Anal Math
https://doi.org/10.1007/s10476-024-00047-4

ON THE HYPERBOLIC GROUP AND SUBORDINATED INTEGRALS AS OPERATORS ON SEQUENCE BANACH SPACES

L. ABADIAS[1], J. E. GALÉ[1], P. J. MIANA[1] and J. OLIVA-MAZA[1,2,*]

[1]Departamento de Matemáticas, Instituto Universitario de Matemáticas y Aplicaciones, Universidad de Zaragoza, 50009 Zaragoza, Spain
e-mails: labadias@unizar.es, gale@unizar.es, pjmiana@unizar.es

[2]Institute of Mathematics, Polish Academy of Sciences, 87-100 Toruń, Poland
e-mail: joliva@impan.pl

(Received September 19, 2023; accepted March 17, 2024)

Abstract. We show that the composition hyperbolic group in the unit disc, once transferred to act on sequence spaces, is bounded on ℓ^p if and only if $p = 2$. We introduce some integral operators subordinated to that group which are natural generalizations of classical operators on sequences. For the description of such operators, we use some combinatorial identities which look interesting in their own.

Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc in the complex plane $\mathbb{C}$ and let $\mathcal{O}(\mathbb{D})$ denote the space of holomorphic functions on $\mathbb{D}$. In [1], the spectral study of integral operators defined by

$$(\mathcal{J}_\delta^{\mu,\nu}\mathfrak{f})(z) := \frac{1}{(1-z)^{\mu+\delta}(1+z)^{\nu+\delta}} \int_z^1 (1-\xi)^\mu (1+\xi)^\nu (\xi - z)^{\delta-1} \mathfrak{f}(\xi)\, d\xi, \ \ z \in \mathbb{D},$$

and

$$(\mathcal{H}_\delta^{\mu,\nu}\mathfrak{f})(z) := \frac{1}{(1-z)^{\mu-\delta+1}(1+z)^{\nu-\delta+1}} \int_{-1}^1 \frac{(1-\xi)^\mu(1+\xi)^\nu}{(1-z\xi)^\delta} \mathfrak{f}(\xi)\, d\xi, \quad z \in \mathbb{D},$$

* Corresponding author.

The authors have been partially supported by Project PID2019-10579GB-I00 and PID2022-137294NB-I00 of the Ministry of Science and Project E26-17R, D.G. Aragón, Universidad de Zaragoza, Spain. The first author has been also supported by Project JIUZ-2019 CIE-01 for Young Researchers, Fundación Ibercaja and Universidad de Zaragoza, Spain.

Key words and phrases: automorphism, hyperbolic group, composition operator, spectrum, subordination, integral operator, generalized Siskakis operator, generalized reduced Hilbert-matrix operator.

Mathematics Subject Classification: 47B38, 40G05, 47A10, 33B15.

Published online: 15 September 2024

for suitable $\mathfrak{f} \in \mathcal{O}(\mathbb{D})$, $\mu, \nu, \delta \in \mathbb{R}$, has been approached on the basis of a detailed analysis of the spectra of weighted hyperbolic composition groups on $\mathbb{D}$ (and their infinitesimal generators). Recall, the (canonical) hyperbolic group $(\varphi_t)_{t\in\mathbb{R}}$ of self-analytic mappings on $\mathbb{D}$ is given by

$$\varphi_t(z) = \frac{(e^t+1)z + e^t - 1}{(e^t-1)z + e^t + 1}, \quad z \in \mathbb{D},\ t \in \mathbb{R}.$$

One can build weighted composition operators $S_\omega(t)$ acting continuously on Banach subspaces X of $\mathcal{O}(\mathbb{D})$, such as Hardy spaces, Bergman spaces, Dirichlet spaces and others, by putting

$$S_\omega(t)(\mathfrak{f}) := \frac{\omega \circ \varphi_t}{\omega}(\mathfrak{f} \circ \varphi_t), \quad \mathfrak{f} \in X,\ t \in \mathbb{R},$$

for appropriate weights ω so that the family $((\omega\circ\varphi_t)\omega^{-1})_{t\in\mathbb{R}}$ becomes a cocycle for $(\varphi_t)_{t\in\mathbb{R}}$.

The integrals defining $\mathcal{J}_\delta^{\mu,\nu}\mathfrak{f}$ and $\mathcal{H}_\delta^{\mu,\nu}\mathfrak{f}$ can be represented as Bochner-convergent vector-valued integrals subordinated to groups $(S_\omega(t))_{t\in\mathbb{R}}$ in the following way. For $\alpha, \beta \in \mathbb{R}$, put $\omega_{\alpha,\beta}(z) := (1-z)^\alpha(1+z)^\beta$, $z \in \mathbb{D}$. Then, for suitable $\mu, \nu, \delta \in \mathbb{R}$,

1) $\mathcal{J}_\delta^{\mu,\nu}\mathfrak{f} = \int_0^\infty \phi_\delta(t) S_{\omega_{\mu+1,\nu+\delta}}(t)\mathfrak{f}\, dt$, $\mathfrak{f} \in X$, with $\phi_\delta(t) := 2^{-\delta}(1-e^{-t})^{\delta-1}$, $t > 0$.

2) $\mathcal{H}_\delta^{\mu,\nu}\mathfrak{f} = \int_{-\infty}^\infty \psi_\delta(t) S_{\omega_{\mu-\delta+1,\nu+1}}(t)\mathfrak{f}\, dt$, $\mathfrak{f}\in X$, $z\in\mathbb{D}$, with $\psi_\delta(t) := \frac{2^{\delta-1}}{(1+e^t)^\delta}$, $t \in \mathbb{R}$.

Through the above representations, one can transfer information from the spectra of $S_{\omega_{\mu+1,\nu+\delta}}(t)$ and $S_{\omega_{\mu-\delta+1,\nu+1}}(t)$ to the spectra of $\mathcal{J}_\delta^{\mu,\nu}$ and $\mathcal{H}_\delta^{\mu,\nu}$, respectively; see [1] for details.

It sounds sensible to investigate if the above facts also hold in a setting of sequence Banach spaces and for operator groups defined by transference of $(S_\omega(t))_{t\in\mathbb{R}}$ to such a setting. Thus, using the standard isometry between the usual Hardy space $H^2(\mathbb{D})$ and the Hilbert space ℓ^2 of square sumable sequences, one defines operator groups $(T_\omega(t))_{t\in\mathbb{R}}$ on ℓ^2 obtained as isometric copies of $(S_{\omega_{\alpha,\beta}}(t))_{t\in\mathbb{R}}$. Then one obtains automatically the bounded integral operators on ℓ^2 given by

$$\mathfrak{J}_\delta^{\mu,\nu} f := \int_0^\infty \phi_\delta(t) T_{\omega_{\mu+1,\nu+\delta}}(t) f\, dt\,, \quad \mathfrak{H}_\delta^{\mu,\nu} f := \int_0^\infty \psi_\mu(t) T_{\omega_{\mu-\delta+1,\nu+1}}(t) f\, dt,$$

$f \in \ell^2$, which in particular enjoy the same spectral picture as the one described in [1] for $\mathcal{J}_\delta^{\mu,\nu}$ and $\mathcal{H}_\delta^{\mu,\nu}$.

However, it turns out that $T_{\omega_{\alpha,\beta}}(t)$, for $t \in \mathbb{R}\setminus\{0\}$ and $\alpha, \beta \in \mathbb{R}$, is not a bounded operator on ℓ^p for every $p \neq 2$. This is proven in Section 2 below, see Theorem 2.4. Thus the question on the boundedness of operators $\mathfrak{J}_\delta^{\mu,\nu}$,

$\mathfrak{H}_\delta^{\mu,\nu}$ on ℓ^p, $p \neq 2$, has not an obvious answer. Then we focus the paper on the Hilbertian case, looking for presenting the above operators directly acting on sequences, in order to get a precise idea of the difficulties which arise in the study of boundedness in the non-Hilbertian case. In this way, our discussion involves combinatorial identities and special functions. Section 3 is devoted to prove such identities. Section 4 gives the integral operators in terms of sequences, for which in particular we use estimates of the hyperbolic group and subordinate integrals on Hardy spaces shown in the first section.

