ACCURATE BIDIAGONAL DECOMPOSITIONS OF CAUCHY-VANDERMONDE MATRICES OF ANY RANK

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Abstract. We present a new decomposition of a Cauchy-Vandermonde matrix as a product of bidiagonal matrices which, unlike its existing bidiagonal decompositions, is now valid for a matrix of any rank. The new decompositions are insusceptible to the phenomenon known as subtractive cancellation in floating point arithmetic and are thus computable to high relative accuracy. In turn, other accurate matrix computations are also possible with these matrices, such as eigenvalue computation amongst others.

1. Introduction

An $n \times n$ Cauchy-Vandermonde (CV) matrix of index l, and parameters $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_l$ is defined as:

$$A = \begin{bmatrix} \frac{1}{x_1+y_1} & \cdots & \frac{1}{x_1+y_l} & 1 & x_1 & \cdots & x_1^{n-l-1} \\ \frac{1}{x_2+y_1} & \cdots & \frac{1}{x_2+y_l} & 1 & x_2 & \cdots & x_2^{n-l-1} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{x_n+y_1} & \cdots & \frac{1}{x_n+y_l} & 1 & x_n & \cdots & x_n^{n-l-1} \end{bmatrix}. \tag{1}$$

At one extreme, $l = 0$, the CV matrix is a Vandermonde matrix and at the other, $l = n$, it is a Cauchy matrix.

When the parameters of a CV matrix are ordered as

$$0 < x_1 \le x_2 \le \cdots \le x_n, \quad 0 < y_1 \le y_2 \le \cdots \le y_l, \tag{2}$$

the CV matrix is *totally nonnegative* (TN) [1], i.e., all of its minors are nonnegative [2]. As such, the decompositions of CV matrices as a product of nonnegative bidiagonal factors become of interest in both the study of the theoretical properties of these matrices [3, 4, 5, 6] as well as accurate and efficient numerical computations with them [7, 8, 9].

The CV matrices appear prominently in computing rational interpolates with prescribed poles [10]. This type of interpolation has applications in control systems [11]. The CV matrices also appear in connection with the numerical solution of singular integral equations [4, 12], as well as in numerical quadrature [13], rational models of regression [14], and E-optimal design [15].

The main issue we address in this paper is that when some of the parameters x_i or y_i coincide, the CV matrix is no longer nonsingular and the existing bidiagonal decomposition (from [1]) no longer exists. To illustrate this, the 3×3 Cauchy matrix (which is a CV matrix of index $l = 3$) with nodes x_1, x_2, x_3 and y_1, y_2, y_3 has a bidiagonal decomposition (see [7])

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$$
\begin{aligned}
&\begin{bmatrix} \frac{1}{x_1+y_1} & \frac{1}{x_1+y_2} & \frac{1}{x_1+y_3} \\ \frac{1}{x_2+y_1} & \frac{1}{x_2+y_2} & \frac{1}{x_2+y_3} \\ \frac{1}{x_3+y_1} & \frac{1}{x_3+y_2} & \frac{1}{x_3+y_3} \end{bmatrix} \\
&\qquad = \begin{bmatrix} 1 & & \\ 0 & 1 & \\ & \frac{x_2+y_1}{x_3+y_1} & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ \frac{x_1+y_1}{x_2+y_1} & 1 & \\ & \frac{(x_1+y_2)(x_2+y_1)(x_2-x_3)}{(x_3+y_1)(x_3+y_2)(x_1-x_2)} & 1 \end{bmatrix} \\
&\qquad \times \begin{bmatrix} \frac{1}{x_1+y_1} & & \\ & \frac{(x_1-x_2)(y_1-y_2)}{(x_1+y_2)(x_2+y_1)(x_2+y_2)} & \\ & & \frac{(x_1-x_3)(x_2-x_3)(y_1-y_3)(y_2-y_3)}{(x_1+y_3)(x_3+y_1)(x_2+y_3)(x_3+y_2)(x_3+y_3)} \end{bmatrix} \\
&\qquad \times \begin{bmatrix} 1 & \frac{x_1+y_1}{x_1+y_2} & \\ & 1 & \frac{(x_1+y_2)(x_2+y_1)(y_2-y_3)}{(x_1+y_3)(x_2+y_3)(y_1-y_2)} \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \\ & 1 & \frac{x_1+y_2}{x_1+y_3} \\ & & 1 \end{bmatrix},
\end{aligned} \tag{3}
$$

which does not exist when $x_1 = x_2$ or $y_1 = y_2$ because $x_1 - x_2$ and $y_1 - y_2$ appear in denominators. This is unfortunate, because the Cauchy matrix itself is very well defined even then.

The main contribution of this paper is to refactor the existing bidiagonal decomposition of any CV matrix as a product of nonnegative bidiagonals, a decomposition that is valid for any complex values of the parameters for which the CV matrix is defined, not just for those that make it nonsingular and TN. Additionally, this paper extends the results of [16] on singularity-free bidiagonal decompositions of CV matrices with one multiple pole (which are of Vandermonde type) to any CV matrix of any index $l \geq 1$ (which are *not* of Vandermonde type).

