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Robust fixed-time distributed optimization with predefined convergence-time bound

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ABSTRACT

This paper introduces a distributed optimization scheme for achieving formation control in multi-agent systems operating under switching networks and external disturbances. The proposed approach utilizes the zero-gradient sum property and consists of two steps. First, it guides each agent towards the minimizer of its respective local cost function. Subsequently, it achieves a formation around the minimizer of the global cost function. The distributed optimization scheme guarantees convergence before a predefined time, even under simultaneous switching networks and external disturbances, distinguishing it from existing finite and fixed-time schemes. Moreover, the algorithm eliminates the need for agents to exchange local gradients or Hessians of the cost functions or even prior knowledge of the number of agents in the network. Additionally, the proposed scheme copes with external disturbances using integral sliding modes. The scheme's effectiveness is validated through an application to distributed source localization, for which several numerical results are provided.

1. Introduction

Recently, there has been a growing interest in consensus-based distributed optimization due to advancements in distributed computing and large-scale networks [1]. This research area has found applications in various domains, including data-based networks, robotics, unmanned aerial vehicles, social and economic networks, smart grids, and epidemic networks. At the same time, the consensus and formation control of Multi-Agent Systems (MAS) have gained significance in control engineering. Motion coordination applications often require multiple agents to achieve and sustain a desired geometric pattern centered around an optimal location for the entire team. These applications encompass various tasks such as exploration, surveillance, robotics, target localization, and satellite formation flight [2].

Distributed source localization is one specific application that highlights the relevance of distributed optimization. This application is studied in various contexts, including pollution source localization, search and rescue missions, gas source localization, acoustic source localization, and more [3,4]. To achieve this task, the use of a single agent often leads to poor performance in the case of large search environments. Therefore, the problem of source localization using multiple agents has been recently investigated [5–7]. As discussed in [8,9], such practical problems can be formulated as a distributed optimization problem, where the objective is to

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cooperatively minimize the sum of local cost functions, a problem which has been studied in [10–13]. Based on the local knowledge of each agent, the optimal value of the global cost function is reached.

Additional requirements should sometimes be considered while solving the distributed optimization problem. For instance, agents should form a specific geometric pattern around the optimal location for the entire team [14]. This leads to the simultaneous optimization and formation control problem [15], which is more general than the consensus problem due to the offset between the final position of the agent and the optimal point according to the cost function.

For many practical applications, the agents should reach the optimal solutions of the corresponding optimization problem within a given time (e.g., the source localization needs to be quickly achieved in the case of disaster) [16]. In order to take into account such convergence time constraints, many algorithms use the fixed-time concept introduced in [17] and further extended in [18]. One can refer to the following papers [19–21]. Despite the interesting properties of such algorithms, it is challenging to design the control gains since the link between the Upper Bound of the Settling Time (*UBST*) and the system parameters is not sufficiently explicit. To deal with this issue, [22–24] introduced the concept of fixed-time stable systems with a predefined *UBST* as a design parameter. As shown in [22–26], one of the main advantages of using this design approach is that it can provide stronger performance guarantees than traditional methods, particularly in distributed optimization [27]. By constraining the time for optimization, the algorithm can be designed to terminate once the solution is obtained rather than continuing to iterate indefinitely. This can significantly reduce the computational resources required and improve the overall scalability of the algorithm. This is particularly important in applications where real-time performance is critical, such as in robotics or autonomous systems.

An increasing amount of work concerning time constraints has been drawn recently in the literature. For instance, a predefined-time optimization algorithm has been proposed in [28] without converging to the ideal global optimum. A three-stage algorithm, which needs an exchange of the gradients of the local cost functions, has been investigated in [29]. In [30], the upper and lower bounds of the Hessians of local cost functions are needed to implement the distributed optimization scheme. A prescribed-time distributed optimization algorithm has been proposed in [16] using time-varying gains that tend to infinity as the time approaches the prescribed convergence time, yielding to inherent robustness and performance limitations [31]. In [27,32,33], without the need for auxiliary variables or time-variable gains, the predefined-time simultaneous formation control and distributed optimization are investigated. It is worth noting that the papers mentioned above assume that the graph topology is fixed and that no external disturbance occurs. However, the problem of varying topologies over time and the presence of external disturbances altering the system dynamics are frequent situations in MAS. In [12], a distributed optimization scheme with adversarial agents over switching topologies is investigated. While this algorithm tackles the consensus problem, it achieves asymptotic convergence to the optimum. A distributed fixed-time optimization algorithm for consensus under time-varying communication topology has been proposed in [20] while requiring the exchange of gradients among agents. In [34], an optimization algorithm for integrator chain systems based on signal generators has been proposed. The algorithm is significantly superior compared to similar algorithms in the literature due to the consideration of matched and unmatched disturbances. However, the settling time depends on the initial condition of the agents, and only fixed topologies are considered. Notice that none of the papers mentioned above address time constraints, switching topologies, and disturbances simultaneously, and this is where our research becomes valuable.

In this paper, the suggested approach is a distributed fixed-time optimization scheme with predefined *UBST* that guarantees formation control for multi-agent systems over switching networks and in the presence of external disturbances. The proposed scheme is divided into two steps. First, the controller guides each agent towards the minimizer of its respective local cost function. Then, the distributed optimization and formation control is achieved. Indeed, all agents converge towards the minimizer of the global cost function. Contrary to many existing algorithms, the proposed scheme satisfies the zero-gradient sum property [35]. Besides, it possesses several noteworthy characteristics, including:

- The proposed algorithm can be used on arbitrarily strongly convex cost functions and requires fewer tuning parameters compared to many works in the literature.
- Contrary to [19,30,36], the proposed algorithm can be used both in formation and consensus control.
- The *UBST* can be assigned according to the desired requirements, contrary to many existing finite and fixed-time schemes.
- Contrary to [29], agents are not required to exchange the local gradients or Hessians of the cost functions.
- Unlike most existing works, the proposed algorithm does not rely on prior knowledge of the number of agents in the network. Furthermore, the scheme deals with switching topologies and handles disturbances through integral sliding modes.

The remainder of this article is organized as follows. Preliminaries and important Lemmas are presented in Section 2. The problem formulation is given in Section 3. Section 4 introduces the proposed predefined-time scheme, including its stability analysis. The applicability for source localization is verified through several numerical experiments in Section 5. Concluding remarks are given in Section 6.

2. Preliminaries

2.1. Notation

Let $\mathbb{R}$ denote the set of real numbers and $\mathbb{R}^n$ the n-dimensional Euclidean space. For $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{x}^\top$ denotes its transpose and $\|\mathbf{x}\|$ its Euclidean norm. Denote $\mathbf{0}_n = [0, \dots, 0]^\top \in \mathbb{R}^n$. $\mathbf{I}_n$ is the identity matrix with dimensions $n \times n$. For a twice differentiable function $f : \mathbb{R}^n \to \mathbb{R}$, ∇f and $\nabla^2 f$ represent the gradient and Hessian of the function, respectively. For matrices $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{B} \in \mathbb{R}^{p\times q}$, $\mathbf{A} \otimes \mathbf{B} \in \mathbb{R}^{mp\times nq}$ denotes the Kronecker product. For any real number h, the function $\|\lfloor\bullet\rceil\|^h : \mathbb{R}^n \to \mathbb{R}^n$ is defined as $\|\lfloor\mathbf{x}\rceil\|^h = \mathbf{x}\|\mathbf{x}\|^{h-1}$ for any $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}_n\}$. Moreover, if $h > 0$, $\|\lfloor\mathbf{0}_n\rceil\|^h = \mathbf{0}_n$.

