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Hypercubical groups

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Abstract

Given a finitely generated group G with a finite generating set Σ, we associate to (G, Σ) a cube complex, the *hypercubical complex of* (G, Σ), that can be thought of as the cubical version of a flag complex, having the Cayley graph $\Gamma(G, \Sigma)$ as 1-skeleton. G is *hypercubical with respect to* Σ if its hypercubical complex is contractible. From this we can deduce some consequences for the group. The aim of this thesis is to introduce the concepts of hypercubical complex of a group and hypercubical groups, and to introduce certain families of hypercubical groups. These families are RAAGs, oriented and twisted RAAGs (two generalizations of RAAGs), and the Borromean cube groups (a family defined inductively, starting from the link group of the Borromean rings). For these groups we will also use the hypercubical complex to deduce properties of cohomological nature.

Sommario

Dati un gruppo finitamente generato G e un sistema finito di generatori Σ, associamo alla coppia (G, Σ) un complesso cubico, il *complesso ipercubico di* (G, Σ), che può essere visto come la versione cubica di un flag complex, avente come 1-scheletro il grafo di Cayley $\Gamma(G, \Sigma)$. G è *ipercubico rispetto a* Σ se il suo complesso ipercubico è contraibile. Da questo possiamo dedurre alcune conseguenze per il gruppo. Lo scopo di questa tesi è di introdurre i concetti di complesso ipercubico di un gruppo e di gruppi ipercubici, e di introdurre alcune famiglie di gruppi ipercubici. Queste famiglie sono quelle dei RAAG, dei RAAG twisted e orientati (due generalizzazioni dei RAAG), e quella dei gruppi cubici di Borromeo (una famiglia definita induttivamente, partendo dal gruppo link degli anelli borromeiani). Utilizzeremo inoltre il complesso ipercubico di questi gruppi per dedurre proprietà di natura coomologica per essi.

Resumen

Dados un grupo finitamente generado G y un conjunto finito de generadores Σ, asociamos a (G, Σ) un complejo cúbico, el *complejo hipercúbico de* (G, Σ), que se puede ver como la versión cúbica de un flag complex, cuyo 1-esqueleto es el grafo de Cayley $\Gamma(G, \Sigma)$. G es *hipercúbico con respeto a* Σ si su complejo hipercúbico es contractible. De esto deduciremos ciertas propiedades de estos grupos. El objetivo de esta tesis es introducir los conceptos de complejo hipercúbico de un grupo y de grupos hipercúbicos, e introducir algunas familias de grupos hipercúbicos. Estas familias son la de los RAAGs, de lo dichos RAAGs twisted y orientados (dos generalizaciones de los RAAGs), y la de los dichos grupos cúbicos de Borromeo (una familia definida inductivamente, empezando por el grupo enlace de los anillos de Borromeo). Además, usaremos el complejo hipercúbico para deducir ciertas propiedades de caracter cohomológico de estos grupos.

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Contents

Chapter 1

Introduction

Geometry, topology and combinatorics can often be powerful instruments for the study of infinite groups. In geometric group theory one can adopt a geometric point of view in two different ways, either by looking at groups as geometric objects on their own, or by studying actions of groups on suitable spaces. This second viewpoint is the general context of this thesis. It is often useful to study actions of groups on spaces that are endowed with a combinatorical structure, compatible with the topology, in such a way that both the combinatorical and the topological structure are preserved by the action. It is the case of cellular actions on CW complexes. In particular, actions on cubical complexes can be of great interest, for many different reasons. On one hand, it is frequently sufficient to understand the combinatorics of a cubical complex, without having to deal explicitly with metric and topological problems. On the other hand, when a cubical complex satisfies certain hypotheses (for instance, being non-positively curved in some sense), many deep results can be proven for groups acting "nicely enough" on it. Such results can have deeply different natures: geometric, cohomological, algorithmic, dynamical and more. This leads us to the topic of this thesis: hypercubical groups. This is a new class of groups, defined by a property that is already known in literature but that has not been studied on its own. A finitely generated group G is said to be *hypercubical* if we can associate to it a certain cubical complex, called the *hypercubical complex of* G, which is contractible. Even though the hypercubical complex has already been used in the past literature (without this name), the concept of hypercubical groups is new and due to the author.

The structure of the thesis is as follows. After briefly recalling, in Chapter 2, the necessary preliminary notions and proving general results that will be needed later on, we give the definitions and general results about hypercubical groups in Chapter 3. Then in Chapter 4 we show that right-angled Artin groups are hypercubical, in fact they are the inspiring example, and we introduce two generalizations of RAAGs: oriented and twisted RAAGs, both of which are hypercubical. Lastly, in

Chapter 5, after a brief introduction about knot and link theory, we show that the link group of the Borromean rings is hypercubical. We then construct inductively a family of groups, the *Borromean cube groups*, that are all hypercubical. Chapter 6 collects the conclusions of this thesis and lists some possible future lines of research about hypercubical groups.

Chapter 2

Preliminary notions

The theme of this thesis is a family of groups, the *hypercubical groups*, whose definition relies on a specific type of cell complex that can be associated to a group, and which pieces of information we are able to deduce about the group itself from such complex. Therefore it seems appropriate to collect some preliminary notions and facts before exploring the definitions and results about hypercubical groups. These preliminaries are exposed briefly and usually without proof, when they are not original results, but references are given.

2.1 CW complexes and cellular homology

Usually in Geometric Group Theory one studies how groups act on sets which admit a topological structure preserved by the group action. Clearly not every action is useful, nor every space. Therefore we need to restrict our attention to specific classes of actions and spaces, one of them being that of cellular actions on CW complexes. We begin by briefly recalling the notions we need about CW complexes. The source for this section is [Hat01].

A *CW complex* (or *cell complex*) X is a topological space that can be constructed inductively as follows:

- start with a discrete set X^0, whose elements will be called the *0-cells* or the *vertices of* X;
- inductively, construct the set X^n, called the *n-skeleton of* X, by attaching n-disks (called *n-cells*) to X^{n-1} along their boundaries. The topology of X^n is given by taking the disjoint union of X_{n-1} and all the n-cells (that have the topology of an n-disk) and giving to X_n the quotient topology;
- $X = \bigcup_n X^n$, where the union is either finite or infinite. X is endowed with

the weak topology: $A \subseteq X$ is open (or closed) in X iff $A \cap X^n$ is open (or closed) in X^n for every n.

The *dimension of* X, denoted $\dim X$, is defined as the maximal dimension of a cell in X, or as infinite if there is no cell of maximal dimension.

Two particular classes of CW complexes are *simplicial complexes* and *cube complexes.*

Definition 2.1. A *simplicial complex* is a CW complex in which all cells are simplices and the intersection between two simplices, if nonempty, is a face of both.

Definition 2.2. A *cube complex* is a CW complex in which all cells are unitary cubes (i.e., copies of $[0,1]^n$ for some n), glued together via isometries along faces.

It is clear from the definition that CW complexes have a remarkable property: they have a strong combinatorial nature, which is the source of many important results. One of the most relevant consequences of this combinatorial nature is that the singular homology of a CW complex is much easier to compute than that of a general topological space. For a CW complex, the singular homology coincides indeed with the so-called cellular homology, which is constructed as follows. The *cellular chain complex* of a CW complex X, denoted $C_\bullet(X)$, has in degree n the free $\mathbb{Z}$-module over the set of cells of dimension n, while the differential sends an n-cell to a linear combination of the $(n-1)$-cells of its boundary, with coefficients that depend on the degrees of the attaching maps. The *cellular homology of* X, $H_\bullet(X)$ is defined as the homology of the cellular chain complex of X. There is an alternative, yet equivalent, definition of cellular homology, which consists in defining the cellular chain complex using portions of long exact sequences for CW pairs. It is a useful approach in order to prove some theoretical results, but it is much more convoluted and we will not need that point of view. As we can see, the calculations only rely on the combinatorial structure of the CW complex, which is much more manageable than the topological one.

The most relevant results, for the purpose of our thesis, are the following (see [Hat01]).

Proposition 2.3. *Let X be a CW complex. Then:*

(i) $H_n(X) = 0$ *if X has no n-cells;*

(ii) if X has k n-cells, then $H_n(X)$ is generated by at most k elements;

(iii) if X has neither $(n-1)$-cells nor $(n+1)$-cells, then $H_n(X)$ is free abelian with basis in one-to-one correspondence with the n-cells of X;

(iv) if X is path-connected, $H_0(X) = \mathbb{Z}$.

Proposition 2.4. *Let X be a CW complex. Then for $n > 1$ the differential of the cellular chain complex is given by*

$$d_n(e^n_\alpha) = \sum_\beta d_{\alpha\beta} e^{n-1}_\beta,$$

where e^{n-1}_β are the $(n-1)$-cells in the boundary of e^n_α, they are finite in number and $d_{\alpha\beta}$ is the degree of the map $S^{n-1}_\alpha \to X^{n-1} \to S^{n-1}_\beta$ given by composing the attaching map of e^n_α with the quotient map that collapses $X^{n-1} \setminus e^{n-1}_\beta$ to a point.

Another family of groups that can be associated to a topological space, thus to a CW complex as well, X is that of the *homotopy groups $\pi_n(X, x_0)$ of X with basepoint x_0*. The case for $n = 1$ is the fundamental group $\pi_1(X, x_0)$.

Definition 2.5. Given a topological space X, the *n-th homotopy group $\pi_n(X, x_0)$ of X with respect to a chosen point x_0* is the set of the homotopy classes of maps $(I^n, \partial I^n) \to (X, x_0)$, where homotopies f_t satisfy $f_t(\partial I^n) = x_0$ for any t. Here I^n is the n-dimensional unit cube $[0, 1]^n$. Equivalently, the elements of $\pi_n(X, x_0)$ can be thought of as homotopy classes of maps $(S^n, s_0) \to (X, x_0)$, which for $n = 1$ is the usual definition. For $n > 1$ such a set is an abelian group with the following operation, which is well-defined on homotopy classes:

$$(f + g)(s_1, \dots, s_n) := \begin{cases} f(2s_1, s_2, \dots, s_n), & \text{if } s_1 \in [0, 1/2] \\ g(2s_1 - 1, s_2, \dots, s_n), & \text{if } s_1 \in [1/2, 1]. \end{cases}$$

For $n = 1$ the same operation gives a group structure but this is not abelian in general. Note that for X path-connected changing the basepoint will produce an isomorphic group, therefore in such cases we will use the notation $\pi_n(X)$.

Proposition 2.6. *Given a CW complex X, the inclusion $X^n \to X$ induces isomorphisms on π_i for $i < n$ and a surjection on π_n.*

The following two results state an important connection between homotopy and homology groups.

Theorem 2.7 (Whitehead)**.** *A weak homotopy equivalence $f\colon X \to Y$ between connected CW complexes, i.e a map inducing isomorphisms $f_*\colon \pi_n(X) \to \pi_n(Y)$ for all n, is a homotopy equivalence. If f is the inclusion of the subcomplex X into Y, then X is a deformation retract of Y.*

Therefore a weakly contractible CW complex (i.e., a CW complex whose homotopy groups are all trivial) is contractible.

Theorem 2.8 (Hurewicz)**.** *For a topological space X,*

$$H_1(X) \simeq \pi_1(X, x_0)/[\pi_1(X, x_0), \pi_1(X, x_0)].$$

Moreover, if X is $(n-1)$-connected for $n \geq 2$ (i.e., $\pi_i(X, x_0) = 0$ for $i < n$), then $H_i(X) = 0$ for $0 < i < n$ and $\pi_n(X) \simeq H_n(X)$.

As a consequence, it is possible to state a homological version of Whitehead's theorem, whose relevance lies in the fact that usually homology groups are easier to compute than homotopy groups.

Corollary 2.9. *Let $f\colon X \to Y$ be a map between simply-connected CW complexes inducing isomorphisms $f_*\colon H_n(X) \to H_n(Y)$ for all n. Then f is a homotopy equivalence.*

2.2 Groups presentations, Cayley graphs and Cayley complexes

Given a group G, it is always possible to find an epimorphism $\phi\colon F \to G$, where F is a free group. Let $N := \ker \phi$, so that $G \simeq F/N$. If $F = F(X)$ is the free group over X and N is the normal closure of a subset $R \subseteq F$, then we can write $G = \langle X \mid R \rangle$, meaning that G is indeed isomorphic to the free group over X modulo the normal closure of R in such free group. This is called a *presentation for* G, X is a set of *generators* for G and the elements of R are called *relators* (they are called *relations* if they are written in the form $v = w$, meaning that vw^{-1} is in R). G is said to be *finitely generated* if it admits a presentation with a finite number of generators, while it is called *finitely presented* if also the number of relators is finite.

Definition 2.10. Let G be a group and S be a symmetric generating set (i.e. $S = S^{-1}$) not containing 1_G. The *Cayley graph of G with respect to S* is the graph $\Gamma(G, S)$ whose vertices are the elements of G and such that two vertices g, h are connected by an oriented edge going from g to h if $h = gs$ for some $s \in S$, in which case such edge is labeled s.

Note that any generating set for G can be made into a symmetric generating set not containing the identity. Note also that this definition of the Cayley graph produces a graph in the sense of Serre (see [Ser03]), but it can be easily adapted to the other definitions of graphs. It is always a connected graph, because S generates G.

Definition 2.11. Let $G = \langle S \mid R \rangle$ be a group with a set of generators S and a finite set of relators R. The *Cayley complex of G with respect to S* is the 2-dimensional CW complex having $\Gamma(G, S)$ as 1-skeleton and a 2-cell for every vertex g and every relator r. Such 2-cell is attached to $\Gamma(G, S)$ along the closed path starting at g and reading the word r.

Note that a Cayley complex is always simply-connected.

The group G acts on every one of its Cayley graphs and Cayley complexes with the action induced by the left regular action. Such action is always free and transitive on the vertices. The quotient of a Cayley complex by the action of the group is what is known as a *presentation complex for G*. It is a complex with one single vertex, an edge for every generator and a 2-cell for every relation. Its fundamental group is the group G itself, but in general the higher homotopy groups are not trivial (in other words, it is not *aspherical*).

2.3 Group cohomology and finiteness conditions

In this section we will introduce group homology and cohomology and some finiteness conditions. The source we suggest is [Bro82].

Group homology and cohomology

In topology, homology and cohomology theories are ways to associate to a topological space a succession of groups that encode some pieces of information about the space itself. The same idea can be applied to groups. We first need to recall some notions of homological algebra.

Given a ring R, an R-module P is *projective* if for any mapping problem of the form

$$\begin{array}{ccccc} & & P & & \\ & {\scriptstyle\psi}\swarrow & \downarrow{\scriptstyle\phi} & \searrow{\scriptstyle 0} & \\ M' & \xrightarrow{i} & M & \xrightarrow{j} & M'' \end{array}$$

where M, M', M'' are R-modules, $j\phi = 0$ and the row is exact there exists a solution, i.e., a map ψ such that $i\psi = \phi$. Note that free modules are projective.

Given an R-module M, we call a *projective resolution for M* an exact sequence of the form

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0,$$

where the P_i's are projective R-modules. If all the P_i's are free, we call it a *free resolution*.

The first step to defining the *homology* of a group G is to take a projective resolution

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \mathbb{Z} \longrightarrow 0,$$

of $\mathbb{Z}$ over $\mathbb{Z}G$, where we see $\mathbb{Z}$ as a trivial $\mathbb{Z}G$-module. Then we apply to each term the *coinvariants* functor $M \to M_G$, where M is a G-module and $M_G := M \,/ < gm - m | m \in M, g \in G >$. In other words, the coinvariants module of a G-module M is the largest quotient of M on which G acts trivially. We obtain the chain complex

$$\cdots \longrightarrow (P_2)_G \longrightarrow (P_1)_G \longrightarrow (P_0)_G \longrightarrow 0.$$

Definition 2.12. The *homology of* G, denoted $H_\bullet(G)$, is defined as the homology of the above chain complex P_G.

It is possible to prove that for every group the trivial module admits a projective resolution and that the homology of the group does not depend on the specific resolution one chooses.

There are other functors that one can apply to a projective resolution in this context, such as the *tensor product* and the *Hom* functor.

Given two left G-modules M and N, we consider N as a right G-module by $ng := g^{-1}n$. Therefore we can define the tensor product $N \otimes_G M$, which is obtained from $N \otimes M = N \otimes_{\mathbb{Z}} M$ by introducing the relations $g^{-1}n \otimes m = n \otimes gm$, or equivalently $n \otimes m = gn \otimes gm$. By fixing a G-module M and by applying the tensor product functor $\_ \otimes_G M$ to a projective resolution $P_\bullet \twoheadrightarrow \mathbb{Z}$ for G, we get the following chain complex:

$$\cdots \longrightarrow P_2 \otimes_G M \longrightarrow P_1 \otimes_G M \longrightarrow P_0 \otimes_G M \longrightarrow 0.$$

Definition 2.13. The *homology of* G *with coefficients in* M is the homology of the above chain complex. It is denoted $H_\bullet(G, M)$.

As $N \otimes_G M = (N \otimes M)_G$, where we consider the diagonal G-action on $N \otimes M$ given by $g(n \otimes m) = gn \otimes gm$, tensoring a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ with $\mathbb{Z}$ is equivalent to applying the coinvariants functor to the projective resolution. Therefore $H_\bullet(G, \mathbb{Z}) = H_\bullet(G)$.

Let now consider a G-module M and the contravariant functor $\mathrm{Hom}_G(\_, M)$. Applying it to a projective resolution $P_\bullet \twoheadrightarrow \mathbb{Z}$ of $\mathbb{Z}$ over $\mathbb{Z}G$ yields a nonnegative cochain complex

$$0 \longrightarrow \mathrm{Hom}_G(P_0, M) \longrightarrow \mathrm{Hom}_G(P_1, M) \longrightarrow \mathrm{Hom}_G(P_2, M) \longrightarrow \dots.$$

Definition 2.14. The *cohomology of* G *with coefficients in* M is the cohomology of the cochain complex above. It is denoted $H^\bullet(G, M)$.

Finiteness conditions

Definition 2.15. Given a group G, a *classifying space for* G is a CW complex whose fundamental group is isomorphic to G and whose higher homotopy groups are trivial. Equivalently, the fundamental group is isomorphic to G and its universal cover is contractible. It is usually denoted by BG or $K(G,1)$, while the universal covering is denoted EG.

Note that as a consequence of the theorems stated in section 2.1, two classifying spaces for the same group are always homotopy equivalent. If X is a BG, then there is a free action induced on the space EG. Equivalently, we could define EG to be a contractible CW complex endowed with a free cellular G-action (i.e., G acts freely on EG by homeomorphisms and every element of G sends each cell into another cell) and define BG as the quotient EG/G. Classifying spaces provide a useful tool as they encode different properties of the group. For example, it is possible to prove that every group has a classifying space and that the cellular chain complex of the universal covering of a classifying space for a group G is in fact a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. The following theorem holds.

Theorem 2.16. *Let G be a group, BG a classifying space for G and EG its universal covering. Then $C_\bullet(EG)_G = C_\bullet(BG)$ and $H_\bullet(G) = H_\bullet(BG)$.*

In fact, some authors consider this as a definition for the homology of G, as the homology of a BG only depends on G.