For example, using the results of this paper, the above Cauchy matrix can instead be decomposed as

$$
\begin{aligned}
&\begin{bmatrix} \frac{1}{x_1+y_1} & \frac{1}{x_1+y_2} & \frac{1}{x_1+y_3} \\ \frac{1}{x_2+y_1} & \frac{1}{x_2+y_2} & \frac{1}{x_2+y_3} \\ \frac{1}{x_3+y_1} & \frac{1}{x_3+y_2} & \frac{1}{x_3+y_3} \end{bmatrix} \\
&\qquad = \begin{bmatrix} 1 & & \\ 0 & 1 & \\ & \frac{x_2+y_1}{x_3+y_1} & x_3 - x_2 \end{bmatrix} \begin{bmatrix} 1 & & \\ \frac{x_1+y_1}{x_2+y_1} & x_2 - x_1 & \\ & \frac{(x_1+y_2)(x_2+y_1)}{(x_3+y_1)(x_3+y_2)} & x_3 - x_1 \end{bmatrix} \\
&\qquad \times \begin{bmatrix} \frac{1}{x_1+y_1} & & \\ & \frac{1}{(x_1+y_2)(x_2+y_1)(x_2+y_2)} & \\ & & \frac{1}{(x_1+y_3)(x_3+y_1)(x_2+y_3)(x_3+y_2)(x_3+y_3)} \end{bmatrix} \\
&\qquad \times \begin{bmatrix} 1 & \frac{x_1+y_1}{x_1+y_2} & \\ & y_2 - y_1 & \frac{(x_1+y_2)(x_2+y_1)}{(x_1+y_3)(x_2+y_3)} \\ & & y_3 - y_1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \\ & 1 & \frac{x_1+y_2}{x_1+y_3} \\ & & y_3 - y_2 \end{bmatrix}.
\end{aligned}
$$

This is a simpler decomposition in that it requires fewer arithmetic operations to compute than the existing one (3). Additionally, the new decomposition is valid for any complex values of the nodes $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_l$ for which the Cauchy–Vandermonde matrix itself is defined.

One benefit of the existing bidiagonal decompositions of CV matrices from [1] is that it allows for numerical computations to be performed with nonsingular TN CV matrices efficiently and to high relative accuracy [1, 7]. The new bidiagonal decompositions remain insusceptible to subtractive cancellation, and thus all of the entries of these decompositions can be computed to high relative accuracy when the matrix is TN. By "high relative accuracy" we mean that the sign and most leading significant digits of each entry are computed correctly (see section 5). Matrix computations can now be performed with TN CV matrices of any rank accurately and efficiently using the methods of [9] – see section 6 for a numerical example.

The efficiency and high relative accuracy is particularly relevant, for example, in eigenvalue computations since the corresponding matrices are unsymmetric. The error bounds for the eigenvalues computed by the conventional algorithms (such as the ones in LAPACK [17, 18]) imply that none of the eigenvalues are guaranteed to be accurate, although the largest ones typically are – see the example in section 6. In contrast, the results of this paper now allow for all eigenvalues to be efficiently computed to high relative accuracy and, in particular, the zero eigenvalues are computed exactly.

The paper is organized as follows. In section 2 we review the existing bidiagonal decompositions of nonsingular TN matrices from [1]. We derive the new singularity-free bidiagonal decomposition of a CV matrix in section 3. For a Cauchy matrix (i.e., a CV matrix of index $l = n$), the results can be derived directly from [16] as we explain in section 4. We discuss accuracy issues in section 5 and present numerical experiments in section 6.

2. The ordinary bidiagonal decomposition of a TN CV matrix

Even though the new decompositions that we derive for the class of CV matrices is valid without any requirement for total nonnegativity, our approach is based on the existing bidiagonal decompositions for nonsingular TN CV matrices, which we review here.

Any nonsingular $n \times n$ TN matrix A can be factored as a product of nonnegative bidiagonal matrices as [6]:

$$A = L^{(1)} L^{(2)} \cdots L^{(n-1)} D U^{(n-1)} U^{(n-2)} \cdots U^{(1)}, \tag{4}$$

where $L^{(i)}$ are $n \times n$ nonnegative and unit lower bidiagonal, D is $n \times n$ nonnegative and diagonal, and $U^{(i)}$ are $n \times n$ nonnegative and unit upper bidiagonal. For the nontrivial entries $l_j^{(k)}$ and $u_j^{(k)}$ of the factors $L^{(i)}$ and $U^{(i)}$, respectively, we have $l_i^{(k)} = u_i^{(k)} = 0$ for $i < n - k$.

Following the terminology of [16], we refer to the decomposition (4) as the *ordinary bidiagonal decomposition* of A to contrast it with the new *singularity-free bidiagonal decomposition* (10) we derive below.

The decomposition (4) occurs naturally in the process of complete Neville elimination when adjacent rows and columns are used for elimination. We refer the reader to [6] and [5] for details on the connection with Neville elimination.

We have exactly n^2 nontrivial entries which parameterize the above decomposition and these are arranged in an $n \times n$ array $M = \mathcal{B}D(A)$, where [7, sec. 4]

$$m_{ij} = L_{i,i-1}^{(n-i+j)}, i > j, \tag{5}$$

$$m_{ij} = U_{j-1,j}^{(n-j+i)}, i < j, \tag{6}$$

$$m_{ii} = D_{ii}. \tag{7}$$

For $i \neq j$, the m_{ij} are the multipliers of the complete Neville elimination with which the (i, j) entry of A is eliminated and m_{ii} are the diagonal entries of D. This particular arrangement of the nontrivial parameters makes it easy to input these parameters to a software as an $n \times n$ matrix M.

