Advances in Mathematics 455 (2024) 109877

Contents lists available at ScienceDirect

Advances in Mathematics

journal homepage: www.elsevier.com/locate/aim

Weighted hyperbolic composition groups on the disc and subordinated integral operators ☆

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A R T I C L E   I N F O

Article history:
Received 13 January 2023
Received in revised form 25 January 2024
Accepted 30 July 2024
Available online 20 August 2024
Communicated by Dan Voiculescu

MSC:
47A25
47B33
47D03

Keywords:
Weighted composition operator
Spectrum
Hyperbolic group
Integral operator

A B S T R A C T

We provide the spectral picture of groups of weighted composition operators, induced by the hyperbolic group of automorphisms of the unit disc, acting on holomorphic functions. Some questions about the spectrum of single weighted hyperbolic composition operators are discussed, and results related with them in the literature are completed or partly extended. Also, our results on the weighted hyperbolic group are applied to the spectral study of two families of multiparameter weighted averaging operators, which generalize both Siskakis' operator and the reduced Hilbert matrix operator.

☆ The authors have been partially supported by Project PID2019-10579GB-I00 and Project PID2022-137294NB-I00 of the Ministry of Science, and Project E26-17R and Project E48-23R of D.G. Aragón, Universidad de Zaragoza, Spain. The fourth author has been also supported by Scholarship BES-2017-081552, MINECO, Spain.

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https://doi.org/10.1016/j.aim.2024.109877

0. Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc in the complex plane $\mathbb{C}$. The purpose of this paper is twofold. In one way, we search for providing a spectral picture of weighted hyperbolic composition groups on $\mathbb{D}$. On the other hand, as an application of the above, we look for giving spectral descriptions of integral operators subordinated to the quoted groups.

Our interest in the above operators and groups has been motivated by several issues arising in different, though connected, ways. There is a vast literature dealing with properties (norm, compactness, spectrum, ...) of families of averaging integral operators acting on Banach spaces X of holomorphic functions in $\mathbb{D}$. Recall, the Cesàro integral operator $\mathcal{C}$ and its equivalent formulation $\mathfrak{C}$ on sequences are defined respectively by

$$(\mathcal{C}f)(z) := \frac{1}{z}\int_0^z \frac{f(w)}{1-w}\,dw\ ; \quad (\mathfrak{C}\widehat{f})(n) := \frac{1}{n+1}\sum_{j=0}^{n} \widehat{f}(j),$$

for $z \in \mathbb{D}$, $n \in \mathbb{N}\cup\{0\}$, $f \in X$, where $\widehat{f} = (\widehat{f}(n))$ denotes the Taylor coefficient sequence of the analytic function f. The corresponding adjoint operators of $\mathcal{C}$ and $\mathfrak{C}$ are given by

$$(\mathcal{C}^*f)(z) := \frac{1}{z-1}\int_1^z f(\xi)\ d\xi\ ; \quad (\mathfrak{C}^*\widehat{f})(n) := \sum_{j=n}^{\infty} \frac{\widehat{f}(j)}{j+1} \quad (z \in \mathbb{D}, n \in \mathbb{N}\cup\{0\}).$$

Let $\mathcal{J}$ denote the operator defined by

$$(\mathcal{J}f)(z) := \frac{1}{1-z}\int_1^z \frac{f(\xi)}{1+\xi}\ d\xi,\ z \in \mathbb{D},$$

which was introduced in [35], where its norm, spectrum and point spectrum in Hardy spaces $H^p(\mathbb{D})$, $p \geq 1$, were studied. Here, we call $\mathcal{J}$ *Siskakis' operator*. Even though it formally looks a weighted version of $\mathcal{C}^*$ (in fact, $\mathcal{J}f = -\mathcal{C}^*\left((1+(\cdot))^{-1}f\right)\big)$ they behave different from a spectral viewpoint. A reason for this is seen below, via certain one-parameter operator families.

Likewise, there are also the so-called Hilbert matrix operator $\mathfrak{H}$ and the *reduced* Hilbert matrix operator $\mathcal{H}$ defined respectively by

$$(\mathfrak{H}f)(z) := \int_0^1 \frac{f(\xi)}{1-z\xi}d\xi, \quad (\mathcal{H}f)(z) := \int_{-1}^1 \frac{f(\xi)}{1-z\xi}d\xi, \qquad z \in \mathbb{D},$$

see [12] for $\mathfrak{H}$. While working on the present paper, the authors have been aware of the fact that A. Aleman, A. Siskakis and D. Vukotic have recently approached the study of the operator $\mathfrak{H}$ using its reduced version $\mathcal{H}$ as a key tool. We are not following this idea here.

In recent times, a line of research has emerged that takes families of (multiparameterized) generalizations of Cesàro operators as study objects. An interesting representative of one of such families is $\mathcal{T}_{\mu,\nu}$, $\mu,\nu \in \mathbb{R}$, given by the formula

$$(\mathcal{T}_{\mu,\nu} f)(z) := z^{\mu-1}(1-z)^{-\nu} \int_0^z \xi^{-\mu}(1-\xi)^{\nu-1} f(\xi)\, d\xi, \qquad z \in \mathbb{D}.$$

The operator $\mathcal{T}_{\mu,\nu}$ generalizes $\mathcal{C}$ (note, $\mathcal{T}_{0,0} = \mathcal{C}$) as well as other operators related with $\mathcal{C}$, see [2,3] and references therein. There are other generalizations of Cesàro operators in the literature, see [2,6,33,37,39,41]; in particular averaging operators of the form $\frac{1}{z}\int_0^z f(\xi)g'(\xi)d\xi$ for generic functions g' of essentially rational type.

In a similar way, it sounds sensible to consider parameterized averaging operators generalizing $\mathcal{J}$ and to investigate their spectral properties. Here we approach the study of the family of operators $\mathcal{J}_\delta^{\mu,\nu}$ given by

$$(\mathcal{J}_\delta^{\mu,\nu} f)(z) := \frac{1}{(1+z)^{\nu+\delta}(1-z)^{\mu+\delta}} \int_z^1 (1+\xi)^{\nu}(1-\xi)^{\mu}(\xi-z)^{\delta-1} f(\xi)\, d\xi, \qquad z \in \mathbb{D}, \tag{0.1}$$

for $z \in \mathbb{D}$, $f \in X$ and suitable values of parameters $\mu,\nu,\delta \in \mathbb{C}$.

This family generalizes Siskakis' operator since $\mathcal{J} = -\mathcal{J}_1^{0,-1}$. For other particular values of μ, ν and δ, operators $\mathcal{J}_\delta^{\mu,\nu}$ are isometric, up to constants, to certain parameterized operators, defined on fractional subspaces of $L^2(0,\infty)$ and $H^2(\mathbb{C}^+)$, considered in [21,29]. The extension of the above operators to arbitrary parameters μ, ν, δ (whenever there is convergence of the integrals) seems to be natural. Weights $(1 \pm z)^{\alpha}$, $\alpha \in \mathbb{R}$, also arise in a natural way if we think of the action of composition operators (see below in this introduction) on spaces like $H^p(\mathbb{D})$ with weights of the same type; see for example [11, Section 4].

As regards generalizations of the reduced Hilbert matrix operator, we deal with the family $\mathcal{H}_\delta^{\mu,\nu}$, for suitable $\mu,\nu,\delta \in \mathbb{C}$, given by

$$(\mathcal{H}_\delta^{\mu,\nu} f)(z) := \frac{1}{(1+z)^{\nu-\delta+1}(1-z)^{\mu-\delta+1}} \int_{-1}^1 (1+\xi)^{\nu}(1-\xi)^{\mu} \frac{f(\xi)}{(1-z\xi)^{\delta}}\, d\xi,$$

for $z \in \mathbb{D}$, $f \in X$. Clearly, $\mathcal{H} = \mathcal{H}_1^{0,0}$. On the other hand, operators $\mathcal{H}_\delta^{\mu,\nu}$ are also a generalization of other operators isometric to the Stieltjes transform or Poisson-like integrals; see [30].

Operators $\mathcal{J}_\delta^{\mu,\nu}$ and $\mathcal{H}_\delta^{\mu,\nu}$ are closely related to groups of automorphisms on the unit disc, in particular with the hyperbolic one, as we explain later on.

For a Banach space X, let $\mathcal{B}(X)$ denote the space of bounded linear operators on X. All families of integral operators quoted above share the property that their elements, say $\mathcal{T}$, can be expressed on appropriate X by subordination to suitable vector-valued functions $V\colon \mathbb{R} \to \mathcal{B}(X)$; that is, $\mathcal{T}$ can be written in the form

$$\mathcal{T}f = \int_{-\infty}^{\infty} g(t)V(t)f\,dt, \quad f \in X, \tag{0.2}$$

where g is locally integrable on $\mathbb{R}$ and $V(t)$ is related with semigroups of composition operators or it is a semigroup itself. We put $V(t) = S(t)$ in this case, and write the semigroup (or group) often as $(S(t))$ with no matter if t runs over the set of nonnegative real numbers or over the set of all real numbers. The above representation (0.2) is relevant for the study of boundedness and norms, spectra and other properties like subnormality, compactness and so on. The idea to exploit subordination, as in (0.2), in the study of properties of $\mathcal{T}$ dates back to [9] at least. A systematic approach to classical averaging operators $\mathcal{T}$ based upon the analysis of the infinitesimal generators of semigroups $S(t)$ was undertaken by A. Siskakis in several papers [12,35,36]. In these works, subordination is mostly restricted to give integral expressions of inverses of generators and, more generally, of resolvent functions. Families $\{\mathcal{J}_\delta^{\mu,\nu}\}$ and $\{\mathcal{H}_\delta^{\mu,\nu}\}$ lie in the framework yield around (0.2). To see this, we need to say some words about composition groups of automorphisms.

Assume that X is a function Banach space continuously contained in the Fréchet space $\mathcal{O}(\mathbb{D})$ of all holomorphic functions on $\mathbb{D}$. Let (ψ_t) be a flow of automorphisms of $\mathbb{D}$. One defines the composition operator $C_{\psi_t}\colon X \to X$ by $C_{\psi_t}(f)(z) := f(\psi_t(z))$ for $f \in X$, $z \in \mathbb{D}$. Frequently, the family (C_{ψ_t}) becomes a C_0-semigroup on X and furthermore it gives rise to *weighted* composition C_0-semigroups $(S(t)) \subseteq \mathcal{B}(X)$, given by

$$[S(t)f](z) = v_t(z)[C_{\psi_t}f](z), \quad f \in X,\ z \in \mathbb{D},$$

where (v_t) is a continuous cocycle for a flow (ψ_t) (see Section 1 for their definitions), with $t \geq 0$ if $(S(t))$ is a semigroup or with $t \in \mathbb{R}$ if $(S(t))$ is indeed a group. In this paper, we are interested in *weighted* composition *groups* $(S(t))$ where (ψ_t) is a group of *hyperbolic* automorphisms. Up to isomorphism, the class of groups of hyperbolic automorphisms of $\mathbb{D}$ is reduced to the hyperbolic flow (φ_t) where

$$\varphi_t(z) := \frac{(e^t+1)z + e^t - 1}{(e^t-1)z + e^t + 1}, \quad z \in \mathbb{D},\ t \in \mathbb{R}. \tag{0.3}$$

The operator $\mathcal{T}_{\mu,\nu}$ as well as other generalizations of Cesàro's operator admit to be represented by subordination, as in (0.2), to semigroups of weighted composition operators, see [38]. In turn, operators $\mathcal{J}_\delta^{\mu,\nu}$ and $\mathcal{H}_\delta^{\mu,\nu}$ can be represented by subordination to a weighted composition group $(u_tC_{\varphi_t})$; namely

$$\mathcal{J}_\delta^{\mu,\nu} = \int_{-\infty}^{\infty} g_\delta(t)\, u_t C_{\varphi_t}\, dt, \qquad \mathcal{H}_\delta^{\mu,\nu} = \int_{-\infty}^{\infty} h_\delta(t)\, u_t C_{\varphi_t}\, dt, \tag{0.4}$$

where, for $t \in \mathbb{R}$ and suitable $\delta \in \mathbb{C}$, $g_\delta(t) = 2^{-\delta}(1 - e^{-t})^{\delta-1}\chi_{(0,\infty)}(t)$ and $h_\delta(t) = 2^{\delta-1}(1 + e^t)^{-\delta}$, see Section 8. Notice that the functions g_δ, h_δ appear on the other hand as subordinating functions in [21], [29], [30]. This fact also suggested considering operators $\mathcal{J}_\delta^{\mu,\nu}$, $\mathcal{H}_\delta^{\mu,\nu}$.

One of the aims in this paper is to describe the fine structure of the spectrum of the operators $\mathcal{J}_\delta^{\mu,\nu}$ and $\mathcal{H}_\delta^{\mu,\nu}$. To do so in a unified way, we connect this question with a functional calculus associated to the group $(u_t C_{\varphi_t})$ and suitable operating functions. More precisely, we adopt Siskakis' view, and therefore we undertake a detailed study of the infinitesimal generator Δ of $(u_t C_{\varphi_t})$. Such a generator is a bisectorial-like operator in the sense of [31], so that we apply the results obtained there on spectral mappings to transfer the information on the spectrum of Δ to the one of $\mathcal{J}_\delta^{\mu,\nu}$ and $\mathcal{H}_\delta^{\mu,\nu}$.

We wish to establish our results here for a class of Banach spaces as large as possible, following a unified approach. Thus we introduce the notion of Banach γ-*space*, depending on a nonnegative parameter γ, which includes classical Banach spaces usually considered in the subject. Among these spaces, one has for instance Hardy spaces, (weighted) Bergman spaces, little Korenblum spaces and the disc algebra, (weighted) Dirichlet spaces and little Bloch spaces.

On the other hand, the study of weighted hyperbolic groups $(u_t C_{\varphi_t})$ has interest in its own. This was another of our aims in the beginning of this work, as well as finding out applications to weighted hyperbolic composition operators, say vC_ψ. Let ψ denote a hyperbolic automorphism and let v denote a weight or multiplier. It is still an open question, in general, whether or not the spectrum $\sigma(vC_\psi)$ is an annulus and, in such a case, which are its radii. Just citing the most recent papers on that question, one has in [8] that, for the classical Dirichlet space ($\mathcal{D}_0^2(\mathbb{D})$ in our notation), v continuous at the fixed points a (attractive) and b (repulsive) of ψ, and vC_ψ invertible,

$$\sigma(vC_\psi) \subseteq \{\lambda \in \mathbb{C} : \min\{|v(a)|, |v(b)|\}\psi'(a) \le |\lambda| \le \max\{|v(a)|, |v(b)|\}\psi'(b)\}.$$

The above inclusion is improved in [17], where it is shown that

$$\sigma(vC_\psi) \subseteq \{\lambda \in \mathbb{C} : \min\{|v(a)|, |v(b)|\} \le |\lambda| \le \max\{|v(a)|, |v(b)|\}\},$$

whenever v is in the disc algebra. It is also conjectured that

$$\sigma(vC_\psi) = \{\lambda \in \mathbb{C} : \min\{|v(a)|, |v(b)|\} \le |\lambda| \le \max\{|v(a)|, |v(b)|\}\}, \tag{0.5}$$

for the Dirichlet space and the Bloch space.

Furthermore, for the spaces $H^p(\mathbb{D})$, $\mathcal{A}_\sigma^p(\mathbb{D})$, $\mathcal{K}_0^{-\gamma}(\mathbb{D})$, $p \ge 1$, and vC_ψ invertible, it is proved in [27] that the spectrum of vC_ψ is contained in the annulus of radii

$\min\{|v(a)|\psi'(a)^{-\gamma}, |v(b)|\psi'(b)^{-\gamma}\}$ and $\max\{|v(a)|\psi'(a)^{-\gamma}, |v(b)|\psi'(b)^{-\gamma}\}$ and that, provided $|v(b)|\psi'(b)^{-\gamma} \leq |v(a)|\psi'(a)^{-\gamma}$,

$$\sigma(vC_\psi) = \{\lambda \in \mathbb{C} : |v(b)|\psi'(b)^{-\gamma} \leq |\lambda| \leq |v(a)|\psi'(a)^{-\gamma}\}, \tag{0.6}$$

as well as, under additional assumptions on u, that $\mathrm{Int}(\sigma(vC_\psi)) \subseteq \sigma_{point}(vC_\psi)$. The question of whether or not the corresponding identity is true in the case $|v(b)|\psi'(b)^{-\gamma} > |v(a)|\psi'(a)^{-\gamma}$ is left open in [27] as a conjecture in the positive.

Every hyperbolic automorphism ψ can be embedded in a hyperbolic flow (ψ_t), in the sense that $\psi = \psi_1$. If the weight v can also be embedded in a cocycle (v_t) for (ψ_t), then the spectrum of the infinitesimal generator Δ of $(v_tC_{\psi_t})$ provides substantial information about the one of $v_1C_{\psi_1} = vC_\psi$. With this method, we prove that conjectures (0.5) and (0.6) are true if the operator vC_ψ can be embedded in a C_0-group $(v_tC_{\psi_t})_{t\in\mathbb{R}}$, and for all the spaces quoted above, see Theorem 7.2. Moreover, the theorem provides information about subspectra of vC_ψ which seems to be of interest, in particular for Dirichlet spaces. The ideas considered in the paper could be helpful to study arbitrary invertible weighted hyperbolic operators uC_{ψ_1} by means of quasi-nilpotent perturbations $uC_{\psi_1} - v_1C_{\psi_1}$, since $uC_{\psi_1} - v_1C_{\psi_1}$ is a quasi-nilpotent operator for a suitable cocycle (v_t) for (ψ_t).

In view of the above, the description of spectra of the infinitesimal generator Δ turns out to be the key point of the paper. Thus another question of importance is to find families of cocycles (u_t) for which the spectral picture of Δ is available. In this respect, it is useful the representation of (u_t) as a coboundary, i.e.

$$u_t = \frac{\omega \circ \varphi_t}{\omega},\ t \in \mathbb{R},$$

for some non-vanishing holomorphic function $\omega : \mathbb{D} \to \mathbb{C}$, see [28,38]. We obtain the notable property that, under fairly mild conditions on (u_t) (namely, that (u_t) is a DW-continuous cocycle, see Section 1), ω presents zeroes or singularities of polynomial type at the Denjoy-Wolf points of (φ_t). This property is crucial (and enough) to give a detailed spectral picture of Δ for Hardy spaces, Bergman spaces, little Korenblum classes and the disc algebra. The case of Dirichlet spaces and little Bloch spaces require an extra condition on ω which does not seem to be strong.

We now outline how the paper is organized.

Section 1 contains basic material about spectra of operators, functional calculus of bisectorial-like operators, semigroups and flows, where we pay special attention on the spectral mapping results of [31]. We also define DW-continuous cocycles and explain that, in most of the paper, we focus on the hyperbolic flow (φ_t) of DW-points 1 and -1. Conditions or properties defining Banach γ-spaces are given in Section 2, together with some lemmas which provide us with a number of such spaces, including the examples quoted above. In particular, condition (**Gam5**) is introduced to place Dirichlet spaces and little Bloch spaces into the setting. For the other examples it is sufficient to recall the well known fact that (**Gam5**) hods for $\varepsilon = 0$. The notion of γ-space covers a range of spaces a bit larger than other systems of axioms do.

Section 3 is devoted to prove that the weight ω associated with a cocycle (u_t) for the flow (φ_t) is tempered at the DW-points -1, 1. The overall argument to prove that is rather involved and culminates with Theorem 3.11. In order to establish our results on spectra in a general form, we also introduce spectrally DW-contractive cocycles, and hyperbolically DW-contractive spaces accordingly (see definitions there), and show that the examples of γ-spaces of Subsection 2.1 are hyperbolically DW-contractive.

In Section 4, estimates on the group $(u_t C_{\varphi_t})$ of asymptotic type related to the spectral radius are given. In Section 5, properties of two helpful integrals related to the resolvent operator are presented, as preparation to Section 6 where the fine structure of the spectrum of Δ is exposed, see Theorem 6.7. This theorem widely extends results of [35]. At this point, it must be said that the ideas behind the results of this paper, in particular in Section 5 and Section 6, have been mainly inspired by papers [2,8,27,33,35]. The level of generality that such ideas present in this paper, in the direction considered here, has been very much facilitated by the quoted Theorem 3.11.

Features of spectra of the generator Δ are transferred, first to the weighted hyperbolic group $u_t C_{\varphi_t} = e^{t\Delta}$ (Theorem 7.1), and then to arbitrary weighted hyperbolic groups $(v_t C_{\psi_t})$ (under corresponding assumptions on (v_t)) by composition with suitable automorphisms, in Section 7, Theorem 7.2. It is to be noticed that Theorem 7.2 gives us information on the full spectrum, essential spectrum, point spectrum and residual spectrum of $v_t C_{\psi_t}$, $t \in \mathbb{R}$. In Remark 7.3, we point out that Theorem 7.2 provides partial solutions, even for Dirichlet and little Bloch spaces, to the conjectures discussed around (0.5) and (0.6).

Finally, in Section 8 the results obtained in preceding sections are applied to the aforementioned integral averaging operators which generalize the Siskakis operator and the reduced Hilbert matrix operator.

Quite frequently through this paper, for a set Y and $h_j \colon Y \to \mathbb{R}$, $j = 1, 2$, we shall write $h_1(y) \lesssim h_2(y)$, $y \in Y$, whenever there exists a parameter $c > 0$ such that $h_1(y) \le c h_2(y)$ for all $y \in Y$. We shall write $h_1(y) \sim h_2(y)$, $y \in Y$, if we have $h_1(y) \lesssim h_2(y) \lesssim h_1(y)$ for all $y \in Y$.

1. Functional calculus, spectra, semigroups, flows

Let X be a Banach space, let $\mathcal{B}(X)$ denote the Banach algebra of bounded linear operators on X and let $\mathcal{C}(X)$ denote the space of closed operators on X. For $A \in \mathcal{C}(X)$, let $\sigma(A)$ be the spectrum of A. Here we collect some results concerning functional calculus of bisectorial-like operators on X, and corresponding spectral mappings.

Given $\theta \in (0, \pi)$ let Σ_θ denote the sector $\Sigma_\theta := \{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \theta\}$ of the complex plane. For every $\theta \in (0, \pi/2]$ and $a \ge 0$, put

$$BS_{\theta,a} := (-a + \Sigma_{\pi-\theta}) \cap (a - \Sigma_{\pi-\theta}), \text{ if } \theta < \pi/2 \text{ or } a > 0; \quad BS_{\pi/2,0} := i\mathbb{R}.$$

Definition 1.1. Let $(\theta, a) \in (0, \pi/2] \times [0, \infty)$ and $A \in \mathcal{C}(X)$. We say that A is a bisectorial-like operator, of angle θ and half-width a, if $\sigma(A) \subseteq \overline{BS_{\theta,a}}$ and

$$\sup\left\{\min\{|\lambda - a|, |\lambda + a|\}\|(\lambda - A)^{-1}\| : \lambda \notin \overline{BS_{\eta,a}}\right\} = K_\eta < \infty, \quad \eta \in (0, \theta).$$

Notice that an operator $A \in \mathcal{C}(X)$ is bisectorial-like if and only if both $-a + A$ and $a - A$ are sectorial of angle $\pi - \theta$ in the sense of [23].

For an open subset Ω of $\mathbb{C}$, let $\mathcal{O}(\Omega)$ denote the algebra of holomorphic functions in Ω. Put $\mathcal{O}[BS_{\theta,a}] := \bigcup_{0<\eta<\theta} \mathcal{O}(BS_{\eta,a})$ for every $(\theta, a) \in (0, \pi/2] \times [0, \infty)$.

Definition 1.2. We say that $f \in \mathcal{O}[BS_{\theta,a}]$ is regular at ∞ if there exists $f(\infty) := \lim_{z\to\infty} f(z)$ in $\mathbb{C}$, where the limit must be understood through the holomorphic domain of f, and

$$\int_{(BS_{\eta',a})\cap\{|z|>R\}} \left|\frac{f(z) - f(\infty)}{z}\right| |dz| < \infty$$

for some $R > 0$, $0 < \eta < \theta$ and all $\eta' \in (\eta, \pi/2]$.

Fix a bisectorial-like operator A of angle $\theta \in (0, \pi/2]$ and half-width $a \in [0, \infty)$. The space of functions in $\mathcal{O}[BS_{\theta,a}]$ which are regular at ∞ and holomorphic at $-a$ and a is denoted here by $\mathcal{E}(A)$. Then it is a matter of fact that

$$\mathcal{E}(A) = \mathcal{E}_0(A) + \frac{1}{b + (\cdot)}\mathbb{C} + \frac{1}{b - (\cdot)}\mathbb{C} + \mathbb{C}\mathbf{1}$$

for any $b \in \mathbb{C} \setminus \overline{BS_{\theta,a}}$, where $\mathbf{1}$ is the constant function with value 1 and $\mathcal{E}_0(A)$ is formed by all elements f in $\mathcal{E}(A)$ with $f(\infty) = f(a) = f(-a) = 0$.

Let us now define the (primary) functional calculus for a bisectorial-like operator A and functions in $\mathcal{E}(A)$ according to the following rules:

$$(\mathbf{1})(A) := A, \ \left(\frac{1}{b + (\cdot)}\right)(A) := (b + A)^{-1}, \ \left(\frac{1}{b - (\cdot)}\right)(A) := (b - A)^{-1}, \quad \forall b \in \mathbb{C} \setminus \overline{BS_{\theta,a}},$$

and

$$f(A) := \frac{1}{2\pi i}\int_{\Sigma_\varepsilon} f(z)(z - A)^{-1} dz, \quad f \in \mathcal{E}_0(A),$$

where Σ_ε is the positively oriented boundary of a bisector $BS_{\eta,a}$ with $\eta' < \eta < \theta$ for some η' such that $f \in \mathcal{O}(BS_{\eta',a})$.

It is not difficult to check that the above integral is well defined in the Bochner sense and independent of Σ_ε, and that the $\mathcal{E}(A)$-calculus is well defined. The calculus just introduced for bisectorial-like operators mimics the primary calculus given in [23] for sectorial operators.

Let $\tilde{\sigma}(A)$ be the extended spectrum of A, which is to say $\tilde{\sigma}(A) := \sigma(A) \cup \{\infty\}$ if A is unbounded and $\tilde{\sigma}(A) := \sigma(A)$ if $A \in \mathcal{B}(X)$. Let $\sigma_{ess}(A)$ denote the Fredholm essential spectrum of A, which is defined as follows. A closed operator T with domain $D(T)$ in X is said to be a Fredholm operator if

$$\dim \ker T < \infty \text{ and } \dim X/\mathrm{Ran}(T) < \infty,$$

see [16, Section I.3]. Then $\sigma_{ess}(T)$ is the subset of $\lambda \in \mathbb{C}$ such that $\lambda - T$ is not a Fredholm operator. The extended essential spectrum $\widetilde{\sigma}_{ess}(T)$ is defined as $\widetilde{\sigma}_{ess}(T) := \sigma_{ess}(T)$ if $\dim X/D(T) < \infty$, and $\widetilde{\sigma}_{ess}(T) := \sigma_{ess}(T) \cup \{\infty\}$ otherwise. Let $\sigma_{point}(A)$ denote the point spectrum of A and $\rho(A)$ the resolvent set of A. By $R(\lambda, A) := (\lambda - A)^{-1}$ we denote the resolvent operator, whenever $\lambda \in \rho(A)$. The approximate spectrum $\sigma_{ap}(A)$ of A is the subset $\sigma_{ap}(A) := \{\lambda \in \mathbb{C} : \lambda - A \text{ is not injective or } \mathrm{Ran}(\lambda - A) \text{ is not closed}\}$. The residual spectrum $\sigma_{res}(A)$ of A is the subset $\sigma_{ap}(A) := \{\lambda \in \mathbb{C} : \mathrm{Ran}(\lambda - A) \text{ is not dense in } X\}$. Finally, let $r(A)$ denote the spectral radius of A, $r(A) := \sup_{\lambda \in \sigma(A)} |\lambda|$. Recall that $r(A) = \lim_{n\to\infty}(\|A^n\|_{\mathcal{B}(X)})^{1/n}$ whenever $A \in \mathcal{B}(X)$.

The following theorem provides spectral mapping results for functions in $\mathcal{E}(A)$, see [31].

