T = \mathbb{Z}_2^2$, $\kappa_0 = (2)$, $\kappa_1 = (2)$, $\gamma_0(g^{(0)})$, $\gamma_1(g^{(1)})$, $\delta = 1$ and relations

$$(g^{(0)})^2 t^{(0)} = (g^{(1)})^2 t^{(1)} = g,$$

then, this is the model from Example 4.2.23 with $(\mathcal{D}, -)$ isomorphic to $(\mathcal{M}_2(\mathbb{F}), {}^t)$, $d_0 = X_{t^{(0)}}$ and $d_1 = X_{t^{(1)}}$. The graded subspaces of $\mathcal{W}$ of degree t are the subspaces of the form $(0, d)$ with $d \in \mathcal{D}$ of degree $(g^{(0)})^{-1} t g^{(1)}$.

5.2.3 Non structurable gradings

In this subsection, we are going to consider the split quartic Cayley algebra $(\mathcal{A},\, -\,)$, over an algebraically closed field $\mathbb{F}$. Our purpose will be to classify up to isomorphism the gradings of $(\mathcal{A},\, -\,)$ which are not gradings of $(\mathcal{A},\, -\,, \Gamma_i)$ for any structurable grading Γ_i. This will be completed in Proposition 5.2.52. We fix a primitive cubic root of the unity ζ. We are going to use the notation of (4.3.2) (denote the subspaces $\mathcal{S}(\mathcal{A},\, -\,)$, $\mathcal{H}(\mathcal{A},\, -\,)$, $\mathcal{K}(\mathcal{A},\, -\,)$ and $\mathcal{M}(\mathcal{A},\, -\,)$ as $\mathcal{S}, \mathcal{H}, \mathcal{K}$ and $\mathcal{M}$ respectively), and denote by Γ_i for $i \in \{1, 2, 3\}$ the structurable gradings defined in Corollary 4.3.14. Our purpose is to classify up to isomorphism the gradings on $(\mathcal{A},\, -\,)$ which are not gradings on $(\mathcal{A},\, -\,, \Gamma_i)$ for any $i \in \{1, 2, 3\}$.

Fix $s \in \mathcal{S}$ such that $s^2 = 1$ and define $\rho_s \in \mathrm{End}_{\mathbb{F}}(\mathcal{M})$ by $\rho_s(m) = sm$ for every $m \in \mathcal{M}$. Due to Lemma 3.2.2 it follows that:

$$\rho_s^2 = \mathrm{id}_{\mathcal{M}}. \tag{5.2.18}$$

Therefore, ρ_s has eigenvalues ± 1, which implies that there is a vector space decomposition:

$$\mathcal{M} = \mathcal{M}_+ \oplus \mathcal{M}_- \tag{5.2.19}$$

where $\mathcal{M}_\sigma = \{m \in \mathcal{M} \mid \rho_s(m) = \sigma m\}$ for $\sigma = \pm$.

Remark 5.2.40. We can notice that

$$\mathcal{M}_\sigma = \frac{1}{2}(1 + (\sigma s))\mathcal{M}$$

for $\sigma = \pm$. Moreover, we can check that both subspaces are nonzero since

$$0 \neq \frac{1}{2}(1 + (\sigma s))x_1$$

for $\sigma = \pm$. We denote $e_+ = \frac{1}{2}(1+s)$ and $e_- = \frac{1}{2}(1-s)$. They satisfy $\overline{e}_+ = e_-$, $\overline{e}_- = e_+$, $e_+^2 = e_+$, $e_-^2 = e_-$ and $e_+e_- = 0 = e_-e_+$.

Lemma 5.2.41 ([BD23]). *For a G-grading Γ, $\deg(s) = e$ if and only if $\mathcal{M}_+$ and $\mathcal{M}_-$ are graded subspaces.*

Proof. In case $\deg(s) = e$, the homomorphisms $m \mapsto \frac{1}{2}(1 \pm s)m$ are homomorphisms of degree e. Hence, since $\mathcal{M}_+$ and $\mathcal{M}_-$ are the images, they are graded.

Conversely, if $\mathcal{M}_+$ and $\mathcal{M}_-$ are graded. Pick $0 \neq m \in \mathcal{M}_+$. Then, $\deg(m) = \deg(sm) = \deg(s)\deg(m)$, which implies that $\deg(s) = e$. □

We denote by $b\colon \mathcal{M}\times\mathcal{M}\to\mathbb{F}$ the bilinear form which satisfies $xy = b(x,y)1+\lambda s+m$ for some $\lambda\in\mathbb{F}$ and $m\in\mathcal{M}$. This is well defined due to (4.3.2) and (4.3.3).

Lemma 5.2.42 ([BD23])**.** *For any G-grading on $\mathcal{A}$, b is a nondegenerate homogeneous bilinear form of degree e.*

Proof. In order to show that b is nondegenerate, we take $0\neq m\in\mathcal{M}$ and we are going to show that there is an element m' such that $b(m,m')\neq 0$. Take $r_1,r_2,r_3\in\mathcal{K}$ such that $m=r_1x_1+r_2x_2+r_3x_3$. Without loss of generality, assume that $r_1\neq 0$ (otherwise, after applying a homomorphism of the form φ^σ as in Corollary 4.3.13 we are in this case). Since r_1 is either a zero divisor or an invertible element, then, either $r_1=\beta e_\sigma$ for $\beta\in\mathbb{F}^\times$ and $\sigma=\pm$ or $r_1\overline{r_1}=\beta 1$ for $\beta\in\mathbb{F}^\times$. In the first case, $b(m,2e_{-\sigma}x_1)=\beta$ and in the second case $b(m,r_1x_1)=\beta$.

b is homogeneous of degree e, due to the fact that $\mathbb{F}1$ is a subspace of $\mathcal{A}_e$ and that $\mathcal{S}\oplus\mathcal{M}$ is a graded subspace. □

Now we are going to define two gradings on $(\mathcal{A},\Gamma)$.

Definition 5.2.43. Let $(\mathcal{A},-)$ be a split quartic Cayley algebra. Let $g_1,g_2\in G$ be such that $g_1^3=e=g_2^3$. We denote by $\Gamma^1_{\mathrm{SQC}}(G,g_1,g_2)$ the grading induced by $\deg(s)=e$, $\deg(e_+(x_1+\zeta x_2+\zeta^2x_3))=g_1$ and $\deg(e_+(x_1+\zeta^2x_2+\zeta x_3))=g_2$.

The previous grading is well defined due to the fact that there are two commuting automorphisms of order 3, $\varphi^\sigma_{\mathbb{F}}\circ\theta_{\mathbb{F}}((\zeta^2e_++\zeta e_-,\zeta^2e_++\zeta e_-),\mathrm{id})$ and $\varphi^\tau_{\mathbb{F}}\circ\theta_{\mathbb{F}}((\zeta^2e_++\zeta e_-,\zeta^2e_++\zeta e_-),\mathrm{id})$, where σ is the permutation $(1,2,3)$, τ is the permutation $(1,3,2)$, and θ is defined as in Lemma 5.1.12.

Definition 5.2.44. Let $(\mathcal{A},-)$ be a split quartic Cayley algebra. Let $g\in G$ be an order 3 element and $h\in G$ an order 2 element. We denote by $\Gamma^2_{\mathrm{SQC}}(G,h,g)$ the grading induced by $\deg(s)=h$ and $\deg(x_1+\zeta x_2+\zeta^2x_3)=g$

The previous grading is well defined due to the fact that there are two commuting automorphisms, the order 3 automorphism $\varphi^\sigma_{\mathbb{F}}$, where σ is the permutation $(1,2,3)$, and the order 2 automorphism $\theta_{\mathbb{F}}((1,1),-)$, where θ is defined as in Lemma 5.1.12.

Lemma 5.2.45. *Let $(\mathcal{A},-)$ be a split quartic Cayley algebra. Let Γ be a grading such that $\deg(s)=e$. If $e_\sigma x_i$ is homogeneous for some $\sigma=\pm$ and $i\in\{1,2,3\}$, then Γ is a grading of $(\mathcal{A},-,\Gamma_i)$.*

Proof. This result is part of the proof [BD23, Theorem 3]. Here we give a proof for completeness.

Since we can apply the automorphisms φ^τ for some $\tau \in \mathrm{Sym}_3$ and $\theta_{\mathbb{F}}(1,1,-)$ (with the notation of Lemma 5.1.12), we can assume that $i=1$ and $\sigma=+$. Denote $\varphi(x)=b(e_+x_1,x)$. $\ker\varphi\cap\mathcal{M}_- = \mathbb{F}e_-x_2\oplus\mathbb{F}e_-x_3$ is a graded subspace due to Lemma 5.2.41 and Lemma 5.2.42.

Since $\mathcal{M}_-$ is a graded subspace, there should be a homogeneous element of the form $x=e_-(x_1+\lambda_2x_2+\lambda_3x_3)$ with $\lambda_2,\lambda_3\in\mathbb{F}$. We can divide the proof into three parts:

(1) In case $\lambda_2=0=\lambda_3$, e_-x_1 is homogeneous and this implies as before that $\mathbb{F}e_+x_2\oplus\mathbb{F}e_+x_3$ is a graded subspace, which, together with the fact that $\mathbb{F}e_-x_2\oplus\mathbb{F}e_-x_3$ is a graded subspace, implies that Γ is a grading of $(\mathcal{A},-,\Gamma_1)$.

(2) If $0\neq\lambda_2,\lambda_3$, since $b(e_+x_1,x)\neq 0$ and $b(x^2,x)\neq 0$, Lemma 5.2.42 implies that e_+x_1 and x^2 have the same degree. Therefore,

$$\frac{1}{2}x^2-\lambda_2\lambda_3e_+x_1=e_+(\lambda_3x_2+\lambda_2x_3)$$

is homogeneous, and it follows that the element

$$(e_+(\lambda_3x_2+\lambda_2x_3))^2=2\lambda_2\lambda_3e_-x_1$$

is homogeneous. Arguing as in (1), it implies that Γ is a grading of $(\mathcal{A},-,\Gamma_1)$.

(3) If $\lambda_2\neq 0$ but $\lambda_3=0$, then, $\frac{1}{2\lambda_2}x^2=e_+x_3$ and $\frac{1}{2\lambda_2}x^2(e_+x_1)=e_-x_2$ are homogeneous. Since $\mathcal{M}_+$ is a graded subspace and the elements e_+x_1 and e_+x_3 are homogeneous, there should be a homogeneous element $w=e_+(\beta_1x_1+x_2+\beta_3x_3)$ with $\beta_1,\beta_3\in\mathbb{F}$. Now, $w(e_+x_3)-\beta_1e_-x_2=e_-x_1$ is homogeneous and, as in (1) Γ is a grading of $(\mathcal{A},-,\Gamma_1)$.