The explicit formulas for the entries (5)–(7) of the ordinary bidiagonal decomposition (4) of a nonsingular TN CV matrix are [1, Prop. 4.1]:

$$m_{ii} = \begin{cases} \frac{1}{x_i+y_i} \prod_{k=1}^{i-1} \frac{1}{(x_i+y_k)(x_k+y_i)} \underline{\underline{\prod_{k=1}^{i-1} (x_i - x_k)(y_i - y_k)}}, & 1 \le i \le l; \\ \prod_{k=1}^{l} \frac{1}{x_i+y_k} \underline{\underline{\prod_{k=1}^{i-1} (x_i - x_k)}}, & l < i \le n; \end{cases}$$

for $1 \le i \le n$,

$$
m_{ij} = \begin{cases} \frac{x_{i-j}+y_j}{x_i+y_j} \prod_{k=1}^{j-1} \frac{x_{i-1}+y_k}{x_i+y_k} \underline{\prod_{k=i-j}^{i-2} \frac{x_i-x_{k+1}}{x_{i-1}-x_k}}, & j \le l; \\[2ex] \prod_{r=1}^{l} \frac{x_{i-1}+y_r}{x_i+y_r} \underline{\prod_{k=i-j}^{i-2} \frac{x_i-x_{k+1}}{x_{i-1}-x_k}}, & j > l; \end{cases} \tag{8}
$$

for $1 \le j < i \le n$, and

$$
m_{ji} = \begin{cases} \frac{x_j+y_{i-j}}{x_j+y_{i-1}} \prod_{k=1}^{j} \frac{x_k+y_{i-1}}{x_k+y_i} \underline{\prod_{r=i-j}^{i-2} \frac{y_i-y_{r+1}}{y_{i-1}-y_r}}, & 1 \le j < i \le l; \\[2ex] \frac{x_j+y_{l-j+1}}{x_j+y_l} \prod_{k=1}^{j} (x_k+y_l) \underline{\prod_{r=l-j+1}^{l-1} \frac{1}{y_l-y_r}}, & i = l+1;\ 1 < j \le l; \\[2ex] x_j, & \begin{array}{l} l+2 \le i \le n, \\ 1 \le j \le i-l-1; \end{array} \\[2ex] x_j + y_{i-j}, & \begin{array}{l} l+2 \le i \le n, \\ i-l \le j \le i-1; \\ i = l+1,\ j = 1; \end{array} \end{cases} \tag{9}
$$

for $1 \le j < i \le n$.

In [1] an algorithm with high relative accuracy and of complexity $\mathcal{O}(n^2)$ for computing the bidiagonal decomposition for nonsingular TN CV matrices was also included.

The trouble with the above expressions when some of the x_i's or some of the y_i's coincide is obvious—the terms in the expressions for m_{ij} and m_{ji} above underlined with a single line have singularities. Of course, the CV matrix itself doesn't have singularities and the reason is that the double underlined terms in the expressions for m_{ii} cancel all problematic denominators.

In our approach below we work the double underlined terms into the expressions for m_{ii} into the bidiagonal factors making all singularities disappear. This results in bidiagonal factors that no longer have unit diagonals for which leads us to a new definition of a *singularity-free bidiagonal decomposition* in the next section.

3. The singularity-free bidiagonal decomposition

Following the approach and notation of [16], we obtain the *singularity-free bidiagonal decomposition* of a matrix into a product of bidiagonal factors where the bidiagonal factors ($L^{(i)}$ and $U^{(i)}$ in (4)) no longer need to have unit diagonals.

The *singularity-free bidiagonal decomposition* of an $n \times n$ matrix A is

$$
A = L_1 L_2 \cdots L_{n-1} D U_{n-1} U_{n-2} \cdots U_1, \tag{10}
$$

where the matrix D is nonnegative and diagonal, the factors L_i and U_i are nonnegative lower and upper bidiagonal, and have the same nonzero patterns as $L^{(i)}$ and $U^{(i)}$ in (4), respectively, $i = 1, 2, \ldots, n-1$. Namely, $(L_k)_{i+1,i} = (U_k)_{i,i+1} = 0$ for $i < n-k$ (see theorem 2.1 in [7]).

Following [9], the nontrivial entries of $\mathcal{SBD}(A)$ are stored in two matrices: B, which is $n \times n$ and C, which is $(n+1) \times (n+1)$:

$$
[B, C] = \mathcal{SBD}(A).
$$

As with $\mathcal{BD}(A)$, the matrix B stores the nontrivial offdiagonal entries of L_i and U_i as well as the diagonal entries of D, exactly as in (6):

$$\begin{aligned} b_{ij} &= (L_{n-i+j})_{i,i-1}, i > j, \\ b_{ij} &= (U_{n-j+i})_{j-1,j}, i < j, \\ b_{ii} &= D_{ii}. \end{aligned}$$

The matrix C stores the diagonal entries of L_i and U_i as

$$c_{ij} = \begin{cases} (L_{n-i+j})_{i-1,i-1}, & i > j, \\ (U_{n-j+i})_{j-1,j-1}, & i < j. \end{cases}$$

In this arrangement, c_{ij}, $i > j$, is the diagonal entry in L_{n-i+j} immediately above b_{ij} and similarly for $i < j$ and U_{n-j+i}. The entries $c_{ii}, i = 1, 2, \ldots, n+1$ as well as the entries $c_{1,n+1}$ and $c_{n+1,1}$ are unused. This is the same construction as the one given in formula (9) in [9], except that, just as in our previous paper [16], we now allow for the (n, n) entry in L_i and U_i to be any nonnegative number and not necessarily equal 1 (see also section 7 of [16]).