Theorem 1.3. *Let A be an unbounded bisectorial-like operator of half-width $a \geq 0$ and angle $\theta \in (0, \pi/2]$. For every $f \in \mathcal{E}(A)$ we have*

(1)

$$\tilde{\sigma}(f(A)) = f(\tilde{\sigma}(A)) \quad \text{and} \quad \tilde{\sigma}_{ess}(f(A)) = f(\tilde{\sigma}_{ess}(A)).$$

(2)

$$f(\sigma_{point}(A)) \subseteq \sigma_{point}(f(A)) \subseteq f(\sigma_{point}(A)) \cup f(\infty).$$

(3) *If, moreover, there exists $c > 0$ such that*

$$|f(z) - f(\infty)| \gtrsim |z|^{-c} \text{ as } z \to \infty \text{ through } BS_{\eta,a}, \tag{1.7}$$

where $0 < \eta < \theta$ is such that $f \in \mathcal{O}(BS_{\eta,a})$, then

$$f(\sigma_{point}(A)) = \sigma_{point}(f(A)).$$

Important examples of bisectorial-like operators are the infinitesimal generators of C_0-groups in $\mathcal{B}(X)$ since such generators are sectorial to the left and to the right. Recall that a family $(S(t)) \subseteq \mathcal{B}(X)$ is said to be a (one-parameter) semigroup if $S(t)$ exists for $t \geq 0$, $S(0)$ is the identity mapping and $S(s + t) = S(s)S(t)$ for all $s, t \geq 0$. If, moreover, $S(t)$ exists for $t \in \mathbb{R}$ and $S(s + t) = S(s)S(t)$ for all $s, t \in \mathbb{R}$ we say that $(S(t))$ is a group. A semigroup $(S(t))$ is called C_0-semigroup when $\lim_{t\to s} T(t)x = T(s)x$ for every $s \geq 0$ and $x \in X$. The infinitesimal generator Δ of a C_0-semigroup $(S(t))$ is the operator defined by $\Delta x := \lim_{t\to 0} t^{-1}(S(t)x - x) = \frac{\partial S(t)x}{\partial t} \mid_{t=0}$ for those $x \in X$ such that the above limit (in norm) exists in X. Put $D(\Delta) := \{x \in X \,:\, \text{ there exists } \Delta x \in X\}$. It is well known that Δ is a closed densely defined linear operator, [18, Section II.1].

Let us assume from now on in this section that $(S(t))$ is a C_0-*group* of bounded operators on X, with infinitesimal generator Δ. Then there exist some $K \geq 0$ and $c \in \mathbb{R}$ such that

$$\|S(t)\|_{\mathcal{B}(X)} \leq K e^{c|t|}, \qquad t \in \mathbb{R},$$

i.e. $(S(t))$ is exponentially bounded, and Δ is bisectorial-like of angle $\pi/2$ and half-width c, see [23, Subsect. 2.1.1].

Let μ be a complex bounded Borel measure on $\mathbb{R}$ such that $\int_{-\infty}^{\infty} e^{c|t|}|d\mu|(t) < \infty$ and let $\mathcal{L}_b(\mu)$ be its bilateral Laplace transform given by

$$\mathcal{L}_b(\mu)(z) := \int_{-\infty}^{\infty} e^{-zt} d\mu(t), \quad z \in BS_{\pi/2,c}.$$

Put $f := \mathcal{L}_b(\mu)(-\,\cdot\,)$.

Next, we state a result on transference of spectra from the generator of an operator group $(S(t))$ to integral operators subordinated to $(S(t))$, which is obtained on the basis of the spectral mappings given in Theorem 1.3.

Theorem 1.4. *For $(S(t))$, μ and f as above, suppose that $f \in \mathcal{E}(\Delta)$. Then*

(1)

$$f(\Delta) = \int_{-\infty}^{\infty} S(t)\, d\mu(t) \in \mathcal{B}(X),$$

whence

$$\widetilde{\sigma}(f(\Delta)) = f(\widetilde{\sigma}(\Delta)) = \{\mathcal{L}_b(\mu)(-z) \,:\, z \in \widetilde{\sigma}(\Delta)\},$$
$$\widetilde{\sigma}_{ess}(f(\Delta)) = f(\widetilde{\sigma}_{ess}(\Delta)) = \{\mathcal{L}_b(\mu)(-z) \,:\, z \in \widetilde{\sigma}_{ess}(\Delta)\},$$
$$\{\mathcal{L}_b(\mu)(-z) \,:\, z \in \sigma_{point}(\Delta)\} \subseteq \sigma_{point}(f(\Delta)) \subseteq \{\mathcal{L}_b(\mu)(-z) \,:\, z \in \sigma_{point}(\Delta)\} \cup \{\mathcal{L}_b(\mu)(\infty)\}.$$

(2) *If, moreover, f satisfies (1.7) one has*

$$\sigma_{point}(f(\Delta)) = f(\sigma_{point}(\Delta)) = \{\mathcal{L}_b(\mu)(-z) \,:\, z \in \sigma_{point}(\Delta)\}.$$

Proof. (1) The integral formula can be shown in a similar way to [23, Prop. 3.3.2] for sectorial operators, as it is noticed in [32, Prop. A.3]. The equalities involving the spectra in (1) and (2) are consequences of the integral identity and Theorem 1.3. □

Remark 1.5. Theorem 1.3 and Theorem 1.4 are given in [31] for the so-called regularized calculus, which involves meromorphic functions. In particular, versions of Theorem 1.4 can be obtained in other cases covering the regularized calculus, sectorial operators and operator semigroups which are not necessarily groups. However, we do not need such results here since our interest is focused on *groups of composition operators* on the unit disc. More precisely, we study weighted hyperbolic groups (acting on a specific but fairly general class of Banach spaces) whose definition is recalled right now.

Let $\mathcal{O}(\mathbb{D})$ be the Fréchet algebra of holomorphic functions on the unit disc $\mathbb{D}$. Let $Aut(\mathbb{D})$ be the group of automorphisms of the disc, that is, $\phi \in Aut(\mathbb{D})$ if and only if $\phi \in \mathcal{O}(\mathbb{D})$ and it is of the form $\phi(z) := e^{i\theta}\phi_\xi(z)$ for all $z \in \mathbb{D}$, where $\xi \in \mathbb{D}$ and $\theta \in [0, 2\pi)$, and where $\phi_\xi(z) = (1 - \overline{\xi}z)^{-1}(z - \xi)$. A family $(\psi_t)_{t\in\mathbb{R}}$ in $Aut(\mathbb{D})$ is said to be a group, or (holomorphic) flow, if

(1) $\psi_0(z) = z$ for all $z \in \mathbb{D}$;
(2) $\psi_{s+t} = \psi_s \circ \psi_t$ for all $s, t \in \mathbb{R}$;
(3) $\psi_t(z)$ is continuous in (t, z) on $\mathbb{R} \times \mathbb{D}$.

Here we use preferably the term *flow* to distinguish such families of automorphisms from groups of operators (on Banach spaces).

The infinitesimal generator of a given flow (ψ_t) is the function Ψ defined by the limit $\Psi(z) := \lim_{t\to 0} t^{-1}(\psi_t(z) - z)$, $z \in \mathbb{D}$. Actually, the limit exists uniformly on $\mathbb{D}$ (see [7, Section 8.2]), the mapping $t \mapsto \psi_t(z)$ is differentiable on $\mathbb{R}$ for every $z \in \mathbb{D}$, and one has $\frac{\partial\psi_t(z)}{\partial t} = \Psi(\psi_t(z))$, $z \in \mathbb{D}$, $t \in \mathbb{R}$. Flows of automorphisms are classified according to their fixed points. Namely, one says that the flow (ψ_t) is: 1) elliptic, if it has a unique fixed point in $\mathbb{D}$; 2) parabolic, if it has a unique fixed point in $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$; 3) hyperbolic, if it has two distinct fixed points in $\mathbb{T}$.

Here we deal with flows of hyperbolic automorphisms. For such a given flow (ψ_t) the well known Denjoy-Wolff theorem states that its fixed points in $\mathbb{T}$ are obtained as

$$a := \lim_{t\to+\infty} \psi_t(z),\, b := \lim_{t\to-\infty} \psi_t(z), \quad z \in \mathbb{D}.$$

Points a and b are called attractive and repulsive DW-points, respectively. There always exists an automorphism ϕ of $\mathbb{D}$ such $\phi(a) = 1$ and $\phi(b) = -1$, so that there exists $c > 0$ for which $\varphi_{ct} := \phi \circ \psi_t \circ \phi^{-1}$, $t \in \mathbb{R}$, where (φ_t) is the hyperbolic flow (0.3) with DW-points 1 (attractive) and -1 (repulsive). The generator G of (φ_t) is given by $G(z) = \frac{1}{2}(1 - z^2)$, $z \in \mathbb{D}$, and one also has

$$\frac{\partial \varphi_t(z)}{\partial t} = G(\varphi_t(z)) = \frac{\partial \varphi_t(z)}{\partial z} G(z), \quad z \in \mathbb{D},\, t \in \mathbb{R}. \tag{1.8}$$

For the above items and other details about flows of self-analytic maps of $\mathbb{D}$, see [4,5,10,35,38].

A family (v_t) of analytic functions $v_t\colon \mathbb{D} \to \mathbb{C}$ is called a continuous cocycle for (ψ_t) if

(1) $v_0(z) = 1$ for all $z \in \mathbb{D}$;
(2) $v_{s+t} = v_t \cdot (v_s \circ \psi_t)$ for all $s, t \in \mathbb{R}$;
(3) the mapping $t \mapsto v_t(z)$ is continuous on $\mathbb{R}$ for every $z \in \mathbb{D}$.

If the mapping $t \mapsto v_t(z)$ is differentiable on $\mathbb{R}$ for every $z \in \mathbb{D}$ the cocycle (v_t) is called differentiable. The infinitesimal generator g of a differentiable cocycle (v_t) is defined by $g(z) := \frac{\partial}{\partial t} v_t(z) \mid_{t=0}$. Suppose that g is analytic in $\mathbb{D}$. Then we define $\omega(z) := \exp\left(\int_0^z \frac{g(\xi)}{\Psi(\xi)}\, d\xi\right)$, $z \in \mathbb{D}$ (note that $\Psi \neq 0$ on $\mathbb{D}$ and the attractive DW-point a of (ψ_t) lies in $\mathbb{T}$). Then one has

$$v_t(z) = \frac{\omega(\psi_t(z))}{\omega(z)}, \quad z \in \mathbb{D},\, t \in \mathbb{R};$$

see [28, Lemma 2.2].

In this paper, we consider cocycles $(v_t)_{t \in \mathbb{R}}$ enjoying the following property:

$$(\forall t \in \mathbb{R}) \ \text{There exist } v_t(b) := \lim_{\mathbb{D} \ni z \to b} v_t(z) \in \mathbb{C}, \ v_t(a) := \lim_{\mathbb{D} \ni z \to a} v_t(z) \in \mathbb{C}; \tag{\textbf{Co1}}$$

see [8,27] for the suitability of this condition when dealing with the spectrum of weighted composition operators on Banach spaces.

Let X be a Banach function space continuously contained in $\mathcal{O}(\mathbb{D})$ (that is, $X \hookrightarrow \mathcal{O}(\mathbb{D})$ for short). Important examples of one-parameter groups in $\mathcal{B}(X)$ are the operator families of the form $(v_t C_{\psi_t})$ where (v_t) is a cocycle for a flow (ψ_t). In fact, that (v_t) is a cocycle is also a necessary condition for $(v_t C_{\psi_t})$ to be a group, see for example [22].

The function spaces X which we are dealing with in this paper satisfy that composition operators $C_\phi : X \to X$ ($C_\phi f = f \circ \phi$), $\phi \in Aut(\mathbb{D})$, are bounded isomorphisms of X, see Remark 2.2. Since multiplication by v_t is decomposed as

$$f \xrightarrow{C_{\psi_t^{-1}}} f \circ \psi_t^{-1} \xrightarrow{v_t C_{\psi_t}} v_t f,$$

we have that $v_t C_{\psi_t}$ is bounded on X if and only if the multiplication operator $f \mapsto v_t f$ is bounded on X which is to say that v_t is a multiplier of X. The space of multipliers of X is denoted by $Mul(X)$. In view of the above, it sounds sensible to consider the following property for a cocycle (v_t):

(**Co2**) The mapping $t \mapsto v_t$ is Bochner-measurable from $\mathbb{R}$ to $Mul(X)$.

Definition 1.6. Let (v_t) be a continuous cocycle for a hyperbolic flow (ψ_t). We say that (v_t) is a *DW-continuous cocycle* (for the flow (ψ_t)) on X if it satisfies conditions (**Co1**) and (**Co2**).

We are interested in groups $(v_t C_{\psi_t})$ where (ψ_t) is a hyperbolic flow and v_t is a DW-continuous cocycle. We have seen before that composition (on the left and on the right) of (ψ_t) with suitable $\phi \in Aut(\mathbb{D})$ turns (ψ_t) into the standard hyperbolic group (φ_t) of generator $G(z) = (1 - z^2)/2$. Let us now see how the action of ϕ affects weighted composition operators, under mild assumptions.

So let (ψ_t) be a hyperbolic flow of $\mathbb{D}$ with DW-points $a, b \in \mathbb{T}$ and let (v_t) be a DW-continuous cocycle for (ψ_t) so that $(v_t C_{\psi_t})$ is a one-parameter group in $\mathcal{B}(X)$. Take $\phi \in Aut(\mathbb{D})$ such that $\phi(a) = 1$, $\phi(b) = -1$. Then there exists some $c > 0$ for which $\varphi_{ct} = \phi \circ \psi_t \circ \phi^{-1}$ for all $t \in \mathbb{R}$, see [5]. Now set $u_t := v_{c^{-1}t} \circ \phi^{-1}$, thus $u_{ct} C_{\varphi_{ct}} = C_{\phi^{-1}} \circ (v_t C_{\psi_t}) \circ C_\phi$. It is readily seen that $t \mapsto u_t$ is measurable if and only if $t \mapsto v_t$ is measurable, hence (u_t) satisfies (**Co2**). Moreover, if there exist $v_t(a) := \lim_{\mathbb{D}\ni z\to a} v_t(z)$ and $v_t(b) := \lim_{\mathbb{D}\ni z\to b} v_t(z)$ in $\mathbb{C}$, then there exist $u_t(-1) := \lim_{\mathbb{D}\ni z\to -1} u_t(z)$, $u_t(1) := \lim_{\mathbb{D}\ni z\to 1} u_t(z)$ in $\mathbb{C}$, for all t, so (u_t) also satisfies (**Co1**), i.e. (u_t) is a DW-continuous cocycle for (φ_t). Since the operators C_ϕ and $C_{\phi^{-1}}$ are isomorphisms, the spectra of $v_t C_{\psi_t}$ and $u_{ct} C_{\varphi_t ct}$ are the same. Thus, from now on, we concentrate our study of spectra of weighted hyperbolic groups on families $(u_t C_{\varphi_t})$ of bounded operators on X where (φ_t) is the hyperbolic flow of (0.3) and (u_t) is a DW-continuous cocycle for (φ_t).

2. γ-conformal spaces

One of the aims of this paper is to study spectra of weighted composition groups $(v_t C_{\psi_t})$ acting on Banach spaces $X \hookrightarrow \mathcal{O}(\mathbb{D})$. In this section, we put up the setting where to work by introducing a number of conditions on X. We also show that most classical holomorphic function spaces satisfy such conditions. The two first of these conditions, namely (**Gam1**) and (**Gam2**), concern *multipliers*. For every open subset $U \subseteq \mathbb{C}$, let $H^\infty(U)$ be the Banach algebra of bounded analytic functions on U endowed with the sup-norm $\|f\|_{H^\infty(U)} := \sup_{z\in U} |f(z)|$, $f \in H^\infty(U)$. If $U = \mathbb{D}$ we write $\|\cdot\|_{H^\infty(\mathbb{D})} = \|\cdot\|_\infty$. Then, set

$$\bigcup_{\overline{\mathbb{D}}\subseteq U open} H^\infty(U) \hookrightarrow Mul(X), \tag{Gam1}$$

where the “hook” arrow on the right means that $\|F\|_{Mul(X)} \leq K_U \|F\|_{H^\infty(U)}$, if $F \in H^\infty(U)$, $\overline{\mathbb{D}} \subseteq U$ open, and K_U is a constant depending on U. By [15, Lemma 11], we have $Mul(X) \hookrightarrow H^\infty(\mathbb{D})$.

Let $\mathcal{P}$ denote the set of functions $f \in \mathcal{O}(\mathbb{D})$ of the form $f(z) = (\lambda z + \mu)^\delta$, $z \in \mathbb{D}$, with $\delta > 0$ and $\lambda, \mu \in \mathbb{C}$ such that $|\mu| \geq |\lambda|$, $\mu \neq 0$. Then, set

$$\mathcal{P} \subseteq Mul(X). \tag{Gam2}$$

The next property is a kind of splitting condition on X related, as we will see, with concentration on DW-points. For the rest of the paper, let ι denote the number -1 or 1. Let $\mathbb{D}_1 := \mathbb{D} \cap \{z : \mathfrak{Re}\, z > 0\}$ and $\mathbb{D}_{-1} := \mathbb{D} \cap \{z : \mathfrak{Re}\, z < 0\}$.

(**Gam3**) There are two Banach spaces $X_1 \hookrightarrow \mathcal{O}(\mathbb{D}_1)$, $X_{-1} \hookrightarrow \mathcal{O}(\mathbb{D}_{-1})$ such that the following holds true

- $X = \{f \in \mathcal{O}(\mathbb{D}) : f|_{\mathbb{D}_\iota} \in X_\iota,\ \iota = -1, 1\}$ (note that the mappings $f \mapsto f|_{\mathbb{D}_\iota}$ are continuous by the closed graph theorem).
- If U is an open set containing $\overline{\mathbb{D}_\iota}$, then $\mathcal{O}(U) \subseteq Mul(X_\iota)$.

In order to take advantage of the theory of C_0-groups, we also assume that

(**Gam4**) The one-parameter group of operators $(C_{\varphi_t})_{t\in\mathbb{R}}$ is strongly continuous on X.

The latter property is a mild assumption since every strongly measurable group of operators is strongly continuous on $\mathbb{R}$ as a consequence of [25, Th. 10.2.3].

Moreover, since (φ_t) is holomorphic in $\overline{\mathbb{D}}$, (**Gam4**) holds if the inclusion $\mathfrak{A}(\mathbb{D}) \hookrightarrow X$ [38, Section 4] is dense. Here, $\mathfrak{A}(\mathbb{D})$ is the disc algebra; that is, the Banach algebra of functions in $\mathcal{O}(\mathbb{D})$ with continuous extension to the closure $\overline{\mathbb{D}}$, endowed with the sup-norm.

Let us set some notation before introducing the two last properties. For $\rho \in \mathbb{R}$ and $\phi \in Aut(\mathbb{D})$ let $C_{\phi,\rho}$ denote the operator on $\mathcal{O}(\mathbb{D})$ given by $C_{\phi,\rho} := (\phi')^\rho C_\phi$, where ϕ' is the derivative of ϕ.

Definition 2.1. Let $\gamma \geq 0$ and let X be a Banach space such that $X \hookrightarrow \mathcal{O}(\mathbb{D})$, which separates points of $\mathbb{D}$, and such that it satisfies properties (**Gam1**)-(**Gam4**). We say that the space X is conformally invariant of index γ and tempered type, or just γ-space for short, if $C_{\phi,\gamma} \in \mathcal{B}(X)$ for all $\phi \in Aut(\mathbb{D})$ and

$$(\forall \varepsilon > 0) \qquad \sup_{\phi \in Aut(\mathbb{D})} (1 - |\phi(0)|)^\varepsilon \|C_{\phi,\gamma}\|_{\mathcal{B}(X)} < \infty. \tag{Gam5}$$

Let $\mathfrak{S}$ be a subset of $\mathcal{O}(\mathbb{D})$ which is invariant for multiplication by functions $z \mapsto (1-z)^\lambda (1+z)^\mu$ for any $\lambda, \mu \in \mathbb{C}$. We say that the pair $(X, \mathfrak{S})$ is a DW-conditioned pair of index γ, or $\gamma - pair$ for short, if X is a γ-space and

$$f \in \mathfrak{S} \text{ such that } |f(z)| \lesssim |(1-z)(1+z)|^{-\gamma+\varepsilon}, z \in \mathbb{D}, \text{ for some } \varepsilon > 0 \implies f \in X. \tag{Gam6}$$

Remark 2.2. (1) Since $\phi \in Aut(\mathbb{D})$ and $C_{\phi,\gamma} \in \mathcal{B}(X)$, it follows from $C_\phi = (\phi')^{-\gamma} C_{\phi,\gamma}$ that C_ϕ is a bounded isomorphism of X.

(2) One obtains from (**Gam5**) that $\sigma(C_{\phi,\gamma}) \subseteq \overline{\mathbb{D}}$. Indeed, if $\phi = \varphi_t$ for some $t \in \mathbb{R} \setminus \{0\}$ (the claim is trivial if $t = 0$), a straightforward calculation gives us

$$\|C^n_{\varphi_t,\gamma}\|_{\mathcal{B}(X)} = \|C_{\varphi_{nt},\gamma}\|_{\mathcal{B}(X)} \lesssim (1-|\varphi_{nt}(0)|)^{-\varepsilon} \lesssim (1+e^{n|t|})^{\varepsilon}, \tag{2.1}$$

for every $\varepsilon > 0$. Then, the spectral radius formula yields $\sigma(C_{\varphi_t,\gamma}) \subseteq \overline{\mathbb{D}}$, and our claim follows. If now ϕ is an arbitrary hyperbolic automorphism one can show, via some $\tilde{\phi} \in Aut(\mathbb{D})$, that the operator $C_{\phi,\gamma}$ is similar to $C_{\varphi_t,\gamma}$ for some $t \in \mathbb{R}$, thus $\sigma(C_{\phi,\gamma}) = \sigma(C_{\varphi_t,\gamma}) \subseteq \overline{\mathbb{D}}$.

Remark 2.3. The definition of γ-pair explicitly involves the canonical hyperbolic flow (φ_t) with DW-points -1 and 1. It must be noticed that such a definition could be also given in terms of an arbitrary hyperbolic flow (ψ_t) with DW-points $a, b \in \mathbb{T}$ instead. Since γ-spaces are C_ϕ-invariant ($\phi \in Aut(\mathbb{D})$), see Remark 2.2(1), all these definitions are indeed equivalent.

2.1. Examples

Here we list several classical Banach spaces which provide examples of γ-pairs.

(1) *Little Korenblum classes and the disc algebra.* For $\gamma \geq 0$, let $\mathcal{K}^{-\gamma}(\mathbb{D})$ be the weighted Korenblum growth class of order γ defined by

$$\mathcal{K}^{-\gamma}(\mathbb{D}) := \{f \in \mathcal{O}(\mathbb{D}) : \|f\|_{\mathcal{K}^{-\gamma}} := \sup_{z\in\mathbb{D}} (1-|z|^2)^{\gamma}|f(z)| < \infty\},$$

which is a Banach space endowed with the norm $\|\cdot\|_{\mathcal{K}^{-\gamma}}$. Note that $\gamma = 0$ corresponds to $H^\infty(\mathbb{D})$. These spaces fulfill all conditions (**Gam1**)-(**Gam6**), except for the strong continuity condition (**Gam4**). Indeed, for $f(z) = (i-z)^{-\gamma}$ if $\gamma > 0$, and $f(z) = (i-z)^i$ if $\gamma = 0$, one can check that the mapping $t \mapsto C_{\varphi_t} f$ is not norm continuous. However, as we pointed out above, the closure of $\mathfrak{A}(\mathbb{D})$ in these spaces satisfies (**Gam4**).
If $\gamma > 0$, then the closure of $\mathfrak{A}(\mathbb{D})$ in $\mathcal{K}^{-\gamma}(\mathbb{D})$ is the Little Korenblum growth class $\mathcal{K}_0^{-\gamma}(\mathbb{D})$ given by

$$\mathcal{K}_0^{-\gamma}(\mathbb{D}) := \{f \in \mathcal{K}^{-\gamma}(\mathbb{D}) \ : \ \lim_{|z|\to 1} (1-|z|^2)^{\gamma}|f(z)| = 0\},$$

with norm $\|\cdot\|_{\mathcal{K}^{-\gamma}}$. Then $(\mathcal{K}_0^{-\gamma}(\mathbb{D}), \mathcal{O}(\mathbb{D}))$ is a γ-pair for every $\gamma > 0$ which satisfies properties (**Gam1**)-(**Gam6**) as we check next.
(**Gam1**) and (**Gam2**): These are clear since $H^\infty(\mathbb{D}) \hookrightarrow Mul(\mathcal{K}_0^{-\gamma}(\mathbb{D}))$.
(**Gam3**): Let $C_0(\mathbb{D}_\iota, (1-|z|^2)^\gamma)$ be the Banach weighted space of continuous functions f on $\mathbb{D}_\iota$ such that

$$\lim_{|z|\to 1, z\in\mathbb{D}_\iota} (1-|z|^2)^{\gamma}|f(z)| = 0 \ \text{ and } \ \|f\|_{\mathcal{K}_\iota^{-\gamma}} := \sup_{z\in\mathbb{D}_\iota} (1-|z|^2)^{\gamma}|f(z)| < \infty.$$

Define

$$\mathcal{K}_0^{-\gamma}(\mathbb{D})_\iota := \mathcal{O}(\mathbb{D}_\iota) \cap C_0(\mathbb{D}_\iota, (1-|z|^2)^\gamma),$$

endowed with the norm $\|\cdot\|_{\mathcal{K}_\iota^{-\gamma}}$, for $\iota = 1, -1$. Since convergence in the norm $\|\cdot\|_{\mathcal{K}_\iota^{-\gamma}}$ implies uniform convergence on compact subsets of $\mathbb{D}_\iota$, it follows that $\mathcal{K}_0^{-\gamma}(\mathbb{D})_\iota$ is closed in the space $C_0(\mathbb{D}_\iota, (1-|z|^2)^\gamma)$. So $\mathcal{K}_0^{-\gamma}(\mathbb{D})_\iota$ is complete. It is also clear that $\mathcal{O}(U) \subseteq Mul(\mathcal{K}_0^{-\gamma}(\mathbb{D})_\iota)$ for all open subset $U \subseteq \mathbb{C}$ containing $\mathbb{D}_\iota$. Then the spaces $\mathcal{K}_0^{-\gamma}(\mathbb{D})_\iota$ satisfy **(Gam3)**.
(Gam4): This holds since the disc algebra $\mathfrak{A}(\mathbb{D})$ is a subspace dense in $\mathcal{K}_0^{-\gamma}(\mathbb{D})$.
(**Gam5**) and (**Gam6**): In fact, we have $\sup_{\phi\in Aut(\mathbb{D})} \|C_{\phi,\gamma}\|_{\mathcal{B}(\mathcal{K}_0^{-\gamma})} = 1$, as it was noted in [2,27]. Also, it is clear that (**Gam6**) holds for every $\gamma > 0$ and $f \in \mathcal{O}(\mathbb{D})$. So $(\mathcal{K}_0^{-\gamma}(\mathbb{D}), \mathcal{O}(\mathbb{D}))$ is a γ-pair for every $\gamma > 0$.
If $\gamma = 0$, when $\mathcal{K}^{-\gamma}(\mathbb{D})$ is $H^\infty(\mathbb{D})$, we have that the closure of the disc algebra $\mathfrak{A}(\mathbb{D})$ in $H^\infty(\mathbb{D})$ is $\mathfrak{A}(\mathbb{D})$ itself. Take $\mathfrak{S}(\mathfrak{A}) := \{f \in \mathcal{O}(\mathbb{D}) : f$ extends continuously to $\overline{\mathbb{D}} \setminus \{1, -1\}\}$. Then one can easily check that $(\mathfrak{A}(\mathbb{D}), \mathfrak{S}(\mathfrak{A}))$ is a 0-pair. For instance, condition **(Gam3)** is satisfied if we consider the Banach spaces of continuous functions $\mathfrak{A}(\mathbb{D})_\iota := \mathcal{O}(\mathbb{D}_\iota) \cap C(\overline{\mathbb{D}_\iota})$ with the sup-norm on $\mathbb{D}_\iota$.

Remark 2.4. Spaces $\mathcal{K}^{-\gamma}(\mathbb{D})$, $\gamma \geq 0$, enjoy the property that, for each $\gamma \geq 0$ and $\varepsilon > 0$, $\mathcal{K}^{-\gamma-\varepsilon}(\mathbb{D})$ contains every Banach space X satisfying (**Gam5**). In effect, in this case, for $f \in X$ one has

$$\begin{aligned}\sup_{z\in\mathbb{D}}(1-|z|^2)^{\gamma+\varepsilon}|f(z)| &= \sup_{\phi\in Aut(\mathbb{D})}(1-|\phi(0)|^2)^{\gamma+\varepsilon}|f(\phi(0))| \\ &= \sup_{\phi\in Aut(\mathbb{D})}(1-|\phi(0)|^2)^{\varepsilon}|(C_{\phi,\gamma}f)(0)| \\ &\lesssim \sup_{\phi\in Aut(\mathbb{D})}(1-|\phi(0)|^2)^{\varepsilon}\|C_{\phi,\gamma}f\|_X \lesssim \|f\|_X,\end{aligned}$$

where Schwarz-Pick's Lemma has been used in the second equality. This bound obviously implies $X \hookrightarrow \mathcal{K}^{-\gamma-\varepsilon}(\mathbb{D})$ as claimed.
Notice that if (**Gam5**) holds for $\varepsilon = 0$, then mimicking the above argument we have $X \hookrightarrow \mathcal{K}^{-\gamma}(\mathbb{D})$.

