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Nonexpansiveness and Fractal Maps in Hilbert Spaces

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Abstract: Picard iteration is on the basis of a great number of numerical methods and applications of mathematics. However, it has been known since the 1950s that this method of fixed-point approximation may not converge in the case of nonexpansive mappings. In this paper, an extension of the concept of nonexpansiveness is presented in the first place. Unlike the classical case, the new maps may be discontinuous, adding an element of generality to the model. Some properties of the set of fixed points of the new maps are studied. Afterwards, two iterative methods of fixed-point approximation are analyzed, in the frameworks of b-metric and Hilbert spaces. In the latter case, it is proved that the symmetrically averaged iterative procedures perform well in the sense of convergence with the least number of operations at each step. As an application, the second part of the article is devoted to the study of fractal mappings on Hilbert spaces defined by means of nonexpansive operators. The paper considers fractal mappings coming from φ-contractions as well. In particular, the new operators are useful for the definition of an extension of the concept of α-fractal function, enlarging its scope to more abstract spaces and procedures. The fractal maps studied here have quasi-symmetry, in the sense that their graphs are composed of transformed copies of itself.

Keywords: nonexpansive maps; iteration; fixed-point theorems; fractal maps; contractions; α-fractal functions; quasi-normed spaces

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1. Introduction

Mean values methods were investigated by important mathematicians like Cesàro or Fejér for the summation of divergent series (see, for instance, the references [1–3]). W.R. Mann proposed in 1953 the use of averaged values for the treatment of nonconvergent iteration processes [4]. The main purpose of their work was the resolution of differential and integral equations modeling several physical problems, by means of the method of successive approximations.

A bit later, Krasnoselskii [5] proved that the Picard iterations of a nonexpansive mapping T defined on certain normed space X may not converge, even if the map has a unique fixed point. However, the sequence defined recursively as:

$$f_{n+1} := \frac{f_n + Tf_n}{2}$$

for $n \geq 0$ and $f_0 \in X$ does converge to it. This finding confirmed that averages method is very useful in the cases where the typical map iteration fails to approach a fixed point. Since then, more sophisticated iterative methods based on means have appeared, as that proposed by Ishikawa [6].

The first aim of this article is the presentation of a new concept of nonexpansiveness, presenting maps that include the usual nonexpansive mappings like a particular case. This will be done in the context of b-metric spaces.

Definition 1. *A b-metric space X is a set endowed with a mapping $d : X \times X \to \mathbb{R}^+$ with the following properties:*

1. $d(x,y) \geq 0$, $d(x,y) = 0$ *if and only if* $x = y$.

2. $d(x, y) = d(y, x)$ for any $x, y \in X$.
3. There exists $s \geq 1$ such that $d(x, y) \leq s(d(x, z) + d(z, y))$ for any $x, y, z \in X$.

The constant s is the index of the b-metric space, and d is called a b-metric.

Example 1. *Let* $X = \mathbb{R}^m$ *be the m-dimensional Euclidean space, and* $u = (u_1, u_2, \ldots, u_m)$, $v = (v_1, v_2, \ldots, v_m)$. *The map* $d : X \times X \to \mathbb{R}^+$ *defined as:*

$$d(u, v) = \sum_{j=1}^{m} |u_j - v_j|^2,$$

is a b-metric with index 2 (see for instance [7]).

Nowadays, a large number of papers are devoted to the study of several types of single and multivalued contractions defined on b-metric spaces. The reader can consult the references [8–11] as a small and far from representative sample of articles on this topic.

The self-maps to be studied here are given by the following definition.

Definition 2. *Let E be a b-metric space, and* $T : E \to E$ *be a self-map such that for any* $f, g \in E$:

$$d(Tf, Tg) \leq ad(f, g) + B \min\{d(f, Tf), d(g, Tg)\}. \quad (1)$$

If $a \leq 1$ *and* $B \geq 0$, *T is a nonexpansive partial contractivity.*

For $B = 0$, we have a nonexpansive mapping.

Section 2 collects some properties of the new operators. In particular, it studies the main characteristics of the set of fixed points, whenever the space of definition owns some specific structure. Section 3 is devoted to the study of the convergence of the Mann iterations (called also Krasnoselskii–Mann iterations) of a nonexpansive partial contractivity, given by the following recursive scheme:

$$f_{n+1} = (1 - a_n)f_n + a_n T f_n. \quad (2)$$

for $n \in \mathbb{N}_0$, $0 \leq a_n \leq 1$ (see for instance [4]). When a_n is a constant, the algorithm is usually called Krasnoselskii iteration [5]. We first prove the convergence of the Mann's method in a b-metric space in the case $as < 1$, where s is the index of the b-metric, whenever the scalars are suitably chosen. Afterwards, the case $a = 1$ is considered in the framework of a Hilbert space. The properties of the sequence of iterates are described and the weak convergence to a fixed point is proved for a wide range of values of the sequence (a_n).

In particular, it is proved that the symmetrically weighted average:

$$f_{n+1} = \frac{f_n + T f_n}{2}, \quad (3)$$

corresponding to the case where $a_n = 1/2$ for all n, converges strongly to a fixed point with asymptotic stability when $as < 1$. If the underlying space is Hilbert and $a = 1$, the average (3) converges weakly to a fixed point. In the latter case, this symmetric pattern is optimum in the sense of performing the least number of operations at each step.

Section 4 is similar to Section 3, but the method studied is a three-steps algorithm called SP-iteration [12]. This is given by the following recursion:

$$f_{n+1} = (1 - a_n)g_n + a_n T g_n, \quad (4)$$

$$g_n = (1 - b_n)h_n + b_n T h_n, \quad (5)$$

$$h_n = (1 - c_n)f_n + c_n T f_n, \quad (6)$$

for $a_n, b_n, c_n \in [0,1]$. In particular, the convergence of the symmetric averaged iterations, corresponding to the case where $a_n = b_n = c_n = 1/2$ is proved. This choice is optimum in the sense of minimum number of operations at each step in the case $a = 1$. Section 5 applies the results obtained to the definition of fractal maps $F^* : I \to V$, where I is a real compact interval and V is a Hilbert space. The mappings constructed are called in this paper nonattracting fractal maps, due to the fact that they cannot be approached by means of the usual Picard iterations, in general. This topic was initiated in the reference [13] in the context of Banach spaces and algebras. Section 6 is devoted to the definition of fractal mappings valued on Hilbert spaces through φ-contractions on the Bochner space of square-integrable mappings. The methods of their successive approximations are also considered. Section 7 constructs mappings of α-fractal type, but in a wider and more abstract setting. These maps generalize the so called α-fractal functions, thoroughly studied in recent mathematical literature (see for instance [13–16] and references therein). Roughly speaking, a geometric object is symmetric if it is composed of similar pieces. The fractal maps studied here own a quasi-symmetry, in the sense that their graphs are composed of transformed copies of itself, that is to say, if G is the graph of a fractal map, it satisfies the set equation:

$$G = \cup_{m=1}^{M} W_m(G),$$

where W_m are set-valued operators to be defined later in this text (see for instance Theorem 5 of [13]).

Remark 1. *In this paper, Tf and $T(f)$ will be used indistinctly to denote the image of an element f by the map T.*

2. Nonexpansive Partial Contractivities

For a basic introduction to b-metric spaces the reader can consult the references [7,17–19], for instance. We extend the definition of partial contractivity presented in the reference [20].

