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Stability of Fixed Points of Partial Contractivities and Fractal Surfaces

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Abstract: In this paper, a large class of contractions is studied that contains Banach and Matkowski maps as particular cases. Sufficient conditions for the existence of fixed points are proposed in the framework of b-metric spaces. The convergence and stability of the Picard iterations are analyzed, giving error estimates for the fixed-point approximation. Afterwards, the iteration proposed by Kirk in 1971 is considered, studying its convergence, stability, and error estimates in the context of a quasi-normed space. The properties proved can be applied to other types of contractions, since the self-maps defined contain many others as particular cases. For instance, if the underlying set is a metric space, the contractions of type Kannan, Chatterjea, Zamfirescu, Ćirić, and Reich are included in the class of contractivities studied in this paper. These findings are applied to the construction of fractal surfaces on Banach algebras, and the definition of two-variable frames composed of fractal mappings with values in abstract Hilbert spaces.

Keywords: partial contractivity; Kirk iteration; fixed-point theorems; fractal maps; contractions; fractal surfaces; fractal frames

MSC: 26A18; 47H10; 28A80; 54H25; 37C25

1. Introduction

M. Fréchet introduced a mapping to measure what he called "l'écart des deux éléments" (distance between two points) in his doctoral thesis [1], presented at the Faculty of Sciences of Paris and published in Italy in 1906. The conditions of this mapping are the axioms of a metric space. The name of metric space, however, is due to F. Hausdorff, who treated the topic in his book "*Grundzüge der Mergenlehre*" of 1914 [2]. Previously, Hilbert [3] and Riemann [4] had shaken the foundations of classical geometry, proposing new axiomatic systems, with precedents in Gauss, Lobachevsky, and Bolyai.

Nowadays, the conditions of a mapping being a distance have been modified in very different ways, giving rise to a great variety of distance spaces (see, for instance, the books [5,6]).

Particularly interesting are the metrics associated with discrete mathematics, that concerns the knowledge and control of complex systems (see [7]). As an example, one may mention the Hamming distance, that measures the number of different bits of two code words, and it quantifies the error of transmission [8].

In this paper, we work with a generalization of a metric space, called in the literature the b-metric or quasi-metric space, that substitutes the triangular inequality by a more general condition. Closely related to metric theory (that gives rise to a class of topological spaces) is fixed-point theory, that establishes conditions for a self-map $T : X \to X$ in order to have a fixed point. The problem of finding a fixed point is intrinsically linked to the sought for solutions of one or several equations, since the equality $x = Tx$ admits the form $x - Tx = 0$, in the case of an underlying vector space X. Important and recent applications of fixed-point theorems can be found in references [9,10], for instance. But these are not the only implications of the theory, since this area of mathematical knowledge has given rise to modern fields of research like fractal theory and others.

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The content of this paper can be summarized as follows. In Section 2, the dynamics of a large class of contractions called $(\varphi - \psi)$-contractivities [11,12] is explored. Some sufficient conditions for the existence of fixed points are proposed, and the convergence of the Picard iterations for their approximation is studied for both cases, single- and multivalued mappings. Some error estimates for the Picard approximation are also given. For a study of multivalued mappings in b-metric spaces the reader may consult the reference [13].

Section 3 studies the stability of the fixed points' $(\varphi - \psi)$-partial contractivities, proving that they are asymptotically stable in the case of their existence.

Section 4 analyzes the iterative algorithm for fixed-point approximation proposed by Kirk in reference [14], when it is applied to a $(\varphi - \psi)$-contractivity defined on a quasi-normed space.

The properties proved can be applied to other types of contractions, since the self-maps considered contain many others as particular cases. For instance, if the underlaying set is a metric space, the contractions of type Kannan, Chatterjea, Zamfirescu, Ćirić, and Reich are included in the class of contractivities studied in this paper (see Corollary 2.2 of reference [11]).

Section 5 considers fractal surfaces whose values lie on Banach algebras. The mappings defining the surfaces are fixed points of an operator on a Bochner functional space. The convergence and stability of Picard and Kirk iterations for their approximation are analyzed, giving in both cases an estimate of the error.

Section 6 studies a particular case, where the vertical contraction is linear and bounded. Fractal convolutions of mappings and operators are defined, and the construction of bivariate fractal frames of the Bochner space of square-integrable mappings on a Hilbert space is undertaken, considering fractal perturbations of standard frames in the same space.

2. Existence of Fixed Points and Convergence of Picard Iterations

In this section, we explore the dynamics of a large class of contractions [11,12]. We provide sufficient conditions for the existence of fixed points, and the convergence of the Picard iterations for their approximation for both single- and multivalued mappings. Some "a priori" error estimates for the Picard approximation are also given.

Let us start with the definition of b-metric space.

Definition 1. *A b-metric space X is a set endowed with a mapping $d : X \times X \to \mathbb{R}^+$ with the following properties:*

1. $d(x, y) \geq 0$, $d(x, y) = 0$ *if and only if* $x = y$.
2. $d(x, y) = d(y, x)$ *for any* $x, y \in X$.
3. *There exists* $s \geq 1$ *such that* $d(x, y) \leq s(d(x, z) + d(z, y))$ *for any* $x, y, z \in X$.

The constant s is the index of the b-metric space, and d is called a b-metric.

Example 1. *The spaces $l^p(\mathbb{R})$ for $0 < p < 1$ are b-metric spaces of index $s = 2^{1/p}$ with respect to the functional*

$$|x - y|_p = (\sum_{i=1}^{\infty} |x_i - y_i|^p)^{1/p}.$$

Other examples can be found in reference [15], for instance.

Given any $x_0, x_1, \dots x_n \in X$, where X is a b-metric space with index s, one has the following inequality for $n \geq 2$:

$$d(x_0, x_n) \leq \sum_{k=0}^{n-2} s^{k+1} d(x_k, x_{k+1}) + s^{n-1} d(x_{n-1}, x_n) \leq \sum_{k=0}^{n-1} s^{k+1} d(x_k, x_{k+1}). \quad (1)$$

The next definition can be read in reference [16].

Definition 2. *A map $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is called a comparison function if it satisfies the following conditions:*

- *φ is increasing.*
- *$\varphi^n(\delta)$ tends to zero when n tends to infinity for any $\delta > 0$.*

Let X be a b-metric space. A map $T : X \to X$ such that $d(Tx, Ty) \leq \varphi(d(x, y))$ for any $x, y \in X$, where φ is a comparison function, is called a φ-contraction or a Matkowski contraction [17].

Example 2. *The maps $\varphi(\delta) = \delta/(1+\delta)$ and $\varphi(\delta) = r\delta$, where $0 < r < 1$, are comparison functions.*

The first aim of this article is the presentation of a new concept of contractivity, presenting maps that include the usual φ-contractions like a particular case, according to the following definition [11,12].