The new singularity-free bidiagonal decomposition of (10) is not unique, but this is inconsequential for the purposes of accurate computations: any accurate decomposition of a TN matrix as a product of nonnegative bidiagonal matrices is an equally good input [7, 9].

To derive the singularity-free bidiagonal decomposition of a CV matrix A, we start with its ordinary bidiagonal decomposition (4). Then, we factor the diagonal factor D into three diagonal factors $D = GEF$, where

$$G_{ii} = \prod_{r=1}^{i-1}(x_i - x_r),$$

$$E_{ii} = \begin{cases} \dfrac{1}{\prod_{r=1}^{i-1}(x_i + y_r)\prod_{k=1}^{i}(x_k + y_i)}, & 1 \le i \le l; \\[2ex] \dfrac{1}{\prod_{r=1}^{l}(x_i + y_r)}, & l+1 \le i \le n, \end{cases} \tag{11}$$

$$F_{ii} = \begin{cases} \prod_{r=1}^{i-1}(y_i - y_r), & 1 \le i \le l; \\ 1, & l+1 \le i \le n, \end{cases} \tag{12}$$

and thus (4) becomes

$$A = \left(L^{(1)}L^{(2)}\cdots L^{(n-1)}G\right)\cdot E \cdot \left(FU^{(n-1)}U^{(n-2)}\cdots U^{(1)}\right). \tag{13}$$

The lower triangular matrix $L^{(1)}L^{(2)}\cdots L^{(n-1)}G$ has a singularity-free bidiagonal decomposition [16, Thm. 4.1]

$$L^{(1)}L^{(2)}\cdots L^{(n-1)}G = L_1 L_2 \cdots L_{n-1}, \tag{14}$$

where

$$L_k = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & s_{n-k+1,1} & x_{n-k+1} - x_{n-k} & & \\ & & & s_{n-k+2,2} & x_{n-k+2} - x_{n-k} & \\ & & & & \ddots & \ddots \\ & & & & s_{nk} & x_n - x_{n-k} \end{bmatrix}. \tag{15}$$

and

$$s_{ij} = \begin{cases} \dfrac{x_{i-j}+y_j}{x_i+y_j} \cdot \displaystyle\prod_{r=1}^{j-1} \dfrac{x_{i-1}+y_r}{x_i+y_r}, & j \le l; \\ \\ \displaystyle\prod_{r=1}^{l} \dfrac{x_{i-1}+y_r}{x_i+y_r}, & j > l. \end{cases}$$

for $i > j$. The quantities s_{ij} above are obtained from their m_{ij} counterparts in (8) by omitting the underlined factors.

The diagonal factor E has no singularities when some of the parameters coincide, thus the only remaining task is to rework the factors of the upper triangular matrix $FU^{(n-1)}U^{(n-2)}\cdots U^{(1)}$. We formulate it in the following result.

Theorem 1. *Let the matrices $U^{(1)}, U^{(2)}, \ldots, U^{(n-1)}$ be defined as in (4) for a TN CV matrix A of index l with nodes $x_1, x_2, \ldots, x_n$, and $y_1, y_2, \ldots, y_l$, and let the matrix F be defined as in (12). Then we have*

$$FU^{(n-1)}U^{(n-2)}\cdots U^{(1)} = U_{n-1}U_{n-2}\cdots U_1, \tag{16}$$

where U_k, $k = 1, 2, \ldots, n-1$, are upper bidiagonal matrices such that

$$\begin{aligned} (U_k)_{tt} &= \begin{cases} y_t - y_{n-k}, & n-k+1 \le t \le l, \\ 1, & \text{otherwise}; \end{cases} \\ (U_k)_{t-1,t} &= \begin{cases} s_{t-n+k,t}, & n-k+1 \le t \le n, \\ 0, & \text{otherwise}, \end{cases} \end{aligned} \tag{17}$$

with the s_{ji} $(1 \le j < i \le n)$ defined as

$$s_{ji} = \begin{cases} \frac{x_j+y_{i-j}}{x_j+y_{i-1}} \prod_{k=1}^{j} \frac{x_k+y_{i-1}}{x_k+y_i}, & 1 \le j < i \le l; \\ \\ \frac{x_j+y_{l-j+1}}{x_j+y_l} \prod_{k=1}^{j} (x_k+y_l), & i = l+1;\ 1 < j \le l; \\ \\ x_j, & \begin{array}{l} l+2 \le i \le n, \\ 1 \le j \le i-l-1; \end{array} \\ \\ x_j + y_{i-j}, & \begin{array}{l} l+2 \le i \le n, \\ i-l \le j \le i-1; \\ i = l+1,\ j = 1. \end{array} \end{cases} \tag{18}$$

Namely,

$$U_k = \begin{bmatrix} 1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & 1 & s_{1,n-k+1} & & & & & \\ & & & y_{n-k+1}-y_{n-k} & \ddots & & & & \\ & & & & \ddots & s_{l,n-k+l} & & & \\ & & & & & y_l - y_{n-k} & \ddots & & \\ & & & & & & 1 & \ddots & \\ & & & & & & & \ddots & s_{kn} \\ & & & & & & & & 1 \end{bmatrix}.$$

Proof. We define $n \times n$ diagonal matrices $D^{(k)}$, $k = 1, 2, \ldots, n$ as:

$$(D^{(k)})_{tt} = \begin{cases} \prod_{r=n-k+1}^{t-1} (y_t - y_r), & n-k+2 \le t \le l, \\ 1, & \text{otherwise}, \end{cases} \tag{19}$$

i.e.,

$$D^{(k)} = \begin{bmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & y_{n-k+2} - y_{n-k+1} & & & & \\ & & & & \ddots & & & \\ & & & & & \prod_{r=n-k+1}^{l-1} (y_l - y_r) & & \\ & & & & & & 1 & & \\ & & & & & & & \ddots & \\ & & & & & & & & 1 \end{bmatrix}$$

and thus, in particular, $D^{(n)} = F$ and

$$D^{(k)} = I \tag{20}$$

for $k = 1, 2, \ldots, n - l + 1$.

