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# Finite multi-utility representable preferences and Pareto orderings

Juan C. Candeal *

*Department of Economics, Faculty of Economics and Business, University of Zaragoza, Spain*

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ABSTRACT

A characterization of the posets that admit a finite multi-utility representation is presented. This result is also connected to the concept of order dimension of a poset as defined by Dushnik and Miller (1941).

## 1. Introduction

In this short paper, we investigate the problem of the existence of a multi-utility representation for an incomplete preference relation defined on a set $X$, using only a finite number of utility functions. This problem has garnered attention since Ok's important contribution in 2002. Characterizing the preference relations that admit such a representation remains an open problem.

The current article builds upon Candeal's previous work. In Candeal (2022), the existence of a bi-utility representation for a partial order on a countable set was characterized. This representation involves two utilities and serves as a special case of a multi-utility representation. In Candeal (2023), the focus shifts to an arbitrary, not necessarily countable, partially ordered set. The article provides further insights into the existence of a two-agent Pareto representation in this broader context. Here, we go a step beyond by allowing $X$ to be an arbitrary set and providing a characterization of the preference relations that admit a multi-utility representation with a finite number of utilities.

The multi-utility representation problem is crucial when modeling incomplete preferences. For instance, it plays a role in analyzing behavioral phenomena such as intransitive choice, multi-criteria decisions, risk and uncertainty models, and game theory. Recent articles, some exploring continuity properties of multi-utility representations, include works by Ok (2002), Evren and Ok (2011), Bosi and Herden (2012), and Giarlotta and Greco (2013). This topic also intersects with mathematics, specifically the concept of order dimension of a poset as defined in Dushnik and Miller's seminal paper (1941). Sprumont (2001) discussed this concept in an economic context, and more recently, Qi (2015, 2016) explored it further.

Our main result relies on the concept of a *Pareto poset*. A Pareto poset is a partially ordered set for which there exists a finite collection of families of subsets of the alternative set $X$ that satisfy certain duality properties. Taking advantage of this concept, we characterize the existence of a finite multi-utility representation in the countable case. By introducing an appropriate separability condition, we extend our characterization to the general case. Unlike most previous contributions, our approach to the problem is more set-theoretic than topological. Additionally, we discuss the relationship between our findings and the classic Dushnik and Miller theorem.

## 2. Preliminaries

We carry on with the notations and standard definitions provided in Candeal (2022). Thus, $X$ will be a nonempty set and a partial order on $X$ will be denoted by $\precsim$. Recall that a *partial order* $\precsim$ on $X$ is a reflexive, antisymmetric and transitive binary relation defined on $X$. A *total order* on $X$ is a complete partial order. Given $x, y \in X$ we say that $x$ and $y$ are *incomparable* in $\precsim$ (or, simply, incomparable) provided that neither $x \precsim y$ nor $y \precsim x$ holds true. In this case, the notation $x \bowtie y$ is used. The asymmetric part of $\precsim$ will be denoted by $\prec$. The pair $(X, \precsim)$ is called a *partially ordered set* or, simply, a *poset*. Throughout the paper it will be assumed that $\precsim$ is a nontrivial partial order on $X$; i.e., $\prec$ will be a nonvoid relation. Given a *binary relation* $R$ on $X$, for any $x \in X$, as is customary, the notations $L_R(x)$, $U_R(x)$ and $N_R(x)$, will stand for the *lower contour set* of $x$, the *upper contour set* of $x$, and the elements of $X$ that are *incomparable* to $x$, respectively. Let $n \in \mathbb{N}$ and $N := \{1, \dots, n\} \subseteq \mathbb{N}$. The natural partial order on the $n$-dimensional Euclidean space $\mathbb{R}^n$ will be denoted by $\leq$; i.e., for any $a = (a_i), b = (b_i) \in \mathbb{R}^n$, $a \leq b$ if and only if $a_i \leq b_i$, for all $i \in N$.

A partial order $\precsim$ defined on $X$ is *finite multi-utility representable* whenever there are $n \in \mathbb{N}$ and a function $u : X \to \mathbb{R}^n$ such that $x \precsim y$ if

* Correspondence to: Department of Economics, Faculty of Economics and Business, University of Zaragoza, Gran Vía, 2, 50005 Zaragoza, Spain
*E-mail address:* candeal@unizar.es.