□

Remark 5.2.46. If $\{i,j,k\}=\{1,2,3\}$, $\sigma=\pm$ and $\lambda_i,\lambda_j\in\mathbb{F}^\times$, then $[e_\sigma(\lambda_ix_i+\lambda_jx_j)]^2=2\lambda_i\lambda_je_{-\sigma}x_k$. Therefore, Lemma 5.2.45, implies that if Γ is a grading of the split quartic Cayley algebra $(\mathcal{A},-)$ but not of $(\mathcal{A},-,\Gamma_i)$ for some $i\in\{1,2,3\}$, then, if $e_\sigma(\lambda_1x_1+\lambda_2x_2+\lambda_3x_3)$ is homogeneous for some $\lambda_1,\lambda_2,\lambda_3\in\mathbb{F}$ and $\sigma=\pm$, it implies that λ_1,λ_2 and λ_3 are nonzero.

Proposition 5.2.47. *Let $(\mathcal{A}, -)$ be the split quartic Cayley algebra over an algebraically closed field $\mathbb{F}$. Let Γ be a G-grading of $(\mathcal{A}, -)$ with $\deg(s) = e$, which is not a grading for $(\mathcal{A}, -, \Gamma_i)$ for any $i \in \{1, 2, 3\}$. Then there are two different elements $g_1, g_2 \in G$ satisfying $g_1^3 = e = g_2^3$, such that $\Gamma \cong \Gamma^1_{\mathrm{SQC}}(G, g_1, g_2)$.*

Proof. This result is part of [BD23, Theorem 3]. Here we give a proof for completeness.

Since $\mathcal{M}_+$ is a graded subspaces of $\mathcal{A}$, take a homogeneous element $x = e_+(\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3)$. Due to Remark 5.2.46, $\lambda_1, \lambda_2, \lambda_3$ are nonzero. Therefore, we can assume that $\lambda_1\lambda_2\lambda_3 = 1$. Take another homogeneous element $y = e_+(\beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3)$ with $\beta_1, \beta_2, \beta_3 \in \mathbb{F}$ and $\beta_1\beta_2\beta_3 = 1$ in such way that $\deg x \neq \deg y$. This should exist, otherwise it would be in contradiction with Lemma 5.2.45. Denote $g_1 = \deg x$ and $g_2 = \deg y$. After a simple computation, we can check that $(x^2)^2 = 8x$ and $(y^2)^2 = 8y$. Therefore, $g_1^3 = g_2^3 = e$. Thus, $\deg x^2 = g_1^{-1}$ and $\deg y^2 = g_2^{-1}$. Additionally, since $g_1 g_2^2 \neq e \neq g_1^2 g_2$, it follows that $b(x, y^2) = 0 = b(y, x^2)$, which after computing it, we get:

$$\lambda_1\lambda_2\beta_3 + \lambda_2\lambda_3\beta_1 + \lambda_3\lambda_1\beta_2 = 0 = \beta_1\beta_2\lambda_3 + \beta_2\beta_3\lambda_1 + \beta_3\beta_1\lambda_2. \tag{5.2.20}$$

Moreover, the fact that $g_1 g_2 \neq g_1^2$ and $g_1 g_2 \neq g_2^2$, we deduce that $x^2 \neq xy \neq y^2$. Since

$$xy = e_-[(\lambda_2\beta_3 + \lambda_3\beta_2)x_1 + (\lambda_1\beta_3 + \lambda_3\beta 1)x_2 + (\lambda_2\beta_1 + \beta_2\lambda_1)x_3],$$

using (5.2.20), we get that

$$xy = e_-(-\lambda_2\lambda_3\beta_1\lambda_1^{-1}x_1 - \lambda_1\lambda_3\beta_2\lambda_2^{-1}x_2 - \lambda_1\lambda_2\beta_3\lambda_3^{-1}x_3).$$

The product of the coefficients is -1. Therefore, if we put $z' = -xy$, we have $(z'^2)^2 = 8z'$. Hence, for $z = \frac{1}{2}z'^2$, it follows that $z^2 = 4z'$ and $(z^2)^2 = 4z'^2 = 8(\frac{1}{2}z'^2) = 8z$. Then, the map sending $x \mapsto e_+(x_1 + \zeta x_2 + \zeta^2 x_3)$, $y \mapsto e_+(x_1 + \zeta^2 x_2 + \zeta x_3)$ and $z \mapsto e_+(x_1 + x_2 + x_3)$, extends to an automorphism. So the grading is isomorphic to $\Gamma^1_{\mathrm{SQC}}(G, g_1, g_2)$. The fact that $g_1 \neq g_2$ implies that g_1, g_2 and $(g_1g_2)^{-1}$ are different, otherwise if for example $g_1 = (g_1g_2)^{-1}$, this would imply that $e = g_1 g_2^2 = g_1 g_2^{-1}$, a contradiction. Thus, since g_1, g_2 and $(g_1g_2)^{-1}$ are different, the grading is not a grading of the algebra with any structurable grading. □

Lemma 5.2.48. *Let $(\mathcal{A}, -)$ be a split quartic Cayley algebra. Let Γ be a grading such that $\deg(s) = h$ for an element of order 2. If rx_i is homogeneous for some $0 \neq r \in \mathcal{K}$, and $i \in \{1, 2, 3\}$, then Γ is a grading of $(\mathcal{A}, -, \Gamma_i)$.*

Proof. This result is part of the proof of [BD23, Theorem 3]. Here we give a proof for completeness.

Assume that rx_i is homogeneous. Since $\deg(rx_i) \neq \deg(s(rx_i))$, it follows that r is not in $\mathbb{F}e_+ \cup \mathbb{F}e_-$. Then, $\mathcal{K}x_i = \mathbb{F}(rx_i) \oplus \mathbb{F}s(rx_i)$ is a graded subspace. Now, since

$$\{x \in \mathcal{M}(\mathcal{A}, -) \mid b(x, \mathcal{K}x_i) = 0\} = \mathcal{K}x_j \oplus \mathcal{K}x_k$$

for $\{i, j, k\} = \{1, 2, 3\}$, then Lemma 5.2.42 implies that $\mathcal{K}x_j \oplus \mathcal{K}x_k$ is a graded subspace. Thus, Γ is a grading of $(\mathcal{A}, -, \Gamma_i)$. □

Lemma 5.2.49. *Let $(\mathcal{A}, -)$ be a split quartic Cayley algebra. Let Γ be a grading such that $\deg(s) = h$ for an element of order 2. If $r_i x_i + r_j x_j$ is homogeneous for $0 \neq r_i, r_j \in \mathcal{K}$, and $i, j \in \{1, 2, 3\}$ with $i \neq j$, then Γ is a grading of $(\mathcal{A}, -, \Gamma_k)$ for some $k \in \{1, 2, 3\}$.*

Proof. This result is part of [BD23, Theorem 3]. Here we give a proof for completeness.

Without loss of generality, assume that $x = r_1x_1 + r_2x_2$ is homogeneous of degree g. Then, since the projection of x^2 on $\mathcal{M}$ is $2(\overline{r_1r_2})x_3$, Lemma 5.2.48 implies that either Γ is a grading of $(\mathcal{A}, -, \Gamma_3)$ or $r_1r_2 = 0$.

In case $r_1r_2 = 0$, since the only divisors of zero in $\mathcal{K}$ are scalar multiples of e_+ and e_-, we can assume that there are $\lambda_1, \lambda_2 \in \mathbb{F}^\times$ such that $x = \lambda_1e_+x_1 + \lambda_2e_-x_2$ is homogeneous. Notice that $e_+x_1 = \lambda_1^{-1}e_+x \in \mathcal{K}x$ and in an analogous way, $e_-x_2 \in \mathcal{K}x$.

Now, $sx = \lambda_1e_+x_1 - \lambda_2e_-x_2$ is another homogeneous element of degree gh. Define $\varphi_x(w) = b(x, w)$ for every $w \in \mathcal{M}(\mathcal{A}, -)$ and define φ_{sx} similarly. The subspace

$$\ker\varphi_x \cap \ker\varphi_{sx} = \{\lambda e_+x_1 + \beta e_-x_2 + tx_3 \mid \lambda, \beta \in \mathbb{F}, \quad t \in \mathcal{K}\}$$

is a graded subspace due to Lemma 5.2.42. Hence, there should be $\beta_1, \beta_2 \in \mathbb{F}$ and $t \in \mathcal{K} \setminus \{0\}$ such that

$$y = \beta_1e_+x_1 + \beta_2e_-x_2 + tx_3$$

is homogeneous. Moreover,

$$sy = \beta_1e_+x_1 - \beta_2e_-x_2 + stx_3.$$

Since

$$xy = \lambda_2\bar{t}e_+x_1 + \lambda_1\bar{t}e_-x_2,$$

Lemma 5.2.48 implies that either t is invertible or Γ is a grading of $(\mathcal{A}, -, \Gamma_k)$ for some $k = 1, 2$ (otherwise either $\bar{t}e_+ = 0$ or $\bar{t}e_- = 0$). Since $b(y, y) = t\bar{t}$, Lemma 5.2.42 implies that $\deg(y)^2 = e$. Moreover, since $0 \neq xy \in \mathcal{K}x$ and the elements x and sx have different degrees, we get that either $xy \in \mathbb{F}x$ or $xy \in \mathbb{F}sx$. Therefore, we deduce that either $\deg(y) = e$ or $\deg(sy) = e$. Without loss of generality, we can assume that $\deg(y) = e$. Since the projection of y^2 on $\mathcal{M}$ is

$$2\beta_2\bar{t}e_+x_1 + 2\beta_1\bar{t}e_-x_2 \in \mathbb{F}e_+x_1 \oplus \mathbb{F}e_-x_2 \subseteq \mathcal{K}x,$$

the fact that y has degree e implies that either $\deg(x) = e$ or $\deg(sx) = e$. We will assume that $\deg(x) = e$, otherwise the proof is analogous. In this case, either

$$z = y - \frac{\beta_1}{\lambda_1}x = tx_3$$

is homogeneous, in which case Lemma 5.2.48 implies that either Γ is a grading of $(\mathcal{A}, -, \Gamma_3)$, or there is $\delta \in \mathbb{F}^\times$ such that

$$z = y - \frac{\beta_1}{\lambda_1}x = \delta e_-x_2 + tx_3,$$

in which case $z^2 = t\bar{t} + 2\delta\bar{t}e_+x_1 \neq 0$ is homogeneous. This implies, that its projection to $\mathcal{M}(\mathcal{A}, -)$, which is $2\delta\bar{t}e_+x_1$ is homogeneous, which, due to Lemma 5.2.48 implies that Γ is a grading of $(\mathcal{A}, -, \Gamma_1)$. □

Remark 5.2.50. For an element $r \in \mathcal{K}$, since there is $\lambda \in \mathbb{F}$ such that $r\bar{r} = \lambda 1$, we can take a choice of $\sqrt{\lambda}$, and denote $\sqrt{r\bar{r}} = \sqrt{\lambda}1$. Whenever we write $\sqrt{r\bar{r}}$ we assume that we have chosen a square root of λ.