With the paramters s_{ji} defined as in (18), we have from (6) and (9) that the offdiagonal entries of the matrix $U^{(k)}$ are

$$U_{t-1,t}^{(k)} = m_{t-n+k,t} = \begin{cases} s_{t-n+k,t} \prod_{r=n-k}^{t-2} \frac{y_t - y_{r+1}}{y_{t-1} - y_r}, & n-k+1 \le t \le l; \\ \\ s_{t-n+k,t} \prod_{r=n-k}^{l-1} \frac{1}{y_l - y_r}, & t = l+1; \\ \\ s_{t-n+k,t}, & l+1 < t \le n. \end{cases} \tag{21}$$

We will prove that

$$D^{(k+1)} U^{(k)} = U_k D^{(k)} \tag{22}$$

for $k = n-1, n-2, \ldots, n-l+1$. This identity will allow us to "work" the matrix $D^{(n)}$ into the product of bidiagonals in (24) below and remove all singularities when some of the parameters defining A coincide.

Since $D^{(k+1)}$ and $D^{(k)}$ are both diagonal and $U^{(k)}$ and U_k are both upper bidiagonal, we have bidiagonals on each side of (22). Thus it suffices to show that the corresponding diagonal and offdiagonal entries are the same.

Since $U^{(k)}$ has a unit diagonal, the diagonal (t,t) entry, $t = 1, 2, \ldots, n$, in the product on the left side of (22) is $D_{tt}^{(k+1)}$. On the right, the corresponding entry is $(U_k)_{tt} D_{tt}^{(k)}$ and these are equal for $t = n-k+1, n-k+2, \ldots, l$, because

$$D_{tt}^{(k+1)} = \prod_{r=n-k}^{t-1} (y_t - y_r) = (y_t - y_{n-k}) \prod_{r=n-k+1}^{t-1} (y_t - y_r) = (U_k)_{tt} D_{tt}^{(k)},$$

since $(U_k)_{tt} = y_t - y_{n-k}$ by (17).

The rest of the diagonal entries on each side of (22) all equal 1, since $D_{tt}^{(k+1)} = D_{tt}^{(k)} = (U_k)_{tt} = 1$ for $t < n-k+1$ and $t > l$.

Thus all diagonal entries on both sides of (22) are equal.

Next, we establish that the offdiagonal, $(t-1,t)$ entries on each side of (22) are equal, i.e., that

$$D^{(k+1)}_{t-1,t-1}U^{(k)}_{t-1,t} = (U_k)_{t-1,t}D^{(k)}_{tt} \tag{23}$$

for $t = 2, 3, \ldots, n$.

Since $U^{(k)}_{t-1,t} = (U_k)_{t-1,t} = 0$ for $t = 2, 3, \ldots, n-k$, we only need to prove (23) for $t \geq n-k+1$.

For $n-k+1 \leq t \leq l$, from (17) and (19),

$$(U_k)_{t-1,t} = s_{t-n+k,t} \quad \text{and} \quad D^{(k)}_{tt} = \prod_{r=n-k+1}^{t-1}(y_t - y_r),$$

which combined with (21) gives

$$\begin{aligned}
(U_k)_{t-1,t}D^{(k)}_{tt} &= s_{t-n+k,t}\prod_{r=n-k+1}^{t-1}(y_t - y_r) \\
&= s_{t-n+k,t}\prod_{r=n-k}^{t-2}(y_t - y_{r+1}) \\
&= s_{t-n+k,t}\prod_{r=n-k}^{t-2}\frac{y_t - y_{r+1}}{y_{t-1} - y_r}\prod_{r=n-k}^{t-2}(y_{t-1} - y_r) \\
&= U^{(k)}_{t-1,t}\cdot D^{(k+1)}_{t-1,t-1},
\end{aligned}$$

so (23) holds.

For $t = l+1$ we have

$$\begin{aligned}
U^{(k)}_{l,l+1}D^{(k+1)}_{ll} &= s_{l+1-n+k,l+1}\prod_{r=n-k}^{l-1}\frac{1}{y_l - y_r}\prod_{r=n-k}^{l-1}(y_l - y_r) \\
&= s_{l+1-n+k,l+1} \\
&= s_{l+1-n+k,l+1}\cdot D^{(k)}_{l+1,l+1},
\end{aligned}$$

since $D^{(k)}_{l+1,l+1} = 1$ and thus (23) holds again.

For $t = l+2, l+3, \ldots, n$ we have $D^{(k+1)}_{t-1,t-1} = D^{(k)}_{tt} = 1$ and $U^{(k)}_{t-1,t} = (U_k)_{t-1,t} = s_{t-n+k,t}$ and (23) holds in that case as well.

With this, (23) and, in turn, (22) are fully established.