Definition 3. *Let X be a b-metric space, and $T : X \to X$ be a self-map such that for any $x, y \in X$,*

$$d(Tx, Ty) \leq \varphi(d(x, y)) + \psi(d(x, Tx)). \tag{2}$$

If φ is a comparison function, ψ is positive, and $\psi(0) = 0$, then T is a $(\varphi - \psi)$-partial contractivity. If $\varphi(\delta) = a\delta$, where $0 < a < 1$, and $\psi(\delta) = B\delta$, with $B \geq 0$, T is a partial contractivity.

If ψ is the null function, we have a standard φ-contraction. If further $\varphi(\delta) = a\delta$, where $0 < a < 1$, then T is a Banach contraction.

Example 3. *Let $T(x) = 1/(1+x)$ be defined in $X = [0, +\infty)$. Then, T is a $(\varphi - \psi)$-partial contractivity with $\varphi(\delta) = \delta/(1+\delta)$ and $\psi(\delta) = B\delta$ for $B \geq 0$.*

Example 4. *Let $X = [0, 1]$ and T be defined as $T(x) = x/4$ if $x \in [0, 1/2]$ and $T(x) = x/6$ if $x \in (1/2, 1]$. T is a $(\varphi - \psi)$-partial contractivity with $\varphi(\delta) = \delta/2$ and $\psi(\delta) = \delta$.*

- *If $x, y \in [0, \frac{1}{2}]$, then*

$$|Tx - Ty| = |\frac{x}{4} - \frac{y}{4}| \leq \frac{1}{2}|x - y|,$$

$$|Tx - Ty| = |\frac{x}{4} - \frac{y}{4}| \leq \frac{1}{2}|x - y|.$$

- *If $x, y \in (\frac{1}{2}, 1]$, then*

$$|Tx - Ty| = |\frac{x}{6} - \frac{y}{6}| \leq \frac{1}{2}|x - y|,$$

$$|Tx - Ty| = |\frac{x}{6} - \frac{y}{6}| \leq \frac{1}{2}|x - y|.$$

If $x \in [0, 1/2]$ and $y \in (1/2, 1]$,

$$|Tx - Ty| = |\frac{x}{4} - \frac{y}{6}| \leq |\frac{x}{4}| + |\frac{y}{6}| \leq \frac{1}{3}|y - \frac{y}{6}| + \frac{1}{3}|x - \frac{x}{4}|,$$

$$|Tx - Ty| \leq \frac{1}{3}|x - y| + \frac{1}{3}|x - \frac{y}{6}| + \frac{1}{3}|x - \frac{x}{4}|,$$

$$|Tx - Ty| \leq \frac{1}{3}|x - y| + \frac{2}{3}|x - \frac{x}{4}| + \frac{1}{3}|\frac{x}{4} - \frac{y}{6}|,$$

and finally,

$$|Tx - Ty| \leq \frac{1}{2}|x - y| + |x - \frac{x}{4}|.$$

In the same way,

$$|Tx - Ty| = |\frac{x}{4} - \frac{y}{6}| \leq \frac{1}{3}|x - \frac{x}{4}| + \frac{1}{3}|y - \frac{y}{6}|,$$

$$|Tx - Ty| \leq \frac{1}{3}|x-y| + \frac{1}{3}|y - \frac{x}{4}| + \frac{1}{3}|y - \frac{y}{6}|,$$

$$|Tx - Ty| \leq \frac{1}{3}|x-y| + \frac{1}{3}|y - \frac{y}{6}| + \frac{1}{3}|\frac{y}{6} - \frac{x}{4}| + \frac{1}{3}|y - \frac{y}{6}|,$$

and

$$|Tx - Ty| \leq \frac{1}{2}|x-y| + |y - \frac{y}{6}|.$$

Remark 1. *Let us note that, unlike the φ-contractive case, a partial contractivity need not be continuous.*

In previous articles, we proved that several well known contractivities, like Zamfirescu or quasi-contractions, belong to the class of $(\varphi - \psi)$-partial contractivities, when the constants associated satisfy some restrictions (see, for instance, [12]). The next result can be read in Proposition 15 of the same reference.

Proposition 1. *Let X be a b-metric space and $T : X \to X$ be a $(\varphi - \psi)$-partial contractivity. If T has a fixed point, it is unique.*

We start with a result concerning the orbit separations in the case where $\psi(t) = Bt$, for $B \geq 0$.

Definition 4. *A functional $\varphi : D \subseteq X \to \mathbb{R}$, where X is a real linear space, is sublinear if*

- $\phi(\delta + \delta') \leq \phi(\delta) + \phi(\delta')$ *for any* $\delta, \delta' \in D$ *such that* $\delta + \delta' \in D$.
- $\phi(\lambda\delta) \leq \lambda\phi(\delta)$ *for any* $\lambda > 0$ *and* $\delta \in D$ *such that* $\lambda\delta \in D$.

Example 5. *The absolute value of a real number $\phi(\delta) = |\delta|$ is a sublinear function. In general, a seminorm is sublinear.*

Proposition 2. *Let X be a b-metric space and $T : X \to X$ be a $(\varphi - \psi)$-partial contractivity, where φ is a sublinear comparison function and $\psi(t) = Bt$, for $B \geq 0$. Then, for all $n \geq 1$,*

$$d(T^n x, T^n y) \leq \varphi^n(d(x,y)) + ((\varphi + B.Id)^n - \varphi^n)(d(x,Tx)), \tag{3}$$

where Id denotes the identity map.

If $Fix(T) \neq \emptyset$ and $x^* \in Fix(T)$, then for any $x, y \in X$,

$$d(T^n x, T^n y) \leq s(\varphi^n(d(x,x^*)) + \varphi^n(d(y,x^*))). \tag{4}$$

Consequently, $\lim_{n\to\infty} d(T^n x, T^n y) = 0$.

Proof. For $n = 1$ the result is clear since

$$d(Tx, Ty) \leq \varphi(d(x,y)) + ((\varphi + B.Id) - \varphi)(d(x,Tx)),$$

by definition of $(\varphi - \psi)$-partial contractivity. Let us assume that Formula (3) is valid for $n = k$ and any $x, y \in X$:

$$d(T^k x, T^k y) \leq \varphi^k(d(x,y)) + ((\varphi + B.Id)^k - \varphi^k)(d(x,Tx)).$$

and let us prove it for $n = k+1$. By definition of $(\varphi - \psi)$-partial contractivity for $T^k x, T^k y$,

$$d(T^{k+1}x, T^{k+1}y) \leq \varphi(d(T^k x, T^k y)) + Bd(T^k x, T^{k+1}x).$$

Applying the subadditivity of φ and the inductive hypothesis in the first term of the last sum,

$$\varphi(d(T^k x, T^k y)) \leq \varphi^{k+1}(d(x,y)) + \varphi((\varphi + B.Id)^k - \varphi^k)(d(x,Tx)). \tag{5}$$

For the second summand, applying the inductive hypothesis for x and Tx, we have

$$Bd(T^k x, T^{k+1}x) \leq B\varphi^k(d(x,Tx)) + B((\varphi + B.Id)^k - \varphi^k)(d(x,Tx)) \leq B(\varphi + BId)^k(d(x,Tx)). \tag{6}$$

Let us consider the map $\varphi((\varphi + B.Id)^k - \varphi^k) + B(\varphi + B.Id)^k$. Developing both binomials, and bearing in mind the property of the combinatorial numbers,

$$\binom{k+1}{j} = \binom{k}{j-1} + \binom{k}{j},$$

for $k \in \mathbb{N}$ and $j = 1, \ldots, k$, we obtain that

$$\varphi((\varphi + B.Id)^k - \varphi^k) + B(\varphi + B.Id)^k = (\varphi + B.Id)^{k+1} - \varphi^{k+1}.$$

Thus, adding (5) and (6),

$$d(T^{k+1}x, T^{k+1}y) \leq \varphi^{k+1}(d(x,y)) + ((\varphi + B.Id)^{k+1} - \varphi^{k+1})(d(x,Tx)).$$

and consequently, the result.

