Research Article

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Using collocation with radial basis functions in a pseudospectral framework to the analysis of laminated plates by the Reissner's mixed variational theorem

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Abstract: In order to predict the static deformations and free vibration behaviour of thin and thick cross-ply laminated plates, we integrate Carrera's unified formulation with a radial basis function collocation technique on a pseudospectral framework, using Reissner's mixed variational theorem. Numerical examples illustrate the precision and effectiveness of this collocation technique for static and vibration problems.

Keywords: laminated plates, collocation, pseudospectrals, RMVT, unified formulation

1 Introduction

Shear deformation theory and numerical methods can be combined to accomplish an analysis of laminated and sandwich plates. The equivalent single-layer (ESL) or layer-wise (LW) theories are common (quasi-) two-dimensional plate theories. One example of an ESL is the first-order shear deformation plate theory known as the Mindlin theory [1]. To mention a few of the most cited, examples of higher-order ESL theories can be found in the studies by Reddy [2] and Kant [3]. Since every layer in an ESL theory is connected to the same degree of freedom, ESL is a popular choice for multilayer laminate analysis. Layer-dependent degrees of freedom in LW theories, in contrast to ESL, restrict their applicability to laminates with few layers. The zig-zag theories are an economical way to combine ESL with LW [4–7]. Certain challenges arise when analysing layered structures in two dimensions since the mechanical properties at each layer interface are discontinuous, which can lead to large shear and normal transverse strains. The Carrera Unified Formulation (CUF), which contains many kernels for different types of theories, helps simplify the process of computing the equations of motion for plate theories. This automatic method [5–7] can be applied in meshless methods based on collocation with radial basis functions (RBFs) [8–12], or weak-form methods like the finite element method [13,14]. Using the principle of virtual displacements (PVD), the CUF may take into account LW theories or comparable ESL theories. Nevertheless, using the LW formulation in conjunction with Reissner's mixed variational theorem (RMVT) is a more intriguing method (although one that requires more computing power). For the transverse stress and displacement variables, the RMVT takes into account two separate fields. This allows for the achievement of *a priori* interlaminar continuous transverse shear and normal stress fields, which is crucial for sandwich-like structures. Carrera [6,7,15] has information on the RMVT. By incorporating zig-zag effects and interlaminar continuity, this method aims to enhance current shear deformation theories of first-order [1,16] or higher-order [17–21]. In the study by Ferreira *et al.* [22], laminated plate analysis was conducted using the RMVT in conjunction with the RBF collocation approach. In this research, the RBF collocation is extended within a pseudospectral (PS) framework. Fernandes *et al.* [23] used PSs to analyse laminated shells using RMVT, which is more general than the current formulation, but at an extra cost in complexity. Here the RMVT for laminated plates is formulated using a new PS technique to compute static deformations and free vibrations. Rather than utilising Chebyshev polynomials, we use RBFs, which enable the analysis of generic geometries. As far as the author is aware, this work closes the knowledge gap in this field of study.

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2 Collocation with RBFs in PS framework

In PS [24,25], a linear combination of some smooth basis functions represents the spatial component $\hat{u}$ of the approximate solution to a particular partial differential equation (PDE), *i.e.*

$$\hat{u}(x) = \sum_{j=1}^{N} c_j \phi_j(x), \quad x \in \mathbb{R}, \tag{1}$$

ϕ_j, $j = 1, \ldots, N$. To get around the restriction to use only tensor-product grids, we use in this work, RBFs rather than polynomials, which enables us to model irregular grids without loss of accuracy when compared to PS techniques. When utilising Kansa's method [26] to solve linear PDEs using an unsymmetrical RBF collocation approach, it is necessary for both the PDE and the applicable linear boundary conditions to be satisfied at a certain set of collocation points. The system of linear algebraic equations that arises from this method is solved for the coefficients c_j in Eq. (1), which can be used to assess the approximate solution $\hat{u}$ at any point x once these coefficients have been identified. Let us consider the linear problem with Dirichlet boundary condition, $u = g$ on $\Gamma = \partial\Omega$, described by the elliptic PDE

