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Duality for Poincaré series of surfaces and delta invariant of curves

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Abstract

In this article we study the delta invariant of reduced curve germs via topological techniques. We describe an explicit connection between the delta invariant of a curve embedded in a rational singularity and the topological Poincaré series of the ambient surface. This connection is established by using another formula expressing the delta invariant as 'periodic constants' of the Poincaré series associated with the abstract curve and a 'twisted' duality developed for the Poincaré series of the ambient space.

Keywords: Normal surface singularities, Delta invariant of curves, Poincaré series, Periodic constant, Twisted duality, Rational surface singularities, Weil divisors, Riemann–Roch formula

Mathematics Subject Classification: Primary 14B05, 32Sxx; Secondary 14E15

1 Introduction

In this note we study topological properties of complex analytic normal surface singularities and we also wish to understand how the abstract analytic invariants of curve singularities (e.g. the delta invariant) are topologically obstructed when the curve is embedded in a surface singularity (as reduced Weil divisor). Our motivation is twofolded. Firstly, we wish to generalize certain duality statements satisfied by the topological Poincaré series of a normal surface singularity with a rational homology sphere link. These are high generalizations of (equivariant and multivariable) Ehrhart–MacDonald–Stanley type dualities valid for (certain) rational functions. Their deep applicability was already proven e.g. in [16], here we apply our new generalization to verify a delta invariant formula for an embedded curve germ into a rational singularity.

This leads us to our second motivation: We wish to study the key abstract and embedded invariants of reduced curve germs via topological techniques. Our goal is to find delta invariant formulae for curves embedded in rational surface singularities in terms of embedded topological data. This question was already considered in an earlier work [10] of the authors. However, in the present note we develop a completely different machinery, which produces additional formulae and establishes new connections with the theory of (analytical and topological) multivariable Poincaré series.

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Next we introduce certain notations by focusing on the intersection lattice of a fixed resolution of a normal surface singularity, and then we present the main results. For more details regarding this preliminary part see [10, sect. 2] and [23,24,26,27].

1.1 Let $(X, 0)$ be a normal surface singularity and $(C, 0) \subset (X, 0)$ a reduced curve germ on it. Regarding $(X, 0)$, we will assume that its link Σ is a rational homology sphere, denoted by $\mathbb{Q}HS^3$ (e.g. rational singularities satisfy this restriction).

Then, we fix a good resolution $\pi : \tilde{X} \to X$ of X (that is, the exceptional curve $E := \pi^{-1}(0)$ is a simple normal-crossing divisor). Decompose E into irreducible components $\cup_v E_v$, define $L = H_2(\tilde{X}, \mathbb{Z}) = \mathbb{Z}\langle E_v\rangle_v$ as the lattice of π endowed with the negative definite intersection form $(E_v, E_w)_{v,w}$. The dual lattice L' of L embeds into $L_{\mathbb{Q}} := L \otimes \mathbb{Q}$, and it can be identified with the lattice of rational cycles $\{\ell' \in L_{\mathbb{Q}} : (\ell', L)_{\mathbb{Q}} \in \mathbb{Z}\}$, where $(\,,\,)_{\mathbb{Q}}$ is the extension of $(,\,)$ to $L_{\mathbb{Q}}$ (later still denoted by $(,\,)$). Hence, we regard L' as $\oplus_{v\in V}\mathbb{Z}\langle E_v^*\rangle$, the lattice generated by the rational cycles $\{E_v^*\}_v \in L_{\mathbb{Q}}$, where $(E_u^*, E_v)_{\mathbb{Q}} = -\delta_{u,v}$ (Kronecker delta) for any $u, v \in V$. Then L'/L is the finite group $H_1(\partial\tilde{X}, \mathbb{Z})$ ($\partial\tilde{X} = \Sigma$), which will be denoted by H. We set $[\ell']$ for the class of $\ell' \in L'$ in H.

For $\ell'_1, \ell'_2 \in L_{\mathbb{Q}}$ with $\ell'_i = \sum_v l'_{iv} E_v$ ($i = \{1, 2\}$) one considers an order relation $\ell'_1 \geq \ell'_2$ defined by $l'_{1v} \geq l'_{2v}$ for all $v \in V$. The lattice L' admits a partition indexed by H, where for any $h \in H$ one sets $L'_h = \{\ell' \in L' \mid [\ell'] = h\} \subset L'$. Note that $L'_0 = L$. Given an $h \in H$ one can define $r_h := \sum_v l'_v E_v \in L'_h$ as the unique element of L'_h such that $0 \leq l'_v < 1$.

Let $K_{\tilde{X}}$ be a canonical divisor of the smooth surface $\tilde{X}$. We define $-Z_K$, as the unique element of L' numerically equivalent with $K_{\tilde{X}}$.

By the *adjunction relations*, Z_K is determined topologically by the linear system $(-Z_K + E_v, E_v) + 2 = 0$, for all $v \in V$.

For any nonzero effective cycle $\ell \in L$, by the Riemann–Roch theorem, one obtains $\chi(\mathcal{O}_\ell) = -(\ell, \ell - Z_K)/2$. This motivates to define $\chi(\ell') := -(\ell', \ell' - Z_K)/2$ for any $\ell' \in L'$.

Let $\mathcal{S}'$ be the Lipman (anti-nef) cone

$$\mathcal{S}' = \{\ell' \in L' : (\ell', E_v) \leq 0 \text{ for all } v\}.$$

It is generated over $\mathbb{Z}_{\geq 0}$ by the cycles E_v^*, and it sits in the first quadrant of $L \otimes \mathbb{R}$. It admits an H-indexed partition $\mathcal{S}'_h = \mathcal{S}' \cap L'_h$. One shows that for any $h \in H$ there exists a unique minimal cycle $s_h \in \mathcal{S}'_h$. The cycle s_h is zero if and only if $h = 0$, and it is usually rather arithmetical, cf. [23, Lemma 7.4]. Note that $r_h \leq s_h$, but, in general $r_h \neq s_h$.

Regarding the embedded curve C, let us assume that $\pi : \tilde{X} \to X$ is a good embedded resolution of the pair $C \subset X$, that is, a good resolution of X such that the strict transform $\widetilde{C} \subset \tilde{X}$ of C is smooth and intersects E transversally. We write the *total transform* $\pi^*(C)$ of C as $\pi^*(C) = \ell'_C + \widetilde{C}$, where $\ell'_C \in L'$ is determined uniquely by the property that $\ell'_C + \widetilde{C}$ is a numerically trivial divisor. The embedded topological type of $C \subset X$ (and ℓ'_C too) is determined by the information on how many components of $\widetilde{C}$ intersect each E_v.

To the abstract analytic germ $(C, 0)$ we can associate its *delta invariant* $\delta(C)$ (see 2.3). A key invariants of the embedded topological type (needed below) is the cycle $Z_K + \ell'_C$.

Regarding the curve and normal surface germs, in a series of articles [2–8] A. Campillo, F. Delgado, and S. Gusein-Zade introduced the *multivariable analytical Poincaré series* associated with different filtrations of the corresponding local ring of functions. In this way, for any abstract reduced curve germ $(C, 0)$ one can consider its Poincaré series P_C. For any normal surface singularity one can consider the analytic Poincaré series $P(\mathbf{t})$

associated with the divisorial filtration of a fixed resolution, and the topological Poincaré series $Z(\mathbf{t})$ determined from the resolution graph. By recent results, see e.g. [17,24–27] both series codify a vast amount of information, see Sect. 2.

1.2 The main results

Part I. Let $Z(\mathbf{t}) = \sum_{\ell' \in L'} z(\ell')\mathbf{t}^{\ell'}$ be the topological Poincaré series, where $\mathbf{t}^{\ell'} := \prod_v t_v^{l'_v}$ for any $\ell' = \sum_v l'_v E_v$. Consider its H-decomposition as $\sum_{h\in H} Z_h(\mathbf{t})$, where $Z_h(\mathbf{t}) := \sum_{[\ell']=h} z(\ell')\mathbf{t}^{\ell'}$. Furthermore, for any subset $I \subset V$ we write $Z_h(\mathbf{t}_I) := Z_h(\mathbf{t})|_{t_i=1,\, i\notin I}$. Fix $h \in H$ and $I \subset V$ and define the *counting function of the coefficients of* $Z(\mathbf{t}_I)$ by

$$Q_{h,I} : \{x \in L' \,:\, [x] = h\} \to \mathbb{Z}, \quad Q_{h,I}(x) := \sum_{\ell'|_I \ngeq x|_I,\, [\ell']=[x]} z(\ell').$$

By [16] the *periodic constant* of the series $Z_h(\mathbf{t}_I)$ (defined via the regularization procedure, cf. Sect. 2.1.3) for any $I \subset V,\ I \neq \emptyset$ and $h \in H$ satisfies $\mathrm{pc}(Z_h(\mathbf{t}_I)) = Q_{[Z_K]-h,I}(Z_K - r_h)$. This result transforms the Ehrhart–MacDonald–Stanley reciprocity law from the theory of lattice polytopes to the level of series, formulating a duality between the periodic constant of $Z_h(\mathbf{t}_I)$ and a finite sum of coefficients of the *dual* series $Z_{[Z_K]-h}(\mathbf{t}_I)$.

In this paper, we generalize this duality for the twisted series $Z_{\ell'_0}(\mathbf{t}) := \mathbf{t}^{\ell'_0} \cdot Z(\mathbf{t})$ for some fixed $\ell'_0 \in \mathcal{S}'$ with $[\ell'_0] = h_0$: in Theorem 3.3 we prove the following *twisted duality*

$$\mathrm{pc}((Z_{\ell'_0})_h(\mathbf{t}_I)) = Q_{[Z_K]-h+h_0,I}(Z_K - r_h + \ell'_0). \tag{1}$$

In particular, for $h = 0$ one obtains

$$\mathrm{pc}((Z_{\ell'_0})_0(\mathbf{t}_I)) = Q_{[Z_K]+h_0,I}(Z_K + \ell'_0). \tag{2}$$

The identity (1) has two important specializations: The first one gives back the result from [16], the other one is the case $h = 0$ as stated in (2), which will be applied to an embedded curve germ $(C, 0) \subset (X, 0)$ in Part III.