2.2. Graph theory

Let $\mathcal{X} = (\mathcal{V}, \mathcal{E})$ denote a graph, where $\mathcal{V} = 1, 2, \ldots, N$ is the set of agents and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges. The corresponding weighted adjacency matrix is $\mathbf{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$ with $a_{ij} > 0$ if $(j, i) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. Assuming that no self-loops are present, $a_{ii} = 0$, $\forall i \in \mathcal{V}$. The neighbor set of agent i is defined as $\mathcal{N}_i = \{j \in \mathcal{V} | (j, i) \in \mathcal{E}\}$. The Laplacian matrix $\mathbf{L} = [l_{ij}] \in \mathbb{R}^{N\times N}$ associated with $\mathcal{X}$ is defined as $l_{ij} = -a_{ij}$ for $i \neq j$ and $l_{ii} = \sum_{j=1,j\neq i}^{N} a_{ij}$. If the graph is undirected and connected, the eigenvalues of the Laplacian matrix $\mathbf{L}$ are $0 < \lambda_2 \leq \cdots \leq \lambda_N$, where λ_2 is the algebraic connectivity of the graph. For more details see [37].

In practice, the network structure may change due to communication issues. This feature can be modeled using a switching dynamic network. Let $\mathcal{F} = \{\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_K\}$ be a collection of K graphs having the same vertex set and $\vartheta : [0, \infty) \to \mathcal{F}$ is a switching signal determining the topology of the dynamic network at each instant of time. For $\vartheta(t) = k$, the adjacency and Laplacian matrices, and the neighbor set of agent i are denoted as $\mathbf{A}_k$, $\mathbf{L}_k$, and $\mathcal{N}_i^k$, respectively.

2.3. Convex analysis

A twice continuously differentiable convex function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be θ-strongly convex and Θ-smooth if the following condition hold

$$\theta \mathbf{I}_n \leq \nabla^2 f(\mathbf{x}) \leq \Theta \mathbf{I}_n. \tag{1}$$

with $\theta, \Theta > 0$. An important consequence of (1) is that the following property is attained [38, Section 9.1.2]:

$$\frac{\theta}{2}\|\mathbf{x}-\mathbf{y}\|^2 \leq f(\mathbf{y}) - f(\mathbf{x}) - \nabla f(\mathbf{x})^\top(\mathbf{y}-\mathbf{x}) \leq \frac{\Theta}{2}\|\mathbf{y}-\mathbf{x}\|^2, \tag{2}$$

If f is a θ-strongly convex function, then its minimizer $\mathbf{x}^* = \arg\min_{\mathbf{x}\in\mathbb{R}^n} f(\mathbf{x})$ is unique. The reader may refer to [38–40] for more details.

2.4. Stability notions

The reader may refer to [41], [25, Section II] and [42, Section III]. Consider the autonomous system

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t); \boldsymbol{\rho}), \quad \mathbf{x}_0 := \mathbf{x}(0) \tag{3}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ is the system state and $\boldsymbol{\rho} \in \mathbb{R}^b$ with $\dot{\boldsymbol{\rho}} = 0$ represents the tunable parameters of the system. Function $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ may be discontinuous, and such that the solutions of (3) exist and are unique in the sense of Filippov. Thus, $\boldsymbol{\Phi}(t, \mathbf{x}_0)$ denotes the solution of (3) and $\mathbf{x} = \mathbf{0}_n$ is the unique equilibrium point.

Definition 1. The origin of (3) is said to be finite-time stable if it is Lyapunov stable and for any $\mathbf{x}_0 \in \mathbb{R}^n$ there exists $0 \leq T < \infty$ such that $\boldsymbol{\Phi}(t, \mathbf{x}_0) = 0$ for all $t \geq T$. The function $T(\mathbf{x}_0) = \inf\left\{T \geq 0 : \boldsymbol{\Phi}(t, \mathbf{x}_0) = 0, \forall t \geq T\right\}$ is called the settling-time function of system (3).

Definition 2. The origin of (3) is fixed-time stable if it is finite-time stable and if there exists a $T_{\max} < \infty$ such that $\sup_{x_0\in\mathbb{R}^n} T(x_0) \leq T_{\max}$.

Definition 3 (*[43], Definition 4*)**.** For the parameter vector $\boldsymbol{\rho}$ of system (3) and an arbitrarily selected constant $T_c := T_c(\boldsymbol{\rho}) > 0$, the origin of (3) is said to be fixed-time stable with T_c as a predefined Upper Bound of the Settling Time (*UBST*) if the settling time function $T(\mathbf{x}_0)$ is uniformly bounded as $T(\mathbf{x}_0) \leq T_c, \forall \mathbf{x}_0 \in \mathbb{R}^n$. In this case, T_c is called a predefined time.

Proposition 1 (*[44],Theorem 3.1*)**.** *If there exists a continuous, positive definite and radially unbounded function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that the time derivative of V along the trajectories of (3) satisfies*

$$\dot{V}(\mathbf{x}(t)) \leq -\frac{1}{\alpha s T_c}\exp(\alpha V(\mathbf{x}(t))^s) V(\mathbf{x}(t))^{1-s}, \tag{4}$$

for $\mathbf{x}(t) \in \mathbb{R}^n \setminus \{\mathbf{0}_n\}$ and constants $T_c := T_c(\boldsymbol{\rho}) > 0$, $\alpha > 0, s \in (0, 1/2]$, then the origin of (3) is fixed-time stable with T_c as a predefined UBST.

2.5. Important lemmas

Lemma 1 (*[45], Theorem 2.1*)**.** *Consider a family of time-dependent switching systems of the form $\dot{\mathbf{x}}(t) = \mathbf{f}_{\vartheta(t)}(\mathbf{x}(t))$ with $\mathbf{f}_{\vartheta(t)} \in \{\mathbf{f}_1, \ldots, \mathbf{f}_n\}$. If all systems in the family share a radially unbounded common Lyapunov function, then the switched system is globally uniformly asymptotically stable regardless of the switching signal $\vartheta(t) \in \{1, \ldots, k\}$.*

Lemma 2 (*[46], Lemma 4.18*). *For a general undirected graph $\mathcal{G}$ with weighted adjacency matrix $\mathbf{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$ and for given vectors $\mathbf{x}_i, \mathbf{y}_i \in \mathbb{R}^n$, it follows that*

$$\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\mathbf{x}_i^\top(\mathbf{y}_i - \mathbf{y}_j) = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}(\mathbf{x}_i - \mathbf{x}_j)^\top(\mathbf{y}_i - \mathbf{y}_j).$$

Lemma 3 (*[27], Lemma 2*). *Let $f(x) = \exp\left(kx^{2s}\right)x^{2(1-s)}$. If $k \geq 0$ and $0 < s < 1/2$, then $f(x)$ is convex for $x > 0$.*

Lemma 4 (*[27], Lemma 3*). *Let $f : \mathbb{R} \to \mathbb{R}$ be convex with $f(0) = 0$ and a set of N^2 numbers v_{ij} with $i, j \in \{1, \dots, N\}$. Let $\mathcal{M}_i \subseteq \{1, \dots, N\}$ be an arbitrary index set. Then,*

$$\frac{1}{N^2}\sum_{i=1}^{N}\sum_{j\in\mathcal{M}_i} f(v_{ij}) \geq f\left(\frac{1}{N^2}\sum_{i=1}^{N}\sum_{j\in\mathcal{M}_i} v_{ij}\right).$$

2.6. Zero-gradient-sum algorithms

Consider a distributed algorithm of the form:

$$\begin{aligned}\dot{\mathbf{x}}_i(t) &= \boldsymbol{\phi}_i\left(\mathbf{x}_i(t), \mathbf{x}_{\mathcal{N}_i}(t)\right)\\ \mathbf{x}_i(0) &= \chi_i, \quad \forall i \in \mathcal{V}\end{aligned} \tag{5}$$

where $\mathbf{x}_i \in \mathbb{R}^n$ represents node i's estimate of the unknown global minimizer $\mathbf{x}^*$ at time t. $\mathbf{x}_{\mathcal{N}_i}(t)$ is a vector obtained by stacking the information obtained from neighboring agents (i.e., $\mathbf{x}_j, \forall j \in \mathcal{N}_i$). $\boldsymbol{\phi}_i : \mathbb{R}^n \times \mathbb{R}^{n|\mathcal{N}_i|} \mapsto \mathbb{R}^n$ is a locally Lipschitz function, and $\chi_i \in \mathbb{R}^n$ is a constant determining the initial state. The agents are endowed with a local cost function $f_i : \mathbb{R}^n \to \mathbb{R}$.