$$\mathcal{L}u = f \quad \text{in } \Omega. \tag{2}$$

The RBF-PS method starts with the following expansion:

$$\hat{u}(\boldsymbol{x}) = \sum_{j=1}^{N} c_j \varphi(\|\boldsymbol{x} - \boldsymbol{\xi}_j\|), \quad \boldsymbol{x} \in \Omega \subseteq \mathbb{R}^s, \tag{3}$$

where the RBF $\phi_j = \varphi(\|\cdot - \boldsymbol{\xi}_j\|)$ has as its *centres* the points ξ_j, $j = 1, \ldots N$. At a set of *collocation points* $\boldsymbol{x}_i$, $i = 1, \ldots, N$, the evaluation of Eq. (3) produces the following results:

$$\hat{u}(\boldsymbol{x}_i) = \sum_{j=1}^{N} c_j \varphi(\|\boldsymbol{x}_i - \boldsymbol{\xi}_j\|), \quad i = 1, \ldots, N,$$

or in the matrix-vector format $\boldsymbol{c} = [c_1, \ldots, c_N]$.

Further details on the current numerical method can be seen in the study of Ferreira and Fasshauer [27] for the computation of an optimal shape parameter, and the computation of the differentiation matrices.

3 Governing equations by RMVT

The RMVT [6,7,15,28] allows us to consider both displacements and transverse shear stresses as primary variables. By using strong-form such as the present RBF-PS methods, we arrive at a coupled system of equations, a set of boundary conditions, and if needed a set of dynamic equations of motion. For all such cases, Carrera proposed an expanding kernel for both displacements and stress, in the form:

$$\begin{aligned} \delta \boldsymbol{u}_s^{k^T}: &\quad \boldsymbol{K}_{uu}^{k\tau s}, \boldsymbol{u}_\tau^k + \boldsymbol{K}_{u\sigma}^{k\tau s} \boldsymbol{\sigma}_{n\tau}^k = \boldsymbol{P}_{u\tau}^k \\ \delta \boldsymbol{\sigma}_{ns}^{k^T}: &\quad \boldsymbol{K}_{\sigma u}^{k\tau s} \boldsymbol{u}_\tau^k + \boldsymbol{K}_{\sigma\sigma}^{k\tau s} \boldsymbol{\sigma}_{n\tau}^k = 0. \end{aligned} \tag{4}$$

The natural boundary conditions (to be imposed on displacements only) can also be expressed as

$$\boldsymbol{\Pi}_u^{k\tau s} \boldsymbol{u}_\tau^k + \boldsymbol{\Pi}_\sigma^{k\tau s} \boldsymbol{\sigma}_{n\tau}^k = \boldsymbol{\Pi}_u^{k\tau s} \bar{\boldsymbol{u}}_\tau^k + \boldsymbol{\Pi}_\sigma^{k\tau s} \bar{\boldsymbol{\sigma}}_{n\tau}^k. \tag{5}$$

Details on the elaboration of the fundamental nucleo can be inspected in the studies by Carrera [6,7,15].

4 Numerical examples

The two examples in this section consider the static analysis of laminated plates, and the free vibration problem for a cross-ply laminated plate.

4.1 Static problems-cross-ply laminated plates

Four equally orientated layers are made up of a simply supported square laminated plate with side a and thickness h, with orientations [0°/90°/90°/0°]. A sinusoidal vertical pressure of the following form is applied to the plate

$$p_z = P \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{a}\right).$$

The coordinate system's origin is situated in the lower left corner of the midplane, and P is the maximum load (note that the load is applied on top of the plate). For every layer, the orthotropic material properties are provided by

$$G_{23} = 0.2E_2 \qquad \nu_{12} = 0.25$$
$$E_1 = 25.0E_2 \qquad G_{12} = G_{13} = 0.5E_2.$$

