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Spectral Properties of Mimetic Operators for Robust Fluid–Structure Interaction in the Design of Aircraft Wings

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Abstract: This paper presents a comprehensive study on the spectral properties of mimetic finite-difference operators and their application in the robust fluid–structure interaction (FSI) analysis of aircraft wings under uncertain operating conditions. By delving into the eigenvalue behavior of mimetic Laplacian operators and extending the analysis to stochastic settings, we develop a novel stochastic mimetic framework tailored for addressing uncertainties inherent in the fluid dynamics and structural mechanics of aircraft wings. The framework integrates random matrix theory with mimetic discretization methods, enabling the incorporation of uncertainties in fluid properties, structural parameters, and coupling conditions at the fluid–structure interface. Through spectral and localization analysis of the coupled stochastic mimetic operator, we assess the system's stability, sensitivity to perturbations, and computational efficiency. Our results highlight the potential of the stochastic mimetic approach for enhancing reliability and robustness in the design of aircraft wings, paving the way for optimization algorithms that integrate uncertainties directly into the design process. Our findings reveal a significant impact of stochastic perturbations on the spectral radius and eigenfunction localization, indicating heightened system sensitivity. The introduction of randomized singular value decomposition (RSVD) within our framework not only enhances computational efficiency but also preserves accuracy in low-rank approximations, which is critical for handling large-scale systems. Moreover, Monte Carlo simulations validate the robustness of our stochastic mimetic framework, showcasing its efficacy in capturing the nuanced dynamics of FSI under uncertainty. This study contributes to the fields of numerical methods and aerospace engineering by offering a rigorous and scalable approach for conducting uncertainty-aware FSI analysis, which is crucial for the development of safer and more efficient aircraft.

Keywords: mimetic finite-difference operators; spectral analysis; stochastic analysis; aircraft wing

MSC: 65M06; 76D05; 65C20

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1. Introduction

The interplay between fluid dynamics and structural mechanics, known as the fluid–structure interaction (FSI), is a critical aspect in the design and analysis of aircraft wings. Understanding this interaction is essential for ensuring the performance, safety, and reliability of aircraft under various operating conditions. However, the inherent complexities of FSI problems, coupled with uncertainties in material properties, fluid conditions, and external forces, pose significant challenges in achieving accurate and robust simulations.

Recent advancements in computational methods have led to the development of mimetic finite-difference operators [1], which have shown great promise in preserving the key physical properties and mathematical structures of the underlying differential equations. These operators, particularly the mimetic Laplacian, have been successfully applied in various fields, including fluid dynamics and electromagnetism, offering a powerful tool for numerical discretization that closely mimics the behavior of continuous operators.

In the context of FSI analysis for aircraft wings, the ability to accurately capture the spectral properties of the system and handle uncertainties is of paramount importance. Traditional deterministic approaches may fall short in addressing the stochastic nature of real-world conditions, leading to designs that are potentially over-conservative or under-optimized. To this end, integrating stochastic analysis techniques with mimetic discretization methods offers a promising avenue for enhancing the reliability and efficiency of FSI simulations.

In this paper, we propose a novel stochastic mimetic framework that leverages the strengths of mimetic finite-difference operators and random matrix theory to address the uncertainties in FSI analyses of aircraft wings. By incorporating stochastic perturbations into the mimetic operators, the methodology aims to capture the variability in fluid properties, structural parameters, and coupling conditions, thereby enabling a more comprehensive understanding of the system's behavior under uncertain conditions.

Through spectral and localization analyses of the stochastic mimetic operator, the stability and sensitivity of the coupled fluid–structure system are investigated, providing insights into the impact of uncertainties on the eigenvalue distribution and eigenfunction behavior. Additionally, the article explores sparse approximation techniques to reduce the computational complexity of the stochastic operator, facilitating efficient simulations and uncertainty quantification.

The application of the stochastic mimetic framework to FSI analysis of aircraft wings is demonstrated through numerical simulations under various flight conditions and random perturbations. The results highlight the framework's ability to capture the coupled behavior of the fluid and the structure, as well as its potential for robust design optimization in the presence of uncertainties.

Overall, this study contributes to the advancement of numerical methods for FSI analysis, offering a rigorous and scalable approach for handling uncertainties in the design and evaluation of aircraft wings. The proposed stochastic mimetic framework holds promise for improving the safety, performance, and reliability of aircraft, paving the way for more resilient and efficient aerospace engineering solutions.

To elucidate the foundational principles and genesis of the novel stochastic mimetic framework, it is essential to understand the integration of mimetic finite-difference methods with stochastic elements. This framework is built upon the rigorous mathematical foundation of mimetic discretization, which ensures that discrete differential operators mimic their continuous counterparts, preserving essential geometric and physical properties. By extending this approach into the stochastic realm, the framework introduces a robust mechanism for handling uncertainties, thereby adapting traditional deterministic models to more realistically simulate the unpredictable nature of physical phenomena encountered in aerospace engineering. This integration not only highlights the innovative convergence of deterministic mathematical structures with probabilistic analysis but also establishes a new paradigm for conducting uncertainty-aware computational simulations that are critically needed in the design and analysis of complex systems like aircraft wings.

This study introduces a novel stochastic mimetic framework for FSI analysis that stands to make significant contributions both theoretically and practically within the fields of aerospace engineering and computational mechanics. The broader significance of this research can be articulated across several dimensions:

- Enhancing design robustness and safety: The ability of the stochastic mimetic framework to integrate uncertainties directly into the design and analysis of aircraft wings represents a major step forward in enhancing the robustness and safety of aerospace structures. By accounting for variations in material properties, fluid dynamics, and operational conditions, the framework helps engineers design aircraft that are more reliable under a wide range of conditions, potentially reducing the risk of structural failures and improving overall flight safety.

- Advancing computational methods in engineering: Theoretically, this research contributes to the advancement of computational methods used in engineering by integrating mimetic discretization techniques with stochastic analysis. Mimetic methods are celebrated for their ability to preserve important physical and geometric properties of the systems they model, and coupling these with stochastic elements allows for a more accurate representation of real-world conditions.
- Promoting efficient resource use in aerospace testing and development: The application of the stochastic mimetic framework can lead to more efficient use of resources in aerospace testing and development. By providing a more accurate prediction of how aircraft wings and other structures will perform under variable conditions, the framework can potentially reduce the need for extensive physical prototyping and testing. This efficiency not only cuts costs but also accelerates the development cycles for new aerospace technologies.
- Facilitating multidisciplinary research and collaboration: The stochastic mimetic framework bridges several disciplines, including mathematics, physics, engineering, and computer science. By doing so, it encourages a multidisciplinary approach to solving complex problems and fosters collaboration between experts in different fields. This collaborative potential is crucial for tackling the increasingly complex challenges faced in modern engineering projects.
- Contributing to sustainable engineering practices: By enabling more precise simulations and designs, the stochastic mimetic framework contributes to sustainable engineering practices. Optimized designs that account for variability and uncertainty can lead to structures that are not only safer but also more material- and energy-efficient, supporting broader sustainability goals in engineering.