If $x^* \in Fix(T)$, then

$$d(T^n x, T^n y) \leq sd(T^n x, x^*) + sd(T^n y, x^*).$$

Applying iteratively the definition of the contractivity,

$$d(T^n x, T^n y) \leq s\varphi^n(d(x,x^*)) + s\varphi^n(d(y,x^*)).$$

The conditions on the comparison function φ imply that

$$d(T^n x, T^n y) \to 0,$$

when n tends to infinity. □

Remark 2. *For the inequality (4), the hypotheses of sublinearity of φ and linearity of ψ are not required.*

Corollary 1. *Let X be a b-metric space and $T : X \to X$ be a partial contractivity, that is to say, $\varphi(\delta) = a\delta$ and $\psi(t) = Bt$, for $B \geq 0$. Then, for all $n \geq 1$,*

$$d(T^n x, T^n y) \leq a^n d(x,y) + ((a+B)^n - a^n)(d(x,Tx)). \tag{7}$$

Consequently, $d(T^n x, T^n y) = O((a+B)^n)$. If $Fix(T) \neq \emptyset$, $d(T^n x, T^n y) = O(a^n)$.

Proof. The rates of orbit separation are straightforward consequences of the expressions (3) and (4). □

Corollary 2. *Let X be a b-metric space and $T : X \to X$ be a partial contractivity, that is to say, $\varphi(\delta) = a\delta$ and $\psi(t) = Bt$, for $B \geq 0$. Then, for all $n \geq 1$,*

$$d(T^n x, T^{n+1}x) \leq (a+B)^n d(x,Tx).$$

If $a + B < 1$, T is asymptotically regular, that it to say,

$$\lim_{n\to\infty} d(T^n x, T^{n+1}x) = 0,$$

and all the orbits are bounded. If $a + B > 1$, the orbit of an element $x \in X$ may be unbounded. For $B = 0$ all the orbits are bounded and they are stable in the sense of Lagrange.

Proof. It suffices to take $y = Tx$ in the inequality (7) to obtain the inequality. The second result comes also from the fact that if $a + B < 1$, T has a fixed point, it is unique (see reference [11]), and all the orbits are convergent to the fixed point, and consequently, they are bounded. □

Remark 3. *In fact, according to (3), T is asymptotically regular if there exists $r \in \mathbb{R}$ with $0 < r < 1$ such that $(\varphi + B.Id)(\delta) \leq r\delta < 1$ for any $\delta > 0$, since substituting $y = Tx$ into inequality (3), $d(T^n x, T^{n+1}x) \leq r^n d(x, Tx)$ and $d(T^n x, T^{n+1}x)$ tends to zero when n tends to infinity.*

Example 6. *The $(\varphi - \psi)$-partial contractivity satisfying the inequality*

$$d(Tx, Ty) \leq \varphi(d(x, y)) + Bd(x, Tx),$$

for $\varphi(\delta) = \delta/(3 + \delta)$ and $B = 1/4$ is asymptotically regular, taking $r = 7/12$.

Proposition 3. *Let X be a b-metric space and $T : X \to X$ be a $(\varphi - \psi)$-partial contractivity. Let us assume that there is a fixed point $x^* \in X$, then for all $n \geq 1$*

$$d(T^n x, x^*) \leq \varphi^n(d(x, x^*)). \tag{8}$$

Consequently, the Picard iterations of any $x \in X$ are convergent to the fixed point and the order of convergence is $\varphi^n(d(x, x^*))$. If $\varphi(\delta) = a\delta$ for $0 < a < 1$, the order of convergence is $O(a^n)$.

Proof. It suffices to apply the definition of $(\varphi - \psi)$-partial contractivity. □

In the following, we give a result of fixed-point existence for $(\varphi - \psi)$-partial contractivities where $\psi(t) = Bt$.

Theorem 1. *If X is a complete b-metric space, $T : X \to X$ is a $(\varphi - \psi)$-partial contractivity where $\psi(\delta) = B\delta$, and there exists a real constant k such that $0 < k < 1$ satisfying the inequality*

$$\varphi_B(\delta) := (\varphi + B.Id)(\delta) \leq k\delta, \tag{9}$$

for all $\delta \geq 0$, then T has a unique fixed point and $T^n x$ tends to x^* for any $x \in X$.

Proof. Defining the sequence $x_n := T^n x$, $x_0 := x$, we have

$$d(x_n, x_{n+1}) = d(T(x_{n-1}), T(x_n)) \leq \varphi(d(x_{n-1}, x_n)) + Bd(x_{n-1}, x_n) = \varphi_B(d(x_{n-1}, x_n)),$$

and

$$d(x_n, x_{n+1}) \leq \varphi_B^n(d(x_0, x_1)). \tag{10}$$

If $\varphi_B(\delta) \leq k\delta$, (x_n) is a Cauchy sequence [18], and consequently, it is convergent to $x^* \in X$, let us see that x^* is a fixed point of T.

$$d(x^*, Tx^*) \leq sd(x_{n+1}, x^*) + sd(Tx_n, x^*) \leq sd(x_{n+1}, x^*) + s\varphi(d(x_n, x^*)) + sBd(x_n, Tx_n).$$

For a comparison function $0 \leq \varphi(\delta) < \delta$, and thus, $lim_{\delta \to 0^+} \varphi(\delta) = 0$. Then, all the right-hand terms tend to zero, and consequently, $x^* = Tx^*$. The uniqueness was proved in Proposition 15 of reference [12]. □

Example 7. *A $(\varphi - \psi)$-partial contractivity where $\varphi(\delta) = \delta/(3 + \delta)$ and $\psi(t) = t/4$ defined on a complete b-metric space has a unique fixed point.*

Remark 4. *For the case $\varphi(\delta) = a\delta$, and $0 < a < 1$, we obtain a partial contractivity, and the sufficient conditions for the existence of fixed point are $a + B < 1$ and X is complete. This result was proved in reference [11].*

In the next theorem, we give an estimation of the error in the fixed-point approximation. Considering the sequence of Picard iterations (x_n) defined as $x_n = T^n x$, it is clear, by definition of $(\varphi - \psi)$-partial contractivity, that

$$d(x_n, x^*) \leq \varphi^n(d(x_0, x^*)). \tag{11}$$

However, it is difficult to know the distance between an element x_0 and the sought fixed point, thus we will give an error estimation in terms of $d(x_0, x_1)$, a quantity easier to find.