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and only if $u(x) \le u(y)$, for any $x, y \in X$. In this case, $u = (u_i)_{i=1}^n$ is said to be a finite multi-utility representation of $\precsim$. If $n = 1$, then the classic concept of a *utility representation* is obtained. If $n = 2$, then we will refer to it as a *bi-utility representation* of $\precsim$.

## 3. Pareto posets and multi-utility representations

In this section, a general result that characterizes the existence of a finite multi-utility representation, for a partially ordered set $(X, \precsim)$, is presented. The set-theoretic approach followed is based on some duality properties of certain families of subsets of $X$. We start with the concept of *inverse* of a given family of subsets. From now on, $2^X$ will denote the powerset of $X$.

Let $\mathcal{A} = (A(x))_{x\in X}$ be a family of subsets of $X$. Then, for every $x \in X$, we will denote by $A^{-1}(x) := \{y \in X : x \in A(y)\}$ and call this subset the *inverse* of $A(x)$. The symbol $\mathcal{A}^{-1}$ is meant for the following family of subsets $\mathcal{A}^{-1} := (A^{-1}(x))_{x\in X}$.

We now introduce two axioms concerning the family $\mathcal{A} = (A(x))_{x\in X}$. The first axiom, called *nonexpansiveness*, refers to a 'shrinking property' that the subsets of the family must satisfy. More precisely, for any $x \in X$, it is required that the subsets of the family corresponding to the points of $A(x)$ must be contained within $A(x)$. As will be seen later, this axiom is closely related to the transitivity of certain binary relation defined on $X$. A natural example of a family satisfying nonexpansiveness is $\mathcal{A} = (L_{\prec}(x))_{x\in X}$, where $\precsim$ is a partial order on $X$. By the way, in this case, it holds that $\mathcal{A}^{-1} = (U_{\prec}(x))_{x\in X}$.

The objective for the second axiom, called *exhaustiveness* is that, for any $x$, $A(x)$ and $A^{-1}(x)$ be a partition of $X \setminus \{x\}$. Therefore, for each $y \in X$, either $y$ is supposed to be in $A(x)$, or $x$ must be in $A(y)$. Thus, a possible interpretation of the above collection is a sort of 'partial covering' and 'partial anti-covering' of $X \setminus \{x\}$ that fully partitions $X \setminus \{x\}$. By 'anti-covering' here we mean a 'winding set', which counts explicitly every $A(y)$ that contains $x$ by keeping track of $y$.

**Definition 1.** A family $\mathcal{A} = (A(x))_{x\in X} \subseteq 2^X$ is said to be:
(i) *nonexpansive* whenever $y \in A(x)$ entails $A(y) \subseteq A(x)$, for any $x, y \in X$,
(ii) *exhaustive* provided that, for any $x \in X$, $\{A(x), A^{-1}(x)\}$ is a partition of $X \setminus \{x\}$.

**Remark 2.** It is straightforward to see that a family $\mathcal{A}$ is nonexpansive (respectively, exhaustive) if and only if so is $\mathcal{A}^{-1}$.

Assume $\mathcal{A} = (A(x))_{x\in X}$ is an exhaustive and nonexpansive family of subsets of $X$. Consider the binary relation $R$ on $X$ defined by:

$x \; R \; y$ if and only if $x \in A(y)$ or $x = y$

**Lemma 3.** *$R$ is a total order on $X$.*

**Proof.** Let us prove that $R$ is transitive. Indeed, let there be given $x, y, z \in X$ such that $x \; R \; y$ and $y \; R \; z$. Assume $x, y, z$ are pairwise distinct because, otherwise, $x \; R \; z$ follows obviously. Thus, $x \in A(y)$ and $y \in A(z)$. Now, because $\mathcal{A}$ is nonexpansive, it holds that $A(y) \subseteq A(z)$. Therefore, $x \in A(y) \subseteq A(z)$, whence $x \; R \; z$ and we are done.

In order to see that $R$ is antisymmetric, let $x, y \in X$ such that $x \; R \; y$ and $y \; R \; x$. Suppose, by way of contradiction, that $x \neq y$. Then, $x \in A(y)$ and $y \in A(x)$. But, by nonexpansiveness, this would imply that $x \in A(x)$ which is not possible because, by exhaustivity, $x \notin A(x)$. Therefore, $x = y$ and $R$ turns out to be antisymmetric.