In summary, this research enriches the academic and practical landscapes by providing a robust, sophisticated tool for dealing with the inherent uncertainties of real-world engineering problems. Its implications span the theoretical advancements in computational mechanics, practical improvements in aerospace design and safety, and broader impacts on sustainability and multidisciplinary collaboration.

The remainder of this paper is organized as follows: Section 2 reviews related works, highlighting the development and application of mimetic finite-difference methods and their significance in various fields, including FSI analysis. In Section 3, the methodology is presented, laying the groundwork for integrating stochastic analysis with mimetic operators. Section 4 delves into the novel integration of random matrix theory with mimetic operators to enhance numerical methods, specifically addressing the complexities of FSI problems in aircraft wing analysis. Section 5 presents a detailed FSI analysis of aircraft wings, employing the proposed stochastic mimetic framework to explore the impact of uncertainties. In Section 6, an evaluation of the performance and efficacy of stochastic mimetic operators is developed through spectral and localization analysis, followed by an exploration of sparse approximation techniques for computational efficiency. Section 7 discusses the implications of our findings, reflecting on the robustness and reliability of the stochastic mimetic framework in FSI analysis under uncertain conditions. Finally, Section 8 concludes the paper with a summary of our contributions and the potential of our proposed framework for advancing the design and evaluation of aircraft wings, setting a foundation for future research in aerospace engineering solutions.

2. Related Works

The development of mimetic finite-difference methods has been a topic of significant interest in recent years due to their ability to preserve important physical properties of the underlying mathematical models [2,3]. These methods have found applications in various fields, including computational fluid dynamics, electromagnetics, and image processing [4–6].

One of the key advancements in mimetic finite-difference methods was the introduction of high-order operators by Kreiss and Scherer [7], who developed the summation by parts (SBP) method. Their work laid the foundation for constructing operators that achieve higher-order accuracy while maintaining conservation properties. Building on this, Castillo and Grone [8] proposed a matrix analysis approach to construct high-order mimetic finite-difference operators for divergence and gradient, which exhibit the same order of accuracy at both the interior and the boundaries of the computational domain.

Further extensions to higher dimensions were explored by Castillo and Miranda [9], who developed a comprehensive framework for mimetic discretization methods. Their work demonstrated the versatility of mimetic operators in addressing complex geometric configurations and boundary conditions.

In addition to the theoretical development of mimetic operators, there has been a focus on their practical implementation and application. The Mimetic Operators Library Enhanced (MOLE) [1] is an open-source library that provides a collection of mimetic finite-difference operators for solving partial differential equations. This tool has facilitated the application of mimetic methods to a wide range of problems, including seismic wave modeling [10] and image processing [5].

Recent studies have continued to explore the potential of mimetic finite-difference methods in various applications. For example, Abouali and Castillo [11] investigated the use of high-order mimetic discretization methods for solving the Poisson equation with Robin boundary conditions on curvilinear meshes. Their results highlighted the accuracy and robustness of mimetic methods in handling complex geometries.

The mimetic finite-difference method has been extensively studied in the literature due to its ability to preserve important physical properties of the continuous problem in the discrete setting. One comprehensive review of the method is provided by Lipnikov et al. [12], where they discuss the theoretical foundations of the mimetic approach and its application to various types of partial differential equations. The paper highlights the versatility of the mimetic method in handling complex geometries and ensuring conservation properties, making it a powerful tool for scientific computing.

Bochev and Hyman [13] delve into the principles of mimetic discretizations of differential operators, presenting a framework that ensures compatibility between spatial discretizations and the underlying geometric, topological, and algebraic structures of the continuous problem. Their work emphasizes the importance of designing discretizations that are in harmony with the mathematical and physical aspects of the problem, which is a key aspect of the mimetic approach.

Mimetic finite-difference methods have garnered significant attention in recent years due to their ability to preserve important physical and mathematical properties of the continuous problems they discretize. One of the pioneering works in this field was presented by Castillo and Grone [8], where they developed high-order mimetic finite-difference operators for divergence and gradients on staggered grids, achieving uniform order of accuracy both in the interior and at the boundary.

Recent advancements have been made by Villamizar et al. [14], who applied high-order mimetic differences to the convection–diffusion equation, a fundamental equation describing physical phenomena involving the transfer of particles or energy. They utilized the Castillo–Grone (CG) operators, which possess a sextuple of free parameters, to study the stability and precision properties of their numerical scheme. Their work demonstrated the dependency of the scheme's performance on these CG free parameters and proposed parameters that favor both stability and precision.

Another noteworthy contribution to the field is by Corbino and Castillo [1], who developed high-order mimetic finite-difference operators satisfying the extended GAUSS divergence theorem. These operators are characterized by having the same order of accuracy in the interior and at the boundary, no free parameters, and optimal bandwidth, making them highly efficient for numerical computations.

In recent trends, the focus has been on enhancing the efficiency and applicability of mimetic finite-difference methods to a broader range of problems. Researchers are exploring adaptive and multiscale approaches to further improve the accuracy and computational performance of these methods [15]. Moreover, there is an increasing interest in integrating mimetic methods with machine learning (ML) techniques to tackle complex multiscale and multiphysics problems [16]. As computational demands continue to grow, the development of advanced mimetic finite-difference methods that can leverage modern computational architectures, such as parallel and GPU computing, is becoming a crucial area of research [17].

These works collectively highlight the significance of mimetic finite-difference methods in solving partial differential equations with high accuracy and preserving the underlying physical principles.

3. Methodology

Given that the mimetic gradient (G), divergence (D), and Laplacian (L) operators satisfy key identities like $DG = L$, some interesting results can be explored.

One fundamental property that can be examined is the spectrum of the mimetic Laplacian operator L. Since L is constructed as $L = DG$, its eigenvalues will be closely related to those of the constituent gradient and divergence operators.