Proposition 5.2.51. *Let $(\mathcal{A}, -)$ be the split quartic Cayley algebra over an algebraically closed field $\mathbb{F}$. Let Γ be a G-grading of $(\mathcal{A}, -)$ with $\deg(s) = h$, which is not a grading for $(\mathcal{A}, -, \Gamma_i)$ for any $i \in \{1, 2, 3\}$. Then there is an order 3 elements $g \in G$ such that $\Gamma \cong \Gamma^2_{\mathrm{SQC}}(G, h, g)$.*

Proof. This result is part of [BD23, Theorem 3]. Here we give a proof for completeness.

Let $x = r_1x_1 + r_2x_2 + r_3x_3$ be a homogeneous element of Γ. Denote $y = \overline{r_2r_3}x_1 + \overline{r_1r_3}x_2 + \overline{r_1r_2}x_3$. y is the projection of $\frac{1}{2}x^2$ on $\mathcal{M}(\mathcal{A}, -)$. Thus, it is also homogeneous.

We can show by contradiction that there is $i \in \{1, 2, 3\}$ such that r_i is invertible. Indeed, assume r_1, r_2, r_3 are zero divisors, then since $\deg(y) \neq \deg(sy)$, by Lemmas 5.2.48 and 5.2.49 there are $a, b, c \in \mathbb{F}^\times$ and $\sigma = \pm$ such that up to a permutation of the indices, $r_1 = ae_\sigma$, $r_2 = be_\sigma$, $r_3 = ce_{-\sigma}$. Thus, $y = abe_{-\sigma}x_3$. This is a contradiction with Lemma 5.2.48. Moreover, if $\{i, j, k\} = \{1, 2, 3\}$, r_i is invertible but r_j and r_k are divisors of zero, we can get a contradiction as before taking y instead of x. Finally, if r_i, r_j are invertible but r_k is a divisor of zero, we get a contradiction taking the projection of $\frac{1}{2}y^2$ in $\mathcal{M}(\mathcal{A}, -)$. Thus, r_1, r_2 and r_3 are invertible.

Denote $\theta_1 = \theta_{\mathbb{F}}(\frac{1}{\sqrt{r_1\overline{r_1}}}\overline{r_1}, \frac{1}{\sqrt{r_2\overline{r_2}}}\overline{r_2}, \mathrm{id})$ (where $\theta_{\mathbb{F}}$ is the morphism defined in Lemma 5.1.12) thus, there is an isomorphic gradings in which $\theta_1(x)$ is homogeneous of the same degree and thus, we can assume that r_1 and r_2 are scalars. If $\deg(x) = g$, since $\mathrm{proj}_{\mathcal{M}(\mathcal{A}, -)}((\mathrm{proj}_{\mathcal{M}(\mathcal{A}, -)}(x^2))^2) = 8(r_1r_2r_3)x$ and the elements x and sx are homogeneous of different degrees, after multiplying by a suitable scalar, we can assume that either $r_1r_2r_3 = 1$ or $r_1r_2r_3 = s$. In the first case $r_3 \in \mathbb{F}$ and $g^3 = e$. In the second case, $(gh)^3 = e$ and we can replace x by sx to assume that $g^3 = e$. Using the previous argument, if we take a homogeneous basis of $\mathcal{M}$ of the form $\{y_1, sy_1, y_2, sy_2, y_3, sy_3\}$, since either $\deg(y_i)^3 = e$ or $\deg(sy_i)^3 = e$ for each $i \in \{1, 2, 3\}$, then, either there is an element whose degree has order 3 or there are three elements whose degree is e. We can rule out this last case because the fact that $\mathcal{K}$ has dimension 2 over $\mathbb{F}$ would lead to the existence of a nonzero homogeneous element of the form $t_1x_1 + t_2x_2$ with $t_1, t_2 \in \mathcal{K}$ which is a contradiction with Lemma 5.2.49. Thus, we can assume that g has order 3. Now, since the projections of x, x^2, x^2x, sx, sx^2 and $s(x^2x)$ on the graded subspace $\mathcal{M}(\mathcal{A}, -)$ have different degrees, they span $\mathcal{M}(\mathcal{A}, -)$.

Now, we can check that the morphism sending $s \mapsto s$ and $x \mapsto x_1 + \zeta x_2 + \zeta^2 x_3$ is an isomorphism sending Γ to $\Gamma^2_{\mathrm{SQC}}(G, h, g)$. □

Proposition 5.2.52. *Given an abelian group G, two order 2 elements $h', h' \in G$, elements $g_1, g_2, g'_1, g'_2 \in G$ such that $g_i^3 = e = g_i'^3$ for $i = 1, 2$, $g_1 \neq g_2$ and $g'_1 \neq g'_2$ and order 3 elements $g, g' \in G$, it follows that*

(1) $\Gamma^1_{\mathrm{SQC}}(G, g_1, g_2)$ is isomorphic to $\Gamma^2_{\mathrm{SQC}}(G, g'_1, g'_2)$ if and only if the triple $(g_1, g_2, (g_1g_2)^{-1})$ is a permutation of $(g'_1, g'_2, (g'_1g'_2)^{-1})$ or $(g_1'^{-1}, g_2'^{-1}, g'_1g'_2)$.

(2) $\Gamma^2_{\mathrm{SQC}}(G, h, g)$ is isomorphic to $\Gamma^2_{\mathrm{SQC}}(G, h', g')$ if and only if $h = h'$ and either $g = g'$ or $g = g'^2$.

Proof. Denote by $\theta_{\mathbb{F}}$ the morphism defined in Lemma 5.1.12.

(1) Let $\varphi \in \mathrm{Aut}(\mathcal{A}, -)$ sending $\Gamma^1_{\mathrm{SQC}}(G, g_1, g_2)$ to $\Gamma^2_{\mathrm{SQC}}(G, g'_1, g'_2)$. If $\varphi(s) = s$ we get that $\varphi(\mathcal{M}_+) = \mathcal{M}_+$, thus, it would follow that $(g_1, g_2, (g_1g_2)^{-1})$ is a permutation of $(g'_1, g'_2, (g'_1g'_2)^{-1})$. If $\varphi(s) = -s$ we get that $\varphi(\mathcal{M}_+) = \mathcal{M}_-$, and it would follow that $(g_1, g_2, (g_1g_2)^{-1})$ is a permutation of $(g_1'^{-1}, g_2'^{-1}, g'_1g'_2)$.

$\Gamma^1_{\mathrm{SQC}}(G, g_1, g_2)$ is isomorphic to $\Gamma^1_{\mathrm{SQC}}(G, g_2, g_1)$ via the morphism $\varphi^{\tau}_{\mathbb{F}}$ where τ is the transposition $(1, 2)$ and we use the notation of Corollary 4.3.13.

$\Gamma^1_{\mathrm{SQC}}(G, g_1, g_2)$ is isomorphic to $\Gamma^1_{\mathrm{SQC}}(G, (g_1g_2)^{-1}, g_2)$ via the automorphism $\theta_{\mathbb{F}}(1, \zeta e_+ + \zeta^2 e_-, \mathrm{id})$.

Finally, if φ denotes the nontrivial automorphism of $\mathcal{K}$, then, the grading $\Gamma^1_{\mathrm{SQC}}(G, g_1, g_2)$ is isomorphic to $\Gamma^1_{\mathrm{SQC}}(G, g_1^{-1}, g_2^{-1})$ via the automorphism $\theta(1, 1, \varphi)$.

Since Sym_3 is generated by the permutations $(1, 2)$ and $(1, 2, 3)$, this is enough to prove (1).

(2) If $\Gamma^2_{\mathrm{SQC}}(G, h, g)$ is isomorphic to $\Gamma^2_{\mathrm{SQC}}(G, h', g')$, $h = \deg(s) = h'$. Moreover, the groups $\langle h, g\rangle$ and $\langle h', g'\rangle$ are the support of each gradings, so they have to be equal and they are isomorphic to C_6, the cyclic group of order 6. Since C_6 only has two order 3 elements, we get that $g = g'$ or $g = g'^2$.

Finally, $\Gamma^2_{\mathrm{SQC}}(G, h, g)$ is isomorphic to $\Gamma^2_{\mathrm{SQC}}(G, h, g^2)$ via the automorphism $\varphi^{\tau}_{\mathbb{F}}$ where τ is the transposition $(2, 3)$.

□

5.2.4 Some special gradings

Let $(\mathcal{A}, -)$ be the split quartic Cayley algebra over an algebraically closed field. Previously, we have found some gradings which are not gradings of $(\mathcal{A}, -, \Gamma_i)$ for any structurable grading Γ_i. Now, due to Proposition 5.1.1 and Theorem 5.1.4, we have an isomorphism of group schemes:

$$\mathbf{Aut}(\mathcal{A}, -, \Gamma_i) \cong \mathbf{Aut}(\mathcal{A}(\mathcal{W}), -, \Delta(\mathcal{W}))$$

where $\mathcal{W} = \mathcal{W}(\mathcal{A}, -, \Gamma_i)$. Thus, the rest of the gradings come from a grading in $(\mathcal{A}(\mathcal{W}), -, \Delta(\mathcal{W}))$. Moreover, as a grading of $(\mathcal{A}, -, \Gamma_i)$ they are isomorphic if and only if they are isomorphic in $(\mathcal{A}(\mathcal{W}), -, \Delta(\mathcal{W}))$. However, although if two gradings of $(\mathcal{A}, -, \Gamma_i)$ are isomorphic, the corresponding gradings on $(\mathcal{A}, -)$ are isomorphic, the converse might not be true. Thus, we are going to tackle this issue in this subsection. In order to do so, we are going to show in Lemma 5.2.53 that this can only happen if the subspaces $\mathcal{K}x_i$ are graded, and classify up to isomorphism the gradings satisfying this property. This will be done in Propositions 5.2.56, 5.2.57 and 5.2.58. We will use the notation from (4.3.2) and from corollaries 4.3.13 and 4.3.14.