Definition 3. *Let E be a b-metric space, and $T : E \to E$ be a self-map such that for any $f, g \in E$:*

$$d(Tf, Tg) \le a d(f, g) + B \min\{d(f, Tf), d(g, Tg)\}. \tag{7}$$

1. *If $a < 1$ and $B \ge 0$, T is a partial contractivity.*
2. *If $a \le 1$ and $B \ge 0$, T is a nonexpansive partial contractivity.*

For $B = 0$ we obtain the classical contractive/nonexpansive mappings.

Example 2. *Let $E = [0,1]$ be endowed with the usual metric, and $T : E \to E$ be defined as $Tx = 0$ for $x \in [0, 1/2)$ and $Tx = x/4$ for $x \in [1/2, 1]$. It is an easy exercise to check that T is a nonexpansive partial contractivity, where $a = 1/4$ and $B = 1/2$.*

Other examples can be found in [19]. In previous works, we proved that several well known types of contractions are partial contractivities whenever the constants satisfy certain conditions. This is true for Zamfirescu and quasi-contractions in particular.

Definition 4. *Let X be a b-metric space and $T : X \to X$ be a map such that there exist constants α, b, k with $0 < \alpha < 1$, $0 < b, k < 1/2$ and for any $x, y \in X$ one of the following conditions is satisfied:*

1. *$d(Tx, Ty) \le \alpha d(x, y)$;*
2. *$d(Tx, Ty) \le b(d(x, Tx) + d(y, Ty))$;*
3. *$d(Tx, Ty) \le k(d(x, Ty) + d(y, Tx))$.*

Then, T is called a Zamfirescu contraction [21].

Proposition 1. *Let X be a b-metric space and T be a Zamfirescu contraction with constants α, b, k. If $b < s^{-1}(s+1)^{-1}$ and $k < s^{-1}/2$, then T is a partial contractivity with constants:*

$$a = \max\{\alpha, \frac{bs}{1-bs^2}, \frac{ks}{1-ks}\} < 1$$

and

$$B = \max\{\frac{b(1+s^2)}{1-bs^2}, \frac{2ks}{1-ks}\}.$$

Proof. See reference [20]. □

Remark 2. *For a metric space, the index s is one, and all the Zamfirescu maps are partial contractivities, according to the last result.*

Something similar happens with quasi-contractions. Let us begin with the definition.

Definition 5. *For a b-metric space (X, d), a self-map $T : X \to X$ is a quasi-contraction if there exists a real constant λ, $0 < \lambda < 1$, such that for all $x, y \in X$:*

$$d(Tx, Ty) \leq \lambda \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}. \tag{8}$$

Proposition 2. *If (X, d) is a b-metric space and $T : X \to X$ is a quasi-contraction with ratio $\lambda > 0$ such that $\lambda < s^{-1}(s+1)^{-1}$, then T is a partial contractivity with:*

$$a = \frac{\lambda s}{1-\lambda s^2}$$

and

$$B = \frac{\lambda s^2}{1-\lambda s^2}.$$

Proof. See reference [19]. □

Remark 3. *For a metric space, a quasi-contraction is a partial contractivity whenever $\lambda < 1/2$, according to the last result.*

Unlike the classical nonexpansive mappings, a nonexpansive partial contractivity need not be continuous (see for instance Example 2). This fact adds an element of generality to partial contractivities. However, if $B = 0$, then it is Lipschitz continuous.

In the following, we give some properties of the set of fixed points of a nonexpansive partial contractivity.

Proposition 3. *Let E be a b-metric space, and T be a continuous nonexpansive partial contractivity. Then, the set of fixed points $Fix(T)$ is a closed set.*

Proof. It is a consequence of the continuity of T and the definition of fixed point. □

In the case of specific normed spaces, we can add some important properties to the set of fixed points of a nonexpansive partial contractivity, where the continuity of T is not needed. Let us start with a definition.

Definition 6. *Let E be a b-metric space and $T : E \to E$. The map T is quasi-nonexpansive if $Fix(T) \neq \emptyset$, and $\forall f \in E$, $\forall f^* \in Fix(T)$.*

$$d(Tf, f^*) \leq d(f, f^*).$$

Example 3. *Let $E = [0, 1]$, and $T : E \to E$ defined as: $Tx = 0$ for $x \in [0, 1/2]$ and $Tx = kx$, where $k < 1$, for $x \in (1/2, 1]$. The only fixed point of T is zero, and T is quasi-nonexpansive and discontinuous in the interval $[0, 1]$.*

Example 4. *Let $E = [0, 1]$, and $T : E \to E$ defined as: $Tx = kx \cos(1/x)$ where $k < 1$, for $x \neq 0$ and $T0 = 0$. T is quasi-nonexpansive and continuous in the interval $[0, 1]$.*

Proposition 4. *Let E be a b-metric space and $T : E \to E$ be a nonexpansive partial contractivity. If $Fix(T) \neq \emptyset$, then T is quasi-nonexpansive and T is continuous on $Fix(T)$.*

Proof. It is a consequence of the definitions of nonexpansive partial contractivity and quasi-nonexpansive maps. □

The next result can be read in the reference [22], Theorem 1.

Theorem 1. *Let E be a strictly convex Banach space, and $C \subseteq E$, such that C is nonempty, closed, and convex. Let $T : C \to C$ be a quasi-nonexpansive self-map. Then, the set $Fix(T)$ is closed and convex.*

Corollary 1. *Let E be a strictly convex Banach space, and $C \subseteq E$, such that C is nonempty, closed, and convex. Let $T : C \to C$ be a nonexpansive partial contractivity. If $Fix(T) \neq \emptyset$, then $Fix(T)$ is a closed and convex set, and T is continuous on it.*

Proof. It is a consequence of Proposition 4 and Theorem 1. □

Corollary 2. *Let E be a strictly convex Banach space, $C \subseteq E$, such that C is nonempty, closed, and convex. Let $T : C \to C$ be a nonexpansive partial contractivity. If $Fix(T) \neq \emptyset$, then $Fix(T)$ is a singleton or it is infinite. If $a + B < 1$ then $Fix(T) = \{f^*\}$.*

Proof. The first part is a consequence of Corollary 1. The second statement was proved in Theorem 2.1 of reference [20]. □

Corollary 3. *If $E = \mathcal{L}^P(I)$, for $1 < p < \infty$, $C \subseteq E$ satisfies the conditions described previously, and $T : C \to C$ is a nonexpansive partial contractivity such that $Fix(T) \neq \emptyset$, then $Fix(T)$ is closed and convex.*

Proof. The Lebesgue spaces where $1 < p < \infty$ are strictly convex, and we have the hypotheses of Corollary 1. □

Let us inquire about the existence of a fixed point with minimal distance to any $g \in E$. The next result can be read in [23], for instance.

Theorem 2. *Let E be a reflexive Banach space, if $F \subseteq E$ is nonempty, closed, and convex, then F is proximal, that is to say, for any $g \in E$ there exists $f^* \in F$ such that $||g - f^*|| = d(g, F) := \inf\{||g - f|| : f \in F\}$. If additionally E is strictly convex then the "best approximation" $f^* \in F$ to g is unique.*

Corollary 4. *Let E be a uniformly convex Banach space, $C \subseteq E$, such that C is nonempty, closed, and convex. Let $T : C \to C$ be a nonexpansive partial contractivity. If $Fix(T) \neq \emptyset$, then for any $g \in E$ there exists a unique $f^* \in Fix(T)$ such that $||g - f^*|| = d(g, Fix(T))$.*

Proof. Uniformly convex Banach spaces are strictly convex. According to Theorem of Milman–Pettis they are reflexive spaces as well (see for instance [24]). By Corollary 1, $Fix(T)$ is closed and convex, and the result is a consequence of Theorem 2. □

In particular, this result holds for nonexpansive partial contractivities defined on Lebesgue spaces $E = \mathcal{L}^p(I)$ with $1 < p < \infty$.