(2) *Hardy spaces of integrable functions.* For $1 \leq p < \infty$, let $H^p(\mathbb{D})$ be the Hardy space on $\mathbb{D}$ formed by all functions $f \in \mathcal{O}(\mathbb{D})$ such that

$$\|f\|_{H^p} := \sup_{0<r<1}\left(\int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi}\right)^{1/p} < \infty,$$

endowed with the norm $\|\cdot\|_{H^p}$.
We claim that $(H^p(\mathbb{D}), \mathcal{O}(\mathbb{D}))$ is a γ-pair for $\gamma = 1/p$. First, $H^\infty(\mathbb{D}) = Mul(H^p(\mathbb{D}))$ and therefore (**Gam1**), (**Gam2**) are fulfilled. **(Gam4)** holds since the disc algebra

$\mathfrak{A}(\mathbb{D})$ is dense in $H^p(\mathbb{D})$. It is well known that they satisfy (**Gam5**) even for $\varepsilon = 0$; in fact, operators $C_{\phi,\gamma}$ are isometries in this case, see [20, Th. 2]. (**Gam6**) is clear. Checking property (**Gam3**) requires a bit more of work:
Given a Banach space Z with norm $\|\cdot\|_Z$ and a set J, let $B(J;Z)$ denote the Banach space of $\|\cdot\|_Z$-bounded Z-valued functions on J, with norm $\|F\|_{Z,\infty} := \sup_{j\in J}\|F(j)\|_Z$. Put $\mathbb{T}_1 := \{z \in \mathbb{T} : \mathfrak{Re}\, z > 0\}$ and $\mathbb{T}_{-1} := \{z \in \mathbb{T} : \mathfrak{Re}\, z < 0\}$, and consider the Banach spaces

$$L^p(\mathbb{T}_{-1}) := \left\{ f\colon \mathbb{T}_{-1} \to \mathbb{C} : \|f\|_{p,-1} = \left(\int_{\pi/2}^{3\pi/2} |f(e^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{1/p} < \infty \right\},$$

$$L^p(\mathbb{T}_{1}) := \left\{ f\colon \mathbb{T}_{1} \to \mathbb{C} : \|f\|_{p,1} = \left(\int_{-\pi/2}^{\pi/2} |f(e^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{1/p} < \infty \right\}.$$

Take the interval $J = (0,1)$ in $\mathbb{R}$ and $Z = L^p(\mathbb{T}_\iota)$, $\iota = -1, 1$. Define

$$H^p(\mathbb{D})_\iota := \mathcal{K}^{-\gamma}(\mathbb{D}_\iota) \cap B((0,1); L^p(\mathbb{T}_\iota)),$$

where $\mathcal{K}^{-\gamma}(\mathbb{D}_\iota) = \{f \in \mathcal{O}(\mathbb{D}_\iota) : \|f\|_{\mathcal{K}_\iota^{-\gamma}} < \infty\}$. In such an intersection, an element $F \in \mathcal{K}^{-\gamma}(\mathbb{D}_\iota)$ is regarded as the family $(F_r)_{0<r<1}$ of functions on $\mathbb{T}$ where $F_r(z) := F(rz)$ for $r \in (0,1)$, $z \in \mathbb{T}$. Thus $F \in H^p(\mathbb{D})_\iota$ means that $F \in \mathcal{K}^{-\gamma}(\mathbb{D}_\iota)$ and $\widetilde{F}\colon (0,1) \to L^p(\mathbb{T}_\iota)$ given by $\widetilde{F}(r) := F_r$ satisfies $\sup_{0<r<1}\|\widetilde{F}(r)\|_{p,\iota} < \infty$. Then the space $H^p(\mathbb{D})_\iota$, provided with the norm

$$\|F\|_{H^p_\iota} := \|F\|_{\mathcal{K}_\iota^{-\gamma}} + \sup_{0<r<1}\|\widetilde{F}(r)\|_{p,\iota},$$

is a Banach space. Since $H^p(\mathbb{D}) \hookrightarrow \mathcal{K}^{-\gamma}(\mathbb{D})$, see the end of Remark 2.4, it is readily seen that $H^p(\mathbb{D})_\iota$ satisfies (**Gam3**).

(3) *Weighted Bergman spaces.* Let $1 \le p < \infty$ and $\sigma > -1$. Let $\mathcal{A}^p_\sigma(\mathbb{D})$ denote the weighted Bergman space formed by all holomorphic functions in $\mathbb{D}$ such that

$$\|f\|_{\mathcal{A}^p_\sigma} := \left(\int_{\mathbb{D}} |f(z))|^p d\mathcal{A}_\sigma(z) \right)^{1/p} < \infty,$$

where $d\mathcal{A}_\sigma(z) = (1-|z|^2)^\sigma\, dA(z)$, and where dA is the Lebesgue measure on $\mathbb{D}$. The space $\mathcal{A}^p_\sigma(\mathbb{D})$, with norm $\|\cdot\|_{\mathcal{A}^p_\sigma}$, is a Banach space such that the pair $(\mathcal{A}^p_\sigma(\mathbb{D}), \mathcal{O}(\mathbb{D}))$ is a γ-pair with for $\gamma = \frac{\sigma+2}{p}$.
Indeed, as in the above examples, $H^\infty(\mathbb{D}) = Mul(\mathcal{A}^p_\sigma(\mathbb{D}))$, so (**Gam1**), (**Gam2**) hold. Define $\mathcal{A}^p_\sigma(\mathbb{D})_\iota := \mathcal{O}(\mathbb{D}_\iota) \cap L^p(\mathbb{D}_\iota, (1-|z|^2)^\sigma)$. Clearly, $\mathcal{A}^p_\sigma(\mathbb{D})_\iota$ endowed with

the usual norm of $L^p(\mathbb{D}_\iota, (1-|z|^2)^\sigma)$ satisfies (**Gam3**). Moreover, $\mathcal{A}^p_\sigma(\mathbb{D})$ satisfies (**Gam4**) since $\mathfrak{A}(\mathbb{D})$ is dense in $\mathcal{A}^p_\sigma(\mathbb{D})$. It is well known that $\mathcal{A}^p_\sigma(\mathbb{D})$ satisfies (**Gam5**); see for instance the proof of [27, Th. 4.6].
Finally, (**Gam6**) is also satisfied. To see this, set $h_\varepsilon(z) := (1-z^2)^{-\gamma+\varepsilon}$, $z \in \mathbb{D}$, for $\varepsilon > 0$. Let us check that h_ε belongs to $\mathcal{A}^p_\sigma(\mathbb{D})$. Note that $h_\varepsilon \in \mathcal{A}^p_\sigma(\mathbb{D})$ if and only if $\int_{\mathbb{D}} |1-z^2|^{-\sigma-2+p\varepsilon} d\mathcal{A}_\sigma(z) < \infty$. Then the finiteness of the integral readily follows by decomposing it in three (finite, eventually) terms corresponding to the (integration) domains $\mathbb{D} \cap D(-1;1/2)$, $\mathbb{D} \setminus (D(-1;1/2) \cup D(1;1/2))$ and $\mathbb{D} \cap D(1;1/2)$ where $D(w;r) := \{z : |z-w| < r\}$, $w \in \mathbb{C}$, $r > 0$.

The two following examples are provided by Dirichlet spaces and Bloch spaces. To deal with them, we introduce the set $\mathfrak{S}_{log}$ of all functions $f \in \mathcal{O}(\mathbb{D})$, zero-free on $\mathbb{D}$, such that

$$(\forall \varepsilon > 0) \qquad \sup_{z\in\mathbb{D}} |(1-z^2)|^{1+\varepsilon} \left| \frac{f'(z)}{f(z)} \right| < \infty.$$

(4) *Weighted Dirichlet spaces.* For $p \geq 1$ and $\sigma > -1$, let $\mathcal{D}^p_\sigma(\mathbb{D})$ denote the weighted Dirichlet space of all functions $f \in \mathcal{O}(\mathbb{D})$ such that $f' \in \mathcal{A}^p_\sigma(\mathbb{D})$ and

$$\|f\|_{\mathcal{D}^p_\sigma} := \left(|f(0)|^p + \|f'\|^p_{\mathcal{A}^p_\sigma} \right)^{1/p} < \infty.$$

Then $\mathcal{D}^p_\sigma(\mathbb{D})$ is a Banach space with norm given by $\|\cdot\|_{\mathcal{D}^p_\sigma}$. When $\sigma > p-1$ one has $\mathcal{D}^p_\sigma(\mathbb{D}) = \mathcal{A}^p_{\sigma-p}(\mathbb{D})$ with equivalent norms, see e.g. [19, Th. 6]. Hence $(\mathcal{D}^p_\sigma(\mathbb{D}), \mathcal{O}(\mathbb{D}))$ is a γ-pair for $\gamma = \frac{\sigma+2}{p} - 1$.

In the case $p-2 \leq \sigma \leq p-1$, we prove that the pair $(\mathcal{D}^p_\sigma(\mathbb{D}), \mathfrak{S}_{log})$ is a γ-pair for $\gamma = \frac{\sigma+2}{p} - 1$. The following lemma concerns multipliers and shows that $\mathcal{D}^p_\sigma(\mathbb{D})$ satisfies properties (**Gam1**) and (**Gam2**).

Lemma 2.5. *Let $\sigma > -1, p \geq 1$ be such that $p-2 \leq \sigma \leq p-1$. Then $H^\infty(U) \hookrightarrow Mul(\mathcal{D}^p_\sigma(\mathbb{D}))$ for every open subset U of $\mathbb{C}$ such that $\overline{\mathbb{D}} \subseteq U$, and also $\mathcal{P} \subseteq Mul(\mathcal{D}^p_\sigma(\mathbb{D}))$.*

Proof. (1) The inclusion $H^\infty(U) \hookrightarrow Mul(\mathcal{D}^p_\sigma(\mathbb{D}))$ is well known. We include here a proof for the sake of completeness. Let U be an open subset of $\mathbb{C}$ such that $\overline{\mathbb{D}} \subseteq U$. Let $h \in H^\infty(U)$. For every $f \in \mathcal{D}^p_\sigma(\mathbb{D})$, one has $\|hf\|^p_{\mathcal{D}^p_\sigma} = |h(0)f(0)|^p + \|(hf)'\|^p_{\mathcal{A}^p_\sigma}$ with

$$\begin{aligned}\|(hf)'\|_{\mathcal{A}^p_\sigma} &\leq \|hf'\|_{\mathcal{A}^p_\sigma} + \|h'f\|_{\mathcal{A}^p_\sigma} \leq \|h\|_\infty \|f'\|_{\mathcal{A}^p_\sigma} + \|h'\|_\infty \|f\|_{\mathcal{A}^p_\sigma} \\ &\lesssim (\|h\|_\infty + \|h'\|_\infty) \|f'\|_{\mathcal{A}^p_\sigma},\end{aligned}$$

where we have used that $\|f\|_{\mathcal{A}^p_\sigma} \lesssim \|f'\|_{\mathcal{A}^p_{\sigma+p}} \le \|f'\|_{\mathcal{A}^p_\sigma}$ for all $f \in \mathcal{D}^p_\sigma(\mathbb{D})$, see for instance [19, Th. 6]. Now, using Cauchy's estimate for the derivative, one has $\|h'\|_\infty \lesssim \|h\|_{H^\infty(U)}$, and we are done.

Let now $g(z) = cz + d$, $z \in \mathbb{D}$, with $c, d \in \mathbb{C}$ such that $|c| \le |d|$ and take $\delta > 0$. If $|c| < |d|$ the function g^δ is a holomorphic function in an open set containing $\overline{\mathbb{D}}$ and therefore it is a multiplier of $\mathcal{D}^p_\sigma(\mathbb{D})$ as seen before. If $|c| = |d|$ one can assume that $g(z) = 1 - z$ since rotations are isometries of $\mathcal{D}^p_\sigma(\mathbb{D})$. Then, for every $f \in \mathcal{D}^p_\sigma(\mathbb{D})$, one has $\|g^\delta f\|^p_{\mathcal{D}^p_\sigma} = |g^\delta(0)f(0)|^p + \|(g^\delta f)'\|^p_{\mathcal{A}^p_\sigma}$ with

$$\begin{aligned}\|(g^\delta f)'\|_{\mathcal{A}^p_\sigma} &\le \|g^\delta\|_\infty \|f'\|_{\mathcal{A}^p_\sigma} + \delta \|g^\delta\|_\infty \|g^{-1} f\|_{\mathcal{A}^p_\sigma} \\ &\le 2^\delta \|f'\|_{\mathcal{A}^p_\sigma} + \delta 2^\delta \left(\int_{\mathbb{D}} |f(z)|^p \rho(z) dA(z) \right)^{1/p},\end{aligned}$$

where $\rho(z) := (1 - |z|^2)^\sigma |1 - z|^{-p}$, $z \in \mathbb{D}$.

Assume first $\sigma > p - 2$. Then, using [24, Th. 1.7], one has

$$\int_{\mathbb{D}} \frac{\rho(\zeta)}{|1 - \overline{\zeta} z|^{\eta+2}} dA(\zeta) = \int_{\mathbb{D}} \frac{(1 - |\zeta|^2)^\sigma}{|1 - \zeta|^p |1 - \overline{\zeta} z|^{\eta+2}} dA(\zeta) \lesssim \frac{\rho(z)}{(1 - |z|)^\eta}, \quad z \in \mathbb{D}.$$

In the terminology of [1], the above inequality implies that $\rho/(1 - |\cdot|)^\eta \in B_1^*(\eta)$, $\eta > \sigma$. Moreover, a few computations show that

$$\|\nabla \rho(z)\|_{\mathbb{R}^2} \le 2\sqrt{2}(|\sigma| + p) \frac{\rho(z)}{1 - |z|^2}, \qquad z \in \mathbb{D},$$

where $\nabla \rho$ denotes the gradient of the differentiable function ρ. In short, ρ satisfies condition (3.21) of [1]. Hence, we can apply [1, Th. 3.2(iv)] in the inequality "$\lesssim$" coming in to obtain

$$\begin{aligned}\int_{\mathbb{D}} |f(z)|^p \rho(z) dA(z) &\lesssim |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^p \rho(z) dA(z) \\ &\le |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^\sigma dA(z) = \|f\|^p_{\mathcal{D}^p_\sigma},\end{aligned}$$

(see also [2, Prop. 3.1]).

Assume now $\sigma = p - 2$ and take $\varepsilon \in (0, \delta)$. One gets

$$\begin{aligned}\|(g^\delta f)'\|_{\mathcal{A}^p_\sigma} &\le \|g^\delta\|_\infty \|f'\|_{\mathcal{A}^p_\sigma} + \delta \|g^{\delta-\varepsilon}\|_\infty \|g^{-(1-\varepsilon)} f\|_{\mathcal{A}^p_\sigma} \\ &\le 2^\delta \|f'\|_{\mathcal{A}^p_\sigma} + \delta 2^{\delta-\varepsilon} \left(\int_{\mathbb{D}} |f(z)|^p \rho_\varepsilon(z) dA(z) \right)^{1/p}\end{aligned}$$

with $\rho_\varepsilon(z) := (1-|z|^2)^\sigma|1-z|^{-p(1-\varepsilon)}$, $z \in \mathbb{D}$. The remainder of the argument goes along the same lines as in the case $\sigma > p-2$, where the weight ρ should be replaced by the weight ρ_ε.

All in all, one has $g^\delta \in Mul(\mathcal{D}^p_\sigma(\mathbb{D}))$ for every $\delta > 0$ and therefore $\mathcal{P} \subseteq Mul(\mathcal{D}^p_\sigma(\mathbb{D}))$. □

Let $\mathcal{D}^p_\sigma(\mathbb{D})_\iota := \{f \in \mathcal{O}(\mathbb{D}_\iota) : f' \in L^p(\mathbb{D}_\iota, (1-|z|^2)^\sigma)\}$ equipped with the norm

$$\|f\|_{\mathcal{D}^p_\sigma,\iota} := \left(|f(\iota/2)|^p + \int\limits_{\mathbb{D}_\iota} |f'(z)|^p \, d\mathcal{A}_\sigma(z)\right)^{1/p},$$

which satisfies (**Gam3**). Note that if (f_n) is a Cauchy sequence in $\mathcal{D}^p_\sigma(\mathbb{D})_\iota$ then there exists $g \in \mathcal{A}^p_\sigma(\mathbb{D})_\iota$ such that $\lim_n f'_n = g$ in $\mathcal{A}^p_\sigma(\mathbb{D})_\iota$. Since $\mathbb{D}_\iota$ is simply connected there exists a primitive function f of g, which we take such that $f(\iota/2) = \lim_n f_n(\iota/2)$. Thus we have that $\lim_n f_n = f$ in $\mathcal{D}^p_\sigma(\mathbb{D})_\iota$ and it follows that this space is complete. Moreover, (**Gam4**) is also satisfied since polynomials are dense in $\mathcal{D}^p_\sigma(\mathbb{D})$, and it is readily seen that the mapping $t \mapsto C_{\varphi_t}Q$ is norm continuous for every polynomial Q. The fact that the Dirichlet space satisfies (**Gam5**) and (**Gam6**) is proved in the following lemma.

Lemma 2.6. *Let $p \geq 1$ and $\sigma > -1$ be such that $p-2 \leq \sigma \leq p-1$. Then $(\mathcal{D}^p_\sigma(\mathbb{D}), \mathfrak{S}_{\log})$ is a γ-pair with $\gamma = \frac{\sigma+2}{p} - 1$.*

Proof. As noticed above, all that is left to prove is that the pair $(\mathcal{D}^p_\sigma(\mathbb{D}), \mathfrak{S}_{\log})$ satisfies properties (**Gam5**) and (**Gam6**). It is known that $\sup_{\phi\in Aut(\mathbb{D})} \|C_{\phi,\gamma}\|_{\mathcal{B}(\mathcal{D}^p_\sigma)} < \infty$ if and only if $\sigma > p-2$ with $\gamma = (\sigma+2)/p - 1$ [2, Prop. 3.1]. Thus $\mathcal{D}^p_\sigma(\mathbb{D})$ satisfies (**Gam5**) with $\gamma = (\sigma+2)/p - 1$ when $\sigma > p-2$. For $\sigma = p-2$, whence $\gamma = 0$, we show that $\mathcal{D}^p_{p-2}$ is a 0-space as follows.

Let $f \in \mathcal{D}^p_{p-2}(\mathbb{D})$ so that $f' \in \mathcal{A}^p_{p-2}(\mathbb{D}) \hookrightarrow \mathcal{K}^{-1}(\mathbb{D})$, see the end of Remark 2.4. Then, since $f(z) = f(0) + \int_0^z f'(\xi)d\xi$ for all $z \in \mathbb{D}$, we have

$$\begin{aligned}|f(z)| &\leq |f(0)| + \int\limits_{[0,z]} \|f'\|_{\mathcal{A}^p_{p-2}} (1-|\xi|)^{-1}|d\xi| \\ &\leq |f(0)| - \|f'\|_{\mathcal{A}^p_{p-2}} \log(1-|z|) \leq \|f\|_{\mathcal{D}^p_{p-2}} (1 - \log(1-|z|)),\end{aligned}$$

for all $z \in \mathbb{D}$ and $f \in \mathcal{D}^p_{p-2}(\mathbb{D})$. Hence, for every $\phi \in Aut(\mathbb{D})$,

$$\begin{aligned}\|f\circ\phi\|_{\mathcal{D}^p_{p-2}} &= \left(|f(\phi(0))|^p + \|\phi'(f'\circ\phi)\|^p_{\mathcal{A}^p_{p-2}}\right)^{1/p} \\ &= \left(|f(\phi(0))|^p + \|f'\|^p_{\mathcal{A}^p_{p-2}}\right)^{1/p} \qquad (2.2) \\ &\leq \|f\|_{\mathcal{D}^p_{p-2}} (1 - \log(1-|\phi(0)|)),\end{aligned}$$

where we have used that $C_{\phi,1}$ is an isometric isomorphism in $\mathcal{A}^p_{p-2}$ and the previous estimate for $|f(z)|$, $z \in \mathbb{D}$. Thus $\mathcal{D}^p_{p-2}(\mathbb{D})$ satisfies (**Gam5**) with $\gamma = 0$.
As for condition (**Gam6**), let $\gamma = \frac{\sigma+2}{p} - 1$ and $f \in \mathfrak{S}_{\log}$ such that, for some $\varepsilon > 0$, we have $|f(z)| \lesssim |1 - z^2|^{-\gamma+\varepsilon}$ for all $z \in \mathbb{D}$. Then

$$|f'(z)| = |f(z)|\frac{|f'(z)|}{|f(z)|} \lesssim |1 - z^2|^{-\gamma+\varepsilon}|1 - z^2|^{-1-\varepsilon/2} = |1 - z^2|^{-\gamma-1+\varepsilon/2},$$

for every $z \in \mathbb{D}$. Since $\gamma + 1 = (\sigma + 2)/p$ one gets $f' \in \mathcal{A}^p_\sigma(\mathbb{D})$; that is, $f \in \mathcal{D}^p_\sigma(\mathbb{D})$, which implies that $(\mathcal{D}^p_\sigma(\mathbb{D}), \mathfrak{S}_{\log})$ is a γ-pair with $\gamma = \frac{\sigma+2}{p} - 1$. □

(5) *Bloch spaces.* For $\delta > 0$, let $B_\delta(\mathbb{D})$ denote the Bloch space, that is, the space of holomorphic functions on $\mathbb{D}$ such that

$$\|f\|_{B_\delta} := |f(0)| + \sup_{z\in\mathbb{D}}(1 - |z|^2)^\delta |f'(z)| < \infty,$$

endowed with the norm $\|\cdot\|_{B_\delta}$. Let $B_{\delta,0}(\mathbb{D})$ denote the little Bloch space, consisting of the closure of polynomials in $B_\delta(\mathbb{D})$. One has indeed

$$B_{\delta,0}(\mathbb{D}) = \{f \in B_\delta(\mathbb{D}) : \lim_{|z|\to 1}(1 - |z|^2)^\delta |f'(z)| = 0\},$$

see [42, Prop. 2]. For $\delta > 1$ these spaces are Korenblum classes; i.e.,

$$B_\delta(\mathbb{D}) = \mathcal{K}^{-(\delta-1)}(\mathbb{D}) \text{ and } B_{\delta,0}(\mathbb{D}) = \mathcal{K}_0^{-(\delta-1)}(\mathbb{D})$$

with corresponding equivalent norms, see [42, Prop. 7].
For $\delta = 1$, $B_1(\mathbb{D})$ fails to satisfy condition (**Gam4**). In fact, the mapping $t \mapsto C_{\varphi_t} f$, with $f(z) = \mathrm{Log}(i - z)$, $z \in \mathbb{D}$, is not norm continuous (where Log is the branch of the logarithm with argument in $[\pi/2, 5\pi/2)$). On the other hand, $B_{1,0}(\mathbb{D})$ satisfies (**Gam4**) since the mapping $t \in \mathbb{R} \mapsto C_{\varphi_t} Q \in B_{1,0}(\mathbb{D})$ is continuous for every analytic polynomial Q and the space of analytic polynomials is dense in $B_{1,0}(\mathbb{D})$.

Let us show that the little Bloch space $B_{1,0}(\mathbb{D})$ is a 0-space and that $(B_{1,0}(\mathbb{D}), \mathfrak{S}_{log})$ is a 0-pair. We know that (**Gam4**) holds. As regards multipliers, we have

$$\begin{aligned} Mul(B_1(\mathbb{D}) &= Mul(B_{1,0}(\mathbb{D})) \\ &= \{f \in H^\infty(\mathbb{D}) : (1 - |\cdot|^2)\log(1 - |\cdot|^2)f' \in H^\infty(\mathbb{D})\}, \end{aligned}$$

see [42, Th. 27], from which (**Gam1**), (**Gam2**) follow.
Define $B_1(\mathbb{D})_\iota := \{f \in \mathcal{O}(\mathbb{D}_\iota) : \sup_{z\in\mathbb{D}_\iota}(1 - |z|^2)|f'(z)| < \infty\}$, with norm $\|f\|_{B_{\sigma,\iota}} := |f(\iota/2)| + \sup_{z\in\mathbb{D}_\iota}(1 - |z|^2)|f'(z)|$, and let $B_{1,0}(\mathbb{D})_\iota$ denote the closure of the polynomials in $B_1(\mathbb{D})_\iota$. If (f_n) is a Cauchy sequence in $B_{1,0}(\mathbb{D})_\iota$, then (f'_n) is convergent to some g in $\mathcal{K}^{-1}(\mathbb{D})_\iota$. Taking $f \in \mathcal{O}(\mathbb{D}_\iota)$ with $f' = g$ and

$f(\iota/2) = \lim_n f_n(\iota/2)$ we get $\lim_n f_n = f$ in $B_{1,0}(\mathbb{D})_\iota$. In short, $B_{1,0}(\mathbb{D})_\iota$ is complete, and it is readily seen that $B_{1,0}(\mathbb{D})_\iota$ satisfies **(Gam3)** for $B_{1,0}(\mathbb{D})$.

Now, for every $\phi \in Aut(\mathbb{D})$,

$$\|f\circ\phi\|_{B_{1,0}} = |f(\phi(0))| + \sup_{z\in\mathbb{D}}(1-|z|^2)|\phi'(z)f'(\phi(z))|,$$

with

$$\begin{aligned}|f(\phi(0))| &\le |f(0)| + \int_0^{\phi(0)} |f'(\xi)||d\xi| \\ &\lesssim \|f\|_{B_{1,0}}\left(1 + \int_0^{\phi(0)} (1-|\xi|)^{-1} d\xi\right) = \|f\|_{B_{1,0}}(1-\log(1-|\phi(0)|)).\end{aligned}$$

On the other hand, using the Schwarz-Pick lemma one has

$$\sup_{z\in\mathbb{D}}(1-|z|^2)|\phi'(z)f'(\phi(z))| \le \sup_{z\in\mathbb{D}}(1-|\phi(z)|^2)|f'(\phi(z))| \le \|f\|_{B_{1,0}}.$$

Thus (**Gam5**) holds. Finally, by an argument like in the case of Dirichlet spaces, it can be seen that $(B_{1,0}(\mathbb{D}), \mathfrak{S}_{\log})$ satisfies (**Gam6**).

3. Cocycles for the hyperbolic group on γ-spaces

Let X be a γ-space for some $\gamma \ge 0$ and let (u_t) be a DW-continuous cocycle for the hyperbolic flow (φ_t) on X. Condition **(Co2)** together with **(Gam4)** imply that the mapping $t \mapsto u_t C_{\varphi_t}$ is strongly measurable, hence $(u_t C_{\varphi_t})$ is a C_0-group of bounded operators on X, see [25, Th. 10.2.3]. This fact implies, along the same lines as in [28, Th. 1], that there exists a holomorphic function $\omega : \mathbb{D} \to \mathbb{C}$ with no zeros such that $u_t = (\omega\circ\varphi_t)/\omega$ for all $t \in \mathbb{R}$.

The first part of this section is devoted to show that functions ω associated to DW-continuous cocycles $(u_t) \subseteq Mul(X)$ as indicated above, present zeroes or singularities of polynomial type at -1 and 1. In the second part, further additional properties of γ-spaces, regarding DW-continuous cocycles, are introduced.

Every measurable subadditive function on $(0,\infty)$ is locally bounded [14, p. 618]. Inspired by this result, we obtain the lemma which follows.

Lemma 3.1. *Let $g\colon (0,\infty) \to \mathbb{R}$ be a measurable function such that*

$$g(s+t) \le g(s) + g(t) + H(s,t) \quad s,t > 0,$$

where H is nondecreasing if s, t increase simultaneously. Then g is locally bounded on $(0,\infty)$.

Proof. Take $a > 0$ and put $F := \{t \in (0,a) : g(t) \geq (g(a) - H(a,a))/2)\}$. For a given $t \in (0,a)$ with $t \notin F$ one has $g(t) < g(a)/2 - H(t,a-t)/2$. Also, $g(a) \leq g(t) + g(a-t) + H(t,a-t)$. All in all,

$$\begin{aligned} g(a-t) &\geq g(a) - g(t) - H(t,a-t) \\ &> g(a) - \frac{g(a) - H(t,a-t)}{2} - H(t,a-t) \geq \frac{g(a)}{2} - \frac{H(a,a)}{2}, \end{aligned}$$

since H is nondecreasing. Hence $t \in a-F$; that is, $(0,a) = F \cup (a-F)$ and so $\mu(F) \geq a/2$.

Suppose now, if possible, that g is unbounded on $[c,d]$ for some $c,d > 0$. Take a sequence (s_n) in $[c,d]$ such that $g(s_n) \geq 2n$ for each $n \in \mathbb{N}$. Put $B_n := \{0 < t < d : g(t) \geq n - H(d,d)\}$, $n \geq 1$. Applying the above argument to $F_n := \{0 < t < s_n : g(t) \geq (g(s_n) - H(s_n,s_n))/2\}$ we get $\mu(B_n) \geq c/2$ since $F_n \subseteq B_n$, for all $n \geq 1$. Then, taking $t \in \bigcap_{n=1}^{\infty} B_n$ one gets $g(t) = \infty$, which is a contradiction.