Lemma 5.2.53. *Let Γ and Γ' be two gradings of $(\mathcal{A}, -, \Gamma_1)$ such that $(\mathcal{A}, -, \Gamma)$ is isomorphic to $(\mathcal{A}, -, \Gamma')$ but $(\mathcal{A}, -, \Gamma_1, \Gamma)$ is not isomorphic to $(\mathcal{A}, -, \Gamma_1, \Gamma')$. Then the subspaces $\mathcal{K}x_i$ are graded subspaces for both gradings for every $i \in \{1, 2, 3\}$.*

Proof. Let φ be an isomorphism $(\mathcal{A}, -, \Gamma) \to (\mathcal{A}, -, \Gamma')$. Due to Proposition 5.1.11, in view of the fact that $(\mathcal{A}, -, \Gamma) \cong (\mathcal{A}, -, \Gamma')$ but $(\mathcal{A}, -, \Gamma_1, \Gamma)$ is not isomorphic to $(\mathcal{A}, -, \Gamma_1, \Gamma')$, there is a permutation $\sigma \in \mathrm{Sym}_3$ such that $\sigma(1) \neq 1$ and an automorphism $\psi \in \mathrm{Aut}(\mathcal{A}, -, \Gamma_{SQ})$ such that $\varphi = \varphi_{\mathbb{F}}^{\sigma} \circ \psi$. Moreover, without loss of generality, we can assume that $\sigma(1) = 2$. Now, $\mathcal{K}x_1$ is a graded subspace of Γ and Γ' due to the fact that these are gradings of Γ_1 and due to the fact that the subspaces $\mathcal{S}(\mathcal{A}, -)$ and $\mathcal{M}(\mathcal{A}, -)$ are graded subspaces. Now, $\varphi(\mathcal{K}x_1) = \mathcal{K}x_2$ so $\mathcal{K}x_2$ is a graded subspace in Γ'. Finally, $\mathcal{K}x_3 = (\mathcal{K}x_2)(\mathcal{K}x_1)$, which is a graded subspace in Γ'. The proof for Γ is analogous. $\square$

We are going to define some gradings in which the subspaces $\mathcal{K}x_i$ are all graded

Definition 5.2.54. Given $g_1, g_2 \in G$, denote by $\Gamma_{SQ}(G, g_1, g_2)$ the grading induced by $\deg(s) = e$, $\deg(e_+x_1) = g_1$, $\deg(e_+x_2) = g_2$, $\deg(e_+x_3) = (g_1g_2)^{-1}$, $\deg(e_-x_1) = g_1^{-1}$, $\deg(e_-x_2) = g_2^{-1}$, $\deg(e_-x_3) = g_1g_2$.

This is the grading associated to the morphism $H^D \to \mathbf{Aut}(\mathcal{A}, -)$ sending, for every unital, commutative associative algebra R, the homomorphism sending $\varphi \in \mathrm{Hom}_{\mathrm{Grp}}(H, R^\times)$ to $\theta_R((r_1e_+ + r_1^{-1}e_-, r_2e_+ + r_2^{-1}e_-), \mathrm{id})$ where H is the group generated by g_1 and g_2, $r_1 = \varphi(g_1)$, $r_2 = \varphi(g_2)$, H^D is the diagonalizable group scheme defined in Example 1.3.3 and θ is the morphism defined in Lemma 5.1.12.

Definition 5.2.55. Given elements $h, g_1, g_2 \in G$ of order at most 2 with $h \neq e$, denote the grading induced by $\deg(s) = h$, $\deg(x_1) = g_1$ and $\deg(x_2) = g_2$, by $\Gamma_{SQ}(G, h, g_1, g_2)$.

This is a grading induced by the morphisms $\theta_{\mathbb{F}}((-1, 1), \mathrm{id})$,$\theta_{\mathbb{F}}((1, -1), \mathrm{id})$ and $\theta_{\mathbb{F}}((1, 1), -)$ where $-$ is the involution on $\mathbb{F} \oplus \mathbb{F}$.

Proposition 5.2.56. *Let Γ be a grading of $(\mathcal{A}, -)$ such that for every $i \in \{1, 2, 3\}$, $\mathcal{K}x_i$ is graded, then either there are $g_1, g_2 \in G$ such that Γ is isomorphic to $\Gamma_{SQ}(G, g_1, g_2)$ or there are elements $h, g_1, g_2 \in G$ of order at most 2 with $h \neq e$, such that Γ is isomorphic to $\Gamma_{SQ}(G, h, g_1, g_2)$.*

Proof. This result is part of [BD23, Theorem 3]. Here we give a proof for completeness.

Assume first that $\deg(s) = e$. In this case, since the grading is compatible with Γ_{SQ}, then for every $i \in \{1, 2\}$, since $\mathcal{K}x_i$ has dimension 2, there is either $g_i \in G$ such that $\mathcal{K}x_i \subseteq \mathcal{A}_{g_i}$, in which case $g_i^{-1} = g_i$ or there are $g_i, g_i' \in G$, both different, such that $(\mathcal{A}_{g_i} \cap \mathcal{K}x_i) \oplus (\mathcal{A}_{g_i'} \cap \mathcal{K}x_i) = \mathcal{K}x_i$, and both $\mathcal{A}_{g_i} \cap \mathcal{K}x_i$ and $\mathcal{A}_{g'_i} \cap \mathcal{K}x_i$ are one dimensional. Since $s\mathcal{A}_{g_i} \subseteq \mathcal{A}_{g_i}$ and $s\mathcal{A}_{g'_i} \subseteq \mathcal{A}_{g'_i}$, we can assume without loss of generality that $\mathcal{A}_{g_i} \cap \mathcal{K}x_i = \mathbb{F}e_+x_i$ and $\mathcal{A}_{g'_i} \cap \mathcal{K}x_i = \mathbb{F}e_-x_i$. Since $b(\mathbb{F}e_+x_i, \mathbb{F}e_-x_i) \neq 0$, Lemma 5.2.42 implies that $g'_i = g_i^{-1}$. Now, since $(e_-x_1)(e_-x_2) = e_+x_3$ and $(e_+x_1)(e_+x_2) = e_-x_3$, it follows that the grading is isomorphic to $\Gamma_{SQ}(G, g_1, g_2)$.

In case that $\deg(s) = h$ for an order 2 element $h \in G$, since $\mathcal{K}x_i$ is graded for every $i \in \{1, 2, 3\}$, it follows that there are nonzero $r_1, r_2, r_3 \in \mathcal{K}$ such that r_ix_i is homogeneous of degree g_i for every $i \in \{1, 2, 3\}$. Moreover, since $\deg(s(r_ix_i)) = h \deg(r_ix_i)$, it follows that $r_i \notin \mathbb{F}e_+ \cup \mathbb{F}e_-$. Thus, it is invertible. After some multiplication by scalar, we can assume that $r_i\overline{r_i} = 1$ for every $i \in \{1, 2, 3\}$. Since $(r_ix_i)^2 = 1$, we get that $g_i^2 = e$. Moreover, since $(r_1x_1)(r_2x_2) = (\overline{r_1r_2})x_3$, we may assume that $r_3 = \overline{r_1r_2}$ and that $g_3 = g_1g_2$ and Γ is isomorphic to $\Gamma_{SQ}(G, h, g_1, g_2)$ via $\theta_{\mathbb{F}}(r_1, r_2, \mathrm{id})$, where θ is the morphism defined in Lemma 5.1.12. □

Proposition 5.2.57. *Let $(\mathcal{A}, -)$ be the split quartic Cayley algebra. Let $g_1, g_2, g'_1, g'_2 \in G$. Then, $\Gamma_{SQ}(G, g_1, g_2)$ is isomorphic to $\Gamma_{SQ}(G, g'_1, g'_2)$ if and only if $(g_1, g_2, (g_1g_2)^{-1})$ is a permutation of $(g'_1, g'_2, (g'_1g'_2)^{-1})$ or a permutation of $(g_1'^{-1}, g_2'^{-1}, (g'_1g'_2))$.*

Proof. Let φ be an automorphism of $(\mathcal{A}, -)$ sending the grading $\Gamma_{SQ}(G, g_1, g_2)$ to $\Gamma_{SQ}(G, g'_1, g'_2)$. In case φ sends $\mathcal{M}_+$ to $\mathcal{M}_+$. Then, since $\mathcal{M}_+$ is generated by homogeneous elements of degree g_1, g_2 and $(g_1g_2)^{-1}$ in $\Gamma_{SQ}(G, g_1, g_2)$ and by homogeneous elements of degree g'_1, g'_2 and $(g'_1g'_2)^{-1}$ in $\Gamma_{SQ}(G, g'_1, g'_2)$, it follows that $(g_1, g_2, (g_1g_2)^{-1})$ is a permutation of $(g'_1, g'_2, (g'_1g'_2)^{-1})$. Similarly, if φ sends $\mathcal{M}_+$ to $\mathcal{M}_-$, we get that $(g_1, g_2, (g_1g_2)^{-1})$ is a permutation of $(g_1'^{-1}, g_2'^{-1}, (g'_1g'_2))$.

Now, denote $g_3 = (g_1, g_2)^{-1}$ and $g'_3 = (g'_1g'_2)^{-1}$. In case $\sigma \in \mathrm{Sym}_3$ is such that $g_{\sigma(i)} = g'_i$ for all $i \in \{1, 2, 3\}$, $\varphi_{\mathbb{F}}^{\sigma}$ is automorphism of $(\mathcal{A}, -)$ sending $\Gamma_{SQ}(G, g_1, g_2)$ to $\Gamma_{SQ}(G, g'_1, g'_2)$. In case $\sigma \in \mathrm{Sym}_3$ is such that $g_{\sigma(i)} = g_i'^{-1}$, using the notation of Lemma 5.1.12, if $-$ is the involution of $\mathcal{K}$, we get that $\varphi_{\mathbb{F}}^{\sigma} \circ \theta(1, 1, -)$ is automorphism of $(\mathcal{A}, -)$ sending $\Gamma_{SQ}(G, g_1, g_2)$ to $\Gamma_{SQ}(G, g'_1, g'_2)$. □

Proposition 5.2.58. *Let $(\mathcal{A}, -)$ be the split quartic Cayley algebra. Let $h, g_1, g_2, h', g'_1, g'_2 \in G$ be elements of order at most 2 with $h, h' \neq e$. Then, $\Gamma_{SQ}(G, h, g_1, g_2)$ is isomorphic to $\Gamma_{SQ}(G, h', g'_1, g'_2)$ if and only if $h = h'$ and there is a map $\alpha \colon \{1, 2, 3\} \to \{e, h\}$ such that $\alpha(1)\alpha(2)\alpha(3) = e$ and such that (g_1, g_2, g_3) is a permutation of $(g'_1\alpha(1), g'_2\alpha(2), g'_3\alpha(3))$ where $g_3 = g_1g_2$ and $g'_3 = g'_1g'_2$.*

Proof. If $\Gamma_{SQ}(G, h, g_1, g_2)$ is isomorphic to $\Gamma_{SQ}(G, h', g'_1, g'_2)$, since $h = \deg(s)$ on the first grading and $h' = \deg(s)$ on the second, $h = h'$. In $\Gamma_{SQ}(G, h, g_1, g_2)$,

$\mathcal{M}$ has a $\mathcal{K}$-basis consisting of homogeneous elements of degree g_1, g_2 and g_3. In $\Gamma_{SQ}(G, h, g_1', g_2')$, $\mathcal{M}$ has a $\mathcal{K}$-basis consisting of homogeneous elements of degree g_1', g_2' and g_3'. Denote by $\overline{g}$ the class of g modulo $\langle h\rangle$. Then the previous argument shows that, there is a permutation $\sigma \in \mathrm{Sym}_3$ such that $\overline{g}_{\sigma(j)} = \overline{g'}_j$ for each $j \in \{1,2,3\}$. Define $\alpha\colon \{1,2,3\} \to \{e,h\}$ by $\alpha(j) = g_{\sigma(j)} g_j'^{-1}$. α is well defined since $\overline{g}_{\sigma(j)} = \overline{g'}_j$. Moreover, $\alpha(1)\alpha(2)\alpha(3) = e$ due to the fact that g_1, g_2, g_1' and g_2' have order at most 2. Hence, (g_1, g_2, g_3) is a permutation of $(g_1'\alpha(1), g_2'\alpha(2), g_3'\alpha(3))$.