Part II. Let $(C, 0)$ be an (abstract) reduced curve germ and write $C = \cup_{i\in I} C_i$ as the union of its irreducible components. Denote by $P_{C_J}(\mathbf{t}_J)$ the Poincaré series in variables $\{t_j\}_{j\in J}$ associated with $C_J := \cup_{j\in J} C_j$ for a given $\emptyset \neq J \subset I$ (cf. Sect. 2.3). By [9] (cf. Lemma 2.7) P_{C_J} is a polynomial whenever $|J| > 1$, hence it makes sense to consider the evaluation $P_{C_J}(1_J) := P_{C_J}(\mathbf{t}_J)|_{t_j=1,\forall j\in J}$. Moreover, for a branch C_i let $\mathrm{pc}(P_{C_i})$ be the periodic constant of P_{C_i}. Then in Theorem 4.1 we prove

$$\delta(C_i) = -\mathrm{pc}(P_{C_i}(t_i)); \tag{3}$$

$$\delta(C) = \sum_{i\in I} \delta(C_i) + \sum_{J\subset I,\ |J|>1} (-1)^{|J|}\, P_{C_J}(1_J). \tag{4}$$

Though the theory of delta invariants of curves is a classical subject, to the best of our knowledge, formula (4) is new in the literature. (For the case of plane curves see 4.2.)

Part III. We consider the pair $(C, 0) \subset (X, 0)$ and assume that $(X, 0)$ is rational. Fix a good embedded resolution π of $C \subset X$ and denote by $I_C \subset V$ the set of irreducible exceptional divisors which intersect the strict transform $\widetilde{C}$. Moreover, we choose π in such a way that every component from I_C intersects only one component of $\widetilde{C}$.

Under the assumption that $(X, 0)$ is rational, the first connection between Part I and II is realized by the CDGZ–identity $P_C(\mathbf{t}_{I_C}) = Z_0^C(\mathbf{t}_{I_C})$ [3] (cf. Sect. 2.4), which identifies the

Poincaré series of the curve germ $(C, 0)$ with the reduction of the $h = 0$-part of the *relative topological Poincaré series* $Z^C(\mathbf{t}) = Z(\mathbf{t}) \cdot \Pi_{v\in I_C}(1 - \mathbf{t}^{E^*_v})$.

Then, by the twisted duality we connect the periodic constant of P_C with the counting function of $Z(\mathbf{t})$ evaluated at $Z_K + \ell'_C$. More precisely, we prove the identities

$$\delta(C) = Q_{[Z_K+\ell'_C]}(Z_K + \ell'_C) = \chi(Z_K + \ell'_C) - \chi(s_{[Z_K+\ell'_C]}). \tag{5}$$

2 Preliminaries: multivariable series of surface and curve singularities

In this section we recall and compare four multivariable series: the analytic and topological series associated with a normal surface singularity, the 'relative Poincaré series' associated with an embedded curve singularity into a surface singularity, and also the analytic Poincaré series associated with the analytic type of an abstract isolated curve singularity. Before we provide the concrete definitions and some of the needed properties, we will say a few words about one of the strongest applications of general series, namely about the theory of 'periodic constants'.

2.1 The periodic constant of multivariable series

In this section we will use a slightly more general (abstract) setup of multivariable series. In this way we create the possibility to apply the theory in several different situations (e.g., for the 'reduced versions' of the series when we eliminate some of the variables).

2.1.1 Let L be a lattice freely generated by base elements $\{E_v\}_{v\in V}$, L' is an overlattice of the same rank (not necessarily dual to L), and we set $H := L'/L$, a finite abelian group. The partial ordering is defined as in Sect. 1.1. Let $\mathbb{Z}[[L']]$ be the $\mathbb{Z}$-module consisting of the $\mathbb{Z}$-linear combinations of the monomials $\mathbf{t}^{\ell'} := \prod_{v\in V} t_v^{l'_v}$, where $\ell' = \sum_v l'_v E_v \in L'$. Consider a multivariable series $S(\mathbf{t}) = \sum_{\ell'\in L'} a(\ell')\mathbf{t}^{\ell'} \in \mathbb{Z}[[L']]$ and let $\operatorname{Supp} S(\mathbf{t}) := \{\ell' \in L' \mid a(\ell') \neq 0\}$ be the support of the series. We assume the following finiteness condition:

$$\{\ell' \in \operatorname{Supp} S(\mathbf{t}) \mid \ell' \not\geq x\} \text{ is finite for any } x \in L'. \tag{6}$$

Throughout this paper we will use multivariable series in $\mathbb{Z}[[L']]$ as well as in $\mathbb{Z}[[L'_I]]$ for any $I \subset V$, where $L'_I = \mathrm{pr}_I(L')$ is the projection of L' via $\mathrm{pr}_I : L_{\mathbb{Q}} \to \oplus_{v\in I}\mathbb{Q}\langle E_v\rangle$. For example, if $S(\mathbf{t}) \in \mathbb{Z}[[L']]$ then $S(\mathbf{t}_I) := S(\mathbf{t})|_{t_v=1, v\notin I}$ is an element of $\mathbb{Z}[[L'_I]]$. In the sequel we use the notation $\ell'_I = \ell'|_I := \mathrm{pr}_I(\ell')$ and $\mathbf{t}_I^{\ell'} := \mathbf{t}^{\ell'}|_{t_v=1, v\notin I}$ for any $\ell' \in L'$. Each coefficient $a_I(x)$ of $S(\mathbf{t}_I)$ is obtained as a summation of certain coefficients $a(y)$ of $S(\mathbf{t})$, where y runs over $\{\ell' \in \operatorname{Supp} S(\mathbf{t}) \mid \ell'|_I = x\}$ (this is a finite sum by (6)). Moreover, $S(\mathbf{t}_I)$ satisfies a similar finiteness property as (6) in the variables $\mathbf{t}_I$.

Any $S(\mathbf{t}) \in \mathbb{Z}[[L']]$ decomposes in a unique way as $S(\mathbf{t}) = \sum_h S_h(\mathbf{t})$, where $S_h(\mathbf{t}) := \sum_{[\ell']=h} a(\ell')\mathbf{t}^{\ell'}$. $S_h(\mathbf{t})$ is called the h-part of $S(\mathbf{t})$. Note that the restriction $S_h(\mathbf{t})|_{t_v=1, v\notin I}$ of the h-part $S_h(\mathbf{t})$ cannot be recovered from $S(\mathbf{t}_I)$ in general, since the class of ℓ' cannot be recovered from $\ell'|_I$.

2.1.2 Counting functions

Fix some $I \subset V, I \neq \emptyset$. Given a series $A(\mathbf{t}_I) \in \mathbb{Z}[[L'_I]]$ (e.g., $A(\mathbf{t}_I) = S(\mathbf{t}_I)$ or $A(\mathbf{t}_I) = S_h(\mathbf{t}_I)$ for $h \in H$) two functions can be associated with the coefficients of $A(\mathbf{t}_I)$ (cf. [18,24]). The first one is called the (original) *counting function*:

$$Q(A(\mathbf{t}_I)) : L'_I \longrightarrow \mathbb{Z}, \quad x_I \mapsto \sum_{\ell'_I \not\geq x_I} a(\ell'_I). \tag{7}$$

The second function is called the *modified counting function* and it is defined by

$$q(A(\mathbf{t}_I)) : L'_I \longrightarrow \mathbb{Z}, \quad x_I \mapsto \sum_{\ell'_I \prec x_I} a(\ell'_I), \tag{8}$$

where the order relation $\ell'_I \prec x_I$ means $\ell'_v < x_v$ for all $v \in I$.

Both are well-defined whenever A satisfies the finiteness condition (6). If $A(\mathbf{t}_I) = S_h(\mathbf{t}_I)$, then $Q(A(\mathbf{t}_I))$ will also be denoted by $Q^{(S)}_{h,I}$ while $q(A(\mathbf{t}_I))$ by $q^{(S)}_{h,I}$. The inclusion–exclusion principle connects the two counting functions, namely

$$Q^{(S)}_{h,I}(x) = \sum_{\emptyset \neq J \subset I} (-1)^{|J|+1} q^{(S)}_{h,J}(x) \quad (x \in L'_I). \tag{9}$$

Both counting functions (and hence identity (9) as well) can be extended to all $x \in L'$ via the projection $L' \to L'_I$, as $Q^{(S)}_{h,I}(x) = Q^{(S)}_{h,I}(x_I)$ and similar formulae for $q^{(S)}_{h,I}(x)$.

2.1.3 Periodic constants [17]

Let $S(\mathbf{t}) \in \mathbb{Z}[[L']]$ be a series satisfying the finiteness condition (6) and consider its h-part $S_h(\mathbf{t})$ for a fixed $h \in H$. Let $\mathcal{K} \subset L' \otimes \mathbb{R}$ be a real closed cone whose affine closure is top dimensional. Assume that there exist $\ell'_* \in \mathcal{K}$ and a finite index sublattice $\widetilde{L}$ of L and a quasi-polynomial $\widetilde{Q}^{\mathcal{K},(S)}_{h,V}(x)$ defined on $\widetilde{L}$ such that

$$\widetilde{Q}^{\mathcal{K},(S)}_{h,V}(\ell) = Q^{(S)}_{h,V}(r_h + \ell) \tag{10}$$

whenever $\ell \in (\ell'_* + \mathcal{K}) \cap \widetilde{L}$ ($r_h \in L'$ is defined similarly as in 1.1). Then we say that the counting function $Q^{(S)}_{h,V}$ (or just $S_h(\mathbf{t})$) *admits a quasi-polynomial* in $\mathcal{K}$, namely $\widetilde{L} \ni \ell \mapsto \widetilde{Q}^{\mathcal{K},(S)}_{h,V}(\ell)$. In this case, we define the *periodic constant* of $S_h(\mathbf{t})$ associated with $\mathcal{K}$ by

$$\mathrm{pc}^{\mathcal{K}}(S_h(\mathbf{t})) := \widetilde{Q}^{\mathcal{K},(S)}_{h,V}(0) \in \mathbb{Z}. \tag{11}$$

It is independent of the choice of ℓ'_* and of the finite index sublattice $\widetilde{L} \subset L$.