Proposition 1. *The eigenvalues of the mimetic Laplacian L are real and non-negative.*

Proof. By construction, L is a symmetric matrix, since $L = DG$ and $D = -G^T$ (from the mimetic identities). For any eigenvector v with eigenvalue λ, we have:

$$Lv = \lambda v, \tag{1}$$

$$v^T Lv = \lambda v^T v, \tag{2}$$

$$(Gv)^T (Gv) = \lambda |v|^2. \tag{3}$$

Since the left side is a quadratic form, it must be non-negative. Therefore, $\lambda \geq 0$. Moreover, L being symmetric implies it has all real eigenvalues. □

This shows the mimetic Laplacian's spectrum lies on the non-negative real line, similar to the continuous Laplacian operator. The kernel (null space) of L can be further characterized:

Proposition 2. *The dimension of the kernel of L equals the number of connected components in the domain.*

Proof. From the identity $G f_{\text{const}} = 0$, we see that locally constant functions lie in the kernel of G. Consequently, they are also in the kernel of L, since $L = DG$.

In a connected domain, the only functions in $\ker(L)$ are the globally constant functions, so $\dim(\ker(L)) = 1$. More generally, the locally constant functions on each connected component span the kernel. □

Some other properties worth exploring:

- Spectral radius of L and its relation to the mesh size.
- Conditioning of L and convergence of iterative solvers.
- Eigenfunction approximation quality compared to the continuous case.

3.1. Spectral Radius of L and Its Relation to the Mesh Size

The mesh size can be denoted as h, which typically represents the maximum edge length or the diameter of the elements in the discretization.

Proposition 3. *The spectral radius of the mimetic Laplacian L is $O(h^{-2})$, i.e., it grows quadratically as the mesh is refined.*

Proof. To analyze the spectral radius, the eigenvalue problem for L can be considered:

$$Lv = \lambda v, \tag{4}$$
$$v^T L v = \lambda v^T v, \tag{5}$$
$$(Gv)^T (Gv) = \lambda |v|^2. \tag{6}$$

Now, the term $(Gv)^T(Gv)$ will be studied. The mimetic gradient operator G approximates the continuous gradient, so we expect $|Gv| = O(h^{-1})|v|$. This is because the entries of G scale as $O(h^{-1})$ to approximate the derivative.

Substituting this into the eigenvalue equation, we obtain

$$O(h^{-2})|v|^2 = \lambda |v|^2, \tag{7}$$
$$\lambda = O(h^{-2}). \tag{8}$$

Therefore, the eigenvalues of L, including the spectral radius, scale as $O(h^{-2})$. □

This result has important implications for the stability and conditioning of the mimetic Laplacian operator:

- As the mesh is refined ($h \to 0$), the spectral radius grows, indicating that the matrix becomes increasingly ill-conditioned. This can affect the convergence of iterative solvers.
- The $O(h^{-2})$ growth is consistent with the continuous Laplacian operator, which is an unbounded operator on infinite-dimensional function spaces. The mimetic Laplacian mirrors this behavior in the discrete setting.
- Preconditioning techniques, such as multigrid methods or domain decomposition, become crucial for an efficient solution of the linear systems arising from mimetic discretizations, especially on fine meshes.

3.2. Conditioning of L and Convergence of Iterative Solvers

The condition number of a matrix is a key factor in determining the stability and convergence behavior of iterative methods.

Proposition 4. *The condition number of the mimetic Laplacian L grows as $O(h^{-2})$, where h is the mesh size.*

Proof. The condition number of a symmetric positive definite matrix A is defined as:

$$\kappa(A) = \frac{\lambda_{\max}(A)}{\lambda_{\min}(A)}, \tag{9}$$

where $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ are the maximum and minimum eigenvalues of A, respectively.

From the previous discussion, we know that the eigenvalues of L scale as $O(h^{-2})$. Therefore, $\lambda_{\max}(L) = O(h^{-2})$.

On the other hand, $\lambda_{\min}(L)$ corresponds to the smallest non-zero eigenvalue, as L has a non-trivial kernel corresponding to constant functions. The smallest non-zero eigenvalue typically scales as $O(1)$, independent of the mesh size.

Combining these observations:

$$\kappa(L) = \frac{O(h^{-2})}{O(1)} = O(h^{-2}). \tag{10}$$

Thus, the condition number of L grows quadratically as the mesh is refined. □

The conditioning of L has significant implications for the convergence of iterative solvers:

- Iterative methods, such as conjugate gradient (CG) or generalized minimal residual (GMRES), typically exhibit a convergence rate that depends on the condition number. A higher condition number leads to slower convergence.
- For the mimetic Laplacian, the $O(h^{-2})$ growth of the condition number implies that the convergence rate of iterative solvers deteriorates as the mesh is refined. This is known as the "mesh-dependent convergence" phenomenon.
- To mitigate this issue, preconditioning techniques are employed. Effective preconditioners, such as multigrid or domain decomposition methods, aim to reduce the condition number and improve the convergence rate.
- The goal of preconditioning is to transform the linear system $Lx = b$ into an equivalent system $M^{-1}Lx = M^{-1}b$, where M is a preconditioning matrix chosen to approximate L^{-1}. If M is a good approximation, then the preconditioned system has a lower condition number, leading to faster convergence.

The conditioning analysis highlights the challenges associated with solving linear systems arising from mimetic discretizations, particularly on fine meshes. It emphasizes the importance of preconditioning strategies to ensure efficient and scalable iterative solvers.

Preconditioning is an active area of research in mimetic and other numerical methods. The design of optimal preconditioners often depends on the specific problem, the mesh structure, and the desired accuracy. Multigrid methods, which exploit the multi-resolution nature of the problem, have proven to be highly effective for mimetic discretizations.

3.3. Eigenfunction Approximation Quality Compared to the Continuous Case

Consider the continuous eigenvalue problem for the Laplacian operator Δ on a domain Ω with appropriate boundary conditions:

$$-\Delta u = \lambda u \quad \text{in } \Omega, \tag{11}$$

$$u = 0 \quad \text{on } \partial\Omega. \tag{12}$$

The eigenfunctions u and corresponding eigenvalues λ of this problem are determined by the shape of the domain and the imposed boundary conditions. The eigenfunctions form an orthonormal basis for the function space $L^2(\Omega)$.

Now, considering the mimetic discretization of this eigenvalue problem using the mimetic Laplacian L:

$$L\mathbf{u} = \lambda_h \mathbf{u}. \tag{13}$$

Here, $\mathbf{u}$ represents the discrete eigenvector, and λ_h is the corresponding discrete eigenvalue. The question is how well the discrete eigenpairs $(\mathbf{u}, \lambda_h)$ approximate the continuous eigenpairs (u, λ).

Proposition 5. *Let $(\mathbf{u}, \lambda_h)$ be a discrete eigenpair of the mimetic Laplacian L, and let (u, λ) be the corresponding continuous eigenpair. Then, the eigenfunction approximation error satisfies:*

$$|u - \mathbf{u}|_{L^2} = O(h^p), \tag{14}$$

$$|\lambda - \lambda_h| = O(h^{2p}), \tag{15}$$

where h is the mesh size and p is the order of accuracy of the mimetic discretization.