The *geometric dimension of* G, denoted $\operatorname{gd} G$, is the minimal dimension of a classifying space for G (therefore it can be infinite if there exists no finite dimensional BG). It can also be defined as the minimal dimension for an EG. Note that if $H \leq G$, then an EG is also an EH, therefore $\operatorname{gd} H \leq \operatorname{gd} G$. Moreover, if $H_n(G) \neq 0$, then $\operatorname{gd} G \geq n$.

We say that G *has cohomological dimension* $\leq n$ for a nonnegative integer n, if G admits a projective resolution of length n, i.e. of the form

$$0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \mathbb{Z} \longrightarrow 0.$$

We write $\operatorname{cd} G \leq n$, where $\operatorname{cd} G$ denotes the *cohomological dimension of* G. The *cohomological dimension of* G is defined as the smallest n for which $\operatorname{cd} G \leq n$. If G does not admit any projective resolution of finite length, then we set $\operatorname{cd} G = \infty$. We now state some results about the cohomological dimension.

Theorem 2.17. *Let G be a group. Then:*

- $\operatorname{cd} G = \inf\{n \mid H^i(G, \_) = 0\, \forall i > n\}$
  $= \sup\{n \mid H^n(G, M) \neq 0$ *for some G-module M*$\}$;

- $\operatorname{cd} G \leq \operatorname{gd} G$*;*
- $\operatorname{cd} G = 0 \iff \operatorname{gd} G = 0 \iff G$ *is trivial;*
- $\operatorname{cd} G = 1 \iff \operatorname{gd} G = 1 \iff G$ *is free and nontrivial;*
- *If* $\operatorname{cd} G < \infty$*, then* $\operatorname{cd} G = \sup\{n \mid H^n(G, F) \neq 0$ *for some free* G*-module* $F\}$*;*
- $\operatorname{cd} H \leq \operatorname{cd} G$ *for any* $H \leq G$*, with equality if* H *has finite index in* G *and* G *is torsion-free (Serre);*
- $\operatorname{cd} G < \infty \implies G$ *is torsion-free;*
- G *admits a free resolution of length* $\operatorname{cd} G$*.*

Theorem 2.18 (Eilenberg-Ganea)**.** *Let* G *be a group. Then* $\operatorname{cd} G = \operatorname{gd} G$*, except possibly if* $\operatorname{cd} G = 2$ *and* $\operatorname{gd} G = 3$*.*

The following result describes a technique that will be used for some computations later on.

Lemma 2.19. *Suppose* $\partial_\bullet \colon X_\bullet \longrightarrow \mathbb{Z}$ *is a free resolution of* $\mathbb{Z}G$*-modules of length* n*, for a certain group* G*. Suppose also that the following hypotheses hold:*

- $X_n = \mathbb{Z}Gc_n \oplus X'_n$*;*
- $X_{n-1} = \mathbb{Z}Gc_{n-1} \oplus X'_{n-1}$*;*
- $\partial_n(X'_n) \subseteq X'_{n-1}$
- $\partial_n(c_n) = gc_{n-1} + z$*, with* $z \in X'_{n-1}$ *and* $g \in G$*.*

Then $\partial'_\bullet \colon X'_\bullet \longrightarrow \mathbb{Z}$ *is a free resolution for* G*, where* $X'_j = X_j$ *for* $j \leq n-2$ *and* $\partial'_\bullet$ *is given by properly restricting* $\partial_\bullet$ *in degree* n *and* $n-1$*.*

Proof. We claim that $X_{n-1} = \operatorname{im}\partial_n \oplus X'_{n-1}$. Clearly the equality holds with the sum. Take $a\partial_n(c_n) = agc_{n-1} + az \in \operatorname{im}\partial_n \cap X'_{n-1}$. Then $agc_{n-1} \in X'_{n-1}$, therefore $a = 0$. This proves the claim.

Let now consider the complex

$$X'_n \xrightarrow{\partial'_n} X'_{n-1} \xrightarrow{\partial'_{n-1}} \ldots \to \mathbb{Z} \to 0.$$

- $\operatorname{im}\partial'_n = \ker\partial'_{n-1}$.
  Indeed, let $x \in \ker\partial'_{n-1}$. Then $x \in \operatorname{im}\partial_n \cap X'_{n-1}$, so we can write $x = \partial_n(y)$ for some $y \in X_n$. Let $a \in \mathbb{Z}G$ and $b \in X'_n$ such that $y = ac_n + b$. Then $x = a\partial_n(c_n) + \partial_n(b) \implies x - \partial_n(b) = a\partial_n(c_n) \in X'_{n-1} \cap \mathbb{Z}G\partial_n(c_n) = 0$, hence $a = 0$ and $x = \partial_n(b) \in \operatorname{im}\partial'_n$.

- $\mathrm{im}\partial'_{n-1} = \ker\partial'_{n-2}$.
  Indeed, $\ker\partial'_{n-2} = \ker\partial_{n-2} = \mathrm{im}\partial_{n-1} = \partial_{n-1}(\mathbb{Z}G\partial_n(c_n) \oplus X'_{n-1}) = \partial_{n-1}(X'_{n-1}) = \mathrm{im}\partial'_{n-1}$.

□

Remark 2.20. If in Lemma 2.19 X'_n is trivial, then the lemma implies that $\mathrm{cd}\, G \leq n-1$.

Definition 2.21. A group G is said to be *of type* FP_n if it admits a projective resolution $P_\bullet \twoheadrightarrow \mathbb{Z}$ where P_i is finitely generated for $0 \leq i \leq n$. G is *of type* FP_∞ if it is of type FP_n for every n, and *of type* FP if it admits a projective resolution of finite length and all the projective modules in the resolution are finitely generated.

Definition 2.22. A group G is said to be *of type* F_n if it admits a classifying space with finite n-skeleton. G is *of type* F_∞ if it is of type F_n for every n, and *of type* F if it admits a finite classifying space.

Remark 2.23. Equivalently, we could define type F_n, F_∞ and F in terms of the action of G on EG. For instance, G is of type F_n if it admits an EG with a finite number of orbits of cells up to dimension n. Another equivalent definition is the following: G is of type F_n if it acts cellularly, freely and cocompactly on an $(n-1)$-connected CW complex (see [Geo08, Prop. 7.2.1]).

Note that the following implications hold.

$$\begin{array}{ccccc} F & \Longrightarrow & F_\infty & \Longrightarrow & F_n \\ \Downarrow & & \Downarrow & & \Downarrow \\ FP & \Longrightarrow & FP_\infty & \Longrightarrow & FP_n \end{array}$$

Also, $F_n \implies F_{n-1}$ and $FP_n \implies FP_{n-1}$ for all n. These implications are strict in general, but it is possible to show that for finitely presented groups being of type F_n is equivalent to being of type FP_n. See for example [BB97]. Moreover, every group is of type F_0 and FP_0, type F_1 and type FP_1 are both equivalent to finite generation and type F_2 is equivalent to finite presentability. Finite groups are of type F_∞ but not of type F.

Other relations among the finiteness conditions we have seen are summarized in the following results.

Proposition 2.24. *(1) Let G be a group and $H \leq G$ of finite index. Then for $0 \leq n \leq \infty$ one has that G is of type $FP_n \iff H$ is of type FP_n.*

(2) If G is of type FP_∞ and $H^n(G, \mathbb{Z}G) = 0$ for some n, then $H^n(G, F) = 0$ for all free $\mathbb{Z}G$-modules F.

(3) If G is of type FP_n and M is a G-module which is finitely generated as an abelian group, then $H_i(G, M)$ and $H^i(G, M)$ are finitely generated abelian groups for $i \leq n$.

(4) A group G is of type FP if and only if $\operatorname{cd} G < \infty$ and G is of type FP_∞ (for the "only if" part it is enough to take G of type FP_n, where $n = \operatorname{cd} G$).

(5) Let G be torsion-free and $H \leq G$ of finite index. Then G is of type $FP \iff H$ is of type FP.

(6) If G is of type FP, then $\operatorname{cd} G = \max\{n \mid H^n(G, \mathbb{Z}G) \neq 0\}$.

2.4 CAT(0) spaces and CAT(0) cube complexes

This section will regard the theory of CAT(0) cube complexes and CAT(0) spaces, which will be relevant in the continuation of this thesis. For the former, a combinatorial approach will be adopted. In order to motivate the combinatorial approach, we will then sum up the fundamentals of the theory of CAT(κ) and CAT(0) metric spaces.

Recall that, given a cube complex X, the *link* of the vertex $v \in X^0$ is the CW complex given by an $(n-1)$-simplex for every n-cube having v as a vertex. It can be seen as the intersection between X and an ε-sphere centered at v, for ε small enough. Recall also that a simplicial complex is called a flag complex if n vertices span an $(n-1)$-simplex if and only if they are pairwise connected by an edge.

Definition 2.25. A *non-positively curved cube complex X* is a cube complex such that the link of every vertex is a flag complex. A simply-connected non-positively curved cube complex is called a *CAT(0) cube complex*.

The relevant characteristic, for the purposes of this thesis, of CAT(0) cube complexes is that they are contractible. But this is not the only significant aspect of these spaces. In fact, a byproduct of proving that a certain cube complex is contractible by showing that it is CAT(0) is that interesting results can be proven for groups acting "sufficiently nicely" on it. The reason why a CAT(0) cube complex is contractible lies in the theory of CAT(κ), and more specifically CAT(0), metric spaces. It is not a coincidence that the same name has been used, as we will see that a CAT(0) cube complex in the sense above is also CAT(0) in the sense of metric spaces. We refer the reader to [BH99] for more details.

Definition 2.26. Let (X, d) be a metric space. A *geodesic segment between two points a and b* in X is the image of an isometric embedding $[0, d(a, b)] \to X$. The metric space (X, d) is said to be *geodesic* if any two points are connected by a geodesic segment, *ρ-geodesic* if any two points at distance less than ρ are connected by a geodesic segment.

Let us consider a metric space (X, d). A geodesic triangle in X is a triangle (i.e., the choice of three vertices and a path between any two of them) in which any side is a geodesic segment between the two vertices of the side (note that in general there might be many geodesic segments between the two vertices). Let M^2_κ be the unique complete simply connected surface with constant curvature κ and let D_κ be the diameter of M^2_κ, which is $+\infty$ if $\kappa \leq 0$ and $\frac{\pi}{\sqrt{\kappa}}$ if $\kappa > 0$. For example, M^2_0 is the Euclidean plane, M^2_1 is the unit sphere S^2 and M^2_{-1} is the hyperbolic plane (actually, each M^2_k can be obtained from one of these examples by scaling the metric).

Definition 2.27. Let Δ be a geodesic triangle in the metric space (X, d). A *comparison triangle* Δ' for Δ in some model space M^2_κ is a triangle with the sides of the same length as the sides of Δ. We say that Δ satisfies the *CAT(κ) inequality* if there exists a comparison triangle Δ' in the model space M^2_κ such that for every $p, q \in \Delta$ the distance between p and q is less or equal than the distance between the two corresponding points p', q' in Δ'.

We recall that such a comparison triangle always exists, provided that the perimeter of Δ is less than $2D_\kappa$, and it is unique up to isometry.

Definition 2.28. For $\kappa \leq 0$, a geodesic space (X, d) is called a *CAT(κ) space* if every geodesic triangle satisfies the CAT(κ) inequality.
For $\kappa > 0$, a D_κ-geodesic space (X, d) is called a *CAT(κ) space* if all geodesic triangles of perimeter less than $2D_\kappa$ satisfy the CAT(κ) inequality.

Definition 2.29. A metric space (X, d) is said to be of *curvature* $\leq \kappa$ if it is locally a CAT(κ) space, i.e., for every $x \in X$ there exists r_x such that the ball $B(x, r_x)$ with the induced metric is a CAT(κ) space. Metric spaces of curvature ≤ 0 are called *non-positively curved.*

The next result is [BH99, Corollary 1.5].

Proposition 2.30. *Any CAT(0) space is contractible.*

Cube complexes can be endowed with a path metric as follows. A *rectifiable path* in the cube complex X is a path that can be divided into subpaths, each of which is contained in some cube and is rectifiable in the classical sense. We then define the *length* of the rectifiable path as the sum of the lengths of the subpaths. The distance between two points $p, q \in X$ is then the infimum of the lengths of all possible rectifiable paths in X with endpoints p and q. If X is finite-dimensional, then this path metric is indeed a metric and X is a complete, geodesic metric space (see [BH99]; the case of infinite-dimensional locally finite complexes was treated by Moussong in [Mou88]). A non-positively curved cube complex is locally CAT(0) (cfr. [Gro87]). As a consequence of Cartan-Hadamard theorem, a simply-connected non-positively curved cube complex is CAT(0) in the sense of metric spaces (see [BH99, theorem 4.1]). In [Lea12] Ian Leary showed that this can be applied also to infinite-dimensional complexes. However, we will only deal with finite-dimensional cube complexes in this thesis.

In the rest of the thesis, when dealing with cube complexes, we will use the two languages about non-positively curved and CAT(0) spaces (the combinatorial one we used at the beginning of this section and the one coming from the general theory of CAT(0) spaces) interchangeably. Sometimes we will call the defining condition of non-positively curved cube complexes the *Gromov's link condition*. Note that this name is usually used, in the theory of CAT(κ) spaces, to denote another, yet equivalent, condition.

Chapter 3

Hypercubical Groups

In this chapter we will see the construction of a cube complex associated to a finitely generated group G with respect to a finite generating set Σ. Such construction is canonical, once the generating set Σ for G is fixed.

3.1 n-cubes

Informally, an n-cube is the n-dimensional version of a cube (e.g., a point for $n = 0$, a segment for $n = 1$, a square for $n = 2$ and a cube for $n = 3$). We consider $[0,1]^n := \underbrace{[0,1] \times \cdots \times [0,1]}_{n \text{ times}}$ as the model of an n-cube. The *interior* of the n-cube is formed by all the points with all coordinates belonging to $(0,1)$, while if some coordinate is either 0 or 1 the point belongs to the *boundary* of the n-cube.

The boundary can be seen as the union of a finite number of cubes of dimension $< n$, called the *faces* of the n-cube.

Proposition 3.1. *In an n-cube the number of m-cubes, for $0 \leq m \leq n$, is*

$$E_{m,n} = 2^{n-m}\binom{n}{m}.$$

In particular, for $m < n$ the m-cubes are in the boundary.

Proof. The proof is combinatorical. For $m = n$ there is nothing to prove, thus let $m < n$. An n-cube has 2^n vertices, the points whose coordinates are all either 0 or 1. For each vertex there are $\binom{n}{m}$ ways to choose m sides of the n-cube defining an m-cube containing it, but in this way each m-cube is counted 2^m times (one for each of its vertices). □

The quantity $E_{m,n}$ satisfies also the following linear recurrence relation:

Proposition 3.2.

$$E_{m,n} = 2E_{m,n-1} + E_{m-1,n-1},$$

where $E_{0,0} = 1$ *and the undefined elements* ($E_{k,l}$ *for* $l < k$, $l < 0$, $k < 0$) *are equal to 0.*

3.2 Hypercubical complex and hypercubical groups

Let (G, Σ) be a finitely generated group with $\Sigma^{-1} = \Sigma$ and $1 \notin \Sigma$. We can associate a cube complex to the Cayley graph of (G, Σ), namely $\Gamma(G, \Sigma)$. This cube complex can be constructed by induction. In order to do this, we first need to define the *cubicalization* $Q(\Gamma)$ of a graph Γ, which is a way to associate a CW complex to Γ by only attaching cubes of different dimensions. The vertices and edges of Γ are respectively called the *0-cells* and *1-cells* of $Q(\Gamma)$.

Let $C_k := [0, 1]^k \, \forall k > 0$, $C_0 = \{0\}$.

Definition 3.3. Set $Q_1 := |\Gamma|$. Two edges e_1, e_2 in Q_1 with a common vertex *satisfy the 2-dimensional cubical link condition* if there exist other two edges e_3, e_4 in Q_1 such that the union of these four edges is isomorphic to the boundary of C_2. We can attach a square, also called a *2-cell*, to Q_1 along the edges $e_1, \ldots, e_4$. Such square is *defined by* $e_1, \ldots, e_4$. We can also say that e_1, e_2 *define the 2-cell* σ, which means that e_1, e_2 satisfy the 2-dimensional cubical link condition and the square that we attach following the procedure above is σ. By attaching all the possible 2-cell that we can get, we define the space Q_2.

Suppose that for $n \geq 3$ we have inductively defined the space Q_{n-1}. Then we say that n edges $e_1, \ldots, e_n$ with a vertex in common satisfy the n-dimensional cubical link condition if:

1. any $n - 1$ of them satisfy the $(n - 1)$-dimensional cubical link condition;
2. there exist edges $e_{n+1}, \ldots, e_{2n}$ such that any $n - 1$ of them satisfy the $(n - 1)$-dimensional cubical link condition;
3. the union of all the $(n - 1)$-cells resulting from the two points above (with the appropriate identifications) is isomorphic to the boundary of C_n.

By attaching all the resulting n-cubes, also called *n-cells*, to Q_{n-1} we define the space Q_n. The definition of an *n-cube defined by n edges* is analogous to the one given for $n = 2$. When this process stops, the resulting space $Q(\Gamma)$ will be called the *cubicalization of* Γ.

Remark 3.4. If Γ is a simplicial graph, then $Q(\Gamma)$ is a cube complex.

We can define the following functions $\forall n$, $\forall i \in \{1, \ldots, n\}$, $\forall j \in \{0, 1\}$:

$$p_{i,j} \colon C_n \to C_n, \quad p_{i,j}(x_1, \ldots, x_n) := (x_1, \ldots, j, \ldots, x_n)$$

where j is in the i-th position. This clearly is the projection on the facet $\{ x_i = j \} \subseteq C_n$.

$$\iota_{i,j} \colon C_{n-1} \to C_n, \quad \iota_{i,j}(x_1, \ldots, x_{n-1}) := (x_1, \ldots, j, \ldots, x_{n-1})$$

where j is in the i-th position. This is the inclusion of C_{n-1} in C_n as the facet $\{ x_i = j \} \subseteq C_n$.

Under the identification of each n-cell with C_n these functions induce analogous functions on each n-cell.

Remark 3.5. For any i, j the map $\iota_{i,j}$ is injective, so it is invertible on its image. The maps $\iota_{i,j}^{-1} \circ p_{i,j}$ coincide for every j, therefore the map $\psi_i := \iota_{i,j}^{-1} \circ p_{i,j}$ is well defined. It is called the *i-th face map.*

Remark 3.6. For any i, j the restriction of $p_{i,j}$ to $\{ x_i = j \}$ is the identity, therefore $\iota_{i,j}^{-1} \circ p_{i,j} \circ \iota_{i,j} = id_{C_{n-1}}$.

Definition 3.7. Given a finitely generated group G and a finite, symmetric generating set Σ not containing 1_G, the *hypercubical complex of* (G, Σ), also called the *hypercubical complex of G with respect to* Σ, is given by $\mathcal{C}_\bullet(G, \Sigma) := Q(|\Gamma(G, \Sigma)|)$. $\mathcal{C}_n(G, \Sigma)$ will be the set of n-cells, while the n-skeleton is denoted $\mathcal{C}_\bullet^n(G, \Sigma)$. When either Σ or (G, Σ) are clear from the context, we shall omit them when it will not cause any confusion.