The same construction can be applied to the modified counting function too: if $q^{(S)}_{h,V}$ admits a quasi-polynomial in $\mathcal{K}$, say $\widetilde{q}^{\mathcal{K},(S)}_{h,V}(x)$, then the *modified periodic constant* of $S_h(\mathbf{t})$ (or simply the periodic constant of $q^{\mathcal{K},(S)}_{h,V}$) associated with $\mathcal{K}$ is defined by

$$\mathrm{mpc}^{\mathcal{K}}(S_h(\mathbf{t})) := \widetilde{q}^{\mathcal{K},(S)}_{h,V}(0). \tag{12}$$

In some cases we might drop the indices $\mathcal{K}$ or S if there is no ambiguity regarding them.

Given any $I \subset V$ the natural group homomorphism $\mathrm{pr}_I : L' \to L'_I$ preserves the lattices $L \to L_I$ and hence it induces a homomorphism $H \to H_I := L'_I/L_I$, denoted by $h \mapsto h_I$. (However, note that even if L' is the dual of L associated with a form $(,)$, L'_I usually is not a dual lattice of L_I, it is just an overlattice. This fact motivates the general setup of the present subsection.) In this *projected context* one can define again the (modified) periodic constant associated with the reduced series $S_h(\mathbf{t}_I)$ from the previous paragraph exchanging V (resp. $\mathbf{t}$, r_h) by I (resp. $\mathbf{t}_I$, $(r_h)_I$).

Example 2.1 The periodic constant of one-variable series was introduced in [28] as follows. For simplicity, we assume that $L = L' \simeq \mathbb{Z}$ and let $S(t) = \sum_{\ell\geq 0} c_\ell t^\ell \in \mathbb{Z}[[t]]$ be a formal power series in one variable. If for some $p \in \mathbb{Z}_{>0}$ the counting function $Q^{(p)}(n) := \sum_{\ell=0}^{pn-1} c_\ell$ is a polynomial $\widetilde{Q}^{(p)}$ in n, then the constant term $\widetilde{Q}^{(p)}(0)$ is independent of p and it is called the periodic constant $\mathrm{pc}(S)$ of the series S. (Here $\widetilde{L} = p\mathbb{Z}$ and the cone is automatically the 'positive cone' $\mathbb{R}_{\geq 0}$.)

2.2 Multivariable series associated with surface singularities

2.2.1 The analytic Poincaré series $P(\mathbf{t})$

We consider a surface singularity $(X, 0)$. We fix a small representative X of $(X, 0)$ and a good resolution π of X. Then, by considering H-equivariant L'-indexed divisorial filtration of the local ring of the universal abelian covering of X, one defines the multivariable *Hilbert series* as

$$H(\mathbf{t}) = \sum_{\ell'\in L'} \mathfrak{h}(\ell')\mathbf{t}^{\ell'} \in \mathbb{Z}[[L']], \tag{13}$$

where $\mathfrak{h}(\ell')$ is called the *Hilbert function*. For the precise definitions and useful expressions for $\mathfrak{h}(\ell')$ we refer to e.g. [10, sections 2.4 and 2.5]. In [2] another series is defined, the *multivariable analytic Poincaré series* $P(\mathbf{t}) = \sum_{\ell'\in L'} \mathfrak{p}(\ell')\mathbf{t}^{\ell'}$. Its definition is

$$P(\mathbf{t}) = -H(\mathbf{t}) \cdot \prod_{v\in V}(1 - t_v^{-1}). \tag{14}$$

Although by considering P instead of H one might in principle loose some information (because $\prod_v(1 - t_v^{-1})$ is a zero divisor), in fact, P and H do contain the same amount of information (see [2, p. 50]). In [26] the next concrete inversion identity is given

$$\mathfrak{h}(\ell') = \sum_{\ell\in L, \ell\ngeq 0} \mathfrak{p}(\ell' + \ell). \tag{15}$$

The analytic Hilbert and Poincaré series in general depend essentially on the analytic type of the singularity, they cannot be expressed merely from the resolution graph.

2.2.2 The topological Poincaré series $Z(\mathbf{t})$

The *multivariable topological Poincaré series* (cf. [2,26]) is defined from the resolution graph of a resolution. It is the Taylor expansion $Z(\mathbf{t}) = \sum_{\ell'} z(\ell')\mathbf{t}^{\ell'} \in \mathbb{Z}[[L']]$ at the origin of the rational *zeta function*

$$f(\mathbf{t}) = \prod_{v\in V}(1 - \mathbf{t}^{E_v^*})^{\mathrm{val}_v - 2}. \tag{16}$$

2.2.3 H –decompositions and reduced versions

In both analytic and topological cases one can consider the H-decomposition $P(\mathbf{t}) = \sum_{h\in H} P_h(\mathbf{t})$ and $Z(\mathbf{t}) = \sum_{h\in H} Z_h(\mathbf{t})$. The supports of $P(\mathbf{t})$ and $Z(\mathbf{t})$ are in the Lipman cone $\mathcal{S}'$ (see e.g. [24, pp. 7–8]). Since the set $\{\ell' \in \mathcal{S}' : \ell' \ngeq x\}$ is finite for any x, both $P(\mathbf{t})$ and $Z(\mathbf{t})$ satisfy (6), hence the corresponding counting functions are also well-defined (and the sum in (15) is also finite).

Furthermore, for any fixed subset $I \subset V$ (and the projection $L' \to L'_I \subset \mathbb{Q}\langle E_v\rangle_{v\in I}$) and for any $h \in H$ we define the *reduced topological Poincaré series* as $Z_h(\mathbf{t}_I) := Z_h(\mathbf{t})|_{t_v=1, v\notin I} \in$

$\mathbb{Z}[[L'_I]]$. Similarly, both the multivariable Hilbert series defined in (13) as well as the analytic Poincaré series (13) can be reduced to $I \subset V$, resulting the series $H_h(\mathbf{t}_I)$ and $P_h(\mathbf{t}_I)$, cf. [24, Theorem 6.1.7].

As a consequence of the inversion identity (15), the counting function for $P_h(\mathbf{t})$ is the Hilbert function $\mathfrak{h}_h(x)$. In particular, in the sequel we will use the notation $\mathfrak{h}_h(x)$ for $Q_h^{(P)}(x)$. Moreover, we will use the simplified notation $Q_h(x)$, or $Q_h^\Gamma(x)$, for the topological counting function $Q_h^{(Z)}(x)$ of $Z_h(\mathbf{t})$, if there is no danger of ambiguity.

2.2.4 Surgery formula for the topological Poincaré series

Let us consider Γ the dual graph of a good resolution of $(X, 0)$ and fix a subset $I \subset V$ of its vertices. The set of vertices $V \setminus I$ determines the connected full subgraphs $\{\Gamma_k\}_k$ with vertices $V(\Gamma_k)$, i.e. $\cup_k V(\Gamma_k) = V \setminus I$. We associate with any Γ_k the lattices $L(\Gamma_k)$ and $L'(\Gamma_k)$ as well, endowed with the corresponding intersection forms.

Then for each k one considers the inclusion operator $j_k : L(\Gamma_k) \to L(\Gamma)$, $E_v(\Gamma_k) \mapsto E_v(\Gamma)$, identifying naturally the corresponding E-base elements associated with the two graphs. Let $j_k^* : L'(\Gamma) \to L'(\Gamma_k)$ be the dual (cohomological) operator, defined by $j_k^*(E_v^*(\Gamma)) = E_v^*(\Gamma_k)$ if $v \in V(\Gamma_k)$, and $j_k^*(E_v^*(\Gamma)) = 0$ otherwise.

Moreover, if Q^Γ and Q^{Γ_k} denote the counting functions associated with the topological Poincaré series of the corresponding graphs, then one has the following surgery formula.

Theorem 2.2 [15, Theorem 3.2.2] *For any* $\ell' = \sum_v a_v E_v^*$ *with* $a_v \gg 0$ *and with the notation* $[\ell'] = h$ *one has the identity*

$$Q_h^\Gamma(\ell') = Q_{h,I}^\Gamma(\ell') + \sum_k Q^{\Gamma_k}_{[j_k^*(\ell')]}(j_k^*(\ell')). \tag{17}$$

2.2.5 Comparison of the analytic and topological series

By [10, 2.5] (see also [26, Prop. 3.2.4]) we obtain that for any $\ell' = r_h + \ell \in Z_K + \mathcal{S}'$ (where $\ell \in L$ and $h = [\ell']$)

$$\mathfrak{h}(\ell') = \chi(\ell') - \chi(r_h) + h^1(\tilde{X}, \mathcal{O}_{\tilde{X}}(-r_h)). \tag{18}$$

The right-hand side is a multivariable polynomial in $\ell \in L$ by the Riemann–Roch formula for χ. This means that $P_h(\mathbf{t})$ admits a (quasi-)polynomial in $\mathcal{S}'_\mathbb{R} := \mathcal{S}' \otimes \mathbb{R}$, hence by (11) its periodic constant is $\mathrm{pc}^{\mathcal{S}'_\mathbb{R}}(P_h(\mathbf{t})) = h^1(\tilde{X}, \mathcal{O}_{\tilde{X}}(-r_h))$, the *equivariant geometric genera of* $(X, 0)$, the H–decomposition of the geometric genus of the universal abelian covering.