In conclusion, g is locally bounded, as we claimed. □

Lemma 3.2. *For (u_t) as above, the mapping $t \mapsto \|u_t\|_{Mul(X)}$ is locally bounded on $\mathbb{R}$.*

Proof. First, we prove that for every $\varepsilon > 0$ there is $K_\varepsilon > 0$ such that

$$\|u_{s+t}\|_{Mul(X)} \leq \|u_s\|_{Mul(X)} \|u_t\|_{Mul(X)} \left(K_\varepsilon e^{\varepsilon \min\{|s|,|t|\}}\right)^2, \qquad s,t \in \mathbb{R}. \tag{3.1}$$

Note that $(u_s \circ \varphi_t) f = C_{\varphi_t,\gamma}(u_s C_{\varphi_{-t},\gamma} f)$ for any $f \in X$, thus $u_s \circ \varphi_t \in Mul(X)$ for every $s,t \in \mathbb{R}$. Moreover, by the cocycle property $u_{s+t} = u_s(u_t \circ \varphi_s) = u_t(u_s \circ \varphi_t)$, hence

$$\begin{aligned} &\|u_{s+t}\|_{Mul(X)} \\ &\leq \min\left\{\|u_s\|_{Mul(X)} \|u_t \circ \varphi_s\|_{Mul(X)}, \|u_t\|_{Mul(X)} \|u_s \circ \varphi_t\|_{Mul(X)}\right\}, \quad s,t \in \mathbb{R}. \end{aligned}$$

In addition, $\|u_s \circ \varphi_t\|_{Mul(X)} \leq \|C_{\varphi_t,\gamma}\|_{\mathcal{B}(X)} \|u_s\|_{Mul(X)} \|C_{\varphi_{-t},\gamma}\|_{\mathcal{B}(X)}$. Since $\|C_{\varphi_t,\gamma}\|_{\mathcal{B}(X)} \leq K_\varepsilon e^{\varepsilon|t|}$ for $t \in \mathbb{R}$ (see (2.1)), the inequality (3.1) follows. Hence, for $s,t \in \mathbb{R}$,

$$\log \|u_{s+t}\|_{Mul(X)} \leq \log \|u_s\|_{Mul(X)} + \log \|u_t\|_{Mul(X)} + 2(\varepsilon \min\{|t|,|s|\} + \log K_\varepsilon). \tag{3.2}$$

Thus applying Lemma 3.1 to $g(t) := \log \|u_t\|_{Mul(X)}$ and $H(s,t) := 2(\varepsilon \min\{|t|,|s|\} + \log K_\varepsilon)$, $s,t > 0$, we obtain that $t \mapsto \|u_t\|_{Mul(X)}$ is bounded on $[c,d]$ if $cd > 0$. So it remains to prove the result for $[c,d]$ with $c < 0$ and $d > 0$.

Fix s big enough so that $s >> |c|$ and $s >> d$. By (3.1)

$$\|u_t\|_{Mul(X)} \leq \|u_s\|_{Mul(X)} \|u_{t-s}\|_{Mul(X)} \left(K_\varepsilon e^{\varepsilon \min\{|s|,|t-s|\}}\right)^2, \qquad t \in [c,d],$$

which is uniformly bounded since $s, t-s$ are bounded away from zero. □

Lemma 3.3. *Let (u_t) be a cocycle as above. Then, u_t has no zero for any $t \in \mathbb{R}$, and the family (u_t^{-1}) is a DW-continuous cocycle for the flow (φ_t) on X.*

Proof. First, for each $t \in \mathbb{R}$, u_t has no zero on $\mathbb{D}$, see [28, Lemma 2.1], so u_t^{-1} is well defined. Moreover, by the cocycle property of (u_t) it follows that $u_t^{-1} = u_{-t} \circ \varphi_t$, $t \in \mathbb{R}$, and then it is readily seen that (u_t^{-1}) is a continuous cocycle for (φ_t).

Now, note that $(u_{-t} \circ \varphi_t)f = C_{\varphi_t}(u_{-t}C_{\varphi_{-t}}f)$, $f \in X$, so that $u_t^{-1} = u_{-t} \circ \varphi_t$ is a multiplier in X since $C_{\varphi_t}, C_{\varphi_{-t}}$ are isomorphisms on X, see Remark 2.2(1). In fact, u_t^{-1} is the inverse multiplier of u_t.

Recall that $Mul(X) \hookrightarrow H^\infty(\mathbb{D})$ as we pointed out in Section 2. This implies that u_t^{-1} is bounded, hence $u_t(1), u_t(-1) \neq 0$ for any $t \in \mathbb{R}$, and as a consequence u_t^{-1} is continuous at the DW-points $-1, 1$, that is, it satisfies (**Co1**). Finally, the mapping $t \mapsto u_t^{-1}$ is measurable since it is the composition of the measurable mapping $t \mapsto u_t$ and the (continuous) inversion map in the group of invertible multipliers of X. Hence, (u_t^{-1}) fulfills (**Co2**). □

Lemma 3.4. *Let (u_t) be a cocycle as above. Then there are $K, w > 0$ such that, for every $t \in \mathbb{R}$,*

$$\sup\left\{\|u_t\|_{Mul(X)}, \|u_t^{-1}\|_{Mul(X)}\right\} \leq Ke^{w|t|},$$
$$\sup\left\{\|u_t\|_\infty, \|u_t^{-1}\|_\infty\right\} \leq Ke^{w|t|}.$$

Proof. By Lemma 3.2 there exists $M > 0$ for which $\sup_{-1\leq t\leq 1} \log \|u_t\|_{Mul(X)} \leq M$. We show by induction that $\log \|u_t\|_{Mul(X)} \leq M + m|t|$ for every $t \in \mathbb{R}$, where $m = 2(\varepsilon + \log K_\varepsilon)$, where $K_\varepsilon, \varepsilon$ are taken as in (3.2). The claim is trivial if $|t| \leq 1$, so assume it holds for all $|t| \leq n$ for some $n \in \mathbb{N}$. Then, for $t \in [n, n+1]$, the inequality (3.2) implies

$$\begin{aligned}\log \|u_t\|_{Mul(X)} &\leq \log \|u_{t-1}\|_{Mul(X)} + \log \|u_1\|_{Mul(X)} + m \\ &\leq M + m|t-1| + m = M + m|t|.\end{aligned}$$

The above inequality is proven analogously for $t \in [-n-1, -n]$, thus the induction holds true and the bound of the lemma follows for $\|u_t\|_{Mul(X)}$.

As regards the inequality for $\|u_t^{-1}\|_{Mul(X)}$, Lemma 3.3 implies that (u_t^{-1}) is a well-defined DW-continuous cocycle for the flow (φ_t), hence the claim follows by what we have already proven for (u_t).

To finish the proof, recall that by [15, Lemma 11], the continuous inclusion $Mul(X) \hookrightarrow H^\infty(\mathbb{D})$ holds, so the inequalities of the claim for $\|u_t\|_\infty, \|u_t^{-1}\|_\infty$ follow from the ones we have already proven. □

The real numbers α_u, β_u found in the following lemma will be called *exponents* of (u_t). They play a central role in our spectral discussion in this paper. Recall that $u_t(1) := \lim_{\mathbb{D}\ni z\to 1} u_t(z)$ and $u_t(-1) := \lim_{\mathbb{D}\ni z\to -1} u_t(z)$.

Lemma 3.5. *There exists* $\alpha_u, \beta_u \in \mathbb{R}$ *such that*

$$|u_t(1)| = e^{\alpha_u t}, \quad |u_t(-1)| = e^{\beta_u t}, \qquad t \in \mathbb{R}.$$

Proof. The mapping $t \mapsto |u_t(\iota)|$ is a group homomorphism for $\iota = -1, 1$ since

$$u_{s+t}(\iota) = \lim_{\mathbb{D}\ni z\to\iota} u_{s+t}(z) = \left(\lim_{\mathbb{D}\ni z\to\iota} u_s(z)\right)\left(\lim_{\mathbb{D}\ni z\to\iota} u_t(\varphi_s(z))\right) = u_s(\iota)u_t(\iota), \quad s,t \in \mathbb{R},$$

where we have used that $\lim_{\mathbb{D}\ni z\to\iota} \varphi_t(z) = \iota$ through $\mathbb{D}$ for all $t \in \mathbb{R}$. It follows from Lemma 3.4 that $t \mapsto |u_t(\iota)|$ is a locally bounded homomorphism from $\mathbb{R}$ to $\mathbb{R}^+$, so it satisfies Cauchy's exponential functional equation. Hence there exists $c_\iota \in \mathbb{R}$ such that $u_t(\iota) = e^{c_\iota t}$, and the claim follows. □

One can deduce from [27, Lemma 4.4] that $\lim_{\mathbb{N}\ni n\to\infty} \|u_n\|_\infty^{1/n} = \max\{|u_1(1)|, |u_1(-1)|\}$ for every DW-continuous cocycle (u_t). We need extensions of this property, which are pointed out in the following lemma.

Lemma 3.6. *Let* $t \in \mathbb{R} \setminus \{0\}$. *Then*

$$\lim_{x\to\infty} \|u_{xt}\|_\infty^{1/x} = \max\{|u_t(1)|, |u_t(-1)|\}.$$

In addition, for $t > 0$ *it holds that*

$$\lim_{x\to\infty} \|u_{xt}\|_{H^\infty(\mathbb{D}_1)}^{1/x} = |u_t(1)|, \quad \lim_{x\to-\infty} \|u_{xt}\|_{H^\infty(\mathbb{D}_{-1})}^{-1/x} = |u_{-t}(-1)|.$$

Proof. The existence of $\lim_{x\to\infty} \|u_{xt}\|_\infty^{1/x}$, as well as the first equality, is a consequence the fact that $t \mapsto \log \|u_t\|_\infty$ is a subadditive function of [27, Lemma 4.4].

The other claims in the statement regarding the limits are obtained similarly to the above and reasoning as in the proof of [27, Lemma 4.4]. □

As it has been said, ω is a zero-free holomorphic function related to (u_t) by $u_t = (\omega \circ \varphi_t)/\omega$. We show in Theorem 3.11 that ω has tempered zeroes or singularities at the DW-points. This property is one of the key facts through our discussion in this paper.

Remark 3.7. In terms of the function ω, Lemma 3.6, second half, reads

$$\lim_{s\to\infty} \left\|\frac{\omega \circ \varphi_s}{\omega}\right\|_{H^\infty(\mathbb{D}_1)}^{1/s} = e^{\alpha_u}, \qquad \lim_{s\to-\infty} \left\|\frac{\omega \circ \varphi_s}{\omega}\right\|_{H^\infty(\mathbb{D}_{-1})}^{-1/s} = e^{-\beta_u}.$$

Lemma 3.8. *Let* ω *be as above, let* $\lambda, \nu \in \mathbb{C}$ *and set* $\rho(z) = \omega(z)(1-z)^\lambda(1+z)^\nu$ *for* $z \in \mathbb{D}$. *Then the cocycle* (v_t) *given by* $v_t = (\rho \circ \varphi_t)/\rho$ *is a* DW*-continuous cocycle for* (φ_t) *on* X *with exponents* $\alpha_v = \alpha_u - \mathfrak{Re}\,\lambda$ *and* $\beta_v = \beta_u + \mathfrak{Re}\,\nu$.

Proof. Given a bounded interval $I \subseteq \mathbb{R}$ and $t \in \mathbb{R}$ there exists an open subset U containing the closed disc $\overline{\mathbb{D}}$ such that the function h_t given by

$$h_t(z) = \left(\frac{1-\varphi_t(z)}{1-z}\right)^{\lambda} \left(\frac{1+\varphi_t(z)}{1+z}\right)^{\nu} = \left(\frac{2}{(e^t-1)z+e^t+1}\right)^{\lambda+\nu} e^{\nu t}, \quad z \in U,$$

is holomorphic in U for all $t \in I$. Then we have that $v_t = u_t h_t$ is a continuous cocycle which is continuous at the DW-points $-1, 1$. Thus it satisfies (**Co1**).

Moreover, U can be chosen for the mapping $t \in I \mapsto h_t \in H^\infty(U)$ to be continuous. Since $H^\infty(U) \hookrightarrow Mul(X)$ by (**Gam1**), it follows that the mapping $t \in I \mapsto v_t \in Mul(X)$ is measurable, so that (v_t) satisfies (**Co2**), that is, (v_t) is a DW-continuous cocycle.

Regarding the exponents of (v_t), a few computations show that $\lim_{z\to 1}(1-\varphi_t(z))/(1-z) = e^{-t}$ and $\lim_{z\to -1}(1+\varphi_t(z))/(1+z) = e^t$, $t \in \mathbb{R}$. In addition, $\lim_{z\to -1}(1-\varphi_t(z))/(1-z) = \lim_{z\to 1}(1+\varphi_t(z))/(1+z) = 1$ since both $-1, 1$ are fixed points of φ_t. Hence we conclude $\lim_{z\to 1}|v_t(z)| = \lim_{z\to 1}|u_t(z)||h_t(z)| = e^{\alpha_u t}|e^{-\lambda t}| = e^{(\alpha_u - \mathfrak{Re}\,\lambda)t}$, i.e. $\alpha_v = \alpha_u - \mathfrak{Re}\,\lambda$. Similarly we obtain $\beta_v = \beta_u + \mathfrak{Re}\,\nu$ and the proof is finished. □

Remark 3.9. According to (1.8), the following equality holds

$$(\varphi_t')^\delta = \frac{G^\delta \circ \varphi_t}{G^\delta}; \quad t \in \mathbb{R},\ \delta \in \mathbb{R},$$

where G is the generator of the flow (φ_t) given by $G(z) = (1-z^2)/2$, $z \in \mathbb{D}$. Whence, it follows by Lemma 3.8 that, for every $\delta \in \mathbb{R}$ and an arbitrary DW-continuous cocycle (u_t) for the flow (φ_t) on X, the family $(u_t(\varphi_t')^\delta)$ is a DW-continuous cocycle for the flow (φ_t) on X. In particular, taking $u_t = \mathbf{1}$ (i.e. the constant function equal to 1) we have that $((\varphi_t')^\delta)$ is a DW-continuous cocycle for the flow (φ_t) on X.

Lemma 3.10. *Let $A \subseteq \mathbb{D}$ be such that $\{-1,1\} \cap \overline{A} = \emptyset$. For ω as above, $\sup_{z\in A}|\omega(z)| < \infty$ and $\inf_{z\in A}|\omega(z)| > 0$.*

Proof. The claim is trivial if ω is a constant function, so let us assume that ω is not constant.

As neither -1 nor 1 belong to $\overline{A}$, it is readily seen that there exists $R > 0$ such that, for any $z \in \overline{A}$ there are (unique) $x \in (-1,1)$ and $t \in [-R,R]$ such that $z = \varphi_t(ix)$. As $\omega(z) = \omega(\varphi_t(ix)) = u_t(ix)\omega(ix)$, one has by Lemma 3.4

$$\sup_{z\in A}|\omega(z)| \le \left(\sup_{x\in(-1,1)}|\omega(ix)|\right)\left(\sup_{x\in(-1,1),\, t\in[-R,R]}|u_t(ix)|\right) \lesssim \sup_{x\in(-1,1)}|\omega(ix)|.$$

Next, we prove $\sup_{x\in(-1,1)}|\omega(ix)| < \infty$ by reaching a contradiction. Thus, suppose for a moment $\sup_{x\in(-1,1)}|\omega(ix)| = \infty$. In this case, for some $d \in \{-1,1\}$, there exists a sequence $(-1,1) \ni x_n \to d$ such that $\lim_{n\to\infty}|\omega(ix_n)| = \infty$. As a consequence, if the limit $\lim_{(-1,1)\ni x\to d}|\omega(ix)|$ existed, it would be equal to ∞. Assume that this is the case with $d=1$ (the argument for $d=-1$ works similarly). Now, for $\theta\in(0,\pi)$, let t_θ denote the unique real number for which $\varphi_{t_\theta}(i)=e^{i\theta}$. A few computations show that

$$t_\theta = 2\tanh^{-1}\left(\frac{-\cos\theta}{1+\sin\theta}\right), \quad \theta\in(0,\pi).$$

Therefore, the mapping $\Phi:[0,1]\times(0,\pi)\to\mathbb{C}$ given by $\Phi(x,\theta)=\varphi_{t_\theta}(ix)$ is continuous. Even more, $\Phi([0,1)\times(0,\pi))\subseteq\mathbb{D}$ and $\Phi(1,\theta)=e^{i\theta}$, so Φ is a continuous family of paths in the sense of [13, pp. 83]. Since there exist $K,w>0$ such that the bound $\|u_t^{-1}\|_\infty\le Ke^{w|t|}$ holds for all $t\in\mathbb{R}$ (see Lemma 3.4), it follows that

$$\lim_{x\to1^-}|\omega(\Phi(x,\theta))| = \lim_{x\to1^-}|\omega(\varphi_{t_\theta}(ix))| = \lim_{x\to1^-}|u_{t_\theta}(ix)||\omega(ix)| = \infty,$$

for all $\theta\in(0,\pi)$, which is absurd by the uniqueness of limits along the family of continuous path Φ, see [13, pp. 83].

Before continuing with the proof, we assume furthermore that $\alpha_u<0$ and $\beta_u>0$. Then, Remark 3.7 implies that there exists $M>0$ such that

$$|\omega(\varphi_s(ix))| < |\omega(ix)|, \qquad \text{for all } |s|\ge M,\ x\in(-1,1). \tag{3.3}$$

We now continue with the proof of the lemma. As $\lim_{(-1,1)\ni x\to d}|\omega(ix)|$ does not exist and in particular is not equal to ∞ for neither $d=-1$ nor $d=1$, there exist $K>0$ and a sequence $(y_n)_{n\in\mathbb{N}}\subseteq(-1,1)$ with accumulation points $-1,1$ and such that $|\omega(iy_n)|\le K$ for all $n\in\mathbb{N}$. One has $\mu:=\sup_{t\in[-M,M]}\|u_t\|_\infty<\infty$ by Lemma 3.4, where $M>0$ is as in (3.3). Take C such that $C>\max\{\mu,1\}$ and $\widetilde{x}:=x_{N_1}$, $\widetilde{y}:=y_{N_2}$, $\widetilde{z}=y_{N_3}$ for $N_1,N_2,N_3\in\mathbb{N}$ such that $|\omega(i\widetilde{x})|>CK$ and $\widetilde{z}<\widetilde{x}<\widetilde{y}$. Let $B\subseteq\mathbb{D}$ be the compact subset

$$B:=\{\varphi_s(ix)\,|\,(x,s)\in[\widetilde{z},\widetilde{y}]\times[-M,M]\}.$$

We now prove that $|\omega|$ reaches its maximum in B in its interior, which contradicts the maximum modulus principle. Let $L=\max_{x\in[\widetilde{z},\widetilde{y}]}|\omega(ix)|$, which is attained in $(\widetilde{z},\widetilde{y})$ since $|\omega(i\tilde{x})|>|\omega(i\tilde{y})|,|\omega(i\tilde{z})|$. Now, notice that

$$\begin{aligned}\max\{|\omega(\varphi_s(i\widetilde{z}))|,\,|\omega(\varphi_s(i\widetilde{y}))|\} &\le C\max\{|\omega(i\widetilde{z})|,\,|\omega(i\widetilde{y})|\}\le CK\\ &< |\omega(i\widetilde{x})|\le L,\ s\in[-M,M],\end{aligned}$$

and, by (3.3),

$$\max\{|\omega(\varphi_{-M}(ix))|,\,|\omega(\varphi_M(ix))|\} < |\omega(ix)|\le L, \quad x\in[\widetilde{z},\widetilde{y}].$$

Hence the maximum of $|\omega|$ in B is not attained in its boundary, reaching a contradiction. Therefore, $\sup_{x\in(-1,1)}|\omega(ix)|<\infty$.

If $\alpha_u \geq 0$ or $\beta_u \leq 0$, we consider the weight $\rho(z) := \omega(z)(1-z)^{-N}(1+z)^M$ and its associated cocycle $v_t := (\rho \circ \varphi_t)/\rho$, where $N > |\alpha_u|$, $M > |\beta_u|$. It follows by Lemma 3.8 that (v_t) is a DW-continuous cocycle with $\alpha_v = \alpha_u - N < 0$ and $\beta_v = \beta_u + M > 0$, so by what we have already proven, $\sup_{x\in(-1,1)} |\rho(ix)| < \infty$, and as a consequence, $\sup_{x\in(-1,1)} |\omega(ix)| \leq 2^{N/2} \sup_{x\in(-1,1)} |\rho(ix)| < \infty$, as we wanted to show.

Finally, consider the DW-continuous cocycle given by (u_t^{-1}), see Lemma 3.3, and let A be a subset as in the statement. Then the weight associated with (u_t^{-1}) is ω^{-1}, whence it follows from the above that $\sup_{z\in A} |\omega(z)^{-1}| < \infty$, that is, $\inf_{z\in A} |\omega(z)| = (\sup_{z\in A} |\omega^{-1}(z)|)^{-1} > 0$. $\square$

Theorem 3.11. *Let ω be the holomorphic function associated with a DW-continuous cocycle (u_t), so $u_t = (\omega \circ \varphi_t)/\omega$. Let α_u, β_u be the exponents of (u_t). Then, for every $\varepsilon > 0$, one has*

$$|\omega(z)| \lesssim |1-z|^{-\alpha_u - \varepsilon} |1+z|^{\beta_u - \varepsilon}, \quad z \in \mathbb{D},$$
$$|\omega(z)| \gtrsim |1-z|^{-\alpha_u + \varepsilon} |1+z|^{\beta_u + \varepsilon}, \quad z \in \mathbb{D}.$$

Proof. By Lemma 3.10, we only have to prove the inequalities of the claim for some arbitrary neighbourhoods $\mathcal{U}_{-1}, \mathcal{U}_1$ of -1, 1 respectively. We prove it for $\mathcal{U}_1$ of 1, being the other one analogous. One has

$$\frac{1-\varphi_s(z)}{1-z} e^s \to 1, \text{ as } z \to 1,$$

uniformly on $s > 0$. On the other hand, by Remark 3.7, for any $\varepsilon' > 0$, there exists some $M > 0$ such that

$$|\omega(\varphi_s(z))| \leq |\omega(z)| e^{s(\alpha_u + \varepsilon')}, \qquad \text{for all } s \geq M,\ z \in \mathbb{D}_1.$$

Hence, for every $\varepsilon > 0$, $C > 1$, there exists a neighbourhood $\mathcal{U}$ of 1, and $M > 0$ such that

$$|\omega(\varphi_s(z))| \leq C|\omega(z)| \left| \frac{1-z}{1-\varphi_s(z)} \right|^{\alpha_u + \varepsilon}, \qquad \text{for all } s \geq M,\ z \in \mathcal{U} \cap \mathbb{D}. \tag{3.4}$$

Since φ_{-M} is analytic at 1 and $\varphi_{-M}(1) = 1$ there is an open subset $\mathcal{V}$ such that $1 \in \mathcal{V} \subseteq \mathcal{U}$ and $\varphi_{-M}(\mathcal{V}) \subseteq \mathcal{U}$. It follows by Lemma 3.10 that ω is bounded on $\mathbb{D}_1 \setminus \mathcal{V}$. Moreover, taking $\mathcal{V}$ such that $\mathbb{D} \setminus \mathcal{U}, \varphi_{-M}(\mathcal{V})$ are two disjoint connected sets, it is easy to see that for all $v \in \mathcal{V} \cap \mathbb{D}$ there is $s(v) \geq M$ such that $\varphi_{-s(v)}(v) \in \mathbb{D} \cap (\mathcal{U} \setminus \mathcal{V})$. But then, (3.4) applied to $z = \varphi_{-s(v)}(v)$ implies, for any $\varepsilon > 0$,

$$|\omega(v)| \leq C|\omega(\varphi_{-s(v)}(v)| \left| \frac{1-\varphi_{-s(v)}(v)}{1-v} \right|^{\alpha_u + \varepsilon} \lesssim |1-v|^{-\alpha_u - \varepsilon}, \quad v \in \mathcal{V},$$

where, in the second inequality, we have used Lemma 3.10 for $|\omega|$, and that $|1-\varphi_{-s(v)}(v)|$, $|\omega(\varphi_{-s_v}(v))|$ are bounded away from zero, since $\varphi_{-s(v)}(v) \notin \mathcal{V}$. As said above, one can analogously obtain that there exists a neighbourhood $\mathcal{U}_{-1} \subseteq \mathbb{D}$ of -1 such that $|\omega(z)| \lesssim |1+z|^{\beta_u-\varepsilon}$, $z \in \mathcal{U}_{-1} \cap \mathbb{D}$. Altogether, one gets $|\omega(z)| \lesssim |1-z|^{-\alpha_u-\varepsilon}|1+z|^{\beta_u-\varepsilon}$, $z \in \mathbb{D}$.

Finally, the inequality $\gtrsim$ of the claim follows by an application of what we have already proven to the DW-continuous cocycle $(v_t) := (u_t^{-1})$ with weight $\rho = \omega^{-1}$, see Lemma 3.3. Indeed, since $\alpha_v = -\alpha_u$ and $\beta_v = -\beta_u$, one has that for any $\varepsilon > 0$, $|\omega(z)^{-1}| = |\rho(z)| \lesssim |1-z|^{\alpha_u-\varepsilon}|1+z|^{-\beta_u-\varepsilon}$ for all $z \in \mathbb{D}$. Thus the proof is concluded. □

Theorem 3.11 is a significant step in our discussion since it shows that, under mild conditions on a cocycle, its associated weight ω must be tempered at DW-points. Besides such a property we next introduce two other conditions of asymptotic type that are needed for the unified approach we carry out in Section 5 and Section 6. Also, recall that by ι we denote either the number -1 or 1.

Definition 3.12. Let X be a γ-space and, for $\iota \in \{-1, 1\}$, let X_ι be Banach spaces for which property (**Gam3**) holds. A DW-continuous cocycle (u_t) for the hyperbolic flow (φ_t) is said to be spectrally DW-contractive (DW-contractive for short) if it satisfies the following conditions:

$$\limsup_{t\to\infty} \|u_{\iota t}\|_{Mul(X)}^{1/t} \leq \max\{|u_\iota(-1)|, |u_\iota(1)|\}; \tag{SpC1}$$

and

$$\limsup_{t\to\infty} \|u_{\iota t} f_t\|_{X_\iota}^{1/t} \leq |u_\iota(\iota)|, \tag{SpC2}$$

for every family $(f_t) \subseteq X$ such that $\limsup_{t\to\infty} \|f_t\|_X^{1/t} \leq 1$.

We say that a γ-space is hyperbolically DW-contractive if every DW-continuous cocycle is spectrally DW-contractive.

Remark 3.13. Similarly to the definition of γ-pair, the hyperbolically DW-contractivity can be equivalently formulated in terms of cocycles (v_t) associated to hyperbolic flows (ψ_t) with arbitrary DW-points $a, b \in \mathbb{T}$. This fact and Remark 2.3 mean that cocycles (v_t) as above satisfy analogous properties to (**SpC1**) and (**SpC2**) when acting on a hyperbolically DW-contractive γ-space X.

Let X be any of the examples of γ-spaces given in Section 2. Next proposition proves that X is hyperbolically DW-contractive. The cases of Hardy spaces, Bergman spaces, little Korenblum classes and the disc algebra are covered by item (1) below.

Proposition 3.14.

(1) *Let X be a γ-space such that the continuous inclusions $Mul(X) \hookrightarrow H^\infty(\mathbb{D})$, $Mul(X_{-1}) \hookrightarrow H^\infty(\mathbb{D}_{-1})$, $Mul(X_1) \hookrightarrow H^\infty(\mathbb{D}_1)$ are bounded below mappings. Then X is hyperbolically DW-contractive.*

(2) *Let either $X = \mathcal{D}^p_\sigma(\mathbb{D})$ for $\sigma > -1, p \geq 1$, and $p-2 \leq \sigma \leq p-1$ or $X = B_{1,0}(\mathbb{D})$. Then X is hyperbolically DW-contractive.*

Proof. (1) By hypothesis, $\|u\|_{Mul(X)} \lesssim \|u\|_\infty$, $\|v\|_{Mul(X_\iota)} \lesssim \|v\|_{H^\infty(\mathbb{D}_\iota)}$ for every $u \in Mul(X)$, $v \in Mul(X_\iota)$ respectively (recall that the embedding $Mul(Y) \hookrightarrow H^\infty(E)$ is continuous for any space Y such that $Y \hookrightarrow \mathcal{O}(E)$, where E is an open subset of $\mathbb{C}$, see [15, Lemma 11]). Let (u_t) be a DW-continuous cocycle for (φ_t). It follows by Lemma 3.6 that

$$\limsup_{t\to\infty} \|u_{\iota t}\|^{1/t}_{Mul(X)} \leq \lim_{t\to\infty} \|u_{\iota t}\|^{1/t}_{H^\infty(\mathbb{D})} = \max\{|u_\iota(1)|, |u_\iota(-1)|\},$$

so that condition (**SpC1**) is fulfilled. Let now $(f_t) \subseteq X$ be such that $\limsup_{t\to\infty} \|f_t\|^{1/t}_X \leq 1$, thus $\limsup_{t\to\infty} \|f_t\|^{1/t}_{X_\iota} \leq 1$ since $X \hookrightarrow X_\iota$. Another application of Lemma 3.6 yields that

$$\limsup_{t\to\infty} \|u_{\iota t} f_t\|^{1/t}_{X_\iota} \leq \limsup_{t\to\infty} \|u_{\iota t}\|^{1/t}_{Mul(X_\iota)} \|f_t\|^{1/t}_{X_\iota} \leq \lim_{t\to\infty} \|u_{\iota t}\|^{1/t}_{H^\infty(\mathbb{D}_\iota)} = |u_\iota(\iota)|,$$

so X satisfies (**SpC2**) and our claim is proven.