To prove that $R$ is complete, let $x, y \in X$ two arbitrary points of $X$. If $\neg(x \; R \; y)$, then, by definition of $R$, $x \neq y$ and $x \notin A(y)$. Thus, by exhaustivity, $x \in A^{-1}(y)$ or, equivalently, $y \in A(x)$. Therefore, $y \in A(x)$ and so $R$ is a total order on $X$. □

We now introduce the fundamental concept of a *Pareto poset*. It involves the existence of a finite collection of nonexpansive and exhaustive families of subsets of $X$ such that, for any $x \in X$, the intersection of the members of the collection associated with $x$ equals $L_{\prec}(x)$; i.e., the set of points which are less preferred than $x$.

**Definition 4.** A partially ordered set $(X, \precsim)$ is said to be a *Pareto poset* provided that there is a finite collection of nonexpansive and exhaustive families of subsets of $X$, $(\mathcal{A}_i)_{i=1}^n = (A_i(x))_{x\in X}$, such that $\bigcap_{i=1}^n A_i(x) = L_{\prec}(x)$, for any $x \in X$.

**Remark 5.** (i) By Lemma 3, for each $i \in N$, the family $\mathcal{A}_i$ defines a total order on $X$. Thus, in particular, for each $i \in N$, and for any distinct $x, y \in X$ it holds that either $x \in A_i(y)$ or $y \in A_i(x)$. Henceforth, the total order on $X$ induced by $\mathcal{A}_i$ will be denoted by $\precsim_i$. Obviously, each $\precsim_i$ is an extension of $\precsim$ (i.e., $x \precsim y$ entails $x \precsim_i y$, for any $x, y \in X$). Further, if, for distinct $x, y \in X$, it holds $x \precsim_i y$, for all $i \in N$, then $x \precsim y$ since $x \in \bigcap_{i=1}^n A_i(y) = L_{\prec}(y)$.
(ii) As a simple illustration of Definition 4, consider the three-point set $X = \{x_1, x_2, x_3\}$ equipped with the following partial order: $x_1 \precsim x_3$, $x_2 \precsim x_3$, and the corresponding self relation of any point to meet reflexivity. Note that $N_{\precsim}(x_1) = x_2$, $N_{\precsim}(x_2) = x_1$, and $N_{\precsim}(x_3) = \emptyset$. Let $(\mathcal{A}_i)_{i=1,2} = (A_i(x_j))_{j=1,2,3}$ be the two following families of subsets of $X$:
(a) $A_1(x_1) = x_2$, $A_2(x_1) = \emptyset$
(b) $A_1(x_2) = \emptyset$, $A_2(x_2) = x_1$
(c) $A_1(x_3) = A_2(x_3) = \{x_1, x_2\}$.

Note that $A_1^{-1}(x_1) = x_3$ and $A_2^{-1}(x_1) = \{x_2, x_3\}$; $A_1^{-1}(x_2) = \{x_1, x_3\}$ and $A_2^{-1}(x_2) = x_3$; $A_1^{-1}(x_3) = A_2^{-1}(x_3) = \emptyset$. Also, it holds that $A_1(x_1) \bigcap A_2(x_1) = \emptyset = L_{\prec}(x_1)$, $A_1(x_2) \bigcap A_2(x_2) = \emptyset = L_{\prec}(x_2)$, and $A_1(x_3) \bigcap A_2(x_3) = \{x_1, x_2\} = L_{\prec}(x_3)$. Thus, it is straightforward to see that the two families $(\mathcal{A}_i)_{i=1,2}$, so-defined, make $(X, \precsim)$ a Pareto poset.

The two total orders on $X$, $\precsim_1$ and $\precsim_2$, induced by these two families are: $x_2 \precsim_1 x_1 \precsim_1 x_3$, and $x_1 \precsim_2 x_2 \precsim_2 x_3$. Obviously, $\precsim_1 \cap \precsim_2 = \precsim$. For other more sophisticated examples, see Section 5 below.

As is usual in utility theory, we now introduce a separability condition that will help us to address the general case.

**Definition 6.** A Pareto poset $(X, \precsim)$, with a finite collection of associated families $(\mathcal{A}_i)_{i=1}^n = (A_i(x))_{x\in X}$, is said to be *separable* provided that there is a coinitial,[1] countable subset $D \subseteq X$ such that, for any $x, y \in X$, any $i \in N$ the following condition holds true: $x \in A_i(y) \Longrightarrow (\exists j \in N)\ (\exists\ d \neq d' \in D)$ such that $x \in A_j(d)$, $d \in A_j(d')$, and $d' \in A_j(y)$.