1. Estimates on Hardy spaces

For $1 \le p < \infty$, let $H^p(\mathbb{D})$ be the Hardy space on $\mathbb{D}$ formed by all functions $\mathfrak{f}$ in $\mathcal{O}(\mathbb{D})$ such that

$$\|\mathfrak{f}\|_p := \sup_{0<r<1} \left(\int_0^{2\pi} |\mathfrak{f}(re^{i\theta})|^p \, \frac{d\theta}{2\pi} \right)^{1/p} < \infty.$$

Let $\alpha, \beta \in \mathbb{R}$ and put $\omega_{\alpha,\beta}(z) := (1-z)^\alpha (1+z)^\beta$, for $z \in \mathbb{D}$. Define the family $(S_{\alpha,\beta}(t))_{t\in\mathbb{R}}$ of weighted composition operators on $\mathcal{O}(\mathbb{D})$ given, for $t \in \mathbb{R}$, $\mathfrak{f} \in \mathcal{O}(\mathbb{D})$ and $z \in \mathbb{D}$, by

$$\begin{aligned}
[S_{\alpha,\beta}(t)\mathfrak{f}](z) &:= \frac{(\omega_{\alpha,\beta} \circ \varphi_t)(z)}{\omega_{\alpha,\beta}(z)} (\mathfrak{f} \circ \varphi_t)(z) \\
&= e^{\beta t} \left(\frac{2}{(e^t-1)z + e^t + 1} \right)^{\alpha+\beta} \mathfrak{f}\left(\frac{(e^t+1)z + e^t - 1}{(e^t-1)z + e^t + 1} \right) \\
&= \frac{(1-\tanh(t/2))^\alpha (1+\tanh(t/2))^\beta}{(1+z\tanh(t/2))^{\alpha+\beta}} \mathfrak{f}\left(\frac{z+\tanh(t/2)}{1+z\tanh(t/2)} \right).
\end{aligned}$$

This family $(S_{\alpha,\beta}(t))_{t\in\mathbb{R}}$ is a C_0-group of bounded operators on $H^p(\mathbb{D})$ (and on quite a number of other Banach spaces X continuously contained in $\mathcal{O}(\mathbb{D})$) for suitable values of α and β, see [1]. Note that $S_{\alpha,\beta}(t) = e^{\beta t} S_{\alpha+\beta,0}(t)$ for all $\alpha, \beta \in \mathbb{R}$ and $t \in \mathbb{R}$. Thus in the sequel we will deal with the group

$$S_\alpha(t) := S_{\alpha,0}(t), \quad t \in \mathbb{R},$$

for simplicity. Here we give the exact estimate of the norm of operators $S_\alpha(t)$, $t \in \mathbb{R}$, on spaces $H^p(\mathbb{D})$ by transferring functions on $\mathbb{D}$ to functions on $\mathbb{C}^+ := \{z \in \mathbb{C} : \mathfrak{Re}\, z > 0\}$.

Let $H^p(\mathbb{C}^+)$ be the Hardy space of all holomorphic functions F on $\mathbb{C}^+$ such that

$$\|F\|_p := \sup_{x>0} \left(\int_{\mathbb{R}} |F(x+iy)|^p \, dy \right)^{1/p} < \infty, \quad 1 \le p < \infty.$$

Let $V\colon H^p(\mathbb{C}^+) \to H^p(\mathbb{D})$ denote the isometric isomorphism given by

$$(VF)(z) = 2^{1/p}(1-z)^{-2/p} F\Big(\frac{1+z}{1-z}\Big), \quad z \in \mathbb{D},\ F \in H^p(\mathbb{C}^+),$$

whose inverse $V^{-1}\colon H^p(\mathbb{D}) \to H^p(\mathbb{C}^+)$ is

$$(V^{-1}f)(w) = 2^{1/p}(1+w)^{-2/p} f\Big(\frac{w-1}{w+1}\Big), \quad w \in \mathbb{C}^+,\ \mathfrak{f} \in H^p(\mathbb{D});$$

see [9, pp. 130–131]. For $\alpha, t \in \mathbb{R}$, define $\widetilde{S}_\alpha(t) := V^{-1}S_\alpha(t)V$. A straightforward computation gives us

$$(1.1) \qquad (\widetilde{S}_\alpha(t)F)(w) = \Big(\frac{1+w}{1+e^t w}\Big)^{\alpha-2/p} F(e^t w), \quad w \in \mathbb{C}^+,\ F \in H^p(\mathbb{C}^+).$$

Proposition 1.1. *Let $1 \le p < \infty$, $\alpha \in \mathbb{R}$ and let $(S_\alpha(t))_{t\in\mathbb{R}}$ be defined on $H^p(\mathbb{D})$ by*

$$S_\alpha(t)\mathfrak{f} = \frac{\omega_{\alpha,0} \circ \varphi_t}{\omega_{\alpha,0}} (\mathfrak{f} \circ \varphi_t), \quad t \in \mathbb{R}.$$

Then $(S_\alpha(t))_{t\in\mathbb{R}}$ is a C_0-group of bounded operators on $H^p(\mathbb{D})$ with

$$\|S_\alpha(t)\|_p = e^{\max\{-t/p,(-\alpha+1/p)t\}}, \quad t \in \mathbb{R}.$$

Proof. Let $w \in \mathbb{C}^+$ and $t \in \mathbb{R}$. It is readily seen that

$$e^{-t} \le \left|\frac{1+w}{1+e^t w}\right| \le 1, \text{ if } t \ge 0, \text{ and } 1 \le \left|\frac{1+w}{1+e^t w}\right| \le e^{-t}, \text{ if } t \le 0.$$

Hence

$$(1.2) \qquad \min\{1, e^{-t}\} \le \left|\frac{1+w}{1+e^t w}\right| \le \max\{1, e^{-t}\}, \quad t \in \mathbb{R},\ w \in \mathbb{C}^+.$$

Let $\widetilde{S}_\alpha(t)$ be as prior to the proposition. By (1.1) and (1.2),

$$\big|(\widetilde{S}_\alpha(t)F)(w)\big| \le K_t |F(e^t w)|, \quad F \in H^p(\mathbb{C}^+),\ t \in \mathbb{R},$$

where $K_t := \max\{1, e^{t(-\alpha+2/p)}\}$. Hence,

$$\|\widetilde{S}_\alpha(t)F\|_p \le K_t \sup_{0<x<\infty} \left\{\int_{-\infty}^{\infty} |F(e^t x + ie^t y)|^p dy\right\}^{1/p}$$

$$= K_t e^{-t/p} \sup_{0<x<\infty} \left\{\int_{-\infty}^{\infty} |F(x+iy)|^p dy\right\}^{1/p} = K_t e^{-t/p}\|F\|_p,$$

and so, through the isometry V, we have $\|S_\alpha(t)\|_p \le e^{\max\{-t/p, t(-\alpha+1/p)\}}$ for $t \in \mathbb{R}$.

On the other hand, it has been shown in [1] that the spectrum of $S_\alpha(t)$, $t \in \mathbb{R}$, is

$$\sigma(S_\alpha(t)) = \left\{ z \in \mathbb{C} : e^{\min\{-t/p, t(-\alpha+1/p)\}} \le |z| \le e^{\max\{-t/p, t(-\alpha+1/p)\}} \right\},$$

so that the spectral radius $r(S_\alpha(t))$ of $S_\alpha(t)$ is $r(S_\alpha(t)) = e^{\max\{-t/p, t(-\alpha+1/p)\}}$. Hence one gets $e^{\max\{-t/p, t(-\alpha+1/p)\}} \le \|S_\alpha(t)\|_p$. All in all, $\|S_\alpha(t)\|_p = e^{\max\{-t/p, t(-\alpha+1/p)\}}$ as claimed.

That $(S_\alpha(t))_{t\in\mathbb{R}}$ is a C_0-group of (bounded) operators on $H^p(\mathbb{D})$ is proved in [13]. □

REMARK 1.2. Asymptotic estimates of $\|S_\omega(t)\|$, $t \in \mathbb{R}$, for weights ω fairly more general than the former $\omega_{\alpha.\beta}$, are given for so called (Banach) γ-spaces in [1], but this is not necessary here. The class of γ-spaces contains the Hardy spaces $H^p(\mathbb{D})$, for $\gamma = 1/p$.

Next, we give upper and lower estimates of norms of integral operators on $H^p(\mathbb{D})$ subordinated to $(S_\alpha(t))_{t\in\mathbb{R}}$. For $1 \le p < \infty$, $t \in \mathbb{R}$, let $\Omega_{\alpha,p}$ denote the vertical strip given by

$$\Omega_{\alpha,p} := \left\{ \lambda \in \mathbb{C} : \min\{-1/p, -\alpha+(1/p)\} \le \Re\lambda \le \max\{-1/p, -\alpha+(1/p)\} \right\}$$

and put

$$M_{\alpha,p}(t) = \max\{-t/p, t(-\alpha+1/p)\}.$$

Let $\phi\colon \mathbb{R} \to \mathbb{C}$ be a measurable function such that $\int_{-\infty}^{\infty} |\phi(t)| e^{M_{\alpha,p}(t)}\, dt < \infty$. Set $\Theta(\phi)\mathfrak{f}$ to denote the Bochner convergent integral defined by

$$\Theta(\phi)\mathfrak{f} := \int_{-\infty}^{\infty} \phi(t) S_\alpha(t)\mathfrak{f}\, dt, \quad \mathfrak{f} \in H^p(\mathbb{D}).$$

PROPOSITION 1.3. *For α, p and $\phi\colon \mathbb{R} \to \mathbb{C}$ as above we have*

$$\sup_{\lambda\in\Omega_{\alpha,p}} \left| \int_{-\infty}^{\infty} \phi(t) e^{\lambda t}\, dt \right| \le \|\Theta(\phi)\|_p \le \int_{-\infty}^{\infty} |\phi(t)| e^{M_{\alpha,p}(t)}\, dt.$$