Note that the technical definition we have given says that in order to construct the hypercubical complex of a finitely generated group we need to attach an n-cube whenever we see $2n$ $(n-1)$-cubes forming the boundary of an n-cube. This can be done also in a non-inductive way, by saying that we attach an n-cube whenever $n2^{n-1}$ edges form the 1-skeleton of an n-cube. In this sense, the hypercubical complex can be seen as the cubical version of a flag complex.

Remark 3.8. The definition of the n-dimensional cubical link condition we have given is useful to define the hypercubical complex in an inductive way, but it can now be reformulated in an equivalent fashion. Indeed, n edges of the hypercubical complex, all incident to the same vertex and pairwise adjacent sides of a square, satisfy the n-dimensional cubical link condition if, and only if, they are edges of an n-cube. This less technical formulation will be useful for some proofs in the continuation of the thesis.

Proposition 3.9. *The hypercubical complex of a finitely generated group G with respect to a generating set Σ is always path-connected.*

Proof. The 1-skeleton is path-connected, being Σ a generating set, and all cells are path-connected. □

Definition 3.10. The *cube number of* (G, Σ) is defined as the highest dimension of a hypercube in $\mathcal{C}_\bullet$.

As the cube number is always finite (there exist at most $2n$ edges coming out from each vertex of the Cayley graph, if n is the cardinality of Σ), $\mathcal{C}_\bullet$ is a finite-dimensional cube complex and therefore it is complete (cfr. [BH99, p. 112]).

Remark 3.11. The maps $\iota_{i,j}$ and $p_{i,j}$ are clearly continuous, as well as the maps they induce on each cell of $\mathcal{C}_\bullet$.

We can now give the definition of the object giving the title to this thesis.

Definition 3.12. Let G be a finitely generated group and Σ a finite generating set not containing 1_G (if Σ is not symmetric, we will in fact consider $\Sigma \cup \Sigma^{-1}$). We say that (G, Σ) *is hypercubical*, or that G *is hypercubical with respect to* Σ, if the hypercubical complex $\mathcal{C}_\bullet(G, \Sigma)$ is contractible. In many cases we simply say that a group is hypercubical, when the generating set is clear from the context or when we do not need to specify the generating set.

The easiest examples are given by free groups, free abelian groups and right-angled Artin groups. The latter is the example that inspired the definition. Due to its importance, we will now just state the results about such groups we are interested in, and we will go back to this family of groups later on.

E.g. 3.13 (Free groups)**.** Let F_n be the free group on n generators and let $\Sigma := \{x_1, \dots, x_n\}$ be a set of free generators. Then $\Gamma(F_n, \Sigma)$ is a tree, which is simply-connected. As a consequence, $\mathcal{C}_\bullet(F_n, \Sigma)$ coincides with $\Gamma(F_n, \Sigma)$, which is contractible. Therefore free groups are hypercubical with respect to any basis.

E.g. 3.14 (Free abelian groups)**.** The Cayley graph of a free abelian group $\mathbb{Z}^n$ with respect to a basis $\Sigma := \{x_1, \dots, x_n\}$ is the square unitary grid in the n-dimensional Euclidean space. As a consequence, $\mathcal{C}_\bullet(\mathbb{Z}^n, \Sigma)$ is the subdivision of $\mathbb{E}^n$ into unitary n-cubes and it is therefore contractible. In other words, free abelian groups are hypercubical with respect to any basis.

E.g. 3.15 (Right-angled Artin group)**.** Right-angled Artin groups, also known as RAAGs, are a family of groups defined by presentations in which the relators are only commutators among some generators. More details will be given in section 4.3. Given a RAAG G with a set of generators Σ, it is possible to associate a cell complex to it, namely the Salvetti complex, which is a classifying space for G. The complex $\mathcal{C}_\bullet(G, \Sigma)$ is the universal covering of the Salvetti complex, therefore it is contractible. This means that every right-angled Artin group is hypercubical with respect to its standard generating set.

More examples will be given in the continuation of the thesis.

3.3 Some general results about hypercubical groups

As we will see, groups of very different natures can be hypercubical, which makes it not likely to be able to prove many general results about hypercubical groups. However, some results can in fact be proven.

Let $\pi\colon F(\Sigma) \to G$ be a finite presentation of a group G, and let $\ell_\Sigma\colon F(\Sigma) \to \mathbb{Z}$ be the canonical length function on the finitely generated free group $F(\Sigma)$, then $\Sigma = W(1)$, where for $k \in \mathbb{Z}$, $k \geq 0$, one defines

$$W(k) := \{\omega \in F(\Sigma) \mid \ell_\Sigma(\omega) = k\}.$$

Definition 3.16. The finitely generated group (G, Σ) is said to be *quadratic* if

$$\ker(\pi) = \langle\langle W(4) \cap \ker(\pi)\rangle\rangle\,,$$

i.e., if it can be generated, as a normal subgroup of G, by words of length 4.

Quadratic groups can be characterized via their hypercubical complex $\mathcal{C}_\bullet$, as follows.

Proposition 3.17. *A finitely generated group (G, Σ) is quadratic if, and only if, the 2-skeleton of its hypercubical complex $\mathcal{C}_\bullet$ is simply-connected.*

Proof. If G is quadratic, then it admits a presentation $G = \langle \Sigma \mid R\rangle$, with $\langle\langle R\rangle\rangle = \langle\langle W(4) \cap \ker(\pi)\rangle\rangle$. Without loss of generality, one can suppose that $R = W(4) \cap \ker(\pi)$. Therefore the 2-skeleton of the hypercubical complex of (G, Σ) coincides with the Cayley complex related to the presentation $G = \langle \Sigma \mid R\rangle$, thus being simply-connected.

The inverse implication holds as one has

$$\pi_1(\mathcal{C}^2_\bullet, 1) \simeq \ker\pi/ \langle\langle W(4) \cap \ker(\pi)\rangle\rangle\,.$$

Indeed, one has that $\pi_1(\mathcal{C}^1_\bullet, 1) \twoheadrightarrow \pi_1(\mathcal{C}^2_\bullet, 1)$ via the cellular approximation theorem. Under the identification $\pi_1(\mathcal{C}^1_\bullet, 1) \simeq \ker\pi$, which holds as $\mathcal{C}^1_\bullet$ is the geometric representation of $\Gamma(G, \Sigma)$, the kernel of the previous surjection clearly is $\langle\langle W(4) \cap \ker(\pi)\rangle\rangle$. If $\mathcal{C}^2_\bullet$ is simply-connected, then $\ker\pi/ \langle\langle W(4) \cap \ker(\pi)\rangle\rangle$ is trivial, thus yielding quadraticity for (G, Σ). □

As a consequence, hypercubical groups have the following property:

Corollary 3.18. *A hypercubical group (G, Σ) is quadratic.*

The left action of a group G on its Cayley graph induces a cellular left action by isometries on the hypercubical complex having that Cayley graph as 1-skeleton. Such action has the properties stated by the following proposition.

Proposition 3.19. *Let G be a finitely generated group and Σ a finite generating set for G. Then the G-action on $\mathcal{C}_\bullet(G,\Sigma)$ is:*

(1) free and transitive on the vertices;

(2) free on the edges, assuming no generator has order 2;

(3) proper;

(4) cocompact;

(5) properly discontinuous.

Moreover the action is free if G is torsion-free. If G is hypercubical with respect to Σ, then also the converse is true.

Proof. By restricting the action of G to the 1-skeleton, we get the G-action on the Cayley graph. This proves (1) and (2). Suppose now that there is a cell C in $\mathcal{C}_\bullet$ fixed by the group action. Let H be the cell stabilizer of C. Then the elements of H permute the vertices of C, therefore there is a homomorphism $H \to S_k$ for a suitable k. The kernel of this homomorphism is trivial, as it fixes every vertex and vertex stabilizers are trivial. Therefore cell stabilizers are finite, which proves (3). Item (4) is due to the fact that any cell is in the same orbit of a cell having 1_G as a vertex. Hence $\mathcal{C}_\bullet$ can be covered by translates of the finite subcomplex given by all the cells having 1_G as a vertex (what is also called the cellular link of 1_G in $\mathcal{C}_\bullet$). Finally, (5) is equivalent to (3) in this context.

Suppose now that an element $g \in G$ fixes a cell C. Then g permutes the vertices of C, which implies that g has finite order, because of (1). On the other hand, if the action is free and G is hypercubical with respect to Σ, then $\mathcal{C}_\bullet(G,\Sigma)$ is the universal covering of a classifying space. As $\mathcal{C}_\bullet$ is finite-dimensional, G is torsion-free. □

Two direct consequences of hypercubicality are the following ones:

Proposition 3.20. *Let G be a torsion-free, hypercubical group. Then G is of type F.*

Proof. As G is torsion-free, the hypercubical complex is an EG, therefore G has a finite classifying space. □

The following is needed in the proof of Proposition 3.22.

Theorem 3.21 ([Geo08, Theorem 7.3.1])**.** *Suppose that, for $n \geq 1$, G acts cellularly and rigidly on an $(n-1)$-connected CW complex X that has finite n-skeleton mod G (the action is* rigid *if every cell stabilizer acts trivially on the cell it fixes). If the stabilizer of each i-cell is of type F_{n-i} for all $i \leq n-1$, then G is of type F_n.*

Proposition 3.22. *Suppose G is hypercubical with respect to Σ. Then G is of type F_∞.*

Proof. Using the notation of Theorem 3.21, let X be the hypercubical complex of G with respect to Σ. Either the action of G on it is rigid, or we pass to the first barycentric subdivision of X which is rigid (as X is a regular CW complex). The hypercubical complex is contractible, therefore it is $(n-1)$-connected for all n. The cell stabilizers are finite (and this holds also for the barycentric subdivision), thus of type F_k for any k. As a consequence, G is of type F_n for every n, which proves the claim. □

Proposition 3.23. *If G is torsion-free and hypercubical with respect to Σ, then $\operatorname{cd} G \leq n$, where n is the cube number of G with respect to Σ.*

Proof. The cube number of (G, Σ) provides an upper bound for the geometric dimension of G, yielding thus an upper bound for the cohomological dimension. □

Remark 3.24. The definition of hypercubical complex and hypercubical groups would work also without restricting our attention to finitely generated groups and finite generating sets. This would allow one to include other examples in the family of hypercubical groups (e.g., infinitely generated free groups). One could think that a possible compromise would be to consider finitely generated group, but allowing for infinite generating sets in the construction of the hypercubical complex. While this seems to be enough for certain arguments, losing the local finiteness of the hypercubical complex might be a problem for many other arguments. Moreover, it is common in geometric group theory to work with finitely generated groups and finite generating sets, therefore we chose to keep the restricted definition for now. However, the next section provides a very interesting example of a finitely generated group that has a contractible hypercubical complex, but with respect to an infinite generating set.

3.3.1 Thompson's group F

The group that is known as Thompson's group F was defined by Richard Thompson in 1965 and was later rediscovered in other areas of mathematics. Brown and Geoghegan proved ([BG84]) that this group is of type FP_∞. It was the first example of an infinite-dimensional, torsion-free group of type FP_∞. It is also an infinitely reiterated HNN extension. We will not dwell on all the details of the theory of this amazing group, but we suggest the following references in case the reader desires a more complete introduction about this group: the PhD thesis of James Belk ([Bel]) and the introductory notes by James Cannon, William Floyd and Walter Parry ([CFP]).

A *dyadic subdivision* of the interval $[0,1]$ is a subdivision that can be obtained by repeatedly cutting some intervals in half, so that every subinterval is of the form $[\frac{k}{2^n}, \frac{k+1}{2^n}]$ for some $k, n \in \mathbb{N}$. Given two dyadic subdivisions $\mathcal{D}, \mathcal{D}'$ with the same number of cuts, we can map $\mathcal{D}$ to $\mathcal{D}'$ with a piecewise-linear homeomorphism $f\colon [0,1] \to [0,1]$ in such a way that each subinterval of $\mathcal{D}$ is sent to the corresponding subinterval of $\mathcal{D}'$ linearly. This piecewise-linear homeomorphism is called a *dyadic rearrangement*. A piecewise-linear homeomorphism is a dyadic rearrangement if, and only if, all its slopes are powers of 2 and the coordinates of the breakpoints are dyadic rational numbers (i.e., of the form $k/2^n$ for some k, n). Dyadic rearrangements form a group under composition.

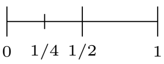
Definition 3.25. Thompson's group F is the group of dyadic rearrangements of $[0,1]$.

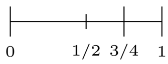
It can also be described in a logical language, relating it to associativity rules (and this in fact was the first description of this group). Moreover, every element of F can be represented by *tree diagrams*, that are pairs of finite binary trees representing the domain and range dyadic subdivisions of $[0,1]$. This paves the way for a combinatorial and geometric approach to the study of this group.

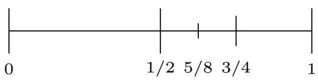
It is possible to prove the following statement.

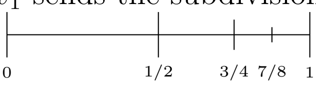
Proposition 3.26. *Thompson's group F is infinite and torsion-free.*

F is finitely presented. It can be generated by the following two elements:

- x_0 sends the subdivision 

to 

;
- x_1 sends the subdivision 

to 

.

A presentation for F having x_0 and x_1 as generators is

$$\left\langle x_0, x_1 \mid (x_1)^{x_0} x_1 = x_1 (x_1)^{x_0^2}, (x_1)^{x_0^2} x_1 = x_1 (x_1)^{x_0^3} \right\rangle,$$

where $a^b := b^{-1}ab$. F can also be defined by an infinite presentation, which is in many senses more convenient than the finite presentation above:

$$\langle x_0, x_1, \cdots \mid x_n^{x_k} = x_{n+1} \text{ for } 0 \le k < n \rangle .$$

It is possible to show (by using the theory of CAT(0) cube complexes or by using combinatorial Morse theory) that the hypercubical complex associated to this last presentation is contractible. This is another formulation of the result proved in [BG84]. As F is torsion-free, the hypercubical complex is the universal covering of a classifying space, but the fact that an infinite number of generators is involved does not allow us to directly derive the results that we stated for hypercubical groups. Nevertheless, it is possible to prove that F is of type F_∞, as it has a classifying space with a single vertex and two cells in every positive dimension.

Chapter 4

Twisted Right-Angled Artin Groups

In this chapter we will first illustrate the results concerning right-angled Artin groups. This will be the starting point to present two generalizations of right-angled Artin groups, namely the families of oriented and twisted right-angled Artin groups, and to show that these groups are hypercubical as well.

4.1 Right-angled Artin groups and two generalizations

Right-angled Artin groups are a specific type of Artin groups, so we will start by introducing Artin groups.

Artin groups, also known as Artin-Tits groups, are defined by a presentation of the form

$$\left\langle v_1, \dots, v_n \mid \underbrace{v_i v_j v_i \dots}_{m_{ij}} = \underbrace{v_j v_i v_j \dots}_{m_{ij}} \forall i, j \right\rangle,$$

where $m_{ij} \in \{2, \dots, \infty\}$ is the length of the word on both sides of the relation and $m_{ij} = \infty$ means that there is no relation between v_i and v_j. Such a presentation can be thought of as induced by a labeled, finite, simplicial graph Γ whose vertices are $v_1, \dots, v_n$ and whose edges are labeled by integers ≥ 2. In this case, the presentation has the vertices of the graph as generators and a relation for every edge, said relation being $\underbrace{v_i v_j v_i \dots}_{m_{ij}} = \underbrace{v_j v_i v_j \dots}_{m_{ij}}$ if the edge between v_i and v_j has label m_{ij}. Relations with $m_{ij} = \infty$ are not written and correspond to the missing edges in the graph Γ. One can associate a Coxeter group to every Artin group by simply specifying that every generator has order 2. This leads to the main distinction among Artin groups: on one hand the so-called *spherical* Artin groups, whose associated Coxeter groups are finite, and on the other hand the Artin groups *of infinite type*, whose associated Coxeter groups are infinite. These two types of groups are dealt with using different techniques and many results are known for spherical Artin groups but still only conjectured for Artin groups of infinite type.

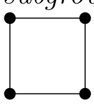
A particular subfamily of Artin groups is that of the so-called *right-angled* Artin groups, sometimes also known as *partially commutative groups* or *graph groups.*

Definition 4.1. Let Γ be a finite, simplicial graph (i.e., a graph with no loops nor multiple edges), with vertex set $V\Gamma$ and edge set $E\Gamma = \mathcal{P}_2(V\Gamma) := \{\text{subsets of } V\Gamma \text{ of cardinality } 2\}$. The *right-angled Artin group*, RAAG for the sake of brevity, A_Γ *associated to* Γ is the group defined by the presentation

$$A_\Gamma := \langle V\Gamma \mid [v_i, v_j] \, \forall \{v_i, v_j\} \in E\Gamma \rangle \, .$$

There is a large number of significant results concerning right-angled Artin groups. This is often due to the fact that these groups have a strong combinatorial and geometric nature. In particular, many properties of a RAAG A_Γ can be determined in terms of the graph Γ, in other words they can be expressed in a graph-theoretical language.

Theorem 4.2 ([Dro87a])**.** *Let A_Γ and A_Λ be two RAAGs. Then $A_\Gamma \simeq A_\Lambda$ if, and only if, $\Gamma \simeq \Lambda$.*

Theorem 4.3 ([Dro87b])**.** *Let A_Γ be a RAAG. Then every finitely generated subgroup of A_Γ is still a RAAG if, and only if, Γ does not contain either the square*  *nor the line on 4 vertices* *as induced subgraphs.*

Right-angled Artin groups are groups of type F, therefore they are torsion-free and have finite cohomological dimension. In particular, for a RAAG A_Γ, one has that $\operatorname{cd} A_\Gamma$ is equal to the *clique number* of Γ, which is defined as the maximal number of vertices of a clique (i.e., a complete subgraph) in Γ. This will be a consequence of Theorem 4.17.

We can now define two generalizations of right-angled Artin groups, namely twisted and oriented RAAGs (respectively TRAAGs and ORAAGs, for the sake of brevity). Oriented RAAGs are a subfamily of twisted RAAGs, but we will start by defining TRAAGs. We refer the reader to [Fon22] for the definitions that follow.

Definition 4.4. A *mixed graph* Γ is a simplicial graph together with a set $\overrightarrow{E\Gamma} \subseteq E\Gamma$ of *oriented edges* and two maps $\mathrm{o}, \mathrm{t} \colon \overrightarrow{E\Gamma} \to V\Gamma$ such that if an edge e has endpoints v, w then $\{\mathrm{o}(e), \mathrm{t}(e)\} = \{v, w\}$. Graphically, an oriented edge e will be represented by an arrow going from $\mathrm{o}(e)$ to $\mathrm{t}(e)$. We will say that such e *goes from* $\mathrm{o}(e)$ *to* $\mathrm{t}(e)$. If Γ is a mixed graph, we can associate a finite simplicial graph $\ddot{\Gamma}$ to it, called the *naïve graph associated to* Γ, which is given by considering oriented edges of Γ as not oriented.