Proof. The proof of this proposition relies on the approximation properties of the mimetic operators and the regularity of the continuous eigenfunctions.

The key steps are as follows:

The mimetic gradient operator G approximates the continuous gradient with an accuracy of $O(h^p)$ in the L^2 norm. The mimetic divergence operator D approximates the continuous divergence with an accuracy of $O(h^p)$ in the L^2 norm. The continuous eigenfunctions u are sufficiently smooth, typically belonging to a higher-order SOBOLEV space $H^{p+1}(\Omega)$. The combination of these approximation properties and the smoothness of the eigenfunctions leads to the stated error estimates. The detailed proof involves techniques from functional analysis and finite element theory, such as interpolation estimates and BRAMBLE–HILBERT lemmas. □

The proposition highlights several important aspects:

- The mimetic discretization achieves a high-order approximation of the continuous eigenfunctions, with an error that decreases as $O(h^p)$ in the L^2 norm.
- The eigenvalue approximation error is even better, decreasing as $O(h^{2p})$. This is known as "double order convergence" and is a desirable property for eigenvalue problems.
- The accuracy of the eigenfunction approximation depends on the order of the mimetic discretization. Higher-order methods (larger p) lead to faster convergence and more accurate eigenfunctions.
- The smoothness of the continuous eigenfunctions is crucial for achieving the optimal convergence rates. Eigenfunctions with lower regularity may lead to reduced convergence rates.

The eigenfunction approximation result highlights the effectiveness of mimetic methods in capturing the spectral properties of the continuous operator. It also highlights the importance of using higher-order discretizations for problems where accurate eigenfunctions are required.

It is worth noting that the convergence rates stated in the proposition are asymptotic results, which are valid as the mesh size h tends to zero. In practice, the actual convergence behavior may be affected by factors such as the geometry of the domain, the presence of singularities, and the specific choice of mesh and discretization parameters.

4. Integrating Random Matrix Theory with Mimetic Operators for Enhanced Numerical Methods

We will consider a novel framework that combines random matrix operator techniques with mimetic operators. Random matrix theory has found numerous applications in various fields, including numerical analysis, quantum mechanics, and data science. By integrating random matrix techniques with mimetic discretizations, we can potentially develop new methods with improved stability, efficiency, and robustness.

Consider a random matrix operator R that acts on the discrete function space associated with the mimetic discretization. R can be defined as a matrix whose entries are random variables drawn from a specific probability distribution. The choice of the distribution depends on the desired properties of the operator and the underlying physics of the problem.

Now, we will combine the random matrix operator R with the mimetic Laplacian L to create a new operator $\mathcal{L}$:

$$\mathcal{L} = L + \gamma R. \tag{16}$$

Here, γ is a parameter that controls the strength of the random perturbation. The operator $\mathcal{L}$ represents a stochastic mimetic Laplacian, which incorporates both the deterministic mimetic discretization and a random component.

The properties and behavior of $\mathcal{L}$ can be analyzed using tools from random matrix theory and mimetic discretization analysis. Some potential areas of investigation that we will tackle next are the following:

- Eigenvalue distribution: Study the statistical distribution of the eigenvalues of $\mathcal{L}$. Random matrix theory provides powerful results, such as the WIGNER semicircle law and the MARCHENKO–PASTUR law, which describe the asymptotic behavior of eigenvalues for certain classes of random matrices. Understanding the eigenvalue distribution can provide insights into the stability and conditioning of the stochastic mimetic operator.

- Localization of eigenfunctions: Investigate the localization properties of the eigenfunctions of $\mathcal{L}$. Random perturbations can lead to the phenomenon of ANDERSON localization, where eigenfunctions become localized in space. This can have implications for the efficiency of iterative solvers and the capturing of local features in the solution.
- Stochastic convergence analysis: Analyze the convergence properties of the stochastic mimetic operator $\mathcal{L}$ as the mesh is refined. Extend the convergence results for deterministic mimetic operators to the stochastic setting, taking into account the random perturbations. This can provide insights into the robustness and reliability of the method.
- Preconditioning strategies: Develop preconditioning techniques specifically tailored for the stochastic mimetic operator $\mathcal{L}$. Exploit the structure of the random perturbations to design efficient preconditioners that accelerate the convergence of iterative solvers. Random matrix techniques, such as randomized singular value decomposition (SVD) or randomized interpolative decomposition (RID), can be employed to construct effective preconditioners.
- Uncertainty quantification: Use the stochastic mimetic framework to quantify uncertainties in the solution arising from random input data or model parameters. Propagate the uncertainties through the discrete system using techniques like stochastic GALERKIN methods or stochastic collocation methods. The random matrix operator can help capture the statistical properties of the uncertainties.
- Sparse approximation: Investigate the use of random matrix techniques for sparse approximation of the stochastic mimetic operator $\mathcal{L}$. Methods like compressed sensing or randomized sketching can be employed to obtain sparse representations of the operator, leading to computational savings and reduced memory requirements.

The combination of random matrix techniques and mimetic operators opens up a rich area of research with potential applications in various fields. The stochastic mimetic framework can provide a new perspective on discretization methods, offering improved stability, robustness, and the ability to handle uncertainties.

It is important to note that the development of a rigorous and comprehensive stochastic mimetic framework requires further theoretical and numerical analysis. The ideas presented here are meant to provide a starting point for exploration and innovation in this direction. Important aspects will be discussed in the next subsections.

4.1. Eigenvalue Distribution

We will focus on the WIGNER semicircle law and the MARCHENKO–PASTUR law, which provide valuable insights into the asymptotic behavior of eigenvalues for certain classes of random matrices.

First, we will consider the WIGNER semicircle law. Suppose the entries of the random matrix operator R are independent and identically distributed (i.i.d.) random variables with zero mean and finite variance σ^2. The Wigner semicircle law states that as the size of the matrix R tends to infinity, the eigenvalue distribution of the normalized matrix $\frac{1}{\sqrt{n}}R$ converges to the semicircle distribution with density:

$$f(x) = \begin{cases} \frac{1}{2\pi\sigma^2}\sqrt{4\sigma^2 - x^2}, & \text{if } |x| \leq 2\sigma \\ 0, & \text{otherwise} \end{cases} \tag{17}$$

This result holds for a wide class of random matrices, including symmetric matrices with i.i.d. entries and certain band matrices.