Finally, we use (22) to work the factor $D^{(n)} = F$ through the product $D^{(n)}U^{(n-1)}U^{(n-2)}\cdots U^{(1)}$ as follows

$$\begin{aligned}
\underline{D^{(n)}U^{(n-1)}}U^{(n-2)}\cdots U^{(1)} &= U_{n-1}\underline{D^{(n-1)}U^{(n-2)}}\cdots U^{(1)} \\
&= U_{n-1}U_{n-2}\underline{D^{(n-2)}U^{(n-3)}}\cdots U^{(1)} \\
&= \cdots \\
&= U_{n-1}\cdots U_{n-l+2}\underline{D^{(n-l+2)}U^{(n-l+1)}}\cdots U^{(1)} \\
&= U_{n-1}\cdots U_{n-l+1}D^{(n-l+1)}U^{(n-l)}\cdots U^{(1)}, \\
&\quad \text{and since } D^{(n-l+1)} = I, \\
&= U_{n-1}U_{n-2}\cdots U_1, \qquad (24)
\end{aligned}$$

where the factors that change on each step are underlined.

Our proof is complete. □

We have thus established our main result.

Theorem 2. *The singularity-free bidiagonal decomposition* (10) *of an* $n \times n$ *CV matrix* A *is*

$$A = L_1 L_2 \cdots L_{n-1} E U_{n-1} U_{n-2} \cdots U_1, \tag{25}$$

for any values of the nodes for which the CV matrix exists, where the matrices A, E, L_k, *and* U_k *are defined in* (1), (11), (15), *and* (17), *respectively.*

Proof. For a nonsingular $n \times n$ TN CV matrix (i.e., when the nodes are ordered and strictly increasing $0 < x_1 < x_2 < \cdots < x_n$, $0 < y_1 < y_2 < \cdots < y_l$ [1]) the result follows directly from [1] and (11), (13), (14), and (16).

For general values of the parameters (when A is not a nonsingular TN matrix), both sides of (25) are defined for any complex values of the parameters x_i and y_j so long as all denominators, $x_i + y_j$, $i = 1, 2, \ldots, n, j = 1, 2, \ldots, l$, in (1) are nonzero.

If we fix the values of all parameters defining the matrix A, except for one, say x_i, the entry a_{ij} of A is a meromorphic function of that parameter x_i on $\mathbb{C}$. When the matrix A is TN and nonsingular, this (i, j)th entry of A equals the (i, j)th entry on the right hand side of (25) on an open interval containing the parameter x_i (e.g., $x_1 \in (0, x_2)$ for $i = 1$, $x_i \in (x_{i-1}, x_{i+1})$ for $2 \le i \le n-1$, and $x_n \in (x_{n-1}, \infty)$).

The Identity Theorem [19, Thm. 3.2.6] then implies that (25) holds for any complex values of the nodes $x_i, i = 1, 2, \ldots, n$, for which the CV matrix A is defined.

By repeating the same argument for any of the parameters y_j in place of the x_i, the proof is complete. □

Algorithm 1 contains the pseudocode[1] to compute with high relative accuracy the singularity-free bidiagonal decomposition of a CV matrix corresponding to Theorem 2. Taking into account that the algorithm mentioned in Section 2 has a computational cost of $\mathcal{O}(n^2)$, this algorithm also has the same computational complexity.

4. Vandermonde and Cauchy matrices

The $n \times n$ Vandermonde

$$V = \left[x_i^{j-1} \right]_{i,j=1}^{n}$$

and $n \times n$ Cauchy matrices

$$C = \left[\frac{1}{x_i + y_j} \right]_{i,j=1}^{n}$$

are the $l = 0$ and $l = n$ cases of an $n \times n$ CV matrix. Both are particular cases of Theorem 2 with the additional convention that indices less than 1 or exceeding n are ignored and empty products equal one.

The singularity-free bidiagonal decomposition of a Vandermonde matrix was obtained in [16]. Since the transpose of a Cauchy matrix with nodes $\{x_i\}_{i=1}^n$ and $\{y_j\}_{j=1}^n$ is a Cauchy matrix with nodes $\{y_j\}_{j=1}^n$ and $\{x_i\}_{i=1}^n$, the singularity-free bidiagonal decomposition of a Cauchy matrix can be obtained directly from (14) without the need of Theorem 2.

5. Numerical accuracy

In the standard "$1 + \delta$" model of floating point arithmetic [21], to which the IEEE 754 double precision arithmetic [22] conforms, the result of any floating point calculation is assumed to satisfy

$$\mathrm{fl}(a \odot b) = (a \odot b)(1 + \delta), \tag{26}$$

where $\odot \in \{+, -, \times, /\}$, $|\delta| \le \varepsilon$, and ε is tiny and is called *machine precision.*

[1]This algorithm is implemented in MATLAB as the routine `STNBDCauchyVandermonde` in the package STNPack [20].