Definition 4.5. Let Γ be a mixed graph. The *twisted right-angled Artin group* A_Γ *associated to* Γ is the group defined by the following presentation:

$$A_\Gamma := \left\langle V\Gamma \mid [v,w]\,\forall\{v,w\} \in E\Gamma \setminus \overrightarrow{E\Gamma}, [v,w\rangle\,\forall e \in \overrightarrow{E\Gamma} \text{ s.t. } v = \mathrm{o}(e), w = \mathrm{t}(e)\right\rangle,$$

where $[v,w] = vwv^{-1}w^{-1}$ and $[v,w\rangle := vwv^{-1}w$.

Remark 4.6. Note that if the relation $[v,w\rangle$ holds, then also the relations $[v^{-1},w\rangle$, $[v,w^{-1}\rangle, [v^{-1},w^{-1}\rangle$ hold.

E.g. 4.7. Here we list two basic examples of twisted right-angled Artin groups.

- Every RAAG is trivially a twisted RAAG, where the subset $\overrightarrow{E\Gamma}$ is empty.
- Let Γ be the graph $\overset{a}{\bullet}\!\longrightarrow\!\overset{b}{\bullet}$, then $A_\Gamma = \langle a,b \mid aba^{-1}b\rangle = \mathbb{Z} \rtimes \mathbb{Z}$ is the fundamental group of the Klein's bottle.

As we will see, in the class of TRAAGs new phenomena show up, that did not show up for RAAGs. In order to better control their behavior and to establish connections with other theories, we might want to focus our attention on a subclass of twisted right-angled Artin groups, that of *oriented RAAGs*. In order to define them, we first need to note that the edges of a mixed graph $\Gamma = (V,E)$ can be divided into two disjoint subsets: that of *special edges*, denoted E_s and consisting of the oriented edges of Γ, and that of *ordinary edges*, denoted E_o and containing the non-oriented edges. We can partition also the set of vertices into two subsets V_o and V_s. For non-isolated vertices, the set V_s of *special vertices* contains the origins of all special edges, while the set V_o of *ordinary vertices* contains the remaining ones. Every isolated vertex can be either ordinary or special (therefore in general this partition is not canonical).

Definition 4.8. Let $\Gamma = (V,E)$ be a mixed graph. We say that Γ is *specially oriented* or *special* if the endpoints of every ordinary edge are ordinary vertices and $\mathrm{t}(e)$ is ordinary $\forall e \in E_s$. The TRAAG A_Γ is then called an *oriented right-angled Artin group*.

4.2 Some results about twisted and oriented RAAGs

There are some results regarding twisted and oriented RAAGs that show how part of the theory of right-angled Artin groups can be extended, but at the same time new phenomena arise. The first main difference is the fact that there are examples of isomorphic TRAAGs defined by non-isomorphic mixed graphs. The following example is taken from [Blu20] (but note that the convention for the orientation of oriented edges is the opposite one compared to the one used here).

E.g. 4.9. Let $\Gamma = \overset{a}{\bullet}\longrightarrow\overset{b}{\bullet}\longleftarrow\overset{c}{\bullet}$ and $\Lambda = \overset{x}{\bullet}\longrightarrow\overset{y}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{z}{\bullet}$. Then the two homomorphisms

$$\phi\colon \begin{cases} a \mapsto x \\ b \mapsto y \\ c \mapsto xz \end{cases}, \qquad \psi\colon \begin{cases} x \mapsto a \\ y \mapsto b \\ z \mapsto a^{-1}c \end{cases}$$

are well defined and one the inverse of the other, therefore they are isomorphisms between A_Γ and A_Λ. Note that the two graphs of this example are special, so Theorem 4.2 cannot be extended to TRAAGs nor to ORAAGs.

Lemma 4.10 ([Fon22])**.** *Let* Γ *be a mixed graph and* $V_t := \mathrm{t}(\overrightarrow{E\Gamma})$*. Then* $A_\Gamma^{\boldsymbol{ab}} = \mathbb{Z}^{|V\Gamma\setminus V_t|} \times (\mathbb{Z}/2\mathbb{Z})^{|V_t|}$.

Proposition 4.11 ([Fon22])**.** *If two mixed graphs* Γ *and* Λ *have either a different number of vertices or a different number of termini of oriented edges, then* $A_\Gamma \not\simeq A_\Lambda$.

As we said in section 4.1, RAAGs are torsion-free. However, this is not always true for TRAAGs, as the following theorem shows. Note that the definitions of clique, induced subgraph and so on extend to the case of mixed graphs: a subgraph of a mixed graph is a clique if it is a clique as a naïve graph.

Theorem 4.12 ([Fon22])**.** *Let* Γ *be a mixed graph. Then* A_Γ *has torsion if, and only if,* Γ *contains a clique whose vertices form an oriented cycle.*

Proof. See [Fon22]. □

Corollary 4.13. *Oriented RAAGs are torsion-free*

Proof. It is straightforward. □

More results about twisted right-angled Artin groups, concerning for instance their normal form and their growth, can be found in [Fon22]. We now state some results about ORAAGs that will appear in a future work in collaboration with S. Blumer, I. Foniqi, C. Quadrelli and T. Weigel ([Blu+ar]). We first need to give two definitions.

Definition 4.14. A special graph Γ is said to be *of elementary type* if it can be constructed starting from a finite number of mixed graphs each consisting of a single vertex and iterating the following two *elementary operations*:

- *disjoint union of graphs*: for two disjoint special graphs $\Lambda_1 = (V_1, E_1), \Lambda_2 = (V_2, E_2)$, their disjoint union is

$$\Lambda_1 \sqcup \Lambda_2 = (V_1 \sqcup V_2, E_1 \sqcup E_2);$$

- *cone*: for a special graph $\Lambda = (V, E)$, the cone on Λ is

$$\nabla\Lambda = (V \sqcup \{w\}, E \sqcup E_\nabla),$$

  where w is a new ordinary vertex and

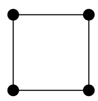
$$E_\nabla := E_1 \sqcup E_2,$$

  where $E_1 = \{ \{v, w\} \mid v \in V_o \}$ is made of non-oriented edges and $E_2 = \{ e = \{v, w\} \mid v \in V_s \}$ is made of oriented edges going from the special vertices of Λ to w.

Sometimes it is not easy to prove that a graph is, or is not, of elementary type. Luckily, there is an easier characterization of special graphs of elementary type.

Proposition 4.15 ([Blu+ar]). *A special graph Γ is of elementary type if, and only if, it does not contain an induced subraph Λ such that:*

- *either $\Lambda =$* 

- *or the naïve graph $\ddot{\Lambda}$ associated to Λ is either the square*  *or the line on 4 vertices* *.*

What follows is the oriented version of Theorem 4.3.

Theorem 4.16 ([Blu+ar]). *Let Γ be a special graph. Then every finitely generated subgroup of A_Γ is an ORAAG if, and only if, Γ is of elementary type.*

4.3 Hypercubicality of RAAGs

As we saw in Example 3.15, right-angled Artin groups are hypercubical. In fact, they are the family of groups that inspired the definition of hypercubicality. The reason for this lies in the geometry of right-angled Artin groups. To any RAAG A_Γ one can associate a cell complex called the *Salvetti complex* $S(\Gamma)$. It is constructed as follows. Note that by n-torus, for $n \geq 1$, we mean $[0, 1]^n$ with identified opposite faces.

1. Start with a single vertex x_0. This is $S(\Gamma)^0$;

2. For any vertex v of Γ, attach to $S(\Gamma)^0$ an oriented loop labeled v. This will produce a bouquet of loops, whose fundamental group is the free group on the vertices of Γ. Denote this bouquet of loops $S(\Gamma)^1$;

3. For any edge $\{v, w\}$ of Γ, attach to $S(\Gamma)^1$ a square, whose boundary is labeled $vwv^{-1}w^{-1}$, along the corresponding loops. This will produce a cube complex consisting in a 2-torus for every edge of Γ and whose fundamental group is A_Γ. Call this cube complex $S(\Gamma)^2$;

4. For any triangle of vertices u, v, w in Γ, attach to $S(\Gamma)^2$ a 3-torus, whose 2-faces correspond to the 3 edges of the triangle, by attaching each face to the corresponding 2-torus in $S(\Gamma)^2$. This defines $S(\Gamma)^3$;

5. For any clique of Γ on n vertices, attach to $S(\Gamma)^{n-1}$ an n-torus as above. This produces $S(\Gamma)^n$;

6. In the end, the resulting cube complex is the *Salvetti complex of* A_Γ.

The following fact is well known in literature, see [CD].

Theorem 4.17. *The Salvetti complex $S(\Gamma)$ of a right-angled Artin group A_Γ is a classifying space for A_Γ.*

Corollary 4.18. *Right-angled Artin groups are of type F, hence torsion-free. Moreover, $\operatorname{cd} A_\Gamma$ coincides with the clique number of Γ.*

Proof. The only thing to prove is the statement about the cohomological dimension. On one hand, each clique having as many vertices as the clique number n of Γ produces a subgroup which is isomorphic to $\mathbb{Z}^n$, which has $\operatorname{cd} = n$. On the other hand, $\dim S(\Gamma) = n$. □

There are different descriptions of the universal covering of the Salvetti complex. We will show that the hypercubical complex is indeed a model for $\widetilde{S(\Gamma)}$, which will imply hypercubicality. We call $V\Gamma$ the *standard generating set of* A_Γ.

Theorem 4.19. *RAAGs are hypercubical with respect to their standard generating sets.*

Proof. Let A_Γ be the RAAG associated to the finite, simplicial graph Γ and let $\mathcal{B}_n$ be the set of n-cells of $\mathcal{C}_\bullet$ having a vertex in 1 and that are spanned by edges of the form $(1, v_i)$, where the v_i's are the vertices of an n-clique of Γ. Then $\mathcal{C}_n = A_\Gamma \mathcal{B}_n$. One inclusion is obvious, the other one is due to the fact that every n-cube in

$\mathcal{C}_\bullet$ can be described as the n-cube spanned by the edges $(g, gx_{i_1}), \dots, (g, gx_{i_n})$ for some $g \in A_\Gamma$ and $x_{i_1}, \dots, x_{i_n} \in V\Gamma$ vertices of an n-clique. Such cube is in the same orbit of a cube of $\mathcal{B}_n$. $\mathcal{C}_\bullet/A_\Gamma$ is the union of a certain number of tori, one for each clique of Γ. Moreover any two tori intersect according to how the corresponding cliques intersect in Γ. This proves that $\mathcal{C}_\bullet/A_\Gamma = S(\Gamma)$.

As the action of A_Γ on $\mathcal{C}_\bullet$ is clearly free and properly discontinuous, the map $\mathcal{C}_\bullet \to \mathcal{C}_\bullet/A_\Gamma$ is a covering map. As A_Γ is quadratic, $\mathcal{C}_\bullet$ is simply-connected, thus being the universal covering of $S(\Gamma)$. As a consequence, $\mathcal{C}_\bullet$ is contractible and A_Γ is hypercubical. □

As we will see in the next section, the hypercubical complex of a TRAAG (hence of a RAAG) is always a CAT(0) cube complex, which gives another proof for the following result, which is classical in the theory of right-angled Artin groups.

Proposition 4.20. *Every RAAG A_Γ acts cellularly, freely, properly discontinuously, cocompactly and by isometries on a CAT(0) cube complex of finite type and of dimension equal to the clique number of Γ.*

4.4 Hypercubicality of TRAAGs

The aim of this section is to state and prove the following theorem, and to show some consequences.

Theorem 4.21. *Let Γ be a mixed graph. Then $\mathcal{C}_\bullet(A_\Gamma, V\Gamma)$ is CAT(0). As a consequence, A_Γ is hypercubical with respect to $V\Gamma$.*

In order to prove this, we need the following:

Definition 4.22. We say that a finitely generated group (G, Σ) satisfies the *cubical link condition* if for $n \geq 3$ every n edges of $\mathcal{C}_\bullet(g, \Sigma)$, all having a vertex in common and being pairwise adjacent in a square, satisfy the n-dimensional cubical link condition.

Lemma 4.23. *The cubical link condition implies Gromov's link condition.*

Proof. Let (G, Σ) be a finitely generated group and $\mathcal{C}_\bullet$ be the associated hypercubical complex. Suppose that (G, Σ) satisfies the cubical link condition and suppose that in the link of a vertex v there are n vertices pairwise adjacent. This means that in $\mathcal{C}_\bullet$ there are n edges incident to v pairwise adjacent in a square. Then these n edges define an n-cell, that corresponds to the desired simplex in the link. Therefore each link is a flag complex. □

Lemma 4.24. *TRAAGs have the cubical link condition with respect to the standard generators.*

Proof. Let Γ be a mixed graph and A_Γ be the associated TRAAG. Suppose there are n edges $e_1, \dots, e_n$ in the hypercubical complex that are all incident to the same vertex and that are pairwise adjacent sides of a square. Without loss of generality, suppose that the common vertex of these edges is 1. We need to show that there is an n-cube having $e_1, \dots, e_n$ as sides. As inverting an edge means inverting a generator, we can fix an orientation for $e_1, \dots, e_n$ and just work with generators and inverses. For instance, suppose that all these edges have origin in 1. As $e_1, \dots, e_n$ pairwise belong to a square, they must be of the form $e_i = (1, x_i)$ for $x_i \in \{v_i, v_i^{-1}\}$, where the v_i's are vertices of an n-clique in Γ. So we only have to prove that an n-clique in Γ of vertices $v_1, \dots, v_n$ corresponds to an n-cube in the hypercubical complex for every choice of $x_i \in \{v_i, v_i^{-1}\}$. Suppose then that Λ is such an n-clique and that a choice for the x_i's has been made. Consider a cube $[0,1]^n$ such that for each i all the sides parallel to direction i are labeled x_i or x_i^{-1}, where the exponent is 1 if the side has a vertex in $\mathbf{0}$ and is yet to be determined otherwise. We think of the sides of the cube as being oriented according to the increasing of the coordinates. Define $k_i :=< \mathbf{w}, \mathbf{z_i} >$, where $\mathbf{w}$ is the vector of the coordinates of a vertex of the cube and $(\mathbf{z_i})_j = 1$ if there is an oriented edge going from v_j to v_i and 0 otherwise. Therefore k_i is the number of components of $\mathbf{w}$ that are $= 1$ and at the same time correspond to origins of oriented edges of Λ having terminus in v_i. Note that if $\mathbf{w}$ is a vertex of the cube which is the origin of an edge with direction i, then $w_i = 0$. Fix $\mathbf{w}$ and a side with origin in $\mathbf{w}$ and direction i. If $w_j = 1$, along every geodesic edge-path in the cube from $\mathbf{0}$ to $\mathbf{w}$ there is a side parallel to the j'th direction. Then k_i is the number of such sides that change the sign of the exponent of x_i. As this does not depend on the specific geodesic edge-path we choose, the exponent of x_i on the fixed side is given by $(-1)^{k_i}$. This defines the exponent of the label x_i of every side of the cube. Let us now consider a square inside such cube, of vertices $\mathbf{w}, \mathbf{w}+\mathbf{u_i}, \mathbf{w}+\mathbf{u_j}, \mathbf{w}+\mathbf{u_i}+\mathbf{u_j}$, where $\{\mathbf{u_i}\}_{1 \le i \le n}$ is the canonical basis of $\mathbb{R}^n$ (see fig. 4.1). We need to show that the sides of this square actually represent a relation. If $[v_i, v_j] = 1$, then $< \mathbf{u_j}, \mathbf{z_i} >=< \mathbf{u_i}, \mathbf{z_j} >= 0$, so that every two parallel sides of the square have the same label. This corresponds to one of the commutators between one of $\{v_i, v_i^{-1}\}$ and one of $\{v_j, v_j^{-1}\}$, which all hold in our TRAAG. If $[v_i, v_j\rangle = 0$, then $< \mathbf{u_j}, \mathbf{z_i} >= 0, < \mathbf{u_i}, \mathbf{z_j} >= 1$, so that the two sides parallel to direction i have the same label, while the other two sides have opposite labels. This corresponds to one of the relations of Remark 4.6. The argument for $[v_j, v_i\rangle$ is analogous. □

We are now ready to prove Theorem 4.21.

Proof of Theorem 4.21. Let Γ be a mixed graph and A_Γ be the associated TRAAG. Let $\mathcal{C}_\bullet$ be the hypercubical complex with respect to the standard generating set of A_Γ. Then, by Lemma 4.24, $\mathcal{C}_\bullet$ is non-positively curved. As A_Γ is quadratic, $\mathcal{C}_\bullet$ is also CAT(0), which proves both claims. □

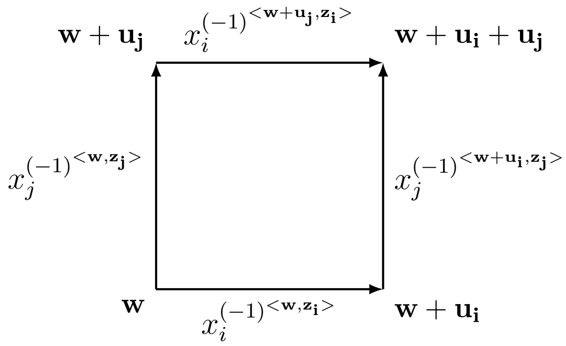


Figure 4.1: The square considered in the proof of Lemma 4.24

Corollary 4.25. *Let* Γ *be a special graph,* A_Γ *the associated ORAAG and* $\mathcal{C}_\bullet$ *the hypercubical complex with respect to the standard generating set of* A_Γ*. Then* $\mathcal{C}_\bullet(A_\Gamma)$ *is a cubical* A_Γ*-complex of finite type of dimension equal to the clique number of* Γ *and it admits a CAT(0) metric. Therefore it is a cubical model for the Borel construction* EA_Γ*. As a consequence,* A_Γ *is of type* F*, thus* FP*, and* $\operatorname{cd} A_\Gamma$ *is equal to the clique number of* Γ

Proof. The only part that is missing is the result about the cohomological dimension. An upper bound is given by the cube number of A_Γ, which is equal to the clique number of Γ. The clique number of Γ is also a lower bound, as Γ being special implies that a clique corresponds to a subgroup isomorphic either to $\mathbb{Z}^r$ or to $\mathbb{Z} \ltimes \mathbb{Z}^{r-1}$, where the action is by inversion and r is the number of vertices of the clique. □

Chapter 5

Borromean Groups

The beginning of this chapter will be dedicated to the proof of the hypercubicality of a group having a totally different origin, the link group of the Borromean rings. This will start with a brief introduction about knot and link theory and the Borromean rings. Then, we will introduce a new family of groups, called the *Borromean groups*, defined via presentations. Such presentations can be constructed by a certain duplication process that we will explain in detail. Then we will show that every Borromean group is hypercubical and provide some consequences of this fact.

5.1 Knot and link theory

The sources followed for this introduction are [Lic97] and [BZ03].