Moreover, if ϕ is nonnegative then

$$\|\Theta(\phi)\|_p \ge \max_{\lambda\in\{-1/p, 1/p-\alpha\}} \int_{-\infty}^{\infty} \phi(t) e^{\lambda t}\, dt.$$

PROOF. The upper estimate holds by the definition of $\Theta(\phi)$ and Proposition 1.1. For the lower estimate, set $\widetilde{\phi}(\lambda) := \int_{-\infty}^{\infty} \phi(t) e^{\lambda t} dt$, which is absolutely convergent for all $\lambda \in \Omega_{\alpha,p}$. The subset $\Omega_{\alpha,p}$ is in fact the spectrum

$\sigma(\Delta_{\alpha,p})$ of the infinitesimal generator $\Delta_{\alpha,p}$ on $H^p(\mathbb{D})$, see [1, Theorem 6.8]. Assume the spectral inclusion $\widetilde{\phi}(\sigma(\Delta_{\alpha,p})) \subseteq \sigma(\Theta(\phi))$ holds. In this case, $\|\Theta(\phi)\|_{H^p\to H^p} \geq r(\Theta(\phi)) \geq \sup_{\lambda\in\Omega_{\alpha,p}} |\widetilde{\phi}(\lambda)|$ from which we get the lower estimate.

Let us prove such a spectral inclusion. Assume first $\alpha = 2/p$. Then $(e^{t/p}S_\alpha(t))_{t\in\mathbb{R}}$ is a uniformly bounded C_0-group by Proposition 1.1, and the claim follows by the spectral mapping theorem for uniformly bounded C_0-groups, see [12, Theorem 3.1] for a slightly more general result.

Assume now $\alpha < 2/p$. By [1, Theorem 6.8], all the points in the interior of $\Omega_{\alpha,p}$ lie in the point spectrum of $\Delta_{\alpha,p}$, and have the function $f_\lambda(z) = \frac{(1+z)^\lambda}{(1-z)^{\lambda+\alpha}}$, $z \in \mathbb{D}$, as eigenvector. By [5, Corollary IV.3.8], it follows that f_λ is an eigenvector of $S_\alpha(t)$ associated to the eigenvalue $e^{\lambda t}$ for all $t \in \mathbb{R}$. As a consequence, one has

$$\Theta(\phi) f_\lambda = \int_{-\infty}^{\infty} \phi(t) S_\alpha(t) f_\lambda \, dt = f_\lambda \int_{-\infty}^{\infty} \phi(t) e^{\lambda t} \, dt = \widetilde{\phi}(\lambda) f_\lambda\,, \ \lambda \in \operatorname{Int}(\Omega_{\alpha,p}),$$

so $\widetilde{\phi}(\operatorname{Int}(\Omega_{\alpha,p})) \subseteq \sigma_{\text{point}}(\Theta(\phi))$. Since $\sigma(\Theta(\phi))$ is a closed subset of $\mathbb{C}$ and $\widetilde{\phi}$ is continuous on $\Omega_{\alpha,p} = \sigma(\Delta_{\alpha,p})$, one gets $\widetilde{\phi}(\Omega_{\alpha,p}) \subseteq \overline{\widetilde{\phi}(\operatorname{Int}(\Omega_{\alpha,p}))} \subseteq \sigma(\Theta(\phi))$, as we wanted to prove.

Finally, assume $\alpha > 2/p$. Again by [1, Theorem 6.8], all the points λ of the interior of $\Omega_{\alpha,p}$ lie in the residual spectrum of $\Delta_{\alpha,p}$ and satisfy that the range space $R_\lambda := \operatorname{Ran}(\lambda - \Delta_{\alpha,p})$ is closed and its codimension is equal to 1. Also, $\operatorname{Ran}(e^{\lambda t} - S_\alpha(t)) \subseteq R_\lambda$ for all $t \in \mathbb{R}$, see for instance [5, Eq. (IV.3.14)]. Fix $\lambda \in \operatorname{Int}(\Omega_{\alpha,p})$. Since R_λ is closed, we have

$$(\widetilde{\phi}(\lambda) - \Theta(\phi)) f = \int_{-\infty}^{\infty} \phi(t) \big(e^{\lambda t} - S_\alpha(t)\big) f \, dt \in R_\lambda, \quad f \in H^p(\mathbb{D}).$$

Thus, we conclude $\operatorname{Ran}(\widetilde{\phi}(\lambda) - \Theta(\phi)) \subseteq R_\lambda$, so $\widetilde{\phi}(\lambda) \in \sigma(\Theta(\phi))$. Reasoning as in the end of the proof for the case $\alpha < 2/p$, we have $\widetilde{\phi}(\Omega_{\alpha,p}) \subseteq \overline{\widetilde{\phi}(\operatorname{Int}(\Omega_{\alpha,p}))} \subseteq \sigma(\Theta(\phi))$, as we wanted to prove.

Now, for nonnegative ϕ, it is clear that

$$\sup_{\lambda\in\Omega_{\alpha,p}} \left| \int_{-\infty}^{\infty} \phi(t) e^{\lambda t} \, dt \right| = \sup_{\lambda\in\mathbb{R}\cap\Omega_{\alpha,p}} \int_{-\infty}^{\infty} \phi(t) e^{\lambda t} \, dt.$$

Then, since the latter integral is convex in the variable λ, it reaches its maximum value at the extreme points of the interval $\mathbb{R} \cap \Omega_{\alpha,p}$, namely $-1/p$ or $1/p - \alpha$. Then our claim follows from what we have already proven. $\square$

Let $\operatorname{supp}(\phi)$ denote the support of ϕ in $\mathbb{R}$.

COROLLARY 1.4. *Let* α, p *and* $\phi\colon \mathbb{R} \to [0,\infty)$ *be as above. Suppose that* $\operatorname{supp}(\phi) \subseteq (-\infty, 0]$ *or* $\operatorname{supp}(\phi) \subseteq [0,\infty)$. *Then*

$$\|\Theta(\phi)\|_p = \int_{-\infty}^{\infty} \phi(t)\ e^{M_{\alpha,p}(t)}\,dt.$$

PROOF. Under the assumption on ϕ it is clear that

$$\max_{\lambda\in\{-1/p,1/p-\alpha\}} \int_{-\infty}^{\infty} \phi(t)e^{\lambda t}\,dt = \int_{-\infty}^{\infty} \phi(t)\ e^{M_{\alpha,p}(t)}\,dt,$$

and the result follows from Proposition 1.3. □

REMARK 1.5. Using similar type of arguments as above, one can show that analogous bounds to the ones given in Proposition 1.3 and Corollary 1.4 also hold for Bergman spaces and little Korenblum spaces.

2. Non-boundedness of the hyperbolic group

Since $H^2(\mathbb{D})$ is isometric to the space ℓ^2 of complex square-summable sequences, one obtains automatically the above estimates for the transferred operators $T_\alpha(t)$ of $S_\alpha(t)$, $t \in \mathbb{R}$, on ℓ^2. The case $p \neq 2$ is very different. In fact, we show that $T_\alpha(t)$ is not bounded on spaces ℓ^p, $p \neq 2$, in this section.

Set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For $\delta \in \mathbb{R}$ and $1 \le p < \infty$, let ℓ^p_δ denote the Banach space of sequences $(f(n))_{n\ge 0}$ such that

$$\|f\|_{p,\delta} := \left(\sum_{n=0}^{\infty} |f(n)|^p (n+1)^{2\delta}\right)^{1/p} < \infty.$$

Put $\ell^p := \ell^p_0$ and $\|f\|_p := \|f\|_{p,0}$. Let Φ denote the correspondence between ℓ^p_δ and $\mathcal{O}(\mathbb{D})$ given by

$$(\Phi f)(z) := \sum_{n=0}^{\infty} f(n) z^n, \quad z \in \mathbb{D},\ f \in \ell^p_\delta.$$

It is well known that Φ defines an isometry from the Hilbert space ℓ^2_δ onto:

– The hilbertian Hardy space $H^2(\mathbb{D})$ when $\delta = 0$,

– the hilbertian Bergman space $A^2(\mathbb{D})$ when $\delta = -1/2$,

– the classical Dirichlet space $\mathcal{D}$ when $\delta = 1/2$.

The family $(S_\alpha(t))_{t\in\mathbb{R}}$ is a C_0-group of bounded operators on $H^2(\mathbb{D})$, $A^2(\mathbb{D})$ and $\mathcal{D}$, see [1]. Hence, the isometric copy $(T_\alpha(t))_{t\in\mathbb{R}}$ of $(S_\alpha(t))_{t\in\mathbb{R}}$ through the isometry Φ, that is,

$$T_\alpha(t) := \Phi^{-1} S_\alpha(t) \Phi, \quad t \in \mathbb{R},$$

is a C_0-group of bounded operators on ℓ^2_δ for $\delta = 0, -1/2, 1/2$ respectively (and on the interpolated Banach spaces in between). Here we restrict ourselves to consider the group $T_\alpha(t))_{t\in\mathbb{R}}$ acting on ℓ^2. It will be enough to show the key points of the matter. Then automatically we have the following consequence of Proposition 1.1.