Algorithm 1

Input: The vector x containing the nodes $\{x_i\}_{1\leq i\leq n}$ and the vector y containing the poles $\{y_i\}_{1\leq i\leq l}$.
Output: $[B, C] = \mathcal{SBD}(A)$

function $[B, C]$=`STNBDCauchyVandermonde`$(y,\, x)$

n=length(x);
l=length(y);
B=ones(n);
C=ones(n);

for j=1:l
  for i=j+1:n
    $B_{i,j} = \dfrac{x_{i-j} + y_j}{x_i + y_j}$;
    for r=1:j-1
      $B_{i,j} = B_{i,j}\dfrac{x_{i-1} + y_r}{x_i + y_r}$;
    end
  end
end

for j=2:n
  for i=j+1:n
    $C_{i,j} = x_{i-1} - x_{i-j}$;
  end
end

j=l+1;
for i=j+1:n
  for r=1:j-1
    $B_{i,j} = B_{i,j}\dfrac{x_{i-1} + y_r}{x_i + y_r}$;
  end
end

for j=l+2:n
  for i=j+1:n
    for r=1:l
      $B_{i,j} = B_{i,j}\dfrac{x_{i-1} + y_r}{x_i + y_r}$;
    end
  end
end

for i=1:l
  for r=1:i-1
    $B_{i,i} = \dfrac{B_{i,i}}{x_i + y_r}$;
  end
  for k=1:i
    $B_{i,i} = \dfrac{B_{i,i}}{x_k + y_i}$;
  end
end

for i=l+1:n
  for r=1:l
    $B_{i,i} = \dfrac{B_{i,i}}{x_i + y_r}$;
  end
end

for i=2:l
  for j=1:i-1
    $B_{j,i} = \dfrac{x_j + y_{i-j}}{x_j + y_{i-1}}$;
    for k=1:j
      $B_{j,i} = B_{j,i} \dfrac{x_k + y_{i-1}}{x_k + y_i}$;
    end
  end
end

for i=2:l
  for j=2:i-1
    $C_{j,i} = -y_{i-j} + y_{i-1}$;
  end
end

i=l+1;
for j=1:i-1
  $B_{j,i} = \dfrac{x_j + y_{i-j}}{x_j + y_{i-1}}$;
  for k=1:j
    $B_{j,i} = B_{j,i} \cdot (x_k + y_{i-1})$;
  end
end

for j=2:i-1
  $C_{j,i} = -y_{i-j} + y_{i-1}$;
end

for j=l+2:n
  for i=1:j-l-1
    $B_{i,j} = x_i$;
  end
  for i=j-l:j-1
    $B_{i,j} = x_i + y_{j-i}$;
  end
end

for i=2:n
  $B_{n,i} = B_{n,i} \cdot \prod_{k=n-i+1}^{n-1} (x_n - x_k)$;
end

For a computed quantity, $\hat{x}$ to *have high relative accuracy*, it means that it satisfies an error bound with its true counterpart, x

$$|\hat{x} - x| \leq \theta |x|,$$

where θ is a modest multiple of ε. In other words, the sign and most significant digits of x must be correct. In particular, if $x = 0$, it must be computed exactly.

The above model directly implies that the accuracy in numerical calculations is lost due to one phenomenon only, known as subtractive cancellation [23]. It occurs when a subtraction of previously rounded off quantities results in the loss of significant digits. Multiplication, division, and addition of same-sign quantities preserve the relative accuracy. The subtractions $x_i - x_j$ as well as $y_i - y_j$ are always computed to high relative accuracy, since the parameters x_i and y_j are initial data and are thus assumed to be exact: (26) tells us the result of those subtractions is computed to high relative accuracy.

Detailed error analysis for the ordinary bidiagonal decompositions of a nonsingular TN CV matrix has already been performed in [1].

The new singularity-free bidiagonal decomposition inherits the same componentwise error bounds and is thus also computable to high relative accuracy: all offdiagonal entries in the matrices L_i and U_i are factors in the corresponding entries in the matrices $L^{(i)}$ and $U^{(i)}$ and thus satisfy the same error bounds as do the elements in the diagonal factor E. The entries on the diagonals of L_i and U_i are either 1 or differences of initial data and have a relative error bounded by ε per (26).

6. Numerical experiments

We performed extensive numerical tests to verify the correctness of the new singularity-free bidiagonal decomposition of CV matrices we derived in this paper. We report on two those here.

For our first numerical experiment we selected a 20×20 CV matrix A of index 6 with nodes

$$\{x_i\}_{i=1}^{14} = \{1, 2, 2, 2, 2, 2, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\} \tag{27}$$

and

$$\{y_j\}_{j=1}^{6} = \{1, 2, 2, 2, 3, 4\}, \tag{28}$$

where the node 2 is repeated in both x and y arrays 6 and 3 times, respectively. This choice of the parameters makes the CV matrix TN (see (2)). It is of rank 15 since the 15×15 submatrix consisting of rows $1, 2$, and 8 through 20 and columns 6 through 20 is a Vandermonde matrix with distinct nodes and thus nonsingular, and rows 3 through 7 are the same as row 2.

We implemented the formulas for the singularity-free bidiagonal decomposition of this CV matrix (Algorithm 1) into the routine `STNBDCauchyVandermonde` in the package STNTool [20]. We then formed the bidiagonal decomposition of the CV matrix with parameters (29) and (28). We then computed the eigenvalues using the algorithm `STNEigenValues` from the same package STNTool, which has complexity of $O(n^3)$ for an $n \times n$ matrix (see also [9] for details on the eigenvalue algorithm) using as input the matrices B and C that are output by `STNBDCauchyVandermonde`. We also computed the eigenvalues using a conventional eigenvalue algorithm (`eig` in MATLAB [24]) in both double precision IEEE floating point arithmetic and, for verification, formed the matrix and then computed its eigenvalues in 40 decimal digit arithmetic. As expected, only the largest eigenvalues are computed accurately by `eig` in double precision arithmetic and all eigenvalues smaller than about 10^{-16} times the largest eigenvalue are lost to roundoff.