Definition 4 (*[47], Definition 1*). A continuous-time distributed algorithm of the form (5) is said to be a Zero-Gradient-Sum algorithm if local cost functions f_i are twice continuously differentiable, θ_i-strongly convex and Θ_i-smooth, $\boldsymbol{\phi}_i\,\forall i \in \mathcal{V}$ is locally Lipschitz, and the following conditions are satisfied:

$$\begin{aligned}&\sum_{i\in\mathcal{V}} \nabla^2 f_i(\mathbf{x}_i(t))\boldsymbol{\phi}_i\left(\mathbf{x}_i(t), \mathbf{x}_{\mathcal{N}_i}(t)\right) = \mathbf{0}_n\\ &\sum_{i\in\mathcal{V}} \mathbf{x}_i(t)^T\nabla^2 f_i(\mathbf{x}_i(t))\boldsymbol{\phi}_i\left(\mathbf{x}_i(t), \mathbf{x}_{\mathcal{N}_i}(t)\right) < 0\\ &\sum_{i\in\mathcal{V}} \nabla f_i(\chi_i) = \mathbf{0}_n.\end{aligned} \tag{6}$$

2.7. Integral sliding modes

Consider a linear uncertain system

$$\dot{x} = Ax + B(u + w), \tag{7}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $w \in \mathbb{R}^m$ which represents an unknown disturbance vector bounded by a known positive constant δ. Assuming that there exists a controller $u^*(x)$, either continuous or discontinuous, which ensures that the nominal system $\dot{x} = Ax + Bu^*$ is stabilized with given properties. Define the sliding surface σ as

$$\begin{aligned}\sigma &= C^T x + z\\ \dot{z} &= -C^T(Ax + Bu^*),\end{aligned} \tag{8}$$

with $z(0) = -C^T x(0)$ and an appropriate matrix C. Using the controller

$$u = u^* - M\,\mathrm{sign}(\sigma),$$

with $M \geq \delta$, the trajectory of the linear uncertain system coincides with the one of the nominal system $\dot{x} = Ax + Bu^*$ without perturbation. Compared to traditional sliding mode control, integral sliding mode approach exhibits system motion on sliding mode that spans the same dimension as the state space. More details can be found in [48].

3. Problem definition

Consider the leaderless MAS with N agents given by

$$\dot{\mathbf{x}}_i(t) = \mathbf{u}_i(t) + \mathbf{w}_i(t) \quad i \in \{1, \dots, N\}, \tag{9}$$

where $\mathbf{x}_i(t), \mathbf{u}_i(t) \in \mathbb{R}^n$ represents the state, the control input of agent $i \in \{1, \dots, N\}$, respectively. In addition, $\mathbf{w}_i(t) \in \mathbb{R}^n$ represents an unknown disturbance vector under the following assumption.

Assumption 1. There exists a known bound $\delta_i \geq 0$ for the disturbance vector $\mathbf{w}_i(t)$ such that $\|\mathbf{w}_i(t)\| \leq \delta_i, \forall t \geq 0$.

Moreover, each agent can exchange information only with its neighbors under a switching network, in which the associated topologies $\mathcal{X}_k$ satisfy the following assumption:

Assumption 2. The switched dynamic network is formed by undirected connected graphs. Furthermore, the switching signal $\vartheta(t)$ is generated exogenously and there is a minimum dwell time between consecutive switchings in such a way that Zeno behavior is excluded.

The agents are endowed with a local cost function $f_i : \mathbb{R}^n \to \mathbb{R}$ and the main objective is to establish a fixed formation around the target point $\mathbf{x}^* \in \mathbb{R}^n$, which minimizes

$$F(\mathbf{x}) := \sum_{i=1}^{N} f_i(\mathbf{x}) \tag{10}$$

before a predefined time T_c. The local cost functions are subject to the following assumption.

Assumption 3. There exists $\theta_i, \Theta_i > 0$ such that the local cost functions $f_i : \mathbb{R}^n \to \mathbb{R}$ are twice differentiable, θ_i-strongly convex and Θ_i-smooth.

Remark 1. Assumption 3 includes many cases such as quadratic, fractional, trigonometric, exponential, logarithmic and other bounded differentiable functions. Some practical examples can be found in economic dispatch, optimal rendezvous of multiple mobile robots, statistics, and machine learning [19].

The formation is defined by fixed displacements $\mathbf{h}_i \in \mathbb{R}^n$, where each agent has a unique displacement that determines their positional offset from the target point. Consequently, agent $i \in \mathcal{V}$ aims to reach the position $\mathbf{x}^* + \mathbf{h}_i$.

In this context, the problem can be written as follows:

$$\min_{\mathbf{x}\in\mathbb{R}^n} F(\mathbf{x}), \quad \text{subject to} \begin{cases} \lim\limits_{t\to T_c} \|\mathbf{p}_i(t) - \mathbf{x}^*\| = 0 \\ \|\mathbf{p}_i(t) - \mathbf{x}^*\| = 0, \quad \forall t \geq T_c, \end{cases} \tag{11}$$

where $\mathbf{x}^* = \arg\min_{\mathbf{x}\in\mathbb{R}^n} F(\mathbf{x})$ and $\mathbf{p}_i(t) = \mathbf{x}_i(t) - \mathbf{h}_i$. Here, $\mathbf{x}_i(t)$ is a local copy of the global optimization variable $\mathbf{x}(t)$. Henceforth, the time dependence of variables will be written only when necessary.

Remark 2. It is worth noting that the considered problem is to simultaneously solve the optimization and formation control problem in a decentralized way. The distributed optimization problem for achieving consensus appears as a special case when $\mathbf{h}_i = \mathbf{0}_n$.