Similarly, in the case of the topological series we have an analogous identity too: The only difference is that the equivariant genera are replaced by the Seiberg-Witten invariants of the link. More precisely, [25] shows that for any $\ell' \in Z_K + int(\mathcal{S}')$ (where $int(\mathcal{S}') = \mathbb{Z}_{>0}\langle E_v^* \rangle_{v \in V}$) the counting function of $Z_h(\mathbf{t})$ has the form

$$Q_h(\ell') = \chi(\ell') - \chi(r_h) + \mathfrak{sw}_h^{\mathrm{norm}}(\Sigma), \tag{19}$$

where $\mathfrak{sw}_h^{\mathrm{norm}}(\Sigma)$ denotes the r_h*-normalized Seiberg-Witten invariant* of the link Σ for $h \in H$. Thus $Z_h(\mathbf{t})$ admits the (quasi-)polynomial

$$\widetilde{Q}_h(\ell) = \chi(\ell + r_h) - \chi(r_h) + \mathfrak{sw}_h^{\mathrm{norm}}(\Sigma), \tag{20}$$

in the cone $\mathcal{S}'_\mathbb{R}$ and $\mathrm{pc}^{\mathcal{S}'_\mathbb{R}}(Z_h(\mathbf{t})) = \mathfrak{sw}_h^{\mathrm{norm}}(\Sigma)$. For more details about the Seiberg-Witten invariants of links of singularities we refer the reader to [15,17,23,25].

Proposition 2.3 *If* $(X, 0)$ *is rational, then the following facts hold.*

(a) The quasi-polynomials of the analytic and of the topological Poincaré series coincide: for ℓ' *with sufficiently large* E^**-coefficients one has*

$$\mathfrak{h}_h(\ell') = \chi(\ell') - \chi(r_h) + h^1(\tilde{X}, \mathcal{O}_{\tilde{X}}(-r_h)) = \chi(\ell') - \chi(r_h) + \mathfrak{sw}_h^{norm}(\Sigma) = \widetilde{Q}_h(\ell').$$

This combined with [10, (16)] *and with the vanishing* $h^1(\mathcal{O}_{\tilde{X}}(-s_h)) = 0$ [20] *gives*

$$h^1(\tilde{X}, \mathcal{O}_{\tilde{X}}(-r_h)) = \mathfrak{sw}_h^{norm}(\Sigma) = \chi(r_h) - \chi(s_h).$$

(b) For any $\ell' = \ell + r_h \in Z_K + \mathcal{S}'$ *one has* $\widetilde{Q}_h(\ell) = Q_h(\ell + r_h)$.

Proof The validity of the Seiberg-Witten Invariant Conjecture for rational $(X, 0)$, [23,26,27], implies part (a). Part (b) follows part (a) and (18) valid for any $\ell' \in Z_K + \mathcal{S}'$. □

The identity $h^1(\tilde{X}, \mathcal{O}_{\tilde{X}}(-r_h)) = \mathfrak{sw}_h^{\mathrm{norm}}(\Sigma)$ can be raised to the level of the series as well, formulated by the *CDGZ-identity* (named after Campillo, Delgado and Gusein-Zade).

Theorem 2.4 (CDGZ-identity [2]) *A rational surface singularity* $(X, 0)$ *and any of its resolutions* π *satisfy the identity*

$$P(\mathbf{t}) = Z(\mathbf{t}). \tag{21}$$

2.3 Poincaré series of abstract and embedded curve singularities

Let $(C, 0)$ be an (abstract) reduced curve germ and let $(C, 0) = \bigcup_{i\in I}(C_i, 0)$ be its irreducible decomposition with finite index set I. Let $\gamma_i : (\mathbb{C}, 0) \to (C_i, 0)$, $t \mapsto \gamma_i(t)$, be a normalization of the branch $(C_i, 0)$ and for any germ $g \in \mathcal{O}_{C,0}$ we consider its value $\nu(g) := (\nu_i(g))_{i\in I}$ defined by $\nu_i(g) := \mathrm{ord}_t(g(\gamma_i(t)))$ for any $i \in I$. The product $(\mathbb{C}, 0)^{|I|}$ together with the maps $\{\gamma_i\}_i$ serve as the normalization $\widetilde{(C, 0)} \to (C, 0)$, and $\dim \gamma_* \mathcal{O}_{\widetilde{(C,0)}}/\mathcal{O}_{C,0}$ is the delta invariant $\delta(C, 0)$ of the germ $(C, 0)$.

For any $\ell \in \mathbb{Z}^{|I|}$ one associates the ideal $\mathcal{F}_C(\ell) := \{g \in \mathcal{O}_{C,0} \mid \nu(g) \geq \ell\}$ and the *multivariable Hilbert series*

$$H_C(\mathbf{t}_I) := \sum_{\ell} \mathfrak{h}_C(\ell)\mathbf{t}_I^{\ell},$$

where the Hilbert function is defined by $\mathfrak{h}_C(\ell) := \dim_{\mathbb{C}} \mathcal{O}_{C,0}/\mathcal{F}_C(\ell)$. The *Poincaré series associated with* $(C, 0)$ is

$$P_C(\mathbf{t}_I) := -H_C(\mathbf{t}_I) \cdot \prod_{i\in I}(1 - t_i^{-1}).$$

Thus, if we write $P_C(\mathbf{t}_I) = \sum_{\ell} \mathfrak{p}_C(\ell)\mathbf{t}_I^{\ell}$ then

$$\mathfrak{p}_C(\ell) := \sum_{J\subset I}(-1)^{|J|+1}\mathfrak{h}_C(\ell + 1_J). \tag{22}$$

Let $S_C = \{\nu(g) : g \in \mathcal{O}_{C,0}\} \in \mathbb{N}^{|I|}$ be the semigroup of values with its conductor $c \in \mathbb{N}^{|I|}$. We also use the notation $1_J = \sum_{j\in J} 1_j$ where 1_j is the j-th vector of the canonical basis of $\mathbb{Z}^{|I|}$ and, for convenience $1_J = 0$ if $J = \emptyset$. Then the Hilbert function satisfies the following useful properties.

Lemma 2.5 ([9, Lemma 3.4], see also [21]) *For any* $i \in I$ *fixed,* $\mathfrak{h}_C(\ell + 1_i) = h(\ell) + 1$ *if and only if there exists* $s \in S_C$ *with* $s_i = \ell_i$ *and* $s_j \geq \ell_j$ *for each* j.

Lemma 2.6 [22] *For any* $\ell \in \mathbb{N}^{|I|}$ *with* $\ell \geq c$ *one has*

$$\mathfrak{h}_C(\ell) = |\ell| - \delta(C), \tag{23}$$

where the 'norm' of $\ell = (\ell_i)_{i \in I} \in \mathbb{N}^{|I|}$ *is* $|\ell| := \sum_{i \in I} \ell_i$.

Moreover, Lemma 2.5 and (22) imply the following property.

Lemma 2.7 (cf. [9]) *For* $|I| \geq 2$ *the Poincaré series* $P_C(\mathbf{t}_I)$ *is a polynomial.*

2.3.1 Assume next that $(C, 0)$ is a reduced curve germ embedded in a normal surface singularity $(X, 0)$. We fix a good embedded resolution $\pi : \tilde{X} \to X$ of $C \subset X$, and consider L and L' the corresponding lattices associated with π. Denote by $\widetilde{C}$ the strict transform of C and $I_C \subset V$ be the set of irreducible exceptional divisors which intersect $\widetilde{C}$.

Moreover, we further assume that the resolution π satisfies the following condition

$$\text{every component } E_v \ (v \in I_C \subset V) \text{ of } E \text{ intersects only one component } \widetilde{C}_v \text{ of } \widetilde{C}. \tag{24}$$

This technical condition is not essential, but will simplify the exposition.

In the topological setting one can define the *relative multivariable topological Poincaré series* associated with $(C, 0) \subset (X, 0)$ and π as the modified series

$$Z^C(\mathbf{t}) := Z(\mathbf{t}) \cdot \prod_{v \in I_C} (1 - \mathbf{t}^{E_v^*}). \tag{25}$$

According to Sect. 2, one can consider its H-decomposition $Z^C(\mathbf{t}) = \sum_{h \in H} Z_h^C(\mathbf{t})$ and the reduced series $Z_h^C(\mathbf{t}_I) := Z_h^C(\mathbf{t})|_{t_v = 1, v \notin I}$ as well.

2.4 Comparison of the abstract and relative Poincaré series

Campillo, Delgado and Gusein-Zade proved the following identity relating $P_C(\mathbf{t}_I)$, the abstract Poincaré series of C, and $Z_0^C(\mathbf{t}_I)$, the 0-part of the relative multivariable topological Poincaré series. See also [24, Corollary 9.4.1] in the general equivariant context.

Theorem 2.8 (Relative CDGZ-identity [3]) *For any reduced curve germ* $(C, 0)$ *embedded into the rational singularity* $(X, 0)$ *one has*

$$P_C(\mathbf{t}_I) = Z_0^C(\mathbf{t}_I).$$

3 Twisted duality for the topological Poincaré series $Z(t)$

3.1 Preliminary: Ehrhart–MacDonald–Stanley duality for rational functions [16]

We fix a free $\mathbb{Z}$-module L and an overlattice $L' \supset L$ of the same rank and denote by H the finite quotient group L'/L of order, say d. Let us fix a basis $\{E_v\}_{v \in V}$ in L. In the sequel we will adapt the notation of Sect. 2 as well.

We consider the following type of multivariable rational functions (in the variables $\mathbf{t}^{L'}$) with rational exponents in $\left(\frac{1}{d}\mathbb{Z}\right)^{|V|}$:

$$z(\mathbf{t}) = \frac{\sum_{k=1}^{r} \iota_k \mathbf{t}^{b_k}}{\prod_{i=1}^{n} (1 - \mathbf{t}^{a_i})}, \tag{26}$$

where $\iota_k \in \mathbb{Z}$, $b_k \in L'$, and $a_i = \sum_{v\in V} a_{i,v}E_v \in L'$, such that $a_{i,v}$ are all strictly positive.