The main characterization result is now presented.

**Theorem 7.** *Let $(X, \precsim)$ be a partially ordered set. Then $\precsim$ is finite multi-utility representable if and only if it is a separable Pareto poset.*

**Proof.** Assume $u : X \to \mathbb{R}^n$, $u = (u_i)_{i=1}^n$, is a finite multi-utility representation of $\precsim$. For each $i \in N$, consider the family of subsets of $X$, $\mathcal{A}_i = (A_i(x))_{x\in X}$, defined as follows:
$A_i(x) = \{y \in X : u_i(y) < u_i(x)\} \bigcup_{j=1}^n \{y \in X : u_i(y) = u_i(x), \ldots, u_{[i+j]}(y) = u_{[i+j]}(x)$ and $u_{[i+j+1]}(y) < u_{[i+j+1]}(x)\} \setminus \{x\}$, where $[i+j]$ denotes the number $i + j$ (module $n$); i.e. the remainder of $i + j$ divided by $n$.

It is routine to check that $\mathcal{A}_i$ is a nonexpansive and exhaustive family of subsets of $X$. Moreover, for any $x \in X$, it holds $\bigcap_{i=1}^n A_i(x) = \{y \in X : u_i(y) \le u_i(x)$ for all $i \in N\} \setminus \{x\} = L_{\prec}(x)$. Thus, $(X, \precsim)$ is a Pareto poset.

To prove that $(X, \precsim)$ is separable, for each $i \in N$, denote by $E_i$ the set of endpoints of $X$ corresponding to the gaps of $u_i(X)$ in $\mathbb{R}$. Note that $E_i$ is countable. In addition, for any $i \in N$ and any pair of rationals $q, q'$ with $q < q'$, pick up, if any, a single point $x(i, q, q') \in X$ such that $q < u_i(x(i, q, q')) < q'$. Let $F_i := \bigcup_{q<q'} \{(x(i, q, q'))\}$ and denote by $D_i := F_i \bigcup E_i$. Clearly, $D_i$ is a countable set of $X$. Finally, define $D := \bigcup_{i=1}^n D_i$. Then, $D$ is a coinitial countable set that satisfies the separability condition. Indeed, $D$ is, obviously, coinitial and countable. In order to see that it satisfies the separability condition let there be

[1] A subset $Y$ of a partially ordered set $(X, \precsim)$ is said to be *coinitial in* $X$ or simply *coinitial*, if $Y$ bounds by below $X$; i.e., for every $x \in X$ there is $y \in Y$ such that $y \precsim x$.

given $i \in N$ and $x \neq y \in X$ such that $x \in A_i(y)$. There are two cases to consider:
(i) $u_i(x) < u_i(y)$. If this situation happens, then either the interval $[u_i(x), u_i(y)]$ includes a gap of $u_i(X)$, or there is a countable dense subset of $u_i(X)$ included in $[u_i(x), u_i(y)]$. In the first case, there are $x', y' \in X$ such that $u_i(x) \leq u_i(x') < u_i(y') \leq u_i(y)$. If $u_i(x) < u_i(x')$ and $u_i(y') < u_i(y)$, then, by definition of $F_i$, either there are points $d, d' \in F_i$ such that $u_i(x) < u_i(d) < u_i(x')$ and $u_i(y') < u_i(d') < u_i(y)$, or the interval $[u_i(x'), u_i(y')]$ defines a gap of $u_i(X)$. If $u_i(x) < u_i(d) < u_i(x')$ and $u_i(y') < u_i(d') < u_i(y)$, then $x \in A_i(d)$, $d \in A_i(d')$, and $d' \in A_i(y)$. If $[u_i(x'), u_i(y')]$ defines a gap of $u_i(X)$, then $x', y' \in E_i$, whence belong to $D_i$. Thus, $x \in A_i(x')$, $x' \in A_i(y')$, and $y' \in A_i(y)$.