If $h = h'$ and (g_1, g_2, g_3) is a permutation of $(g_1'\alpha(1), g_2'\alpha(2), g_3'\alpha(3))$, let $\sigma \in \mathrm{Sym}_3$ be such that $g_{\sigma(j)} = g_j'\alpha(j)$ for each $j \in \{1,2,3\}$. Denote $r_j = is$ if $\alpha(j) = h$ and $r_j = 1$ if $\alpha(j) = e$. Using the notation of Lemma 5.1.12, we have that $\varphi_{\mathbb{F}}^{\sigma} \circ \theta_{\mathbb{F}}(r_1, r_2, \mathrm{id})$ is an automorphism of $(\mathcal{A}, -)$ sending $\Gamma_{SQ}(G, h, g_1, g_2)$ to $\Gamma_{SQ}(G, h', g_1', g_2')$. □

5.2.5 Gradings on structurable algebras

Up to now, we found an isomorphism between the automorphism group scheme of algebras with a structurable grading and some associative algebras with a 3-grading and an involution (see Proposition 5.1.1 and Theorem 5.1.4). We found that over algebraically closed fields, except in the case of the split quartic Cayley algebra, this isomorphism is between the structurable algebra and the associative algebra with a 3-grading and an involution. Thus, in order to classify their gradings it is enough to classify the gradings on the associative setting, which is something we have already done.

On the split quartic Cayley algebra, there are some gradings which don't come from the associative setting, we have found those and classified these up to isomorphism (see Proposition 5.2.47, 5.2.51 and 5.2.52). There are some others which come from the associative setting as in Definition 5.2.38, but due to Lemma 5.2.53, it might not enough to classify them up to isomorphism in the associative setting in order to classify them up to isomorphism. We have also given a classification up to isomorphism on this case (see Propositions 5.2.56, 5.2.57 and 5.2.58). And finally, there are some which come from the associative setting and we can classify them up to isomorphism using only their classification on the associative algebra.

In the subsequent paragraphs, we are going to connect the gradings we know in the split quartic Cayley algebra with the gradings on the associative algebras with the 3-grading and the involution. Then, we are going to use this knowledge to classify the gradings up to isomorphism in this algebra (Theorem 5.2.63). Finally, we are going to give a classification up to isomorphism of the gradings on the structurable algebras with the uniquenes property (Theorem 5.2.65).

Denote by π_1 the projection of $\mathbb{Z} \times G$ onto $\mathbb{Z}$.

Remark 5.2.59. Given a group G, an elementary 2-subgroup T, a nondegenerate alternating bicharacter $\beta\colon T \times T \to \mathbb{F}$, κ_0, κ_1, γ_0, γ_1 and $g \in G$ as in (5.2.3), (5.2.4), (5.2.5) and (5.2.6) and $\delta = \pm 1$, let M be the graded algebra with involution $\mathcal{M}(G, T, \beta, \kappa_0, \kappa_1, \gamma_0, \gamma_1, \delta, g)$. In this case, $L(\mathbf{W}(M))$ is isomorphic to $\mathcal{M}(G, T, \beta, \kappa_0, \gamma_0, \delta, g)$.

Proposition 5.2.60. *Let $(\mathcal{A}, -)$ be the split quartic Cayley algebra and let $\mathbb{F}$ be algebraically closed. Given a group G, an elementary 2-subgroup T, a nondegenerate alternating bicharacter $\beta\colon T \times T \to \mathbb{F}$, κ_0, κ_1, γ_0, γ_1 and $g \in G$ as in* (5.2.3), (5.2.4), (5.2.5) *and* (5.2.6) *and $\delta = \pm 1$, then there are elements $h, g_1, g_2 \in G$ of order at most 2 with $h \neq e$ such that*

$$\mathbf{GrA}(\mathcal{M}(G, T, \beta, \kappa_0, \kappa_1, \gamma_0, \gamma_1, \delta, g)) \cong (\mathcal{A}, -, \Gamma_{\mathrm{SQ}}(G, h, g_1, g_2))$$

if and only if either

(1) $T \cong \mathbb{Z}_2^2$, $\kappa_0 = \kappa_1 = (1)$, $\gamma_0 = (g^{(0)})$, $\gamma_1 = (g^{(1)})$, $\delta = 1$ and $(g^{(0)})^2 = (g^{(1)})^2$.

(2) $T = \{e\}$, $\kappa_0 = (1, 1)$, $\kappa_1 = (1, 1)$, $\gamma_0 = (g_1^{(0)}, g_2^{(0)})$, $\gamma_1 = (g_1^{(1)}, g_2^{(1)})$, $\delta = 1$, they satisfy $g_1^{(0)}(g_1^{(1)})^{-1} = g_2^{(0)}(g_2^{(1)})^{-1}$ and the relations are:

$$(g_1^{(0)})^2 = (g_2^{(0)})^2 = (g_1^{(1)})^2 = (g_2^{(1)})^2 = g.$$

Proof. Denote $g_3 = g_1 g_2$. Denote just by M the graded algebra with involution $\mathcal{M}(G, T, \beta, \kappa_0, \kappa_1, \gamma_0, \gamma_1, \delta, g)$. Assume first that

$$\mathbf{GrA}(M) \cong (\mathcal{A}, -, \Gamma_{\mathrm{SQ}}(G, h, g_1, g_2))$$

by an isomorphism φ. Then there is a structurable grading $\Gamma_i = \mathcal{E}_i \oplus \mathcal{W}_i$ of $(\mathcal{A}, -)$ such that $\varphi(\mathbf{W}(M)) = \mathcal{W}_i$ and $\varphi(L(\mathbf{W}(M))) = \mathcal{E}_i$. We may assume that $i = 1$. There are three possibilities: $g_1 \neq e, h$, $g_1 = h$ or $g_1 = e$.

In case $g_1 \neq e, h$, since $(L(\mathbf{W}(M)), -) \cong (\mathcal{E}_1, -)$, then $(L(\mathbf{W}(M)), -)$ is a graded division algebra with involution, since $1, s, x_1$ and sx_1 are homogeneous of different degrees, of dimension 4 and the involutions is orthogonal. Thus, Remark 5.2.59 implies that $T \cong \mathbb{Z}_2^2$, $\kappa_0 = (1)$ and $\delta = 1$. Since $\mathcal{W}_1$ has dimension 4 and it is isomorphic to $\mathbf{W}(M)$, it follows that $\kappa_1 = (1)$. Now, the relations are:

$$(g^{(0)})^2 t^{(0)} = (g^{(0)})^2 t^{(1)} = g$$

for some $t^{(0)}, t^{(1)} \in T$. Now, $\mathbf{GrA}(M)$ is the algebra from item (7) in Example 5.2.39 therefore, since there should be an element $(0, d) \in \mathbf{W}(M)$ which we can identify with x_2 in $\mathcal{A}$, it should satisfy that $(1, 0) = (0, d)^2 = (d(X_{t^{(1)}})^{-1}\widehat{d}(X_{t^{(0)}}), 0)$, comparing degrees in $\mathcal{D}_{\text{inv}}(T, \beta)$, we find that $t^{(0)} = t^{(1)}$, which implies $(g^{(0)})^2 = (g^{(1)})^2$.

In case $g_1 = h$, due to proposition 5.2.58, we can assume that $g_1 = e$.

In case $g_1 = e$, since $(L(\mathbf{W}(M)), -) \cong (\mathcal{E}_1, -)$ which is not a graded division algebra and it has dimension 4, it follows that $T = \{e\}$. Since the involution is orthogonal $\delta = 1$. Moreover, since $\operatorname{Supp} L(\mathbf{W}(M)) = \{e, h\}$, it follows that $\kappa_0 = (1, 1)$ and $\gamma_0 = (g_1^{(0)}, g_2^{(0)})$. The relation

$$g_1^{(0)} g_2^{(0)} = g$$

is not possible since in this case, the space of skew symmetric elements of $L(\mathbf{W}(M))$ would be spanned by the element

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

which has degree e. However, in $(\mathcal{E}_1, -)$ it has degree h. Thus the relation

$$(g_1^{(0)})^2 = (g_2^{(0)})^2 = g$$

holds and the skew hermitian element has degree $g_1^{(0)}(g_2^{(0)})^{-1}$. Now, since $\mathcal{W}_i$ has dimension 4, and it is isomorphic to $\mathbf{W}(M)$, then κ_1 is either (2) or (1, 1). In case $\kappa_1 = (2)$, this is the graded algebra from item (1) in Example 5.2.39. Using this model, there is no element on the graded subspaces described in (5.2.12) and (5.2.13) whose square is 1. However, in $\mathcal{W}_i$ the element x_{i+1} satisfies this property. Therefore, this case is not possible. In case $\kappa_1 = (1, 1)$, $\gamma_1 = (g_1^{(1)}, g_2^{(1)})$ and the relations:

$$(g_1^{(0)})^2 = (g_2^{(0)})^2 = g_1^{(1)} g_2^{(1)} = g$$

we are in item (2) of Example 5.2.39. Since we have two graded subspaces given in (5.2.14) and (5.2.15), which contain all the homogeneous elements, checking the product of an element on this subspace with itself, we can check that there is no homogeneous element in $\mathbf{W}(M)$ whose square is 1, so arguing as before, this case is not possible.

Finally, the relations

$$(g_1^{(0)})^2 = (g_2^{(0)})^2 = (g_1^{(1)})^2 = (g_2^{(1)})^2 = g$$

imply that we are in item (6) of Example 5.2.39. Since $\operatorname{Supp}\mathcal{W}_1 = \{e, h, g_1, hg_1\}$, this implies that $g_1 = e$ or $g_1 = h$. Thus, $\operatorname{Supp}\mathcal{W}_1$ has two elements. Therefore, as shown in the example, unless $g_1^{(0)}(g_1^{(1)})^{-1} = g_2^{(0)}(g_2^{(1)})^{-1}$, this case is not possible.

Conversely, if $T \cong \mathbb{Z}_2^2$ and $\kappa_0 = \kappa_1 = (1)$, we just need to show that x_1, x_2 and x_3 are homogeneous in $\mathbf{GrA}(M)$. We can take the choice from Example 4.3.17.