4. Main results

To solve problem (11), the following distributed scheme is proposed, in which controller $\mathbf{u}_i(t)$ is comprised of two parts: a sliding mode controller $\mathbf{u}_i^{\mathrm{d}}(t)$ and a two-step controller for the optimization process $\mathbf{u}_i^*(t)$. Mathematically,

$$\mathbf{u}_i(t) = \mathbf{u}_i^{\mathrm{d}}(t) + \mathbf{u}_i^*(t) \tag{12}$$

$$\mathbf{u}_i^{\mathrm{d}}(t) = -\frac{1}{2sT_\sigma} \exp(\|\boldsymbol{\sigma}_i\|^{2s}) \left\lfloor\boldsymbol{\sigma}_i\right\rceil^{1-2s} - \rho_i \left\lfloor\boldsymbol{\sigma}_i\right\rceil^{0} \tag{13}$$

where

$$\begin{aligned} \boldsymbol{\sigma}_i(t) &= \mathbf{x}_i(t) - \mathbf{z}_i(t) \\ \dot{\mathbf{z}}_i(t) &= \mathbf{u}_i^*(t) \end{aligned} \tag{14}$$

with $\mathbf{z}_i(0) = \mathbf{b}_i \in \mathbb{R}^n$, and

$$\mathbf{u}_i^*(t) = \begin{cases} -c_1 \left(\nabla^2 f_i(\mathbf{x}_i)\right)^{-1} \exp(\|\nabla f_i(\mathbf{x}_i)\|^{2s}) \left\lfloor\nabla f_i(\mathbf{x}_i)\right\rceil^{1-2s}, & \forall t \leq \mu T_c \\ -c_2 \left(\nabla^2 f_i(\mathbf{p}_i)\right)^{-1} \sum\limits_{j\in\mathcal{N}_i^k} \sqrt{c_3 a_{ij}} \exp(\|\mathbf{p}_{ij}\|^{2s}) \left\lfloor\mathbf{p}_{ij}\right\rceil^{1-2s}, & \forall t > \mu T_c \end{cases} \tag{15}$$

along with the following definitions and parameter design rules:

$$\begin{aligned} &\bar{\Theta} = \max_{i\in\mathcal{V}}\{\Theta_i\}, \quad \underline{\lambda}_2 = \min_{k\in F}\{\lambda_2^k\}, \qquad \mathbf{p}_{ij} = \sqrt{c_3 a_{ij}}(\mathbf{p}_i - \mathbf{p}_j), \\ &c_1 = \frac{1}{2s\mu T_c}, \qquad c_2 = \frac{2}{s(1-\mu)T_c}, \quad c_3 \geq \frac{\bar{\Theta}}{2\underline{\lambda}_2}. \end{aligned} \tag{16}$$

Here, a_{ij} is the element of the adjacency matrix $\mathbf{A}_k$ and λ_2^k is the algebraic connectivity of $\mathbf{L}_k$, with $k \in \{1, \ldots, K\}$. The parameter μ is used to establish the duration of each step in terms of the total optimization time T_c. Hence, $0 < \mu < 1$. T_σ denotes the predefined-time for reaching the sliding surface $\boldsymbol{\sigma} = \mathbf{0}_n$, provided that $T_\sigma < \mu T_c$. As usual, solutions to the closed loop system are understood in the sense of Filippov due to the introduction of the discontinuous term $\left\lfloor\boldsymbol{\sigma}_i\right\rceil^0$ in (13).

Remark 3. Some remarks are made for the previously defined controller:

- The control input $\mathbf{u}_i^{\mathrm{d}}(t)$ has the form of an integral sliding mode controller, with the purpose of eliminating the typical reaching phase by enforcing the sliding mode from $t = 0$ through setting $\boldsymbol{b}_i = \mathbf{x}_i(0)$. This approach has the advantage of avoiding significant deviations of the states of the agents due to the disturbances during the reaching phase. More information can be found in [24,48–52].
- The term $\mathbf{u}_i^*(t)$ is a two-step controller with the purpose of leading the agents toward the global minimizer, under the scope of the zero-gradient-sum theory *ZGS*, proposed in [47].
- Notice that the lower bound of the gain c_3 depends on system parameters $\bar{\Theta}$ and $\underline{\lambda}_2$. Nevertheless, the control parameters are computed beforehand and then remain constant for $t \geq 0$. In addition, it is possible to have estimations, average values or bounds of the system parameters such that they can be used to calculate c_3. Once the gain is set, only local information is needed to achieve consensus control while minimizing the global cost function.

Proposition 2. *Let Assumption 1 hold. Consider system (9) under the controller defined in (12) and (13) with $\rho_i \geq \delta_i$. Then, the surface $\boldsymbol{\sigma}_i(t) = \mathbf{0}_n$ is fixed time stable with T_σ as a predefined UBST, regardless of the choice of $\mathbf{b}_i$ in (14). If $\boldsymbol{b}_i = \mathbf{x}_i(0)$ then $\boldsymbol{\sigma}_i(t) \equiv \mathbf{0}_n, \forall t \geq 0$.*

Proof. Define a Lyapunov candidate of the form $V_i(\boldsymbol{\sigma}_i) = \ \|\boldsymbol{\sigma}_i\|^2$, such that its time derivative yields

$$\begin{aligned}\dot{V}_i(\boldsymbol{\sigma}_i) &= 2\boldsymbol{\sigma}_i^\top \dot{\boldsymbol{\sigma}}_i = 2\boldsymbol{\sigma}_i^\top(\dot{\mathbf{x}}_i - \mathbf{u}_i^*) = 2\boldsymbol{\sigma}_i^\top(\mathbf{u}_i^{\mathrm{d}} + \mathbf{w}_i)\\ &= 2\boldsymbol{\sigma}_i^\top\left(-\frac{1}{2sT_\sigma}\exp(\|\boldsymbol{\sigma}_i\|^{2s})\left\lfloor\boldsymbol{\sigma}_i\right\rceil^{1-2s} - \rho_i\left\lfloor\boldsymbol{\sigma}_i\right\rceil^{0} + \mathbf{w}_i\right)\\ &= -\frac{1}{sT_\sigma}\exp(\|\boldsymbol{\sigma}_i\|^{2s})\|\boldsymbol{\sigma}_i\|^{2(1-s)} - 2\rho_i\|\boldsymbol{\sigma}_i\| + 2\boldsymbol{\sigma}_i^\top\mathbf{w}_i.\end{aligned} \tag{17}$$

By applying the Cauchy–Schwarz inequality to the last term on the right hand side, one get $\boldsymbol{\sigma}_i^\top\mathbf{w}_i \leq \|\boldsymbol{\sigma}_i\|\delta_i$ using Assumption 1. Hence

$$\dot{V}_i(\boldsymbol{\sigma}_i) \leq -\frac{1}{sT_\sigma}\exp(\|\boldsymbol{\sigma}_i\|^{2s})\|\boldsymbol{\sigma}_i\|^{2(1-s)} - 2(\rho_i - \delta_i)\|\boldsymbol{\sigma}_i\|.$$

Finally, since $\rho_i \geq \delta_i$ by assumption,

$$\dot{V}_i(\boldsymbol{\sigma}_i) \leq -\frac{1}{sT_\sigma}\exp(V_i(\boldsymbol{\sigma}_i)^s)V_i(\boldsymbol{\sigma}_i)^{1-s}, \tag{18}$$

meaning that the system follows Proposition 1 with $\alpha = 1$, and T_σ as predefined *UBST*. The last part of the proposition follows by noting that $\boldsymbol{\sigma}_i(t) = \mathbf{0}_n$ is a stable equilibrium due to the previous reasoning, and $\mathbf{b}_i = \mathbf{x}_i(0)$ ensures $\boldsymbol{\sigma}_i(0) = \mathbf{0}_n$, completing the proof. □

Proposition 3. *Let Assumptions 1 and 3 hold. Consider system (9) under the controller defined in (12), (13), (15) with $\rho_i \geq \delta_i$. If $\mathbf{b}_i = \mathbf{x}_i(0)$. Then, $\mathbf{x}^* = \arg\min_{\mathbf{x}\in\mathbb{R}^n} F(\mathbf{x})$ is a fixed-time stable equilibrium with μT_c as a predefined UBST.*