(2) Property (**SpC1**) is essentially proved in [17, Th. 5.2] for $\mathcal{D}^2_0(\mathbb{D})$. The proof for arbitrary σ, p as in the statement, as well as for $B_{1,0}(\mathbb{D})$ and property (**SpC2**), runs similarly. □

4. Estimates of hyperbolic composition groups

Let X be a γ-space with $\gamma \geq 0$ and let (u_t) be a DW-continuous cocycle for the hyperbolic flow (φ_t) on X given by (0.3). Then, as seen before, there exists a zero-free holomorphic function $\omega : \mathbb{D} \to \mathbb{C}$ such that $u_t = (\omega \circ \varphi_t)/\omega$, $t \in \mathbb{R}$. Define

$$S_\omega(t) := u_t C_{\varphi_t} \quad t \in \mathbb{R}.$$

Proposition 4.1. *For (u_t) and ω as above, the family $(S_\omega(t))$ is a C_0-group in $\mathcal{B}(X)$.*

Proof. It follows that $(S_\omega(t))$ is strongly measurable since (u_t) is strongly measurable by (**Co2**), and C_{φ_t} is strongly continuous on X by (**Gam4**). Hence, $(S_\omega(t))$ is strongly continuous since every strongly measurable group is strongly continuous [25, Th. 10.2.3]. □

Here we deal with asymptotic estimates of the norm of operators $S_\omega(t)$, $t \in \mathbb{R}$. For the sake of convenience we set $\alpha := \alpha_u$, $\beta := \beta_u$ where α_u, β_u are the exponents of the cocycle (u_t) obtained in Lemma 3.5; that is, $|u_t(1)| = e^{\alpha t}$, $|u_t(-1)| = e^{\beta t}$ for $t \in \mathbb{R}$.

Proposition 4.2. *Let X be a hyperbolically DW-contractive γ-space for some $\gamma \geq 0$. For $(S_\omega(t))$ as above,*

$$\lim_{t\to\infty} \|S_\omega(t)\|_{\mathcal{B}(X)}^{1/t} \leq \max\{e^{\beta-\gamma}, e^{\alpha+\gamma}\},$$

and

$$\lim_{t\to\infty} \|S_\omega(-t)\|_{\mathcal{B}(X)}^{1/t} \leq \max\{e^{-\beta+\gamma}, e^{-\alpha-\gamma}\}.$$

Proof. Let $\varepsilon > 0$ and $\iota = -1, 1$. Since X is a γ-space we have $\|C_{\varphi_{\iota t},\gamma}\|_{\mathcal{B}(X)} \leq K_\varepsilon e^{\varepsilon t}$, for $t > 0$; see (2.1). On the other hand, $S_\omega(t) = u_t(\varphi_t')^{-\gamma} C_{\varphi_t,\gamma}$, $t \in \mathbb{R}$, where $(u_t(\varphi_t')^{\gamma})$ is a DW-continuous cocycle for the flow (φ_t) with exponents $\alpha_{u(\varphi')^{-\gamma}} = \alpha + \gamma$ and $\beta_{u(\varphi')^{-\gamma}} = \beta - \gamma$, see Lemma 3.8 and Remark 3.9. As a consequence,

$$\|S_\omega(\iota t)\|_{\mathcal{B}(X)} \leq \|u_{\iota t}(\varphi_{\iota t}')^{-\gamma}\|_{Mul(X)} \|C_{\varphi_{\iota t},\gamma}\|_{\mathcal{B}(X)} \leq \|u_{\iota t}(\varphi_{\iota t}')^{-\gamma}\|_{Mul(X)} K_\varepsilon e^{\varepsilon t}, \quad t > 0.$$

Since X satisfies (**SpC1**), it follows that

$$(\forall \varepsilon > 0) \qquad \lim_{t\to\infty} \|S_\omega(\iota t)\|_{\mathcal{B}(X)}^{1/t} \leq \max\{e^{\iota(\beta-\gamma)}, e^{\iota(\alpha+\gamma)}\} e^{\varepsilon}.$$

Then, making $\varepsilon \to 0$ one obtains the result. □

The following result is about localization at the DW-points of the norm of the hyperbolic group. For $\delta < 0$, set $\mathfrak{X}_{-1}^\delta := \{f \in X \,:\, G_{-1}^\delta f \in X\}$ and $\mathfrak{X}_1^\delta := \{f \in X \,:\, G_1^\delta f \in X\}$, where $G_{-1}(z) := (1+z)$, $G_1(z) := (1-z)$ for $z \in \mathbb{D}$.

Proposition 4.3. *For X, (u_t), ω, α and β as above, assume $\beta - \alpha < 2\gamma$. Then*

(i) $\lim_{t\to\infty} \|S_\omega(t)f\|_X^{1/t} \leq e^{\beta-\gamma}$ *for all* $f \in \mathfrak{X}_1^{\beta-\alpha-2\gamma}$.
(ii) $\lim_{t\to\infty} \|S_\omega(-t)f\|_X^{1/t} \leq e^{-\alpha-\gamma}$ *for all* $f \in \mathfrak{X}_{-1}^{\beta-\alpha-2\gamma}$.

Proof. (i) For $\delta < 0$ and $f \in \mathfrak{X}_1^\delta$, put $f_{\delta,1} := G_1^\delta f$. Then, for $t > 0$,

$$\begin{aligned} S_\omega(t)f &= \frac{(\omega \circ \varphi_t)(G_1^{-\delta} \circ \varphi_t)}{\omega} (f_{\delta,1} \circ \varphi_t) \\ &= G_1^{-\delta} \frac{(G_1^{-\delta}\omega) \circ \varphi_t}{G_1^{-\delta}\omega} (f_{\delta,1} \circ \varphi_t) = G_1^{-\delta}(S_{G_1^{-\delta}\omega}(t) f_{\delta,1}). \end{aligned}$$

The cocycle (v_t) given by $v_t = ((G_1^{-\delta}\omega)\circ\varphi_t)/(G_1^{-\delta}\omega)$ is a DW-continuous cocycle with exponents $\alpha+\delta$ and β (associated to the DW-points $1,-1$ respectively) by Lemma 3.8. Moreover, $G_1^{-\delta}\in Mul(X)$ by (**Gam2**). Hence, an application of Proposition 4.2 to the group $(S_{G_1^{-\delta}\omega}(t))$ yields that

$$\lim_{t\to\infty}\|S_{G_1^{-\delta}\omega}(t)\|_{\mathcal{B}(X)}^{1/t}\le\max\{e^{\alpha+\delta+\gamma},e^{\beta-\gamma}\},$$

and then

$$\lim_{t\to\infty}\|S_\omega(t)f\|_X^{1/t}\le\lim_{t\to\infty}\left(\|G_1^{-\delta}\|_{Mul(X)}^{1/t}\|S_{G_1^{-\delta}\omega}(t)\|_{\mathcal{B}(X)}^{1/t}\|f_{\delta,1}\|_X^{1/t}\right)\le\max\{e^{\alpha+\delta+\gamma},e^{\beta-\gamma}\}.$$

Taking now $\delta=\beta-\alpha-2\gamma$ one obtains $\lim_{t\to\infty}\|S_\omega(t)f\|_X^{1/t}\le e^{\beta-\gamma}$ for every $f\in\mathfrak{X}_1^{\beta-\alpha-2\gamma}$, as we wanted to show.

(ii) The argument to prove this part is similar to the preceding one. We leave it to the reader. □

5. Two useful integrals

Through this section, let X be a hyperbolically DW-contractive γ-space and let $(S_\omega(t))$ be a weighted composition group as in Section 4, with α,β the exponents of $((\omega\circ\varphi_t)/\omega)$. Inspired by some ideas exposed within [33], which were further developed in [2], we introduce two integral operators which play a key role in the study of the spectrum of $(S_\omega(t))$ in Section 6.

For $z\in\mathbb{D}$, $f\in\mathcal{O}(\mathbb{D})$ and $\lambda\in\mathbb{C}$ (and $\iota=-1,1$), set

$$(\Lambda_\omega^{\lambda,\iota}f)(z):=\frac{-2}{\omega(z)}\left(\frac{1+z}{1-z}\right)^\lambda\int_\iota^z\frac{(1-\xi)^{\lambda-1}}{(1+\xi)^{\lambda+1}}\,\omega(\xi)f(\xi)\,d\xi,\quad z\in\mathbb{D},\tag{5.1}$$

where the integration path is to be understood as any simple path in $\mathbb{D}\cup\{\iota\}$ going from ι to z and leaving ι non-tangentially (it will be seen next that the value of the integral is independent of the chosen path), and

$$L_\omega^\lambda f:=\int_{-1}^{1}\frac{(1-\xi)^{\lambda-1}}{(1+\xi)^{\lambda+1}}\,\omega(\xi)f(\xi)\,d\xi,\tag{5.2}$$

where the integral is understood on any path in $\mathbb{D}$ between -1 and 1 touching $-1,1$ non-tangentially.

The convergence of the above integrals is considered right now.

Lemma 5.1. *Let $f\in X$, $z\in\mathbb{D}$, $\lambda\in\mathbb{C}$. Then, the following holds.*

- *$(\Lambda_\omega^{\lambda,-1} f)(z)$ converges (absolutely) if $\mathfrak{Re}\,\lambda < \beta - \gamma$.*
- *$(\Lambda_\omega^{\lambda,1} f)(z)$ converges (absolutely) if $\mathfrak{Re}\,\lambda > \alpha + \gamma$.*

In any of the above cases, the value of $(\Lambda_\omega^{\lambda,\iota} f)(z)$ is independent on the integration path taken, whenever it is a simple path in $\mathbb{D} \cup \{\iota\}$ leaving ι non-tangentially. Also, the function $\Lambda_\omega^{\lambda,\iota} f$ is holomorphic in the disc.

Proof. Let us show the claims for $\Lambda_\omega^{\lambda,-1}$. Let θ_0 be a fixed angle such that $|\theta_0| < (\pi/2)$. Then, for $\xi \in \mathbb{D}$ such that $1 + \xi = |1 + \xi|e^{i\theta}$ with $|\theta| \le |\theta_0|$ and $|1 + \xi| < \cos\theta_0$, one has $|\xi|^2 = |1 + \xi|^2 + 1 - 2|1 + \xi|\cos\theta$, whence $1 - |\xi|^2 = |1 + \xi|(2\cos\theta - |1 + \xi|) \ge |1 + \xi|(2\cos\theta_0 - |1 + \xi|) > (\cos\theta_0)|1 + \xi|$. In short,

$$1 - |\xi|^2 > (\cos\theta_0)|1 + \xi|, \tag{5.3}$$

for every ξ in the sector $-1 + \sum_{\theta_0}$ of angle θ_0, with vertex at -1 and symmetric with respect to $(-1, \infty)$, such that $|1 + \xi| < \cos\theta_0$.

Let $f \in X$ and $\varepsilon > 0$. By Remark 2.4 one has that $|f(\xi)| \lesssim (1 - |\xi|^2)^{-\gamma-\varepsilon}\|f\|_X$. Hence $|f(\xi)| \lesssim (\cos\theta_0)|1 + \xi|^{-\gamma-\varepsilon}\|f\|_X$ for all ξ as in (5.3). Also, ω has exponent β at -1 and so $|\omega(\xi)| \lesssim |1 + \xi|^{\beta-\varepsilon}$ for ξ as before, see Theorem 3.11.

Altogether,

$$|\omega(\xi)f(\xi)||1 + \xi|^{-\mathfrak{Re}\,\lambda-1} \lesssim |1 + \xi|^{\beta-\gamma-\mathfrak{Re}\,\lambda-1-2\varepsilon},$$

for every $\xi \in (-1 + \sum_{\theta_0})$ such that $|1 + \xi| < \cos\theta_0$, which readily implies the convergence of $\Lambda_\omega^{\lambda,-1} f$ on any path touching -1 non-tangentially, provided $\mathfrak{Re}\,\lambda < \beta - \gamma$.

The statement for $\Lambda_\omega^{\lambda,1} f$, that is, $\Lambda_\omega^{\lambda,1} f$ converges provided $\mathfrak{Re}\,\lambda > \alpha + \gamma$, is proven using analogous argument to the above one. It is left to the reader.

Let us now assume $\mathfrak{Re}\,\lambda < \beta - \gamma$ and let τ be a closed path in $\mathbb{D}$ joining -1 and a fixed $z \in \mathbb{D}$, and being non-tangential (to $\mathbb{T}$) at -1. For $\delta > 0$ small enough, we can assume that the circle $\{\xi \in \mathbb{C} : |1 + \xi| < \delta\}$ intersects τ exactly twice. So let C_δ be the arc in $\mathbb{D}$ of such circle joining these two intersection points. Let τ_1, τ_{-1} be paths defined by $\tau_1 := (\tau \cap \{\xi \in \mathbb{D} : |1 + \xi| \ge \delta\}) \cup C_{\delta,-}$ and $\tau_{-1} := (\tau \cap \{\xi \in \mathbb{D} : |1 + \xi| < \delta\}) \cup C_{\delta,+}$, where $C_{\delta,-}$ (respectively $C_{\delta,+}$) is C_δ negatively (positively) orientated. Then we have $\int_{\tau_1} \frac{(1-\xi)^{\lambda-1}}{(1+\xi)^{\lambda+1}}\, \omega(\xi)f(\xi)\, d\xi = 0$ by Cauchy's theorem and therefore

$$\begin{aligned}
\int_{\tau} \frac{(1-\xi)^{\lambda-1}}{(1+\xi)^{\lambda+1}} \omega(\xi)f(\xi)\, d\xi &= \int_{\tau_{-1}} \frac{(1-\xi)^{\lambda-1}}{(1+\xi)^{\lambda+1}}\, \omega(\xi)f(\xi)\, d\xi \\
&= \int_{\tau_{-1}} \chi_{(\tau_{-1}\setminus C_{\delta,+})}(\xi) \frac{(1-\xi)^{\lambda-1}}{(1+\xi)^{\lambda+1}}\, \omega(\xi)f(\xi)\, d\xi + \int_{C_{\delta,+}} \frac{(1-\xi)^{\lambda-1}}{(1+\xi)^{\lambda+1}}\, \omega(\xi)f(\xi)\, d\xi
\end{aligned}$$

where $\chi_{(\tau_{-1}\setminus C_{\delta,+})}$ is the characteristic function of $\tau_{-1}\setminus C_{\delta,+}$. The first term of the two latter integrals tends to zero as $\delta\to 0$ by the dominated convergence theorem. As regards the second one, it is bounded up to a constant by $\int_{C_{\delta,+}} |1+\xi|^{\beta-\gamma-\mathfrak{Re}\,\lambda-1-2\varepsilon}|d\xi|$, which in turn, using the parameterization $1+\xi=\delta e^{i\theta}$, $\theta_1\le\theta\le\theta_2$, where θ_1 and θ_2 are the arguments of the extreme points of the arc $C_{\delta,+}$, equals

$$\int_{\theta_1}^{\theta_2} \delta^{\beta-\gamma-\mathfrak{Re}\,\lambda-1-2\varepsilon}\delta\, d\theta \le \pi\delta^{\beta-\gamma-\mathfrak{Re}\,\lambda-2\varepsilon}$$

(with ε small enough).

In conclusion, one has $\int_\tau \frac{(1-\xi)^{\lambda-1}}{(1+\xi)^{\lambda+1}}\omega(\xi)f(\xi)\,d\xi=0$ and so the integral which defines $(\Lambda_\omega^{\lambda,-1}f)(z)$ is independent of paths in $\mathbb{D}$ joining -1 and $z\in\mathbb{D}$ non-tangentially at -1. The case $\Lambda_\omega^{\lambda,1}f$ is proven in the same way.

Finally, it is readily seen that, under the above hypothesis, the functions $\Lambda_\omega^{\lambda,\iota}f$, $\iota=-1,1$, are holomorphic in $\mathbb{D}$. □

In the following corollary, we extend the values of λ for which $\Lambda_\omega^{\lambda,\iota}f$ is well defined in the case that f belongs to the subspaces $\mathfrak{X}_\iota^\delta$ introduced prior to Proposition 4.3.

Corollary 5.2. *Assume $\beta-\alpha<2\gamma$. Let $f\in X$ and $z\in\mathbb{D}$. Then, on every path as in Lemma 5.1, $(\Lambda_\omega^{\lambda,-1}f)(z)$ converges (absolutely) if $\mathfrak{Re}\,\lambda<\gamma+\alpha$ and $f\in\mathfrak{X}_{-1}^{\beta-\alpha-2\gamma}$; and $(\Lambda_\omega^{\lambda,1}f)(z)$ converges (absolutely) if $\mathfrak{Re}\,\lambda>\beta-\gamma$ and $f\in\mathfrak{X}_1^{\beta-\alpha-2\gamma}$.*

Moreover, the value of $\Lambda_\omega^{\lambda,\iota}f$ is independent on the integration path taken, whenever it is a simple path in $\mathbb{D}\cup\{\iota\}$ leaving ι non-tangentially. Also, the function $\Lambda_\omega^{\lambda,\iota}f$ is holomorphic in $\mathbb{D}$.

Proof. The statement is an immediate consequence of Lemma 5.1 applied to the function $(1+\cdot)^{\beta-\alpha-2\gamma}f$ if $f\in\mathfrak{X}_{-1}^{\beta-\alpha-2\gamma}$, and to the function $(1-\cdot)^{\beta-\alpha-2\gamma}f$ if $f\in\mathfrak{X}_1^{\beta-\alpha-2\gamma}$. □

We show now the relationship between the integrals of (5.1) and the group $(S_\omega(t))$.

Proposition 5.3. *Let $f\in X$. Then*

(i) $\Lambda_\omega^{\lambda,1}f=\int_0^\infty e^{-\lambda t}S_\omega(t)f\,dt$, *in X, provided $\mathfrak{Re}\,\lambda>\max\{\beta-\gamma,\alpha+\gamma\}$.*

(ii) $\Lambda_\omega^{\lambda,-1}f=-\int_0^\infty e^{\lambda t}S_\omega(-t)f\,dt$, *in X, provided $\mathfrak{Re}\,\lambda<\min\{\beta-\gamma,\alpha+\gamma\}$.*

Assume furthermore $\beta-\alpha<2\gamma$. Then

(iii) $\Lambda_\omega^{\lambda,1} f = \int_0^\infty e^{-\lambda t} S_\omega(t) f \, dt$, *in* X, *provided* $\mathfrak{Re}\,\lambda > \beta - \gamma$ *and* $f \in \mathfrak{X}_1^{\beta-\alpha-2\gamma}$.

(iv) $\Lambda_\omega^{\lambda,-1} f = -\int_0^\infty e^{\lambda t} S_\omega(-t) f \, dt$, *in* X, *provided* $\mathfrak{Re}\,\lambda < \alpha + \gamma$ *and* $f \in \mathfrak{X}_{-1}^{\beta-\alpha-2\gamma}$.

Proof. (i) Let $f \in X$. The map $t \in [0,\infty) \mapsto S_\omega(t) f \in X$ is norm continuous and

$$\|S_\omega(t) f\|_X \le K_\varepsilon \max\{e^{(\beta-\gamma+\varepsilon)t}, e^{(\alpha+\gamma+\varepsilon)t}\},$$

for $\varepsilon > 0$, by Proposition 4.2. Hence, choosing ε small enough, one obtains that the integral $\int_0^\infty e^{-\lambda t} S_\omega(t) f \, dt$ is Bochner-convergent in X for $\mathfrak{Re}\,\lambda > \max\{\beta-\gamma, \alpha+\gamma\}$.

Now, for $z \in \mathbb{D}$, we apply Lemma 5.1 with the path $\xi = \frac{z+r}{1+rz}$, $0 \le r \le 1$, and make the variable change $r = \tanh(t/2)$, to obtain

$$\begin{aligned} &\frac{2}{\omega(z)} \left(\frac{1+z}{1-z}\right)^\lambda \int_z^1 \frac{(1-\xi)^{\lambda-1}}{(1+\xi)^{\lambda+1}} \omega(\xi) f(\xi) \, d\xi \\ &= \frac{2}{\omega(z)} \int_0^1 \frac{(1-r)^{\lambda-1}}{(1+r)^{\lambda+1}} \omega\left(\frac{z+r}{1+zr}\right) f\left(\frac{z+r}{1+zr}\right) dr \qquad (5.4) \\ &= \int_0^\infty e^{-\lambda t} \frac{(\omega(\varphi_t(z))}{\omega(z)} f(\varphi_t(z)) dt = \int_0^\infty e^{-\lambda t} (S_\omega(t) f)(z) \, dt, \end{aligned}$$

for every $\lambda \in \mathbb{C}$ such that $\mathfrak{Re}\,\lambda > \alpha + \gamma$. Since the latter integral, regarded as a vector-valued integral, is Bochner convergent for $\mathfrak{Re}\,\lambda > \max\{\beta-\gamma, \alpha+\gamma\}$ we get the wished-for result.

(ii) This part follows along the same lines as before, by applying Proposition 4.2 to the semigroup $(S_\omega(-t))_{t\ge 0}$.

Items (iii) and (iv) are obtained with an analogous argument. Corollary 5.2 states that $\Lambda_\omega^{\lambda,1} f$, $\Lambda_\omega^{\lambda,-1} f$ are well-defined in these cases, and the sharper asymptotic bounds for $\|S_\omega(t) f\|_X$ as $t \to \iota\infty$, $\iota = -1, 1$, given in Proposition 4.3 imply that the integrals of the statement are convergent in the Bochner sense. □

The following lemma is significant to study the residual spectrum of the infinitesimal generator of the C_0-group $(S_\omega(t))$.

Lemma 5.4. *Assume* $\beta - \alpha > 2\gamma$ *and* $\gamma + \alpha < \mathfrak{Re}\,\lambda < \beta - \gamma$. *Then the mapping* $L_\omega^\lambda : X \to \mathbb{C}$ *given by* (5.2) *is a continuous linear functional on* X. *Moreover, for every* $f \in \ker L_\omega^\lambda$ *one has that* $\Lambda_\omega^{\lambda,1} f$ *and* $\Lambda_\omega^{\lambda,-1} f$ *lies in* X *and* $\Lambda_\omega^{\lambda,1} f = \Lambda_\omega^{\lambda,-1} f$.

Proof. Let $\varepsilon > 0$. By Remark 2.4, we have $\sup_{z\in\mathbb{D}}(1-|z|)^{\gamma+\varepsilon}|f(z)| \lesssim \|f\|_X$ for all $f \in X$. Moreover, $|\omega(z)| \lesssim |1-z|^{-\alpha-\varepsilon}|1+z|^{\beta-\varepsilon}$ for all $z \in \mathbb{D}$ by Theorem 3.11. Therefore,

$$|L_\omega^\lambda f| \le \int_{-1}^{1} \left| \frac{(1-\xi)^{\lambda-1}}{(1+\xi)^{\lambda+1}} \omega(\xi) f(\xi) \right| d\xi$$
$$\lesssim \|f\|_X \int_{-1}^{1} (1-\xi)^{\mathfrak{Re}\,\lambda-\alpha-\gamma-2\varepsilon-1}(1+\xi)^{-\mathfrak{Re}\,\lambda+\beta-\gamma-2\varepsilon-1}\, d\xi.$$

The last integral is finite for $\varepsilon > 0$ small enough, hence L_ω^λ is a well-defined bounded functional on X.

Now, it follows by Lemma 5.1 that $\Lambda_\omega^{\lambda,1} f, \Lambda_\omega^{\lambda,-1} f \in \mathcal{O}(\mathbb{D})$ for each $f \in X$. Moreover, a simple computation shows that

$$(\Lambda_\omega^{\lambda,-1} f)(z) = (\Lambda_\omega^{\lambda,1} f)(z) - \frac{2}{\omega(z)} \left(\frac{1+z}{1-z}\right)^{\lambda} L_\omega^\lambda f, \quad z \in \mathbb{D},\ f \in X.$$

Hence $\Lambda_\omega^{\lambda,1} f = \Lambda_\omega^{\lambda,-1} f$ if $f \in \ker L_\omega^\lambda$ as claimed.

Next we prove that $\Lambda_\omega^{\lambda,1} f \in X_1$, where X_1 is the subspace of $\mathcal{O}(\mathbb{D}_1)$ associated with X through **(Gam3)**. Note that the identity (5.4) holds whenever $\mathfrak{Re}\,\lambda > \alpha+\gamma$. Moreover

$$(\Lambda_\omega^{\lambda,1} f)(z) = \int_0^\infty e^{-\lambda t}(S_\omega(t)f)(z)\, dt = \int_0^\infty e^{-\lambda t} u_t(z)(\varphi_t'(z))^{-\gamma}(C_{\varphi_t,\gamma} f)(z)\, dt, \quad z \in \mathbb{D},$$

with $\lim_{t\to\infty} \|C_{\varphi_t,\gamma} f\|_X^{1/t} \le 1$ by (**Gam5**). Since X is hyperbolically DW-contractive and $(u_t(\varphi_t')^{-\gamma})$ is a DW-continuous cocycle with exponents $\alpha+\gamma$, $\beta-\gamma$ (see Lemma 3.8 and Remark 3.9), it follows by condition (**SpC2**) that for any $\varepsilon > 0$,

$$\|e^{-\lambda t} u_t(\varphi_t')^{-\gamma} C_{\varphi_t,\gamma} f\|_{X_1} \lesssim e^{-\mathfrak{Re}\,\lambda t} e^{\varepsilon t} |u_1(1)(\varphi_1'(1))^{-\gamma}|^t = e^{(-\mathfrak{Re}\,\lambda+\gamma+\alpha+\varepsilon)t}, \qquad t \ge 0.$$

Therefore, the integral $\int_0^\infty e^{-\lambda t} S_\omega(t) f\, dt$ is Bochner-convergent in the Banach space X_1, the equality $\Lambda_\omega^{\lambda,1} f = \int_0^\infty e^{-\lambda t} S_\omega(t) f\, dt \in X_1$ holds and, in particular, $\Lambda_\omega^{\lambda,1} f \in X_1$.

Reasoning along similar lines, one obtains $\Lambda_\omega^{\lambda,-1} f \in X_{-1}$. Hence $\Lambda_\omega^{\lambda,1} f \in X$ since $X = \mathcal{O}(\mathbb{D}) \cap X_{-1} \cap X_1$ (see condition **(Gam3)**), and the proof is finished. □

Remark 5.5. Under the conditions of Lemma 5.4, the kernel of the functional L_ω^λ is not the whole space X, i.e. $L_\omega^\lambda \ne 0$. Indeed, assume that $L_\omega^\lambda = 0$, and let us see that we reach a contradiction.

Take a non-zero $f \in X$. Since $|\omega(z)| \lesssim |1-z|^{-\alpha+\varepsilon}|1+z|^{\beta+\varepsilon}$ (Theorem 3.11) and $f \in \mathcal{K}^{-\gamma-\varepsilon}(\mathbb{D})$ (Remark 2.4), one has that the function $(1-\cdot)^\lambda(1+\cdot)^{-\lambda}\omega f$ is a continuous function when restricted to the real interval $[-1, 1]$. By the density of polynomials in

$C([-1,1])$ (the set of continuous complex-valued functions on $[-1,1]$), it follows that the functional $L : C([-1,1]) \to \mathbb{C}$ given by $g \mapsto \int_{-1}^{1} g(1-\cdot)^{\lambda}(1+\cdot)^{-\lambda}\omega f$ is the zero functional, hence the function $(1-\cdot)^{\lambda}(1+\cdot)^{-\lambda}\omega f$ is the zero function, which is nonsense.

Remark 5.6. Under the conditions of Lemma 5.4, fix $f \in X$. Using a similar reasoning as in the beginning of the proof of Lemma 5.4, one gets that the mapping from $\mathbb{D} \cup \{-1, 1\}$ to $\mathbb{C}$ given by

$$z \mapsto \int_0^z \frac{(1-\xi)^{\lambda-1}}{(1+\xi)^{\lambda+1}}\, \omega(\xi) f(\xi)\, d\xi,$$

is continuous whenever z approaches $-1, 1$ via non-tangential paths.

6. Spectra of the generator

Let X be a hyperbolically DW-contractive γ-space, $\gamma \geq 0$, and let $\mathfrak{S}$ be a subset of $\mathcal{O}(\mathbb{D})$ such that $(X, \mathfrak{S})$ is a γ-pair. Let $\omega \in \mathcal{O}(\mathbb{D})$ be non-vanishing, let $(S_\omega(t))$ be a weighted composition group as in Section 4, and let Δ_ω denote its infinitesimal generator. The aim of this section is to describe the fine structure of the spectrum of Δ_ω. For $c, d \in \mathbb{R}$, we set $|c, d| = \{z \in \mathbb{C} : \min\{c, d\} \leq \mathfrak{Re}\, z \leq \max\{c, d\}\}$.