The normalised forms of the in-plane and transverse shear stresses, normal stresses, and transverse displacements are as follows:

$$w = \frac{10^2 w_{(a/2,a/2,0)} h^3 E_2}{Pa^4}, \quad \sigma_{xx} = \frac{\sigma_{xx(a/2,a/2,h/2)} h^2}{Pa^2}$$
$$\sigma_{yy} = \frac{\sigma_{yy}(a/2, a/2, h/4) h^2}{Pa^2}, \quad \tau_{xz} = \frac{\tau_{xz(0,a/2,0)} h}{Pa}.$$

We show findings for the current RMVT technique employing 13 × 13 up to 21 × 21 points in Table 1. Our

Table 1: [0°/90°/90°/0°] square laminated plate: Transverse displacements and stresses

$\frac{a}{h}$	Method	w	σ_{xx}	σ_{yy}	τ_{zx}
4	Higher-order shear deformation theory (HSDT) [2]	1.8937	0.6651	0.6322	0.2064
	FSDT [30]	1.7100	0.4059	0.5765	0.1398
	Elasticity [29]	1.954	0.720	0.666	0.270
	Present (13 × 13 grid)	1.9784	0.6766	0.5872	0.2332
	Present (17 × 17 grid)	1.9783	0.6766	0.5872	0.2332
	Present (21 × 21 grid)	1.9783	0.6765	0.5872	0.2332
10	HSDT [2]	0.7147	0.5456	0.3888	0.2640
	FSDT [30]	0.6628	0.4989	0.3615	0.1667
	Elasticity [29]	0.743	0.559	0.403	0.301
	Present (13 × 13 grid)	0.7326	0.5627	0.3909	0.3321
	Present (17 × 17 grid)	0.7325	0.5627	0.3908	0.3321
	Present (21 × 21 grid)	0.7325	0.5627	0.3908	0.3321
100	HSDT [2]	0.4343	0.5387	0.2708	0.2897
	FSDT [30]	0.4337	0.5382	0.2705	0.1780
	Elasticity [29]	0.4347	0.539	0.271	0.339
	Present (13 × 13 grid)	0.4308	0.5432	0.2731	0.3774
	Present (17 × 17 grid)	0.4307	0.5431	0.2730	0.3771
	Present (21 × 21 grid)	0.4307	0.5431	0.2730	0.3768

findings are contrasted with those of Reddy [2], Pagano [29], and Reddy and Chao [30] for their first-order shear deformation theory (FSDT) solutions. When compared to the precise solutions, the current RMVT meshless technique yields superior results for transverse displacements, normal stresses, and transverse shear stresses for all ratios. It is evident that thick laminates are unsuitable for use with the FSDT. The solution of stresses σ_{xx} for $a/h = 10$ and $a/h = 4$, respectively, is shown in Figures 1 and 2. The solution of stresses τ_{xz} for $a/h = 10$ and $a/h = 4$, respectively, is shown in Figures 3 and 4. Figures 1 through 4 make use of 21×21 points. Note that each interface's transverse shear stresses are determined straight from the constitutive equations.

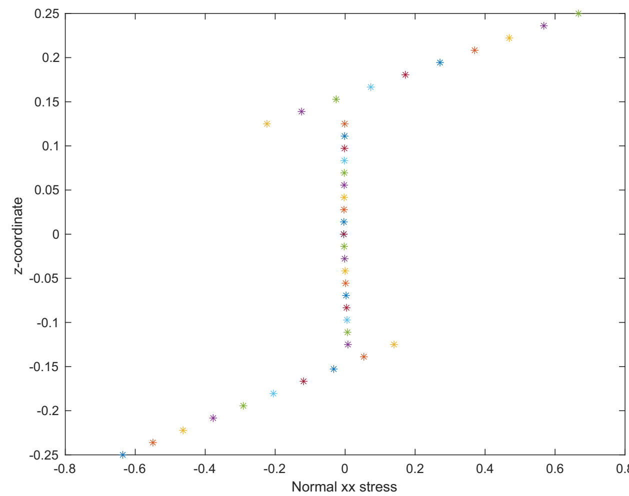

Figure 2: Normalised normal σ_{xx} stress for $a/h = 4$, 21×21 points.