In case $T = \{e\}$, $\kappa_0 = (1,1)$, $\kappa_1 = (1,1)$, $\gamma_0 = (g_1^{(0)}, g_2^{(0)})$, $\gamma_1 = (g_1^{(1)}, g_2^{(1)})$, $\delta = 1$, such that $g_1^{(0)}(g_2^{(1)})^{-1} = g_2^{(0)}(g_1^{(1)})^{-1}$ and the relations

$$(g_1^{(0)})^2 = (g_2^{(0)})^2 = (g_1^{(1)})^2 = (g_2^{(1)})^2 = g$$

are satisfied, we can take the choice from Example 4.3.16. □

Proposition 5.2.61. *Let $(\mathcal{A}, -)$ be the split quartic Cayley algebra and let $\mathbb{F}$ be algebraically closed. Given a group G, an elementary 2-subgroup T, a nondegenerate alternating bicharacter $\beta\colon T \times T \to \mathbb{F}$, κ_0, κ_1, γ_0, γ_1 and $g \in G$ as in* (5.2.3), (5.2.4), (5.2.5) *and* (5.2.6) *and $\delta = \pm 1$, then there are elements $g_1, g_2 \in G$ such that*

$$\mathbf{GrA}(\mathcal{M}(G, T, \beta, \kappa_0, \kappa_1, \gamma_0, \gamma_1, \delta, g)) \cong (\mathcal{A}, -, \Gamma_{\mathrm{SQ}}(G, h, g_1, g_2))$$

if and only if either:

(1) There are $g'^{(0)}, g''^{(0)}, g'^{(1)}, g''^{(1)} \in G$ such that $T = \{e\}$, $\kappa_0 = (1,1)$, $\kappa_1 = (1,1)$, $\gamma_0 = (g'^{(0)}, g''^{(0)})$, $\gamma_1 = (g'^{(1)}, g''^{(1)})$, $\delta = 1$ and the relations are:

$$g'^{(0)}g''^{(0)} = g'^{(1)}g''^{(1)} = g.$$

(2) There are $g'^{(0)}, g''^{(0)}, g^{(1)} \in G$ such that $T = \{e\}$, $\kappa_0 = (1,1)$, $\kappa_1 = (2)$, $\gamma_0 = (g'^{(0)}, g''^{(0)})$, $\gamma_1 = (g^{(1)})$, $\delta = 1$ and the relations are:

$$g'^{(0)}g''^{(0)} = (g^{(1)})^2 = g.$$

(3) There are $g_1^{(0)}, g_2^{(0)}, g_1^{(1)}, g_2^{(1)} \in G$ such that $T = \{e\}$, $\kappa_0 = (2)$, $\kappa_1 = (1,1)$, $\gamma_0 = (g^{(0)})$, $\gamma_1 = (g'^{(1)}, g''^{(1)})$, $\delta = 1$ and the relations are:

$$(g^{(0)})^2 = g'^{(1)}g''^{(1)} = g.$$

(4) $T = \{e\}$, $\kappa_0 = (2)$, $\kappa_1 = (2)$ and $\delta = 1$.

Proof. Denote $\mathcal{M}(G, T, \beta, \kappa_0, \kappa_1, \gamma_0, \gamma_1, \delta, g)$ just by M. Assume first that

$$\mathbf{GrA}(M) \cong (\mathcal{A}, -, \Gamma_{SQ}(G, g_1, g_2))$$

by an isomorphism φ. Then there is a structurable grading $\Gamma_i = \mathcal{E}_i \oplus \mathcal{W}_i$ of $(\mathcal{A}, -)$ such that $\varphi(\mathbf{W}(M)) = \mathcal{W}_i$ and $\varphi(L(\mathbf{W}(M))) = \mathcal{E}_i$. Assume that $i = 1$.

Since $(\mathcal{E}_1, -)$ is isomorphic to $(L(\mathbf{W}(M)), -)$, and we know that the involution is orthogonal, $L(\mathbf{W}(M))$ has dimension 4, it is not a graded-division algebra and $\operatorname{Supp} L(\mathbf{W}(M)) = \{e, g_1, g_1^{-1}\}$, it follows that $\delta = 1$, $T = \{e\}$ and either $\kappa_0 = (1, 1)$ if $g_1 \neq e$, or $\kappa_0 = (2)$ if $g_1 = e$. In the first case, since the skew-hermitian elements of $(\mathcal{E}_1, -)$, have degree e, then, the skew-hermitian elements of $(L(\mathbf{W}(M)), -)$ should have degree e. Thus, if $\gamma_0 = (g'^{(0)}, g''^{(0)})$, it should happen that

$$g'^{(0)} g''^{(0)} = g.$$

In case $\kappa_0 = (1, 1)$ and $\kappa_1 = (1, 1)$, We are in either item (3) or item (4) in Example 5.2.39. We use the same notation as in the examples and $\gamma_1 = (g'^{(1)}, g''^{(1)})$. Now, there is a homogeneous element $y_1 \in \mathcal{W}_1 \cap \mathcal{M}_+$ of degree g_2 and a homogeneous element $y_2 \in \mathcal{W}_1 \cap \mathcal{M}_-$ of degree g_2^{-1} such that $y_1 + y_2$ can be identified with x_2. Since x_2^2, it follows that on the model, either $y_1 = (0, E_{1,1})$ and $y_2 = (0, E_{2,2})$ or $y_1 = (0, E_{1,2})$ and $y_2 = (0, E_{2,1})$. Thus, since the degree of y_1 is the inverse of the degree of y_2, from the possible equalities, we get:

$$g'^{(0)} g''^{(0)} = g'^{(1)} g''^{(1)} = g.$$

In case $\kappa_0 = (2)$, since $g_1 = e$, it follows that the support of $\mathcal{W}_1$ is $\{g_2, g_2^{-1}\}$. In case $g_2 = g_2^{-1}$, it follows that the only possibility is $\kappa_1 = (2)$, and in case $g_2 \neq g_2^{-1}$, it follows that it follows that the only possibility is $\kappa_1 = (1, 1)$, with $\gamma_1 = (g'^{(1)}, g''^{(1)})$. If we denote $\gamma_0 = (g^{(0)})$, in this case, we have that the support of $\mathcal{W}_1$ is $\{g^{(0)}(g'^{(1)})^{-1}, g^{(0)}(g''^{(1)})^{-1}\}$. Since $g^{(0)}(g'^{(1)})^{-1} = (g^{(0)}(g''^{(1)})^{-1})^{-1}$ and $(g^{(0)})^2 = g$, the relations:

$$(g^{(0)})^2 = g'^{(1)} g''^{(1)} = g$$

must hold.

Thus, if we have the isomorphism then it has to happen either (1), (2), (3) or (4).

Now, if we have (1), (2), (3) or (4), we need to show that on $\mathbf{GrA}(M)$ we have homogeneous elements which we can identify with e_+x_2, e_-x_2, e_+x_3, e_-x_3. In case (4) it is clear since in $\mathbf{W}(M)$ every element is homogeneous. In the other cases it it proved in Example 5.2.39. □

Definition 5.2.62. Given a group G, an elementary 2-subgroup T, a non-degenerate alternating bicharacter $\beta\colon T\times T\to\mathbb{F}$, κ_0, κ_1, γ_0, γ_1 and $g\in G$ as in (5.2.3), (5.2.4), (5.2.5) and (5.2.6) and $\delta=\pm1$, we say that

$$(G,T,\beta,\kappa_0,\kappa_1,\gamma_0,\gamma_1,1,g)$$

has the **property** SQ, if it satisfies any of the properties from Propositions 5.2.60 and 5.2.61

Theorem 5.2.63. *Let $(\mathcal{A},-)$ be the split quartic Cayley algebra and let Γ be a grading. Then, either:*

(1) There are $g_1,g_2\in G$ such that $(\mathcal{A},-,\Gamma)$ is isomorphic to the split quartic Cayley algebra with the gradings $\Gamma_{SQ}(G,g_1,g_2)$.

(2) There are elements $h,g_1,g_2\in G$ of order at most 2 with $h\neq e$, such that $(\mathcal{A},-,\Gamma)$ is isomorphic to the split quartic Cayley algebra with the grading $\Gamma_{SQ}(G,h,g_1,g_2)$.

(3) There are g_1,g_2 of order 3 satisfying that g_1,g_2 and $(g_1g_2)^{-1}$ are different and such that $(\mathcal{A},-,\Gamma)$ is isomorphic to the split quartic Cayley algebra with the grading $\Gamma^1_{\mathrm{SQC}}(G,g_1,g_2)$.

(4) There is an order 2 element h and an order 3 element g such that that $(\mathcal{A},-,\Gamma)$ is isomorphic to the split quartic Cayley algebra with the grading $\Gamma^2_{\mathrm{SQC}}(G,h,g)$.

(5) $(\mathcal{A},-,\Gamma)$ is isomorphic to $\mathbf{GrA}(\mathcal{M}(G,T,\beta,\kappa_0,\kappa_1,\gamma_0,\gamma_1,1,g))$ where either:

(a) $T=\{e\}$, $\kappa_0=(2)$, $\kappa_1=(1,1)$, $\gamma_0=(g^{(0)})$, $\gamma_1=(g_1^{(1)},g_2^{(1)})$ and the relations are:
$$(g^{(0)})^2=(g_1^{(1)})^2=(g_2^{(1)})^2=g.$$

(b) $T=\{e\}$, $\kappa_0=(1,1)$, $\kappa_1=(2)$, $\gamma_0=(g_1^{(0)},g_2^{(0)})$ and $\gamma_1=(g^{(1)})$ and the relations are:
$$(g_1^{(0)})^2=(g_2^{(0)})^2=(g^{(1)})^2=g.$$

(c) $T=\{e\}$, $\kappa_0=(1,1)$, $\kappa_1=(1,1)$, $\gamma_0=(g'^{(0)},g''^{(0)})$ and $\gamma_1=(g_1^{(1)},g_2^{(1)})$ and the relations are:

$$g'^{(0)}g''^{(0)}=(g_1^{(1)})^2=(g_2^{(1)})^2=g.$$

(d) $T=\{e\}$, $\kappa_0=(1,1)$, $\kappa_1=(1,1)$, $\gamma_0=(g_1^{(0)},g_2^{(0)})$ and $\gamma_1=(g'^{(1)},g''^{(1)})$ and the relations are:

$$(g_1^{(0)})^2=(g_2^{(0)})^2=g'^{(1)}g''^{(1)}=g.$$

(e) $T = \{e\}$, $\kappa_0 = (1,1)$, $\kappa_1 = (1,1)$, $\gamma_0 = (g_1^{(0)}, g_2^{(0)})$, $\gamma_1 = (g_1^{(1)}, g_2^{(1)})$, *satisfying* $g_1^{(0)}(g_1^{(1)})^{-1} \neq g_2^{(0)}(g_2^{(1)})^{-1}$, *and the relations are:*

$$(g_1^{(0)})^2 = (g_2^{(0)})^2 = (g_1^{(1)})^2 = (g_2^{(1)})^2 = g.$$

(f) $T \cong \mathbb{Z}_2^2$, $\kappa_0 = (1)$, $\kappa_1 = (1)$, $\gamma_0 = (g^{(0)})$, $\gamma_1 = (g^{(1)})$, *satisfying* $(g^{(0)}) \neq (g(1))^2$, *and the relations are:*

Moreover, the graded algebras in different item are non isomorphic and

(i) *If* $(\mathcal{A}, -)$ *is the split quartic Cayley algebra and* $g_1, g_2, g_1', g_2' \in G$. *Then,* $(\mathcal{A}, -, \Gamma_{SQ}(G, g_1, g_2))$ *is isomorphic to* $(\mathcal{A}, -, \Gamma_{SQ}(G, g_1', g_2'))$ *if and only if* $(g_1, g_2, (g_1g_2)^{-1})$ *is a permutation of* $(g_1', g_2', (g_1'g_2')^{-1})$ *or* $(g_1'^{-1}, g_2'^{-1}, (g_1'g_2'))$.