3. Mann Iterations for Nonexpansive Partial Contractivities

As noticed in the introduction, for a nonexpansive self-map, the Picard iterations may not converge, even if the fixed point exists and is unique [5]. In this case, some other approximation methods may be necessary. In the following, we consider one of the simplest iteration procedures for fixed point approaching, in the context of quasi-normed and normed spaces.

Definition 7. *If E is a real linear space, the mapping $|\cdot|_s : E \times E \to \mathbb{R}^+$ is a quasi-norm of index s if:*

1. $|f|_s \geq 0$; $f = 0$ *if and only if* $|f|_s = 0$.
2. $|\lambda f|_s = |\lambda||f|_s$.
3. *There exists $s \geq 1$ such that $|f+g|_s \leq s(|f|_s + |g|_s)$ for any $f, g \in E$.*

The space $(E, |\cdot|_s)$ is a quasi-normed space. If E is complete with respect to the b-metric induced by the quasi-norm, then E is a quasi-Banach space. Obviously, if $s = 1$, then E is a normed space.

Example 5. *Let $\mathcal{L}^p(I)$ be a Lebesgue space with $0 < p < 1$, and let the map $d : X \times X \to \mathbb{R}^+$ be defined as:*

$$d(f,g) = (\int_I |f-g|^p d\mu)^{1/p}.$$

$\mathcal{L}^p(I)$ is a quasi-Banach space with index $s = 2^{1/p-1}$.

Let E be a quasi-normed space, $C \subseteq E$ be nonempty, closed, and convex, and $T : C \to C$ be a nonexpansive partial contractivity. Let us assume that $Fix(T) \neq \emptyset$. The Mann iterative scheme [4] is given by:

$$f_{n+1} = (1-a_n)f_n + a_n T f_n. \tag{9}$$

for $n \in \mathbb{N}_0$, $0 \leq a_n \leq 1$. When $a_n = 1$ for all n we obtain the Picard iteration, as a particular case. If $a_n = \lambda$ for any n we have the Krasnoselskii [5] iteration:

$$f_{n+1} = (1-\lambda)f_n + \lambda T f_n, \tag{10}$$

In the following, we prove that for some values of a_n the Mann iterations associated with a partial contractivity T converge to a fixed point if $Fix(T) \neq \emptyset$. In reference [20], it was proved that if the ratio a is lower than one and there exists a fixed point, it is unique.

Let (f_n) be the sequence of Mann iterates (9) of an element $f := f_0 \in C$ and $f^* \in Fix(T)$. Then:

$$|f_{n+1} - f^*|_s = |(1-a_n)(f_n - f^*) + a_n(T(f_n) - f^*)|_s \leq s(1-a_n)|f_n - f^*|_s + s a_n |T(f_n) - f^*|_s.$$

Due to the contractivity condition (7):

$$|T(f_n) - f^*|_s \leq a|f_n - f^*|_s,$$

and hence:

$$|f_{n+1} - f^*|_s \leq (s(1-a_n) + s a_n a)|f_n - f^*|_s = s(1 - a_n(1-a))|f_n - f^*|_s. \tag{11}$$

Repeating the argument for consecutive values of n, we obtain:

$$|f_{n+1} - f^*|_s \leq s^{n+1}(\prod_{j=0}^{n}(1 - a_j(1-a))|f_0 - f^*|_s.$$

Let us assume in the first place that $as < 1$. Let us choose a number k such that $s(1 - a_j(1 - a)) < k$ for any j. Then:

$$\frac{1 - ks^{-1}}{1 - a} < a_j \leq 1. \tag{12}$$

In order to satisfy the condition:

$$\frac{1 - ks^{-1}}{1 - a} < 1$$

it is necessary that $as < k$. Then, let us take $k \in \mathbb{R}$ such that $as < k < 1$.

We have that:

$$|f_{n+1} - f^*|_s \leq k^{n+1}|f_0 - f^*|_s, \tag{13}$$

Consequently, the Mann iterations (f_n) converge to the fixed point f^* with global asymptotical stability for values (a_n) satisfying the condition (12) for $j \geq 0$. Additionally, any ball $B(f^*, r)$ centered at the fixed point f^* is an invariant set for the Mann iterations (see reference [25]). In the normed case ($s = 1$), it suffices that the sequence (a_n) be such that $\inf a_n > 0$.

Case $a = 1$

Let us consider now that $s = 1$, that is to say, E is a normed space, and assume that T is a nonexpansive partial contractivity where $a = 1$. We will denote the norm on E with the usual notation $|| \cdot ||$. Let us consider the following definition about a sequence in E and a self-map T (see for instance the references [26,27]).

Definition 8. *Let E be a normed space, $C \subseteq E$ and $T : C \to C$ be such that $Fix(T) \neq \emptyset$ and a sequence $(f_n) \subseteq C$. We say that (f_n) has:*

1. *The limit existence property (LE property for short) if $\lim_{n\to\infty} ||f_n - f^*||$ exists and is finite for any $f^* \in Fix(T)$.*
2. *The approximate fixed point property (AF for short) if $\lim_{n\to\infty} ||f_n - Tf_n|| = 0$.*

Proposition 5. *Let E be a normed space, $C \subseteq E$ nonempty, closed, and convex, and $T : C \to C$ be a nonexpansive partial contractivity such that $Fix(T) \neq \emptyset$. For any $f_0 \in C$, the Mann iteration (f_n) has the LE property for any choice of the scalars (a_n).*

Proof. If (f_n) is the sequence of Mann iterates for any (a_n), according to (11) for $a = s = 1$:

$$||f_{n+1} - f^*|| \leq ||f_n - f^*||.$$

The sequence of real numbers $||f_n - f^*||$ is decreasing and bounded, and consequently, it is convergent for any $f_0 \in C$. Hence, the sequence has the LE property. □

Let use see that choosing the scalars (a_n) suitably, the Mann iteration has the AF property as well. For it, we need a previous Lemma, that can be read in the reference [28].

Lemma 1. *Let E be a uniformly convex Banach space, and a sequence $(\lambda_n) \subseteq E$ be such that there exist $p, q \in \mathbb{R}$ satisfying the condition $0 < p \leq \lambda_n \leq q < 1$ for all $n \in \mathbb{N}$. Let (x_n), (y_n) be sequences of E such that $\limsup_{n\to\infty} ||x_n|| \leq r$, $\limsup_{n\to\infty} ||y_n|| \leq r$, and $\limsup_{n\to\infty} ||\lambda_n x_n + (1 - \lambda_n)y_n|| = r$ for some $r \geq 0$. Then, $\lim_{n\to\infty} ||x_n - y_n|| = 0$.*

Proposition 6. *Let E be a uniformly convex Banach space, $C \subseteq E$ nonempty, closed, and convex, and $T : C \to C$ a nonexpansive partial contractivity such that $Fix(T) \neq \emptyset$. If the scalars a_n are chosen such that $0 < \inf a_n \leq \sup a_n < 1$, the Mann iteration has the AF property for any $f_0 \in C$.*

Proof. For $f_0 \in C$, let (f_n) be the Mann iteration with scalars chosen as described in the statement of this proposition. Let $f^* \in Fix(T)$, by Proposition 5 $\lim ||f_n - f^*|| = l \in \mathbb{R}$. The definition of nonexpansive partial contractivity (7) implies that:

$$||Tf_n - f^*|| \leq ||f_n - f^*||$$

and

$$\limsup_{n\to\infty} ||f_n - f^*|| = l,$$

$$\limsup_{n\to\infty} ||Tf_n - f^*|| \leq l.$$

Moreover:

$$\limsup_{n\to\infty} ||(1-a_n)(f_n - f^*) + a_n(Tf_n - f^*)|| = \limsup_{n\to\infty} ||f_{n+1} - f^*|| = l.$$

According to the previous lemma, $\lim_{n\to\infty} ||f_n - Tf_n|| = 0$, and (f_n) has the AF property. □

For the next definition the reader can consult the reference [29].