Lemma 6.1 below can be considered standard. Recall that the function $G(z) = \frac{\partial \varphi_t(z)}{\partial t}\big|_{t=0} = (1-z^2)/2$, $z \in \mathbb{D}$, is the generator of the hyperbolic flow (φ_t) given in (0.3).

Lemma 6.1. *The infinitesimal generator Δ_ω of the C_0-group $(S_\omega(t))$ is given by the differential operator*

$$\Delta_\omega(f) := \frac{\omega'}{\omega} G f + G f', \quad f \in \mathcal{D}(\Delta_\omega),$$

where $\mathcal{D}(\Delta_\omega) = \{f \in X : (\omega'/\omega)Gf + Gf' \in X\}$.

Proof. The proof mimics the one of an analogous result for Hardy spaces $H^p(\mathbb{D})$ given in [35]. □

Lemma 6.2. *The spectrum $\sigma(\Delta_\omega)$ of the infinitesimal generator Δ_ω satisfies*

$$\sigma(\Delta_\omega) \subseteq |\beta - \gamma, \gamma + \alpha|.$$

Moreover,

$$R(\lambda, \Delta_\omega) f = \Lambda_\omega^{\lambda,\iota} f, \quad f \in X, \tag{6.1}$$

for $\iota = 1$ if $\mathfrak{Re}\,\lambda > \max\{\beta - \gamma, \gamma + \alpha\}$ and for $\iota = -1$ if $\mathfrak{Re}\,\lambda < \min\{\beta - \gamma, \gamma + \alpha\}$.

Proof. By the spectral mapping inclusion for C_0-semigroups (see e.g. [18, Th. IV.3.6]) we have $e^{t\sigma(\Delta_\omega)} \subseteq \sigma(S_\omega(t))$ for $t \in \mathbb{R}$. Also, $r(S_\omega(t)) \le e^{\max\{(\beta-\gamma)t,(\gamma+\alpha)t\}}$ and $r(S_\omega(t)^{-1}) = r(S_\omega(-t)) \le e^{\max\{-(\beta-\gamma)t,-(\alpha+\gamma)t\}}$ by Proposition 4.2. Hence we obtain $\sigma(\Delta_\omega) \subseteq |\beta - \gamma, \gamma+\alpha|$ as claimed.

Let now $\mathfrak{Re}\,\lambda > \max\{\beta-\gamma, \gamma+\alpha\}$. Using the integral representation of the resolvent operator of Δ_ω in terms of the semigroup $(S_\omega(t))_{t\ge 0}$ (see e.g. [18, Th. II.1.10]) and Proposition 5.3(i), one has

$$R(\lambda, \Delta_\omega)f = \int_0^\infty e^{-\lambda t} S_\omega(t)f\,dt = \Lambda_\omega^{\lambda,1} f, \qquad f \in X,\, \mathfrak{Re}\,\lambda > \max\{\alpha+\gamma, \beta-\gamma\}.$$

If $\mathfrak{Re}\,\lambda < \min\{\beta-\gamma, \gamma+\alpha\}$, it suffices to apply the integral representation of the resolvent of $-\Delta_\omega$ in terms of the semigroup $(S_\omega(-t))_{t\ge 0}$ and Proposition 5.3(ii) to obtain the result. $\square$

In the remainder of the section, we describe several spectral sets of Δ_ω. For a suitable understanding of the arguments we divide the overall proof in a series of results and remarks.

Proposition 6.3. *The point spectrum of the infinitesimal generator Δ_ω is given by*

$$\sigma_{point}(\Delta_\omega) = \{\lambda \in \mathbb{C} \,:\, g_\lambda \in X\}, \quad g_\lambda(z) := \frac{1}{\omega(z)}\left(\frac{1+z}{1-z}\right)^\lambda,\ z \in \mathbb{D}.$$

The eigenspace of each $\lambda \in \sigma_{point}(\Delta_\omega)$ is one-dimensional and generated by g_λ. If in addition $\omega^{-1} \in \mathfrak{S}$, then $\sigma_{point}(\Delta_\omega)$ satisfies the following inclusions:

$$\{\lambda \in \mathbb{C} \,:\, \beta-\gamma < \mathfrak{Re}\,\lambda < \alpha+\gamma\} \subseteq \sigma_{point}(\Delta_\omega) \subseteq \{\lambda \in \mathbb{C} \,:\, \beta-\gamma \le \mathfrak{Re}\,\lambda \le \alpha+\gamma\},$$

if $\beta-\alpha \le 2\gamma$; and $\sigma_{point}(\Delta_\omega) = \emptyset$ if $\beta - \alpha > 2\gamma$.

Proof. The identity $\sigma_{point}(\Delta_\omega) = \{\lambda \in \mathbb{C} \,:\, g_\lambda \in X\}$ can be proved for arbitrary γ-spaces as it is proven in [35, Th. 3] for Hardy spaces.

By Theorem 3.11, for every $\varepsilon > 0$ we have

$$|1-z|^{\alpha+\varepsilon}|1+z|^{-\beta+\varepsilon} \lesssim |\omega(z)|^{-1} \lesssim |1-z|^{\alpha-\varepsilon}|1+z|^{-\beta-\varepsilon}, \quad z \in \mathbb{D}.$$

Thus, for $\gamma' > \gamma$,

$$|1-z^2|^{\gamma'}|g_\lambda(z)| \gtrsim |1-z|^{\gamma'+\alpha+\varepsilon-\mathfrak{Re}\,\lambda}|1+z|^{\gamma'-\beta+\varepsilon+\mathfrak{Re}\,\lambda}, \quad z \in \mathbb{D}.$$

Hence $\sup_{z\in\mathbb{D}} |1-z^2|^{\gamma'}|g_\lambda(z)| = \infty$ for some $\gamma' > \gamma$, provided $\mathfrak{Re}\,\lambda < \beta-\gamma$ or $\mathfrak{Re}\,\lambda > \alpha+\gamma$. It follows that $g_\lambda \notin \mathcal{K}^{-\gamma'}(\mathbb{D})$ and therefore $g_\lambda \notin X$, see Remark 2.4. This implies the inclusion $\sigma_{point}(\Delta_\omega) \subseteq \{\lambda \in \mathbb{C} \,:\, \beta-\gamma \le \mathfrak{Re}\,\lambda \le \alpha+\gamma\}$.

Now, fix $\lambda \in \mathbb{C}$ with $\beta - \gamma < \mathfrak{Re}\,\lambda < \alpha + \gamma$. Then $g_\lambda \in \mathfrak{G}$ since, for any $\lambda \in \mathbb{C}$, $\mathfrak{G}$ is invariant by multiplication with the function $z \mapsto (1+z)^\lambda (1-z)^{-\lambda}$. Then Theorem 3.11 implies, for $\delta > 0$ small enough, that $|g_\lambda(z)| \lesssim |1 - z^2|^{-\gamma+\delta}$, $z \in \mathbb{D}$. Therefore, $g_\lambda \in X$ by property (**Gam6**), so that $\lambda \in \sigma_{point}(\Delta_\omega)$. Thus $\{\lambda \in \mathbb{C} : \beta - \gamma < \mathfrak{Re}\,\lambda < \alpha + \gamma\} \subseteq \sigma_{point}(\Delta_\omega)$ as we wanted to prove. □

The assumption $\omega^{-1} \in \mathfrak{G}$ in Proposition 6.3 is superfluous when $X = H^p(\mathbb{D})$, $\mathcal{A}^p_\sigma(\mathbb{D})$, $\mathcal{K}_0^{-\gamma}(\mathbb{D})$ for $1 \le p < \infty$, $\sigma > -1$ and $\gamma > 0$, since in any of these examples $\mathfrak{G}$ is the set $\mathcal{O}(\mathbb{D})$ of all holomorphic functions in the disc $\mathbb{D}$. The next result shows that such an assumption is also redundant for the disc algebra $\mathfrak{A}(\mathbb{D})$. We conjecture that there exist subsets $\mathfrak{G}(\mathcal{D}^p_\sigma)$, $\mathfrak{G}(B_{1,0})$ such that $(\mathcal{D}^p_\sigma(\mathbb{D}), \mathfrak{G}(\mathcal{D}^p_\sigma))$ and $(B_{1,0}(\mathbb{D}), \mathfrak{G}(B_{1,0}))$ are γ-pairs and the assumptions $\omega^{-1} \in \mathfrak{G}(\mathcal{D}^p_\sigma), \mathfrak{G}(B_{1,0})$ are redundant as well.

Proposition 6.4. *Let (u_t) be a DW-continuous cocycle for the flow (φ_t) on the disc algebra $\mathfrak{A}(\mathbb{D})$ with weight ω, i.e. $u_t = (\omega \circ \varphi_t)/\omega$. Then $\omega^{-1} \in \mathfrak{G}(\mathfrak{A})$.*

Proof. Recall that $\mathfrak{G}(\mathfrak{A})$ is the subset of functions of $\mathcal{O}(\mathbb{D})$ which can be continuously extended to $\overline{\mathbb{D}} \setminus \{-1, 1\}$.

First note that ω can be extended to almost every point of $\mathbb{T} \setminus \{-1, 1\}$ via non-tangential limits. Indeed, the holomorphic function $z \mapsto (1 - z^2)^\lambda \omega(z)$ lies in $H^\infty(\mathbb{D})$ for $\lambda > 0$ big enough by Theorem 3.11, thus $(1 - (\cdot)^2)^\lambda \omega$ can be extended a.e. via non-tangential limits to $\mathbb{T}$ (see for instance [26, p.38]), whence the same holds true for ω in $\mathbb{T} \setminus \{-1, 1\}$. Moreover, such non-tangential limits are never equal to 0 by Theorem 3.11.

We claim that such limits exist for every point in $\mathbb{T} \setminus \{-1, 1\}$. To see this, fix $v \in \mathbb{T} \setminus \{-1, 1\}$ with $\mathfrak{Im}\,v > 0$ such that the (non-tangential) limit $\lim_{z \to v} \omega(z)$ exists. Notice that $u_t = (\omega \circ \varphi_t)/\omega \in Mul(\mathfrak{A}(\mathbb{D})) = \mathfrak{A}(\mathbb{D})$ for each $t \in \mathbb{R}$. Since $\varphi_t \in Aut(\mathbb{D})$, it follows from $\omega \circ \varphi_t = u_t \omega$ that the limit $\lim_{z \to \varphi_t(v)} \omega(z)$ exists, that is, ω has non-tangential limits at $\{\varphi_t(v) : t \in \mathbb{R}\} = \{z \in \mathbb{T} : \mathfrak{Im}\,z > 0\}$. After repeating the argument with $v \in \mathbb{T}$ such that $\mathfrak{Im}\,v < 0$, we obtain that ω has non-tangential limits at every point in $\mathbb{T} \setminus \{-1, 1\}$.

Next we show that the extension of ω to $\mathbb{D} \setminus \{-1, 1\}$ via non-tangential limits is continuous when restricted to $\mathbb{T} \setminus \{-1, 1\}$. Note that the mapping $t \in \mathbb{R} \mapsto u_t = S_\omega(t)\mathbf{1} \in \mathfrak{A}(\mathbb{D})$ is continuous, where $\mathbf{1}$ denotes the constant function $\mathbf{1}(z) = 1$. As a consequence, the mapping $t \mapsto u_t(v)$ is continuous for any $v \in \overline{\mathbb{D}}$. Hence the mapping $t \mapsto \omega(\varphi_t(v)) = u_t(v)\omega(v)$ is also continuous. Note also that $t \mapsto \varphi_t(v)$ is a homeomorphism from $\mathbb{R}$ to $\{z \in \mathbb{T} : \operatorname{sgn} \mathfrak{Im}\,z = \operatorname{sgn} \mathfrak{Im}\,v\}$ for every $v \in \mathbb{T} \setminus \{-1, 1\}$. Thus ω is continuous on $\mathbb{T} \setminus \{-1, 1\}$.

Taking λ as at the beginning of the proof, we obtain that the function $(1 - (\cdot)^2)^\lambda \omega$ is holomorphic and bounded on $\mathbb{D}$, and that it can be extended to every point in $\overline{\mathbb{D}}$ via non-tangential limits, being such an extension continuous when restricted to the boundary $\mathbb{T}$. Using the Poisson kernel, one gets $(1 - (\cdot)^2)^\lambda \omega \in \mathfrak{A}(\mathbb{D})$. Since ω has no zeros in $\overline{\mathbb{D}} \setminus \{-1, 1\}$, we conclude that $\omega^{-1} \in \mathfrak{G}(\mathfrak{A})$ and the proof is finished. □

Remark 6.5. We now study the range space of the operator $\lambda - \Delta_\omega : \mathcal{D}(\Delta_\omega) \to X$ for a fixed $\lambda \in \mathbb{C}$. To begin with, a few computations show that all the solutions $(g_{f,K})_{K\in\mathbb{C}} \in \mathcal{O}(\mathbb{D})$ of the differential equation $(\lambda - G\omega'/\omega)g - Gg' = f$, $f \in \mathcal{O}(\mathbb{D})$, are given by

$$g_{f,K}(z) = \frac{1}{\omega(z)} \frac{(1+z)^\lambda}{(1-z)^\lambda} \left(K - 2\int_0^z \frac{(1-\xi)^{\lambda-1}}{(1+\xi)^{\lambda+1}} \omega(\xi) f(\xi)\, d\xi \right), \quad z \in \mathbb{D}, K \in \mathbb{C}. \quad (6.2)$$

Thus, we have by Lemma 6.1 that a function $f \in X$ lies in the range of $\lambda - \Delta_\omega$ if and only if there exists some $K \in \mathbb{C}$ such that the function $g_{f,K}$ given in (6.2) belongs to X. Indeed, if this is the case, then $g_{f,K} \in \mathcal{D}(\Delta_\omega)$ and $(\lambda - \Delta_\omega)g_{f,K} = f$.

The lemma below gives the range space $\mathrm{Ran}(\lambda - \Delta_\omega)$ when $\beta - \alpha \neq 2\gamma$. Notice that, by Lemma 6.2, $\lambda - \Delta_\omega$ is a surjective (moreover, invertible) operator whenever $\lambda \notin |\beta - \gamma, \alpha + \gamma|$.

Lemma 6.6. *Let $\lambda \in \mathbb{C}$. We have*

$$\mathrm{Ran}(\lambda - \Delta_\omega) = \begin{cases} X, & \text{if } \beta - \alpha < 2\gamma, \text{ and } \beta - \gamma < \mathfrak{Re}\,\lambda < \alpha + \gamma, \\ \ker L_\omega^\lambda \subsetneq X, & \text{if } \beta - \alpha > 2\gamma, \text{ and } \alpha + \gamma < \mathfrak{Re}\,\lambda < \beta - \gamma. \end{cases}$$

Proof. Assume first $\beta - \alpha < 2\gamma$ and $\beta - \gamma < \mathfrak{Re}\,\lambda < \alpha + \gamma$. Let $m \in \mathbb{N}$ be such that $m \geq 2(2\gamma + \alpha - \beta)$. For $f \in X$, set

$$f_j(z) := 2^{-m} \binom{m}{j} (1-z)^j (1+z)^{m-j} f(z), \quad z \in \mathbb{D}, 0 \leq j \leq m. \quad (6.3)$$

Notice that $(1 + \iota z)^\delta \in \mathcal{P} \subseteq Mul(X)$ for all $\delta \geq 0, \iota \in \{-1, 1\}$ by (**Gam2**), so $f_j \in X$ for all $0 \leq j \leq m$. Moreover, $f_j \in \mathfrak{X}_{-1}^{\beta-\alpha-2\gamma}$ if $j \leq m/2$, and $f_j \in \mathfrak{X}_1^{\beta-\alpha-2\gamma}$ otherwise. It follows from Proposition 5.3(iii) and (iv) that $\Lambda_\omega^{\lambda,c_j} f_j \in X$ for all j, where $c_j = -1$ if $j \leq m/2$ and $c_j = 1$ otherwise. Set

$$K_j := -2\int_{c_j}^0 \frac{(1-\xi)^{\lambda-1}}{(1+\xi)^{\lambda+1}} \omega(\xi) f_j(\xi)\, d\xi, \qquad 0 \leq j \leq m.$$

Corollary 5.2 shows that the complex numbers K_j, $0 \leq j \leq m$, are well defined and that $g_{f_j,K_j} = \Lambda_\omega^{\lambda,c_j} f_j \in X$ for all $0 \leq j \leq m$. Hence, by Remark 6.5 we have $g_{f_j,K_j} = \Lambda_\omega^{\lambda,c_j} f_j \in \mathcal{D}(\Delta_\omega)$ and $(\lambda - \Delta_\omega)g_{f_j,K_j} = f_j$, that is $f_j \in \mathrm{Ran}(\lambda - \Delta_\omega)$ for all $0 \leq j \leq m$. Since $f = \sum_{j=0}^m f_j$, it follows that $f \in \mathrm{Ran}(\lambda - \Delta_\omega)$ and we conclude that $\mathrm{Ran}(\lambda - \Delta_\omega) = X$.

Assume now $\beta - \alpha > 2\gamma$ and $\alpha + \gamma < \mathfrak{Re}\,\lambda < \beta - \gamma$. By Lemma 5.4, L_ω^λ is a continuous functional on X, and $\Lambda_\omega^{\lambda,1} f = \Lambda_\omega^{\lambda,-1} f \in X$ if $f \in \ker L_\omega^\lambda$. By Lemma 5.1,

$$K := -2\int_{-1}^{0} \frac{(1-\xi)^{\lambda-1}}{(1+\xi)^{\lambda+1}}\omega(\xi)f(\xi)\,d\xi$$

is well defined and $g_{f,K} = \Lambda_\omega^{\lambda,-1} f \in X$. By Remark 6.5, $g_{f,K} \in \mathcal{D}(\Delta_\omega)$ and $(\lambda - \Delta_\omega)g_{f,K} = f$, so that $f \in \operatorname{Ran}(\lambda - \Delta_\omega)$. Thus, $\ker L_\omega^\lambda \subseteq \operatorname{Ran}(\lambda - \Delta_\omega)$.

Let now $f \in X \setminus \ker L_\omega^\lambda$. The mapping $z \mapsto \int_0^z (1-\xi)^{\lambda-1}(1+\xi)^{-\lambda-1}\omega(\xi)f(\xi)\,d\xi$ is continuous from $\mathbb{D} \cup \{-1, 1\}$ to $\mathbb{C}$, see Remark 5.6. Hence, for all $K \in \mathbb{C}$,

$$\begin{aligned} K - 2\int_0^z \frac{(1-\xi)^{\lambda-1}}{(1-\xi)^{\lambda+1}}\omega(\xi)f(\xi)\,d\xi &\xrightarrow[z\to 1]{} c_{K,+} \in \mathbb{C}, \\ K - 2\int_0^z \frac{(1-\xi)^{\lambda-1}}{(1-\xi)^{\lambda+1}}\omega(\xi)f(\xi)\,d\xi &\xrightarrow[z\to -1]{} c_{K,-} \in \mathbb{C}, \end{aligned} \tag{6.4}$$

whenever z tends to $-1, 1$ non-tangentially. Since $c_{K,+} - c_{K,-} = -2L_\omega^\lambda f \neq 0$, either $c_{K,+} \neq 0$ or $c_{K,-} \neq 0$. By Theorem 3.11, we have for each $\varepsilon > 0$, $|\omega(z)^{-1}| \gtrsim |1-z|^{\alpha+\varepsilon}|1+z|^{-\beta+\varepsilon}$ for all $z \in \mathbb{D}$. Applying this bound in (6.2) one gets that, for each $K \in \mathbb{C}$, either $|g_{f,K}(x)| \gtrsim |1-x|^{-\gamma'}$ as $x \to 1$ or $|g_{f,K}(x)| \gtrsim |1+x|^{-\gamma'}$ as $x \to -1$, for some $\gamma' > \gamma$. In any case, $g_{f,K} \notin \mathcal{K}^{-\gamma-\delta}(\mathbb{D})$ for some $\delta > 0$ and therefore $g_{f,K} \notin X$, see Remark 2.4. As a consequence, $f \notin \operatorname{Ran}(\lambda - \Delta_\omega)$ by Remark 6.5. Thus $\operatorname{Ran}(\lambda - \Delta_\omega) \subseteq \ker L_\omega^\lambda$, and the proof is finished. □

The following theorem gives the spectrum of the generator Δ_ω.

Theorem 6.7. *Let α and β be the exponents of ω, and suppose $\omega^{-1} \in \mathfrak{S}$. Then*

$$\sigma(\Delta_\omega) = |\beta - \gamma, \alpha + \gamma|.$$

Proof. First assume $\beta - \alpha \neq 2\gamma$. The inclusion $\sigma(\Delta_\omega) \subseteq |\beta - \gamma, \alpha + \gamma|$ is in Lemma 6.2. On the other hand, $\operatorname{Int}(|\beta-\gamma, \alpha+\gamma|) \subseteq \sigma(\Delta_\omega)$ by Proposition 6.3 in the case $\beta - \alpha < 2\gamma$, and by Lemma 6.6 if $\beta - \alpha > 2\gamma$. Therefore $\sigma(\Delta_\omega) = |\beta - \gamma, \alpha + \gamma|$ since the spectrum of a closed operator is a closed subset of $\mathbb{C}$.

In the case $\beta - \alpha = 2\gamma$ one cannot directly use the results obtained in Section 5. Instead, we use the invariance of ω in the sense of Lemma 3.8 to slightly modify the exponents α, β and then take advantage of what has been already proved for $\beta - \alpha \neq 2\gamma$.

As above, one has $\sigma(\Delta_\omega) \subseteq |\beta-\gamma, \alpha+\gamma|$ by Lemma 6.2. To prove the reverse inclusion, take $\lambda \in \mathbb{C}$ in $|\alpha + \gamma, \beta - \gamma|$, which is to say $\mathfrak{Re}\,\lambda = \alpha + \gamma = \beta - \gamma$. Recall

$$g_\lambda(z) = \omega(z)^{-1}(1+z)^\lambda(1-z)^{-\lambda}, \quad z \in \mathbb{D}.$$

If $g_\lambda \in X$ then $\lambda \in \sigma_{point}(\Delta_\omega)$ by Proposition 6.3, and we are done. Thus we assume $g_\lambda \notin X$ and $\lambda - \Delta_\omega$ injective. Under this assumption we show next, by contradiction, that $\lambda - \Delta_\omega$ is not surjective, whence $\lambda \in \sigma(\Delta_\omega)$ and the proof will be finished.

Thus suppose that $\lambda - \Delta_\omega$ is a surjective operator. As noticed in Remark 6.5, this implies that, for every $f \in X$, there exists $K \in \mathbb{C}$ for which the function $g_{f,K}$ in (6.2) lies in X. Since $g_\lambda \in \mathcal{O}(\mathbb{D}) \setminus X$, we have that either $g_\lambda \notin X_1$ or $g_\lambda \notin X_{-1}$ (meaning that the restriction of g_λ to $\mathbb{D}_1$ or $\mathbb{D}_{-1}$ is not in X_1 or X_{-1} respectively), where X_{-1}, X_1 are the Banach spaces given in (**Gam3**).

Suppose $g_\lambda \notin X_{-1}$ without loss of generality. For $c > 0$, set $\omega_c(z) := \omega(z)/(1-z)^c$, $z \in \mathbb{D}$, and $v_t := (\omega_c \circ \varphi_t)/\omega_c$, $t \in \mathbb{R}$. Then (v_t) is a DW-continuous cocycle for the flow (φ_t) on X with exponents $\alpha_c = \alpha + c > \alpha$ and $\beta_c = \beta$, see Lemma 3.8. In particular $\beta_c - \alpha_c = 2\gamma - c < 2\gamma$, so we conclude that $\sigma(\Delta_{\omega_c}) = |\beta_c - \gamma, \alpha_c + \gamma| = |\beta - \gamma, \alpha + \gamma + c|$ by the first part of this proof. In particular, $\lambda \in \sigma(\Delta_{\omega_c})$ and so $\lambda - \Delta_{\omega_c}$ is either not injective or not surjective.

If $\lambda - \Delta_{\omega_c}$ is not injective Lemma 6.3 implies that the holomorphic function $g_\lambda(1-(\cdot))^c$ is in X, and therefore its restriction to $\mathbb{D}_{-1}$ is in X_{-1}. However, the function $(1-(\cdot))^{-c}$ is in $Mul(X_{-1})$ since it is holomorphic in an open set containing $\overline{\mathbb{D}_{-1}}$, see (**Gam3**). Hence we have $g_\lambda \in X_{-1}$, which is a contradiction since we have assumed the opposite. Therefore $\lambda - \Delta_{\omega_c}$ must be an injective operator, which implies in turn that $\lambda - \Delta_{\omega_c}$ is not surjective by the preceding paragraph. However, we shall show next that $\lambda - \Delta_{\omega_c}$ is also surjective, reaching again a contradiction.

By a similar trick as after (6.3), one gets $X = \mathfrak{X}_{-1}^{-c} + \mathfrak{X}_1^{-c}$, and then it is enough to show that $\mathfrak{X}_{-1}^{-c}$ and $\mathfrak{X}_1^{-c}$ are subspaces of $\mathrm{Ran}(\lambda - \Delta_{\omega_c})$. Take $f \in \mathfrak{X}_{-1}^{-c} = \mathfrak{X}_{-1}^{\beta_c - \alpha_c - 2\gamma}$. By Proposition 5.3(iv), $\Lambda_{\omega_c}^{\lambda,-1} f \in X$ and, as in the proof of Lemma 6.6, case $\beta - \alpha < 2\gamma$, one obtains $(\lambda - \Delta_{\omega_c})\Lambda_{\omega_c}^{\lambda,-1} f = f$. Thus $f \in \mathrm{Ran}(\lambda - \Delta_{\omega_c})$ and then it follows that $\mathfrak{X}_{-1}^{-c} \subseteq \mathrm{Ran}(\lambda - \Delta_{\omega_c})$. Take now $f \in \mathfrak{X}_1^{-c}$ and define $f_c \in X$ by $f_c(z) = (1-z)^{-c} f(z)$ for $z \in \mathbb{D}$. As we have assumed $\lambda - \Delta_\omega$ is surjective on X, there exists $K \in \mathbb{C}$ such that $g_{f_c,K} \in X$, see Remark 6.5. Since $(1-(\cdot))^c \in \mathcal{P} \subseteq Mul(X)$ by (**Gam2**) one has $(1-(\cdot))^c g_{f_c,K} \in X$. Using again Lemma Remark 6.5 with the weight ω_c instead ω one gets $f \in \mathrm{Ran}(\lambda - \Delta_{\omega_c})$. So $\mathfrak{X}_1^c \subseteq \mathrm{Ran}(\lambda - \Delta_{\omega_c})$.

Therefore, $\lambda - \Delta_{\omega_c}$ is surjective, hence invertible, reaching the forecasted contradiction since $\lambda \in \sigma(\Delta_{\omega_c})$. We finally conclude that our assumption $\lambda \notin \sigma(\Delta_\omega)$ is incorrect, and the proof is finished. □

The overall discussion carried out in preceding places of this paper leads to the following detailed description of $\sigma(\Delta_\omega)$. Recall that the approximate spectrum and residual spectrum of a closed operator A are denoted by $\sigma_{ap}(A)$ and $\sigma_{res}(A)$ respectively.