(ii) *If* $(\mathcal{A}, -)$ *is the split quartic Cayley algebra and* $h, g_1, g_2, h', g_1', g_2' \in G$ *are elements of order at most* 2 *with* $h, h' \neq e$. *Then, the graded algebra with involution* $(\mathcal{A}, -, \Gamma_{SQ}(G, h, g_1, g_2))$ *is isomorphic to* $(\mathcal{A}, -, \Gamma_{SQ}(G, h', g_1', g_2'))$ *if and only if* $h = h'$ *and there is a map* $\alpha\colon \{1,2,3\} \to \{e, h\}$ *such that* $\alpha(1)\alpha(2)\alpha(3) = h$ *and such that the triple* (g_1, g_2, g_3) *is a permutation of* $(g_1'\alpha(1), g_2'\alpha(2), g_3'\alpha(3))$ *where* $g_3 = g_1g_2$ *and* $g_3' = g_1g_2$.

(iii) *Given order* 3 *elements* $g_1, g_2, g_1', g_2' \in G$ *such that* g_1, g_2 *and* $(g_1g_2)^{-1}$ *are different and* g_1', g_2' *and* $(g_1'g_2')^{-1}$ *are different, the split quartic Cayley algebra with the grading* $\Gamma^1_{\mathrm{SQC}}(G, g_1, g_2)$ *is isomorphic to the split quartic Cayley algebra with the grading* $\Gamma^1_{\mathrm{SQC}}(G, g_1', g_2')$ *if and only if* $(g_1, g_2, (g_1g_2)^{-1})$ *is a permutation of* $(g_1', g_2', (g_1'g_2')^{-1})$ *or* $(g_1'^{-1}, g_2'^{-1}, g_1'g_2')$.

(iv) *For two order* 2 *elements* $h, h' \in G$ *and two order* 3 *elements* $g, g' \in G$ *the split quartic Cayley algebra with the grading* $\Gamma^2_{\mathrm{SQC}}(G, h, g)$ *is isomorphic to the split quartic Cayley algebra with the grading* $\Gamma^1_{\mathrm{SQC}}(G, h', g')$ *if and only if* $h = h'$ *and either* $g = g'$ *or* $g = g'^{-1}$.

(v) *The graded algebra* $\mathbf{GrA}(\mathcal{M}(G, T, \beta, \kappa_0, \kappa_1, \gamma_0, \gamma_1, 1, g))$ *is isomorphic to* $\mathbf{GrA}(\mathcal{M}(G, T', \beta', \kappa_0', \kappa_1', \gamma_0', \gamma_1', 1, g'))$ *if and only if there is* $T = T'$, $\beta = \beta'$, $f \in G$ *such that* $\Xi(\kappa_0, \gamma_0) = f\Xi(\kappa_0', \gamma_0')$, $\Xi(\kappa_1, \gamma_1) = f\Xi(\kappa_1', \gamma_1')$ *and* $g = g'f^{-1}$.

Proof. Due to Propositions 5.2.47 and 5.2.51, it follows that if Γ is not a grading of $(\mathcal{A}, -, \Gamma_i)$ for one of the structurable gradings $\Gamma_i\colon \mathcal{A} = \mathcal{E}_i \oplus \mathcal{W}_i$, we are in case (1) or (2). In other case, due to Proposition 5.1.1, Theorem 5.1.4, Proposition 4.2.29 and the fact that $\mathcal{E}_i$ is simple, there are, an elementary 2-subgroup T of G, a nondegenerate alternating bicharacter $\beta\colon T \times T \to \mathbb{F}$, κ_0, κ_1, γ_0, γ_1 and $g \in G$ as in (5.2.3), (5.2.4), (5.2.5) and (5.2.6) and $\delta = \pm 1$. If $(G, T, \beta, \kappa_0, \kappa_1, \gamma_0, \gamma_1, 1, g)$ has the property *SQ*, due to Propositions 5.2.60 and 5.2.61, we are in cases (3) and (4). Otherwise, since $\mathcal{A}$ is the split quartic

Cayley algebra if and only if $\mathcal{W}_i$ has dimension 4, $\mathcal{E}_i$ has dimension 4 and the involution of $\mathcal{E}_i$ is orthogonal, we get that $\delta = 1$, $T = \{e\}$ and some of the possibilities in $(a), (b), (c), (d)$ or (e) holds.

For the second part, we see that each item is not isomorphic due to the fact that in items (i) and (ii) the gradings are not gradings of $(\mathcal{A}, -, \Gamma_i)$ for any of the structurable gradings Γ_i. Moreover, in item (1), the skew symmetric elements have degree e and in item (2) they have degree of order 2. Now, the gradings in item (5) are not isomorphic to the gradings in (3) and (4) due to Propositions 5.2.60 and 5.2.60. Finally, the gradings in item (3) satisfy that the skew symmetric elements have degree e and in item (4) they have degree of order 2. Finally, $(i) - (iv)$ are a consequence of Propositions 5.2.52, 5.2.57, 5.2.58, and (v) is a consequence of Lemma 5.2.53, the fact that $(G, T, \beta, \kappa_0, \kappa_1, \gamma_0, \gamma_1, 1, g)$ doesn't have the Property SQ and Theorem 5.2.37. □

Definition 5.2.64. Given a group G, an elementary 2-subgroup T, a nondegenerate alternating bicharacter $\beta \colon T \times T \to \mathbb{F}$, κ_0, κ_1, γ_0, γ_1 and $g \in G$ as in (5.2.3), (5.2.4), (5.2.5) and (5.2.6) and $\delta = \pm 1$, we say that

$$(G, T, \beta, \kappa_0, \kappa_1, \gamma_0, \gamma_1, 1, g)$$

is a **split quartic-uple** if either:

(1) $\delta = 1$, $T = \mathbb{Z}_2^2$ and $\kappa_0 = (1) = \kappa_1$

(2) $\delta = 1$, $T = \{e\}$ and $\kappa_0, \kappa_1 \in \{(2), (1, 1)\}$.

Theorem 5.2.65. *Let $(\mathcal{A}, -, \Gamma)$ be a central simple structurable algebra with the uniqueness property and a G-grading. Then one of the following holds:*

(1) There is an elementary 2-subgroup T of G, a nondegenerate alternating bicharacter $\beta \colon T \times T \to \mathbb{F}$, $k_0, k_1 > 0$, $\gamma_0 \in G^{k_0}$ and $\gamma_1 \in G^{k_1}$ consisting on distinct elements modulo T, $\kappa_0 \in \mathbb{Z}_{>0}^{k_0}$ and $\kappa_1 \in \mathbb{Z}_{>0}^{k_1}$ such that $(\mathcal{A}, -, \Gamma)$ is isomorphic to $\mathbf{GrA}(\mathcal{M}(G, \mathcal{D}(T, \beta), \kappa_0, \kappa_1, \gamma_0, \gamma_1)^{\mathrm{ex}})$.

(2) There are, an elementary 2-subgroup T of G, a nondegenerate alternating bicharacter $\beta \colon T \times T \to \mathbb{F}$, κ_0, κ_1, γ_0, γ_1 and $g \in G$ as in (5.2.3), (5.2.4), (5.2.5) and (5.2.6) and $\delta = \pm 1$ satisfying that $(G, T, \beta, \kappa_0, \kappa_1, \gamma_0, \gamma_1, \delta, g)$ is not a split quartic-uple, such that $(\mathcal{A}, -, \Gamma)$ is isomorphic to $\mathbf{GrA}(\mathcal{M}(G, T, \beta, \kappa_0, \kappa_1, \gamma_0, \gamma_1, \delta, g))$.

(3) There are, an elementary 2-subgroup T of G, a nondegenerate alternating bicharacter $\beta \colon T \times T \to \mathbb{F}$, an order 2 element $t \in G \setminus T$, κ_0, κ_1, γ_0, γ_1 and $g \in G$ as in (5.2.3), (5.2.4), (5.2.5) and (5.2.6), and $\delta = \pm 1$ such that $(\mathcal{A}, -, \Gamma)$ is isomorphic to $\mathbf{GrA}(\mathcal{M}(G, T, \beta, t, \kappa_0, \kappa_1, \gamma_0, \gamma_1, 1, g))$.

Moreover, the graded algebras on different item are non isomorphic and:

(i) The graded algebra $\mathbf{GrA}(\mathcal{M}(G, \mathcal{D}(T, \beta), \kappa_0, \kappa_1, \gamma_0, \gamma_1)^{\mathrm{ex}})$ *is isomorphic to* $\mathbf{GrA}(\mathcal{M}(G, \mathcal{D}(T', \beta'), \kappa'_0, \kappa'_1, \gamma'_0, \gamma'_1)^{\mathrm{ex}})$ *if and only if either:*

  (a) $T = T'$, $\beta = \beta'$ *and there is* $f \in G$ *such that* $\Xi(\kappa_0, \gamma_0) = f\Xi(\kappa'_0, \gamma'_0)$ *and* $\Xi(\kappa_1, \gamma_1) = f\Xi(\kappa'_1, \gamma'_1)$ *or*

  (b) $T = T'$, $\beta = \beta'$ *and there is* $f \in G$ *such that* $\Xi(\kappa_0, \gamma_0) = f\Xi(\kappa'_0, \gamma_0'^{-1})$ *and* $\Xi(\kappa_1, \gamma_1) = f\Xi(\kappa_1'^{-1}, \gamma'_1)$.

(ii) The graded algebra $\mathbf{GrA}(\mathcal{M}(G, T, \beta, \kappa_0, \kappa_1, \gamma_0, \gamma_1, \delta, g))$ *is isomorphic to* $\mathbf{GrA}(\mathcal{M}(G, T', \beta', \kappa'_0, \kappa'_1, \gamma'_0, \gamma'_1, \delta', g'))$ *if and only if* $T = T'$, $\beta = \beta'$, $\delta = \delta'$ *and there is* $f \in G$ *such that* $\Xi(\kappa_0, \gamma_0) = f\Xi(\kappa'_0, \gamma'_0)$, $\Xi(\kappa_1, \gamma_1) = f\Xi(\kappa'_1, \gamma'_1)$ *and* $g = g'f^{-1}$.