A knot, for example in S^3, can be defined as an embedding of S^1 into S^3. However, as we will work with tame knots, that are equivalent via an ambient isotopy to simple closed polygons in S^3, it seems easier to restrict the discussion to the piecewise linear case.

Definition 5.1. A *link L of m components* (see for example fig. 5.1) is a subset of S^3 or $\mathbb{R}^3$ consisting in m disjoint, piecewise linear, simple closed curves. A link of one component is called a *knot.*

A link can be considered to be either inside $\mathbb{R}^3$ or inside S^3. This seems somehow just a technical detail and it does not affect the intuitive understanding of what a link is, as one can see S^3 as $\mathbb{R}^3 \sqcup \{\infty\}$. The compactness of S^3 however can be useful in many different arguments. The request of piecewise linearity means that each curve composing L is made up of a finite number of straight line segments placed end to end, where "straight" is to be intended either in the linear structure of $\mathbb{R}^3$ or in the structure of one of the 3-simplices of a triangulation of S^3. When

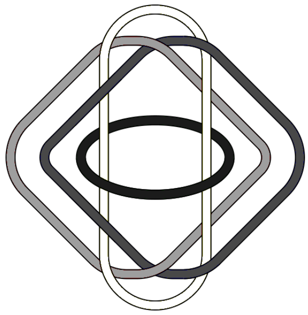


Figure 5.1: An example of a class of links knowns as *Brunnian links*.

we draw the diagram of a link or a knot we implicitely assume that these straight segments are so many and so small to give the impression of roundedness of the curves. The reason why we require having a *finite* number of straight segments only is to avoid pathological behaviours, like having infinitely many kinks getting smaller and smaller while converging to a point (those links are called *wild*). There are other ways to avoid this wildness, but they seem to be more technical and less manageable, although giving rise to an equivalent theory.

Definition 5.2. Two links L_1 and L_2 are *equivalent* (see for instance fig. 5.2) if there is an orientation preserving piecewise linear homeomorphism $h\colon S^3 \to S^3$ such that $h(L_1) = L_2$.

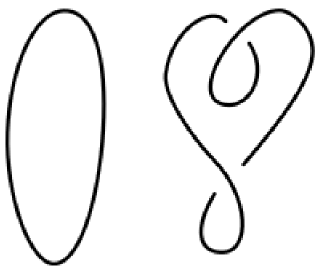


Figure 5.2: Two equivalent *unknots*.

In this case the piecewise linear condition means that, if we consider the two copies of S^3 triangulated, then, up to subdivisions of the simplices of these triangulations into possibly very many smaller simplices, h maps simplices in simplices in a linear way. If the links are oriented (i.e., every curve of the links has a fixed orientation) or the components are ordered, h is asked to preserve such attributes. A basic theorem of piecewise linear topology states what follows.

Proposition 5.3. *Under these hypotheses, h is isotopic to the identity. In other words, there exist $h_t\colon S^3 \to S^3$ for every $t \in [0, 1]$ such that* *(i)* $h_0 = id_{S^3}$*;*

(ii) $h_1 = h$*;*

(iii) $(x, t) \mapsto (h_t(x), t)$ *is a piecewise linear homeomorphism of* $S^3 \times [0, 1]$ *to itself.*

The reason why this definition is stated in these terms is that defining equivalence simply in terms of continuously distorting L_1 into L_2 could permit knots to be pulled tighter and tighter in order to make any complication disappear at a single point.

A link L can always be represented by a diagram in $\mathbb{R}^2$. Up to equivalence, we can consider L to be in general position with respect to the standard projection $\mathbb{R}^3 \to \mathbb{R}^2$. This means that:

- each line segment of L projects to a line segment in $\mathbb{R}^2$;
- the projections of two line segments intersect in at most one point and this point is not an endpoint if the two segments are disjoint;
- no points belong to the projection of three segments.

For every crossing (i.e, a point in which the projections of two disjoint segments intersect) one needs to specify some "under and over" information, referring to the relative heights above $\mathbb{R}^2$ of the two preimages of the crossing. The image of the link L together with this "under and over" information at the crossings (graphically represented by breaks in the under-passing segments) is known as a *link diagram of L* (see fig. 5.3).

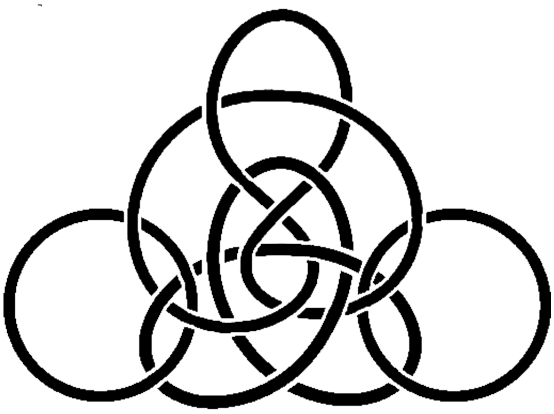


Figure 5.3: An example of a link diagram.

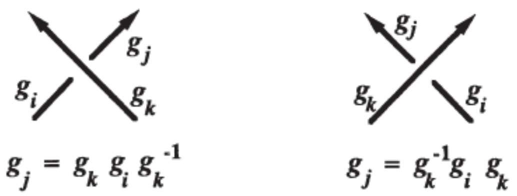
One of the most important themes in knot theory is the classification of knots and links. Sometimes it is easy to directly understand whether two links or knots are equivalent, but usually this is not the case. Many different tools have been developed in order to determine whether two links or knots are equivalent. The main one is the so called *link group* G_L of a link L, namely the fundamental group $\pi_1(S^3 \setminus L)$ of its complement $S^3 \setminus L$. The noticeable fact about this group is that two equivalent knots have homeomorphic complements in S^3, therefore the link group is invariant under equivalence of the links.

Given a link L, its link group G_L has a presentation which can be easily computed starting from a diagram of L, the so-called *Wirtinger presentation.* The procedure is the following one:

1. Select an orientation for L;

2. Take a group generator g_i for each segment of the diagram (i.e., each longest possible "over-passing" section of the link traversing some number of under-passes);

3. For each crossing take a relation as follows: suppose g_i is the under-passing arc approaching the crossing, g_j is the under-passing arc leaving the crossing, g_k is the over-passing arc. If for the over-passing arc pointing upward the under-passing arc points right, the relation is $g_j g_k = g_k g_i$, otherwise it is $g_k g_j = g_i g_k$, as the following picture shows.



The symbol g_i represents a loop that starts from a base point above the diagram (for example, the eye of the reader), goes straight to the i^{th} arc, encircles it according to the right-hand rule (that is, going down on the right-hand side and coming back up from the left-hand side of the arc, if the arc points upwards) and goes back straight to the base point (see fig. 5.4). Therefore the relation given by a crossing can be easily deduced by this remark.

E.g. 5.4. Let us consider the following knot K, known as the *trefoil knot.* The generators of the knot group G_K are a, b, c and the relations are $bc = ab$ from the

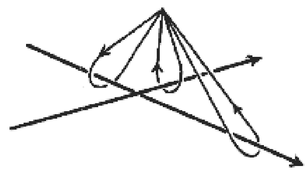
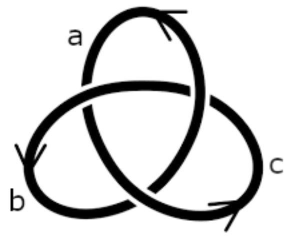


Figure 5.4: Loops in the Wirtinger presentation.



top-left crossing, $ca = ab$ from the top-right crossing and $bc = ca$ from the bottom crossing. Therefore one has the following Wirtinger presentation:

$$G_K =< a, b, c \mid ab = bc, bc = ca >,$$

as we can omit one of the relations being direct consequence of the other two. Note that this group cannot be hypercubical, as the squares given by the three relations $ab = bc, bc = ca, ca = ab$ form a 2-sphere in the Cayley complex that is not the boundary of a 3-cube (see fig. 5.5), therefore it results in a nontrivial element in the second homotopy group of the associated hypercubical complex.

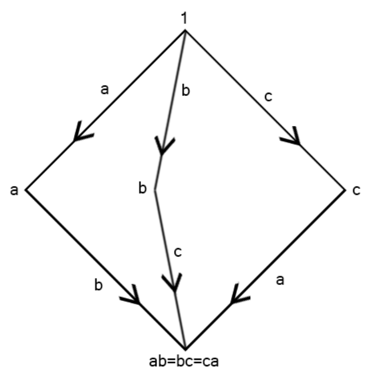


Figure 5.5: A part of the Cayley graph of G_K

5.2 The Borromean rings and the group G_3

The Borromean rings (see fig. 5.6) are a 3-components link in which the removal of every component leaves the other two unlinked. This is a characteristic of a more general class of links, called *Brunnian links* (see fig. 5.1).

The name of the Borromean rings comes from the fact that they were used on the coat of arms of the aristocratic Borromeo family in Northern Italy, but the link itself is much older, dating back to the 7^{th} century in the form of the *valknut* (a symbol made of three interlocking triangles with the same property as the Borromean rings) on Norse image stones. They have been used to represent many different things, for example in religion or in art, usually related to the idea of strength in unity.

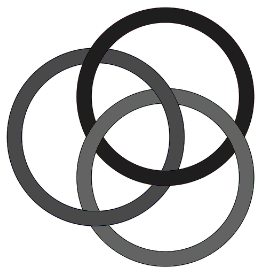
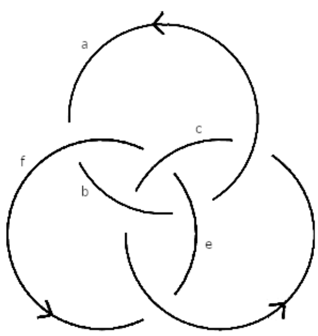


Figure 5.6: A representation of the *Borromean rings B*.

The Wirtinger presentation of the link group G_3 (this notation will make sense later on) of the Borromean rings can be easily calculated from the following diagram.



The generators are a, b, c, d, e, f and the relations coming from the crossings are $ae = eb, af = fb, da = ac, db = bc, fc = ce, fd = de$, therefore the Wirtinger representation for G_3 is

$$G_3 =< a, b, c, d, e, f \mid ae = eb, af = fb, da = ac, db = bc, fc = ce, fd = de > .$$

It is possible to graphically represent this presentation with an oriented labeled graph of the 1-skeleton of a 3-cube as in fig. 5.7, where the labels of the edges correspond to the generators and on each 2-face of the cube we can read a relation.

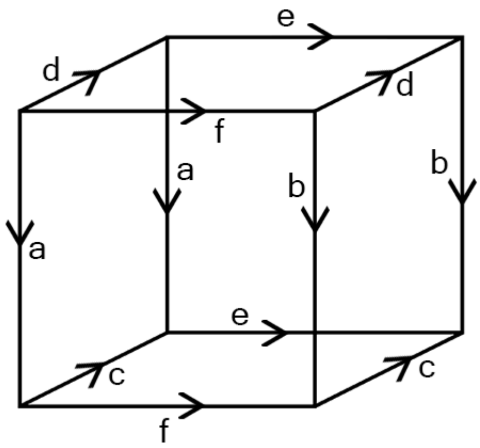


Figure 5.7: The presentation of G_3 on a 3-cube.

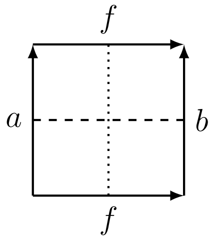
Lemma 5.5. *The link complement $S^3 \setminus B$ of the Borromean rings B is a classification space BG_3 for G_3*

Proof. Being a classification space for G_3 is equivalent to being aspherical with fundamental group the link group G_3 of B, and this last request is clearly obviously verified by definition of G_3. The asphericity is a consequence of the fact that $S^3 \setminus B$ admits a complete hyperbolic metric of finite volume, as stated for example in [GLO15]. □

Lemma 5.6. *In the hypercubical complex $\mathcal{C}_\bullet$ of G_3 with respect to the Wirtinger presentation the 2-cells are translates of the squares defined by the relations in the Wirtinger presentation (up to cyclic conjugates and inverses).*

Proof. The generators and relations carry certain combinatorics, as we will see. Note, indeed, that the generators a, b, c, d, e, f can be divided into three subsets

$S_1 := \{a, b\}, S_2 := \{c, d\}, S_3 := \{e, f\}$ such that every relation can be written as $xy = yz$, where $\{x, z\}$ is one of the three subsets above and y belongs to another subset. We will say that *y transforms x into z* (or that *y transforms z into x*, or also that *y transforms x*). Note that for $i \neq j$, every element of S_i produces the same effect on the two elements of S_j, either it transforms them or not (in fact, in the first case, each element of S_i transforms the elements of S_j into each other). In particular, S_1 transforms S_2, S_2 transforms S_3 and S_3 transforms S_1. We can represent each S_i as a midline in each square representing a relation involving some elements of S_i, such midline being perpendicular to the sides labeled by elements of S_i. We say that the midline representing *S_i has type i*. For instance, the midlines of the square representing the relation $af = fb$ is

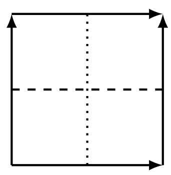


,

where the horizontal dashed line is a midline of type 1 and the vertical dotted line is a midline of type 3.

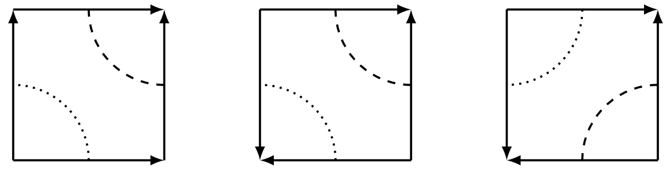
Consider now a relation of length 4, draw a Van Kampen diagram for it and draw the midlines corresponding to the four sides of the boundary of the diagram. The midline of the top side will end in one of the other sides of the boundary and the other midline will connect the other two sides. Call these two midlines the *principal midlines of the diagram* and note that each of them is the midline of a strip. Note also that every other midline in the diagram is a closed simple curve. As Möbius strips cannot be embedded in the plane, the orientation of the sides crossing a midline must be preserved along the midline. Therefore, up to rotations and reflections of the diagram, we are reduced to one of the following two cases:

1. each of the two principal midlines connects the opposite sides of the diagram. In this case there is a *vertical* midline, connecting the top and the bottom sides, and a *horizontal* midline, connecting the left and the right sides;



2. each of the two principal midlines connects a side of the diagram with an adjacent side. In this case, the midline relative to the top side ends either

in the left or in the right side of the boundary of the diagram and the other midline connects the remaining sides.



Let us begin with case 1. Observe that the type of the components of a midline is constant and also that crossings only happen between midlines of different types. Suppose that the horizontal midline is of type 1 and the vertical midline is of type 2, the other cases being analogous. Note that the previous remarks imply that the principal midlines must have different types, since they cross. Let us consider the first crossing, along the horizontal midline and going from left to right, of the two principal midlines. Let us call it the *distinguished crossing.* This crossing represents a relation which is either $da = ac$ or $db = bc$. Moreover, it divides each of the two midlines into two branches. We want to study how the midlines (principal and non-principal) of the diagram cross the principal ones. We need to divide the analysis into three cases, according to the type of midline crossing the principal midlines:

Type 1: A non-principal midline of type 1 is a closed simple curve and the distinguished crossing is necessarily contained in the exterior of the curve (as the extremities of a principal midline are contained in the exterior of the curve and each branch of a principal midline is a path from the distinguished crossing to one of the extremities of a principal midline). As a consequence, the number of crossings of a non-principal midline of type 1 with each branch of the principal midlines is even. The right-hand branch of the principal midline of type 1 can cross both branches of the vertical midline, but always an even amount of times.

Type 2: For a non-principal midline of type 2, an argument analogous to the one above holds, resulting in an even number of crossings on each branch of the principal midlines. Both branches of the principal midline of type 2 can only cross the right-hand branch of the horizontal midline, but always an even amount of times.

Type 3: A midline of type 3 is always non-principal, therefore it is a closed simple curve. Either the distinguished crossing lies in the interior of the curve or it lies in the exterior. If it lies in the exterior, then the midline crosses each branch of the principal midlines an even amount of time. If it lies in the interior, then the midline crosses each of the four branches of the principal midlines an odd number of times.

On the top and the bottom branches of the vertical midline we have an even number of crossings with a midline of type 1 and a certain number of crossings with midlines of type 3. In both cases, the top side of the boundary of the diagram has the same label as the top side of the relation represented by the distinguished crossing. On the left-hand and right-hand branches of the horizontal midline we have an even number of crossings with a midline of type 2 and a certain number of crossings with midlines of type 3, but the parity of the number of crossings with a midline of type 3 is the same on the two branches. Therefore the label of the left-hand side of the boundary of the diagram changes, with respect to the label on the left-hand side of the relation represented by the distinguished crossing, if and only if the same happens on the right-hand side. Consequently, the relation on the boundary of the diagram is one of the relations in the presentation.

Let us approach now case 2. In this case the two principal midlines always cross an even number of times and on every principal midline there is an even number of crossings with every non-principal midline of the other two types. Therefore the boundary relation is a trivial relation that does not identify a square in the hypercubical complex. □

Theorem 5.7. *The link group G_3 of the Borromean rings described above is hypercubical.*

Proof. Lemma 5.5 shows that the universal covering of $S^3 \setminus B$ is contractible. Therefore we only need to show that the hypercubical complex of G_3 with respect to the generators of the Wirtinger presentation is a model for the universal covering of $S^3 \setminus B$. Let us denote the hypercubical complex by $\mathcal{C}_\bullet$.

Lemma 5.6 shows that the only squares that we see in $\mathcal{C}_\bullet$ are those coming from the relations in the presentation. We claim that in the Cayley graph of G_3 one finds only one orbit of 3-cubes and no n-cubes for $n \geq 4$. More precisely, by focusing the attention on the subcomplex having 1 as a vertex, we will check that one has that the only 3-cubes obtained are the ones given by the 8 copies of the cube in fig. 5.7, each one having 1 in a different vertex. Clearly the same holds for every vertex in the Cayley graph. To prove this, consider a cube with a vertex in one. Then we can suppose, without loss of generality, that one of the three 2-faces of the cube intersecting in 1 is the one corresponding to a cyclic conjugate of the relator $afb^{-1}f^{-1}$. By looking for the possible combination of 3 edges compatible with this condition, one sees that there is only one possible cube, if we fix the vertex of the relator above corresponding to 1. These eight cubes are adjacent only by either a single vertex or a single edge. They lay in the same orbit, as they can be translated one into the other via multiplication by suitable elements of the group. Moreover, no cube of higher dimension can be obtained. Indeed, if one had a 4-cube in the hypercubical complex, then there would be two 3-cubes with a square in common, which is not possible.

As the action of G_3 on $\mathcal{C}_\bullet$ satisfies the hypotheses of [Hat01, Prop. 1.40], the projection is a covering. The hypercubical complex $\mathcal{C}_\bullet$ actually is the universal covering of the link complement (as the link complement can easily be seen as the quotient of the cube in figure fig. 5.7 under the action of G_3): indeed it is simply connected, as its 2-skeleton coincides with the Cayley complex of the link group. Moreover it is weakly contractible, because of the previous lemma and [Hat01, Prop 4.1], and thus contractible, having the homotopy type of a CW complex. This leads to the thesis. □

5.3 The Borromean cube groups G_n

In this section we are going to define a family of groups generalizing the link group of the Borromean rings. This will be done by producing a presentation for each of them in an inductive way.