Theorem 6.8. *Let $\gamma \geq 0$ and let X be a γ-space which is hyperbolically DW-contractive, and let $\mathfrak{S}$ be such that $(X, \mathfrak{S})$ is a γ-pair. Let (u_t) be a hyperbolically DW-continuous cocycle for (φ_t), so that $(u_t C_{\varphi_t})$ is a C_0-group in $\mathcal{B}(X)$. Let α, β be the exponents of (u_t), and let ω be an associated weight to (u_t). Let Δ_ω be the infinitesimal generator of $(S_\omega(t)) := (u_t C_{\varphi_t})$, and assume $\omega^{-1} \in \mathfrak{S}$. Then one has the following.*

i) *The full spectrum $\sigma(\Delta_\omega)$ of Δ_ω is the strip $|\alpha + \gamma, \beta - \gamma|$.*

ii) *The essential spectrum of* Δ_ω *is the boundary of* $\sigma(\Delta_\omega)$*, that is,* $\sigma_{ess}(\Delta_\omega) = \partial(|\alpha + \gamma, \beta - \gamma|)$*.*

iii) *The approximate spectrum of* Δ_ω *is given by*

$$\sigma_{ap}(\Delta_\omega) = \begin{cases} |\alpha+\gamma, \beta-\gamma|, & \text{if} \quad \beta - \alpha \le 2\gamma; \\ \partial(|\alpha+\gamma, \beta-\gamma|), & \text{if} \quad \beta - \alpha > 2\gamma. \end{cases}$$

iv) *The point spectrum* $\sigma_{point}(\Delta_\omega)$ *of* Δ_ω *satisfies*

$$\{\lambda \in \mathbb{C} \,:\, \beta - \gamma < \mathfrak{Re}\,\lambda < \alpha + \gamma\} \subseteq \sigma_{point}(\Delta_\omega) \subseteq \{\lambda \in \mathbb{C} \,:\, \beta - \gamma \le \mathfrak{Re}\,\lambda \le \alpha + \gamma\}.$$

The eigenspace of $\lambda \in \sigma_{point}(\Delta_\omega)$ *is the one-dimensional subspace* $\mathbb{C} g_\lambda$*.*

v) *The residual spectrum* $\sigma_{res}(\Delta_\omega)$ *of* Δ_ω *on* X *satisfies*

$$\{\lambda \in \mathbb{C} \,:\, \alpha + \gamma < \mathfrak{Re}\,\lambda < \beta - \gamma\} \subseteq \sigma_{res}(\Delta_\omega), \quad \beta - \alpha > 2\gamma;$$
$$\sigma_{res}(\Delta_\omega) \subseteq \{\lambda \in \mathbb{C} \,:\, \mathfrak{Re}\,\lambda = \alpha + \gamma \ \text{ or } \ \mathfrak{Re}\,\lambda = \beta - \gamma\}, \quad \beta - \alpha \le 2\gamma.$$

Proof. i) This is Theorem 6.7.

ii) Let $\lambda \in \sigma(\Delta_\omega) = |\alpha + \gamma, \beta - \gamma|$. The kernel of $\lambda - \Delta_\omega$ is at most one-dimensional by Proposition 6.3, so $\dim(\ker(\lambda - \Delta_\omega)) < \infty$. In addition, if $\lambda \in \mathrm{Int}(|\alpha + \gamma, \beta - \gamma|)$, then $\dim(X/\mathrm{Ran}(\lambda - \Delta_\omega)) \le 1 < \infty$ by Lemma 6.6, so we conclude that $\mathrm{Int}(|\alpha + \gamma, \beta - \gamma|) \cap \sigma_{ess}(\Delta_\omega) = \emptyset$.
Now let $\lambda \in \partial(|\alpha + \gamma, \beta - \gamma|)$. By item i), λ is an accumulation point of both the resolvent set $\rho(\Delta_\omega)$ and the spectrum $\sigma(\Delta_\omega)$. As a consequence, $\lambda \in \sigma_{ess}(\Delta_\omega)$, see for example [16, Th. I.3.25].

iii) First, the inclusion $\partial\sigma(A) \subseteq \sigma_{ap}(A)$ holds for an arbitrary closed operator A, see for example [18, IV.1.10]. Now take $\lambda \in \mathrm{Int}(\sigma(\Delta_\omega)) = \mathrm{Int}(|\beta - \gamma, \alpha + \gamma|)$. Then $\mathrm{Ran}(\lambda - \Delta_\omega)$ is a closed subspace by Lemma 6.6, and $\lambda - \Delta_\omega$ is not injective if and only if $\beta - \alpha < 2\gamma$, see Proposition 6.3.

iv) This is Proposition 6.3.

v) This is a direct consequence of Lemma 6.6. □

Remark 6.9. (1) From item i) in the theorem above, (6.1) gives the resolvent $R(\lambda, \Delta_\omega)$ for all $\lambda \in \rho(\Delta_\omega)$.

(2) For Hardy spaces, weighted Bergman spaces, Little Korenblum spaces, and the disc algebra, condition $\omega^{-1} \in \mathfrak{S}$ in Theorem 6.8 is superfluous, in view of Proposition 6.4 and the comment prior to Proposition 6.4.

7. Spectra of weighted hyperbolic composition groups

Let $\omega, (S_\omega(t))$ be as in Section 4. The spectral analysis of the infinitesimal generator Δ_ω given in Theorem 6.8 is here transferred to the group $(S_\omega(t))$.

Theorem 7.1. *Let X, $\mathfrak{S}$ and $S_\omega(t)$ be as in Theorem 6.8. Let $t \in \mathbb{R} \setminus \{0\}$. Then*

i) *The full spectrum of $S_\omega(t)$ is the annulus*

$$\sigma(S_\omega(t)) = \{\lambda \in \mathbb{C} \,:\, e^{\min\{(\alpha+\gamma)t,(\beta-\gamma)t\}} \leq |\lambda| \leq e^{\max\{(\alpha+\gamma)t,(\beta-\gamma)t\}}\}.$$

ii) *The essential spectrum of $S_\omega(t)$ coincides with the full spectrum, i.e.,*

$$\sigma_{ess}(S_\omega(t)) = \sigma(S_\omega(t)).$$

iii) *The point spectrum $\sigma_{point}(S_\omega(t))$ of $S_\omega(t)$ satisfies*

$$\begin{aligned} \operatorname{Int}(\sigma(S_\omega(t))) \subseteq \sigma_{point}(S_\omega(t)), &\quad \textit{if} \ \ \beta - \alpha < 2\gamma; \\ \sigma_{point}(S_\omega(t)) = \emptyset, &\quad \textit{if} \ \ \beta - \alpha > 2\gamma. \end{aligned}$$

Moreover, the eigenspace of λ is:

$$\overline{\operatorname{span}}\{g_\mu \,:\, \mu \in W_\lambda\}, \ \textit{if } \lambda \in \operatorname{Int}(\sigma_{point}(S_\omega(t)))$$

and

$$\overline{\operatorname{span}}\{g_\mu \,:\, \mu \in W_\lambda \ \textit{and} \ g_\mu \in X\}, \ \textit{if } \lambda \in \partial(\sigma_{point}(S_\omega(t))),$$

where $W_\lambda = \{\mu \in \mathbb{C} \,:\, e^{\mu t} = \lambda\}$.

iv) *The residual spectrum $\sigma_{res}(S_\omega(t))$ of $S_\omega(t)$ on X satisfies*

$$\begin{aligned} \operatorname{Int}(\sigma(S_\omega(t))) \subseteq \sigma_{res}(S_\omega(t)), &\quad \textit{if} \ \ \beta - \alpha > 2\gamma; \\ \sigma_{res}(S_\omega(t)) \subseteq \partial(\sigma(S_\omega(t))), &\quad \textit{if} \ \ \beta - \alpha \leq 2\gamma. \end{aligned}$$

If $\lambda \in \operatorname{Int}(\sigma_{\mathrm{res}}(\mathrm{S}_\omega(\mathrm{t})))$ then $\operatorname{Ran}(\lambda - S_\omega(t)) \subseteq \bigcap_{\mu \in W_\lambda} \ker L_\omega^\mu$.

Proof. i) We have $e^{t\sigma(\Delta_\omega)} \subseteq \sigma(S_\omega(t))$ for any $t \in \mathbb{R}$ by the spectral mapping inclusion for C_0-semigroups, see [18, IV.3.6]. Thus the inclusion $\supseteq$ of the statement follows from Theorem 6.8. The reverse inclusion $\subseteq$ follows from the spectral radius theorem together with the asymptotic bounds for $\|S_\omega(t)\|_{\mathcal{B}(X)}$ given in Proposition 4.2.

ii) If $\lambda \in \partial(\sigma(S_\omega(t)))$, then item i) shows that λ is an accumulation point of both the resolvent set $\rho(S_\omega(t))$ and the spectrum $\sigma(S_\omega(t))$. As a consequence, $\lambda \in \sigma_{ess}(S_\omega(t))$, see [16, Th. I.3.25].

Now let $\lambda \in \operatorname{Int}(\sigma(S_\omega(t)))$. One can assume $\beta - \alpha \neq 2\gamma$ since otherwise $\operatorname{Int}(\sigma(S_\omega(t))) = \emptyset$ by item i). If $\beta - \alpha < 2\gamma$ then $\dim(\ker(\lambda - S_\omega(t))) = \infty$, as we see below in the proof of item iii), so $\lambda \in \sigma_{ess}(S_\omega(t))$. On the other hand, if $\beta - \alpha > 2\gamma$, then

$$\operatorname{Ran}(\lambda - S_\omega(t)) \subseteq \cap_{\mu\in W_\lambda} \operatorname{Ran}(\mu - \Delta_\omega) = \cap_{\mu\in W_\lambda} \ker L^\mu_\omega, \tag{7.1}$$

by [18, Equation (IV.3.14)] and Lemma 6.6.

Moreover, $\{L^\mu_\omega\}$ is linearly independent in the dual space of X since L^μ_ω is an eigenvector associated to the eigenvalue μ of the adjoint operator of Δ_ω, see Lemma 6.6. Therefore the subspace $\cap_{\mu\in W_\lambda} \ker L^\mu_\omega$ has infinite codimension [34, Lemma 3.9], and we conclude that $\lambda \in \sigma_{ess}(S_\omega(t))$, as we wanted to prove.

This proves the claim made at iv) about $\operatorname{Ran}(\lambda - S_\omega(t))$ since $\operatorname{Ran}(\mu - \Delta_\omega) = \ker L^\mu_\omega$ for all $\mu \in W_\lambda$ by Lemma 6.6.

iii) & iv) We have $\sigma_{point}(S_\omega(t)) = e^{t\sigma_{point}(\Delta_\omega)}$ and $\sigma_{res}(S_\omega(t)) = e^{t\sigma_{res}(\Delta_\omega)}$, $t \in \mathbb{R}$, see for instance [18, Th. IV.3.7]. Thus the given inclusions for the respective spectra are immediate consequences of Theorem 6.8. The claim about the eigenspaces follows from the fact that the kernel of $\lambda - S_\omega(t)$ is the closure of the linear span of the eigenspaces of $\mu - \Delta_\omega$, where $\mu \in W_\lambda$, see e.g. [18, Corollary IV.3.8]. The claim made about $\operatorname{Ran}(\lambda - S_\omega(t))$ follows from (7.1). □

As a consequence of Theorem 7.1, one obtains the fine spectrum of weighted composition groups of the form $(v_t C_{\psi_t})$ where (ψ_t) is an arbitrary hyperbolic flow.

Theorem 7.2. *Let $(X, \mathfrak{S})$ be a γ-pair with $\gamma \geq 0$ such that X is hyperbolically DW-contractive. Let (ψ_t) be a hyperbolic flow with DW-points a (attractive), b (repulsive) $\in \mathbb{T}$, and let (v_t) be a DW-continuous cocycle for (ψ_t) on X. Let ϖ be such that $v_t = (\varpi \circ \psi_t)/\varpi$ and assume $\varpi^{-1} \in C_\phi(\mathfrak{S})$, where $\phi \in Aut(\mathbb{D})$ is such that $\phi(a) = 1$, $\phi(b) = -1$. Then, for $t \in \mathbb{R} \setminus \{0\}$,*

i) *The full spectrum of $v_t C_{\psi_t}$ is the set*

$$\sigma(v_t C_{\psi_t}) = \left\{ \lambda \in \mathbb{C} \,:\, \min\left\{ \frac{|v_t(a)|}{\psi_t'(a)^\gamma}, \frac{|v_t(b)|}{\psi_t'(b)^\gamma} \right\} \leq |\lambda| \leq \max\left\{ \frac{|v_t(a)|}{\psi_t'(a)^\gamma}, \frac{|v_t(b)|}{\psi_t'(b)^\gamma} \right\} \right\}.$$

ii) *The essential spectrum of $v_t C_{\psi_t}$ coincides with its full spectrum, i.e., $\sigma_{ess}(v_t C_{\psi_t}) = \sigma(v_t C_{\psi_t})$.*

iii) *The point spectrum of $v_t C_{\psi_t}$ satisfies*

$$\operatorname{Int}(\sigma(v_t C_{\psi_t})) \subseteq \sigma_{point}(v_t C_{\psi_t}), \quad \textit{if } \frac{|v_1(b)|}{\psi_1'(b)^\gamma} < \frac{|v_1(a)|}{\psi_1'(a)^\gamma};$$

$$\sigma_{point}(v_t C_{\psi_t}) = \emptyset, \quad \textit{if } \frac{|v_1(b)|}{\psi_1'(b)^\gamma} > \frac{|v_1(a)|}{\psi_1'(a)^\gamma}.$$

Moreover, the eigenspace of λ is:

$$\overline{\operatorname{span}}\{\widetilde{g}_\mu \,:\, \mu \in \widetilde{W_\lambda}\}, \ \textit{if } \lambda \in \operatorname{Int}(\sigma_{point}(v_t C_{\psi_t})),$$

where $\widetilde{g}_\mu(z) := \frac{1}{\varpi(z)} \frac{(b-z)^\mu}{(a-z)^\mu}$, $z \in \mathbb{D}$, and

$$\overline{\text{span}}\{\widetilde{g}_\mu \,:\, \mu \in \widetilde{W_\lambda} \text{ and } \widetilde{g}_\mu \in X\}, \text{ if } \lambda \in \partial(\sigma_{point}(v_t C_{\psi_t})),$$

where $\widetilde{W_\lambda} = \{\mu \in \mathbb{C} \,:\, \psi_t'(a)^\mu = \lambda^{-1}\}$.

iv) *The residual spectrum of* $v_t C_{\psi_t}$ *satisfies*

$$\text{Int}(\sigma(v_t C_{\psi_t})) \subseteq \sigma_{res}(v_t C_{\psi_t}), \quad \text{if } \frac{|v_1(b)|}{\psi_1'(b)^\gamma} > \frac{|v_1(a)|}{\psi_1'(a)^\gamma};$$

$$\sigma_{res}(v_t C_{\psi_t}) \subseteq \partial(\sigma(v_t C_{\psi_t})), \quad \text{if } \frac{|v_1(b)|}{\psi_1'(b)^\gamma} \le \frac{|v_1(a)|}{\psi_1'(a)^\gamma}.$$

If $\lambda \in \text{Int}(\sigma_{res}(v_t C_{\psi_t}))$, *then* $\text{Ran}(\lambda - v_t C_{\psi_t}) \subseteq \cap_{\mu \in \widetilde{W_\lambda}} \ker \tilde{L}^\mu_\varpi$, *where* $\tilde{L}^\mu_\varpi : X \to \mathbb{C}$ *is the continuous functional on* X *given by*

$$\tilde{L}^\mu_\varpi f = \int_b^a \frac{(a-\xi)^{\mu-1}}{(b-\xi)^{\mu+1}} \varpi(\xi) f(\xi)\, d\xi, \qquad f \in X. \tag{7.2}$$

Here, we can take any simple integration path in $\mathbb{D}$ *from* b *to* a *such that approaches both* b, a *non-tangentially.*

Proof. There is $c > 0$ such that $v_t C_{\psi_t} = C_\phi (u_{ct} C_{\varphi_{ct}}) C_{\phi^{-1}}$, $t \in \mathbb{R}$, where $(u_t) := (v_{c^{-1}t} \circ \phi^{-1})$ is a DW-continuous cocycle for (φ_t), see the end of Section 1. Therefore, it is enough to obtain the spectral sets for the operator $u_{ct} C_{\varphi_{ct}}$.

It is readily seen that $u_t = ((\varpi \circ \phi^{-1}) \circ \varphi_t)/(\varpi \circ \phi^{-1})$. Hence $u_t C_{\varphi_t} = S_\omega(t)$, $t \in \mathbb{R}$, in the notation of Section 6, where $\omega := \varpi \circ \phi^{-1}$. Thus $\omega^{-1} \in \mathfrak{S}$ and we have that the hypotheses of Theorem 7.1 are satisfied.

Therefore we can apply Theorem 7.1 to $S_\omega(ct)$. So all that we have to prove is $e^{(\alpha+\gamma)ct} = |v_t(a)| \psi_t'(a)^{-\gamma}$ and $e^{(\beta-\gamma)ct} = |v_t(b)| \psi_t'(b)^{-\gamma}$, where α, β are the exponents of the DW-continuous cocycle (u_t), see Lemma 3.5. From here, our claims regarding the spectra of $u_{ct} S_\omega(ct)$ follow immediately. Let us see.

On the one hand, $e^{\alpha ct} = e^{\alpha(ct)} = \lim_{z\to 1} |u_{ct}(z)| = \lim_{z \to a} |v_t(z)| = |v_t(a)|$. On the other hand, $\psi_t'(a) = (\phi^{-1} \circ \varphi_{ct} \circ \phi)'(a) = \varphi_{ct}'(1) = e^{-ct}$, $t \in \mathbb{R}$, and then $e^{c\gamma t} = \psi_t'(a)^{-\gamma}$. Thus $e^{(\alpha+\gamma)ct} = |v_t(a)| \psi_t'(a)^{-\gamma}$. The identity $e^{(\beta-\gamma)ct} = |v_t(b)| \psi_t'(b)^{-\gamma}$ can be obtained analogously.

Next we prove the claim made on the eigenspaces of $v_t C_{\psi_t}$. Let $\lambda \in \text{Int}(\sigma_{point}(v_t C_{\psi_t})) = \text{Int}(\sigma_{point}(S_\omega(ct)))$. By Theorem 7.1, the eigenspace of $S_\omega(ct)$ associated with the eigenvalue λ is $\overline{\text{span}}\{g_\nu \,:\, (\psi_t'(a))^\nu = \lambda^{-1}\} = \overline{\text{span}}\{g_\nu \,:\, \nu \in \widetilde{W_\lambda}\}$. Therefore the eigenspace of $v_t C_{\psi_t}$ associated to the eigenvalue λ is $\overline{\text{span}}\{g_\nu \circ \phi \,:\, \nu \in \widetilde{W_\lambda}\}$. It is readily seen that the linear fractional mapping $(1+\phi)/(1-\phi)$ has one zero at $z = b$ and one pole at $z = a$, so that it is equal to $(b - (\cdot))/(a - (\cdot))$ up to a constant. Thus $\mathbb{C}\widetilde{g}_\nu = \mathbb{C}(g_\nu \circ \phi)$, that is, the eigenspaces of $v_t C_{\psi_t}$ are as claimed in the statement. The case $\lambda \in \partial(\sigma_{point}(v_t C_{\psi_t}))$ runs similarly.

It only remains to prove the claim made about the range space $\mathrm{Ran}(\lambda - v_t C_{\psi_t})$. Take $\lambda \in \mathrm{Int}(\sigma_{res}(v_t C_{\psi_t}))$. By Theorem 7.1, $\mathrm{Ran}(\lambda - v_t C_{\psi_t}) = C_\phi(\mathrm{Ran}(\lambda - S_\omega(ct))) \subseteq C_\phi(\ker L^\mu_\omega) = \ker(L^\mu_\omega C_{\phi^{-1}})$ for all $\mu \in \widetilde{W}_\lambda$, where L^μ_ω is a continuous functional on X, see Lemma 5.4. Now, we are going to prove that $\tilde{L}^\mu_\varpi = k L^\mu_\omega C_{\phi^{-1}}$ for some $k \in \mathbb{C} \setminus \{0\}$, and the proof will be done.

Recall that Ψ denotes the generator (ψ_t). One has $\Psi(z) = \frac{c}{a-b}(a-z)(b-z) = G(\phi(z))/\phi'(z)$ for $z \in \mathbb{D}$, see [5, Th. 1.6]. As a consequence, the change of variable $z = \phi^{-1}(\xi)$ in the integral below yields

$$\begin{aligned} L^\mu_\omega C_{\phi^{-1}} f &= \int_{-1}^{1} \frac{(1-\xi)^{\mu-1}}{(1+\xi)^{\mu+1}} \omega(\xi)(f \circ \phi^{-1}(\xi))\, d\xi \\ &= k \int_b^a \frac{(a-z)^{\mu-1}}{(b-z)^{\mu+1}} \varpi(z) f(z)\, dz = k\, \tilde{L}^\mu_\varpi f, \qquad f \in X, \end{aligned}$$

as we wanted to prove. $\square$

Remark 7.3. (1) As it has been shown in Section 2, Section 3 and Section 6, spaces $H^p(\mathbb{D})$, $\mathcal{A}^p_\sigma(\mathbb{D})$, $\mathcal{K}_0^{-\gamma}(\mathbb{D})$, $\mathfrak{A}(\mathbb{D})$, $\mathcal{D}^p_\sigma(\mathbb{D})$ and $B_{1,0}(\mathbb{D})$, for $p \geq 1$, $\sigma > -1$, $\gamma > 0$, satisfy the conditions assumed on X in Theorem 7.2. Furthermore, for $H^p(\mathbb{D})$, $\mathcal{A}^p_\sigma(\mathbb{D})$, $\mathcal{K}_0^{-\gamma}(\mathbb{D})$ and $\mathfrak{A}(\mathbb{D})$ the hypothesis $\varpi^{-1} \in C_\phi(\mathfrak{S})$ is superfluous, see Remark 6.9(2). For $\mathcal{D}^p_\sigma(\mathbb{D})$, we conjecture that there exist a subset $\mathfrak{S}(\mathcal{D}^p_\sigma)$ defined in terms of Carleson measures such that $(\mathcal{D}^p_\sigma(\mathbb{D}), \mathfrak{S}(\mathcal{D}^p_\sigma))$ is a γ-pair and that the assumption $\varpi^{-1} \in C_\phi(\mathfrak{S}(\mathcal{D}^p_\sigma))$ is redundant as well.

(2) Theorem 7.2 answers in the positive the conjectures established in [8,17,27] about the spectrum of a weighted hyperbolic invertible operator vC_ψ on γ-spaces in the case that v can be embedded in a cocycle for (ψ_t), where $\psi_1 = \psi$ (see the Introduction).

Remark 7.4. Nonseparable Korenblum spaces, H^∞ in particular, and Bloch spaces are not under the scope of the paper since weighted composition groups are not strongly continuous on them. These cases will be specifically approached in a forthcoming paper.

8. Weighted averaging operators

Here, we make use of the theory developed in the preceding sections to study the boundedness and spectral sets of two families of weighted averaging operators acting on γ-spaces. Throughout all this section, $(X, \mathfrak{S})$ denotes a γ-pair for some $\gamma \geq 0$ such that X is hyperbolically DW-contractive and such that the constant function $\mathbf{1}$ lies in $\mathfrak{S}$. In particular, it applies to any of the γ-spaces listed in the examples of Subsection 2.1.

From now on, we denote by $B(\cdot,\cdot)$, $\Gamma(\cdot)$ the Beta function and the Gamma function respectively. The following estimate for the Gamma function will be used in the sequel.

For $\lambda \in \mathbb{C}$, one has

$$\frac{\Gamma(z+\lambda)}{\Gamma(z)} = z^{\lambda}\left(1 + \frac{\lambda(\lambda+1)}{2z} + O(|z|^{-2})\right) = z^{\lambda}\left(1 + O(|z|^{-1})\right), \quad z \in \mathbb{C},\ |z| \to \infty, \tag{8.1}$$

whenever $z \neq 0, -1, -2, \ldots$ and $z \neq -\lambda, -\lambda-1, -\lambda-2\ldots$, see [40] for more details.

8.1. Siskakis type operators

Let $\mu, \nu, \delta \in \mathbb{C}$. Here we analyze the weighted averaging operators given by

$$(\mathcal{J}_{\delta}^{\mu,\nu} f)(z) = \frac{1}{(1+z)^{\nu+\delta}(1-z)^{\mu+\delta}} \int_{z}^{1} (1+\xi)^{\nu}(1-\xi)^{\mu}(\xi - z)^{\delta-1} f(\xi)\, d\xi, \qquad z \in \mathbb{D}. \tag{8.2}$$

Proposition 8.1. *Let $\mathfrak{Re}\,\mu - \gamma + 1 > 0$, $\gamma - \mathfrak{Re}\,(\nu+\delta) > 0$ and $\mathfrak{Re}\,\delta > 0$. Let $\omega(z) = (1+z)^{\nu+\delta}(1-z)^{\mu+1}$ for $z \in \mathbb{D}$. Then,*

$$\mathcal{J}_{\delta}^{\mu,\nu} f = 2^{-\delta} \int_{0}^{\infty} (1-e^{-t})^{\delta-1} S_{\omega}(t) f\, dt, \qquad f \in X, \tag{8.3}$$

where the integral is Bochner-convergent. In particular, $\mathcal{J}_{\delta}^{\mu,\nu}$ is a bounded operator on X.

Proof. Set $(u_t) = ((\omega \circ \varphi_t)/\omega)$, so (u_t) is a DW-continuous cocycle for the hyperbolic flow (φ_t) on X with exponents $\alpha = -\mathfrak{Re}\,\mu - 1$, $\beta = \mathfrak{Re}\,(\nu+\delta)$, see Lemma 3.8. By Proposition 4.2, for every $\varepsilon \in (0, \min\{\mathfrak{Re}\,\mu - \gamma + 1,\ \gamma - \mathfrak{Re}\,(\nu+\delta)\})$, there exists $K_{\varepsilon} > 0$ such that

$$\|S_{\omega}(t)\|_{\mathcal{B}(X)} \leq K_{\varepsilon} e^{-t\min\{\gamma - \mathfrak{Re}\,(\nu+\delta),\, \mathfrak{Re}\,\mu - \gamma + 1\} + \varepsilon t}, \qquad t \geq 0.$$

Hence,

$$\begin{aligned}
&\left\| \int_{0}^{\infty} (1-e^{-t})^{\delta-1} S_{\omega}(t)\, dt \right\|_{\mathcal{B}(X)} \\
&\leq K_{\varepsilon} \int_{0}^{\infty} (1-e^{-t})^{\mathfrak{Re}\,\delta - 1} e^{t(\varepsilon - \min\{\gamma - \mathfrak{Re}\,(\nu+\delta),\, \mathfrak{Re}\,\mu - \gamma + 1\})}\, dt \\
&= K_{\varepsilon} B(\mathfrak{Re}\,\delta, \min\{\gamma - \mathfrak{Re}\,(\nu+\delta),\, \mathfrak{Re}\,\mu - \gamma + 1\} - \varepsilon) < \infty.
\end{aligned}$$

As a consequence, the integral $\int_0^\infty (1-e^{-t})^{\delta-1} S_\omega(t)dt$ is strongly convergent in the Bochner sense and it defines a bounded operator on X. Moreover, for $f \in X$ and $z \in \mathbb{D}$,

$$\begin{aligned}
&\int_0^\infty (1-e^{-t})^{\delta-1}(S_\omega(t)f)(z)\,dt \\
&= \int_0^\infty (1-e^{-t})^{\delta-1}\left(\frac{1+\varphi_t(z)}{1+z}\right)^{\nu+\delta}\left(\frac{1-\varphi_t(z)}{1-z}\right)^{\mu+1} f(\varphi_t(z))\,dt \\
&= \frac{2^\delta}{(1+z)^{\nu+\delta}(1-z)^{\mu+\delta}}\int_z^1 (1+\xi)^\nu(1-\xi)^\mu(\xi-z)^{\delta-1}f(\xi)\,d\xi = 2^\delta(\mathcal{J}_\delta^{\mu,\nu}f)(z),
\end{aligned}$$

where we used the change of variable $\xi = \varphi_t(z)$, and the proof is done. □

Theorem 8.2. *Let $\mathfrak{Re}\,\mu-\gamma+1>0$, $\gamma-\mathfrak{Re}\,(\nu+\delta)>0$ and $\mathfrak{Re}\,\delta>0$. Then the spectrum, essential spectrum and point spectrum of $\mathcal{J}_\delta^{\mu,\nu}$ on X are*

$$\begin{aligned}
\sigma(\mathcal{J}_\delta^{\mu,\nu}) &= \left\{2^{-\delta}B(\delta,\lambda)\,:\,\lambda\in|\gamma-\mathfrak{Re}\,(\nu+\delta),\mathfrak{Re}\,\mu-\gamma+1|\right\}\cup\{0\},\\
\sigma_{ess}(\mathcal{J}_\delta^{\mu,\nu}) &= \left\{2^{-\delta}B(\delta,\lambda)\,:\,\mathfrak{Re}\,\lambda=\gamma-\mathfrak{Re}\,(\nu+\delta)\ \text{or}\ \mathfrak{Re}\,\lambda=\mathfrak{Re}\,\mu-\gamma+1\right\}\cup\{0\},\\
\sigma_{point}(\mathcal{J}_\delta^{\mu,\nu}) &= \left\{2^{-\delta}B(\delta,\lambda)\,:\,\lambda\in\mathbb{C}\ \textit{such that}\ \left[\xi\mapsto(1+\xi)^{\lambda-\nu-\delta}(1-\xi)^{\mu-\lambda+1}\right]\in X\right\}.
\end{aligned}$$

In particular,

$$\begin{aligned}
&\{2^{-\delta}B(\delta,\lambda)\,:\,\mathfrak{Re}\,\mu-\gamma+1<\mathfrak{Re}\,\lambda<\gamma-\mathfrak{Re}\,(\nu+\delta)\}\subseteq\sigma_{point}(\mathcal{J}_\delta^{\mu,\nu}),\\
&\quad \textit{if}\ \mathfrak{Re}\,(\mu+\nu+\delta)<2\gamma-1,
\end{aligned}$$

and

$$\sigma_{point}(\mathcal{J}_\delta^{\mu,\nu})=\emptyset,\qquad \textit{if}\ \mathfrak{Re}\,(\mu+\nu+\delta)>2\gamma-1.$$

Proof. Set $\rho=\mathfrak{Re}\,(\nu+\delta-\mu-1)/2$ and $\omega(z)=(1+z)^{\nu+\delta}(1-z)^{\mu+1}$ for $z\in\mathbb{D}$. By Proposition 8.1, one has

$$\mathcal{J}_\delta^{\mu,\nu}=2^{-\delta}\int_0^\infty(1-e^{-t})^{\delta-1}S_\omega(t)\,dt=\int_{-\infty}^\infty e^{-\rho t}S_\omega(t)\,d\tilde{\mu}(t),$$

where $d\tilde{\mu}(t)=e^{\rho t}2^{-\delta}(1-e^{-t})^{\delta-1}\chi_{(0,\infty)}(t)\,dt$.