Proof. As a direct consequence of Proposition 2 and the definition in (14), $\dot{\boldsymbol{\sigma}}_i = \mathbf{0}_n = \dot{\mathbf{x}}_i - \mathbf{u}_i^*$ for $t \geq 0$. On sliding mode, the proposed controller cancels the effect of the unknown disturbance $\mathbf{w}_i$ for $t \geq 0$. Therefore, the closed-loop system takes the form

$$\dot{\mathbf{x}}_i = \mathbf{u}_i^* = -c_1\left(\nabla^2 f_i(\mathbf{x}_i)\right)^{-1}\exp(\|\nabla f_i(\mathbf{x}_i)\|^{2s})\left\lfloor\nabla f_i(\mathbf{x}_i)\right\rceil^{1-2s}. \tag{19}$$

For agent $i \in \mathcal{V}$, one can define the Lyapunov function candidate $V_i(\mathbf{x}_i) = \|\nabla f_i(\mathbf{x}_i)\|^2$, which is positive, radially unbounded and equal to zero if and only if $\nabla f_i(\mathbf{x}_i) = \mathbf{0}_n$ due to Assumption 3. The time derivative of $V_i(\mathbf{x}_i)$ takes the form

$$\dot{V}_i(\mathbf{x}_i) = 2\nabla f_i(\mathbf{x}_i)^\top\nabla^2 f_i(\mathbf{x}_i)\dot{\mathbf{x}}_i.$$

Then it follows from the definition of $\dot{\mathbf{x}}_i$

$$\begin{aligned}\dot{V}_i(\mathbf{x}_i) &= -2c_1\exp(\|\nabla f_i(\mathbf{x}_i)\|^{2s})\nabla f_i(\mathbf{x}_i)^\top\left\lfloor\nabla f_i(\mathbf{x}_i)\right\rceil^{1-2s}\\ &= -\frac{1}{s\mu T_c}\exp(\|\nabla f_i(\mathbf{x}_i)\|^{2s})\|\nabla f_i(\mathbf{x}_i)\|^{2(1-s)}\\ &= -\frac{1}{s\mu T_c}\exp(V_i(\mathbf{x}_i)^s)V_i(\mathbf{x}_i)^{1-s}.\end{aligned} \tag{20}$$

which completes the proof by using Proposition 1 with μT_c as *UBST*. □

Remark 4. The zero-gradient-sum approach requires that the initial condition of all agents is located at their local minimum, meaning that all gradients must be equal to zero at $t = 0$. In practical scenarios, this condition is not realistic and so there must be a prior step before consensus, in which all agents reach their local minima before a predefined time, starting from any initial condition.

Remark 5. Notice that both controllers $\mathbf{u}_i^*(t)$ and $\mathbf{u}_i^{\mathrm{d}}(t)$ act simultaneously, contrary to [53], who proposed an algorithm in which the reaching phase is done prior to the optimization phase. In our case, the reaching phase is eliminated under the assumption of knowledge of $\mathbf{x}_i(0)$, which is reasonable in practice due to the knowledge of the local state $\mathbf{x}_i(t)$ for all time. Still, even if $\mathbf{x}_i(0)$ is not accurately available due to the presence of uncertainty, the reaching phase is ensured to last at most T_σ units of time after the initial instant as we show in Proposition 2.

Theorem 1. *Let Assumptions 1–3 hold. Consider system (9) under the controller defined in (12), (13), (15) with $\rho_i \geq \delta_i$ and (16). Then, $\mathbf{p}_i(t) = \mathbf{x}^*$ is fixed-time stable with $(1-\mu)T_c$ as a predefined UBST.*

Proof. As a consequence of Propositions 2 and 3, for $t \geq \mu T_c$, the closed-loop system becomes

$$\dot{\mathbf{x}}_i = \mathbf{u}_i^* = -c_2\left(\nabla^2 f_i(\mathbf{p}_i)\right)^{-1}\sum_{j\in\mathcal{N}_i^k}\sqrt{c_3 a_{ij}}\exp(\|\mathbf{p}_{ij}\|^{2s})\left\lfloor\mathbf{p}_{ij}\right\rceil^{1-2s}, \tag{21}$$

which depends on the switching network $\mathcal{X}_k$. Consider a common Lyapunov function for all the topologies associated to the network of the form

$$V(\mathbf{p}) = \sum_{i=1}^{N} f_i(\mathbf{x}^*) - f_i(\mathbf{p}_i) - \nabla f_i(\mathbf{p}_i)^\top(\mathbf{x}^* - \mathbf{p}_i), \tag{22}$$

with $\mathbf{p} = \left[\mathbf{p}_1^\top, \dots, \mathbf{p}_N^\top\right]^\top$. Note that Assumption 3 implies that

$$V(\mathbf{p}) \geq \sum_{i=1}^{N}\frac{\theta_i}{2}\|\mathbf{p}_i - \mathbf{x}^*\|^2 \geq 0.$$

Hence, $V(\mathbf{p})$ is positive, radially unbounded and equal to zero if and only if $\mathbf{p}_i = \mathbf{x}^*$, i.e. when formation is achieved. The time derivative of (22) becomes

$$\dot{V}(\mathbf{p}) = \sum_{i=1}^{N}\mathbf{p}_i^\top\nabla^2 f_i(\mathbf{p}_i)\dot{\mathbf{x}}_i - (\mathbf{x}^*)^\top\sum_{i=1}^{N}\nabla^2 f_i(\mathbf{p}_i)\dot{\mathbf{x}}_i. \tag{23}$$

The assumption of undirected graphs implies that $\mathbf{p}_{ij} = -\mathbf{p}_{ji}$. Therefore, according to (6) in Definition 4

$$(\mathbf{x}^*)^\top\sum_{i=1}^{N}\nabla^2 f_i(\mathbf{p}_i)\dot{\mathbf{x}}_i = -c_2(\mathbf{x}^*)^\top\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i^k}\sqrt{c_3 a_{ij}}\exp(\|\mathbf{p}_{ij}\|^{2s})\left\lfloor\mathbf{p}_{ij}\right\rceil^{1-2s} = 0. \tag{24}$$

Then

$$\dot{V}(\mathbf{p}) = \sum_{i=1}^{N}\mathbf{p}_i^\top\nabla^2 f(\mathbf{p}_i)\dot{\mathbf{x}}_i = -c_2\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i^k}\exp(\|\mathbf{p}_{ij}\|^{2s})\sqrt{c_3 a_{ij}}\mathbf{p}_i^\top\left\lfloor\mathbf{p}_{ij}\right\rceil^{1-2s}. \tag{25}$$

Applying Lemma 2 and pre-multiplying by N^2/N^2

$$\begin{aligned}\dot{V}(\mathbf{p}) &= -\frac{c_2}{2}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i^k}\exp(\|\mathbf{p}_{ij}\|^{2s})\sqrt{c_3 a_{ij}}(\mathbf{p}_i - \mathbf{p}_j)^\top\left\lfloor\mathbf{p}_{ij}\right\rceil^{1-2s}\\ &= -\frac{c_2}{2}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i^k}\exp(\|\mathbf{p}_{ij}\|^{2s})\mathbf{p}_{ij}^\top\left\lfloor\mathbf{p}_{ij}\right\rceil^{1-2s}\\ &= -\frac{c_2}{2}\frac{N^2}{N^2}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i^k}\exp(\|\mathbf{p}_{ij}\|^{2s})\|\mathbf{p}_{ij}\|^{2(1-s)}.\end{aligned} \tag{26}$$