As we have seen, the Wirtinger presentation for the link group of the Borromean rings can be represented on an oriented cubical graph, in which one vertex is the *source* or *expander* and the opposite one is the *attractor*. The orientation is given by orienting all possible paths of length 3 from the source to the attractor. See fig. 5.7. Call such cube the *defining cube for* G_3.

We can notice two things. First of all, a 4-cube can be obtained by a 3 cube via a duplication process consisting in taking two copies of a 3-cube and joining every vertex in one of them to the corresponding vertex in the other cube by an edge. Second, every edge of the cube is paired to a parallel edge and two paired edges have the same label and the same orientation. We see also that no two faces of the defining cube have the same boundary labels.

The third remark will be useful later on. On the other hand, the first two remarks suggest a possible way to define a family of groups that will probably be hypercubical. The approach is inductive, being G_3 the base case.

Definition 5.8. Given the definitions of the link group of the Borromean rings, we can define inductively a family of finitely presented groups called the *Borromean cube groups*, denoted G_n for $n \geq 3$. The first step of this induction is the link group of the Borromean rings, which is the Borromean 3-cube group G_3. The Borromean $(n+1)$-cube group will be defined by a *defining* $(n+1)$*-cube*, representing its generators and relators, and such $(n+1)$-cube is constructed from the defining n-cube as follows:

1. Suppose the defining n-cube is labeled with small letters. Then, duplicate it and label the edges with the same letters, but capital. Call the first cube the *small* or the *inside* n*-cube* and the latter the *capital* or the *outside* n*-cube*.

2. Draw the *diagonal* edges in such a way that every edge goes from a vertex of the outside n-cube to the corresponding vertex in the inside n-cube. In this way, all the n-cubes inside the $(n+1)$-cube are isomorphic as oriented graphs. The n-expander of the outside n-cube becomes the $(n+1)$-expander, or *global expander*, and the n-attractor of the inside n-cube becomes the $(n+1)$-attractor, or *global attractor*.

3. Consider $[0,1]^{n+1}$ with the coordinates written in reverse order (i.e., if the coordinate are $z_1, \ldots, z_{n+1}$, we write them as $(z_{n+1}, \ldots, z_1)$). Identify the $(n+1)$-cube with $[0,1]^{n+1}$ in such a way that

   - the coordinate z_{n+1} is 0 on the outside n-cube and 1 on the inside n-cube,
   - the $(n+1)$-expander corresponds to $\mathbf{0}$, while the $(n+1)$-attractor corresponds to $\mathbf{1}$ (therefore the n-attractor of the outside n-cube will be $(0, 1, \ldots, 1)$ and the n-expander of the inside n-cube will be $(1, 0, \ldots, 0)$).

   Under such identification, two diagonal edges will be paired, and thus labeled with the same letter, if their origins are represented by two consecutive binary numbers the smallest of which is even. See fig. 5.8 for the case $n = 3$.

Therefore G_{n+1} is generated by the labels of the edges, subject to the relations defined by the squares in the defining $(n+1)$-cube. We call this presentation the *borromic presentation of* G_{n+1}.

Short exact sequences

For every $n \geq 4$ there are also other two groups that we can define:

- $\widehat{G_n}$ is the group obtained by applying to G_{n-1} the same duplication procedure as above, but giving all different labels to the diagonal edges.
- $\widetilde{G_n}$ is the group obtained by applying to G_{n-1} the same duplication process, but giving to all the diagonal edges the same label. It is clearly the free product $\mathbb{Z} * G_{n-1}$, and also an HNN-extension of $G_{n-1} * G_{n-1}$ (where the stable letter conjugates every generator of one of the two factors to the corresponding generator of the other factor).

The groups G_{n-1}, $\widetilde{G_n}$ and $\widehat{G_n}$ are involved in short exact sequences arising from the following composition of surjections:

$$\widehat{G_n} \twoheadrightarrow G_n \twoheadrightarrow \widetilde{G_n} \twoheadrightarrow G_{n-1}.$$

The kernels of this maps are given by respectively pairing the diagonal generators in the proper way, sending all the diagonal generators to the only diagonal generator of $\widetilde{G_n}$ and quotienting out the single diagonal generator. By composing some of the surjections above, we get other short exact sequences.

Proposition 5.9. *The Borromean cube groups are all distinct.*

Proof. From the composition of surjections above, we can write the following chain of surjective homomorphisms:

$$\cdots \twoheadrightarrow \widetilde{G_4} \twoheadrightarrow G_4 \twoheadrightarrow \widetilde{G_3} \twoheadrightarrow G_3 \twoheadrightarrow 0.$$

By passing to the abelianizations, the induced homomorphisms are onto and every $\widetilde{G_n}$ has abelianization of rank given by $1 + \operatorname{rk} G_n^{ab}$, so at each step $\operatorname{rk} G_{n+1}^{ab} \geq \operatorname{rk} \widetilde{G_n}^{ab} > \operatorname{rk} G_n^{ab}$. □

5.4 The case of G_4

Let us now focus our attention on G_4, the first of these groups generalizing the link group of the Borromean rings.

The borromic presentation of the Borromean 4-cube group is the one given by the defining 4-cube represented in fig. 5.8. It is given by two copies of the defining 3-cube, namely the exterior and interior 3-cubes, which are isomorphic as labeled oriented graphs. Note that in fig. 5.8 the exterior 3-cube is the one on the left and the interior one is that on the right. Every vertex of the exterior 3-cube is connected to the corresponding vertex of the interior cube by an oriented edge going from outside to inside. In this way, the expander of the cube on the outside becomes the global expander, while the attractor of the cube on the inside becomes the global attractor. Certain diagonal edges are paired according to Definition 5.8. Note that both the interior and the exterior 3-cubes are isomorphic to the defining 3-cube of the Borromean link group.

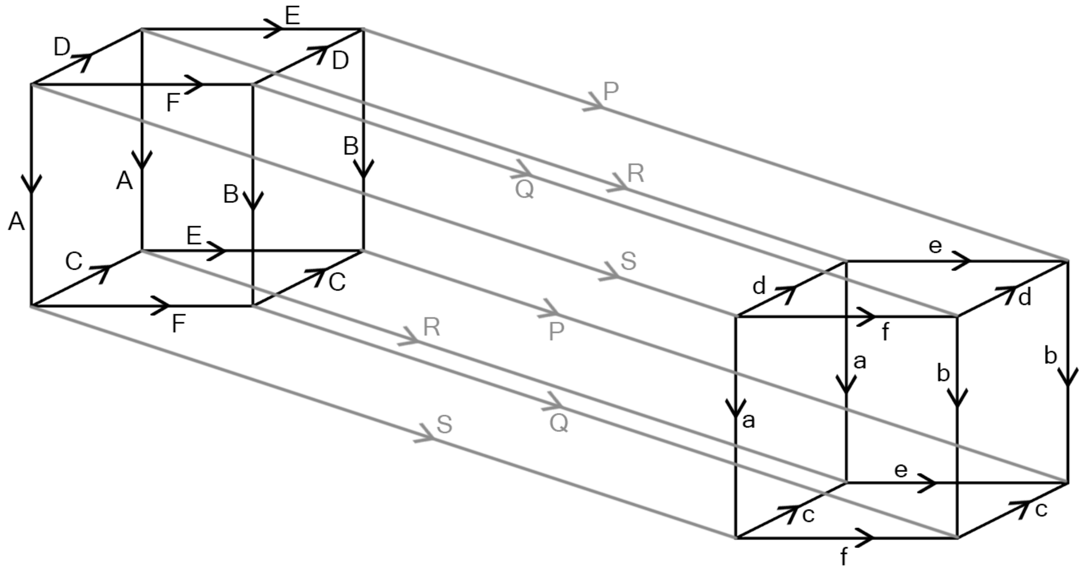


Figure 5.8: The presentation of G_4 on a 4-cube.

Explicitly, the presentation is the following one:

- Generators
  - Small generators: a, b, c, d, e, f (they generate a copy of G_3);
  - Capital generators: A, B, C, D, E, F (they generate a copy of G_3);
  - Diagonal generators: P, Q, R, S.
- Relations:
  - Relations among small generators: $ae = eb$, $af = fb$, $da = ac$, $db = bc$, $fc = ce$, $fd = de$;
  - Relations among capital generatos: $AE = EB$, $AF = FB$, $DA = AC$, $DB = BC$, $FC = CE$, $FD = DE$;
  - Relations of conjugation between a small and a capital generator: $AS = Sa$, $AR = Ra$, $BP = Pb$, $BQ = Qb$;
  - Other relations among small and capital generators: $CR = Sc$, $CP = Qc$, $DR = Sd$, $DP = Qd$, $EP = Re$, $FQ = Sf$.

As we will see, G_4 is hypercubical with respect to the standard generators. In order to prove this, we need some results first.

5.4.1 Useful homomorphisms

In this section we list some homomorphisms that will be used to prove that G_4 is hypercubical. The first homomorphism we define is

$$\psi\colon G_4 \to G_3$$

which sends the diagonal generators P, Q, R, S to 1, fixes the small generators a, b, c, d, e, f and sends each capital generator A, B, C, D, E, F to its corresponding small generator.

We now introduce a family of homomorphisms. Let $\phi\colon G_4 \to \mathbb{Z}$ be a homomorphism. For such a homomorphism to be defined, it needs to satisfy:

$$\phi(a) = \phi(b) = \phi(A) = \phi(B),\ \phi(c) = \phi(d),\ \phi(e) = \phi(f),\ \phi(C) = \phi(D),\ \phi(E) = \phi(F).$$

We also note that

$$\phi(P) = \phi(R) \iff \phi(e) = \phi(E) = \phi(F) = \phi(f) \iff \phi(S) = \phi(Q)$$

and

$$\phi(P) = \phi(Q) \iff \phi(c) = \phi(C) = \phi(D) = \phi(d) \iff \phi(R) = \phi(S).$$

As every morphism of groups is completely determined by the images of the generators, we will list some homomorphisms. They are classified by the images of the diagonal generators, fixing which there are infinite possibilities for the images of the other generators. So the homomorphisms that we list are to be meant as examples of subfamilies of morphisms $G_4 \to \mathbb{Z}$.

- $\phi_0(P) = \phi_0(Q) = \phi_0(R) = \phi_0(S) = 0$ and all the other generators are sent to 1;
- $\phi_1(P) = \phi_1(Q) = \phi_1(R) = \phi_1(S) = 1$ and all the other generators are sent to 0;
- $\phi_{PQ}(P) = \phi_{PQ}(Q) = 1$, $\phi_{PQ}(R) = \phi_{PQ}(S) = 0$. In this case, $\phi_{PQ}(c) = \phi_{PQ}(C)$ and $\phi_{PQ}(e) = 1 + \phi_{PQ}(E)$. For the remaining generators, e, f are sent to 1, all the others to 0;
- $\phi_{PR}(P) = \phi_{PR}(R) = 1$, $\phi_{PR}(Q) = \phi_{PR}(S) = 0$. In this case, $\phi_{PR}(e) = \phi_{PR}(E)$, $\phi_{PR}(c) = 1 + \phi_{PR}(C)$. For the remaining generators, c, d are sent to 1, all the others to 0;
- $\phi_{QS}(Q) = \phi_{QS}(S) = 1$, $\phi_{QS}(P) = \phi_{QS}(R) = 0$. In this case, $\phi_{QS}(e) = \phi_{QS}(E)$, $\phi_{QS}(C) = 1 + \phi_{QS}(c)$. For the remaining generators, C, D are sent to 1, all the others to 0;

- $\phi_{RS}(R) = \phi_{RS}(S) = 1$, $\phi_{RS}(P) = \phi_{RS}(Q) = 0$. In this case, $\phi_{RS}(c) = \phi_{RS}(C)$, $\phi_{RS}(E) = 1 + \phi_{RS}(e)$. For the remaining generators, E, F are sent to 1, all the others to 0.

- The augmentation map $\varepsilon\colon G_4 \to \mathbb{Z}$, sending all generators to 1.

The last homomorphism that we need to define is the following. Let Γ be the segment graph of vertices x, y and the edge labelled 3. Let A_Γ be the associated Artin group, i.e., $A_\Gamma =< x, y | xyx = yxy >$. It is straightforward that the following homomorphism is well defined:

$$\rho\colon G_4 \longrightarrow A_\Gamma$$

$$a, \dots, f \mapsto x$$

$$A, \dots, F \mapsto y$$

$$P, Q, R, S \mapsto xy.$$

5.4.2 Generators of G_4

The following result is an easy application of the homomorphisms of the previous section.

Proposition 5.10. *The generators of the borromic presentation of G_4 and their inverses are all different and they all have infinite order.*

Proof. The maps $\psi, \phi_{PQ}, \phi_{QS}, \phi_{PR}, \phi_{RS}$ allow us to deduce that:

- all generators have infinite order;

- $\{P^{\pm 1}, Q^{\pm 1}, R^{\pm 1}, S^{\pm 1}\} \cap \{a^{\pm 1}, \dots, f^{\pm 1}, A^{\pm 1}, \dots, F^{\pm 1}\} = \emptyset$;

- $P^{\pm 1}, Q^{\pm 1}, R^{\pm 1}, S^{\pm 1}, c^{\pm 1}, d^{\pm 1}, e^{\pm 1}, f^{\pm 1}, C^{\pm 1}, D^{\pm 1}, E^{\pm 1}, F^{\pm 1}$ are all distinct;

- $\{a^{\pm 1}, b^{\pm 1}, A^{\pm 1}, B^{\pm 1}\}$ are distinct from all the other generators;

- $\{a^{\pm 1}, A^{\pm 1}\} \cap \{b^{\pm 1}, B^{\pm 1}\} = \emptyset$.

By using ρ one also sees that $A \neq a$ and $B \neq b$. Moreover, if $A = a^{-1}$, then $\forall \phi\colon G_4 \to \mathbb{Z} \quad \phi(a) = \phi(A) = -\phi(a)$, so $\phi(a) = 0$, which is not true (take for example the augmentation map). Analogously, $B \neq b^{-1}$. □

Corollary 5.11. *G_4 is not the trivial group. The only words of length 2 which are =1 in G_4 are free cancellations, so there is no non-trivial loop of length 2 in the Cayley graph of G_4.*

5.4.3 Hypercubicality

Lemma 5.12 is a particular case of Lemma 5.25. We state it here as we need it to prove that G_4 is hypercubical, but we refer the reader to the general case for the proof.

Lemma 5.12. *The only non-trivial relations of length 4 in G_4 are those of the borromic presentation (up to inversion and cyclic conjugation).*

Lemma 5.13. $\dim \mathcal{C}_\bullet(G_4) = 4$.

Proof. As the defining 4-cube appears in the Cayley graph of G_4, the hypercubical complex has dimension at least 4. Suppose there is a 5-cube C in $\mathcal{C}_\bullet$ and fix a vertex of C. Then either all the 5 edges incident to the fixed vertex are either all small or all capital generators (or inverses), or there is exactly one diagonal generator. To see this, note that by Lemma 5.12 there are no relators of length 4 with a subword xy, where x is capital and y small (or the other way around) or where both are diagonal. By eventually restricting our attention to the subcube spanned by non-diagonal edges, this will produce either a 4-cube or a 5-cube in the hypercubical complex for G_3, which is not possible. □

Lemma 5.14. *The map $\psi\colon G_4 \to G_3$ extends to a cellular map $\psi\colon \mathcal{C}_\bullet(G_4) \to \mathcal{C}_\bullet(G_3)$.*

Proof. Let C be either a 3-cube or a 4-cube in $\mathcal{C}_\bullet(G_4)$. If it contains a diagonal generator, Lemma 5.12 implies that the sides labeled by diagonal generators must be perpendicular to the subcubes labeled by small and capital generators. By definition, ψ collapses the sides labeled by diagonal generators and identifies the subcube labeled by small generators with the subcube labeled by capital generators. Therefore $\psi(C)$ is either a 2-cube or a 3-cube. □

Theorem 5.15. *The group G_4 satisfies the following two properties:*

(i) the combinatorial link of 1 (i.e., the subcomplex made of all the cubes having a vertex in 1) consists of 16 copies of the defining 4-cube, each one obtained by putting 1 in a different vertex of the defining cube;

(ii) any two such 4-cubes in the combinatorial link of 1 intersect in a subcube of codimension at least 2.

As a consequence, G_4 is hypercubical.

Proof. By Lemma 5.12, all loops in the Cayley graph of G_4 are made of squares and all such squares come from the relations that are already written in the presentation. Let C be an m-cube, with $m \in \{3, 4\}$. We will see that C is always contained in a copy of the defining 4-cube. Suppose first of all that $m = 3$. Then $\psi(C)$ is either a cube or a square. In the first case, C is made either only of small or only of capital generators (as there is no length 4 relation mixing them without diagonal generators) and therefore C is a copy of the defining cube for G_3 contained in a copy of the defining 4-cube. If $\psi(C)$ is a square, it is a copy of a face of the defining cube for G_3 and there is a face in C mapping onto $\psi(C)$ with either all small or all capital generators. Fix a vertex in C. Up to inversion, the three edges incident to it are exactly one diagonal generator and either two small or two capital generators. Suppose they are small generators and without loss of generality suppose that they have positive exponent when pointing away from the fixed vertex. This implies that the diagonal generator has a positive exponent when pointing towards the fixed vertex. We first need to look for couple of small generators x, y such that there exist a diagonal generator T so that Tx, Ty are subwords of some of the relations and such that x, y are involved in some nontrivial relation among small generators of the form $xz_1 = yz_2$. By looking for such couples x, y and for the diagonal generators T for each couple, one sees that the only possibilities are $\{a, e, R\}, \{a, f, S\}, \{a, d, S\}, \{b, d, Q\}, \{c, f, S\}, \{d, f, S\}$ (where the small generators are pointing away from the fixed vertex and the diagonal generator is pointing towards it). In all these cases there is only one possibility to complete the obtained configuration to a 3-cube, all of them being contained in a copy of the defining 4-cube. The case of two capital and one diagonal generators is analogous. Therefore every 3-cube in $\mathcal{C}_\bullet(G_4)$ is contained in a copy of the defining 4-cube. Suppose now that $m = 4$. In this case, $\psi(C)$ is a 3-cube and C contains a subcube mapping isomorphically onto $\psi(C)$ made either of only capital or only small generators, being therefore a copy of the defining 3-cube of G_3. Therefore at every vertex in C there are one diagonal and three non-diagonal generators incident to it, the non-diagonals being either all small or all capital. Fix a vertex of C and suppose that the incident edges are labeled by a diagonal generator (with positive exponent when pointing toward the fixed vertex) and three small generators (with positive exponents when pointing away from the fixed vertex). Then by an argument analogous to that of the 3-dimensional case, the only possible combination is for the three small generators to be a, d, f and the diagonal generator to be S. In this case as well there is only one possibility of completing it to a 4-cube, resulting thus in a copy of the defining 4-cube. The case with the capital generators is analogous. This proves (i).