Corollary 2.1. *The family $(T_\alpha(t))_{t\in\mathbb{R}}$ is a C_0-group of bounded operators on ℓ^2 with*

$$\|T_\alpha(t)\|_2 = e^{\max\{-t/2,(-\alpha+1/2)t\}}, \quad t \in \mathbb{R}.$$

Now, we express the group $(T_\alpha(t))_{t\in\mathbb{R}}$ acting on sequences.

Proposition 2.2. *Let $\alpha \in \mathbb{R}$, $f \in \ell^2$ and $t \in \mathbb{R}$. We have*

$$(T_\alpha(t)f)(n) = \sum_{k=0}^{\infty} a^\alpha_{n,k}(t) f(k), \quad n \in \mathbb{N}_0, \tag{2.1}$$

where

$$a^\alpha_{n,k}(t) := \frac{2^\alpha (\tanh(t/2))^{n+k}}{(e^t+1)^\alpha} \sum_{j=0}^{\min\{n,k\}} \binom{-k-\alpha}{n-j}\binom{k}{j} (\tanh(t/2))^{-2j}$$

$$= 2^\alpha \sum_{j=0}^{\min\{n,k\}} \binom{-k-\alpha}{n-j}\binom{k}{j} \frac{(e^t-1)^{n+k-2j}}{(e^t+1)^{n+k+\alpha-2j}}, \quad n,k \in \mathbb{N}_0.$$

Proof. Let $z \in \mathbb{D}$, $t \in \mathbb{R}$ and $f \in \ell^2$. Then

$$\sum_{n=0}^{\infty} (T_\alpha(t)f)(n) z^n = (\Phi T_\alpha(t) f)(z)$$

with

$$(\Phi T_\alpha(t) f)(z) = (S_\alpha(t)\Phi f)(z) = \left(\frac{1-\tanh(t/2)}{1+z\tanh(t/2)}\right)^\alpha (\Phi f)\left(\frac{z+\tanh(t/2)}{1+z\tanh(t/2)}\right)$$

$$= (1-\tanh(t/2))^\alpha \sum_{k=0}^{\infty} f(k) \frac{(z+\tanh(t/2))^k}{(1+z\tanh(t/2))^{k+\alpha}}$$

$$= (1-\tanh(t/2))^\alpha \sum_{k=0}^{\infty} f(k) \sum_{n=0}^{k} \binom{k}{n} z^n (\tanh(t/2))^{k-n}$$

$$\times \sum_{j=0}^{\infty} \binom{-k-\alpha}{j} (z\tanh(t/2))^j$$

$$= (1-\tanh(t/2))^{\alpha} \sum_{n=0}^{\infty} z^n \sum_{k=0}^{\infty} f(k) \sum_{j=0}^{\min\{n,k\}} \binom{-k-\alpha}{n-j} \binom{k}{j} (\tanh(t/2))^{n+k-2j},$$

whence the statement follows. □

REMARK 2.3. For fixed $k \in \mathbb{N}_0$, taking the function $\mathfrak{f}(z) := z^k$ in the above proposition one directly gets

(2.2)
$$\sum_{n=0}^{\infty} a_{n,k}^{\alpha}(t) z^n = \left(\frac{1-\tanh(t/2)}{1+z\tanh(t/2)}\right)^{\alpha} \left(\frac{z+\tanh(t/2)}{1+z\tanh(t/2)}\right)^{k}, \quad z \in \mathbb{D},\ \alpha, t \in \mathbb{R}.$$

By Corollary 2.1, $T_\alpha(t)$ is bounded on ℓ^2 and its norm can be exactly computed. These facts are not simple to prove directly from formula (2.1).

Since ℓ^p and $H^p(\mathbb{D})$ are not isomorphic for $p \neq 2$, the above argument is useless when we wonder whether or not T_α defines a C_0 group on ℓ^p. Actually, the next result gives a negative answer for all $p \neq 2$.

THEOREM 2.4. *Let $1 \le p < \infty$ with $p \neq 2$, and let $\alpha, t \in \mathbb{R}$, $t \neq 0$. Then $T_\alpha(t)$ does not define a bounded operator on ℓ^p.*

PROOF. Fix $t \neq 0$ throughout all the proof. Let $\beta \in \mathbb{R}$ and set $\mathfrak{g}_\beta(z) := (1 + z\tanh(t/2))^{-\beta}$ for $z \in \mathbb{D}$. Then $\mathfrak{g}_\beta$ is a holomorphic function in the open disc of radius $|1/\tanh(t/2)| > 1$ centered at $z = 0$ and therefore the sequence of its Taylor coefficients $g_\beta(n) = a_{n,0}^{\beta}$, $n \in \mathbb{N}_0$, belongs to ℓ^1 for all $\beta \in \mathbb{R}$. Since $1/\mathfrak{g}_\beta = \mathfrak{g}_{-\beta}$, the convolution operator $\mathcal{K}_\beta$ defined by $\mathcal{K}_\beta f := \Phi^{-1}(\mathfrak{g}_\beta\ \Phi(f))$, $f \in \ell^p$, is an invertible bounded operator on ℓ^p for all $p \in [1, \infty]$.

On the other hand, from the definition of $T_\alpha(t)$ prior to Corollary 2.1 it follows readily that

$$(\forall \alpha, \beta \in \mathbb{R}) \quad T_\alpha(t) = (1 - \tanh(t/2))^{\alpha-\beta} \mathcal{K}_{\alpha-\beta} T_\beta(t).$$

Hence, given a fixed $\alpha \in \mathbb{R}$, one has that $T_\alpha(t)$ is a bounded operator on ℓ^p if and only if $T_\beta(t)$ is a bounded operator on ℓ^p for all $\beta \in \mathbb{R}$. Next, we proceed differently depending on p.

(1) Case $p = 1$. It is known that a composition mapping

$$f \mapsto \Phi^{-1} \circ (\Phi(f) \circ \mathfrak{g})$$

is bounded on ℓ^1 if and only if $\mathfrak{g}(z) = cz$, with $|c| = 1$, see [11, p. 38]. Then, it follows that $T_0(t)$ is not a bounded operator on ℓ^1.

(2) Case $1 < p < 2$. If $T_0(t)$ were a bounded operator on ℓ^p, then its adjoint operator $T_0(t)^*$, which is given by

$$(T_0(t)^* f)(k) = \sum_{n=0}^{\infty} a^0_{n,k}(t) f(n), \quad k \in \mathbb{N}_0,$$

would be a bounded operator on $\ell^{p'}$ (where $1/p + 1/p' = 1$), with norm $\|T_0(t)^*\|_{p'} = \|T_0(t)\|_p < \infty$. For $k \in \mathbb{N}_0$, let ev_k denote the evaluation mapping $ev_k f = f(k)$. It is readily seen that

$$\|ev_k T_0(t)^*\|_{\ell^{p'} \to \mathbb{C}} = \left(\sum_{n=0}^{\infty} |a^0_{n,k}(t)|^p \right)^{1/p} =: C_{k,p},$$

so $C_{k,p} \le \|T_0(t)^*\|_{p'}$ for all $k \in \mathbb{N}_0$. But, we also have by (2.2) and [14, Theorem 1] that $C_{k,p} \sim k^{(2/p)-1} \to \infty$ as $k \to \infty$, whence one gets a contradiction.

(3) Case $2 < p < \infty$. Assume T_1 is a bounded operator on ℓ^p. Then we have

$$\|ev_n T_1(t)\|_{\ell^p \to \mathbb{C}} = \left(\sum_{k=0}^{\infty} |a^1_{n,k}(t)|^{p'} \right)^{1/p'} =: D_{n,p'}\,,$$

so $D_{n,p'} \le \|T_1(t)\|_p$ for all $n \in \mathbb{N}_0$. It is readily seen that

$$a^1_{n,k}(t) = (-1)^{n+k} a^1_{k,n}(t)$$

for all $n, k \in \mathbb{N}_0$, so $D_{n,p'} = \left(\sum_{k=0}^{\infty} |a^1_{k,n}(t)|^{p'} \right)^{1/p'}$. Reasoning as at the beginning of the proof and using (2.2), one obtains

$$A_{p'} \left(\sum_{k=0}^{\infty} |a^0_{k,n}(t)|^{p'} \right)^{1/p'} \le \left(\sum_{k=0}^{\infty} |a^1_{k,n}(t)|^{p'} \right)^{1/p'}$$
$$\le B_{p'} \left(\sum_{k=0}^{\infty} |a^0_{k,n}(t)|^{p'} \right)^{1/p'}, \quad n \in \mathbb{N}_0,$$

for some constants $A_{p'}, B_{p'} > 0$ independent of n. By [14, Theorem 1] again, $D_{n,p'} \sim C_{n,p'} \to \infty$ as $n \to \infty$, which is a contradiction. □

3. Some identities of sums of binomial coefficients

In this section we give results about finite series involving binomial coefficients which seem to be unknown and will be useful to describe the integral operators subordinated to the group $(T_\alpha(t))_{t\in\mathbb{R}}$ presented in Section 4.