Four equally orientated layers are made up of a simply supported square laminated plate with side a and thickness h, with orientations [0°/90°/90°/0°]. A sinusoidal vertical pressure of the following form is applied to the plate

$$p_z = P \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{a}\right).$$

The coordinate system's origin is situated in the lower left corner of the midplane, and P is the maximum load (note that the load is applied on top of the plate). For every layer, the orthotropic material properties are provided by

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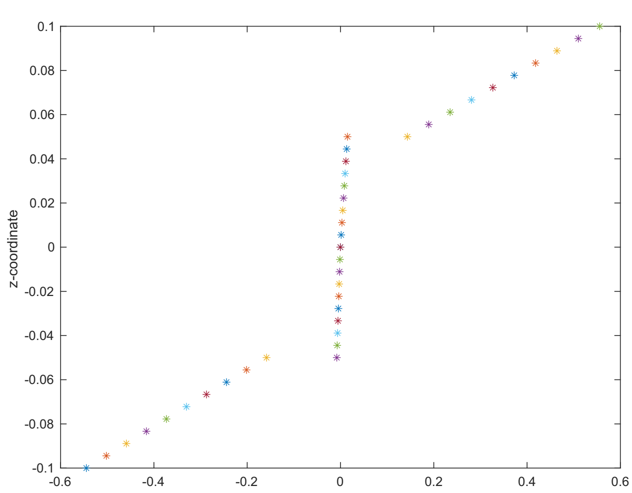

Figure 1: Normalised normal σ_{xx} stress for $a/h = 10$, 21×21 points.

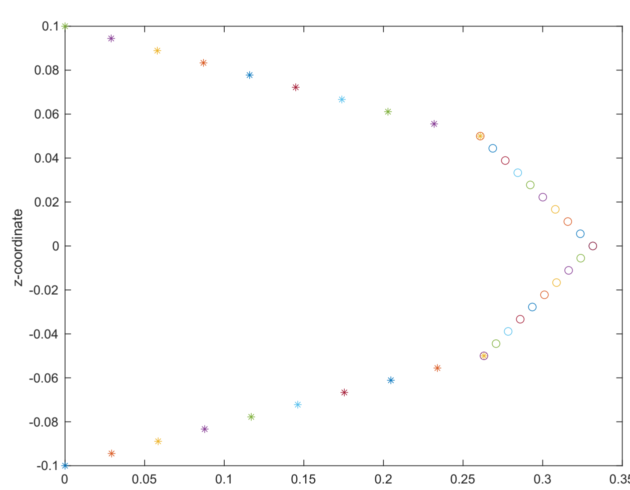

Figure 3: Normalised normal σ_{xz} stress for $a/h = 10$, 21×21 points.

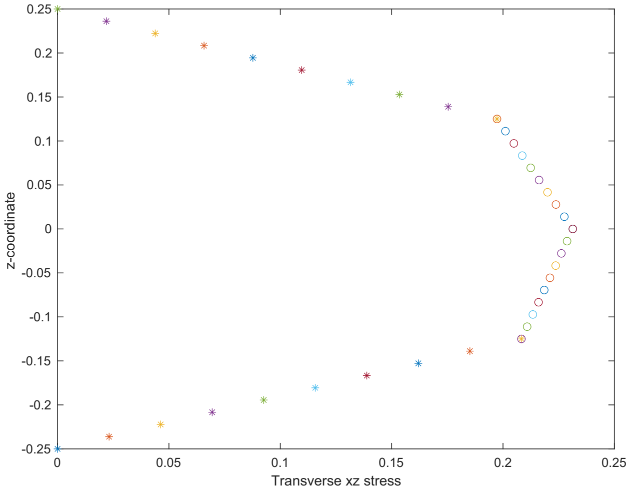

Figure 4: Normalised transverse τ_{xz} stress for $a/h = 4$, 21 × 21 points.