To prove (ii), observe first that each cube with a vertex in 1 corresponds to translating the defining 4-cube by a certain element. Two such cubes intersect in a subcube C if and only if there are two identical copies of C in the defining 4-cube. As only the faces corresponding to the relations $EP = Re$ and $FQ = Sf$ appear twice in the defining 4-cube, no 3-subcube of the defining 4-cube appears twice. This means that two copies of the defining 4-cube intersect in a subcube of codimension at least 2.

For what concerns the last part of the statement, G_4 is quadratic, so that $\mathcal{C}_\bullet$ is simply-connected by Proposition 3.17. Moreover the two conditions above imply Gromov's link condition (see Definition 2.25), therefore $\mathcal{C}_\bullet$ is a CAT(0) cube complex, hence the claim. □

5.4.4 The action on the hypercubical complex

The action of G_4 on its hypercubical complex satisfies the following conditions.

Proposition 5.16. *The action of G_4 on its hypercubical complex is free.*

Proof. Suppose that there is $g \in G_4 \setminus \{1\}$ that fixes a point of a cell C. As the action on the vertices of the hypercubical complex is free, we must have that $\dim C \geq 1$. Moreover the action is cellular, so C is globally fixed by g and g permutes the vertices of C. Therefore g has finite order. The cell C must be a translation of the defining 4-cube or of one of its subcubes, so choosing the expander vertex we see that there must be a vertex h of C such that every other vertex can be written as hw, for w a positive word. Then $h^{-1}C$ is a cell with a vertex in 1 and such that all the vertices are described by positive words. The element $\bar{g} := g^h$ fixes $h^{-1}C$ and $\bar{g} = \bar{g}1 = w$, where w is the positive word describing one of the vertices of $h^{-1}C$. As for all positive words u either $\phi_0(u) \geq 1$ or $\phi_1(u) \geq 1$, one between $\phi_0(\bar{g}) = \phi_0(w)$ and $\phi_1(\bar{g}) = \phi_1(w)$ is not 0. Therefore $\bar{g}$ has infinite order and so has g, thus producing a contradiction. □

Proposition 5.17. *$\mathcal{C}_\bullet(G_4)$ is the universal covering of a classifying space for G_4. As a consequence, G_4 is torsion-free.*

Proof. $\mathcal{C}_\bullet(G_4)$ is CAT(0) and therefore contractible, moreover G_4 acts cellularly and freely on it. □

The following corollary is a direct yet important consequence of the two propositions above.

Corollary 5.18. *G_4 is cocompactly cubulated (or, in other words, it is a CAT(0) cubical group), meaning that it acts properly, cocompactly and by isometries on a CAT(0) cube complex.*

5.4.5 Cohomology

In this section we will describe the cellular chain complex associated to $\mathcal{C}_\bullet(G_4)$ and we will use it for some computations in homology and cohomology. The main result is the computation of the cohomological dimension of G_4.

Lemma 5.19. $2 \leq \mathrm{cd}(G_4) \leq 4$.

Proof. G_3 has cohomological dimension 2 and $G_3 \leq G_4$. Moreover G_4 has a classifying space of dimension 4. □

In order to apply Lemma 2.19 and Remark 2.20, we need a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G_4$. The hypercubical complex $\mathcal{C}_\bullet(G_4)$ provides such free resolution, as implied by [Bro82, Prop 4.1]. We now give a detailed description of the modules and differentials of the complex.

The augmented cellular chain complex of $\mathcal{C}_\bullet(G_4)$ is

$$0 \longrightarrow X_4 \xrightarrow{\partial_4} X_3 \xrightarrow{\partial_3} \dots \xrightarrow{\partial_1} X_0 \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0,$$

where:

- $X_0 = \mathbb{Z}G_4 x_0$ is a single orbit of vertices,
- $X_1 = \bigoplus \mathbb{Z}G_4\sigma_x$ for $x \in \{a, \dots, f, A, \dots, F, P, Q, R, S\}$ has rank 16,
- $X_2 = \bigoplus_{i=1}^{6} \mathbb{Z}G_4 r_i \oplus \bigoplus_{j=1}^{6} \mathbb{Z}G_4 \bar{r_j} \oplus \bigoplus_{l=1}^{10} \mathbb{Z}G_4 s_l$ has rank 22,
- $X_3 = \bigoplus_{i=1}^{8} \mathbb{Z}G_4 C_i$ has rank 8,
- $X_4 = \mathbb{Z}G_4 K$ is the single orbit of 4-cubes.

The generators of the module X_2 are in bijective correspondence with the squares in the defining 4-cube of G_4, such correspondence will be made explicit when describing the differentials of the chain complex. The same will happen for the generators of X_3, that corresponds to the eight 3-subcubes of the defining 4-cube.

For what concerns the differentials, they are defined as follows:

- $\varepsilon x_0 = 1$
- $\partial_1 \sigma_x = (x - 1)x_0$
- $\partial_2 r_1 = \sigma_a + a\sigma_e - e\sigma_b - \sigma_e$
  $\partial_2 r_2 = \sigma_a + a\sigma_f - f\sigma_b - \sigma_f$
  $\partial_2 r_3 = \sigma_d + d\sigma_a - a\sigma_c - \sigma_a$
  $\partial_2 r_4 = \sigma_d + d\sigma_b - b\sigma_c - \sigma_b$
  $\partial_2 r_5 = \sigma_f + f\sigma_c - c\sigma_e - \sigma_c$
  $\partial_2 r_6 = \sigma_f + f\sigma_d - d\sigma_e - \sigma_d$
  $\partial_2 \bar{r_1} = \sigma_A + A\sigma_E - E\sigma_B - \sigma_E$
  $\partial_2 \bar{r_2} = \sigma_A + A\sigma_F - F\sigma_B - \sigma_F$
  $\partial_2 \bar{r_3} = \sigma_D + D\sigma_A - A\sigma_C - \sigma_A$

$\partial_2\bar{r_4} = \sigma_D + D\sigma_B - B\sigma_C - \sigma_B$
$\partial_2\bar{r_5} = \sigma_F + F\sigma_C - C\sigma_E - \sigma_C$
$\partial_2\bar{r_6} = \sigma_F + F\sigma_D - D\sigma_E - \sigma_D$
$\partial_2 s_1 = \sigma_A + A\sigma_S - S\sigma_a - \sigma_S$
$\partial_2 s_2 = \sigma_A + A\sigma_R - R\sigma_a - \sigma_R$
$\partial_2 s_3 = \sigma_B + B\sigma_P - P\sigma_b - \sigma_P$
$\partial_2 s_4 = \sigma_B + B\sigma_Q - Q\sigma_b - \sigma_Q$
$\partial_2 s_5 = \sigma_C + C\sigma_R - S\sigma_c - \sigma_S$
$\partial_2 s_6 = \sigma_C + C\sigma_P - Q\sigma_c - \sigma_Q$
$\partial_2 s_7 = \sigma_D + D\sigma_R - S\sigma_d - \sigma_S$
$\partial_2 s_8 = \sigma_D + D\sigma_P - Q\sigma_d - \sigma_Q$
$\partial_2 s_9 = \sigma_E + E\sigma_P - R\sigma_e - \sigma_R$
$\partial_2 s_{10} = \sigma_F + F\sigma_Q - S\sigma_f - \sigma_S$

- $\partial_3 C_1 = dr_1 - r_2 - r_3 + fr_4 + ar_5 - r_6$
  $\partial_3 C_2 = D\bar{r_1} - \bar{r_2} - \bar{r_3} + F\bar{r_4} + A\bar{r_5} - \bar{r_6}$
  $\partial_3 C_3 = -Sr_3 + \bar{r_3} + s_1 - Ds_2 + As_5 - s_7$
  $\partial_3 C_4 = -Qr_4 + \bar{r_4} - Ds_3 + s_4 + Bs_6 - s_8$
  $\partial_3 C_5 = Sr_6 - \bar{r_6} - s_7 + Fs_8 - Ds_9 + s_{10}$
  $\partial_3 C_6 = Sr_5 - \bar{r_5} - s_5 + Fs_6 - Cs_9 + s_{10}$
  $\partial_3 C_7 = -Rr_1 + \bar{r_1} - s_2 + Es_3 + (1-A)s_9$
  $\partial_3 C_8 = -Sr_2 + \bar{r_2} - s_1 + Fs_4 + (1-A)s_{10}$
- $\partial_4 K = -SC_1 + C_2 + C_3 - FC_4 - C_5 + AC_6 - DC_7 + C_8$

Lemma 5.20. $\mathrm{cd}(G_4) \leq 3$.

Proof. Apply Lemma 2.19 and Remark 2.20 with $c_{n-1} = C_8$ and $X'_{n-1} = \bigoplus_{i=1}^{7} \mathbb{Z}G_4C_i$. □

Theorem 5.21. $\mathrm{cd}(G_4) = 2$. *In particular,* $H_1(G_4) = \mathbb{Z}^6$ *and* $H_2(G_4) = \mathbb{Z}^5$.

Proof. Thanks to Lemma 5.19, we only need to prove that $\mathrm{cd}\, G_4 \leq 2$. In order to do so, we will produce a free resolution of length 2. It suffices to repeatedly apply Lemma 2.19 to the free resolution

$$0 \longrightarrow X'_3 \longrightarrow \cdots \longrightarrow X_0 \longrightarrow \mathbb{Z} \longrightarrow 0,$$

where $X'_3 = \bigoplus_{i=1}^{7} \mathbb{Z}G_4C_i$, in order to get the free resolution

$$0 \longrightarrow X'_2 \longrightarrow \cdots \longrightarrow X_0 \longrightarrow \mathbb{Z} \longrightarrow 0,$$

where $X'_2 = \bigoplus_{i=1}^{5} \mathbb{Z}G_4r_i \oplus \bigoplus_{j=1}^{5} \mathbb{Z}G_4\bar{r_j} \oplus \mathbb{Z}G_4s_3 \oplus \mathbb{Z}G_4s_4 \oplus \mathbb{Z}G_4s_6 \oplus \mathbb{Z}G_4s_9 \oplus \mathbb{Z}G_4s_{10}$ (the last step involves the application of Remark 2.20). The explicit calculations are as follows.

(1) $s_2 = -Rr_1 + \bar{r_1} + Es_3 + (1-A)s_9 - \partial_3 C_7$, which yields

$$0 \longrightarrow \bigoplus_{i=1}^{6} \mathbb{Z}G_4C_i \longrightarrow \bigoplus_{i=1}^{6} \mathbb{Z}G_4r_i \oplus \bigoplus_{i=1}^{6} \mathbb{Z}G_4\bar{r_i} \oplus \mathbb{Z}G_4s_1 \oplus \bigoplus_{i=3}^{10} \mathbb{Z}G_4s_i \longrightarrow X_1 \longrightarrow \dots$$

(2) $s_5 = Sr_5 - \bar{r_5} + Fs_6 - Cs_9 + s_{10} - \partial_3 C_6$, which yields

$$0 \longrightarrow \bigoplus_{i=1}^{5} \mathbb{Z}G_4C_i \longrightarrow \bigoplus_{i=1}^{6} \mathbb{Z}G_4r_i \oplus \bigoplus_{i=1}^{6} \mathbb{Z}G_4\bar{r_i} \oplus \bigoplus_{\substack{i=1\\ i\neq 2,5}}^{10} \mathbb{Z}G_4s_i \longrightarrow X_1 \longrightarrow \dots$$

(3) $s_7 = Sr_6 - \bar{r_6} + Fs_8 - Ds_9 + s_{10} - \partial_3 C_5$, which yields

$$0 \longrightarrow \bigoplus_{i=1}^{4} \mathbb{Z}G_4C_i \longrightarrow \bigoplus_{i=1}^{6} \mathbb{Z}G_4r_i \oplus \bigoplus_{i=1}^{6} \mathbb{Z}G_4\bar{r_i} \oplus \bigoplus_{\substack{i=1\\ i\neq 2,5,7}}^{10} \mathbb{Z}G_4s_i \longrightarrow X_1 \longrightarrow \dots$$

(4) $s_8 = -Qr_4 + \bar{r_4} - Ds_3 + s_4 + Bs_6 - \partial_3 C_4$, which yields

$$0 \longrightarrow \bigoplus_{i=1}^{3} \mathbb{Z}G_4C_i \longrightarrow \bigoplus_{i=1}^{6} \mathbb{Z}G_4r_i \oplus \bigoplus_{i=1}^{6} \mathbb{Z}G_4\bar{r_i} \oplus \bigoplus_{\substack{i=1\\ i\neq 2,5,7,8}}^{10} \mathbb{Z}G_4s_i \longrightarrow X_1 \longrightarrow \dots$$

(5) $r_6 = dr_1 - r_2 - r_3 + fr_4 + ar_5 - \partial_3 C_1$, which yields

$$0 \longrightarrow \bigoplus_{i=2}^{3} \mathbb{Z}G_4C_i \longrightarrow \bigoplus_{i=1}^{5} \mathbb{Z}G_4r_i \oplus \bigoplus_{i=1}^{6} \mathbb{Z}G_4\bar{r_i} \oplus \bigoplus_{\substack{i=1\\ i\neq 2,5,7,8}}^{10} \mathbb{Z}G_4s_i \longrightarrow X_1 \longrightarrow \dots$$

(6) $\bar{r_6} = D\bar{r_1} - \bar{r_2} - \bar{r_3} + F\bar{r_4} + A\bar{r_5} - \partial_3 C_2$, which yields

$$0 \longrightarrow \mathbb{Z}G_4C_3 \longrightarrow \bigoplus_{i=1}^{5} \mathbb{Z}G_4r_i \oplus \bigoplus_{i=1}^{5} \mathbb{Z}G_4\bar{r_i} \oplus \bigoplus_{\substack{i=1\\ i\neq 2,5,7,8}}^{10} \mathbb{Z}G_4s_i \longrightarrow X_1 \longrightarrow \dots$$

(7) $s_1 = -Sr_2 + Fs_4 + (1-A)s_{10} - S\partial_3 C_1 + \partial_3 C_2 + \partial_3 C_3 - \partial_3 C_5 + A\partial_3 C_6 - D\partial_3 C_7$, so that we can apply Remark 2.20 and get the following free resolution of length 2:

$$0 \longrightarrow \bigoplus_{i=1}^{5} \mathbb{Z}G_4r_i \oplus \bigoplus_{i=1}^{5} \mathbb{Z}G_4\bar{r_i} \oplus \bigoplus_{\substack{i=3\\ i\neq 5,7,8}}^{10} \mathbb{Z}G_4s_i \longrightarrow X_1 \longrightarrow \mathbb{Z}G_4x_0 \longrightarrow \mathbb{Z} \longrightarrow 0.$$

In order to compute the homology, it suffices to apply the coinvariant functor to the resolution of length 2 that we have just obtained, and write the differential ∂_2 as a matrix. By using the SAGE command `elementary_divisors()`, one gets that the Smith normal form of the matrix representing ∂_2 is:

$$\begin{pmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}.$$

From this one can deduce that the second homology group is the desired one, while H_1 is isomorphic to the abelianization of G_4, which can be seen to be $\mathbb{Z}^6$ by direct computations.

□

5.5 The general case

In this section we prove that G_n is hypercubical for any $n \geq 3$ and deduce some results about these groups. The first step will be to describe certain combinatorics of the generators and relations of G_n.

First of all, note that we can divide the set of generators into equivalence classes given by the parallelism relation in the defining n-cube. The class of generators labeling edges parallel to the i-th axis is called *of type* i. Given a square of the defining n-cube, the opposite sides belong to the same class by definition. If two opposite sides of a square belong to the class of type i, the segment connecting their midpoints is called a *midline of type* i. Note that every square has two crossing midlines of different types and that for any i the midline of type i is given by the intersection of the square with the hyperplane consisting of the points with i-th coordinate equal to 1/2, i.e., $\{z_i = 1/2\}$ (here z_i is the i-th coordinate, as in

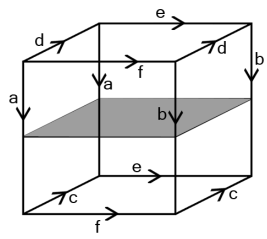
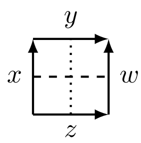


Figure 5.9: The defining 3-cube and a hyperplane

Definition 5.8). To see this, consider the square 

, where x, w are of type i and z, y are of type j (which means that the dashed horizontal midline is of type i and the dotted vertical midline is of type j). For every point in the square the other coordinates are fixed, while the i-th and the j-th can vary between 0 and 1. In particular, the horizontal midline corresponds to the points with $z_i = 1/2$, analogously for the vertical midline. If we walk along one of the two midlines, say the horizontal one, from one side to the other of the square (for example if we move from the left to the right), we start from an edge labeled x and we end up in an edge labeled w. The generators x, w are in the same parallelism class and this holds for every square in the defining n-cube (as we are crossing a square from one side to the opposite one, which is clearly parallel to the first). This implies that if we fix a type, say type i, then every time a midline of type $j \neq i$ crosses a midline of type i the generators of the class of type i are permuted (but they are not mixed with generators of other types). As every midline of type j is given by the intersection between a square in the defining n-cube and the hyperplane of equation $z_j = 1/2$, on each square of the defining n-cube with two opposite sides of type j the permutation described above coincides with the one induced by the reflection across the hyperplane $\{z_j = 1/2\}$ on the labels of the edges of type $\neq j$ of the square.

For instance, consider the case $n = 3$. In fig. 5.9, the edges labeled $\{a, b\}$ have direction z_1, those labeled $\{c, d\}$ have direction z_2 and those labeled $\{e, f\}$ have direction z_3. Therefore the generators of type 1 are $\{a, b\}$, those of type 2 are $\{c, d\}$ and those of type 3 are $\{e, f\}$. In gray is represented the intersection between the hyperplane $\{z_1 = 2\}$ and the defining 3-cube. Consider for example the parallelism class $\{e, f\}$. On this parallelism class, crossing a midline of type 1 induces the trivial permutation, while crossing a midline of type 2 induces the permutation that swaps e and f.