The following proposition can be proved by a combinatorial argument based on induction on $k \in \mathbb{N}_0$. Instead, we have chosen to give a complex variable proof.

LEMMA 3.1. *Let* $\nu \in \mathbb{R}$ *and* $k \in \mathbb{N}_0$, $\lambda \in \mathbb{R}$ *such that* $\lambda \neq 0, -1, -2, \dots, -k$. *Then*

$$\sum_{j=0}^{k} \binom{\nu+j}{k} \binom{k}{j} \frac{(-1)^j}{j+\lambda} = \frac{1}{k+1} \binom{\nu-\lambda}{k} \binom{\lambda+k}{k+1}^{-1}. \tag{3.1}$$

PROOF. Let ν, k, λ be as in the statement. It is readily seen that

$$\binom{\nu+j}{k}\binom{k}{j}(-1)^j = \binom{-\nu-1}{j}\binom{\nu}{k-j} \quad \text{for } j = 0, 1, \dots, k,$$

so that proving (3.1) is equivalent to prove

$$(k+1)\binom{\lambda+k}{k+1} \sum_{j=0}^{k} \binom{-\nu-1}{j}\binom{\nu}{k-j}\frac{1}{\lambda+j} = \binom{\nu-\lambda}{k}, \quad j = 0, 1, \dots, k. \tag{3.2}$$

The two members, on the left and on the right, of equality (3.2) are poynomials in λ (also in ν) of degree k. Thus, in order to prove (3.2), it is enough to show that (3.2) holds for $k+1$ different λ in $\mathbb{R}$. In fact, we are going to show that (3.2) is fulfilled by every $\lambda \in \mathbb{N}$.

So let N be a natural number. Note that

$$(k+1)\binom{N+k}{k+1} = \frac{(k+N)\cdots(k+1)}{\Gamma(N)}.$$

Define

$$F_N(z) := \frac{1}{\Gamma(N)} \sum_{k=0}^{\infty}\sum_{j=0}^{k} \binom{-\nu-1}{j}\binom{\nu}{k-j}\frac{(k+1)\cdots(k+N)}{N+j}\, z^k, \quad z \in \mathbb{D}.$$

We see below that indeed $F_N(z)$ is an absolutely convergent series in $\mathbb{D}$.

Clearly, (3.2) for $\lambda = N$ is equivalent to the equality $F_N(z) = (1+z)^{\nu-N}$, $z \in \mathbb{D}$. The claim of the proposition will be established as soon as we prove the latter equality.

Let G_N be in $\mathcal{O}(\mathbb{D})$ given by

$$G_N(z) := \sum_{k=0}^{\infty}\sum_{j=0}^{k} \binom{-\nu-1}{j}\binom{\nu}{k-j}\frac{1}{N+j}\, z^{k+N}, \quad z \in \mathbb{D},$$

so that $F_N(z) = \frac{1}{\Gamma(N)} G_N^{(N)}(z)$, for every $z \in \mathbb{D}$. Without loss of generality we can assume $\nu < 0$ since (3.1) is polynomial in ν.

For $z \in \mathbb{D}$ we have

$$\begin{aligned} G_N(z) &:= z^N \sum_{k=0}^{\infty} \sum_{j=0}^{k} \binom{-\nu-1}{j} \binom{\nu}{k-j} \frac{1}{N+j} z^k \\ &= z^N \left(\sum_{k=0}^{\infty} \binom{\nu}{k} z^k \right) \left(\sum_{k=0}^{\infty} \binom{-\nu-1}{k} \frac{1}{N+k} z^k \right) \\ &= (1+z)^\nu \sum_{k=0}^{\infty} \binom{-\nu-1}{k} \frac{1}{N+k} z^{k+N} =: (1+z)^\nu h_N(z), \end{aligned}$$

with

$$\begin{aligned} h_N'(z) &= \sum_{k=0}^{\infty} \binom{-\nu-1}{k} z^{k+N-1} = z^{N-1}(1+z)^{-\nu-1} \\ &= (1+z-1)^{N-1}(1+z)^{-\nu-1} \\ &= \sum_{m=0}^{N-1} \binom{N-1}{m} (1+z)^m (-1)^{N-1-m} (1+z)^{-\nu-1} \\ &= \sum_{m=0}^{N-1} \binom{N-1}{m} (-1)^{N-1-m} (1+z)^{m-\nu-1}. \end{aligned}$$

Hence,

$$h_N(z) = \sum_{m=0}^{N-1} \binom{N-1}{m} (-1)^{N-1-m} \frac{(1+z)^{m-\nu}}{m-\nu} + C_N(\nu), \quad z \in \mathbb{D},$$

where $C_N(\nu)$ is constant in z.

Thus we have, for $z \in \mathbb{D}$,

$$G_N(z) = \sum_{m=0}^{N-1} \binom{N-1}{m} (-1)^{N-1-m} \frac{(1+z)^{m-\nu}}{m-\nu} + C_N(\nu)(1+z)^\nu,$$

and therefore

$$F_N(z) = \frac{1}{\Gamma(N)} G_N^{(N)}(z) = \frac{1}{\Gamma(N)} C_N(\nu)\, \nu(\nu-1)\cdots(\nu-N+1)(1+z)^{\nu-N}.$$

Moreover, since $h_N(0)=0$, we get

$$C_N(\nu) = -\sum_{m=0}^{N-1}\binom{N-1}{m}(-1)^{N-1-m}\frac{1}{m-\nu},$$

whence

$$C_N(\nu) = (-1)^N\sum_{m=0}^{N-1}\binom{N-1}{m}(-1)^m\int_0^1 t^{m-\nu-1}\,dt$$

$$= (-1)^N\int_0^1\sum_{m=0}^{N-1}\binom{N-1}{m}(-t)^m t^{-\nu-1}dt = (-1)^N\int_0^1(1-t)^{N-1}t^{-\nu-1}\,dt$$

$$= (-1)^N B(N;-\nu) = (-1)^N\frac{\Gamma(N)\Gamma(-\nu)}{\Gamma(N-\nu)}.$$

As a consequence,

$$F_N(z) = \frac{(-1)^N}{\Gamma(N)}\frac{\Gamma(N)\Gamma(-\nu)}{\Gamma(N-\nu)}\ \nu\cdots(\nu-N+1)(1+z)^{\nu-N} = (1+z)^{\nu-N},$$

as desired.The proof is finished. □

LEMMA 3.2. *Let* $n,k\in\mathbb{N}_0$ *and* $\lambda\in\mathbb{R}$ *such that* $\frac{|n-k|+\lambda}{2}\neq 0,-1,\dots,$ $-\min\{n,k\}$. *Then*

$$\sum_{i=0}^{\min\{n,k\}}\binom{n+k-i}{k}\binom{k}{i}\frac{(-1)^i}{n+k-2i+\lambda}$$

$$= \frac{(-1)^{\min\{n,k\}}}{2(\min\{n,k\}+1)}\binom{\frac{n+k-\lambda}{2}}{\min\{n,k\}}\binom{\frac{n+k+\lambda}{2}}{\min\{n,k\}+1}^{-1}.$$

PROOF. First of all, note that $\binom{n+k-i}{k}\binom{k}{i}=\binom{n+k-i}{n}\binom{n}{i}$. Thus we have

$$\binom{n+k-i}{k}\binom{k}{i} = \binom{n+k-i}{\min\{n,k\}}\binom{\min\{n,k\}}{i}.$$

Applying the change of index $j=\min\{n,k\}-i$, one has

$$\sigma := \sum_{i=0}^{\min\{n,k\}}\binom{n+k-i}{\min\{n,k\}}\binom{\min\{n,k\}}{i}\frac{(-1)^i}{n+k-2i+\lambda}$$

$$= \frac{(-1)^{\min\{n,k\}}}{2} \sum_{j=0}^{\min\{n,k\}} \binom{\max\{n,k\}+j}{\min\{n,k\}} \binom{\min\{n,k\}}{j} \frac{(-1)^j}{j+\frac{|n-k|+\lambda}{2}}.$$

Now, by Lemma 3.1,

$$\sigma = \frac{(-1)^{\min\{n,k\}}}{2(\min\{n,k\}+1)} \binom{\max\{n,k\}-\frac{|n-k|+\lambda}{2}}{\min\{n,k\}} \binom{\frac{|n-k|+\lambda}{2}+\min\{n,k\}}{\min\{n,k\}+1}^{-1}$$

$$= \frac{(-1)^{\min\{n,k\}}}{2(\min\{n,k\}+1)} \binom{\frac{n+k-\lambda}{2}}{\min\{n,k\}} \binom{\frac{n+k+\lambda}{2}}{\min\{n,k\}+1}^{-1}.$$