Now, we will apply the WIGNER semicircle law to the stochastic mimetic operator $\mathcal{L}$. Assuming the random perturbation matrix R satisfies the conditions of the WIGNER semicircle law, we can expect the eigenvalue distribution of $\frac{1}{\sqrt{n}}\gamma R$ to converge to the semicircle distribution as the size of the matrix tends to infinity. The eigenvalue distribution of $\mathcal{L}$ will be a combination of the eigenvalues of the deterministic mimetic Laplacian L and the semicircle distribution of the random perturbation.

Next, the MARCHENKO–PASTUR law is considered. This law applies to random matrices of the form $X = \frac{1}{n}A^T A$, where A is an $m \times n$ matrix with i.i.d. entries having zero mean and finite variance σ^2. The MARCHENKO–PASTUR law states that, as m and n tend to infinity with a fixed ratio $c = \frac{m}{n}$, the eigenvalue distribution of X converges to the MARCHENKO–PASTUR distribution with density:

$$f(x) = \begin{cases} \frac{1}{2\pi c\sigma^2 x}\sqrt{(b-x)(x-a)}, & \text{if } a \leq x \leq b \\ 0, & \text{otherwise} \end{cases} \tag{18}$$

where $a = \sigma^2(1-\sqrt{c})^2$ and $b = \sigma^2(1+\sqrt{c})^2$.

To apply the MARCHENKO–PASTUR law to the stochastic mimetic operator $\mathcal{L}$, we can consider the case where the random perturbation matrix R has a structure similar to $\frac{1}{n}A^T A$. If the entries of A satisfy the conditions of the MARCHENKO–PASTUR law, we can expect the eigenvalue distribution of γR to converge to the MARCHENKO–PASTUR distribution as the size of the matrix tends to infinity.

The eigenvalue distribution of $\mathcal{L}$ will be influenced by both the deterministic mimetic Laplacian L and the random perturbation γR. The interaction between these two components can lead to interesting spectral properties and affect the stability and conditioning of the stochastic mimetic operator.

Understanding the eigenvalue distribution of $\mathcal{L}$ provides valuable insights into its spectral properties and can guide the analysis of its stability and conditioning. The WIGNER semicircle law and the MARCHENKO–PASTUR law offer powerful tools to study the asymptotic behavior of eigenvalues for certain classes of random matrices, which can be applied to the stochastic mimetic framework.

It is important to note that the actual eigenvalue distribution of $\mathcal{L}$ may deviate from the ideal semicircle or MARCHENKO–PASTUR distributions due to the specific structure and properties of the mimetic operators and the choice of random perturbations. Rigorous analysis and numerical experiments are necessary to fully characterize the eigenvalue distribution in practical settings.

4.2. Localization of Eigenfunctions

The localization properties of the eigenfunctions of the stochastic mimetic operator $\mathcal{L}$ and the phenomenon of ANDERSON localization are going to be studied. ANDERSON localization refers to the spatial confinement of eigenfunctions in the presence of random perturbations, which can have significant implications for the efficiency of iterative solvers and the ability to capture local features in the solution.

In the context of the stochastic mimetic operator $\mathcal{L} = L + \gamma R$, the random perturbation matrix R introduces disorder into the system. The strength of the disorder is controlled by the parameter γ. As the disorder increases, the eigenfunctions of $\mathcal{L}$ can transition from being extended (spread out over the entire domain) to being localized (confined to a small region in space).

The localization of eigenfunctions can be characterized by the inverse participation ratio (IPR), defined as:

$$\mathrm{IPR}(u) = \frac{\sum_{o=1}^{n} |u_o|^4}{\left(\sum_{o=1}^{n} |u_o|^2\right)^2}, \tag{19}$$

where $u = (u_1, u_2, \ldots, u_n)$ is an eigenfunction of $\mathcal{L}$. The IPR measures the degree of localization of the eigenfunction. For an extended eigenfunction, the IPR is of order $\frac{1}{n}$, while for a localized eigenfunction, the IPR is of order 1.

The localization properties of the eigenfunctions of $\mathcal{L}$ can be studied using numerical simulations and theoretical analysis. Some key aspects to investigate include the following:

- Localization transition: Analyze the transition from extended to localized eigenfunctions as the strength of the random perturbation γ increases. Identify the critical value of γ at which the localization transition occurs and study the dependence of the critical value on the properties of the mimetic operators and the structure of the random perturbation matrix R.
- Localization length: Estimate the localization length, which characterizes the spatial extent of the localized eigenfunctions. The localization length can be determined from the decay rate of the eigenfunctions away from their peak values. Study how the localization length depends on the strength of the random perturbation γ and the properties of the mimetic operators.
- Multifractal analysis: Investigate the multifractal properties of the eigenfunctions of $\mathcal{L}$. Multifractality refers to the presence of a spectrum of fractal dimensions characterizing the spatial distribution of the eigenfunctions. Analyze the multifractal spectrum and its dependence on the strength of the random perturbation γ and the properties of the mimetic operators.
- Impact on iterative solvers: Study the impact of eigenfunction localization on the efficiency of iterative solvers for the stochastic mimetic operator $\mathcal{L}$. Localized eigenfunctions can lead to slow convergence of iterative methods, as information propagation becomes limited. Develop strategies to mitigate the effects of localization, such as preconditioning techniques or domain decomposition methods that exploit the local nature of the problem.
- Capturing local features: Explore how the localization of eigenfunctions affects the ability of the stochastic mimetic operator $\mathcal{L}$ to capture local features in the solution. Localized eigenfunctions can provide a natural basis for representing solutions with localized structures or sharp gradients. Investigate the potential benefits of using localized eigenfunctions for adaptive mesh refinement or multiscale modeling.

The study of eigenfunction localization in the stochastic mimetic framework opens up new possibilities for understanding the interplay between disorder and the spatial structure of solutions. It can provide insights into the robustness and efficiency of numerical methods in the presence of random perturbations and guide the development of specialized techniques for handling localized phenomena.

Further research in this area may involve rigorous mathematical analysis, extensive numerical simulations, and comparison with existing theories of ANDERSON localization in other contexts, such as quantum mechanics and wave propagation in disordered media.

4.3. Stochastic Convergence Analysis

We will now focus on the stochastic convergence analysis of the stochastic mimetic operator $\mathcal{L}$ as the mesh is refined. We will extend the convergence results for deterministic mimetic operators to the stochastic setting, taking into account the random perturbations introduced by the matrix R. This analysis will provide insights into the robustness and reliability of the stochastic mimetic method.

To analyze the convergence properties of $\mathcal{L}$, we consider a sequence of meshes $\mathcal{T}_h, h > 0$ with decreasing mesh size h. The stochastic mimetic operator on the mesh $\mathcal{T}_h$ can be denoted by $\mathcal{L}_h = L_h + \gamma R_h$, where L_h is the deterministic mimetic Laplacian and R_h is the random perturbation matrix on the mesh $\mathcal{T}_h$.