Lemma 5.22. *For each n, i, the reflection across the hyperplane $\{z_i = 1/2\}$ induces a well defined permutation σ of the labels of the edges of the defining n-cube so that, for each subsquare of types i, j, σ corresponds to the action of the i-midline on the j-generators.*

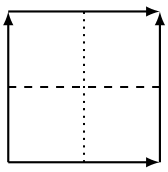
Proof. It is enough to prove that the reflection across $\{z_i = 1/2\}$ sends edges with the same label to edges with the same label. We proceed by induction, $n = 3$ being the base case. For $n = 3$ the claim can be verified directly. Suppose the claim holds for G_{n-1} and fix $i \in \{1, \dots, n\}$. If $i = n$, then the reflection across $\{z_n = 1/2\}$ swaps every edge of the outside $(n-1)$-cube with the corresponding edge of the inside $(n-1)$-cube. As the two $(n-1)$-cubes are isomorphic, σ is well defined. Suppose now that $i \neq n$. Every vertex of the defining n-cube can be represented by a binary sequence $\varepsilon\omega$, where $\varepsilon = 0$ on the outside $(n-1)$-cube and $\varepsilon = 1$ on the inside one, as in Definition 5.8. Note that the reflection across $\{z_i = 1/2\}$ sends each $\varepsilon\omega$ to $\varepsilon\bar{\omega}$, where $\bar{\omega}$ differs from ω only in position i. By inductive hypothesis, the restriction of such reflection to the outside $(n-1)$-cube induces a well defined permutation on the set of capital generators, analogously for the set of small generators. For what concerns diagonal edges, if two of them have the same label then by definition they issue from vertices described by two binary sequences $0\omega0$ and $0\omega1$ for some ω. The reflection across $\{z_i = 1/2\}$ either sends these two vertices to $0\bar{\omega}0$ and $0\bar{\omega}1$, if $i \neq 1$, or swaps them for $i = 1$. In both cases, the diagonal edges issuing from these vertices have the same label, hence the claim. □

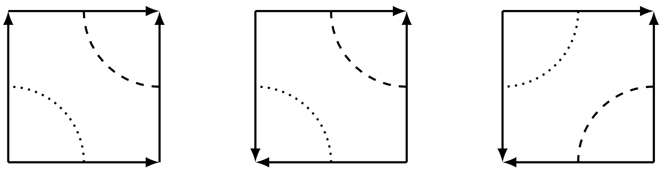
Remark 5.23. For each i, let σ_i be the permutation induced by the reflection $\{z_i = 1/2\}$ as in Lemma 5.22. Then σ_i has order 2 and the orbits of the action of $\langle\sigma_i\rangle$ on the set of generators of type $\neq i$ have length at most 2.

Remark 5.24. For each i, let τ_i denote the reflection across $z_i = 1/2$. Then the Coxeter group generated by $\tau_1, \dots, \tau_n$ is $(\mathbb{Z}/2)^n$. As a consequence, for any i, j, k pairwise distinct, the permutations induced by the midlines of type i and j on the set of generators of type k commute.

Lemma 5.25. *The only non-trivial relations of length 4 in G_n are those of the borromic presentation (up to inversion and cyclic conjugation).*

Proof. Consider a relation of length 4, draw a Van Kampen diagram K for it and draw the midlines corresponding to the four sides of the boundary . Call them the *principal midlines* of the diagram. Note that, as Möbius strips cannot be embedded in the plane, the orientation of the edges crossed by a midline has to be preserved along the midline. Therefore, up to rotations and reflections of the Van Kampen diagram K, we are reduced to consider 4 cases:

(1) 

(2) 

Case (2) is easily ruled out, as all the other midlines have to cross the principal ones an even number of times and also the principal ones have to cross each other an even number of times. Taking into account that the permutations induced by the crossings commute with each other and have order 2 (by Remarks 5.23 and 5.24), we deduce that the boundary relation has form $xy = xy, x^{-1}y = x^{-1}y$ or $x^{-1}x = y^{-1}y$, which do not identify squares in the hypercubical complex.

For what concerns case (1), suppose that the horizontal midline has type i and the vertical one has type j. Consider the first crossing, along the horizontal midline and going from left to right, of the two principal midlines. This distinguished crossing corresponds to a relation of the presentation and it divides every principal midline into two branches. Except for the distinguished crossing, the two principal midlines have to cross each other an even number of times on every branch. Non-principal midlines of type i can cross the principal midline of type j an even number of times on each branch, same for non-principal midlines of type j and the principal midline of type i. Non-principal midlines of type $k \neq i, j$ can cross each branch of the principal midlines either an even number of times (if the distinguished crossings is contained in the exterior of the curve) or an odd number of times (if the distinguished vertex lies in the interior of the curve), and the parity is the same on the four branches. By Remarks 5.23 and 5.24, an even number of crossings of a principal midline with a midline of a fixed type does not produce any change in the relation and the order is not relevant. Therefore we only need to focus on the crossings with midlines of type $k \neq i, j$ that are closed simple curves whose interior contains the distinguished crossing. Note that, for every such midline, the parity of the number of crossings on each of the four branches is always odd and, as the associated permutations have order two, we may assume that this number is always one. Moreover, the order of the crossings on the branches of the principal midlines is not relevant, therefore we can suppose that the order is the same on each branch, so that the effect of the crossings is as in fig. 5.10 (note that the X in the middle indicates the distinguished crossing and the colored closed simple curves around it are non-principal midlines that actually produce an effect on the relation associated to the distinguished crossing). As this means that we are applying reflections to the

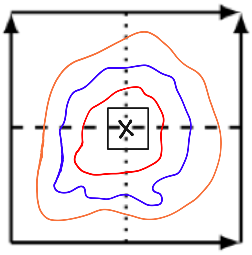


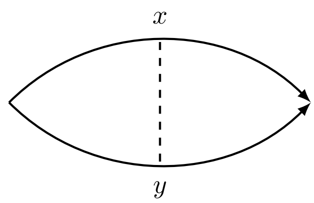
Figure 5.10: Representing the effect of the crossings in the proof of Lemma 5.25

square of the defining n-cube lying around the distinguished crossing with respect to hyperplanes not intersecting it, the image will always be another square of the defining n-cube. That image is precisely the boundary of the Van Kampen diagram K, therefore the boundary relation is a relation we already see in the presentation of G_n. □

The following generalizes Proposition 5.10. Note that the homomorphisms defined in section 5.4.1 for G_4 can be defined analogously for G_n for any $n \geq 4$.

Lemma 5.26. *All the generators of G_n and their inverses are pairwise distinct.*

Proof. We have to check that there is no length 2 relation. Assume there is and consider a Van Kampen diagram K for it. As we noted at the beginning of the proof of Lemma 5.25, the orientation of the edges at the extremities of the principal midline of K has to be the same and the generators associated to them have to be in the same parallelism class. Therefore, the K will have the following form:



The generators x, y belong to the same parallelism class. All the other midlines of the diagram are non-principal, therefore they are closed simple curves and they cross the principal midline an even number of times. As this crossings commute with each other and en even number of crossings with a midline of the same type does not produce any effect (by Remarks 5.23 and 5.24), $x = y$ and the relation is trivial. □

Lemma 5.27. $\dim \mathcal{C}_\bullet(G_n) = n$.

Proof. The proof is by induction, $n = 4$ being the base case. Suppose the claim holds for G_{n-1}. The same argument used to show that $\dim \mathcal{C}_\bullet(G_4) = 4$ (see Lemma 5.13) can be applied to prove the induction. □

Lemma 5.28. *The homomorphism $\psi\colon G_n \to G_{n-1}$ extends to a cellular map $\psi\colon \mathcal{C}_\bullet(G_n) \to \mathcal{C}_\bullet(G_{n-1})$.*

Proof. The proof is by induction, $n = 4$ being the base case, and it mimics the proof of Lemma 5.14. □

Theorem 5.29. *Suppose that for $n \geq 4$ the group G_{n-1} satisfies the following two properties:*

A(n-1): the combinatorial link of 1 (i.e., the subcomplex made of all the cubes having a vertex in 1) consists of 2^{n-1} copies of the defining $(n-1)$-cube, each one obtained by putting 1 in a different vertex of the defining cube;

B(n-1): any two such $(n-1)$-cubes in the combinatorial link of 1 intersect in a subcube of codimension at least 2.

Then G_n satisfies $A(n)$ and $B(n)$. As a consequence, G_n is hypercubical.

Proof. As G_n is quadratic, and by Lemma 5.25, all loops in its Cayley graph are made of squares and all such squares come from the relations that are already written in the presentation. Let C be an m-cube with a vertex in 1, with $m \leq n$. The sides of C having a vertex in 1 are either all capital or all small, or there is exactly one diagonal generator. If there is a diagonal generator, then $\psi(C)$ is an $(m-1)$-subcube of the defining $(n-1)$-cube by hypothesis. Then there is only one way to complete such $(m-1)$-cube to an m-cube by using diagonal generators, therefore C is contained in the defining n-cube. If C does not contain any diagonal generator, then $\psi(C)$ is a copy of C in $\mathcal{C}_\bullet(G_{n-1})$, therefore by hypothesis it is a subcube of the defining $(n-1)$-cube, thus of the defining n-cube. This proves $A(n)$. To prove $B(n)$, suppose there exist two copies C_1 and C_2 of the defining n-cube in $\mathcal{C}_\bullet(G_n)$ such that $C_1 \cap C_2$ has codimension 1. If $C_1 \cap C_2$ contains diagonal edges, then $\psi(C_1 \cap C_2)$ is a codimension 1 subcube of both $\psi(C_1)$ and $\psi(C_2)$, both of them being copies of the defining $(n-1)$-cube. This produces a contradiction. If $C_1 \cap C_2$ does not contain any diagonal edge, then $\psi(C_1 \cap C_2) = \psi(C_1) = \psi(C_2)$. This means that we can find two copies of $C_1 \cap C_2$ in the defining n-cube that can be completed to the defining n-cube in two different ways. The two copies must be either both the capital $(n-1)$-cube or the small one, but there is only a way to complete them to the defining n-cube. In both cases, we get to a contradiction

For what concerns the last part of the statement, G_n is quadratic and the two conditions above imply Gromov's link condition. □

Corollary 5.30. *G_n is hypercubical, for every $n \geq 3$, with respect to the generators of the borromic presentation.*

Proof. The statement is already known for $n = 3$ and $n = 4$. For the general case, proceed by induction on n, $n = 4$ being the base case. □

Propositions 5.16 and 5.17 and Corollary 5.18 can be generalized to the whole family of Borromean cube groups, as the arguments proceed verbatim. This proves the following.

Proposition 5.31. *For any $n \geq 3$, the group G_n acts freely on its hypercubical complex, is torsion free and cocompactly cubulated.*

Lemma 5.32. *For any $n \geq 4$, $2 \leq \operatorname{cd} G_n \leq n - 1$.*

Proof. G_3 is a subgroup of G_n for any n and $\operatorname{cd} G_3 = 2$. Moreover $\mathcal{C}_\bullet(G_n)$ is an EG_n of dimension n, therefore $\operatorname{cd} G_n \leq n$. In the augmented cellular chain complex, the module in dimension n is a free module of rank 1, generated say by c_n. Then either $\partial_n c_n = 0$, which implies that we can reduce the length of the chain complex by 1, or in the expression of $\partial_n c_n$ there is at least a summand of the form gc_{n-1} with $g \in G_n$. We can now apply Lemma 2.19 to get that $\operatorname{cd} G_n \leq n - 1$. □

Corollary 5.33. *G_n is torsion-free for any $n \geq 3$.*

Chapter 6

Conclusions

Conclusions

Hypercubical groups are a class of groups defined by the property that their hypercubical complex is contractible, for some fixed finite generating set. The results contained in this thesis are mainly of two types. One the one hand, we have proven that RAAGs, oriented and twisted RAAGs, and Borromean cube groups are hypercubical, and we have also proven some results about these families of groups. On the other hand, we have stated and proven general results about hypercubical groups, such as the fact that they are either of type F_∞ or F, properties of the action on the hypercubical complex and an upper bound on the cohomological dimension in the torsion-free case. In addition to these results, the main contribution of this thesis is to have given a unified context to study groups whose hypercubical complex is contractible (as some sporadic examples were already known in literature, even though it was not stated in these terms). This is interesting in that these groups have a rich geometric nature and the hypercubical complex can be used for homological and cohomological computations as well.

Geometric group theory has a wide variety of connections with other branches of mathematics, and there are numerous open questions that can be approached for hypercubical groups. Some of them are listed below.

(Q.1) Is the hypercubical complex of a hypercubical group always CAT(0)?

(Q.2) Some techniques would allow one to prove general results about hypercubical groups also when the generating set used in the construction of the hypercubical complex is infinite, therefore it might be significant to extend the definition to this case.

(Q.3) More geometric aspects about the hypercubical complex of a hypercubical group and the hypercubical group itself can be studied, for instance the topology of the boundary (at least when the hypercubical complex is CAT(0)).

(Q.4) The properties of the hypercubical complex of a group strongly depend on the specific generating set one chooses. How are different hypercubical complexes for the same group related?

(Q.5) The definition of Borromean cube groups depends on the specific pattern used for the labeling of the defining cubes. It would be interesting to study more general patterns, either on n-cubes or on other graphs (as long as the resulting presentation is quadratic).

(Q.6) One could use the hypercubical complex to show cohomological properties for other classes of hypercubical groups.

(Q.7) Is it possible to produce answers to algorithmic problems by using the hypercubical complex of a group?

(Q.8) What can we say about the nature of the group $\mathrm{Aut}(\mathcal{C}_\bullet(G, \Sigma))$ for some group G and generating set Σ? What is the relation with G?

(Q.9) One could study the BNSR-invariants for some families of hypercubical groups.

(Q.10) Is it possible to find relations between (certain subfamilies of) hypercubical groups and other important families of groups often studied in geometric group theory?

Conclusioni

I gruppi ipercubici sono una classe di gruppi definiti dalla proprietà che il loro complesso ipercubico è contraibile, per un qualche sistema finito di generatori fissato. I risultati contenuti in questa tesi si dividono principalmente in due categorie. Da una parte, abbiamo provato che i RAAG, i RAAG twisted e orientati e i gruppi borromeiani cubici sono ipercubici. Dall'altra parte, abbiamo enunciato e dimostrato risultati generali riguardo ai gruppi ipercubici, quali ad esempio il fatto che sono di tipo F_∞ o F, alcune proprietà dell'azione sul complesso ipercubico e un limite superiore della dimensione coomologica. In aggiunta a questi risultati, il contributo principale di questa tesi è di aver introdotto un contesto unificato per lo studio dei gruppi il cui complesso ipercubico è contraibile (esempi sporadici di gruppi con questa proprietà erano già noti in letteratura, anche se non in questi termini). L'interesse di ciò risiede nel fatto che questi gruppi hanno una ricca natura geometrica e che è possibile utilizzare il complesso ipercubico di questi gruppi anche per calcoli di tipo omologico e coomologico.

La teoria geometrica dei gruppi ha una grande varietà di connessioni con altri rami della matematica, pertanto anche le domande aperte che si possono affrontare per i gruppi ipercubici sono numerose. Alcune di queste sono elencate qui di seguito.

(Q.1) Dato un gruppo ipercubico, il suo complesso ipercubico è sempre CAT(0)?

(Q.2) Alcune tecniche permettono di dimostrare risultati generali sui gruppi ipercubici anche nel caso in cui il sistema di generatori usato nella costruzione del complesso ipercubico sia infinito, sarebbe quindi significativo estendere la definizione a questo caso.

(Q.3) Aspetti più geometrici del complesso ipercubico di un gruppo ipercubico, e del gruppo ipercubico stesso, potrebbero essere di interesse, quali ad esempio la topologia della sua frontiera (almeno nel caso in cui il complesso ipercubico sia CAT(0)).

(Q.4) Le proprietà del complesso ipercubico di un gruppo dipendono strettamente dal sistema di generatori considerato. Che relazione c'è tra differenti complessi ipercubici dello stesso gruppo?

(Q.5) La definizione di gruppi borromeiani cubici è legata allo specifico pattern utilizzato per etichettare i cubi di definizione. Sarebbe interessante studiare pattern più generali, su n-cubi o su altri grafi (sempre che la presentazione risultante sia quadratica).

(Q.6) Si potrebbe usare il complesso ipercubico per dimostrare proprietà coomologiche per altre classi di gruppi ipercubici.

(Q.7) É possibile rispondere a problemi di tipo algoritmico usando il complesso ipercubico di un gruppo?

(Q.8) Che cosa possiamo dire sulla natura del gruppo $\mathrm{Aut}(\mathcal{C}_\bullet(G,\Sigma))$ per un gruppo G e un sistema di generatori Σ? Che relazione sussiste con G?

(Q.9) Si potrebbero studiare gli invarianti BNSR per alcune famiglie di gruppi ipercubici.

(Q.10) É possibile trovare relazioni tra (certe sottofamiglie di) gruppi ipercubici e altre importanti famiglie di gruppi studiate nell'ambito della teoria geometrica dei gruppi?

Conclusiones

Los grupos hipercúbicos son una clase de grupos definidos por la propiedad de que su complejo hipercúbico es contractible para algún conjunto finito de generadores fijado. Los resultados contenidos en esta tesis son principalmente de dos tipos. Por un lado, hemos demostrado que los RAAGs, los RAAGs orientados y twisted y los grupos cúbicos de Borromeo son hipercúbicos. Por otro lado hemos demostrado también resultados generales sobre los grupos hipercúbicos, por ejemplo que los grupos hipercúbicos son de tipo F_∞ o de tipo F, unas propiedades de la acción sobre el complejo hipercúbico y una cota para la dimensión cohomológica. Además de estos resultados, la contribución principal de esta tesis es haber proporcionado un contexto unificado para el estudio de los grupos cuyo complejo hipercúbico es contractible (algunos ejemplos esporádicos ya se conocían en la literatura, aunque no en estos términos). Lo interesante de esto es que estos grupos tienen una rica naturaleza geométrica y el complejo hipercúbico también se puede utilizar para cálculos homológicos y cohomológicos.

La teoría geométrica de grupos tiene una amplia variedad de conexiones con otras ramas de las matemáticas, por lo tanto, hay numerosas preguntas abiertas que se pueden abordar para los grupos hipercúbicos. Algunas de ellas se enumeran a continuación.

(Q.1) ¿Es el complejo hipercúbico de un grupo hipercúbico siempre CAT(0)?

(Q.2) Algunas técnicas permiten demostrar resultados generales sobre los grupos hipercúbicos también cuando el conjunto de generadores usato en la construcción del complejo hipercúbico es infinito, por lo tanto, podría ser significativo extender la definición a este caso.

(Q.3) Se pueden estudiar aspectos más geométricos sobre el complejo hipercúbico de un grupo hipercúbico y sobre el grupo hipercúbico mismo, por ejemplo la topología de su frontera (al menos cuando el complejo hipercúbico es CAT(0)).

(Q.4) Las propiedades del complejo hipercúbico de un grupo dependen fuertemente del conjunto de generador específico que se elija. ¿Cómo están relacionados distintos complejos hipercúbicos para el mismo grupo?

(Q.5) La definición de grupos cúbicos de Borromeo dependen del patrón específico usado para el etiquetado de los cubos definitorios. Sería interesante estudiar patrones más generales, ya sea en n-cubos o en otros grafos (siempre que la presentación resultante sea cuadrática).

(Q.6) Se podría usar el complejo hipercúbico para mostrar propiedades cohomológicas para otras clases de grupos hipercúbicos.

(Q.7) ¿Se puede usar el complejo hipercúbico de un grupo para solucionar problemas algorítmicos?

(Q.8) ¿Qué podemos decir sobre la naturaleza del grupo $\mathrm{Aut}(\mathcal{C}_\bullet(G,\Sigma))$ para algún grupo G y conjunto de generadores Σ? ¿Qué relación tiene con G?

(Q.9) Se podrían estudiar los invariantes BNSR de algunas familias de grupos hipercúbicos.

(Q.10) ¿Se pueden encontrar relaciones entre (ciertas familias de) grupos hipercúbicos y otras familias importantes de grupos estudiados en teoría geométrica de grupos?

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