By Proposition 4.2 and Lemma 6.1, the infinitesimal generator $\Delta_\omega-\rho$ of the C_0-group $(e^{-\rho t}S_\omega(t))$ is bisectorial-like of angle $\pi/2$ and half-width c, for each $c>|\mathfrak{Re}\,(\mu+\nu+\delta)-2\gamma+1|/2$; see for instance [23, Subsection 2.1.1]. Moreover, c can be taken such that $\int_{-\infty}^\infty e^{c|t|}\,|d\tilde{\mu}|(t)<\infty$ (see the proof of Proposition 8.1).

Define $f \in \mathcal{O}(\mathbb{D})$ by

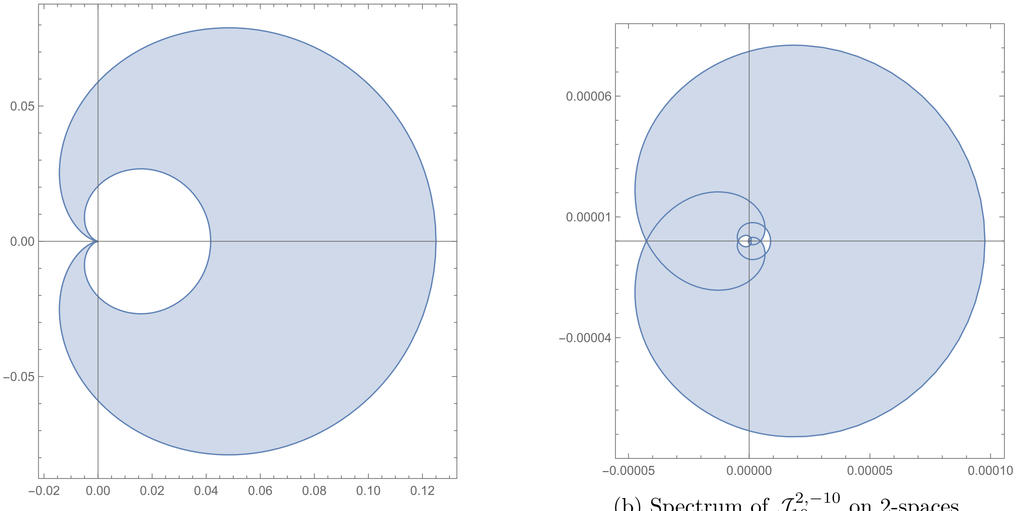
$$f(z) = \mathcal{L}_b(\tilde{\mu})(-z) = \int_{-\infty}^{\infty} e^{zt} d\tilde{\mu}(t) = 2^{-\delta} \int_{0}^{\infty} (1 - e^{-t})^{\delta-1} e^{(z+\rho)t}\, dt = 2^{-\delta} B(\delta, -z-\rho),$$

for all $z \in \mathbb{C}$ with $|\mathfrak{Re}\, z| < c$. Note that f can be analytically extended to the bisector $BS_{\theta,c}$ for any $\theta \in (0, \pi/2)$. Also, by (8.1),

$$\begin{aligned} f(z) &= 2^{-\delta} \frac{\Gamma(\delta)\Gamma(-\rho - z)}{\Gamma(\delta - \rho - z)} \\ &= 2^{-\delta}\Gamma(\delta)(-\rho - z)^{-\delta}(1 + O(|z+\rho|^{-1}))^{-1}, \quad |z| \to \infty \ (z \in BS_{\theta,c}). \end{aligned}$$

Thus, f has regular limit (equal to 0) at ∞ satisfying (1.7), which implies $f \in \mathcal{E}(\Delta_\omega - \rho)$. Hence, we can apply Theorem 1.4 to get $\widetilde{\sigma}(\mathcal{J}_\delta^{\mu,\nu}) = f(\tilde{\sigma}(\Delta_\omega - \rho))$, $\tilde{\sigma}_{ess}(\mathcal{J}_\delta^{\mu,\nu}) = f(\tilde{\sigma}_{ess}(\Delta_\omega - \rho))$ and $\sigma_{point}(\mathcal{J}_\delta^{\mu,\nu}) = f(\sigma_{point}(\Delta_\omega - \rho))$. Now, it suffices to apply Proposition 6.3 and Theorem 6.8 to obtain the claim. (Note that $\infty \in \tilde{\sigma}_{ess}(\Delta_\omega - \rho)$ since $\tilde{\sigma}_{ess}(A)$ is a closed subset of the Riemann sphere $\mathbb{C} \cup \{\infty\}$ for any closed operator A with non-empty resolvent, see for instance [31].) $\square$

We depict in Fig. 1 the spectrum and essential spectrum of two Siskakis type operators acting on two different γ-Banach spaces.



(a) Spectrum of $\mathcal{J}_2^{2,-2}$ on 1-spaces

(b) Spectrum of $\mathcal{J}_{10}^{2,-10}$ on 2-spaces

Fig. 1. Spectral pictures for two Siskakis type operators. The bold lines depict the essential spectrum.

Corollary 8.3. *Let $0 < \gamma < 1$. The Siskakis operator $\mathcal{J}$ is a bounded operator on X, and the following holds true.*

- *$\sigma(\mathcal{J})$ is the region between the circles $C_1 := \{z \in \mathbb{C} : |z + 1/\gamma| = 1/\gamma\}$ and $C_2 := \{z \in \mathbb{C} : |z + 1/(1-\gamma)| = 1/(1-\gamma)\}$.*
- *$\sigma_{ess}(\mathcal{J}) = C_1 \cup C_2$.*
- *If $\gamma > 1/2$, then $\mathrm{Int}(\sigma(\mathcal{J})) \subseteq \sigma_{point}(\mathcal{J})$. If $\gamma < 1/2$, then $\sigma_{point}(\mathcal{J}) = \emptyset$.*

8.2. Reduced Hilbert type operators

Let $\mu, \nu, \delta \in \mathbb{C}$. In this subsection, we study the spectrum of the multiparameter family of operators $(\mathcal{H}_\delta^{\mu,\nu})$, with

$$(\mathcal{H}_\delta^{\mu,\nu} f)(z) = \frac{1}{(1+z)^{\nu-\delta+1}(1-z)^{\mu-\delta+1}} \int_{-1}^{1} (1+\xi)^\nu (1-\xi)^\mu \frac{f(\xi)}{(1-z\xi)^\delta}\, d\xi, \quad z \in \mathbb{D}. \tag{8.4}$$

The next result gives sufficient conditions on μ, ν, δ for the boundedness of $\mathcal{H}_\delta^{\mu,\nu}$ on X.

Proposition 8.4. *Assume $\mathfrak{Re}\,\mu > \gamma - 1$, $\mathfrak{Re}\,\nu > \gamma - 1$, $\mathfrak{Re}\,(\delta - \mu) > 1 - \gamma$ and $\mathfrak{Re}\,(\delta - \nu) > 1 - \gamma$. Set $\omega(z) = (1+z)^{\nu+1}(1-z)^{\mu-\delta+1}$ for $z \in \mathbb{D}$. Then*

$$\mathcal{H}_\delta^{\mu,\nu} f = \int_{-\infty}^{\infty} \frac{2^{\delta-1}}{(1+e^t)^\delta} S_\omega(t) f\, dt, \qquad f \in X,$$

where the integral is Bochner-convergent. In particular, $\mathcal{H}_\delta^{\mu,\nu}$ is a bounded operator on X.

Proof. The proof is similar to the proof of Proposition 8.1.

Here, the DW-continuous cocycle $((\omega \circ \varphi_t)/\omega)$ has exponents $\alpha = \mathfrak{Re}\,(\delta - \mu) - 1$, $\beta = \mathfrak{Re}\,\nu + 1$. Fix $\varepsilon > 0$ small enough, and set $\rho := \varepsilon + \max\{\mathfrak{Re}\,\nu - \gamma + 1, \mathfrak{Re}\,(\delta - \mu) + \gamma - 1\}$ and $\widetilde{\rho} := \varepsilon + \max\{\mathfrak{Re}\,(\delta - \nu) + \gamma - 1, \mathfrak{Re}\,\mu - \gamma + 1\}$. Then, there exists $K_\varepsilon > 0$ such that

$$\begin{aligned}
&\left\| \int_{-\infty}^{\infty} \frac{1}{(1+e^t)^\delta} S_\omega(t)\, dt \right\|_{\mathcal{B}(X)} \\
&\leq K_\varepsilon \left(\int_0^\infty \frac{e^{t(\varepsilon + \max\{\beta-\gamma, \alpha+\gamma\}}}{(1+e^t)^{\mathfrak{Re}\,\delta}}\, dt + \int_{-\infty}^{0} \frac{e^{-t(\varepsilon+\max\{\gamma-\beta, -\alpha-\gamma\})}}{(1+e^t)^{\mathfrak{Re}\,\delta}}\, dt \right)
\end{aligned}$$

$$= K_\varepsilon \left(\int_1^\infty \frac{x^{\rho-1}}{(1+x)^{\mathfrak{Re}\,\delta}}\,dx + \int_1^\infty \frac{x^{\widetilde{\rho}-1}}{(1+x)^{\mathfrak{Re}\,\delta}}\,dx \right) < \infty,$$

where we have used $\mathfrak{Re}\,\delta > \max\{\rho, \widetilde{\rho}\}$ in the last step, and we have used the change of variables $e^t = x$ and $e^{-t} = x$, respectively in each integral sign, in the second-to-last step. We conclude that $\int_{-\infty}^{\infty}(1+e^t)^{-\delta} S_\omega(t)\,dt$ is Bochner-strongly convergent, whence it defines a bounded operator. Similar computations as in the proof of Proposition 8.1 give us

$$\begin{aligned}
&\int_{-\infty}^{\infty} \frac{2^{\delta-1}}{(1+e^t)^\delta}(S_\omega(t)f)(z)\,dt \\
&= \int_{-\infty}^{\infty} \frac{2^{\delta-1}}{(1+e^t)^\delta} \left(\frac{1-\varphi_t(z)}{1-z}\right)^{\mu-\delta+1} \left(\frac{1+\varphi_t(z)}{1+z}\right)^{\nu+1} f(\varphi_t(z))\,dt \\
&= \frac{1}{(1+z)^{\nu-\delta+1}(1-z)^{\mu-\delta+1}} \int_{-1}^{1} (1+\xi)^\nu (1-\xi)^\mu \frac{f(\xi)}{(1-z\xi)^\delta}\,dw \\
&= (\mathcal{H}_\delta^{\mu,\nu} f)(z), \quad z \in \mathbb{D},\ f \in X,
\end{aligned}$$

and the proof is finished. □

Now, we obtain the spectra of operators $\mathcal{H}_\delta^{\mu,\nu}$. First, we prove the following lemma.

Lemma 8.5. *Assume* $\mathfrak{Re}\,\mu > \gamma-1$, $\mathfrak{Re}\,\nu > \gamma-1$, $\mathfrak{Re}\,(\delta-\mu) > 1-\gamma$ *and* $\mathfrak{Re}\,(\delta-\nu) > 1-\gamma$. *Then* $\mathcal{H}_\delta^{\mu,\nu}$ *is an injective operator on* X.

Proof. Let $f \in X$, put $g(\xi) := (1+\xi)^\nu(1-\xi)^\mu f(\xi)$ for $\xi \in (-1,1)$, and fix $\varepsilon > 0$ small enough. Then $|f(\xi)| \lesssim (1-\xi^2)^{-\gamma-\varepsilon}$ for all $\xi \in (-1,1)$ by Remark 2.4. Hence

$$\int_{-1}^{1} |g(\xi)|\,dt \lesssim \int_{-1}^{1} (1+\xi)^{\mathfrak{Re}\,\nu-\gamma-\varepsilon}(1-\xi)^{\mathfrak{Re}\,\mu-\gamma-\varepsilon}\,dt < \infty,$$

that is, $g \in L^1(-1,1)$.

Now, assume furthermore $f \in \ker \mathcal{H}_\delta^{\mu,\nu}$, and let $K^\delta(n)$, $n \in \mathbb{N}_0$ be such that $(1-z)^{-\delta} = \sum_{n=0}^{\infty} K^\delta(n) z^n$, $z \in \mathbb{D}$. One has

$$(\mathfrak{H}_\delta^{\mu,\nu} f)(z) = \int_{-1}^{1} (1+\xi)^\nu (1-\xi)^\mu \frac{f(\xi)}{(1-z\xi)^\delta}\,d\xi = \int_{-1}^{1} \frac{g(\xi)}{(1-z\xi)^\delta}\,d\xi$$

$$= \int_{-1}^{1} g(\xi) \sum_{n=0}^{\infty} K^\delta(n)(z\xi)^n \, d\xi = \sum_{n=0}^{\infty} z^n K^\delta(n) \int_{-1}^{1} \xi^n g(\xi) \, d\xi = 0, \quad z \in \mathbb{D},$$

where have used Fubini's theorem since

$$\sum_{n=0}^{\infty} \int_{-1}^{1} \left| K^\delta(n) z^n \xi^n g(\xi) \, d\xi \right| \le \|g\|_{L^1(-1,1)} (1 - |z|)^\delta < \infty.$$

As a consequence, $K^\delta(n) \int_{-1}^{1} \xi^n g(\xi) \, d\xi = 0$, $n \in \mathbb{N}_0$, which implies $\int_{-1}^{1} \xi^n g(\xi) \, d\xi = 0$, $n \in \mathbb{N}_0$ (note that $\mathfrak{Re}\, \delta > 0$ by the hypotheses assumed and so $K^\delta(n) \neq 0$, $n \in \mathbb{N}_0$). In short, $g = 0$, thus $f = 0$ and our claim follows. $\square$

Theorem 8.6. *Assume* $\mathfrak{Re}\, \mu > \gamma - 1$, $\mathfrak{Re}\, \nu > \gamma - 1$, $\mathfrak{Re}\,(\delta - \mu) > 1 - \gamma$ *and* $\mathfrak{Re}\,(\delta - \nu) > 1 - \gamma$. *Then the spectrum, essential spectrum and point spectrum of* $\mathcal{H}_\delta^{\mu,\nu}$ *are*

$$\begin{aligned}
\sigma(\mathcal{H}_\delta^{\mu,\nu}) &= \{2^{\delta-1} B(z, \delta - z) \, : \, z \in [\mathfrak{Re}\, \nu - \gamma + 1, \mathfrak{Re}\,(\delta - \mu) + \gamma - 1]\} \cup \{0\}, \\
\sigma_{ess}(\mathcal{H}_\delta^{\mu,\nu}) &= \{2^{\delta-1} B(z, \delta - z) \, : \, \mathfrak{Re}\, z = \mathfrak{Re}\, \nu - \gamma + 1 \text{ or } \mathfrak{Re}\, z = \mathfrak{Re}\,(\delta - \mu) + \gamma - 1\} \\
&\quad \cup \{0\}, \\
\sigma_{point}(\mathcal{H}_\delta^{\mu,\nu}) & \\
= \{2^{\delta-1} B(z, \delta - z) &\, : \, z \in \mathbb{C} \text{ such that } \left[\xi \mapsto (1+\xi)^{z-\nu-1}(1-\xi)^{\mu-\delta-z+1}\right] \in X\}.
\end{aligned}$$

In particular,

$$\begin{aligned}
&\{2^{\delta-1} B(z, \delta - z) \, : \, \mathfrak{Re}\, \nu - \gamma + 1 < \mathfrak{Re}\, z < \mathfrak{Re}\,(\delta - \mu) + \gamma - 1\} \subseteq \sigma_{point}(\mathcal{H}_\delta^{\mu,\nu}), \\
&\quad \textit{if } \mathfrak{Re}\,(\mu + \nu - \delta) < 2(\gamma - 1),
\end{aligned}$$

and

$$\sigma_{point}(\mathcal{H}_\delta^{\mu,\nu}) = \emptyset, \qquad \textit{if } \mathfrak{Re}\,(\mu + \nu - \delta) > 2(\gamma - 1).$$

Proof. The proof runs along similar lines as Theorem 8.2.

For $\rho = \mathfrak{Re}\,(\nu + \delta - \mu)/2$, we have $\mathcal{H}_\delta^{\mu,\nu} = \int_{-\infty}^{\infty} e^{-\rho t} S_\omega(t) \, d\tilde{\mu}(t)$, where $d\tilde{\mu}(t) = 2^{\delta-1} e^{\rho t}(1 + e^t)^{-\delta} \, dt$ for $t \in \mathbb{R}$, see Proposition 8.4. On the other hand, it follows by Proposition 4.2 and Lemma 6.1 that, for all $c > |\mathfrak{Re}\,(\delta - \mu - \nu) + 2(\gamma - 1)|/2$, the infinitesimal generator $\Delta_\omega - \rho$ of $(e^{-\rho t} S_\omega(t))$ is bisectorial-like of angle $\pi/2$ and half-width c, see [23, Subsection 2.1.1].

Define $f \in \mathcal{O}(\mathbb{D})$ by

$$f(z) = (\mathcal{L}_b \tilde{\mu})(-z) = \int_{-\infty}^{\infty} e^{zt} d\tilde{\mu}(t) = 2^{\delta-1} \int_{-\infty}^{\infty} \frac{e^{(z+\rho)t}}{(1 + e^t)^\delta} \, dt$$

$$= 2^{\delta-1}B(z+\rho, \delta - z - \rho), \quad |\mathfrak{Re}\, z| < c.$$

Note that f can be analytically extended to a bisector $BS_{\theta,c}$ for any $\theta \in (0, \pi/2)$. We claim that there exists $K > 0$ for which $|f(z)| \lesssim e^{-K|z|}$ as $z \to \infty$ through $BS_{\theta,c}$. This is true if $\delta = 1$ since in this case $f(z) = \frac{\pi}{\sin \pi(z+\rho)}$ for all $z \in \mathbb{C} \setminus \{-\rho, -\rho-1, -\rho-2, ...; \rho-1, \rho-2, ...\}$, and $|\sin \pi(z+\rho)| \gtrsim e^{\pi \sin\theta |z|}$ as $z \to \infty$ through $BS_{\theta,c}$. If $\delta \neq 1$, note that

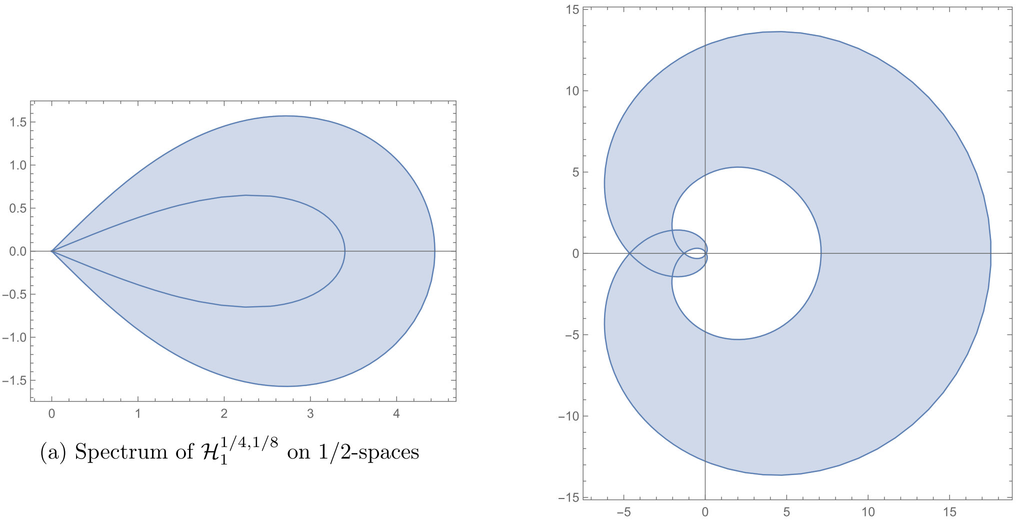
$$f(z) = B(z+\rho, \delta - z - \rho) = \frac{\Gamma(\delta - z - \rho)}{\Gamma(\delta-1)\Gamma(1-z-\rho)} \frac{\pi}{(\delta-1)\sin \pi(1-z-\rho)},$$

for all $z \in \mathbb{C} \setminus \{-\rho, -\rho-1, -\rho-2, ...; \rho-1, \rho-2, ...\}$. Thus, it follows by (8.1) that

$$f(z) = \frac{(-z-\rho)^{\delta-1}}{\Gamma(\delta-1)} \frac{\pi}{(\delta-1)\sin \pi(1-z-\rho)} (1 + O(|z+\rho|^{-1}))^{-1}, \quad z \in BS_{\theta,c},$$

obtaining the fore-mentioned inequality. Thus f is regular at ∞ with $f(\infty) = 0$, $f \in \mathcal{E}(\Delta_\omega - \rho)$ and the hypotheses of Theorem 1.4 are satisfied. As a consequence, $\widetilde{\sigma}(\mathcal{H}_\delta^{\mu,\nu}) = f(\widetilde{\sigma}(\Delta_\omega - \rho))$, $\widetilde{\sigma}_{ess}(\mathcal{H}_\delta^{\mu,\nu}) = f(\widetilde{\sigma}_{ess}(\Delta_\omega - \rho))$ and $f(\sigma_{point}(\Delta_\omega - \rho)) \subseteq \sigma_{point}(\mathcal{H}_\delta^{\mu,\nu}) \subseteq f(\sigma_{point}(\Delta_\omega - \rho)) \cup \{0\}$. The statement follows since $\mathcal{H}_\delta^{\mu,\nu}$ is injective by Lemma 8.5, and the different spectra of Δ_ω were given in Theorem 6.8. $\square$

We provide in Fig. 2 the spectral picture of two reduced Hilbert type operators acting on two different γ-Banach spaces.



(a) Spectrum of $\mathcal{H}_1^{1/4,1/8}$ on 1/2-spaces

(b) Spectrum of $\mathcal{H}_{10}^{8,3/2}$ on 1-spaces

Fig. 2. Spectral pictures for two reduced Hilbert type operators. The bold lines depict the essential spectrum.

References

[1] A. Aleman, O. Constantin, Spectra of integration operators on weighted Bergman spaces, J. Anal. Math. 109 (1) (2009) 199–231.
[2] A. Aleman, A. Persson, Resolvent estimates and decomposable extensions of generalized Cesàro operators, J. Funct. Anal. 258 (1) (2010) 67–98.
[3] S. Ballamoole, T.L. Miller, V.G. Miller, A class of integral operators on spaces of analytic functions, J. Math. Anal. Appl. 414 (1) (2014) 188–210.
[4] E. Berkson, R. Kaufman, H. Porta, Möbius transformations of the disc and one-parameter groups of isometries of H^p, Trans. Am. Math. Soc. 199 (1974) 223–239.
[5] E. Berkson, H. Porta, Semigroups of analytic functions and composition operators, Mich. Math. J. 25 (1) (1978) 101–115.
[6] O. Blasco, M. Contreras, S. Díaz-Madrigal, J. Martínez, M. Papadimitrakis, A. Siskakis, Semigroups of composition operators and integral operators in spaces of analytic functions, Ann. Acad. Sci. Fenn., Math. 38 (2013) 67–90.
[7] F. Bracci, M.D. Contreras, S. Díaz-Madrigal, Continuous Semigroups of Holomorphics Self-Maps of the Unit Disc, Springer, Cham, 2020.
[8] I. Chalendar, E. Gallardo-Gutiérrez, J. Partington, Weighted composition operators on the Dirichlet space: boundedness and spectral properties, Math. Ann. 363 (2015) 1265–1279.
[9] C. Cowen, Subnormality of the Cesàro operator and a semigroup of composition operators, Indiana Univ. Math. J. 33 (2) (1984) 305–318.
[10] C. Cowen, B. MacCluer, Composition Operators on Spaces of Analytic Functions, CRC Press, Boca Raton, 1995.
[11] M. Cwikel, J. McCarthy, T. Wolff, Interpolation between weighted Hardy spaces, Proc. Am. Math. Soc. 116 (2) (1992) 381–388.
[12] E. Diamantopoulos, A. Siskakis, Composition operators and the Hilbert matrix, Stud. Math. 140 (2) (2000) 191–198.
[13] E.P. Dolzhenko, G.T. Tumarkin, NN Luzin and the theory of boundary properties of analytic functions, Russ. Math. Surv. 40 (3) (1985) 79–95.
[14] N. Dunford, J.T. Schwartz, Linear Operators I: General Theory, vol. VII), John Wiley & Sons, New York, 1958.
[15] P. Duren, B. Romberg, A. Shields, Linear functionals on H^p spaces with $0 < p < 1$, J. Reine Angew. Math. 238 (1969) 32–60.
[16] D. Edmunds, W. Evans, Spectral Theory and Differential Operators, Oxford University Press, Oxford/New York, 1987.
[17] T. Eklund, M. Lindström, P. Mleczko, Spectral properties of weighted composition operators on the Bloch and Dirichlet spaces, Stud. Math. 232 (2016) 1–18.
[18] K.J. Engel, R. Nagel, One-Parameter Semigroups for Linear Evolution Equations, Graduate Texts in Mathematics, vol. 194, Springer, New York, 2000.
[19] T.M. Flett, The dual of an inequality of Hardy and Littlewood and some related inequalities, J. Math. Anal. Appl. 38 (1972) 746–765.
[20] F. Forelli, The isometries of H^p, Can. J. Math. 16 (1964) 721–728.
[21] J.E. Galé, P.J. Miana, L. Sánchez–Lajusticia, RKH spaces of Brownian type defined by Cesaro–Hardy operators, Anal. Math. Phys. 11 (3) (2021) 1–34.
[22] E. Gallardo-Gutiérrez, A. Siskakis, D. Yakubovich, Generators of C_0-semigroups of weighted composition operators, Isr. J. Math. 255 (1) (2021) 63–80.
[23] M. Haase, The Functional Calculus for Sectorial Operators, Oper. Theory Adv. Appl., vol. 169, Birkhäuser, Basel, 2006.
[24] H. Hedenmalm, B. Korenblum, K. Zhu, Theory of Bergman Spaces, vol. 199, Springer-Verlag, New York, 2000.
[25] E. Hille, R.S. Phillips, Functional Analysis and Semi-Groups, vol. 31, revised and expanded edition, American Mathematical Society, New York, 1957, p. 42.
[26] K. Hoffman, Banach Spaces of Analytic Functions, Prentice-Hall, Englewood Cliffs, 1962.
[27] O. Hyvärinen, M. Lindström, I. Nieminen, E. Saukko, Spectra of weighted composition operators with automorphic symbols, J. Funct. Anal. 265 (8) (2013) 1749–1777.
[28] W. König, Semicocycles and weighted composition semigroups on H^p, Mich. Math. J. 37 (1990) 469–476.
[29] C. Lizama, P.J. Miana, R. Ponce, L. Sánchez-Lajusticia, On the boundedness of generalized Cesàro operators on Sobolev spaces, J. Math. Anal. Appl. 419 (1) (2014) 373–394.

[30] P.J. Miana, J. Oliva-Maza, Integral operators on Sobolev–Lebesgue spaces, Banach J. Math. Anal. 15 (3) (2021) 1–30.

[31] J. Oliva-Maza, Spectral mapping theorems for essential spectra and regularized functional calculi, Proc. R. Soc. Edinb., Sect. A (2023) 1–23.

[32] J. Oliva-Maza, M. Warma, Introducing and solving generalized Black-Scholes PDEs through the use of functional calculus, J. Evol. Equ. 23 (1) (2022) 1–40.

[33] A.M. Persson, On the spectrum of the Cesàro operator on spaces of analytic functions, J. Math. Anal. Appl. 340 (2) (2008) 1180–1203.

[34] W. Rudin, Functional Analysis, 2 ed., McGraw-Hill, Singapore, 1991.

[35] A. Siskakis, Weighted composition semigroups on Hardy spaces, Linear Algebra Appl. 84 (1986) 359–371.

[36] A. Siskakis, Composition semigroups and the Cesàro operator on H^p, J. Lond. Math. Soc. (2) 36 (1) (1987) 153–164.

[37] A. Siskakis, The Koebe semigroup and a class of averaging operators on $H^p(D)$, Trans. Am. Math. Soc. 339 (1) (1993) 337–350.

[38] A. Siskakis, Semigroups of composition operators on spaces of analytic functions, a review, Contemp. Math. 213 (1998) 229–252.

[39] K. Stempak, Cesàro averaging operators, Proc. R. Soc. Edinb., Sect. A 124 (1) (1994) 121–126.

[40] F. Tricomi, A. Erdélyi, et al., The asymptotic expansion of a ratio of gamma functions, Pac. J. Math. 1 (1) (1951) 133–142.

[41] J. Xiao, Cesàro operators on Hardy, BMOA and Bloch spaces, Arch. Math. 68 (1997) 398–406.

[42] K. Zhu, Bloch type spaces of analytic functions, Rocky Mt. J. Math. 23 (3) (1993) 1143–1177.