The normalised forms of the in-plane and transverse shear stresses, normal stresses, and transverse displacements are as follows:

$$w = \frac{10^2 w_{(a/2,a/2,0)} h^3 E_2}{Pa^4}, \quad \sigma_{xx} = \frac{\sigma_{xx(a/2,a/2,h/2)} h^2}{Pa^2}$$
$$\sigma_{yy} = \frac{\sigma_{yy}(a/2, a/2, h/4) h^2}{Pa^2}, \quad \tau_{xz} = \frac{\tau_{xz(0,a/2,0)} h}{Pa}.$$

We show findings for the current RMVT technique employing 13 × 13 up to 21 × 21 points in Table 1. Our findings are contrasted with those of Reddy [2], Pagano [29], and Reddy and Chao [30] for their FSDT solutions. When compared to the precise solutions, the current RMVT meshless technique yields superior results for transverse displacements, normal stresses, and transverse shear stresses for all a/h ratios. It is evident that thick laminates are unsuitable for use with the FSDT. The solution of stresses σ_{xx} for $a/h = 10$ and $a/h = 4$, respectively, is shown in Figures 1 and 2. The solution of stresses τ_{xz} for $a/h = 10$ and $a/h = 4$, respectively, is shown in Figures 3 and 4. Figures 1 through 4 make use of 21 × 21 points. Note that each interface's transverse shear stresses are determined straight from the constitutive equations.

Table 2: Normalised fundamental frequency of the simply-supported cross-ply laminated square plate [0°/90°/90°/0°] ($\bar{w} = (wa^2/h)\sqrt{\rho/E_2}$, $h/a = 0.2$)

Method	**Grid**	E_1/E_2			
		10	**20**	**30**	**40**
Liew [32]		8.2924	9.5613	10.320	10.849
Exact (Khdeir and Librescu) [31]		8.2982	9.5671	10.326	10.854
Present ($\nu_{23} = 0.49$)	11 × 11	8.2866	9.5391	10.2676	10.7590
	13 × 13	8.2863	9.5388	10.2673	10.8035
	17 × 17	8.2862	9.5387	10.2672	10.8034

4.2 Free vibration problems-cross-ply laminated plates

It is assumed that the laminate's layers are all composed of the same linearly elastic composite material, have the same thickness, and density. Each layer's material parameters are utilised as follows:

$$\frac{E_1}{E_2} = 10, 20, 30 \quad \text{or} \quad 40; \nu_{12} = 0.25$$
$$G_{12} = G_{13} = 0.6E_2; \quad G_3 = 0.5E_2.$$

In the material properties above, subscripts 1 and 2 indicate the directions normal and transverse to the fibre direction. Every layer's ply angle is calculated by measuring from the fibre direction to the global x-axis.

A simply supported square plate of cross-ply lamination [0°/90°/90°/0°] is taken into consideration. h and a represent the thickness and length of the plate, respectively. In the computation, the thickness-to-span ratio $h/a = 0.2$ is used. The fundamental frequency of the simply supported laminate composed of several E_1/E_2 modulus ratios is listed in Table 2. It can be concluded that the current meshless findings match extremely well with the values in the study by Khdeir and Librescu [31] and the meshfree results in the study by Liew *et al.* [32]. The small variations might result from the current formulation taking through-the-thickness deformations into account.

5 Conclusion

In this study, we developed a PS framework for the prediction of the static deformations and free vibration behaviour of thin and thick cross-ply laminated plates by combining CUF with an RBF collocation technique. Excellent results in terms of transverse displacements, direct transverse stresses at each layer interface, and free vibrations were obtained for the first time when the RMVT and the RBF-PS collocation were combined. There is a knowledge gap in this field of research that is filled by this unique combination of methodologies.

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