Theorem 2. *Let X be a b-metric space, and T be a $(\varphi - \psi)$-partial contractivity, such that $\psi(\delta) = B\delta$. Let us assume that the maps*

$$\phi_n(\delta) := \sum_{k=0}^{\infty} s^k \varphi_B^{n+k}(\delta) \tag{12}$$

are such that $\phi_n(\delta) < \infty$ for all $n = 1, 2, \ldots$. Then, if $x^* \in Fix(T)$ and $x_n := T^n x$,

$$d(x_n, x^*) \leq s^2 \phi_n(d(x_0, x_1)).$$

Proof. Inequality (1) implies that

$$d(x_n, x_{n+j}) \leq \sum_{k=0}^{j-2} s^{k+1} d(x_{n+k}, x_{n+k+1}) + s^{j-1} d(x_{n+j-1}, x_{n+j})$$

Then, using (10),

$$d(x_n, x_{n+j}) \leq s\Big(\sum_{k=0}^{j-2} s^k \varphi_B^{n+k} d(x_0, x_1)\Big) + s^{j-1} \varphi_B^{n+j-1}(d(x_0, x_1)). \tag{13}$$

If $x^* \in Fix(T)$, then

$$d(x_n, x^*) \leq s d(x_n, x_{n+j}) + s d(x_{n+j}, x^*).$$

Using (13) in the first term of the right-hand side,

$$d(x_n, x^*) \leq s^2\Big(\sum_{k=0}^{j-2} s^k \varphi_B^{n+k} d(x_0, x_1)\Big) + s^j \varphi_B^{n+j-1}(d(x_0, x_1)) + s d(x_{n+j}, x^*).$$

Taking limits when j tends to infinity, the second and third terms of the right-hand side tend to zero, due to the hypothesis of the theorem and inequality (11), respectively. Then,

$$d(x_n, x^*) \leq s^2 \phi_n(d(x_0, x_1)) = s^2 \sum_{k=0}^{\infty} s^k \varphi_B^{n+k}(d(x_0, x_1)).$$

□

Corollary 3. *In the particular case where $\varphi(\delta) = a\delta$ with $0 < a < 1$, we have the following "a priori" error estimation for partial contractivities such that $(a + B)s < 1$:*

$$d(x_n, x^*) \leq s^2 \sum_{k=0}^{\infty} s^k (a + B)^{n+k} d(x_0, x_1) = s^2 (a + B)^n \frac{d(x_0, x_1)}{1 - s(a + B)}.$$

In the case where $B = 0$, T is a Banach contraction, and the former expression generalizes the inequality given by Bakhtin [19,20] for this type of map.

The next result concerns set-valued maps satisfying a condition of partial contractivity type. Let us start with some definitions. Given a metric space X, let us consider the set of all nonempty bounded subsets of X, $\mathcal{B}(X)$. Let us define the functional $D : \mathcal{B}(X) \times \mathcal{B}(X) \to \mathbb{R}^+$,

$$D(A, B) = \sup\{d(a, b) : a \in A, b \in B\}.$$

Let us consider the distance of a point to a set $d(a, B) = \inf\{d(a, b) : b \in B\}$.

Theorem 3. *Let X be a complete b-metric space, and let $\tau : X \to \mathcal{B}(X)$ be a set-valued map. Assume that there exist $a \in \mathbb{R}, 0 < a < 1$, and $B > 0$ such that $a + B < 1$, satisfying the inequality*

$$D(\tau(x), \tau(y)) \leq ad(x, y) + Bd(x, \tau(x)) \tag{14}$$

for any $x, y \in X$. Then, τ has a fixed point $x^* \in X$, $\tau(x^*) = \{x^*\}$ and there exists a partial contractivity $T : X \to X$ whose unique fixed point is x^*. Additionally,

$$d(T^n x, x^*) \leq (a + B)^n d(x, x^*)$$

and, if $s(a + B) < 1$,

$$d(T^n x, x^*) \leq s^2 (a + B)^n \frac{d(x, Tx)}{1 - s(a + B)}.$$

Proof. Let us define a map $T : X \to X$ such that $Tx \in \tau(x)$ and let us see that T is a partial contractivity:

$$d(Tx, Ty) \leq D(\tau(x), \tau(y)) \leq ad(x, y) + Bd(x, \tau(x)) \leq ad(x, y) + Bd(x, T(x)),$$

for any $x, y \in X$. Consequently, T is a partial contractivity. Since $a + B < 1$, T has a fixed point $x^* \in X$. By definition of T, $x^* \in \tau(x^*)$, and consequently, it is a fixed point of τ as well. Applying the contractivity condition (14),

$$D(\tau(x^*), \tau(x^*)) \leq ad(x^*, x^*) + Bd(x^*, \tau(x^*)).$$

Since $x^* \in \tau(x^*)$ the terms of the right-hand side are null, and $\tau(x^*)$ reduces to the single point x^*.

The error estimate of the statement was proved in Corollary 3. □

Example 8. *Let us consider the closed unit ball in $\mathbb{R}$, $X = \overline{B}(0, 1) = [-1, 1]$ and $\tau : X \to \mathcal{B}(X)$ defined as $\tau(x) = \overline{B}(\frac{x}{8}, |\frac{x}{8}|)$. Then, $d(x, \tau(x)) = |x - \frac{x}{4}| = |\frac{3x}{4}|$. The map τ fulfills the inequality*

$$D(\tau(x), \tau(y)) \leq \frac{1}{4}|x - y| + \frac{1}{3}d(x, \tau(x))$$

for any $x, y \in X$. The constants a, B satisfy the conditions of Theorem 3 and τ has as a unique fixed point $x = 0$, since $0 \in \tau(0) = \{0\}$ and, if $x \neq 0$, x does not belong to $\tau(x)$.

3. Stability of Fixed Points of a $(\varphi - \psi)$-Partial Contractivity

In this paragraph, we study the fixed-point stability $(\varphi - \psi)$-partial contractivities. We consider $(\varphi - \psi)$-contractivities as described in Definition 3. Let $Fix(T)$ denote the set of fixed points of T. Let us remember that if a $(\varphi - \psi)$-partial contractivity has a fixed point, it is unique. Let $N(x^*)$ denote the set of neighborhoods of $x^* \in X$.