Observing that the function inside the double sum in (26) is convex (Lemma 3) and that $\|\mathbf{p}_{ij}\|^2 = c_3 a_{ij}\|\mathbf{p}_i - \mathbf{p}_j\|^2$, Lemma 4 can be applied to obtain

$$\begin{aligned}\dot{V}(\mathbf{p}) \leq &-\frac{c_2 N^2}{2}\exp\left(\left(\frac{c_3}{N^2}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i^k}a_{ij}\|\mathbf{p}_i - \mathbf{p}_j\|^2\right)^s\right)\\ &\times\left(\frac{c_3}{N^2}\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i^k}a_{ij}\|\mathbf{p}_i - \mathbf{p}_j\|^2\right)^{1-s}.\end{aligned} \tag{27}$$

Since

$$\sum_{i=1}^{N}\sum_{j\in\mathcal{N}_i^k}a_{ij}\|\mathbf{p}_i - \mathbf{p}_j\|^2 = 2\mathbf{p}^\top(\mathbf{L}_k\otimes\mathbf{I}_n)\mathbf{p},$$

it follows that

$$\dot{V}(\mathbf{p}) \leq -\frac{c_2 N^2}{2} \exp\left(\left(\frac{2c_3}{N^2}\mathbf{p}^\top(\mathbf{L}_k \otimes \mathbf{I}_n)\mathbf{p}\right)^s\right)\left(\frac{2c_3}{N^2}\mathbf{p}^\top(\mathbf{L}_k \otimes \mathbf{I}_n)\mathbf{p}\right)^{1-s}. \tag{28}$$

Following the mathematical deduction given in [19] and knowing that $\lambda_2^k \geq \underline{\lambda}_2$, then:

$$V(\mathbf{p}) \leq \frac{\bar{\Theta}}{\lambda_2^k}\mathbf{p}^\top(\mathbf{L}_k \otimes \mathbf{I}_n)\mathbf{p} \leq \frac{\bar{\Theta}}{\underline{\lambda}_2}\mathbf{p}^\top(\mathbf{L}_k \otimes \mathbf{I}_n)\mathbf{p}, \tag{29}$$

After replacing (29) into (28) and rearranging some terms, it yields

$$\dot{V}(\mathbf{p}) \leq -\frac{c_2}{2N^{-2s}} \exp\left(N^{-2s}\left(\frac{2c_3\underline{\lambda}_2}{\bar{\Theta}}V(\mathbf{p})\right)^s\right)\left(\frac{2c_3\underline{\lambda}_2}{\bar{\Theta}}V(\mathbf{p})\right)^{1-s}. \tag{30}$$

Finally, based on the definition of c_2 and c_3, (30) reduces to

$$\dot{V}(\mathbf{p}) \leq -\frac{1}{\alpha s(1-\mu)T_c}\exp(\alpha V(\mathbf{p})^s)V(\mathbf{p})^{1-s}. \tag{31}$$

where $\alpha = N^{-2s}$. Note that the dynamics of the defined common Lyapunov function are independent of the topology. Hence, according to Proposition 1 and Lemma 1, the switched system is fixed-time stable with $(1-\mu)T_c$ as *UBST*. □

Corollary 1. *In the absence of information about parameters $\underline{\lambda}_2$ and $\bar{\Theta}$, the MAS will converge in fixed-time to the global optimum with an UBST of the form $\frac{T_c}{c_3}\frac{\bar{\Theta}}{2\underline{\lambda}_2}$, provided that $c_3 > 0$.*

Proof. From (30) and the definition of c_2, let $\alpha = N^{-2s}$ and $\gamma = \frac{2c_3\underline{\lambda}_2}{\bar{\Theta}}$. Then

$$\dot{V}(\mathbf{p}) \leq -\frac{\gamma^{1-s}}{\alpha s T_c}\exp\left(\alpha\left(\gamma V(\mathbf{p})\right)^s\right)\left(V(\mathbf{p})\right)^{1-s}. \tag{32}$$

Since all parameters are positive, $\dot{V}(\mathbf{p}) \leq 0$, ensuring convergence to the global minimum. Moreover, the solution of the differential inequality (32) implies

$$V(\mathbf{p}) \leq \left[\frac{1}{\alpha\gamma^s}\ln\left(\frac{1}{\frac{\gamma}{T_c}(t-t_0)+\exp\left(-\alpha\gamma^s V_0^s\right)}\right)\right]^{1/s}, \quad V_0 = V(\mathbf{p}(t_0)).$$

Notice that $V(\mathbf{p}) = 0$ if $\frac{\gamma}{T_c}(t-t_0)+\exp\left(-\alpha\gamma^s V_0^s\right) = 1$, thus

$$t - t_0 \equiv T(V_0) \leq \frac{T_c}{\gamma}\left[1-\exp\left(-\alpha\gamma^s V_0^s\right)\right].$$

Since $0 < \exp\left(-\alpha\gamma^s V_0^s\right) \leq 1$, the fraction $\frac{T_c}{\gamma} = \frac{T_c}{c_3}\frac{\bar{\Theta}}{2\underline{\lambda}_2}$ is an *UBST*.

Notice that if $c_3 \geq \frac{\bar{\Theta}}{2\underline{\lambda}_2}$, then the *UBST* $\frac{T_c}{\gamma} \leq T_c$, which is consistent with Theorem 1. □

5. Application to source localization

This section presents the application of the results derived from previous sections to address the source localization problem. The objective of source localization is to determine the location of a nearby isotropic source by deploying a set of sensors in the workspace. These sensors measure the presence of the source, which could be an emitter such as a radio signal or a thermal, sound, or pollution source. In any case, it is detectable through suitable sensing technology. The sensor provides a scalar quantity, such as the energy or strength of the emitted signal, which diminishes with the distance between the sensor and the source.

5.1. Approximate source localization

Exact or optimal source localization is a highly intricate problem, especially in distributed settings. Consequently, approximate solutions have been proposed [8,9]. In these approaches, the primary function of sensors is to determine whether the source is nearby by comparing the measured scalar quantity with a predefined threshold. By placing the sensors uniformly across the region of interest in the workspace, the point that minimizes the distance to all activated sensors can serve as a good indicator of the source location. In practical scenarios such as search and rescue operations, where promptly identifying the source position is of utmost importance, this technique can be employed to obtain a rough estimate of the source location. Subsequently, more sophisticated techniques can be utilized in a smaller portion of the workspace around the rough estimate to find the exact location of the source.

In the distributed setting, the agents modeled by (9) do not have access to all the sensors. This limited accessibility can be represented by a $N \times r$ binary matrix $\mathbf{D} = [d_{i\ell}]$, where $d_{i\ell} = 1$ indicates that agent $i \in \mathcal{V}$ can access an activated sensor positioned at $\mathbf{s}_\ell \in \mathbb{R}^n$ $(\ell = 1, \dots, r)$ and $d_{i\ell} = 0$ otherwise. Motivated by the previous discussion, the main objective is establishing a fixed formation

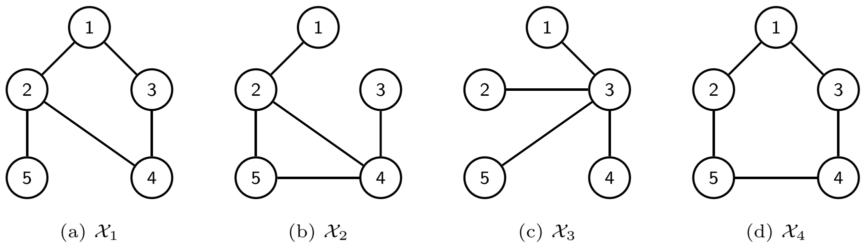


Fig. 1. Topologies associated to the switching network.

around the target point $\mathbf{x}^*$, which minimizes the cumulative distance between the agents' positions $\mathbf{x}_i$ and the corresponding sensors before a predefined time T_c. Hence, the local cost functions can be defined as:

$$f_i(\mathbf{x}_i) = \sum_{\ell=1}^{r} d_{i\ell} \|\mathbf{x}_i - \mathbf{s}_\ell\|^2. \tag{33}$$

Note that (33) satisfies Assumption 3 with

$$\Theta_i = 2 \sum_{\ell=1}^{r} d_{i\ell}, \tag{34}$$

since $\nabla^2 f_i(\mathbf{x}_i) = \left(2 \sum_{\ell=1}^{r} d_{i\ell}\right)\mathbf{I}_n = \Theta_i \mathbf{I}_n$. Therefore, the controllers developed in this work can be used for the agents to reach a formation around the approximate source position $\mathbf{x}^*$.