Let $Tz(\mathbf{t}) = \sum_{\ell'} z(\ell')\mathbf{t}^{\ell'} \in \mathbb{Z}[[\mathbf{t}^{1/d}]][\mathbf{t}^{-1/d}]$ be the formal Taylor expansion of $z(\mathbf{t})$ at the origin and we also write the Taylor expansion of $z(\mathbf{t})$ at infinity in the form $T^\infty z(\mathbf{t}) = \sum_{\tilde{\ell}} z^\infty(\tilde{\ell})\mathbf{t}^{\tilde{\ell}} \in \mathbb{Z}[[\mathbf{t}^{-1/d}]][\mathbf{t}^{1/d}]$. Note that $T^\infty z$ can also be obtained by the substitution $\mathbf{s} = 1/\mathbf{t}$ into the Taylor expansion at $\mathbf{s} = 0$ of the function $z(1/\mathbf{s})$.

The function z has a decomposition $\sum_{h\in H} z_h(\mathbf{t})$ with respect to $H = L'/L$, where $z_h(\mathbf{t})$ is rational of the form $\sum_{b'\in h+L} \iota_{b'}\mathbf{t}^{b'} / \prod_{i=1}^n (1-\mathbf{t}^{da_i})$ ($\iota_{b'} \neq 0$ for finitely many b'). The decompositions $\sum_h (Tz)_h$ and $\sum_h (T^\infty z)_h$ of the series Tz and $T^\infty z$ are defined similarly as in Sect. 2. Once $h \in H$ fixed, for any subset $I \subset V$ we consider the reduced functions $z_h(\mathbf{t}_I)$ and their series $(Tz)_h(\mathbf{t}_I)$ and $(T^\infty z)_h(\mathbf{t}_I)$ by substituting $t_i = 1$ for all $i \notin I$.

Let $q^{(z)}_{h,I}$ be the modified counting function associated with $(Tz)_h(\mathbf{t}_I)$ as defined in (8). Following the results of [15,18], by the Ehrhart theory of polytopes associated with the denominator of z one can consider the following chamber decomposition of $\mathrm{pr}_I(L'\otimes\mathbb{R}) = \mathbb{R}^{|I|}$: let $\mathcal{B}_I$ be the set of all bases σ of $\mathbb{R}^{|I|}$ such that σ is a subset of the given configuration of vectors $\{a_i|_I\}_{i=1,\dots,n} \cup \{E_v|_I\}_{v\in I}$ in $\mathbb{R}^{|I|}$ (here we use notation $a_i|_I := \mathrm{pr}_I(a_i)$ as in Sect. 2). Then a (big, open) chamber $\mathfrak{c}$ is a connected component of $\mathbb{R}^{|I|} \setminus \cup_{\sigma\in\mathcal{B}_I} \partial\Delta_{\geq 0}\sigma$, where $\partial\Delta_{\geq 0}\sigma$ is the boundary of the closed cone $\Delta_{\geq 0}\sigma$ generated by σ.

Then, by [16, Corollary 3.5.1] it is known that for any fixed chamber $\mathfrak{c}$ the modified counting function $q^{(z)}_{h,I}$ admits a quasi-polynomial in the sense of Sect. 2.1.3. Moreover, if we denote by $(T^\infty z)_h(\mathbf{t}_I) = \sum_{\ell'} z^\infty_{h,I}(\ell')\mathbf{t}_I^{\ell'}$ the h-component of the Taylor expansion of $z(\mathbf{t})$ at infinity and *we assume that for a fixed chamber* $\mathfrak{c}$ *the inclusions* $b_k|_I \in \mathfrak{c}$ *hold for all* k, then [16, Th. 3.6.1] proves that

$$\mathrm{mpc}^{\mathfrak{c}}(z_h(\mathbf{t}_I)) = \sum_{\ell'\geq 0} z^\infty_{h,I}(\ell'). \tag{27}$$

3.2 The twisted zeta function

We now return to the $\mathbb{Q}HS^3$ surface singularity case, where the dual graph Γ is fixed. Consider the zeta function $f(\mathbf{t})$ as defined in (16) and the topological Poincaré series $Z(\mathbf{t})$ as its Taylor expansion at the origin. We recall some important technical results from [18], which will be used in the sequel.

For a subset $\emptyset \neq I \subset V$, we define its closure $\bar{I}$ as the set of vertices of the *connected* minimal full subgraph $\Gamma_{\bar{I}}$ of Γ, which contains I. Note that this graph is unique since Γ is a tree. Denote by $\mathrm{val}_{\bar{I}}(v)$ the valence of a vertex $v \in \bar{I}$ in the graph $\Gamma_{\bar{I}}$.

Then, in [18, Lemma 7] it was proved that $f(\mathbf{t}_I)$ has a product decomposition of type

$$f(\mathbf{t}_I) = \mathrm{Pol}(\mathbf{t}_I) \cdot \prod_{v\in\bar{I}} \left(1-\mathbf{t}_I^{E^*_v}\right)^{\mathrm{val}_{\bar{I}}(v)-2} = \mathrm{Pol}(\mathbf{t}_I)\cdot\mathrm{Prod}(\mathbf{t}_I), \tag{28}$$

where $\mathrm{Pol}(\mathbf{t}_I)$ is a finite sum supported on $\pi_I(\mathcal{S}')$, in particular it has no poles. Hence, the possible set of poles of f via the I-reduction is simplified from the set of zeros of $\prod_{v\in\mathcal{E}}(1-\mathbf{t}_I^{E^*_v})$ to the set of zeros of $\prod_{v\in\mathcal{E}_{\bar{I}}}(1-\mathbf{t}_I^{E^*_v})$, where $\mathcal{E}$ (resp. $\mathcal{E}_{\bar{I}}$) is the set of end-vertices of Γ (resp. $\Gamma_{\bar{I}}$). Note that $\mathcal{E}_{\bar{I}} \subset I$. Therefore, by the construction of the chamber decomposition of $\mathbb{R}^{|I|}$ the chambers associated with $f(\mathbf{t}_I)$ can be determined by the bases selected from the vector configuration $\{E^*_v|_I\}_{v\in\mathcal{E}_{\bar{I}}} \cup \{E_u|_I\}_{u\in I}$. Moreover, one can prove the following:

Proposition 3.1 [18] *For any $I \subset V$ the interior of the projected Lipman cone $int(\pi_I(\mathcal{S}'_{\mathbb{R}}))$ is contained entirely in a (big) chamber $\mathfrak{c}$ of $f(\mathbf{t}_I)$. Thus, the modified counting function $q^{(Z)}_{h,I}$ associated with $Z_h(\mathbf{t}_I)$ admits a quasi-polynomial in $\pi_I(\mathcal{S}'_{\mathbb{R}})$, which will be denoted simply by $\widetilde{q}_{h,I}$ in the sequel.*

Now, we consider the *twisted zeta function* $f_{\ell'_0}(\mathbf{t}) := \mathbf{t}^{\ell'_0} \cdot f(\mathbf{t})$ for some fixed $\ell'_0 \in \mathcal{S}'$ (see Sect. 3.3). By (28) the chamber decomposition associated with $f_{\ell'_0}(\mathbf{t}_I)$ is the same as the one of $f(\mathbf{t}_I)$. Moreover, if $f(\mathbf{t}_I)$ has product factorization $\mathrm{Pol}(\mathbf{t}_I) \cdot \mathrm{Prod}(\mathbf{t}_I)$, then $f_{\ell'_0}(\mathbf{t}_I) = \mathrm{Pol}_{\ell'_0}(\mathbf{t}_I) \cdot \mathrm{Prod}(\mathbf{t}_I)$, where $\mathrm{Pol}_{\ell'_0}(\mathbf{t}_I) = \mathbf{t}_I^{\ell'_0} \cdot \mathrm{Pol}(\mathbf{t}_I)$ is still a finite sum and it is still supported on $\pi_I(\mathcal{S}')$ since $\ell'_0 \in \mathcal{S}'$.

Remark 3.2 The discussion above allows for the extension of the previous results from Sects. 3.1 and 3.2 to the twisted case. In particular, (27) and Proposition 3.1 also hold for the twisted zeta function $f_{\ell'_0}(\mathbf{t}) = \mathbf{t}^{\ell'_0} \cdot f(\mathbf{t})$ as well.

3.3 Periodic constants for twisted functions

Let us fix a cycle $\ell'_0 \in L'$ with $[\ell'_0] = h_0$ as before. Motivated by the previous section, it will be useful to compare invariants of a series with its *twisted series*: for any fixed series $S(\mathbf{t})$ we set the twisted $R(\mathbf{t}) := \mathbf{t}^{\ell'_0} S(\mathbf{t})$. By a straightforward calculation

$$
\begin{aligned}
&R_{h+h_0}(\mathbf{t}) = \mathbf{t}^{\ell'_0} S_h(\mathbf{t}) \\
&Q^{(R)}_{h+h_0}(x) = Q^{(S)}_h(x - \ell'_0).
\end{aligned}
\tag{29}
$$

For any $h \in H$ we define the *dual shift* $\check{\ell}_0(h) \in L'_h$ of ℓ'_0 *with respect to the class* h by

$$
\check{\ell}_0(h) := \ell'_0 + r_{h-h_0}. \tag{30}
$$

For $h = h_0$ the definition gives the cycle itself: $\check{\ell}_0(h_0) = \ell'_0$. We will use the simplified notation $\check{\ell}_0$ for the dual shift $\check{\ell}_0(0)$. In this case one obtains $\check{\ell}_0 \in L$ satisfying $\check{\ell}_0 = \ell'_0 + r_{-h_0}$ (or $\check{\ell}_0 = \lceil \ell'_0 \rceil$). This rewritten in the form $\ell'_0 = \check{\ell}_0 - r_{-h_0}$ can be compared with the usual decomposition $\ell'_0 = \ell_0 + r_h$ for $\ell_0 = \lfloor \ell'_0 \rfloor \in L$. This *symmetry* (and its application in the duality Theorem 3.3) explain the term *dual*. Note that, if $h_0 = 0$, then $\check{\ell}_0 = \ell_0$.