Definition 5. *Le X be a b-metric space, $T : X \to X$ and $G \subseteq X$. Then, G is positively invariant if $T(G) \subseteq G$.*

Proposition 4. *If X is a b-metric space, and T is a $(\varphi - \psi)$-partial contractivity with a fixed point x^*, then any (open or closed) ball centered at the fixed point x^* is a positively invariant set.*

Proof. Let $x \in B(x^*, r)$ for $r > 0$, then applying the definition of $(\varphi - \psi)$-partial contractivity,

$$d(Tx, x^*) \leq \varphi(d(x, x^*)) < d(x, x^*) < r,$$

since a comparison function satisfies the inequality $\varphi(\delta) < \delta$ for all $\delta > 0$ (see, for instance, [16]). Hence, $T(B(x^*, r)) \subseteq B(x^*, r)$. □

Definition 6. *Le X be a b-metric space, $T : X \to X$ and $x^* \in Fix(T)$. Then, x^* is stable if for any $U \in N(x^*)$ if there exists $V \in N(x^*)$ such that $T^n(V) \subseteq U$ for all $n \geq 0$.*

If x^* is stable and there exists $V \in N(x^*)$ such that $\lim_{n\to\infty} T^n x = x^*$ for any $x \in V$, then x^* is asymptotically stable.

Proposition 5. *Le X be a b-metric space, and $T : X \to X$ be a $(\varphi - \psi)$-partial contractivity such that $Fix(T) \neq \emptyset$. Then, the Picard iterations $(T^n x)$ converge for any $x \in X$ to the fixed point with asymptotical stability.*

Proof. Let $U \in N(x^*)$ and $r > 0$ such that $B(x^*, r) \subseteq U$, Then, if $x \in B(x^*, r)$,

$$d(T^n x, x^*) \leq \varphi^n(d(x, x^*)) \leq d(x, x^*) < r.$$

Consequently, $T^n(B(x^*, r)) \subseteq B(x^*, r) \subseteq U$ for all $n \geq 0$, and x^* is stable. Moreover,

$$d(T^n x, x^*) \leq \varphi^n(d(x, x^*)) \to 0,$$

due to the definition of the comparison function. Hence, x^* is asymptotically stable, and it is a global attractor, as proved previously. □

4. Convergence and Stability of the Kirk Iterations

In this section, the iterative algorithm for fixed-point approximation proposed by Kirk in reference [14] is analyzed for when it is applied to a $(\varphi - \psi)$-contractivity defined on a quasi-normed space.

Definition 7. *If E is a real linear space, the mapping $|\cdot|_s : E \times E \to \mathbb{R}^+$ is a quasi-norm of index s if*

1. *$|f|_s \geq 0$; $f = 0$ if and only if $|f|_s = 0$.*
2. *$|\lambda f|_s = |\lambda||f|_s$.*
3. *There exists $s \geq 1$ such that $|f + g|_s \leq s(|f|_s + |g|_s)$ for any $f, g \in E$.*

The space $(E, |\cdot|_s)$ is a quasi-normed space. If E is complete with respect to the b-metric induced by the quasi-norm, then E is a quasi-Banach space. Obviously, if $s = 1$ then E is a normed space.

The index of a quasi-norm is called sometimes the modulus of concavity of X.

Example 9. *The spaces $l^p(\mathbb{R})$ for $0 < p < 1$ are quasi-normed spaces of index $s = 2^{1/p}$ with respect to the functional*

$$|x|_p = (\sum_{i=1}^{\infty} |x_i|^p)^{1/p}.$$

Kirk's algorithm [14] is given by the scheme of order k:

$$y_{n+1} = \sum_{i=0}^{k} \alpha_i T^i y_n, \tag{15}$$

where $k \in \mathbb{N}$, $\alpha_k > 0$, $\alpha_i \geq 0$, for $i = 1, 2, \ldots, k-1$, $\sum_{i=0}^{k} \alpha_i = 1$, and $y_0 \in X$. For $k = 1$, the algorithm agrees with the Krasnoselskii method [21]. If additionally the coefficients change at every step, one has the Mann iteration [22].

Let us define the Kirk operator $T_K : X \to X$, where X is a quasi-normed space, as

$$T_K x = \sum_{i=0}^{k} \alpha_i T^i x.$$

Kirk proved that the set of fixed points of a nonexpansive mapping T in a Banach space agrees with the set of fixed points of T_K, that is to say, $Fix(T) = Fix(T_K)$. He proved also that if $Fix(T) \neq \emptyset$ and X is uniformly convex, then T_K is asymptotically regular, that is to say,

$$\lim_{n\to\infty} ||T_K^{n+1}x - T_K^n x|| = 0.$$

Let us study the convergence and stability of this algorithm for the approximation of the fixed point of a $(\varphi - \psi)$-partial contractivity T in a quasi-normed space.

If $y_0 := y \in X$ and $x^* \in Fix(T)$,

$$|y_{n+1} - x^*|_s = |\sum_{i=0}^{k} \alpha_i (T^i y_n - x^*)|_s \leq \sum_{i=0}^{k-1} \alpha_i s^{i+1} |T^i y_n - x^*|_s + \alpha_k s^k |T^k y_n - x^*|_s.$$

Applying the definition of $(\varphi - \psi)$-partial contractivity,

$$|y_{n+1} - x^*|_s \leq \sum_{i=0}^{k-1} \alpha_i s^{i+1} \varphi^i(|y_n - x^*|_s) + \alpha_k s^k \varphi^k(|y_n - x^*|_s), \tag{16}$$

defining $\varphi^0 := Id$ as always. Let us assume that the map $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ defined as

$$\phi = \sum_{i=0}^{k-1} \alpha_i s^{i+1} \varphi^i + \alpha_k s^k \varphi^k. \tag{17}$$

is a comparison function. In this case, we have

$$|y_{n+1} - x^*|_s \leq \phi(|y_n - x^*|_s),$$

and, in general,

$$|y_n - x^*|_s \leq \phi^n(|y_0 - x^*|_s). \tag{18}$$

Consequently, the Kirk iterations are convergent to the fixed point x^* with asymptotic stability as in the previous section. For the particular case where $\varphi(\delta) = a\delta$, assuming that

$$r := \sum_{i=0}^{k-1} \alpha_i s^{i+1} a^i + \alpha_k s^k a^k < 1,$$

the Kirk algorithm is convergent and stable since, from (18), we have

$$|y_n - x^*|_s \leq r^n |y_0 - x^*|_s. \tag{19}$$

The order of convergence of the iteration is $O(r^n)$. In the normed case, where $s = 1$, this is always true since

$$r = \sum_{i=0}^{k} \alpha_i a^i < 1,$$

due to the conditions on α_i, and we have the following theorem:

Theorem 4. *If X is a normed space, $T : X \to X$ is a $(\varphi - \psi)$-partial contractivity where $\phi(\delta) = a\delta$, $0 < a < 1$, and $x^* \in X$ is a fixed point, the Kirk iteration $(T_K^n x)$ is convergent, asymptotically stable and asymptotically regular for any values of α_i and $x \in X$.*

Kirk proved that the iterates of his algorithm converge weakly to a fixed point of a nonexpansive mapping, according to the next theorem [14].

Theorem 5. *Let X be a uniformly convex Banach space, K be a closed, bounded, and convex subset of X, and $T : K \to K$ be a nonexpansive mapping. Then, for $x \in K$ the sequence $(T_K^n x)$ converges weakly to a fixed point of T.*

We give in the following a variant of this theorem.