In the second case, where there is a countable dense subset of $u_i(X)$ included in $[u_i(x), u_i(y)]$, it is clear, due to the definition of $D_i$, that there exist points $d, d' \in D_i$ such that $u_i(x) < u_i(d) < u_i(d') < u_i(y)$. Thus, $x \in A_i(d)$, $d \in A_i(d')$, $d' \in A_i(y)$, and we are done.
(ii) There is $j \in N$ such that $u_i(x) = u_i(y), \ldots, u_{[i+j]}(y) = u_{[i+j]}(x)$ and $u_{[i+j+1]}(x) \leq u_{[i+j+1]}(y)$. In this situation, by considering now $u_{[i+j+1]}$ instead of $u_i$ and replacing $D_i$ with $D_{[i+j+1]}$, an entirely similar argument to the one used above applies.

Conversely, let $(X, \precsim)$ be a separable Pareto poset. Consider the total orders on $X$, $(\precsim_i)_{i=1}^n$, provided by Lemma 3 (see, also, Remark 5(i)). Note that the separability condition can be rephrased, in terms of $(\precsim_i)_{i=1}^n$, as follows: for any $x, y \in X$, any $i \in N$
$x \prec_i y \Longrightarrow (\exists j \in N)\ (\exists\, d, d' \in D)$ such that $x \precsim_j d \prec_j d' \precsim_j y$.

For each $i \in N$, consider the restriction of $\precsim_i$ to the coinitial countable set $D$, denoted by $\precsim_i|_D$. By Cantor's theorem (see Cantor (1895)) there is a utility function, defined on $D$, that represents $\precsim_i|_D$. Let us denote by $(v_i)_{i=1}^n$ such utility functions. We may assume, without loss of generality, that $v_i$ is bounded, for any $i \in N$. Note, in addition, that, for any distinct $d_1, d_2 \in D$ such that $d_1 \precsim d_2$, it holds $v_i(d_1) < v_i(d_2)$, for any $i \in N$. Define, for each $i \in N$, the real-valued function $u_i : X \to \mathbb{R}$ given by: $u_i(x) = \sup_{d \precsim_i x, d \in D} v_i(d)$. Let us see that the function $u : X \to \mathbb{R}^n$, $u = (u_i)_{i=1}^n$, is a multi-utility representation of $\precsim$. First of all, note that, for every $i \in N$, $u_i$ is well-defined because $D$ bounds by below $X$ and $v_i$ is bounded. Let $x, y \in X$ such that $x \precsim y$. Then, $x \precsim_i y$ holds true for all $i \in N$ because $\precsim_i$ extends $\precsim$ (see Remark 5(i)). This, obviously, entails $u_i(x) \leq u_i(y)$, for all $i \in N$. Thus, $u(x) \leq u(y)$.

Suppose now that, for all $i \in N$, $u_i(x) \leq u_i(y)$ holds true for some distinct $x, y \in X$. Let us prove that $x \precsim_i y$, for all $i \in N$. Assume, otherwise, that there is $p \in N$ such that $y \precsim_p x$ (actually, $y \prec_p x$). Then, by definition of $u_p$, there are no $d, d' \in D$ such that $y \precsim_p d \prec_p d' \precsim_p x$ because, otherwise, it would follow that $u_p(y) \leq u_p(d) = v_p(d) < v_p(d') = u_p(d') \leq u_p(x)$, which contradicts the fact that $u_p(x) \leq u_p(y)$. Note that, a fortiori, it would have $u_p(y) = u_p(x)$. Now, by the separability condition, there would exist $q \in N$, $q \neq p$, and $d, d' \in D$ so that $y \precsim_q d \prec_q d' \precsim_q x$. But this possibility entails $u_q(y) < u_q(x)$, which contradicts the fact that $u_q(x) \leq u_q(y)$. Therefore, $x \precsim_i y$ holds true for all $i \in N$, whence, by Remark 5(i) again, $x \precsim y$ which ends the proof. □

## 4. Relation with Dushnik and Miller theorem

Now, the relationship between our approach and the classic Dushnik and Miller theorem is briefly discussed. For that purpose, the following definition is needed.

**Definition 8.** A partial order $\precsim$ defined on $X$ has the *finite Dushnik and Miller property*, (*finite DM property*, for short) provided that it is the intersection of a finite number of total orders on $X$.

The next result establishes that, for a partially ordered set, the latter property amounts to be a Pareto poset.