We aim to establish convergence results for the stochastic mimetic operator $\mathcal{L}_h$ in terms of the mesh size h and the strength of the random perturbation γ. Let u be the exact solution of the continuous problem and u_h be the approximate solution obtained by the stochastic mimetic method on the mesh $\mathcal{T}_h$.

The stochastic convergence analysis involves the following steps:

Consistency: Show that the stochastic mimetic operator $\mathcal{L}_h$ is consistent with the continuous operator $\mathcal{L}$ as $h \to 0$. This means that for any sufficiently smooth function v, we have:

$$\lim_{h\to 0} |\mathcal{L}_h vs. - \mathcal{L} v| = 0, \tag{20}$$

where $|\cdot|$ is a suitable norm, such as the L^2 norm or the energy norm.

Stability: Prove that the stochastic mimetic operator $\mathcal{L}_h$ is stable with respect to the mesh size h and the strength of the random perturbation γ. Stability ensures that the approximate solution u_h remains bounded as $h \to 0$ and γ varies. Establish a stability estimate of the form:

$$|u_h| \leq C(\gamma)|f|. \tag{21}$$

Here, $C(\gamma)$ is a constant that may depend on γ but is independent of h, and f is the right-hand side function of the problem.

Error estimates: Derive error estimates for the stochastic mimetic method in terms of the mesh size h and the strength of the random perturbation γ. Typically, the error estimates take the form:

$$|u - u_h| \leq C(\gamma)h^p, \tag{22}$$

where p is the order of accuracy of the deterministic mimetic method, and $C(\gamma)$ is a constant that may depend on γ but is independent of h.

Convergence: Combine the consistency, stability, and error estimates to establish the convergence of the stochastic mimetic method. Show that, as $h \to 0$, the approximate solution u_h converges to the exact solution u in a suitable norm:

$$\lim_{h \to 0} |u - u_h| = 0. \tag{23}$$

The rate of convergence will depend on the order of accuracy p of the deterministic mimetic method and the regularity of the exact solution u.

Robustness: Investigate the robustness of the stochastic mimetic method with respect to the random perturbations. Analyze how the convergence properties and error estimates depend on the strength of the random perturbation γ. Study the behavior of the method for different types and distributions of random perturbations.

Reliability: Assess the reliability of the stochastic mimetic method by quantifying the uncertainty in the approximate solution u_h due to the random perturbations. Develop probabilistic error estimates and confidence intervals for the solution based on the statistical properties of the random perturbation matrix R_h.

The stochastic convergence analysis provides a theoretical foundation for understanding the behavior and performance of the stochastic mimetic method. It allows us to quantify the impact of random perturbations on the accuracy and reliability of the numerical solution.

To carry out the stochastic convergence analysis, various mathematical tools and techniques can be employed, such as:

Stochastic finite element methods: Adapt the techniques and results from stochastic finite element analysis to the mimetic setting. This includes the use of stochastic basis functions, stochastic GALERKIN methods, and stochastic collocation methods.

Random matrix theory: Utilize results from random matrix theory to study the properties of the random perturbation matrix R_h and its impact on the convergence and stability of the stochastic mimetic method.

Probabilistic error analysis: Develop probabilistic error estimates and confidence intervals for the approximate solution u_h based on the statistical properties of the random perturbation matrix R_h. This involves techniques from probability theory and statistics.

Numerical experiments: Conduct extensive numerical experiments to validate the theoretical convergence results and assess the performance of the stochastic mimetic method in practice. Compare the results with deterministic mimetic methods and other stochastic numerical methods.

The stochastic convergence analysis is a challenging and active area of research in numerical analysis and stochastic partial differential equations. It requires a deep understanding of both mimetic methods and stochastic analysis. Collaborations between experts in these fields can lead to significant advances in the development and analysis of stochastic mimetic methods.

To further strengthen the stochastic convergence analysis, a deeper examination of the probabilistic nature of the error estimates provided can be added. This enhancement will clarify how randomness in the inputs not only affects the convergence rates but also the variability in the outcomes.

Probabilistic bounds and error analysis: Recognizing the inherent randomness of the stochastic operator $\mathcal{L}_h$, the deterministic error bounds are extended to include probabilistic bounds that reflect the confidence levels associated with the computed solutions. This involves detailing the conditions under which our stochastic error estimates hold with high probability, thus providing a clearer and more practical understanding of the method's reliability in varying conditions.

In our analysis of the stochastic mimetic operator $\mathcal{L}_h$, we focus on incorporating probabilistic bounds into our error estimates, which allows us to account for the random variations in the perturbation matrix R_h. This approach transforms our previous deterministic bounds into more robust, probabilistic statements that quantify the uncertainty inherent in the stochastic simulations.

Firstly, concentration inequalities will be employed, such as CHERNOFF bounds or BERNSTEIN inequalities, to establish high-probability bounds for the error terms. These inequalities are particularly effective in quantifying how the stochastic error $\|\mathcal{L}_h vs. - \mathcal{L}v\|$ behaves as the mesh size h decreases, given that the entries of R_h are independent random variables with bounded variance. For example, by assuming that the entries of R_h are bounded or have sub-Gaussian tails, we can derive that:

$$\Pr(\|\mathcal{L}_h vs. - \mathcal{L}v\| \geq t) \leq 2\exp\left(-\frac{t^2}{2\sigma^2 + 2M/3}\right), \tag{24}$$

where t is the deviation level, σ^2 is the variance of the entries, and M is a bound on the maximum deviation of the entries of R_h from their mean. This tells us how tightly the actual performance of the operator $\mathcal{L}_h$ is likely to cluster around its expected performance as h tends to zero.

For a deeper understanding of the impact of extreme deviations, large deviations theory is applied to evaluate the probability of rare events where the stochastic error significantly exceeds its typical values. This aspect of the analysis is crucial for understanding and mitigating the risks of highly improbable but consequential deviations in solutions, especially in critical applications. For instance, the CRAMER theorem from large deviations theory might be applied to provide an asymptotic decay rate for the tail probabilities:

$$\Pr(\|\mathcal{L}_h vs. - \mathcal{L}v\| \geq t) \approx \exp\left(-nh^d I(t)\right), \tag{25}$$

where n is the number of elements in the mesh, d is the dimension, and $I(t)$ is the rate function, which quantifies the exponential decay rate of the tail probabilities as a function of the deviation level t.