Theorem 6. *Let X be a quasi-Banach space, K be a closed and convex subset of X, and $T : K \to K$ be a $(\varphi - \psi)$-contraction where ψ is the null function. Then, $Fix(T) \neq \emptyset$, $Fix(T) = \{x^*\}$, and the Kirk iterations converge strongly to the fixed point x^* for any $x \in K$ if the map ϕ defined in (17) is a comparison function.*

Proof. A $(\varphi - \psi)$-contractivity where $\psi = 0$ is a nonexpansive mapping, since

$$||Tx - Ty|| \leq \varphi(||x - y||) \leq ||x - y||,$$

for any $x, y \in K$. But, according to the hypotheses, T is also a φ-contraction on a complete b-metric space, consequently it has a single fixed point [23] and the Picard iterations are strongly convergent to it. Due to (18), the Kirk iterations have the same properties if ϕ is a comparison function. □

5. Banach-Valued Fractal Surfaces

In this section, we define fractal surfaces whose values lie on Banach algebras. The convergence and stability of Picard and Kirk iterations for their approximation are also analyzed, giving in both cases an estimate of the error.

The mappings defining the surfaces are fixed points of an operator on the space of bivariate p-integrable maps on a Banach algebra $\mathbb{A}$, $\mathcal{B}^p(I \times J, \mathbb{A})$, where I and J are real compact intervals. For $1 \leq p < \infty$ this space is Banach with respect to the norm

$$|f|_p = (\int_{I \times J} ||f(x,y)||^p dxdy)^{1/p},$$

where $|| \cdot ||$ is the norm in $\mathbb{A}$. For $0 < p < 1$, the space is quasi-Banach with modulus of concavity $s = 2^{1/p-1}$. Consequently, in all the cases $\mathcal{B}^p(I \times J, \mathbb{A})$ is a complete b-metric space.

Let us consider partitions for the intervals I and J, $x_0 < x_1 < \ldots < x_M$ for $I = [x_0, x_M]$, and $y_0 < y_1 < \ldots < y_N$ for $J = [y_0, y_N]$, $M, N > 1$. Let us consider subintervals $I_i = [x_{i-1}, x_i)$ for $i = 1, 2, \ldots, M-1$, $I_M = [x_{M-1}, x_M]$ and $J_j = [y_{j-1}, y_j)$ for $j = 1, 2, \ldots, N-1$, $J_N = [y_{N-1}, y_N]$. Let us define the affine maps

$$u_i(x) = c_i x + d_i, \qquad v_j(y) = e_j y + f_j,$$

satisfying the conditions

$$u_i(x_0) = x_{i-1}, \qquad u_i(x_M) = x_i, \qquad v_j(y_0) = y_{j-1}, \qquad v_j(y_N) = y_j, \tag{20}$$

for $i = 1, 2, \ldots, M$ and $j = 1, 2, \ldots, N$. Given two maps $f, g \in \mathcal{B}^p(I \times J, \mathbb{A})$, let us define

$$F_{ij}(x, y, A) = f(u_i(x), u_j(y)) + R_{ij}(x, y, A) - R_{ij}(x, y, g(x, y)),$$

for $(x, y) \in I \times J$ and $A \in \mathbb{A}$, with the same ranges of indexes. The case where $\mathbb{A} = \mathbb{R}$ and $R_{ij}(x, y, z) = \alpha_{ij} z$ was treated in reference [24].

Let us assume that the operator $S_{ij}:\mathcal{B}^p(I\times J,\mathbb{A})\to\mathcal{B}^p(I\times J,\mathbb{A})$ defined for $h\in\mathcal{B}^p(I\times J,\mathbb{A})$ as

$$S_{ij}(h)(x,y)=R_{ij}(x,y,h(x,y)) \tag{21}$$

is a φ_{ij}-contraction with respect to $|\cdot|_p$ for $i=1,2,\dots,M$, $j=1,2,\dots,N$. It is an easy exercise to prove that $\varphi(t):=\max_{ij}\{\varphi_{ij}(t)\}$ is also a comparison function.

Let us define the operator

$$Th(x,y)=F_{ij}(u_i^{-1}(x),v_j^{-1}(y),h(u_i^{-1}(x),v_j^{-1}(y))),$$

for $x\in I_i$, and $y\in J_j$. In order to simplify the notation, let us define $H_{ij}(x,y)=(u_i^{-1}(x),v_j^{-1}(y))$ for any i,j and write

$$Th(x,y)=F_{ij}(H_{ij}(x,y),h\circ H_{ij}(x,y))) \tag{22}$$

for $(x,y)\in I_i\times J_j$. Let us see that T is a φ-contraction:

$$|Th-Th'|_p^p=\sum_{i=1}^{M}\sum_{j=1}^{N}\int_{I_i\times J_j}||R_{ij}(H_{ij}(x,y),h\circ H_{ij}(x,y))-R_{ij}(H_{ij}(x,y),h'\circ H_{ij}(x,y))||^p dxdy.$$

With the change $u_i^{-1}(x)=x'$, and $v_j^{-1}(y)=y'$, and renaming the variables, we have

$$|Th-Th'|_p^p=\sum_{i=1}^{M}\sum_{j=1}^{N}c_ie_j\int_{I\times J}||R_{ij}(x,y,h(x,y))-R_{ij}(x,y,h'(x,y))||^p dxdy.$$

By definition of the operator S_{ij} (21),

$$|Th-Th'|_p^p=\sum_{i=1}^{M}\sum_{j=1}^{N}c_ie_j\int_{I\times J}||S_{ij}(h)(x,y)-S_{ij}(h')(x,y)||^p dxdy.$$

Using the fact that S_{ij} is a φ-contraction,

$$|Th-Th'|_p^p=\sum_{i=1}^{M}\sum_{j=1}^{N}c_ie_j|S_{ij}(h)-S_{ij}(h')|_p^p\le\sum_{i=1}^{M}\sum_{j=1}^{N}c_ie_j(\varphi(|h-h'|_p)^p. \tag{23}$$

Consequently,

$$|Th-Th'|_p\le(\sum_{i=1}^{M}\sum_{j=1}^{N}c_ie_j)^{1/p}\varphi(|h-h'|_p).$$

But $\sum_{i=1}^{M}\sum_{j=1}^{N}c_ie_j=1$ due to conditions (20), and thus,

$$|Th-Th'|_p\le\varphi(|h-h'|_p),$$

T is a φ-contraction, and consequently, is a $(\varphi-\psi)$-contractivity with $\psi=0$. Since $\mathcal{B}^p(I\times J,\mathbb{A})$ is quasi-Banach, and thus, a complete b-metric space, T has a single fixed point $f^\varphi\in\mathcal{B}^p(I\times J,\mathbb{A})$, and the Picard iterations of any point are convergent to it. The graph of f^φ has a fractal structure (see Theorem 5 of reference [25]). The order of convergence is

$$|T^nh-f^\varphi|_p\le\varphi^n(|h-f^\varphi|_p),$$

for any $h\in\mathcal{B}^p(I\times J,\mathbb{A})$. In the particular case where $\varphi(t)=at$ for $0<a<1$, according to Corollary 3,

$$|T^nh-f^\varphi|_p\le\frac{|Th-h|_p}{1-a}a^n, \tag{24}$$

for $1 \le p < \infty$, and

$$|T^n h - f^\varphi|_p \le \frac{s^2|Th - h|_p}{1 - as} a^n \tag{25}$$

for $0 < p < 1$, if $as < 1$, with $s = 2^{1/p-1}$.