**Theorem 9.** *A partially ordered set $(X, \precsim)$ is a Pareto poset if and only if $\precsim$ has the finite DM property.*

**Proof.** Suppose that $(X, \precsim)$ is a Pareto poset. Then, by Lemma 3 and Remark 5(i), it holds that $\precsim = \bigcap_{i=1}^n \precsim_i$. Thus, $\precsim$ has the finite DM property.

Conversely, assume there are total orders on $X$, say $(\precsim^i)_{i=1}^n$, so that $\precsim = \bigcap_{i=1}^n \precsim^i$, and let us prove that $(X, \precsim)$ is a Pareto poset. To that end, for every $i \in N$, $x \in X$, define $A_i(x) := L_{\prec^i}(x)$. Then, it is immediate to check that, for every $i \in N$, $\mathcal{A}_i = (A_i(x))_{x \in X}$ is a nonexpansive and exhaustive family of subsets of $X$ such that $\bigcap_{i=1}^n A_i(x) = \bigcap_{i=1}^n L_{\prec^i}(x) = L_{\prec}(x)$, for any $x \in X$. Therefore, $(X, \precsim)$ is a Pareto poset and we are done. □

In view of Theorems 7 and 9 for a countable poset the following conclusion holds.

**Corollary 10.** *For a partially ordered countable set $(X, \precsim)$ the following assertions are equivalent:*
*(i) $\precsim$ is finite multi-utility representable,*
*(ii) $(X, \precsim)$ is a Pareto poset,*
*(iii) $\precsim$ has the finite DM property.*

**Remark 11.** The countability hypothesis cannot be dropped from the statement of Corollary 10. Indeed, whereas the existence of a finite multi-utility representation entails the fulfillment of the finite DM property, the converse is not true, at large. As an example consider the totally ordered set $(\mathbb{R}^2, \leq_{lex})$, where $\leq_{lex}$ stands for the usual lexicographic order on $\mathbb{R}^2$. Obviously, $(\mathbb{R}^2, \leq_{lex})$ satisfies the finite DM property. However, it is not finite multi-utility representable because, as is well-known, it does not admit a utility function. The same counterexample demonstrates that, in general, a Pareto poset does not necessarily have a multi-utility representation.

## 5. Examples

We now present some examples to illustrate the content and scope of the main concepts and results of the article.
(1) The underlying idea in a Pareto poset suggests the existence of a procedure to determine all finite multi-utility representable posets. We offer an example of a six-point poset through which the main properties analyzed in the paper are developed.

Let there be given $X = \{x_1, \ldots, x_6\}$ and $\precsim := \{(x_1, x_1), \ldots, (x_6, x_6), (x_1, x_4), (x_1, x_5), (x_2, x_5), (x_2, x_6), (x_3, x_4), (x_3, x_6)\}$. It is straightforward to see that $\precsim$, so-defined, is a partial order on $X$. It is not bi-utility representable (see, e.g., Fishburn (1997) or Sprumont (2001)). However, it admits a representation in $\mathbb{R}^3$. Let us provide it by showing that $(X, \precsim)$ is a Pareto poset for $n = 3$. To that end, note that $N = \{1, 2, 3\}$, $i = 1, 2, 3$, and $x_j \in X$, $j = 1, \ldots, 6$. For every $i \in N$, consider the following family of subsets of $X$: $\mathcal{A}_i = A_i(x_j)_{j \in \{1, \ldots, 6\}}$ which has been obtained in accordance with the duality properties given in Definition 4:
$(a_1)$ $A_1(x_1) = \{x_2, x_3, x_6\}$,
$(b_1)$ $A_1(x_2) = \{x_3\}$,
$(c_1)$ $A_1(x_3) = \emptyset$,
$(d_1)$ $A_1(x_4) = \{x_1, x_2, x_3, x_5, x_6\}$,
$(e_1)$ $A_1(x_5) = \{x_1, x_2, x_3, x_6\}$, and
$(f_1)$ $A_1(x_6) = \{x_2, x_3\}$.

$(a_2)$ $A_2(x_1) = \{x_2\}$,
$(b_2)$ $A_2(x_2) = \emptyset$,
$(c_2)$ $A_2(x_3) = \{x_1, x_2, x_5\}$,
$(d_2)$ $A_2(x_4) = \{x_1, x_2, x_3, x_5\}$,
$(e_2)$ $A_2(x_5) = \{x_1, x_2\}$, and
$(f_2)$ $A_2(x_6) = \{x_1, x_2, x_3, x_4, x_5\}$.