The proof is finished. $\square$

Lemma 3.3. *Let $n,k \in \mathbb{N}_0$. We have*

$$\sum_{i=0}^{\min\{n,k\}} \binom{n+k-i}{k} \binom{k}{i} \frac{(-1)^i}{n+k-2i+1} = \frac{(-1)^{\min\{n,k\}}}{n+k+1}.$$

Proof. Applying Lemma 3.2 with $\lambda = 1$ one obtains

$$\sum_{i=0}^{\min\{n,k\}} \binom{n+k-i}{k} \binom{k}{i} \frac{(-1)^i}{n+k-2i+1}$$

$$= \frac{(-1)^{\min\{n,k\}}}{2(\min\{n,k\}+1)} \binom{\frac{n+k-1}{2}}{\min\{n,k\}} \binom{\frac{n+k+1}{2}}{\min\{n,k\}+1}^{-1}$$

$$= \frac{(-1)^{\min\{n,k\}}}{2} \frac{1}{\frac{n+k+1}{2}} = \frac{(-1)^{\min\{n,k\}}}{n+k+1},$$

as we wanted to show. $\square$

4. Integrals subordinated to the hyperbolic group

Let $\alpha, \beta, \mu \geq 0$. Here we deal with the operators formally given by

$$\mathfrak{I}_\delta^{\mu,\nu} = 2^{-\delta} \int_0^\infty (1-e^{-t})^{\delta-1} e^{(\nu+\delta)t} T_{\mu+\nu+\delta+1}(t)\, dt,$$

$$\mathfrak{H}_\delta^{\mu,\nu} = \int_{-\infty}^\infty \frac{2^{\delta-1}}{(1+e^t)^\delta} e^{(\nu+1)t} T_{\mu+\nu-\delta+2}(t)\, dt.$$

Versions of the above operators have been treated in [1] (with $S_\alpha(t)$ instead $T_\alpha(t)$ in $\mathfrak{I}_\delta^{\mu,\nu}$ and $\mathfrak{H}_\delta^{\mu,\nu}$) or in [6] (with $\widetilde{S}_\alpha(t)$ instead $T_\alpha(t)$ in $\mathfrak{I}_\delta^{\mu,\nu}$). We next discuss the boundedness of the above operators on the space ℓ^2, and give their expressions in terms of sequences.

Let B denote the Beta function and ${}_2F_1$ the gaussian hypergeometric function of integral representation

$$ {}_2F_1(a,b;c;z) := \frac{\Gamma(c)}{\Gamma(c-b)\Gamma(b)} \int_0^1 s^{b-1}(1-s)^{c-b-1}(1-zs)^{-a}\, ds, $$

$z \in \mathbb{C} \setminus [1,+\infty)$, for $a,b,c \in \mathbb{C}$ with $\mathfrak{Re}\, c > \mathfrak{Re}\, b > 0$; see for example [7, Formula 9.111].

4.1. Siskakis type operators on sequences. For all $\rho > 0$, $\nu \in \mathbb{R}$, let $M_{\nu,\rho}$ be given by

$$ M_{\nu,\rho} := \lim_{\mathbb{D} \ni z \to -1} \sum_{j=0}^{\infty} \binom{\nu+j}{j} \frac{z^j}{\rho+j}, $$

where the series converges absolutely for all $z \in \mathbb{D}$. By [7, Formula 9.102], one has

$$ (\forall \nu < 1) \quad M_{\nu,\rho} = \sum_{j=0}^{\infty} \binom{\nu+j}{j} \frac{(-1)^j}{\rho+j}, $$

with absolute convergence if $\nu < 0$, and non-absolute otherwise.

PROPOSITION 4.1. *Let $\mu > -1/2$, $\delta > 0$, $\nu + \delta < 1/2$. Then $\mathfrak{J}_\delta^{\mu,\nu}$ is a bounded operator on ℓ^2 with*

$$ \|\mathfrak{J}^{\alpha}_{\delta,+}\|_{\ell^2 \to \ell^2} = 2^{-\delta} B(\delta, 1/2 + \min\{\mu, -\nu-\delta\}) $$

such that, for all $n \in \mathbb{N}_0$ and $f \in \ell^2$,

$$ (\mathfrak{J}^{\alpha}_{\delta,+} f)(n) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{\min\{n,k\}} \binom{-k-\mu-\nu-\delta-1}{n-j} \binom{k}{j} E(n,k,j,\mu,\nu,\delta) \right) f(k), $$

where, for $k \in \mathbb{N}_0$ and $0 \le j \le \min\{n,k\}$,

$$ E(n,k,j,\mu,\nu,\delta) := B(n+k+\delta-2j, \mu+1) $$
$$ \times\, {}_2F_1(-\nu, n+k+\delta-2j; n+k+\mu+\delta-2j+1; -1). $$

PROOF. The boundedness of $\mathfrak{J}_\delta^{\mu,\nu}$ on ℓ^2 for $\mu > -1/2$, $\delta > 0$, $\nu + \delta < 1/2$ and the exact value of its norm are direct application of Corollary 1.4, with $p=2$ and $\phi(t)=2^{-\delta}(1-e^{-t})^{\delta-1}e^{(\nu+\delta)t}$, $t\in\mathbb{R}$, since $S_{\mu+\nu+\delta+1}(t)$ and $T_{\mu+\nu+\delta+1}(t)$ are unitarily equivalent for every $t\in\mathbb{R}$. In effect,

$$\begin{aligned}\|\mathfrak{J}_\delta^{\mu,\nu}\|_{\ell^2\to\ell^2}=\|\mathcal{J}_\delta^{\mu,\nu}\|_{H^2\to H^2}&=2^{-\delta}\int_0^\infty(1-e^{-t})^{\delta-1}e^{-t(1/2+\min\{\mu,-\nu-\delta\})}\,dt\\&=2^{-\delta}\int_0^\infty(1-x)^{\delta-1}x^{1/2+\min\{\mu,-\nu-\delta\}-1}\,dx\\&=2^{-\delta}B(\delta,1/2+\min\{\mu,-\nu-\delta\}).\end{aligned}$$

Now, by Proposition 2.2 we have, for $f\in\ell^2$ and $n\in\mathbb{N}_0$,

$$(\mathfrak{J}_\delta^{\mu,\nu}f)(n)=2^{-\delta}\sum_{k=0}^\infty\bigg(\int_0^\infty(1-e^{-t})^{\delta-1}e^{(\nu+\delta)t}a_{n,k}^{\mu+\nu+\delta+1}(t)\,dt\bigg)f(k),$$

where

$$\begin{aligned}&\int_0^\infty(1-e^{-t})^{\delta-1}e^{(\nu+\delta)t}a_{n,k}^{\mu+\nu+\delta+1}(t)\,dt\\&=2^{\mu+\nu+\delta+1}\int_0^\infty\sum_{j=0}^{\min\{n,k\}}\binom{-k-\mu-\nu-\delta-1}{n-j}\binom{k}{j}\\&\quad\times\frac{(e^t-1)^{n+k-2j}}{(e^t+1)^{n+k+\mu+\nu+\delta+1-2j}}(1-e^{-t})^{\delta-1}e^{(\nu+\delta)t}\,dt\\&=2^{\mu+\nu+\delta+1}\sum_{j=0}^{\min\{n,k\}}\binom{-k-\mu-\nu-\delta-1}{n-j}\binom{k}{j}\\&\quad\times\int_0^\infty e^{-(\mu+1)t}\frac{(1-e^{-t})^{n+k+\delta-2j-1}}{(1+e^{-t})^{n+k+\mu+\nu+\delta+1-2j}}\,dt\\&=2^{\mu+\nu+\delta+1}\sum_{j=0}^{\min\{n,k\}}\binom{-k-\mu-\nu-\delta-1}{n-j}\binom{k}{j}\\&\quad\times\int_0^1 x^\mu\frac{(1-x)^{n+k+\delta-2j-1}}{(1+x)^{n+k+\mu+\nu+\delta+1-2j}}\,dx,\end{aligned}$$

with

$$\int_0^1 x^\mu\frac{(1-x)^{n+k+\delta-2j-1}}{(1+x)^{n+k+\mu+\nu+\delta+1-2j}}\,dx$$

$$= 2^{-\mu-\nu-1}B(n+k+\delta-2j,\mu+1)$$
$$\times\ {}_2F_1(-\nu,n+k+\delta-2j;n+k+\mu+\delta-2j+1;-1),$$

by [7, Formula 3.197(8)] and [7, Formula 9.131(1)]. □

We now take a closer look at the case $\mu=0$, $\nu=-\delta$.

Corollary 4.2. *Let* $\delta>0$, $f\in\ell^2$, $n\in\mathbb{N}_0$, *and set* $\mathfrak{J}_\delta:=\mathfrak{J}_\delta^{0,-\delta}$. *Then*

$$(\mathfrak{J}_\delta f)(n)=\sum_{k=0}^{\infty}\frac{(-1)^{\min\{n,k\}+n}}{2(\min\{n,k\}+1)}c(n,k)f(k),$$

where

$$c(n,k):=\lim_{\mathbb{D}\ni z\to-1}\sum_{j=0}^{\infty}\binom{\delta-1+j}{j}\binom{(n+k-\delta-j)/2}{\min\{n,k\}}\binom{(n+k+\delta+j)/2}{\min\{n,k\}+1}^{-1}z^j.$$

Moreover, the series in j *converges absolutely for all* $z\in\mathbb{D}$. *The limit in* z *can be intertwined with the series in* j *for* $\delta<2$, *and in this case the series in* j *converges absolutely for* $\delta<1$.