The fixed point f^φ satisfies the functional equation

$$f^\varphi(x,y) = f(x,y) + R_{ij}(H_{ij}(x,y), f^\varphi \circ H_{ij}(x,y)) - R_{ij}(H_{ij}(x,y), g \circ H_{ij}(x,y)), \tag{26}$$

for $(x,y) \in I_i \times J_j$.

Let us consider now the Kirk iteration, and the case where $\varphi_{ij}(\delta) = a_{ij}\delta$. Let us define $a := \max_{ij} a_{ij}$ and assume that

$$r := \sum_{i=0}^{k-1} \alpha_i s^{i+1} a^i + \alpha_k s^k a^k < 1. \tag{27}$$

Denoting the Kirk iterates as $\hat{h}_n$, bearing in mind the estimation (19),

$$|\hat{h}_n - f^\varphi|_p \le r^n |\hat{h}_0 - f^\varphi|_p,$$

for $n \ge 0$. Thus, the Kirk iterations are convergent with a rate of convergence $O(r^n)$. This is always true in the normed case ($1 \le p < \infty$) since

$$r = \sum_{i=0}^{k} \alpha_i a^i < 1,$$

due to the definition of α_i.

Consequently, the Kirk iteration is convergent in the normed case for any values of the coefficients, with asymptotic stability. Since the Kirk operator T_K is also a Banach contraction if $r < 1$, we obtain error estimates for Kirk iterations as well:

$$|T_K^n h - f^\varphi|_p \le \frac{|T_K h - h|_p}{1 - r} r^n, \tag{28}$$

for $1 \le p < \infty$, and

$$|T_K^n h - f^\varphi|_p \le \frac{s^2|T_K h - h|_p}{1 - rs} r^n \tag{29}$$

for $0 < p < 1$, if $rs < 1$, where $s = 2^{1/p-1}$, and r is defined as in (27) in this second case.

6. Fractal Surfaces with Linear Vertical Contractions

Let us consider in this section the case where the vertical contraction operator $S_{ij} : \mathcal{B}^p(I \times J, \mathbb{A}) \to \mathcal{B}^p(I \times J, \mathbb{A})$, defined as $S_{ij}(h)(x,y) = R_{ij}(x,y,h(x,y))$ (see (21)), is linear and bounded.

We will define fractal convolutions of mappings and operators, and we will construct bivariate fractal frames of the Hilbert space $\mathcal{B}^2(I \times J, \mathbb{A})$ as fractal perturbations of standard frames in this space.

For the first part of (23), we have

$$|Th - Th'|_p \le \Big(\sum_{i=1}^{M}\sum_{j=1}^{N} c_i e_j |S_{ij}|_p^p\Big)^{1/p} |h - h'|_p. \tag{30}$$

If

$$C := \Big(\sum_{i=1}^{M}\sum_{j=1}^{N} c_i e_j |S_{ij}|_p^p\Big)^{1/p} < 1, \tag{31}$$

then T is a Banach contraction in a quasi-Banach space. Consequently, it has a fixed point f^{φ} that is a bivariate mapping $f^{\varphi}(x,y)$ whose values are in the algebra $\mathbb{A}$.

Some choices for a linear operator may be $S_{ij}(g) = \lambda_{ij}(g \circ c)$, where $c : I \times J \to I \times J$ and $\lambda_{ij} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ or $S_{ij}(g) = \lambda_{ij}(g \cdot v)$, with $v : I \times J \to \mathbb{A}$, where $\cdot$ represents the product in the algebra $\mathbb{A}$. In the first case, for $c = Id$ we have the classical vertical contraction of the fractal interpolation functions.

Remark 5. *Let us notice that we use the same notation for the norms of maps and linear operators ($|\cdot|_p$) in order to simplify the text.*

The estimation of the Picard iterations for the approximation of the fixed point f^{φ} (24) and (25) holds in this case as well, substituting a by C, and considering the hypothesis $Cs < 1$ in the second error estimation ($s \neq 1$).

We can define the operator $\mathcal{F}^{\varphi} : \mathcal{B}^p(I \times J, \mathbb{A}) \to \mathcal{B}^p(I \times J, \mathbb{A})$ that applies every map into its fractal perturbation: $\mathcal{F}^{\varphi}(f) = f^{\varphi}$, considering the partition, functions u_i, v_j, operators S_{ij}, and mapping $g \in \mathcal{B}^p(I \times J, \mathbb{A})$. It is an easy exercise to check that

$$|\mathcal{F}^{\varphi}(f) - f|_p \leq \frac{Cs}{1 - Cs}|f - g|_p,$$

whenever $Cs < 1$. If, additionally, f and g are related by a linear and bounded operator $\mathcal{L}$, that is to say, $g = \mathcal{L}f$, then $\mathcal{F}^{\varphi}$ is linear and bounded and

$$|\mathcal{F}^{\varphi}|_p \leq (1 + \frac{Cs|Id - \mathcal{L}|_p}{1 - Cs}).$$

This operator was defined by the author and studied for single-variable real maps and several functional spaces in reference [26].

In the case where $R_{ij}(x,y,A) = \alpha_{ij}(x,y).A$ and $\max_{ij} |\alpha_{ij}|_p < 1$, we obtain an α-fractal surface [24].

Modulating the distance between a mapping $f(x,y)$ and its fractal associated $f^{\varphi}(x,y)$, one can obtain fractal bases of several functional spaces (see, for instance, [25]). Given that f^{φ} is obtained by the action of two bivariate mappings f and g, it is possible to define a binary internal operation $*_{\varphi}$ in the space $\mathcal{B}^p(I \times J, \mathbb{A})$ as

$$f *_{\varphi} g := f^{\varphi}.$$

This was called in the real case the fractal convolution of f and g. This operation is linear, that is to say,

$$(\lambda f + \mu f') *_{\varphi} (\lambda g + \mu g') = \lambda(f *_{\varphi} g) + \mu(f' *_{\varphi} g'),$$

for any $f, f', g, g' \in \mathcal{B}^p(I \times J, \mathbb{A})$ and $\lambda, \mu \in \mathbb{R}$. It is also idempotent, that is to say,

$$f *_{\varphi} f = f,$$

for any $f \in \mathcal{B}^p(I \times J, \mathbb{A})$. Additionally, the map $P : \mathcal{B}^p(I \times J, \mathbb{A}) \times \mathcal{B}^p(I \times J, \mathbb{A}) \to \mathcal{B}^p(I \times J, \mathbb{A})$ defined as $P(f,g) = f *_{\varphi} g$ is bounded. The details are similar to the real univariate case.