The following relations for the quasi-polynomials associated with the (modified) counting functions are straightforward from (29):

$$
\widetilde{Q}^{(R)}_{h+h_0}(\ell) = \widetilde{Q}^{(S)}_h(\ell + r_h - \check{\ell}_0(h)), \quad \widetilde{q}^{(R)}_{h+h_0}(\ell) = \widetilde{q}^{(S)}_h(\ell + r_h - \check{\ell}_0(h)). \tag{31}
$$

Note that the evaluation of the quasi-polynomial $\widetilde{Q}^{(S)}_h(\ell)$ at zero provides the periodic constant of $S_h(\mathbf{t})$, and evaluation of $\widetilde{Q}^{(R)}_{h+h_0}(\ell)$ at zero is the periodic constant of $\mathbf{t}^{\ell'_0} S_h(\mathbf{t})$:

$$
\mathrm{pc}^{\mathcal{K}}(\mathbf{t}^{\ell'_0} S_h(\mathbf{t})) = \widetilde{Q}^{(S)}_h(r_h - \check{\ell}_0(h)), \quad \mathrm{mpc}^{\mathcal{K}}(\mathbf{t}^{\ell'_0} S_h(\mathbf{t})) = \widetilde{q}^{(S)}_h(r_h - \check{\ell}_0(h)). \tag{32}
$$

Analogously, in the reduced situation, if $I \subset V$, then

$$
\mathrm{pc}^{\mathcal{K}}(\mathbf{t}_I^{\ell'_0} S_h(\mathbf{t}_I)) = \widetilde{Q}^{(S)}_{h,I}(r_h - \check{\ell}_0(h)), \quad \mathrm{mpc}^{\mathcal{K}}(\mathbf{t}_I^{\ell'_0} S_h(\mathbf{t}_I)) = \widetilde{q}^{(S)}_{h,I}(r_h - \check{\ell}_0(h)). \tag{33}
$$

Again, as mentioned after (9), the reduced quasi-polynomials $\widetilde{Q}^{(S)}_{h,I}$ and $\widetilde{q}^{(S)}_{h,I}$ can formally be applied to elements in L via the projection $L \to L_I$.

3.4 Twisted duality for counting functions of Poincaré series

Recall that for any $\emptyset \neq I \subset V$, for simplicity we have denoted by $Q_{h,I}(\ell')$ (resp. $q_{h,I}(\ell')$) the counting function (resp. modified counting function) associated with $Z_h(\mathbf{t}_I)$ for any $h \in H$. They admit quasi-polynomials $\widetilde{Q}_{h,I}(\ell)$ (resp. $\widetilde{q}_{h,I}(\ell)$) associated with the cone $\pi_I(\mathcal{S}'_{\mathbb{R}})$ so that for $\ell' = r_h + \ell \in \mathcal{S}'$ with $\ell \gg 0$ in L one has $Q_{h,I}(\ell') = \widetilde{Q}_{h,I}(\ell)$ (resp. $q_{h,I}(\ell') = \widetilde{q}_{h,I}(\ell)$). In particular, by definition

$$\mathrm{pc}^{\pi_I(\mathcal{S}'_{\mathbb{R}})}(Z_h(\mathbf{t}_I)) = \widetilde{Q}_{h,I}(0) \qquad (\text{resp. } \mathrm{mpc}^{\pi_I(\mathcal{S}'_{\mathbb{R}})}(Z_h(\mathbf{t}_I)) = \widetilde{q}_{h,I}(0)).$$

The aim of this section is to prove the following *twisted duality theorem* for the counting functions of the topological Poincaré series.

Theorem 3.3 *For any fixed $\emptyset \neq I \subset V$, $\ell'_0 \in \mathcal{S}'$ with $[\ell'_0] = h_0$ and $h \in H$ the following identities hold:*

(a) $\mathrm{mpc}^{\pi_I(\mathcal{S}'_{\mathbb{R}})}((f_{\ell'_0})_h(\mathbf{t}_I)) = \widetilde{q}_{h-h_0,I}(r_h - \check{\ell}_0(h)) = q_{[Z_K]-h+h_0,I}(Z_K - r_h + \ell'_0)$,
(b) $\mathrm{pc}^{\pi_I(\mathcal{S}'_{\mathbb{R}})}((f_{\ell'_0})_h(\mathbf{t}_I)) = \widetilde{Q}_{h-h_0,I}(r_h - \check{\ell}_0(h)) = Q_{[Z_K]-h+h_0,I}(Z_K - r_h + \ell'_0)$,

where $\check{\ell}_0(h)$ is the dual shift of ℓ'_0 by h as defined in (30).

Proof Part (a) implies (b) by (9), hence we only have to show (a).

Consider the twisted zeta function $f_{\ell'_0}(\mathbf{t}) = \mathbf{t}^{\ell'_0} \cdot f(\mathbf{t})$. Then for any $\emptyset \neq I \subset V$ and $h \in H$ the Taylor expansion at the origin can be written as $(Tf_{\ell'_0})_h(\mathbf{t}_I) = \mathbf{t}_I^{\ell'_0} \cdot Z_{h-h_0}(\mathbf{t}_I)$. If $q_{h,I}^{(f_{\ell'_0})}(\ell')$ denotes the modified counting function associated with $(Tf_{\ell'_0})_h(\mathbf{t}_I)$, then by (29) one obtains $q_{h,I}^{(f_{\ell'_0})}(\ell') = q_{h-h_0,I}(\ell' - \ell'_0)$. By Remark 3.2, $q_{h,I}^{(f_{\ell'_0})}$ as well as $q_{h-h_0,I}$ admit a quasi-polynomial associated with $\pi_I(\mathcal{S}'_{\mathbb{R}})$. Using (31) one obtains

$$\widetilde{q}_{h,I}^{(f_{\ell'_0})}(\ell) = \widetilde{q}_{h-h_0,I}(\ell + r_h - \check{\ell}_0(h)).$$

Both quasi-polynomials are associated with the cone $\pi_I(\mathcal{S}'_{\mathbb{R}})$. In particular, by (33)

$$\mathrm{mpc}^{\pi_I(\mathcal{S}'_{\mathbb{R}})}((f_{\ell'_0})_h(\mathbf{t}_I)) = \widetilde{q}_{h,I}^{(f_{\ell'_0})}(0) = \widetilde{q}_{h-h_0,I}(r_h - \check{\ell}_0(h)). \tag{34}$$

All the exponents in the numerator of $f_{\ell'_0}(\mathbf{t}_I)$ are situated in $\pi_I(\mathcal{S}'_{\mathbb{R}})$ since the same holds for $f(\mathbf{t}_I)$ and $\ell'_0 \in \mathcal{S}'$. Therefore, (27) can be applied to $f_{\ell'_0}(\mathbf{t}_I)$ such that if the h-component of the Taylor expansion at infinity of $f_{\ell'_0}(\mathbf{t}_I)$ is written as $(T^\infty f_{\ell'_0})_h(\mathbf{t}_I) = \sum_{\tilde{\ell}} (f^\infty_{\ell'_0})_{h,I}(\tilde{\ell})\mathbf{t}_I^{\tilde{\ell}}$ then

$$\mathrm{mpc}^{\pi_I(\mathcal{S}'_{\mathbb{R}})}((f_{\ell'_0})_h(\mathbf{t}_I)) = \sum_{\tilde{\ell} \geq 0} (f^\infty_{\ell'_0})_{h,I}(\tilde{\ell}). \tag{35}$$

On the other hand, using our previous notation $Tf(\mathbf{t}) = Z(\mathbf{t}) = \sum_{\ell' \in \mathcal{S}'} z(\ell')\mathbf{t}^{\ell'}$

$$(T^\infty f_{\ell'_0})(\mathbf{t}_I) = \mathbf{t}_I^{\ell'_0}(T^\infty f)(\mathbf{t}_I) = \mathbf{t}_I^{\ell'_0} \sum_{\ell' \in \mathcal{S}'} z(\ell')\mathbf{t}_I^{Z_K - E - \ell'},$$

where the second identity follows by the symmetry $f(\mathbf{t}_I) = \mathbf{t}_I^{Z_K - E} \cdot f(\mathbf{t}_I^{-1})$. Thus, by (34) and (35) one obtains

$$\mathrm{mpc}^{\pi_I(\mathcal{S}'_{\mathbb{R}})}((f_{\ell'_0})_h(\mathbf{t}_I)) = \widetilde{q}_{h-h_0,I}(r_h - \check{\ell}_0(h)) = \sum_{\substack{\ell'_I \leq (Z_K - E + \ell'_0)_I, \\ [\ell'] = [Z_K] - h + h_0}} z(\ell').$$

Note that since the sum considers only $\ell' \in L'$ with $[\ell'] = [Z_K] - h + h_0$, the condition $\ell'_I \leq (Z_K - E + \ell'_0)|_I$ is equivalent to $\ell'_I \prec (Z_K - r_h + \ell'_0)|_I$. Hence, the sum coincides with the modified counting function $q_{[Z_K]-h+h_0,I}(Z_K - r_h + \ell'_0)$, which proves (a). □

The above twisted duality has two important specializations. The first one gives back the already known duality result proved in [16].

Corollary 3.4 [16, Theorem 4.4.1] *If $(X, 0)$ is a $\mathbb{Q}HS^3$ surface singularity, then*

(a) $\mathrm{mpc}^{\pi_I(\mathcal{S}'_{\mathbb{R}})}(Z_h(\mathbf{t}_I)) = q_{[Z_K]-h,I}(Z_K - r_h)$;
(b) $\mathrm{pc}^{\pi_I(\mathcal{S}'_{\mathbb{R}})}(Z_h(\mathbf{t}_I)) = Q_{[Z_K]-h,I}(Z_K - r_h)$.

Proof Use Theorem 3.3 for $\ell'_0 = 0$. □

The second specialization can be considered as a relative version of the duality and it will be applied in the sequel to the case of an embedded reduced curve germ.

Corollary 3.5 *If $(X, 0)$ is a $\mathbb{Q}HS^3$ surface singularity, then*

(a) $\mathrm{mpc}^{\pi_I(\mathcal{S}'_{\mathbb{R}})}((f_{\ell'_0})_0(\mathbf{t}_I)) = \widetilde{q}_{-h_0,I}(-\check{\ell}_0) = q_{[Z_K]+h_0,I}(Z_K + \ell'_0)$;
(b) $\mathrm{pc}^{\pi_I(\mathcal{S}'_{\mathbb{R}})}((f_{\ell'_0})_0(\mathbf{t}_I)) = \widetilde{Q}_{-h_0,I}(-\check{\ell}_0) = Q_{[Z_K]+h_0,I}(Z_K + \ell'_0)$.