Definition 9. *A normed space E satisfies the Opial's condition if for any sequence $(f_n) \subseteq E$ such that (f_n) converges weakly to $f \in E$:*

$$\liminf_{n\to\infty} ||f_n - f|| < \liminf_{n\to\infty} ||f_n - g||. \quad (14)$$

for any $g \neq f$.

Proposition 7. *Let E be a Hilbert space, $C \subseteq E$ be nonempty, closed, and convex, and $T : C \to C$ be a nonexpansive partial contractivity. If a sequence $(f_n) \subseteq C$ converges weakly to $f \in C$ and it satisfies the AF property with respect to T, then $f \in Fix(T)$.*

Proof. Let (f_n) be a sequence satisfying the condition AF (Definition 8), and f be the weak limit of (f_n). Since E is a Hilbert space, it satisfies the Opial condition (see for instance Lemma 1 of reference [29]). If $Tf \neq f$:

$$\liminf_{n\to\infty} ||f_n - f|| < \liminf_{n\to\infty} ||f_n - Tf|| \leq \liminf_{n\to\infty}(||f_n - Tf_n|| + ||Tf_n - Tf||).$$

The AF property implies that $\liminf_{n\to\infty} ||f_n - Tf_n|| = 0$. Applying the contractivity condition in the second sunmmand, we have:

$$\liminf_{n\to\infty} ||f_n - f|| < \liminf_{n\to\infty}(||f_n - f|| + B||Tf_n - f_n||) = \liminf_{n\to\infty} ||f_n - f||.$$

This is a contradiction, and consequently, $f = Tf \in Fix(T)$. □

Theorem 3. *Let E be a Hilbert space, $C \subseteq E$ be nonempty, closed, bounded, and convex, and $T : C \to C$ be a nonexpansive partial contractivity such that $Fix(T) \neq \emptyset$. Then, the Mann iterates such that $0 < \inf a_n \leq \sup a_n < 1$ converge weakly to a fixed point f^* of T for any $f_0 \in C$.*

If additionally $(||f_n||)$ converges to $||f^*||$, or f^* is a cluster point of (f_n), then (f_n) converges strongly to f^*.

Proof. According to Propositions 5 and 6, the Mann iterates satisfy the LE and AF properties. Since C is nonempty, closed, bounded, and convex in a Hilbert space, then it is weakly compact [30]. For $f_0 \in C$, let us consider the Mann iterates with scalars satisfying the condition described, $(f_n) \subseteq C$. Since C is weakly compact, there exists a weakly convergent subsequence (f_{n_j}). This subsequence satisfies the LE and AF properties as well. Let f^* be the weak limit of (f_{n_j}). Since C is weakly closed, $f^* \in C$.

Arguing as in the previous proposition for the sequence (f_{n_j}), $f^* \in Fix(T)$.

If there exists another subsequence (f_{n_k}) converging weakly to g^* and $f^* \neq g^*$, using similar arguments we have that $g^* \in Fix(T)$. Using Opial and LE conditions:

$$\lim_{n\to\infty} ||f_n - f^*|| = \lim_{n\to\infty} ||f_{n_j} - f^*|| < \lim_{n\to\infty} ||f_{n_j} - g^*|| = \lim_{n\to\infty} ||f_n - g^*||,$$

and

$$\lim_{n\to\infty} ||f_n - g^*|| = \lim_{n\to\infty} ||f_{n_k} - g^*|| < \lim_{n\to\infty} ||f_{n_k} - f^*|| = \lim_{n\to\infty} ||f_n - f^*||.$$

This is a contradiction, consequently, $f^* = g^*$ and (f_n) converges weakly to f^*. If $(||f_n||)$ converges to $||f^*||$, let us consider that:

$$||f_n - f^*||^2 =< (f_n - f^*), (f_n - f^*) >= ||f_n||^2 - 2 < f_n, f^* > + ||f^*||^2.$$

The last quantity converges to:

$$||f^*||^2 - 2 < f^*, f^* > + ||f^*||^2 = 0,$$

and consequently, $\lim_{n\to\infty} ||f_n - f^*|| = 0$.

If f^* is a cluster point of (f_n), due to the LE property, the convergence is also strong. □

Corollary 5. *With the hypotheses of the previous theorem, if (f_n) is the sequence of Mann iterates and f^* is its weak limit, then:*

$$< f_n, g > \to < f^*, g >,$$

for any $g \in E$.

If E is the real Lebesgue space $\mathcal{L}^2(I)$, where I is a compact interval:

$$\int_I f_n g d\mu \to \int_I f^* g d\mu,$$

for any $g \in \mathcal{L}^2(I)$ and, in particular:

$$\int_I f_n d\mu \to \int_I f^* d\mu.$$

This fact makes the Mann method suitable to approximate the integral of a fixed point mapping in $\mathcal{L}^2(I)$.

Remark 4. *The results given in this section are true in particular for Krasnoselskii iteration (10) whenever $0 < \lambda < 1$.*

Remark 5. *The reader may note that the Picard iteration (particular case of Mann iteration when $a_n = 1$ for all n) does not lie in the range of convergence given in Theorem 3. The choice $a_n = 1/2$ for all n is optimum in the sense of performing the least number of operations at each step.*

Corollary 6. *Let E be a Hilbert space, and $T : C \to C$, where C is nonempty, bounded, closed, and convex, be a nonexpansive mapping. Then, $Fix(T) \neq \emptyset$ and the Mann iterates such that $0 < \inf a_n \leq \sup a_n < 1$ converge weakly to a fixed point of T for any $f_0 \in C$.*

Proof. A nonexpansive mapping is a nonexpansive partial contractivity with $B = 0$. With the conditions given on E, C, and T, Browder's Theorem implies that $Fix(T) \neq \emptyset$ (see reference [31]). Applying Theorem 3 we obtain the weak convergence of the Mann iterates to a fixed point of T. □

4. SP-Algorithm for Nonexpansive Partial Contractivities

We study now the SP-algorithm (see, for instance, [12]) for nonexpansive partial contractivities. This is a three-step iterative procedure given for $n \geq 0$ by:

$$f_{n+1} = (1 - a_n)g_n + a_n T g_n, \tag{15}$$

$$g_n = (1 - b_n)h_n + b_n T h_n, \tag{16}$$

$$h_n = (1 - c_n)f_n + c_n T f_n, \tag{17}$$

for $a_n, b_n, c_n \in [0, 1]$.

For $a_n = c_n = 1$ for all n, we have the Karakaya algorithm (see, for instance, [20,32]). In reference [19], we studied the convergence of this method for a quasi-normed space with index s under the hypothesis $as < 1$. We obtained that, if $f^* \in Fix(T)$, for values of the scalars such that:

$$\frac{1 - s^{-1}}{1 - a} < \frac{1 - k}{1 - a} < a_n, b_n, c_n \leq 1, \tag{18}$$

where the SP-iterates satisfy the following inequality:

$$|f_{n+1} - f^*|_s \leq (ks)^3 |f_n - f^*|_s. \tag{19}$$

where k is a constant such that $a < k < s^{-1}$. Thus, the algorithm is convergent and the order is $O((ks)^{3n})$.