(iii) The graded algebra $\mathbf{GrA}(\mathcal{M}(G, T, \beta, t, \kappa_0, \kappa_1, \gamma_0, \gamma_1, 1, g))$ *is isomorphic to* $\mathbf{GrA}(\mathcal{M}(G, T', \beta', t', \kappa'_0, \kappa'_1, \gamma'_0, \gamma'_1, 1, g'))$ *if and only if* $T\langle t\rangle = T'\langle t'\rangle$, $t = t'$, $\beta^{[t]} = \beta^{[t']}$ *and there is* $f \in G$ *such that* $\Xi(\kappa_0, \gamma_0) = f\Xi(\kappa'_0, \gamma'_0)$, $\Xi(\kappa_1, \gamma_1) = f\Xi(\kappa'_1, \gamma'_1)$ *and* $g = g'f^{-1}$.

Proof. Let $\Gamma^S \colon \mathcal{A} = \mathcal{E} \oplus \mathcal{W}$ be a structurable grading of $\mathcal{A}$. Let $(\mathcal{B}, \bar{\varphi}, \Delta_{\mathbb{Z}}) = \mathbf{A}(\mathfrak{W}(\mathcal{A}, \varphi, \Gamma^S))$. In case, Γ is a grading of $(\mathcal{A}, \varphi, \Gamma^S)$, then due to Theorem 5.1.7, we have that there is a G-grading

$$\Delta_G \colon \mathcal{B} = \bigoplus_{g \in G} B_g$$

of $(\mathcal{B}, \bar{\varphi}, \Delta_{\mathbb{Z}})$ such that the grading Γ is induced by $\deg(w) = g$ for all $w \in \mathcal{W}$ such that $w \in \mathcal{B}_g$. If we denote by

$$\Delta_{\mathbb{Z}\times G} \colon \mathcal{B} = \bigoplus_{(i,g) \in \mathbb{Z}\times G} \mathcal{B}_{(i,g)}$$

Where for each $i \in \mathbb{Z}$ and $g \in G$, $\mathcal{B}_{(i,g)})\mathcal{B}_i \cap \mathcal{B}_g$, we get that

$$(\mathcal{A}, -, \Gamma) \cong \mathbf{GrA}(\mathcal{B}, \bar{\varphi}, \Delta_{\mathbb{Z}\times G})$$

Thus, in view of Theorem 5.2.37 and Propositions 5.2.60, 5.2.61 and the fact that $(\mathcal{A}, -)$ is a split quartic Cayley algebra if and only if the involution restricted to $\mathcal{E}$ is orthogonal and $\dim_{\mathbb{F}} \mathcal{E} = 4 = \dim_{\mathbb{F}} \mathcal{W}$ as shown in subsection 4.3.1, it follows that the algebra falls into one of the cases $(1), (2)$ or (3).

The graded algebras with involution on item (1) are not isomorphic to the rest due to the fact that they are not graded simple algebras. The graded algebras with involution on item (2) are not isomorphic to the algebras in (3) due to the fact that they are not simple algebras.

Finally, $(i), (ii)$ and (iii) follow from Theorem 5.2.37. □

Conclusions

The main results we have tackled in this thesis can be divided into two:

(1) Results related with r-fold cross products.

(2) Results related with structurable algebras related to an hermitian form.

- **r-fold cross products**

  Our main contribution to this topic has been the determination of the automorphism group scheme of each r-fold cross product (V, X, b) on dimension n. These are the following:

  ⋆ (Case n even $r = 1$) In this case $\mathbf{Aut}(V, X) = \mathbf{Cent}_{\mathbf{GL}(V)}(X)$ and $\mathbf{Aut}(V, X, b) = \mathbf{U}(V, h)$ for a hermitian form $h\colon V \times V \to \mathbb{K} = \mathbb{F}\mathrm{id} \oplus \mathbb{F}X$ given by:

  $$h(u, v) = b(u, v)\mathrm{id} - b(X(u), v)X$$

  ⋆ (Case $n \geq 3$, $r = n - 1$) In this case $\mathbf{Aut}(V, X) = \tilde{\mathbf{O}}(V, b)$ and $\mathbf{Aut}(V, X, b) = \mathbf{O}^+(V, b)$.

  ⋆ (Case $n = 2$, $r = 7$) In this case $\mathbf{Aut}(V, X) = \mathbf{Aut}(V, X, b)$ is a simple affine group scheme of type G_2.

  ⋆ (Case $n = 3$, $r = 8$) In this case $\mathbf{Aut}(V, X) = \mathbf{Aut}(V, X, b)$ is isomorphic to $\mathbf{Spin}(\mathcal{C}_0, -n)$ for a Cayley algebra $\mathcal{C}$.

  Finally, we have determined the gradings up to isomorphism. From there we have found that we have fine gradings universal groups given in the following list:

  ⋆ (Case n even $r = 1$) In this case we have a fine grading with universal group $\mathbb{Z}^n$.

- ⋆ (Case $n \geq 3$, $r = n-1$) In this case we have fine gradings with universal groups $\mathbb{Z}^q \times \mathbb{Z}/(q-1)$ (where $2q = n$), $\mathbb{Z}^q \times \mathbb{Z}/(2q-1)$ (where $n = 2q+1$) and $\mathbb{Z}^q \times \mathbb{Z}/(2n-4) \times (\mathbb{Z}/(2))^{p-2}$ (where $p + 2q = n$ and $p > 1$).
- ⋆ (Case $n = 2$, $r = 7$) In this case we have fine gradings with universal groups $\mathbb{Z}^2$ and $\mathbb{Z}_2^3$.
- ⋆ (Case $n = 3$, $r = 8$) In this case we have fine gradings with universal groups $\mathbb{Z}^3$ and $\mathbb{Z}_2^4$.

- **Structurable algebras**

  In this case we had a few contribulions:

  - ⋆ We have defined the concept of a structurable grading and we have shown how important it is by finding an equivalence of categories between the category of central simple algebras with a structurable gradings and a categroy of 3-graded associative algebras with an involution which inverts the degrees.
  - ⋆ We found that over an algebraically closed field, the split quartic Cayley algebra is the only algebra which has more than one structurable gradings. Concretely, it has 3.
  - ⋆ We found that the automorphism group scheme of the split quartic Cayley algebra is isomorphic to $(\mathbf{G}_{\mathrm{m}} \times \mathbf{G}_{\mathrm{m}}) \rtimes (\mathrm{C}_2 \times \mathrm{Sym}_3)$.
  - ⋆ We found that the automorphism group scheme of a central simple algebra with a structurable grading is isomorphic to the automorphism group scheme of the 3-graded associative algebras with an involution which inverts the degrees.
  - ⋆ We used these results in order to give a complete classification up to isomorphism of the gradings on central simple structurable algebras related to an hermitian form.

Future work

After classifying these algebras, the only class of central simple structurable algebras in which gradings have not been classified is the class of structurable algebras with skew dimension 1. Thus, as a future work we will classify their gradings up to isomorphism.

Conclusiones

Los principales resultados obtenidos en esta tesis se pueden dividir en dos:

(1) Resultados relacionados con r-fold cross products.

(2) Resultados relacionados con álgebras estructurables relacionadas con una forma hermítica.

- **r-fold cross products**

  Nuestra principal contribución a este tema ha sido la determinación de los esquemas de automorphismos para cada r-fold cross product (V, X, b) en dimensión n. Estos son los siguientes:

  ⋆ (Caso n par $r = 1$) En este caso $\mathbf{Aut}(V, X) = \mathbf{Cent}_{\mathbf{GL}(V)}(X)$ and $\mathbf{Aut}(V, X, b) = \mathbf{U}(V, h)$ para una forma hermítica $h\colon V \times V \to \mathbb{K} = \mathbb{F}\mathrm{id} \oplus \mathbb{F}X$ dada como:

  $$h(u, v) = b(u, v)\mathrm{id} - b(X(u), v)X$$

  ⋆ (Caso $n \geq 3$, $r = n - 1$) En este caso $\mathbf{Aut}(V, X) = \tilde{\mathbf{O}}(V, b)$ y $\mathbf{Aut}(V, X, b) = \mathbf{O}^+(V, b)$.

  ⋆ (Caso $n = 2$, $r = 7$) En este caso $\mathbf{Aut}(V, X) = \mathbf{Aut}(V, X, b)$ es un esquema en grupo simple de tipo G_2.

  ⋆ (Caso $n = 3$, $r = 8$) En este caso $\mathbf{Aut}(V, X) = \mathbf{Aut}(V, X, b)$ es isomorfo a $\mathbf{Spin}(\mathcal{C}_0, -n)$ para un álgebra de Cayley $\mathcal{C}$.

  Finalmente, hemos determinado sus graduaciones salvo isomorfismo. Con ello, hemos encontrado las graduaciones finas, cuyos grupos universales vienen dados en la siguiente lista:

  ⋆ (Caso n par $r = 1$) En este caso tenemos una graduación fina con grupo universal $\mathbb{Z}^n$.

- ⋆ (Caso $n \geq 3$, $r = n - 1$) En este caso tenemos graduaciones finas con grupos universales $\mathbb{Z}^q \times \mathbb{Z}/(q-1)$ (donde $2q = n$), $\mathbb{Z}^q \times \mathbb{Z}/(2q-1)$ (donde $n = 2q+1$) y $\mathbb{Z}^q \times \mathbb{Z}/(2n-4) \times (\mathbb{Z}/(2))^{p-2}$ (donde $p + 2q = n$ y $p > 1$).
- ⋆ (Caso $n = 2$, $r = 7$) En este caso tenemos graduaciones finas con grupos universales $\mathbb{Z}^2$ y $\mathbb{Z}_2^3$.
- ⋆ (Caso $n = 3$, $r = 8$) En este caso tenemos graduaciones finas con grupos universales $\mathbb{Z}^3$ y $\mathbb{Z}_2^4$.

- **Álgebras estructurables**

  En este caso, tenemos las siguientes contribuciones:

  - ⋆ Hemos definido el concepto de graduación estructurable y hemos enseñado cómo de importante es dado que hemos encontrado una equivalencia de categorías entre la categorías de algebras con una graduación estructurable centrales y simples y una categoría de algebras asociativas 3-graduadas con una involución que invierte los grados.
  - ⋆ Hemos probado que sobre cuerpos algebraicamente cerrados, la split quartic Cayley algebra es él único álgebra que tiene más de una graduación estructurable. Concretamente, tiene 3.
  - ⋆ Hemos probado que el esquema de automorfismos de la split quartic Cayley algebra es isomorfo a $(\mathbf{G}_{\mathrm{m}} \times \mathbf{G}_{\mathrm{m}}) \rtimes (\mathrm{C}_2 \times \mathrm{Sym}_3)$.
  - ⋆ central y simple con una graduación estructurable es isomorfo al esquema de automorfismos del álgebra 3-graduadas con involución correspondiente
  - ⋆ Hemos usado estos resultados para dar una clasificción completa salvo isomorfismos de las graduaciones en álgebras centrales y simples relacionadas con una forma hermítica.

Future work

Después de clasificar estas álgebras, la única clase de álgebras estructurables centrales y simples que quedan es la clase de álgebras con dimensión antisimétrica 1. Por tanto, como trabajo futuro, podremos encontrar una clasificación salvo isomorfismo de las álgebras de esta clase.

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