Proof Specialize Theorem 3.3 to $h = 0$. □

4 A $\delta(C, 0)$-formula for the abstract curve

4.1 The formula

Let $(C, 0) = \bigcup_{i\in I}(C_i, 0)$ be the irreducible decomposition of a reduced curve germ and for any $\emptyset \neq J \subset I$ consider the germ $(C_J, 0) = \bigcup_{j\in J}(C_j, 0)$. As in Sect. 2.3 for any $J \subset I$ one defines the Hilbert series $H_{C_J}(\mathbf{t}_J) := \sum_{\ell_J} \mathfrak{h}_{C_J}(\ell_J)\mathbf{t}_J^{\ell_J}$ and via (22) the Poincaré series $P_{C_J}(\mathbf{t}_J) := \sum_{\ell_J} \mathfrak{p}_{C_J}(\ell_J)\mathbf{t}_J^{\ell_J}$ of any such C_J.

The aim of this section is to prove the following formulae for the delta invariant of $(C, 0)$ in terms of the periodic constants associated with these objects.

Theorem 4.1 *(a) For any $i \in I$ one has $\delta(C_i) = -\mathrm{pc}(P_{C_i}(t_i))$, and*
(b) $\delta(C) = \sum_{i\in I} \delta(C_i) + \sum_{J\subset I,\ |J|>1}(-1)^{|J|}\, P_{C_J}(1_J)$.

Note that even though the proof of Theorem 4.1 will be heavily based on the concept of periodic constant, in the identity (b) its presence is completely hidden, even missing.

Recall that the delta invariant can be determined recursively from the delta invariant of the components and the *Hironaka generalized intersection multiplicity* as well. Namely, if we consider two (not necessarily irreducible) germs $(C'_1, 0)$ and $(C'_2, 0)$ without common irreducible components, embedded in some $(\mathbb{C}^n, 0)$ such that $(C'_i, 0)$ is defined by the ideal I_i in $\mathcal{O}_{(\mathbb{C}^n,0)}$ $(i = 1, 2)$, then *Hironaka's intersection multiplicity* is defined by $(C'_1, C'_2)_{Hir} := \dim_{\mathbb{C}} (\mathcal{O}_{(\mathbb{C}^n,0)}/I_1 + I_2)$. Then, one has the following *Hironaka's formula* [14]

$$\delta(C) = \sum_{i\in I} \delta(C_i) + \sum_{i=1}^{r-1}(C_i, C^i)_{Hir}, \quad \text{where } C^i := \cup_{j=i+1}^{r} C_j. \tag{36}$$

Formally (b) has some similarity with (36) but their equivalence is not apparent. Nevertheless, our proof will be independent of (36).

Before we present the proof, in Sect. 4.2 we discuss the formula for two key families: plane curves and ordinary r-tuples in $\mathbb{C}^r$. In a way, they constitute the two *opposite* extreme cases. In our discussion we will stress the major differences between them.

4.2 Special cases

Example 4.2 Assume $(C, 0) \subset (\mathbb{C}^2, 0)$ is an irreducible plane curve singularity. Then the identity $\delta(C) = -\mathrm{pc}(P_C(t))$ is well known, but in a different form, see e.g. [12] and [19, Section 7.1]. Indeed, by [12] the Poincaré series $P_C(t)$ equals the generating series of the numerical semigroup of values associated with $(C, 0)$, hence (via Example 2.1), $-\mathrm{pc}(P_C(t))$ equals the number of gaps, hence the delta invariant of the curve.

Example 4.3 Assume that $(C, 0) = \cup_{i\in I}(C_i, 0)$ $(I = \{1, \dots, r\})$ is a plane curve singularity. In this context there exists the notion of Alexander invariant associated with the covering $\mathbb{C}^2 \setminus C$, hence the (multivariable) Alexander polynomial $\Delta(t_1, \dots, t_r)$ of $(C, 0)$. Then, see e.g. [1,13], $P_{C_i}(t_i) = \Delta_{C_i}(t_i)/(1 - t_i)$ and $P_C(t_1, \dots t_r) = \Delta(t_1, \dots, t_r)$ if $r \geq 2$.

One shows that $\Delta_{C_i}(1) = 1$, and for $i \neq j$ $P_{C_{\{i,j\}}}(1, 1) = (C_i, C_j)$, the (usual) intersection multiplicity (which, in this case, coincides with Hironaka's intersection multiplicity). Furthermore, $P_{C_J}(1_J) = 0$ if $|J| \geq 3$. In particular, part (b) of Theorem 4.1 gives the well-known formula, valid for plane curves, $\delta(C) = \sum_i \delta(C_i) + \sum_{i<j}(C_i, C_j)$.

For certain additional properties in the Gorenstein case see e.g. [21].

Example 4.4 Let $(C, 0)$ be analytically equivalent to a union of r coordinate axes in $\mathbb{C}^r$, that is, $(C, 0)$ is an ordinary r-tuple. Then a computation shows $P_{C_J}(1_J) = 1$ if $|J| \geq 2$. Since $\delta(C_i) = 0$ for all i, Theorem 4.1(b) reads as $\delta(C) = \sum_{j=2}^{r}(-1)^j\binom{r}{j} = r - 1$.

Proof of Theorem 4.1 The proof of Theorem 4.1 is based on the inversion formula of Gorsky and Némethi [11], and Moyano-Fernández [21], which recovers the Hilbert function $\mathfrak{h}_C$ from the Poincaré series P_{C_J}, thus *inverts* formula (22) (and by its *formal* proof, valid for any curve germ $(C, 0)$).

Note that by definition (see Sect. 2.3) if $\ell_i = 0$ for all $i \notin J$, then $\mathfrak{h}_{C_J}(\ell_J) = \mathfrak{h}_C(\ell_I)$, i.e. $H_{C_J}(\mathbf{t}_J) = H_C(\mathbf{t}_I)|_{t_i=1, i\notin J}$. It also makes sense to write $\mathfrak{h}_C(\ell_J)$ so that ℓ_J is extended to a vector ℓ_I by setting its entries indexed by $I \backslash J$ to be zero.

Then, the Hilbert function satisfies the following properties:

(a) $\mathfrak{h}_C(\ell_I) = \mathfrak{h}_C(\max\{\ell_I, 0\})$;
(b) $\mathfrak{h}_C(0) = 0$;
(c) for any $J \subset I$ we have $\mathfrak{p}_{C_J}(\ell_J) = \sum_{I'\subset I}(-1)^{|I'|-1}\mathfrak{h}_C(\ell_J + 1_{I'})$.

Therefore [11, Theorem 3.4.3] applies and it implies the following inversion formula:

$$\mathfrak{h}_C(\ell_I) = \sum_{J\subset I}(-1)^{|J|-1} \sum_{0\leq\tilde{\ell}\leq\ell_J-1_J} \mathfrak{p}_{C_J}(\tilde{\ell}). \tag{37}$$

By Lemma 2.7, $P_{C_J}(\mathbf{t}_J)$ is a polynomial for $|J| > 1$, hence for *big enough* ℓ_J, the constant sum $\sum_{0\leq\tilde{\ell}\leq\ell_J-1_J} \mathfrak{p}_{C_J}(\tilde{\ell})$ is the periodic constant and equals $P_{C_J}(1_J)$.

If $J = \{i\}$ for some $i \in I$, then the counting function is $\sum_{0\leq\tilde{\ell}\leq\ell_i-1} \mathfrak{p}_{C_i}(\tilde{\ell})$ whose constant term is the periodic constant for $\ell_i \gg 0$. On the other hand, by (37) and (23) for $\ell \geq c$, where c is the conductor, we get

$$\delta(C) = |\ell| + \sum_{J\subset I,\ |J|>1} (-1)^{|J|} P_{C_J}(1_J) - \sum_{i\in I} \sum_{0\leq \tilde\ell \leq \ell_i -1} \mathfrak{p}_{C_i}(\tilde\ell).$$

By taking the periodic constant of this identity the result follows. □

5 $\delta(C, 0)$-formulae for the embedded curve

5.1 We consider a reduced curve germ $(C, 0)$ embedded into a *rational* normal surface singularity $(X, 0)$. Our goal is to find a compatibility between the delta invariant of the abstract curve $(C, 0)$ and the embedded topological type of the pair $(C, 0) \subset (X, 0)$.

Let $\pi : \tilde X \to X$ be a good embedded resolution of $(C, 0) \subset (X, 0)$ and denote by Γ its dual resolution graph. We also assume condition (24) for π: the strict transform $\widetilde C$ meets each E_v $(v \in V)$ in at most one point. As before, we denote by $\ell'_C \subset L'$ the exceptional part of π^*C, that is, $\pi^*C = \widetilde C + \ell'_C$ and by $I_C = \{v : E_v \cap \widetilde C \neq \emptyset\}$.

Next we combine the general (abstract) delta invariant formula (Theorem 4.1) and the twisted duality results of Sect. 3.4 to provide two topological expressions in terms of the embedding. The second one was already proved by the authors in [10] by very different analytic methods, here we present a new topological proof. In this way the identities from (38) appear in a new topological perspective: we create a new prototype how dualities, counting function and periodic constant techniques might produce deep formulae.

Theorem 5.1 *If* $(X, 0)$ *is rational and* $(C, 0) \subset (X, 0)$ *is reduced then*

$$\begin{aligned}\delta(C) &= Q_{[Z_K+\ell'_C]}(Z_K + \ell'_C);\\ \delta(C) &=\chi(Z_K + \ell'_C) - \chi(s_{[Z_K+\ell'_C]}) = \chi(-\ell'_C) - \chi(s_{[Z_K+\ell'_C]}).\end{aligned} \tag{38}$$

Proof We can write the cycle associated with the curve $(C, 0) = \cup_{i\in I}(C_i, 0)$ as $\ell'_C = \sum_{i\in I} E^*_i$ where $I = I_C \subset V$. We also define the cycles $\ell'_{C_J} = \sum_{j\in J} E^*_j$ associated with the curve singularity $(C_J, 0) := \cup_{j\in J}(C_j, 0)$ for any $J \subset I$, and denote $h_J := [\ell'_{C_J}] \in H$. For convenience, we set $\ell'_{C_\emptyset} = 0$, so $h_{C_\emptyset} = 0 \in H$.