Case $a = 1$

In the next theorem, we consider $f^* \in Fix(T)$ in the case $a = s = 1$.

Theorem 4. *Let E be a uniformly convex Banach space, $C \subseteq E$ nonempty, closed, and convex, and $T : C \to C$ be a nonexpansive partial contractivity such that $Fix(T) \neq \emptyset$. The SP-algorithm has the LE and AF properties whenever the scalars a_n, b_n, c_n are chosen such that:*

$$0 < \inf a_n \leq \sup a_n < 1,$$

$$0 < \inf b_n \leq \sup b_n < 1,$$

$$0 < \inf c_n \leq \sup c_n < 1.$$

Proof. By (33) of [19], the SP-iterates of any $f_0 \in C$ satisfy the inequality:

$$||f_{n+1} - f^*|| \leq ||f_n - f^*||. \tag{20}$$

Then, the sequence (f_n) is decreasing and bounded and consequently convergent. Thus, the SP-iterates of a nonexpansive partial contractivity have the LE property (see Definition 8). Let us define:

$$l := \lim_{n \to \infty} ||f_n - f^*||. \tag{21}$$

Additionally, for the definition of nonexpansive partial contractivity:

$$||T f_n - f^*|| \leq ||f_n - f^*||, \tag{22}$$

and consequently:

$$\limsup_{n \to \infty} ||T f_n - f^*|| \leq l. \tag{23}$$

Applying the expression (32) of reference [19]:

$$||T g_n - f^*|| \leq ||g_n - f^*|| \leq ||f_n - f^*||. \tag{24}$$

Consequently:

$$\limsup_{n\to\infty} ||Tg_n - f^*|| \leq \limsup_{n\to\infty} ||g_n - f^*|| \leq l. \tag{25}$$

Moreover:

$$l = \lim_{n\to\infty} ||f_{n+1} - f^*|| = ||(1-a_n)(g_n - f^*) + a_n(Tg_n - f^*)||. \tag{26}$$

Lemma 1 implies that the sequence (g_n) has the AF property:

$$\lim_{n\to\infty} ||g_n - Tg_n|| = 0. \tag{27}$$

Let us check now that (g_n) has the LE property as well.:

$$||f_{n+1} - f^*|| = ||(1-a_n)(g_n - f^*) + a_n(Tg_n - f^*)|| \leq ||g_n - f^*|| + a_n||Tg_n - g_n||,$$

then:

$$l \leq \liminf_{n\to\infty} ||g_n - f^*||,$$

and thus, by (25):

$$\lim_{n\to\infty} ||g_n - f^*|| = l. \tag{28}$$

Consequently, (g_n) has the LE property. Let us check now the properties of the sequence (h_n):

$$l = \lim_{n\to\infty} ||g_n - f^*|| = \lim_{n\to\infty} ||(1-b_n)(h_n - f^*) + b_n(Th_n - f^*)||.$$

Using the definitions of nonexpansive partial contractivity, h_n and the expression (31) of reference [19]:

$$||Th_n - f^*|| \leq ||h_n - f^*|| = ||(1-c_n)(f_n - f^*) + c_n(Tf_n - f^*)|| \leq ||f_n - f^*||, \tag{29}$$

and thus:

$$\limsup_{n\to\infty} ||Th_n - f^*|| \leq \limsup_{n\to\infty} ||h_n - f^*|| \leq l. \tag{30}$$

By (28):

$$l = \lim_{n\to\infty} ||g_n - f^*|| = \lim_{n\to\infty} ||(1-b_n)(h_n - f^*) + b_n(Th_n - f^*)||.$$

This equality along (30) implies that:

$$\lim_{n\to\infty} ||h_n - Th_n|| = 0, \tag{31}$$

and (h_n) has the AF property. Then:

$$||g_n - f^*|| = ||(1-b_n)(h_n - f^*) + b_n(Th_n - f^*)|| \leq ||h_n - f^*|| + b_n||Th_n - h_n||,$$

and thus:

$$l = \liminf_{n\to\infty} ||g_n - f^*|| \leq \liminf_{n\to\infty} ||h_n - f^*||. \tag{32}$$

By (30) and (32), (h_n) has the LE property and:

$$l = \lim_{n\to\infty} ||h_n - f^*|| = \lim_{n\to\infty} ||(1-c_n)(f_n - f^*) + c_n(Tf_n - f^*)||.$$

According to Lemma 1, the last equality along with (21) and (23) provide the AF property of $(f_n) : \lim_{n\to\infty} ||f_n - Tf_n|| = 0$.

Thus, we have proved that the SP-iterates have the LE and AF properties for any $f_0 \in C$. $\square$

As a consequence, we have the following results.

Theorem 5. *Let E be a Hilbert space, and $T : C \to C$, where C is nonempty, closed, bounded, and convex, be a nonexpansive partial contractivity such that $Fix(T) \neq \emptyset$. Then, the SP-algorithm with $0 < \inf a_n \leq \sup a_n < 1$, $0 < \inf b_n \leq \sup b_n < 1$, $0 < \inf c_n \leq \sup c_n < 1$, converges weakly to a fixed point of T for any $f_0 \in C$.*

Corollary 7. *Let E be a Hilbert space, and $T : C \to C$, where C is nonempty, bounded, closed, and convex, be a nonexpansive mapping. Then, $Fix(T) \neq \emptyset$ and the SP-algorithm with $0 < \inf a_n \leq \sup a_n < 1$, $0 < \inf b_n \leq \sup b_n < 1$, $0 < \inf c_n \leq \sup c_n < 1$, converges weakly to a fixed point of T for any $f_0 \in C$.*

The proofs are similar to the given for the Mann iteration. If additionally f^* is a cluster point of (f_n) or $(||f_n||)$ converges to $||f^*||$, then (f_n) converges strongly to f^*, for the reasons given in Theorem 3.

Corollary 5 is true for SP-iterations as well. The choice $a_n = b_n = c_n = 1/2$ for all n is optimum in the sense of performing the least number of operations at each step.

5. Nonattracting Fractal Mappings on a Hilbert Space

In this section, we define fractal maps through an operator on the Hilbert space of square-integrable Hilbert-valued maps $\mathcal{B}^2(I, V)$, where $I = [0, 1]$ and V is a real Hilbert space, associated with an Iterated Function System. $\mathcal{B}^2(I, V)$ is endowed with an inner product defined for $F, G \in \mathcal{B}^2(I, V)$, as:

$$< F, G >= \int_I < F(t), G(t) > dt,$$

and norm:

$$||F||_2 = (\int_I ||F(t)||^2 dt)^{1/2},$$

where $|| \cdot ||$ is the norm associated with the inner product in V. $\mathcal{B}^2(I, V)$ is a Hilbert space. It agrees with the Lebesgue space $\mathcal{L}^2(I, V)$, when $V = \mathbb{R}$.

Since the operator to be defined is nonexpansive, the fractal mappings cannot be approached, in general, by the Picard iterations of the operator, and in this sense we call them nonattracting fractal functions.