Proof. Let d_k be the term multiplying $f(k)$ in Proposition 4.1. First we notice

$$E(n,k,i,0,-\delta,\delta)=B(n+k+\delta-2i,1)$$
$$\times\ {}_2F_1(\delta,n+k+\delta-2i;n+k+\delta-2i+1;-1)=M_{\delta-1,n+k+\delta-2i},$$

see [7, Subsection 9.10]. Then

$$d_k=\sum_{i=0}^{\min\{n,k\}}\binom{-k-1}{n-i}\binom{k}{i}E(n,k,i,0,-\delta,\delta)$$

$$=\sum_{i=0}^{\min\{n,k\}}(-1)^{n-i}\binom{n+k-i}{k}\binom{k}{i}\lim_{\mathbb{D}\ni z\to-1}\sum_{j=0}^{\infty}\binom{\delta-1+j}{j}\frac{z^j}{n+k+\delta-2i+j}$$

$$=\lim_{\mathbb{D}\ni z\to-1}\sum_{j=0}^{\infty}\binom{\delta-1+j}{j}(-1)^n z^j\sum_{i=0}^{\min\{n,k\}}\binom{n+k-i}{k}\binom{k}{i}\frac{(-1)^i}{n+k+\delta-2i+j}$$

$$=\lim_{\mathbb{D}\ni z\to-1}\sum_{j=0}^{\infty}\binom{\delta-1+j}{j}z^j\frac{(-1)^{\min\{n,k\}+n}}{2(\min\{n,k\}+1)}\binom{\frac{n+k-\delta-j}{2}}{\min\{n,k\}}\binom{\frac{n+k+\delta+j}{2}}{\min\{n,k\}+1}^{-1},$$

where we have applied Lemma 3.2 in the latter equality. The convergence criterion as $z\to-1$ is derived from the definition of $M_{\nu,\mu}$ prior to Proposition 4.1. □

REMARK 4.3. The above operators $\mathfrak{J}_\delta^{\mu,\nu}$ are bounded on ℓ^2 since they are subordinated, through L^1-functions, to the group $(T_{\mu+\nu+\delta+1}(t))_{t\in\mathbb{R}}$ which is formed by bounded operators on ℓ^2. However, operators $T_{\mu+\nu+\delta+1}(t)$, $t \in \mathbb{R}$, are not bounded on ℓ^p, for $p \neq 2$ according to Theorem 2.4. Thus the following question is in order.

QUESTION. *Are operators* $\mathfrak{J}_\delta^{\mu,\nu}$ *bounded on* ℓ^p *for* $1 \le p \le \infty$*?*

This question does not seem to be simple in view of the formulae in Proposition 4.1 or Corollary 4.2.

We further illustrate this question with the case $\mu = 0$, $\delta = 1$, $\nu = -1$. For these values in $\mathfrak{J}_\delta^{\mu,\nu}$ one obtains the operator $\mathfrak{J}_1^{0,-1}$ which, up to a constant, corresponds to the Siskakis operator $\mathcal{J}$ given by

$$\mathcal{J} f(z) := \frac{1}{1-z} \int_1^z \frac{f(\xi)}{1+\xi}\, d\xi, \quad z \in \mathbb{D}.$$

on $H^p(\mathbb{D})$; see [1] and [13]. By using the Taylor series of $(1+\xi)^{-1}$ in the integral, then integrating and then dividing by $(1-z)$ one gets that, in terms of sequences, the operator $\mathfrak{J}$ corresponding to $\mathcal{J}$ is $\mathfrak{J}\colon (a_n) \mapsto (b_n)$ where

$$b_n = \sum_{k=n}^{\infty} (-1)^{k+1} \frac{1}{k+1} \sum_{j=0}^{k} (-1)^j a_j, \quad n \ge 0.$$

Thus, since $\mathcal{J}$ is bounded on $H^2(\mathbb{D})$ [13], or by Proposition 4.1 above, we have

$$\left(\sum_{n=0}^{\infty} \left| \sum_{k=n}^{\infty} \frac{(-1)^{k+1}}{k+1} \sum_{j=0}^{k} (-1)^j a_j \right|^2 \right)^{1/2} \le 2 \left(\sum_{n=0}^{\infty} |a_n|^2 \right)^{1/2}.$$

We wonder if the inequality remains true if one changes exponents 2 and 1/2 by p and $1/p$ respectively, $p \neq 2$, in it, and replaces the (best) constant 2 with a suitable constant C_p.

4.2. Hilbert matrix type operators on sequences. For $\mu, \nu, \delta \in \mathbb{R}$, let $\mathfrak{H}_\delta^{\mu,\nu}$ be the weighted Hilbert type operator given by

$$\mathfrak{H}_\delta^{\mu,\nu} := \int_{-\infty}^{\infty} 2^{\delta-1} \frac{e^{(\nu+1)t}}{(1+e^t)^\delta} T_{\mu+\nu-\delta+2}(t)\, dt.$$

PROPOSITION 4.4. *Let* $\min\{\mu,\nu\} > -1/2$, $\min\{\delta-\mu, \delta-\nu\} > 1/2$. *Then* $\mathfrak{H}_\delta^{\mu,\nu}$ *is a bounded operator on* ℓ^2 *with*

$$\max_{\lambda\in\{\mu,\nu\}} 2^{\delta-1} B(\lambda+1/2, \delta-\lambda-1/2) \le \|\mathfrak{H}_\delta^{\mu,\nu}\|_{\ell_2\to\ell_2}$$

$$\leq 2^{\delta-1}\left(\frac{{}_2F_1(\delta,\gamma;\gamma+1;-1)}{\gamma}+\frac{{}_2F_1(\delta,\widetilde{\gamma};\widetilde{\gamma}+1;-1)}{\widetilde{\gamma}}\right),$$

where $\gamma=\min\{\mu+1/2,\delta-\nu-1/2\}$ *and* $\widetilde{\gamma}=\min\{\nu+1/2,\delta-\mu-1/2\}$, *such that*

$$(\mathfrak{H}^{\alpha}_{\mu,\delta}f)(n) \tag{4.1}$$

$$=2^{\mu+\nu+1}\sum_{k=0}^{\infty}\left(\sum_{j=0}^{\min\{n,k\}}\binom{-k-\mu-\nu+\delta-2}{n-j}\binom{k}{j}D(n,k,j,\mu,\nu,\delta)\right)f(k)$$

for all $n\in\mathbb{N}_0$ *where, for* $k\in\mathbb{N}_0$ *and* $0\leq j\leq\min\{n,k\}$,

$$\begin{gathered}D(n,k,j,\mu,\nu,\delta)=(-1)^{n+k}B(n+k-2j+1,\nu+1)\\ \times\ {}_2F_1(n+k+\mu+\nu-2j+2,\nu+1;n+k+\nu-2j+2;-1)\\ +B(n+k-2j+1,\mu+1)\\ \times\ {}_2F_1(n+k+\mu+\nu-2j+2,\mu+1;n+k+\mu-2j+2;-1).\end{gathered}$$

PROOF. As regard the estimates for the norm of $\mathfrak{H}^{\mu,\nu}_{\delta}$, note first that, by Corollary 1.4, one has

$$\|\mathfrak{H}^{\mu,\nu}_{\delta}\|_{\ell^2\to\ell^2}\leq 2^{\delta-1}\int_{-\infty}^{\infty}\frac{e^{(\nu+1)t}}{(1+e^t)^{\delta}}e^{M_{\mu+\nu-\delta+2,1/2}(t)}\,dt$$

$$=2^{\delta-1}\left(\int_{-\infty}^{0}\frac{e^{-t\max\{-\nu-1/2,\mu-\delta+1/2\}}}{(1+e^t)^{\delta}}\,dt+\int_{0}^{\infty}\frac{e^{t\max\{\nu+1/2,\delta-\mu-1/2\}}}{(1+e^t)^{\delta}}\,dt\right)$$

$$=2^{\delta-1}\left(\int_{1}^{\infty}\frac{x^{\delta-\widetilde{\gamma}-1}}{(1+x)^{\delta}}\,dx+\int_{1}^{\infty}\frac{x^{\delta-\gamma-1}}{(1+x)^{\delta}}\,dx\right)$$

$$=2^{\delta-1}\left(\frac{{}_2F_1(\delta,\gamma;\gamma+1;-1)}{\gamma}+\frac{{}_2F_1(\delta,\widetilde{\gamma};\widetilde{\gamma}+1;-1)}{\widetilde{\gamma}}\right),$$

where γ and $\widetilde{\gamma}$ are as in the statement and we have applied [7, Formula 3.194(2)] in the last step. So we conclude that the integral

$$\int_{-\infty}^{\infty}e^{(\nu+1)t}(1+e^t)^{-\delta}T_{\mu+\nu-\delta+2}(t)\,dt$$

converges in the Bochner sense for μ,ν,δ as in the statement, and obtain the claimed upper bound for $\|\mathfrak{H}^{\mu,\nu}_{\delta}\|_{\ell^2\to\ell^2}$. For the lower bound of the norm we just apply directly Proposition 1.3.