$(a_3)$ $A_3(x_1) = \{x_3\}$,
$(b_3)$ $A_3(x_2) = \{x_1, x_3, x_4\}$,
$(c_3)$ $A_3(x_3) = \emptyset$,
$(d_3)$ $A_3(x_4) = \{x_1, x_3\}$,
$(e_3)$ $A_3(x_5) = \{x_1, x_2, x_3, x_4\}$, and

$(f_3)$ $A_3(x_6) = \{x_1, x_2, x_3, x_4, x_5\}$.

It is straightforward to see that the families $(\mathcal{A}_i)_{i=1}^3 = (A_i(x_j))_{j\in\{1,\dots,6\}}$ satisfy the conditions of Definition 4 and, therefore, $(X, \precsim)$ is a Pareto poset. The total orders $(\precsim_i)_{i=1}^3$ are easily calculated and given by:

(i) $x_3 \precsim_1 x_2 \precsim_1 x_6 \precsim_1 x_1 \precsim_1 x_5 \precsim_1 x_4$,

(ii) $x_2 \precsim_2 x_1 \precsim_2 x_5 \precsim_2 x_3 \precsim_2 x_4 \precsim_2 x_6$,

(iii) $x_3 \precsim_3 x_1 \precsim_3 x_4 \precsim_3 x_2 \precsim_3 x_5 \precsim_3 x_6$.

Note that $\precsim = \bigcap_{i=1}^3 \precsim_i$. Moreover, the values of the finite multi-utility representation provided by Theorem 7 are: $u(x_1) = (3,1,1)$, $u(x_2) = (1,0,3)$, $u(x_3) = (0,3,0)$, $u(x_4) = (5,4,2)$, $u(x_5) = (4,2,4)$ and $u(x_6) = (2,5,5)$.[2]

(2) We now exhibit an example for which no finite multi-utility representation does exist.[3] Because it is well-known that any finite poset has a finite multi-utility representation, the example necessarily involves an infinite poset. In fact, it is a denumerable (i.e., countably infinite) poset. Let $X = (x_m)_{m\in\mathbb{Z}\setminus\{0\}}$. Define the following binary relation $\precsim$ on $X$: $x_p \precsim x_q$, provided that either $p = q$ or ($p < 0$, $q > 0$ and $q \neq -p$). Clearly, $(X, \precsim)$ is a poset. Suppose, by way of contradiction, that a finite multi-utility representation does exist. Then, by Corollary 10, $(X, \precsim)$ is a Pareto poset. Thus, there is a collection of subsets of $X$, say $(\mathcal{A}_i)_{i=1}^n = (A_i(x))_{x\in X}$, that satisfy Definition 4. Recall that, for each $i \in N$, and for any distinct $x, y \in X$ it holds that either $x \in A_i(y)$ or $y \in A_i(x)$ (see Remark 5(i)). Now, because $x_{-1} \bowtie x_1$ (i.e., $x_{-1}$ is incomparable to $x_1$), it follows that there is $i_1 \in N$ such that $x_1 \in A_{i_1}(x_{-1})$. Indeed, if for all $i \in N$, $x_{-1} \in A_i(x_1)$, then $x_{-1} \in \bigcap_{i=1}^n A_i(x_1) = L_{\prec}(x_1)$. Thus, $x_{-1} \prec x_1$, which, by definition of $\precsim$, is impossible. Moreover, because for any $i \in N$, $x \in X$, it holds that $L_{\prec}(x) \subseteq A_i(x)$, we have that $x_{-p} \in A_{i_1}(x_1)$, $x_1 \in A_{i_1}(x_{-1})$, and $x_{-1} \in A_{i_1}(x_p)$, for all $p > 1$. By nonexpansiveness of $(A_{i_1}(x))_{x\in X}$, the latter expression entails $x_{-p} \in A_{i_1}(x_p)$, for any $p > 1$. In particular, $x_{-(n+1)} \in A_{i_1}(x_{n+1})$ holds true. Now, because $x_p \in A_{i_1}^{-1}(x_{-p})$, it follows, by exhaustiveness, that there is $i_p \neq i_1 \in N$ such that $x_p \in A_{i_p}(x_{-p})$, for any $p > 1$. In particular, $x_{-(n+1)} \in A_{i_p}(x_{n+1})$ holds true, for any $i_p$. Because the numbers $\{i_p : p = 1, \dots, n\}$ are pairwise distinct, after $n$ steps a rearrangement of $N$ is obtained. Thus, it follows that $x_{-(n+1)} \in \bigcap_{p=1}^n A_{i_p}(x_{n+1}) = \bigcap_{i=1}^n A_i(x_{n+1}) = L_{\prec}(x_{n+1})$, whence $x_{-(n+1)} \prec x_{n+1}$. But this contradicts the fact that $x_{-(n+1)} \bowtie x_{n+1}$. Therefore, $(X, \precsim)$ cannot be a Pareto poset and thus, a finite multi-utility representation cannot exist. Note, however, that an infinite multi-utility representation does exist for $\precsim$. Indeed, for each $p < 0$, define $u(x_p)(k) = 1$, if $k = -p$, and $u(x_p)(k) = 0$, otherwise. For each $p > 0$, define $u(x_p)(k) = 0$, if $k = p$, and $u(x_p)(k) = 1$, otherwise. It is straightforward to see that $u : X \longrightarrow \{0,1\}^{\mathbb{N}\cup\{0\}}$, so-defined, is an infinite multi-utility representation of $\precsim$, the corresponding utilities $(u_i)_{i=0}^\infty$ being the $i$-projections of $u$.