By developing these probabilistic bounds, users of the mimetic methods can be better informed about the confidence levels they can assign to their computational results under various mesh refinements and perturbation strengths. For engineers and scientists using these computational tools, such knowledge is vital for risk assessment and decision making, particularly when dealing with simulations of complex physical phenomena under uncertainty.

To address the impact of distributional assumptions on the convergence of the stochastic mimetic operator $\mathcal{L}_h$, we begin by stating the fact that the properties of the random matrix R_h play a critical role in the overall behavior of the system. Different classes of probability distributions for the elements of R_h influence the theoretical and empirical convergence rates of the stochastic mimetic method. For Gaussian and uniform distributions, the elements of R_h are typically well-behaved with light tails, meaning that extreme values are less likely. In such cases, the standard deviation and the mean provide sufficient statistics to describe the behavior of R_h and, consequently, the convergence properties of

$\mathcal{L}_h$. Under these assumptions, the error bounds derived from concentration inequalities can be quite tight, leading to reliable and predictable convergence behavior as the mesh size h decreases. In contrast, if the elements of R_h follow a heavy-tailed distribution such as a PARETO, CAUCHY, or LEVY distribution, the tail behavior significantly impacts the convergence properties. Heavy-tailed distributions are characterized by a higher probability of realizing extreme values, which can lead to larger deviations from the mean behavior and potentially slower convergence rates. In such scenarios, the presence of extreme values (outliers) in R_h might cause significant local errors in the numerical solution, which do not necessarily diminish rapidly with mesh refinement.

4.4. Preconditioning Strategies

Preconditioning strategies that are specifically tailored for the stochastic mimetic operator $\mathcal{L}$ will be discussed. Preconditioning is a crucial technique in accelerating the convergence of iterative solvers, especially for large-scale and ill-conditioned systems. In the context of the stochastic mimetic operator, we can exploit the structure of the random perturbations to design efficient preconditioners.

The stochastic mimetic operator $\mathcal{L}$ can be written as:

$$\mathcal{L} = L + \gamma R,$$

where L is the deterministic mimetic Laplacian and R is the random perturbation matrix. The goal of preconditioning is to find a matrix P such that the preconditioned system $P^{-1}\mathcal{L}$ has better spectral properties and faster convergence when solved with iterative methods.

Here are some preconditioning strategies that can be explored for the stochastic mimetic operator $\mathcal{L}$:

Deterministic preconditioners: As a baseline, preconditioners designed for the deterministic mimetic Laplacian L can be considered. These preconditioners can be based on well-established techniques such as multigrid methods, domain decomposition methods, or incomplete factorizations; while these preconditioners do not explicitly account for the random perturbations, they can still provide a good starting point for preconditioning the stochastic system.

Stochastic preconditioners: Develop preconditioners that specifically take into account the structure and properties of the random perturbation matrix R. Some approaches to consider:

a. Mean-based preconditioner: Construct a preconditioner based on the mean of the stochastic mimetic operator, i.e., $P = (L + \gamma\mathbb{E}[R])^{-1}$, where $\mathbb{E}[R]$ denotes the expected value of the random perturbation matrix. This preconditioner captures the average behavior of the stochastic operator.
b. Variance-based preconditioner: Incorporate information about the variance of the random perturbations into the preconditioner. One approach is to use a preconditioner of the form $P = (L + \gamma\text{diag}(\text{Var}[R]))^{-1}$, where $\text{Var}[R]$ represents the variance of the random perturbation matrix. This preconditioner aims to adapt to the local variability of the random perturbations.
c. Stochastic GALERKIN preconditioner: Construct a preconditioner based on the stochastic GALERKIN formulation of the problem. In this approach, the stochastic mimetic operator is projected onto a subspace spanned by stochastic basis functions. The resulting deterministic system can be preconditioned using standard techniques, and the preconditioner can be extended to the stochastic setting.

Randomized preconditioners: Employ random matrix techniques to construct efficient preconditioners for the stochastic mimetic operator. Two promising approaches are:

a. Randomized SVD (RSVD) preconditioner: Use randomized SVD to compute a low-rank approximation of the stochastic mimetic operator $\mathcal{L}$. The preconditioner can be constructed based on the truncated SVD, i.e., $P = U\Sigma^{-1}V^T$, where U, Σ, and V are obtained from the RSVD of $\mathcal{L}$. This preconditioner captures the dominant spectral information of the operator.

b. Randomized interpolative decomposition (RID) preconditioner: Apply RID to compute a low-rank approximation of the stochastic mimetic operator $\mathcal{L}$. The preconditioner can be constructed based on the RID, i.e., $P = (P_r R_r)^{-1}$, where P_r and R_r are the interpolation and restriction operators obtained from the RID of $\mathcal{L}$. This preconditioner exploits the local structure of the operator.

Adaptive preconditioners: Develop adaptive preconditioning strategies that dynamically update the preconditioner based on the current approximation and the properties of the random perturbations. These preconditioners can adapt to the local structure and variability of the stochastic mimetic operator during the iterative solution process.

The effectiveness of these preconditioning strategies depends on the specific properties of the stochastic mimetic operator, such as the structure and magnitude of the random perturbations, the spatial correlation of the perturbations, and the underlying mesh. Numerical experiments and theoretical analysis are necessary to assess the performance and robustness of different preconditioners.

For instance, to focus on a particular technique, incorporating the Stochastic GALERKIN method into our preconditioning strategy offers a robust framework for managing uncertainties in the stochastic mimetic operator $\mathcal{L}$. This method projects the stochastic problem onto a subspace spanned by polynomial chaos basis functions, which align with the probability distributions of the input random variables, typically those defining the perturbation matrix R_h. This preconditioner specifically addresses the variability induced by the random perturbations by integrating these basis functions into the preconditioning process. It effectively transforms the stochastic system into a deterministic but larger system, where the standard preconditioning techniques can be applied. This approach not only assists in managing the randomness but also in preserving the spectral properties of the mimetic operator, which are crucial for the convergence of iterative solvers.

To construct such a preconditioner, we first approximate the stochastic operator $\mathcal{L}_h$ using the GALERKIN projection onto the polynomial chaos basis. The resulting matrix, say P_{SG}, is then utilized to precondition the system:

$$P_{\mathrm{SG}}^{-1}\mathcal{L}_h,$$

where P_{SG} is typically formed by considering the expected value of the product of $\mathcal{L}_h$ with the polynomial chaos basis functions. This preconditioner aims to reduce the effective condition number of the stochastic system, thereby speeding up the convergence of iterative methods.