Let us denote by $L(\mathcal{B}^p(I \times J, \mathbb{A}))$ the space of linear and bounded operators on $\mathcal{B}^p(I \times J, \mathbb{A})$. Based on the fractal convolution of maps, it is possible to define an internal operation in the set of operators as

$$(U *_{\varphi} V)f = (Uf) *_{\varphi} (Vf),$$

for any $f \in \mathcal{B}^p(I \times J, \mathbb{A})$, and $U, V \in L(\mathcal{B}^p(I \times J, \mathbb{A}))$. The linearity and boundedness of U, V, and P imply that the operator is well defined.

In the following, we consider the case $1 \leq p < \infty$.

Theorem 7. *Let $U, V \in L(\mathcal{B}^p(I \times J, \mathbb{A}))$ and let us assume that U is invertible such that $|Vf|_p \leq |Uf|_p$ for any $f \in \mathcal{B}^p(I \times J, \mathbb{A})$. Then, $U *_\varphi V$ is invertible and*

$$|(U *_\varphi V)|_p \leq \frac{1+C}{1-C}|U|_p, \tag{32}$$

$$|(U *_\varphi V)^{-1}|_p \leq \frac{1+C}{1-C}|U^{-1}|_p, \tag{33}$$

Proof. The proof is similar to that given in Theorem 8 of reference [25]. □

In the case where $p = 2$ and $\mathbb{A} = H$ is a Hilbert space, $\mathcal{B}^2(I \times J, H)$ is Hilbert as well, with respect to the inner product:

$$< f, g >= \int_{I \times J} < f(x,y), g(x,y) > dxdy.$$

In the following, we consider $p = 2$ and we prove that given a frame $(f_m)_{m=0}^\infty$ of bivariate functions $f_m(x,y) \in H$, we can construct Hilbert-valued fractal frames with two variables of type

$$(U(f_m) *_\varphi V(f_m))_{m=0}^\infty.$$

Let us start by defining a frame in a Hilbert space.

Definition 8. *A sequence $(f_m)_{m=0}^\infty \subseteq X$, where X is a Hilbert space is a frame if there exist real positive constants A, B such that*

$$A||f||^2 \leq \sum_{m=0}^\infty | < f, f_m > |^2 \leq B||f||^2, \tag{34}$$

for any $f \in X$, where $|| \cdot ||$ denotes the norm associated with the inner product in X. A and B are the bounds of the frame.

Theorem 8. *Let $U, V \in L(\mathcal{B}^2(I \times J, H))$ satisfy the hypotheses given in Theorem 7, and $(f_m)_{m=0}^\infty \subseteq \mathcal{B}^2(I \times J, H)$ be a frame with bounds A, B. Then, $((U *_\varphi V)f_m)_{m=0}^\infty$ is also a frame with bounds A_φ and B_φ, defined as*

$$A_\varphi := A(\frac{1+C}{1-C})^{-2}|U^{-1}|_2^{-2}$$

and

$$B_\varphi := B(\frac{1+C}{1-C})^2|U|_2^2.$$

Proof. Let $(U *_\varphi V)^+$ be the adjoint operator of $(U *_\varphi V)$, and $g \in \mathcal{B}^2(I \times J, H)$. Applying the frame inequalities of $(f_m)_{m=0}^\infty$ for $f = (U *_\varphi V)^+ g$ one obtains

$$A|(U *_\varphi V)^+ g|_2^2 \leq \sum_{m=0}^\infty | < (U *_\varphi V)^+ g, f_m > |_2^2 \leq B|(U *_\varphi V)^+ g|_2^2 \leq B|(U *_\varphi V)|_2^2 |g|_2^2. \tag{35}$$

Since

$$\sum_{m=0}^\infty | < (U *_\varphi V)^+ g, f_m > |_2^2 = \sum_{m=0}^\infty | < g, (U *_\varphi V) f_m > |_2^2, \tag{36}$$

using (35) and (36) one has

$$A|(U *_\varphi V)^+ g|_2^2 \leq \sum_{m=0}^\infty | < g, (U *_\varphi V) f_m > |_2^2 \leq B|(U *_\varphi V)|_2^2 |g|_2^2, \tag{37}$$

and we have the right inequality for the sequence $((U *_{\varphi} V)f_m)_{m=0}^{\infty}$. For Theorem 7 we know that $(U *_{\varphi} V)$ is an invertible operator. Then,

$$|g|_2^2 = |((U *_{\varphi} V)^{+})^{-1} \circ (U *_{\varphi} V)^{+} g|_2^2 \leq |((U *_{\varphi} V)^{+})^{-1}|_2^2 |(U *_{\varphi} V)^{+} g|_2^2,$$

and

$$A|((U *_{\varphi} V)^{+})^{-1}|_2^{-2} |g|_2^2 \leq A|(U *_{\varphi} V)^{+} g|_2^2.$$

Since $|((U *_{\varphi} V)^{+})^{-1}|_2 = |(U *_{\varphi} V)^{-1}|_2$, then, by (37),

$$A|(U *_{\varphi} V)^{-1}|_2^{-2} |g|_2^2 \leq \sum_{m=0}^{\infty} | < g, (U *_{\varphi} V)f_m > |_2^2 \leq B|(U *_{\varphi} V)|_2^2 |g|_2^2. \quad (38)$$

Hence, by (32) and (33), $(U(f_m) *_{\varphi} V(f_m))$ is also a frame and its bounds are A_{φ} and B_{φ}, defined as

$$A_{\varphi} = A\left(\frac{1+C}{1-C}\right)^{-2} |U^{-1}|_2^{-2} \leq A|(U *_{\varphi} V)^{-1}|_2^{-2}$$

and

$$B_{\varphi} = B\left(\frac{1+C}{1-C}\right)^{2} |U|_2^2 \geq B|(U *_{\varphi} V)|_2^2.$$

□

7. Conclusions

This article delves into the concept of $(\varphi - \psi)$-partial contractivity, defined by the author in previous references in the framework of b-metric and quasi-normed spaces. In particular, it provides sufficient conditions for the existence of fixed points for these maps. The convergence and stability of the Picard iterations for the approximation of the fixed points are proved, giving "a priori" error estimates as well.

Kirk's algorithm for the same purpose is analyzed, giving conditions for its convergence and stability. In particular, it is proved that the method enjoys these properties if the underlying space is a normed space and the comparison function φ is linear, in the case of the existence of a fixed point. Some error estimates are also given.

The properties proved can be applied to other types of contractions, since the self-maps considered contain many others as particular cases. For instance, if the underlying set is a metric space, the contractions of type Kannan, Chatterjea, Zamfirescu, Ćirić, and Reich are included in the class of contractivities studied in this paper (see Corollary 2.2 of reference [11]).

These facts are applied to the definition of new fractal surfaces, more general than those studied so far. The construction of fractal frames composed of bivariate mappings is performed, belonging to very general Hilbert functional spaces.

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