(3) The third example includes the ordered sum of two totally ordered sets. Let there be given two totally ordered sets $(C, \precsim_C)$ and $(D, \precsim_D)$ such that $C \cap D = \emptyset$. Consider the ordered pair $(X = C \cup D, \precsim_X)$, where $\precsim_X$ is defined as follows: $x \precsim_X y$ if and only if $[(x, y \in C) \wedge (x \precsim_C y)]$, or $[(x, y \in D) \wedge (x \precsim_D y)]$. Obviously, $\precsim_X$, so-defined, is a partial order on $X$. Then, $(X, \precsim_X)$ is a Pareto poset for the two following families of subsets $(\mathcal{A}_i)_{i=1,2} = (A_i(x))_{x\in X}$, where $A_1(x) = L_{\prec_X}(x)$, if $x \in C$, and $A_1(x) = L_{\prec_X}(x) \bigcup C$, if $x \in D$. Similarly, $A_2(x) = L_{\prec_X}(x) \bigcup D$, if $x \in C$, and $A_2(x) = L_{\prec_X}(x)$, if $x \in D$.

The two total orders on $X$, $\precsim_1$ and $\precsim_2$, associated with the families $(\mathcal{A}_i)_{i=1,2}$ are: $x \precsim_1 y$ if and only if $[(x, y \in C) \wedge (x \precsim_C y)]$, or $[(x, y \in D) \wedge (x \precsim_D y)]$, or $[(x \in C) \wedge (y \in D)]$; $x \precsim_2 y$ if and only if $[(x, y \in C) \wedge (x \precsim_C y)]$, or $[(x, y \in D) \wedge (x \precsim_D y)]$, or $[(x \in D) \wedge (y \in C)]$. Note that $\precsim_X = \precsim_1 \cap \precsim_2$. In addition, if $\precsim_C$ and $\precsim_D$ are both representable, on $C$ and $D$, respectively, by corresponding utility functions, then, clearly, $(X, \precsim_X)$ is a separable Pareto poset, and $\precsim_1$ and $\precsim_2$ admit also corresponding utility functions on $X$. Let $u, v$ denote these utility functions. Then, the pair $(u, v)$ is a bi-utility representation of $\precsim_X$ on $X$.

## 6. Conclusions

This article provides a comprehensive characterization of partial orders defined on an arbitrary set that admit a finite multi-utility representation. This represents a significant advancement in the literature because it generalizes certain partial results that have recently emerged. Additionally, the article explores the connection between the multi-utility representation problem and the concept of dimension in a partially ordered set (poset), as established by Dushnik and Miller (1941). The article also includes illustrative examples that highlight the main results obtained. Partial orders allowing for multi-utility representation play a crucial role in various fields, including theoretical economics, decision sciences, and optimization.

## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

## Data availability

No data was used for the research described in the article.


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[2] Because $X$ is finite, these values are calculated by taking the cardinality of the corresponding $A_i(x)$.

[3] This example was inspired by Hack et al. (2022), Proposition 7).