Let us consider a natural number $N > 1$ and a partition of the interval $I = [0, 1]$, $0 = t_0 < t_1 < \ldots < t_N = 1$. Let us define subintervals $I_m = [t_{m-1}, t_m)$ for $m = 1, \ldots N - 1$, and $I_N = [t_{N-1}, t_N]$. Consider affine maps $L_m(t) = c_m t + d_m$, satisfying the so-called join-up conditions:

$$L_{m-1}(t_0) = t_{m-1}, \qquad L_m(t_N) = t_m. \tag{33}$$

Let $F_m : I \times V \to V$, where V is a Hilbert space with associated norm $|| \cdot ||$, be defined as:

$$F_m(t, v) = R_m(t, v) + Q_m(t), \tag{34}$$

where $R_m : I \times V \to V$ and $Q_m : I \to V$. Let us assume that:

$$||F_m(t, v)|| \leq K, \qquad ||Q_m(t)|| \leq K', \tag{35}$$

for $t \in I$, $v \in V$, $m = 1, \cdots, N$ and $K + K' \leq 1$. Let us assume that R_m, Q_m are Bochner square-integrable maps. Let us assume that R_m are uniformly nonexpansive in the second variable, that is to say:

$$||R_m(t, v) - R_m(t, v')|| \leq ||vs. - v'||, \tag{36}$$

for any $t \in I$, $m = 1, \dots N$ and $v, v' \in V$. The mappings $W_m(t,v) = (L_m(t), F_m(t,v),)$ for $m = 1, 2, \dots, N$, compose an iterated function system whose attractor is the graph of an integrable map (see for instance [13]).

Let $C \subseteq \mathcal{B}^2(I,V)$ be nonempty, closed, and convex and let us define $T : C \to C$ as:

$$TG(t) = F_m(L_m^{-1}(t), G \circ L_m^{-1}(t)), \tag{37}$$

for $t \in I_m$. The conditions given on F_m imply that $T : C \to C$, where $C = \bar{B}(0,1)$ is the closed unit ball (see reference [13] for details). The operator T is nonexpansive, due to condition (36).

The next result, due to F.E. Browder in an article of 1965, gives sufficient conditions for the existence of fixed points of nonexpansive mappings on subsets of Hilbert spaces [31].

Theorem 6. *Let C be a nonempty closed bounded convex subset of a Hilbert space H, and $T : C \to C$ be a nonexpansive map, then T has a fixed point in C.*

According to this result, the operator T defined by the expression (37) satisfies the conditions of Browder's Theorem and it has some fixed point $F^* \in \bar{B}(0,1) \subseteq \mathcal{B}^2(I,V)$. The graph of $F^* : I \to V$ has a fractal structure (see Theorem 5 of the reference [13]). The map F^* need not be unique, and the Picard iterations of T may fail to converge to it, and in this sense, we call F^* a nonattracting Hilbert-valued fractal mapping.

The results obtained in the second section provide the following theorems, which collect several properties of the set of fixed points of T and the convergence of the iterations studied.

Theorem 7. *Let $C = \bar{B}(0,1) \subseteq \mathcal{B}^2(I,V)$ and $T : C \to C$, defined as in (37). Let $\mathcal{N}_T := Fix(T)$ be the set on nonattracting fractal maps associated with T. Then:*

1. *$\mathcal{N}_T$ is nonempty, closed, and convex.*
2. *$\mathcal{N}_T$ is either a singleton or is infinite.*
3. *For any $G \in C$, there exists a unique $F^* \in \mathcal{N}_T$ with minimal distance to G, that is to say, $||G - F^*||_2 = \inf\{||G - F||_2 : F \in \mathcal{N}_T\}$.*
4. *If a sequence $(F_n) \subseteq C$ converges weakly to $F \in C$ and it satisfies the AF property, then $F \in \mathcal{N}_T$, that is to say, F is a nonattracting fractal function associated with T.*

Theorem 8. *Let $C = \bar{B}(0,1) \subseteq \mathcal{B}^2(I,V)$ and $T : C \to C$, defined as in (37).*

1. *The Mann iterates chosen such that $0 < \inf a_n \leq \sup a_n < 1$, converge weakly to a nonattracting fractal function F^* for any $G_0 \in C$. If additionally, $||G_n||_2$ converges to $||F^*||_2$ either F^* is a cluster point of (G_n), the convergence is strong.*
2. *If (G_n) is the quoted Mann iterates and F^* is its weak limit, then:*

$$< G_n, G > \to < F^*, G >,$$

   for any $G \in C$.
3. *For $V = \mathbb{R}$, that is to say, E is the real Lebesgue space $\mathcal{L}^2(I)$ then:*

$$\int_I G_n G d\mu \to \int_I F^* G d\mu,$$

   and in particular:

$$\int_I G_n d\mu \to \int_I F^* d\mu.$$

4. *Krasnoselskii iterations with $0 < \lambda < 1$ satisty the properties (1)–(3).*
5. *SP-iterations with scalars such that $0 < \inf a_n \leq \sup a_n < 1$, $0 < \inf b_n \leq \sup b_n < 1$, $0 < \inf c_n \leq \sup c_n < 1$, satisfy the properties (1)–(3).*

6. Fractal Mappings Defined through φ-Contractions

In this section, we consider fractal functions defined by an operator T satisfying an inequality of φ-contractive type. That is to say:

$$||TG - TG'||_2 \leq \varphi(||TG - TG'||_2), \tag{38}$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies certain conditions, and $G, G' \in \mathcal{B}^2(I, V)$. In reference [19], a $(\varphi - \psi)$ partial contractivity was defined in a b-metric space X, through the inequality:

$$d(Tx, Ty) \leq \varphi(d(x, y)) + \psi(d(x, Tx)),$$

for any $x, y \in X$, whenever φ and ψ meet specific conditions. A mapping T fulfilling (38) is a $(\varphi - \psi)$ partial contractivity for $\psi = 0$.

There is a great number of results about the existence of fixed points of mappings satisfying (38). We recall important results from Matkowski [33] and Boyd and Wong [34] as a couple of examples. The next theorem can be read in reference [33].

Theorem 9. *Let (X, d) be a complete metric space, and $T : X \to X$ satisfying:*

$$d(Tx, Ty) \leq \varphi(d(x, y)),$$

for any $x, y \in X$, where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is nondecreasing and such that $\lim_{n\to\infty} \varphi^n(\delta) = 0$ for any $\delta > 0$. Then, T has a unique fixed point $x^* \in X$ and the Picard iterations of any point converge to it.

Remark 6. *The order of convergence of the Picard iterations is obviously $d(T^n x, x^*) \leq \varphi^n(d(x_0, x^*))$.*

Definition 10. *A self-map T satisfying the conditions given in Theorem 9 is a φ-contraction, and φ is a comparison function.*

Example 6. *The map $\varphi(\delta) = \delta/(1 + \delta)$ for $\delta \geq 0$ is a comparison function.*

Proposition 8. *If T is a φ-contraction on a complete metric space, then T is nonexpansive, and consequently, it is quasi-nonexpansive and Lipschitz continuous.*

Proof. The conditions imposed to a comparison function imply that $\varphi(\delta) < \delta$ for any $\delta > 0$. By Matkowski's Theorem, T has a fixed point x^*. Then, for any $x \in X$:

$$d(Tx, x^*) \leq \varphi(d(x, x^*)) \leq d(x, x^*),$$

and T is quasi-nonexpansive. For all $x, y \in X$:

$$d(Tx, Ty) \leq \varphi(d(x, y)) \leq d(x, y),$$

and consequently, T is nonexpansive and Lipschitz continuous. □

Definition 11. *A comparison function φ that satisfies the condition $\delta - \varphi(\delta) \to \infty$ when δ tends to infinity is called a strict comparison function, and T is a strict φ-contraction.*

For a strict comparison function we have a different way to evaluate the order of convergence of the Picard iterations of a φ-contraction. This result is due to Russ in the reference [35].