5.2. Numerical example

Consider a group of five agents whose individual dynamics follows (9), with $\mathbf{x} = [x_1, x_2, x_3]^\top \in \mathbb{R}^3$. Their communication is described by a switching network composed of four topologies, as shown in Fig. 1 and, without loss of generality, the weights a_{ij} were set equal to one. The algebraic connectivity associated to each graph is $\lambda_2(\mathcal{X}_1) = 1.83$, $\lambda_2(\mathcal{X}_2) = 0.70$, $\lambda_2(\mathcal{X}_3) = 1.00$ and $\lambda_2(\mathcal{X}_4) = 1.38$. Hence, $\underline{\lambda}_2 = 0.70$. These agents interact with four sensors located at $\mathbf{s}_1 = [3, 0, 1]^\top, \mathbf{s}_2 = [0, 2, 1]^\top, \mathbf{s}_3 = [-1, -1, 1]^\top, \mathbf{s}_4 = [4, 3, 1]^\top$. Thus, the local function for each agent can be written as $f_i(\mathbf{x}_i) = \sum_{\ell=1}^{4} d_{i\ell} \|\mathbf{x}_i - \mathbf{s}_\ell\|^2$ and the global function as $\sum_{i=1}^{5} f_i(\mathbf{x}_i)$. Given an arbitrary accessibility matrix

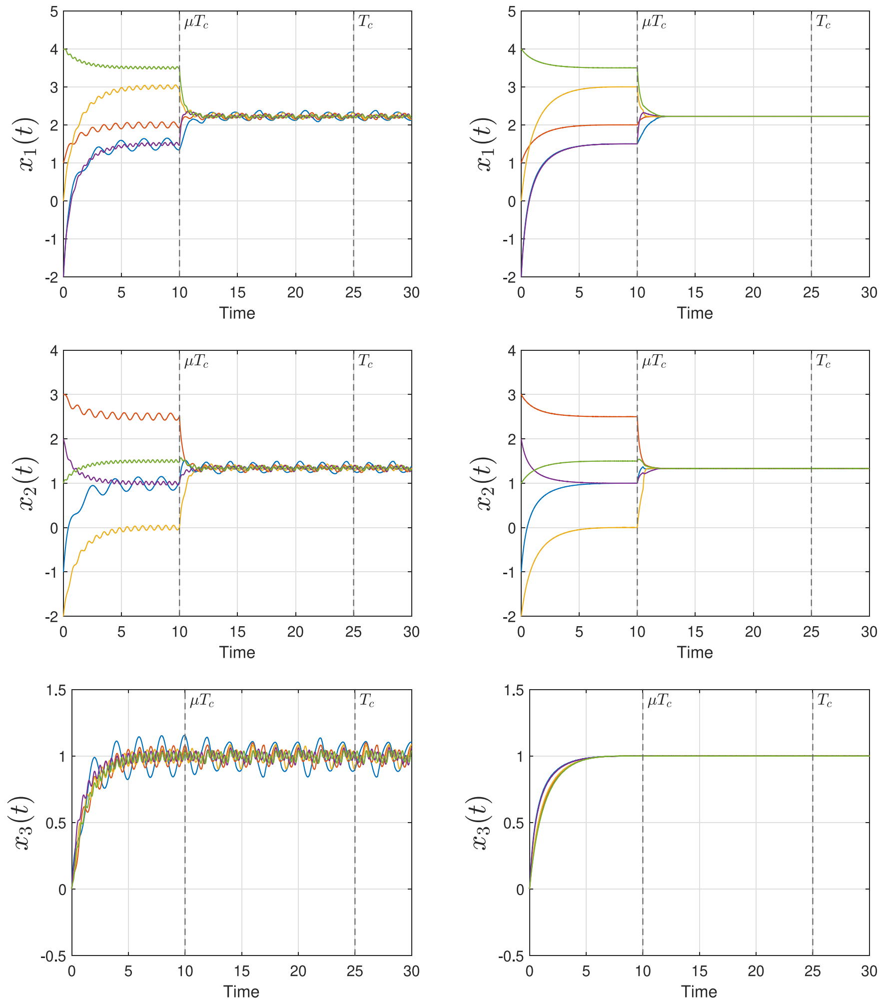
$$\mathbf{D} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

and taking into consideration (34), it follows that $\Theta_1 = \Theta_2 = \Theta_4 = 4$ and $\Theta_3 = \Theta_5 = 2$, hence, $\bar{\Theta} = 4$. The initial state of the agents is randomly selected in the interval $[-4, 4]$ for coordinates x_1 and x_2 and set to zero for coordinate x_3. The rest of the parameters are defined as $s = 0.2$, $\mu = 0.4$, and $T_c = 25s$. Consider persistent sinusoidal external disturbances affecting each coordinate on every agent, described by the local disturbance vector $\mathbf{w}_i(t) = 0.5[\sin(i\pi t), \cos(i\pi t), \sin(-i\pi t)]^\top$. Thus, $\|\mathbf{w}_i(t)\| \leq \delta_i = 0.5\sqrt{2}$. Then, we set the disturbance rejection parameter as $\rho_i = \delta_i$.

Fig. 2 shows the trajectories described by each agent (represented by different colors) w.r.t time on each coordinate x_1, x_2, x_3 and for pure consensus, i.e., $\mathbf{h}_i = \mathbf{0}_n$. Figs. 2(a) show the system's behavior when no disturbance rejection occurs. In contrast, Figs. 2(b) show the behavior when the sliding controller $\mathbf{u}_i^d$ is active and considering known initial conditions, i.e., $\mathbf{z}_i(0) = \mathbf{x}_i(0)$. Under this premise, $\boldsymbol{\sigma}_i(0) = \mathbf{0}_n$, which translates in no significant deviations of the states of the agents for the optimization time, as pointed out in Remark 3. On the other hand, Figs. 3 simulate the behavior of the system with uncertainty in the initial conditions of the states, where $\mathbf{z}_i(0)$ is set to zero. Here, the satisfactory performance of controller $\mathbf{u}_i^d$ is evident in bringing the system to the sliding phase $\boldsymbol{\sigma}_i = \mathbf{0}_n$ before the predefined time $T_\sigma = 1s$ and following a smooth trajectory for the rest of the optimization time.

When comparing Figs. 2 and 3, is evident the excellent performance of the controller in rejecting the bounded disturbance $\mathbf{w}_i(t)$ for both steps of the optimization process. From these figures, one can easily see that each agent is guided toward the minimizer of its respective local cost function before $\mu T_c = 10s$. Then, consensus toward the minimizer of the global function is obtained before a predefined time $T_c = 25s$. The satisfactory disturbance rejection is even more apparent in the 3D representation of the trajectories found in Fig. 4.