In contrast, all nonzero eigenvalues computed by `STNEigenValues` were correct to at least 14 correct decimal digits when compared with those computed in 40 decimal digit precision using Mathematica. Their magnitude ranging from about 10^{15} to 10^{-5} meant 40 digit arithmetic was sufficient to get the nonzero eigenvalues accurate to 16 digits. No amount of extra precision will allow us to reliably compute the zero eigenvalues using conventional algorithms. The reason we know the algorithm computed the correct number of zero eigenvalues (5) is because we know the rank of the matrix from theory as described earlier in this section[2] – see Figure 1.

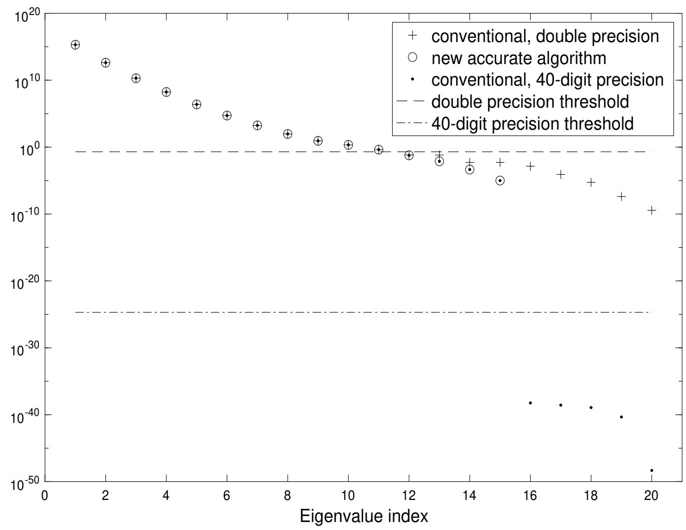

Figure 1. The eigenvalues of the 20×20 Cauchy-Vandermonde matrix A as computed by various algorithms.

For our second numerical experiment, we chose a nonsingular TN Cauchy-Vandermonde matrix of index 0 with (distinct) nodes

$$\{x_i\}_{i=1}^{9} = \{1/2, 1, 5/2, 3, 10/3, 4, 11/2, 17/3, 6\} \tag{29}$$

and no poles. This matrix is also TN Vandemonde, which allowed us to compare our algorithms with those from our previous work [16] as well as the algorithms of [7]. We chose the nodes carefully so that the matrix is very ill conditioned (condition number 4.7×10^{11}), but not too ill conditioned, so that the conventional matrix algorithms (e.g., LAPACK [17] as implemented in MATLAB [24]) would compute even the smallest eigenvalues with some relative accuracy.

We compared the accuracy of all computed eigenvalues with those computed by Maple in 50 decimal digit floating point arithmetic, which (because the condition number is 4.7×10^{11}) was sufficient to ensure they accurate to at least 16 correct decimal digits in each.

In the second column of Table 1, we observe, as expected, that our new algorithms compute eigenvalues that agree with the ones computed by Maple to at least 15 digits, i.e., to high relative accuracy. We computed the bidiagonal decomposition using Algorithm 1 as implemented by `STNBDCauchyVandermonde` in the package STNTool [20], followed by a call to `STNEigenvalues` in the same package.

In the third column, we observe, again, as expected, that our new algorithm computed the eigenvalues just as accurately as the algorithms of our previous work [16] using instead the routine `STNVandermonde` from STNTool [20] for the bidiagonal decomposition.

The fourth column demonstrates the importance of deriving the accurate formulas for the bidiagonal decomposition of a TN Cauchy-Vandermonde matrix and that those are better than just using Neville elimination to obtain those bidiagonal decompositions (which is susceptible to subtractive cancellation). By computing the bidiagonal decomposition using Neville elimination using the routine `TNBD` from the package

[2]Since the graph is log-scale, the zero eigenvalues are not depicted and the eigenvalues computed negative or complex by the conventional algorithms are displayed by their absolute values.

TNTool [20], instead of the results of this paper, we caused irreparable damage – the eigenvalues of the thus decomposed matrix, even though computed to high relative accuracy using the routine `TNEigenvalues`, differ to various degrees from the true ones. This example underscores the importance of computing the bidiagonal decompositions accurately in the first place, which is the main contribution of this paper.

λ_i	`STNBDCauchyVandermonde` `+ STNEigenValues`	`STNBDVandermonde` `+ STNEigenValues`	`TNBD` `+ TNEigenValues`	`eig`
1.88193e+06	0	0	2.79604e-14	1.23719e-16
1.38376e+04	6.57263e-16	6.57263e-16	7.74256e-14	1.70888e-15
3.11538e+02	0	0	9.17778e-14	9.28726e-14
1.83012e+01	1.94124e-16	1.94124e-16	2.00724e-13	2.23437e-13
2.43185e+00	1.82614e-16	1.82614e-16	2.71912e-13	6.13932e-11
1.11330e+00	1.99447e-16	1.99447e-16	2.46317e-13	6.28627e-11
2.19017e-01	7.60369e-16	7.60369e-16	3.84240e-13	1.45762e-10
3.11176e-02	3.34484e-16	3.34484e-16	7.34416e-13	1.91914e-10
4.90302e-04	2.21129e-16	2.21129e-16	2.61574e-12	1.21019e-07

TABLE 1. The eigenvalues of a TN Vandermonde matrix and the relative error in each as computed by various algorithms.

Finally, in the last column, we report the results of MATLAB's command `eig` where, expectedly, only the largest eigenvalues are computed accurately then progressively losing relative accuracy in the smaller ones.

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Conflict of interest statement

The authors have no conflicts of interest to disclose.

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