Theorem 10. *If T is a strict φ-contraction and $x^* \in X$ is its fixed point then, for any $x \in X$:*

$$d(T^n x, x^*) \leq \varphi^n(\delta_0),$$

where $\delta_0 := \sup\{\delta \in \mathbb{R}^+ : \delta - \varphi(\delta) \leq d(x, Tx)\}$.

We recall another important result of fixed point existence due to Boyd and Wong [34]. Let us denote, for a metric d, the range of d as P, that is to say: $P := \{d(x,y) : x, y \in X\}$, and let $\bar{P}$ be its closure.

Theorem 11. *Let (X, d) be a complete metric space, and $T : X \to X$ satisfying:*

$$d(Tx, Ty) \leq \varphi(d(x,y)),$$

for any $x, y \in X$, where $\varphi : \bar{P} \to \mathbb{R}^+$ is upper semicontinuous from the right and such that $\varphi(\delta) < \delta$ for any $\delta > 0$. Then, T has a unique fixed point and the Picard iterations of any point converge to it.

Turning to the fractal mappings, we consider now the interval, partition, and maps L_m as in Section 5, and $F_m(t,v) = R_m(t,v) + Q_m(t)$, where R_m is such that the operator $H_m : \mathcal{B}^2(I,V) \to \mathcal{B}^2(I,V)$, defined as $H_m(G)(t) = R_m(t, G(t))$, is a φ-contraction for $m = 1, \dots N$, that is to say:

$$||H_m(G) - H_m(G')||_2 \leq \varphi(||G - G'||_2),$$

for any $G, G' \in \mathcal{B}^2(I,V)$. Then, the operator defined by $TG(t) = F_m(L_m^{-1}(t), G \circ L_m^{-1}(t))$ for $t \in I_m$ is also a φ-contraction since:

$$||TG - TG'||_2^2 = \sum_{m=1}^{N} \int_{I_m} ||R_m(L_m^{-1}(t), G \circ L_m^{-1}(t)) - R_m(L_m^{-1}(t), G' \circ L_m^{-1}(t))||^2 dt,$$

$$||TG - TG'||_2^2 = \sum_{m=1}^{N} c_m \int_I ||R_m(t, G(t)) - R_m(t, G'(t))||^2 dt,$$

$$||TG - TG'||_2^2 = \sum_{m=1}^{N} c_m \int_I ||H_m(G)(t) - H_m(G')(t)||^2 dt = \sum_{m=1}^{N} c_m ||H_m(G) - H_m(G')||_2^2$$

and thus:

$$||TG - TG'||_2^2 \leq (\varphi(||G - G'||_2))^2,$$

since $\sum_{m=1}^{N} c_m = 1$ due to conditions (33). Consequently, T is a φ-contraction, it has a unique fixed point $F^* \in \mathcal{B}(I,V)$ and the Picard iterations are convergent (Theorem 9).

As said previously, the conditions given for the function φ imply that $\varphi(\delta) < \delta$ for $\delta > 0$, and consequently, T is nonexpansive and continuous (Proposition 8). The results given in Section 5 are applicable to it. In this case we have a result of strong convergence for Krasnoselskii algorithm specific for Hilbert spaces.

Theorem 12. *Let us assume that the comparison function φ is such that $\varphi(t) \leq at$ for any $t > 0$ and $a < 1$. Then, the Krasnoselskii iterations of the operator T are strongly convergent to the fixed point for $0 < \lambda < 1$ with order of convergence $O(k^n)$, where:*

$$k := ((1-\lambda)^2 + 2a\lambda(1-\lambda) + \lambda^2)^{\frac{1}{2}}.$$

Proof. Let us consider the Krasnoselskii operator:

$$\bar{T}(G) = (1-\lambda)G + \lambda TG,$$

for $G \in \mathcal{B}^2(I,V)$. Then, for any $G, G' \in \mathcal{B}^2(I,V)$,

$$||\bar{T}G - \bar{T}G'||_2^2 = (1-\lambda)^2 ||G - G'||_2^2 + 2\lambda(1-\lambda) < TG - TG', G - G' > + \lambda^2 ||TG - TG'||_2^2,$$

Applying the Cauchy–Schwartz inequality in the second summand, and the conditions on T and φ:

$$||\bar{T}G - \bar{T}G'||_2^2 \leq ((1-\lambda)^2 + 2a\lambda(1-\lambda) + \lambda^2)||G - G'||_2^2.$$

It is an easy exercise to prove that:

$$k := ((1-\lambda)^2 + 2a\lambda(1-\lambda) + \lambda^2)^{\frac{1}{2}} < 1.$$

Consequently, $\bar{T}$ is a Banach contraction with ratio k, and its iterations are convergent. □

7. Mappings of α-Fractal Type

Let us consider the maps L_m, defined as in Section 5 and $F_m(t,v) = R_m(t,v) + Q_m(t)$. Given two mappings $F, F' \in \mathcal{B}^2(I,V)$, we now define Q_m as:

$$Q_m(t) = F \circ L_m(t) - R_m(t, F'(t)),$$

for $m = 1, \dots N$. Let us assume that R_m are as in the previous section, that is to say, the operators H_m defined as $H_m(G)(t) = R_m(t, G(t))$ are $\varphi-$ contractions on $\mathcal{B}^2(I,V)$:

$$||H_m(G) - H_m(G')||_2 \leq \varphi(||G - G'||_2),$$

where φ is a comparison function. As seen before, T is a φ-contraction as well. This iterated function system defines a map $F^\varphi : I \to V$ as the unique fixed point of the operator T. In previous papers, $F^\varphi : I \to V$ has been called the fractal convolution of F and F' when R_m are linear contractions, and denoted as $F^\varphi = F * F'$ (see for instance [13]).

Figures 1 and 2 illustrate the action of the system $\{(L_m(t), R_m(t,v))\}$ on the real function $F(t) = e^{-t}\sin(4\pi t)$ in the interval $I = [0,1]$. In this example, $V = \mathbb{R}$, $F'(t) = \cos(\frac{\pi(1-2t)}{2})$, the number of evenly sampled subintervals is $N = 10$, and $R_m(t,v) = t\cos(Nv)/(N+1)$.

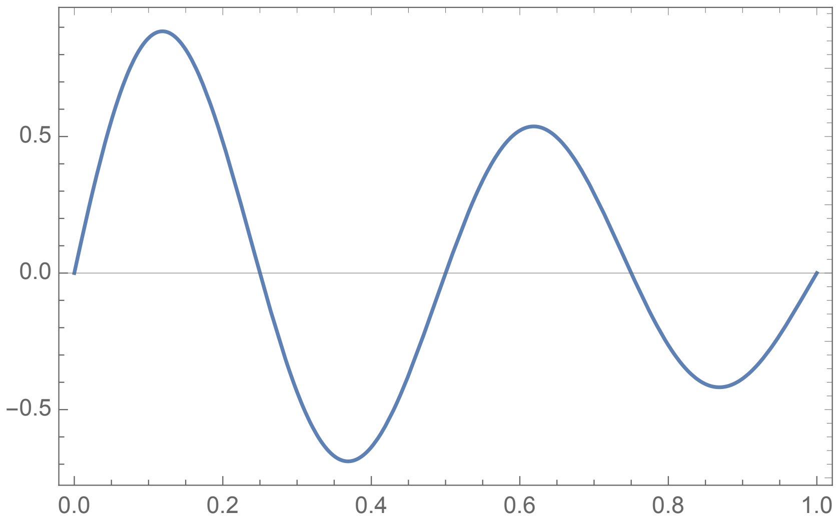

Figure 1. Graph of the function $F(t) = e^{-t}\sin(4\pi t)$ in the interval $I = [0,1]$.