For the case of formation control, and inspired by [29], we defined the desired pattern as a pentagon embedded in a circle of radius R = 0.3, i.e., $\mathbf{h}_i = 0.3\left[\sin\left(\frac{2\pi}{5}(1-i)\right), \cos\left(\frac{2\pi}{5}(1-i)\right), 0\right]^\top$. A 3D representation of the trajectories for this case can be found in Fig. 5.



(a) Without disturbance rejection (b) With disturbance rejection and $\mathbf{z}_i(0) = \mathbf{x}_i(0)$

Fig. 2. Evolution of the states of the agents with time.

Additionally, Fig. 6 shows the norm of the control input for every agent (top) when the random switching signal $\vartheta(t)$ (bottom) induces a change in topology every 0.5 s. It is worth noting that there is no communication among agents before μT_c. Up to this point, their task is to reach their corresponding minima individually. At $t = \mu T_c$, there is a commutation on the controller and formation (or consensus) control takes place.

6. Conclusion

This study presented a novel fixed-time distributed optimization scheme for achieving formation control in first-order systems with a predefined convergence-time bound. Contrary to many existing algorithms in the literature, the proposed algorithm exhibits robustness in the presence of external disturbances and communication issues modeled by a switching network while eliminating the need for agents to exchange local gradients or Hessians of the cost functions, thereby minimizing communication overhead. The results underlined their potential for complex control problems and their adaptability to various system dynamics, highlighting their practical importance in real applications. To this end, its applicability for source localization was verified through several numerical experiments. Although the scheme proved satisfactory, two critical challenges still need to be addressed, including broader classes of functions by relaxing the strongly convex condition and the unconstrained nature of the optimization problem. These limitations will be the main focus of future works.

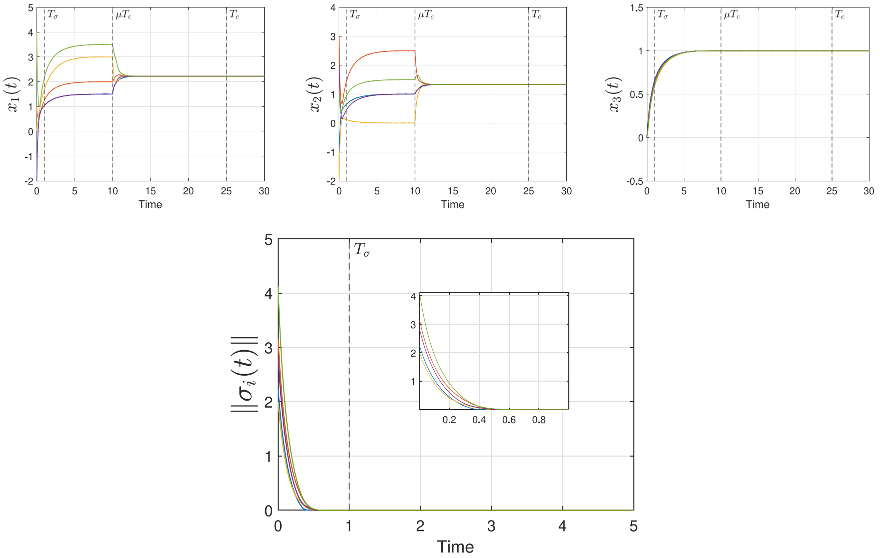
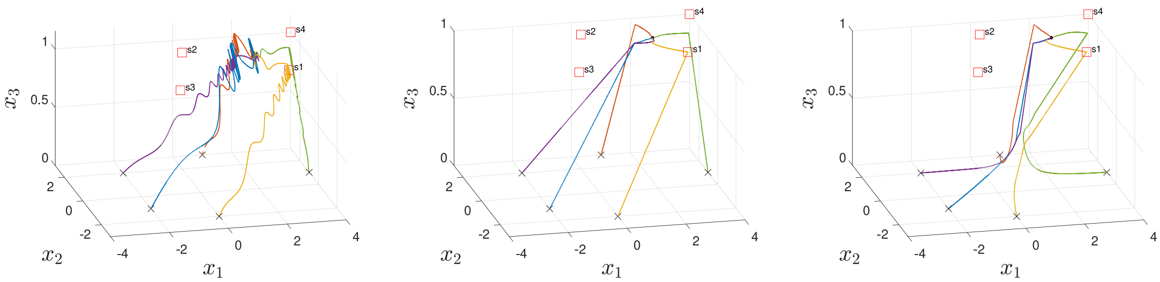


Fig. 3. Evolution of the states of the agents with time (top) and behavior of the sliding surfaces (bottom) with disturbance rejection and $\mathbf{z}_i(0) = \mathbf{0}_n$.



(a) Without disturbance rejection

(b) With disturbance rejection and $\mathbf{z}_i(0) = \mathbf{x}_i(0)$

(c) With disturbance rejection and $\mathbf{z}_i(0) = \mathbf{0}_n$

Fig. 4. 3D plot of the evolution of the states of the agents toward the estimated source.

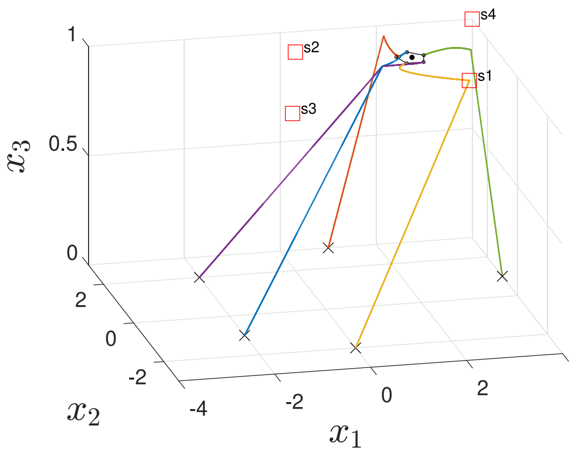


Fig. 5. 3D plot of the evolution of the fixed formation of the agents around the estimated source.

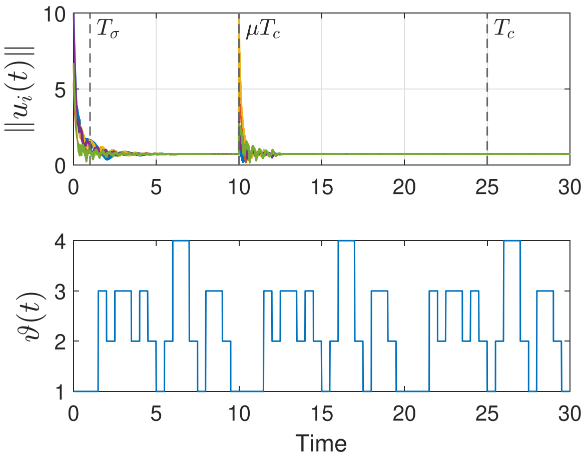


Fig. 6. Behavior of the Control input (top) under the dynamic network $\mathcal{X}_{\vartheta(t)}$ with random switching signal $\vartheta(t)$ (bottom).

CRediT authorship contribution statement

P. De Villeros: Conceptualization, Formal analysis, Methodology, Writing – original draft. **R. Aldana-López:** Formal analysis, Methodology, Writing – original draft. **J.D. Sánchez-Torres:** Conceptualization, Methodology, Supervision, Writing – original draft, Writing – review & editing. **M. Defoort:** Conceptualization, Methodology, Supervision, Writing – original draft, Writing – review & editing. **A.G. Loukianov:** Conceptualization, Methodology, Supervision, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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