As for the action of $\mathfrak{H}_\delta^{\mu,\nu}$ on sequences we have, for $f \in \ell^2$ and $n \in \mathbb{N}_0$,

$$(\mathfrak{H}_\delta^{\mu,\nu} f)(n) = 2^{\mu+\nu-1} \sum_{k=0}^{\infty} f(k) \sum_{j=0}^{\min\{n,k\}} \binom{-k-\mu-\nu+\delta-2}{n-j} \binom{k}{j}$$
$$\times \int_{-\infty}^{\infty} e^{(\nu+1)t} \frac{(e^t-1)^{n+k-2j}}{(1+e^t)^{n+k+\mu+\nu+2-2j}}\, dt,$$

with

$$\int_{-\infty}^{\infty} \frac{e^{(\nu+1)t}(e^t-1)^{n+k-2j}}{(1+e^t)^{n+k+\mu+\nu+2-2j}}\, dt$$
$$= \int_0^1 ((-1)^{n+k} x^\nu + x^\mu) \frac{(1-x)^{n+k-2j}}{(1+x)^{n+k+\mu+\nu+2-2j}}\, dx.$$

Then the formula of the statement follows by [7, Formula 3.197(8)]. □

REMARK 4.5. Take $\mu = \nu = 0$ and $\delta = 1$ in the formula of $\mathfrak{H}_\delta^{\mu,\nu}$ on sequences. Since ${}_2F_1(\alpha, m; \alpha, z) = {}_2F_1(m, \alpha; \alpha, z) = (1-z)^{-m}$, see [7, Formula 9.121(1)]), we have

$$D(n,k,j,0,0,1) = \frac{1}{2} \frac{(-1)^{n+k}+1}{n+k-2j+1}.$$

Then by Proposition 4.4 we obtain

$$(\mathfrak{H}_1^{0,0} f)(n) = \sum_{k=0}^{\infty} f(k) \sum_{j=0}^{\min\{n,k\}} \binom{-k-1}{n-j} \binom{k}{j} \frac{(-1)^{n+k}+1}{n+k-2j+1}$$
$$= \sum_{k=0}^{\infty} f(k) \sum_{j=0}^{\min\{n,k\}} (-1)^{n-j} \binom{n+k-j}{n-j} \binom{k}{j} \frac{(-1)^{n+k}+1}{n+k-2j+1}$$
$$= \sum_{k=0}^{\infty} f(k)((-1)^n + (-1)^k) \sum_{j=0}^{\min\{n,k\}} \binom{n+k-j}{k} \binom{k}{j} \frac{(-1)^j}{n+k-2j+1}$$
$$= \sum_{k=0}^{\infty} f(k)((-1)^n + (-1)^k) \frac{(-1)^{\min\{n,k\}}}{n+k+1} = \sum_{k=0}^{\infty} \frac{1+(-1)^{n+k}}{n+k+1} f(k),$$

where we have used Lemma 3.3 in the last-but-one equality. That is, the operator $\mathfrak{H}_1^{0,0}$ is the so called reduced Hilbert matrix operator [1,2], whose corresponding integral formula on functions $\mathfrak{f}$ in $\mathcal{O}(\mathbb{D})$ is

$$\mathcal{H}_1^{0,0}\mathfrak{f}(z) := \int_{-1}^{1} \frac{\mathfrak{f}(\xi)}{1 - z\xi}\, d\xi, \quad z \in \mathbb{D}.$$

Thus we have by Proposition 4.4 that $\mathfrak{H}_1^{0,0}$ is bounded on ℓ^2. Nonetheless, a better result is known: let $\mathcal{H} \equiv \mathfrak{H}$ be the Hilbert matrix operator given by

$$(\mathcal{H}\mathfrak{f})(z) = \int_0^1 \frac{\mathfrak{f}(w)}{1 - zw}\, dw, \qquad z \in \mathbb{D},$$

or, alternatively, by

$$(\mathfrak{H} f)(n) = \sum_{k=0}^{\infty} \frac{1}{n+k+1} f(k) \quad n \in \mathbb{N}_0,$$

for $\mathfrak{f} \in \mathcal{O}(\mathbb{D})$ with Taylor coeficients $(f(n))_{n=0}^{\infty}$ (provide the integral converges), see [3,4,10]. The Hilbert inequality tells us that $\mathfrak{H}$ is a bounded operator on ℓ^p for every $1 < p < \infty$, see [8]. Define the operator $\Omega\colon \mathcal{O}(\mathbb{D}) \to \mathcal{O}(\mathbb{D})$ given by $(\Omega f)(z) := f(-z)$. Then $(\Phi^{-1}\Omega\Phi f)(n) = (-1)^n f(n)$, for $n \in \mathbb{N}_0$ and $f \in \ell^2$ where Φ is as in Section 2. It is readily seen that $\mathcal{H}_1^{0,0} = \mathcal{H} + \Omega\mathcal{H}\Omega$, whence it follows that

$$(\mathfrak{H}_1^{0,0} f)(n) = (\Phi^{-1}\mathcal{H}_1^{0,0}\Phi f)(n)$$
$$= ((\mathfrak{H} + (\Phi^{-1}\Omega\Phi)\mathfrak{H}(\Phi^{-1}\Omega\Phi))f)(n) = \sum_{k=0}^{\infty} \frac{1 + (-1)^{n+k}}{n+k+1} f(k),$$

for all $n \in \mathbb{N}_0$. In consequence, $\mathfrak{H}_1^{0,0}$ extends from $\ell^2 \cap \ell^p$ to a bounded operator from ℓ^p into itself.

Let us call the operator $\mathfrak{H}_\delta^{\mu,\nu}$ given in (4.1) the generalized reduced Hilbert matrix operator of indexes μ, ν, δ.

QUESTION. *Is the operator $\mathfrak{H}_\delta^{\mu,\nu}$ bounded on ℓ^p for $1 < p < \infty$?*

In other words, we wonder if there is a multiparameterized extension of the classical Hilbert inequality for sequences which would apply to $\mathfrak{H}_\delta^{\mu,\nu}$ in (4.1).

References

[1] L. Abadías, J. E. Galé, P. J. Miana and J. Oliva-Maza, Weighted hyperbolic comoposition groups on the unit disc and subordinated integral operators, *Adv. Math.*, **455** (2024), Paper No. 109877, 56 pp.

[2] A. Aleman, A. G. Siskakis and D. Vukotić, Spectrum of the Hilbert matrix on Banach spaces of analytic functions, preprint.

[3] E. Diamantopoulos and A. Siskakis, Composition operators and the Hilbert matrix, *Studia Math.*, **140** (2000), 191–198.

[4] M. Dostanić, M. Jevtić and D. Vukotić, Norm of the Hilbert matrix on Bergman and Hardy spaces and a theorem of Nehari type, *J. Funct. Anal.*, **254** (2008), 2800–2815.

[5] K.-J. Engel and R. Nagel, *One-parameter Semigroups for Linear Evolution Equations*, Graduate Texts in Math., vol. 194, Springer (New York, 2000).

[6] J. E. Galé, P. J. Miana and L. Sánchez-Lajusticia, RKH spaces of Brownian type defined by Cesàro–Hardy operators, *Anal. Math. Phys.*, **11** (2021), Paper No. 119, 34 pp.

[7] I. Gradshteyn and I. Ryzhik, *Table of Integrals, Series, and Products*, Academic Press (Amsterdam, 2014).

[8] G. H. Hardy, Note on a theorem of Hilbert concerning series of positive terms, *Proc. London Math. Soc.*, **23** (1925), 45–46.

[9] K. Hoffman, *Banach Spaces of Analytic Functions*, Prentice-Hall, Inc. (Englewood Cliffs, NJ, 1962).

[10] W. Magnus, On the spectrum of Hilbert's matrix, *Amer. J. Math.*, **72** (1950), 699–704.

[11] D. Newman, Homomorphisms of ℓ_+, *Amer. J. Math.*, **91** (1969), 37–46.

[12] H. Seferoğlu, A spectral mapping theorem for representations of one-parameter groups, *Proc. Amer. Math. Soc.*, **134** (2006), 2457–2463.

[13] A. Siskakis, Weighted composition semigroups on Hardy spaces, *Linear Algebra Appl.*, **84** (1986), 359–371.

[14] O. Szehr and R. Zarouf, ℓ^p-norms of Fourier coefficients of powers of a Blaschke factor, *J. Anal. Math.*, **140** (2020), 1–30.

Funding Open access funding provided thanks to the CRUE-CSIC agreement with Springr Nature.