For a linear contraction $R_m(t,v) = \alpha_m(t)v$ (where α_m is a real function such that $-1 < |\alpha_m(t)| < 1$), $H_m(G)(t) = \alpha_m(t)G(t)$ is a φ-contraction for $\varphi(t) = At$, and $A = \sup\{|\alpha_m(t)| : m = 1\dots, N; t \in I\}$, assuming that A exists. For this kind of fractal maps, it is easy to establish the distance between F and its fractal counterpart, usually denoted as F^α. We have (see for instance [13,14]):

$$||F^\alpha - F||_2 \leq \frac{A}{1-A}||F - F'||_2. \tag{39}$$

However, in the general case, this bound is not that easy. We will approach this problem with the help of approximate operators. We begin with a definition.

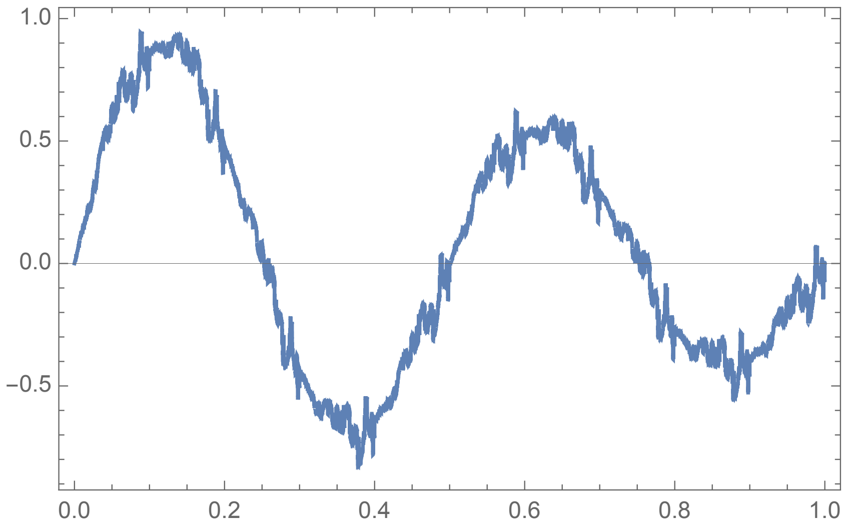

Figure 2. Graph of the fractal version of the function $F(t) = e^{-t}\sin(4\pi t)$ in $I = [0,1]$.

Definition 12. *Let be given a metric space (X,d), two self-maps $T, U : X \to X$ are approximate operators if there exists $k \geq 0$ such that $d(Tx, Ux) \leq k$ for any $x \in X$.*

The bound k determines the distance between the fixed points of T and U, according to the next result, which can be read in the reference [30], Theorems 7.5 and 7.6.

Theorem 13. *Let be given a metric space (X,d), and two approximate operators $T, U : X \to X$ be such that $d(Tx, Ux) \leq k$ for any $x \in X$.*

If T is a strict φ-contraction whose fixed point if $x_T \in X$ and $x_U \in Fix(U)$, then:

$$d(x_T, x_U) \leq \delta_k, \tag{40}$$

where $\delta_k = \sup\{\delta \in \mathbb{R}^+ : \delta - \varphi(\delta) \leq k\}$.

If T is a φ-contraction, where φ is subadditive and $\sum_{n=0}^{\infty} \varphi^n(\delta) < \infty$ for any $\delta > 0$, then:

$$d(x_T, x_U) \leq s(k), \tag{41}$$

where $s(k) = \sum_{n=0}^{\infty} \varphi^n(k)$.

Let us consider the operator of fractal interpolation, for the maps F_m described at the beginning of this section. Then:

$$TG(t) = F(t) + R_m(L_m^{-1}(t), G \circ L_m^{-1}(t)) - R_m(L_m^{-1}(t), F' \circ L_m^{-1}(t)),$$

for $t \in I_m$. Let us define $U(G) = F$ for all $G \in \mathcal{B}^2(I,V)$. Then, T and U are approximate operators in the case where $k := \sup_\delta \varphi(\delta) < \infty$ since, arguing as in Section 6:

$$||TG - UG||_2 = ||TG - F||_2 \leq \varphi(||G - F'||_2) \leq k.$$

Let F^φ be the fixed point of T. According to Theorem 13:

1. If T is a strict φ-contraction, then:

$$||F^{\varphi} - F||_2 \leq \delta_k = \sup\{\delta \in \mathbb{R}^+ : \delta - \varphi(\delta) \leq k\}.$$

2. If T is a φ-contraction, where φ is subadditive and $\sum_{n=0}^{\infty} \varphi^n(\delta) < \infty$ for any $\delta > 0$, then:

$$||F^{\varphi} - F||_2 \leq s(k) = \sum_{n=0}^{\infty} \varphi^n(k).$$

If φ is unbounded, but $T : C \to C$ where C is bounded and closed, we can consider $k := \varphi(diam(C))$ where $diam(C) := \sup\{||G - G'||_2 : G, G' \in C\}$.

A different approach for real continuous functions can be read in the reference [14].

8. Conclusions

The first aim of this article was the presentation of a new type of nonexpansiveness, which includes the usual nonexpansive mappings like a particular case. The new self-maps $T : E \to E$ so defined have been called nonexpansive partial contractivities. Some poperties of the set of their fixed points $Fix(T)$ have been studied. For instance, $Fix(T)$ is convex and closed if the space E is a strictly convex Banach space. Consequently, the set of fixed point is empty, a singleton, or infinite. If additionally E is uniformly convex, then $Fix(T)$ is a Chebyshev set, that is to say, for every element of the space, there is a fixed point of minimal distance.

Afterwards, two iterative procedures for the approximation of fixed points in the context of normed spaces have been studied: Mann and SP-algorithms. For every method, two different cases have been considered. The fist one concerns quasi-normed spaces, and a ratio a of the self-map linked to the partial contractivity. It has been proved that if $as < 1$, where s is the index of the quasi-norm, and the scalars are suitably chosen, the algorithms converge strongly to a fixed point, with asymptotic statiblity. In a second instance, the nonexpansive case has been studied ($a = 1$). It has been proved that the sequence of iterates satisfy the LE and AF properties, whenever the space E is a uniformly convex Banach space, and the scalars associated with the algorithms take intermediate values. If E is a Hilbert space, the algorithms converge weakly to a fixed point. In particular, it is proved that the symmetrically averaged iterative procedures perform well in the sense of convergence with the least number of operations at each step, in the case where $a = 1$ and E is Hilbert.

These findings have been used for the definition of fractal mappings of type $F^* : I \to V$, where I is a real compact interval and V is a Hilbert space. These maps are fixed points of a nonexpansive operator defined on the space of square-integrable Bochner mappings, which contain the space $\mathcal{L}^2(I)$ as a particular case (for $V = \mathbb{R}$). In this text, they are called nonattracting fractal mappings, in the sense that they cannot be approximated by the typical Picard iterations of the operator in general.

In a very different approach, other fractal mappings on Hilbert spaces have been defined through φ-contractions. Apart of their existence, the strong convergence of the Picard and Krasnoselskii iterations has been proved.

Finally, mappings of α-fractal type have been considered, in a framework much more general than the usual real case. The maps here defined are not linked to a scale factor, and they are defined by means of φ-contractions. The new maps F^{φ} are fractal perturbations of mappings $F : I \to V$, where I is a real compact interval and V is a Hilbert space. The distance between F^{φ} and F has been bounded with the use of approximate operators. The fractal maps studied here own a quasi-symmetry, in the sense that their graphs are composed of transformed copies of itself.

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