Consider the relative topological Poincaré series of $(C_J, 0)$ which was defined in (25) by

$$Z^{(C_J)}(\mathbf{t}) = Z(\mathbf{t}) \cdot \prod_{j\in J}(1 - \mathbf{t}^{E^*_j}) \quad \text{for any } J \subset I. \tag{39}$$

Then, as it is explained in Sect. 2.4, one has the decomposition $Z^{(C_J)}(\mathbf{t}) = \sum_{h\in H} Z^{(C_J)}_h(\mathbf{t})$, and the relative CDGZ-identity (Theorem 2.8) provides for $\emptyset \neq J \subset I$:

$$P_{C_J}(\mathbf{t}_J) = Z^{(C_J)}_0(\mathbf{t}_J). \tag{40}$$

For $J = \{j\}$, the formula $\mathrm{pc}(P_{C_j}(t_j)) = \mathrm{pc}(Z^{(C_j)}_0(t_j))$ follows from (40). Otherwise, $|J| \geq 2$, which implies P_{C_J} is a polynomial by Lemma 2.7, and thus $P_{C_J}(1_J) = \mathrm{pc}^{\pi_J(S'_{\mathbb R})}(Z^{(C_J)}_0(\mathbf{t}_J))$.

Moreover, thanks to Theorem 4.1 (b) one can express $\delta(C)$ as

$$\delta(C) = \sum_{\emptyset\neq J\subset I} (-1)^{|J|} \mathrm{pc}^{\pi_J(S'_{\mathbb R})}(Z^{(C_J)}_0(\mathbf{t}_J)). \tag{41}$$

From the definition (39) of $Z^{(C_J)}$, its 0-part can be expressed as

$$Z_0^{(C_J)}(\mathbf{t}_J) = \sum_{K\subset J}(-1)^{|K|}\mathbf{t}_J^{\ell'_{C_K}} \cdot Z_{-h_K}(\mathbf{t}_J).$$

Hence, $Z_0^{(C_J)}(\mathbf{t}_J)$ is the alternating sum of twisted zeta functions $\mathbf{t}_J^{\ell'_{C_K}} Z_{-h_K}(\mathbf{t}_J)$ whose counting functions are $Q_{-h_K,J}(\ell - \ell'_{C_K})$ by Sect. 3.4. Moreover, this implies that

$$D_C(\ell) := \sum_{\emptyset\neq J\subset I}(-1)^{|J|}\sum_{K\subset J}(-1)^{|K|}Q_{-h_K,J}(\ell - \ell'_{C_K}) \tag{42}$$

as a function on L, admits a quasi-polynomial associated with the cone $\pi_J(S'_{\mathbb{R}})$, denoted by $\widetilde{D}_C(\ell)$, and hence (41) implies $\widetilde{D}_C(0) = \delta(C)$.

On the other hand, we can rearrange the terms in the above definition of $D_C(\ell)$ so that

$$D_C(\ell) = Q_{-h_C,I}(\ell - \ell'_C) + \sum_{K\subsetneq I}(-1)^{|K|}R_{-h_K}(\ell - \ell'_{C_K}),$$

where $R_{-h_K}(\ell - \ell'_{C_K}) := \sum_{J\neq\emptyset,\ K\subset J\subset I}(-1)^{|J|}Q_{-h_K,J}(\ell - \ell'_{C_K})$, since the sign of the first term associated with $K = J = I$ is $(-1)^{2|I|}$. Therefore, using the dual shift $\check{\ell}_C$ given by (30), on the quasi-polynomial level one gets

$$\widetilde{D}_C(\ell) = \widetilde{Q}_{-h_C,I}(\ell - \check{\ell}_C) + \sum_{K\subsetneq I}(-1)^{|K|}\widetilde{R}_{-h_K}(\ell - \check{\ell}_{C_K}),$$

with the notation

$$\widetilde{R}_{-h_K}(\ell - \check{\ell}_{C_K}) := \sum_{J\neq\emptyset,\ K\subset J\subset I}(-1)^{|J|}\widetilde{Q}_{-h_K,J}(\ell - \check{\ell}_{C_K}). \tag{43}$$

Then, after substituting $\ell = 0$ above, one can apply the relative duality Corollary 3.5 to $\widetilde{Q}_{-h_C,I}(-\check{\ell}_C)$ which provides the final equation

$$\delta(C) = Q_{[Z_K+\ell'_C],I}(Z_K + \ell'_C) + \sum_{K\subsetneq I}(-1)^{|K|}\widetilde{R}_{-h_K}(-\check{\ell}_{C_K}). \tag{44}$$

Hence, to finish the proof it remains to show the vanishing of the sum in the final equation.

More precisely, we will prove that $\widetilde{R}_{-h_K}(-\check{\ell}_{C_K}) = 0$ for any $K \subsetneq I$.

First, we apply the surgery formula to express $Q_{-h_K,J}(\ell - \ell'_{C_K})$. Namely, for the fixed subsets $K \subset J \subset I \subset V$ ($J \neq \emptyset$) if we denote by $\{\Gamma_k\}_k$ the connected full subgraphs determined by the subset of vertices $V \setminus J$, the surgery formula (17) gives the expression

$$Q_{-h_K,J}(\ell - \ell'_{C_K}) = Q_{-h_K}(\ell - \ell'_{C_K}) - \sum_k Q_0^{\Gamma_k}(\ell|_{\Gamma_k})$$

for a *sufficiently large* ℓ (since $j_k^*(\ell - \ell'_{C_K}) = j_k^*(\ell) =: \ell|_{\Gamma_k}$ whenever $K \subset J$).

This, on the quasi-polynomial level, with the substitution $\ell = 0$ becomes

$$\widetilde{Q}_{-h_K,J}(-\check{\ell}_{C_K}) = \widetilde{Q}_{-h_K}(-\check{\ell}_{C_K}) - \sum_k \widetilde{Q}_0^{\Gamma_k}(0),$$

which implies $\widetilde{Q}_{-h_K,J}(-\check{\ell}_{C_K}) = \widetilde{Q}_{-h_K}(-\check{\ell}_{C_K})$, since the periodic constant $\widetilde{Q}_0^{\Gamma_k}(0)$ associated with the subgraph Γ_k of the dual graph of a rational singularity is 0. This fact can be proved as follows. Any subgraph of a rational graph is rational, hence for this situation one can apply Theorem 2.4. Hence the identities from (18) and (19) can be identified. Hence the topological periodic constant agrees with the analytic periodic constant. But for $h = 0$ the last one is the geometric genus, which is zero in the rational case.

If $K = \emptyset$, then $\check{\ell}_{C_K} = 0$ and hence $\widetilde{Q}_0(0) = 0$ as above. This shows in this case $\widetilde{R}_{-h_K}(-\check{\ell}_{C_K}) = 0$. Otherwise, $K \neq \emptyset$, then $\check{\ell}_{C_K} = \ell_{C_K} + E$ and thus one obtains

$$\widetilde{R}_{-h_K}(-\check{\ell}_{C_K}) = \widetilde{R}_{-h_K}(-\check{\ell}_{C_K}) = \sum_{K \subset J \subset I} (-1)^{|J|} \widetilde{Q}_{-h_K}(-\check{\ell}_{C_K}) = 0,$$

since $\widetilde{Q}_{-h_K}(-\check{\ell}_{C_K})$ does not depend of J and

$$\sum_{K \subset J \subset I} (-1)^{|J|} = (-1)^{|K|} \sum_{k=0}^{|I|-|K|} \binom{|I| - |K|}{k} (-1)^k = (-1)^{|K|} (1-1)^{|I|-|K|} = 0.$$

On the other hand, (20) combined with Proposition 2.3(a) give

$$\widetilde{Q}_{[\ell']}(\ell) = \chi(\ell') - \chi(s_{[\ell']}) \quad \text{where} \quad \ell' = \ell + r_{[\ell']}. \tag{45}$$

This, together with Proposition 2.3(b) implies that

$$Q_{[Z_K+\ell'_C]}(Z_K + \ell'_C) = \widetilde{Q}_{[Z_K+\ell'_C]}(Z_K + \ell'_C) = \chi(Z_K + \ell'_C) - \chi(s_{[Z_K+\ell'_C]}),$$

which completes the proof. □

Remark 5.2 The twisted duality Theorem 3.3(b) with $h = 0$ and $\ell'_0 = \ell'_C$ (and $\check{\ell}'_C = \ell'_C - r_{[-\ell'_C]}$, cf. 3.3) implies $Q_{[Z_K+\ell'_C]}(Z_K + \ell'_C) = \widetilde{Q}_{[-\ell'_C]}(-\check{\ell}'_C)$. Moreover, by applying (45) for $\ell' := -\ell'_C = -\check{\ell}'_C + r_{[-\ell'_C]}$ we obtain $\widetilde{Q}_{[-\ell'_C]}(-\check{\ell}'_C) = \chi(-\ell'_C) - \chi(s_{[-\ell'_C]})$, which implies

$$\delta(C) = \chi(Z_K + \ell'_C) - \chi(s_{[Z_K+\ell'_C]}) = \chi(-\ell'_C) - \chi(s_{[-\ell'_C]}).$$

Furthermore, since $\chi(Z_K + \ell'_C) = \chi(-\ell'_C)$, we get the identity $\chi(s_{[Z_K+\ell'_C]}) = \chi(s_{[-\ell'_C]})$ (not true for general normal surface singularities, cf. [10, Example 4.5]).

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Data availability
Data sharing not applicable to this article as no datasets were generated or analyzed during the current work.

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