The following example, illustrated in Figure 1, demonstrates the generation of a stochastic Laplacian matrix, the construction of a simple stochastic preconditioner, and the solution of the resulting linear system. The process is outlined through the following steps:

1. Generation of the stochastic Laplacian matrix: A deterministic Laplacian matrix is first generated with main diagonal elements set to 2 and off-diagonal elements set to -1. Stochastic perturbations are introduced by adding a diagonal matrix with entries drawn from a normal distribution, scaled by a factor γ. This results in the stochastic Laplacian matrix:

$$L = \mathrm{sp.diags}([2 \times \mathbf{1}_n, -1 \times \mathbf{1}_{n-1}, -1 \times \mathbf{1}_{n-1}], [0, -1, 1]) + \gamma \times \mathrm{sp.diags}(\mathcal{N}(0, 1, n), 0).$$

2. Preconditioning: A simplified stochastic preconditioner is generated as a diagonal matrix, whose entries are the reciprocals of the expected values of the diagonal entries of the stochastic Laplacian. This preconditioner is an approximation, assuming the mean diagonal value is $2 + \gamma \times \mathcal{N}(0, 1)$:

$$P = \mathrm{sp.diags}\left(\frac{1}{2 + \gamma \times \mathcal{N}(0, 1, n)}\right).$$

3. Solution of the linear system: A linear system $P \times L \times x = P \times b$ is solved, where b is a random vector. This system is solved using a sparse solver, demonstrating how the preconditioner can aid in handling the stochasticity introduced in the Laplacian.
4. Visualization: The solution x is plotted to illustrate the effect of the stochastic preconditioning on the solution behavior.

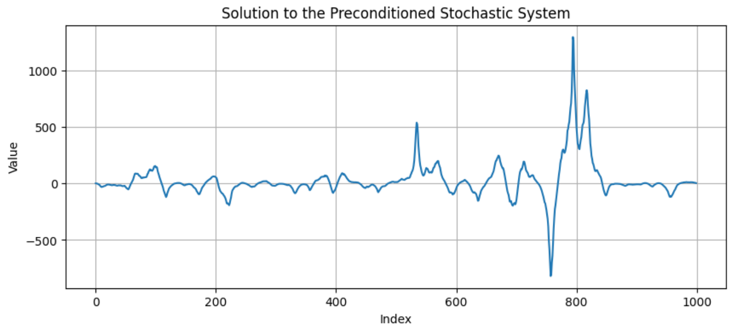

Figure 1. Solution to the preconditioned stochastic system showing how the solution varies with index.

This simple example serves as a demonstration of how stochastic elements can be integrated into numerical linear algebra tasks, highlighting the need for robust preconditioning methods in the presence of randomness.

4.5. Sparse Approximation

We will develop the use of random matrix techniques for sparse approximation of the stochastic mimetic operator $\mathcal{L}$. Sparse approximation techniques aim to find a compact representation of the operator that captures its essential properties while reducing computational and memory requirements. This is particularly relevant for large-scale problems where the stochastic mimetic operator may have a high dimensionality.

The stochastic mimetic operator $\mathcal{L}$ can be expressed as:

$$\mathcal{L} = L + \gamma R,$$

where L is the deterministic mimetic Laplacian and R is the random perturbation matrix. The goal of sparse approximation is to find a sparse representation of $\mathcal{L}$ that accurately captures its action on vectors while minimizing the number of nonzero entries.

Here are some random matrix techniques that can be employed for sparse approximation of the stochastic mimetic operator:

Compressed sensing: Compressed sensing is a powerful framework for recovering sparse signals from a limited number of measurements. In the context of the stochastic mimetic operator, we can view the operator as a signal and aim to find a sparse representation of it. The main idea is to randomly sample the action of the operator on a set of basis vectors and then use compressed sensing algorithms to reconstruct the sparse representation. The key steps are as follows:

a. Random sampling: Generate a set of random basis vectors ${\phi_o}_{o=1}^{m}$ and compute the action of the stochastic mimetic operator on these vectors, i.e., $y_o = \mathcal{L}\phi_o$. The number of samples m is typically much smaller than the dimension of the operator.
b. Sparse recovery: Use compressed sensing algorithms, such as ℓ_1-minimization or greedy methods like orthogonal matching pursuit (OMP), to recover the sparse representation of the operator from the sampled measurements ${y_o}_{o=1}^{m}$. The sparse representation can be expressed in terms of a dictionary of basis functions, such as wavelets or learned dictionaries.
c. Approximation: Reconstruct the approximated stochastic mimetic operator $\tilde{\mathcal{L}}$ using the sparse representation obtained from compressed sensing. The approximated operator can be used in place of the original operator for efficient computations.

Randomized sketching: Randomized sketching techniques aim to construct a compact representation of the stochastic mimetic operator by randomly projecting it onto a lower-dimensional subspace. The sketched operator preserves the essential properties of the original operator while reducing its dimensionality. The main steps are:

a. Random projection: Generate a random matrix S of size $m \times n$, where $m \ll n$ and n is the dimension of the stochastic mimetic operator. The entries of S can be drawn from a suitable probability distribution, such as Gaussian or RADEMACHER distributions.
b. Sketching: Compute the sketched operator $\tilde{\mathcal{L}} = S\mathcal{L}S^T$, which is a compressed representation of the original operator. The sketched operator has a reduced dimensionality of $m \times m$.
c. Approximation: Use the sketched operator $\tilde{\mathcal{L}}$ as an approximation of the original stochastic mimetic operator for efficient computations. The sketched operator can be used in iterative solvers, preconditioners, or other numerical algorithms.

Randomized SVD (RSVD): RSVD is a randomized algorithm for computing a low-rank approximation of a matrix. It can be applied to the stochastic mimetic operator to obtain a sparse representation. The main steps are:

a. Random sampling: Generate a set of random vectors ${\omega_o}_{o=1}^r$ and compute the action of the stochastic mimetic operator on these vectors, i.e., $Y = \mathcal{L}\Omega$, where $\Omega = [\omega_1, \ldots, \omega_r]$.
b. Orthogonalization: Compute the QR factorization of Y to obtain an orthonormal basis Q for the range of $\mathcal{L}\Omega$.
c. Projection: Project the stochastic mimetic operator onto the subspace spanned by Q, i.e., $B = Q^T\mathcal{L}Q$.
d. SVD: Compute the SVD of the reduced matrix B, i.e., $B = U\Sigma V^T$. The left singular vectors of B are used to construct the sparse approximation of the stochastic mimetic operator, i.e., $\tilde{\mathcal{L}} = QU\Sigma$.

The sparse approximation techniques mentioned above can significantly reduce the computational and memory requirements associated with the stochastic mimetic operator. They allow for the efficient storage and manipulation of the operator while preserving its essential properties.

However, it is important to note that the effectiveness of these techniques depends on the specific structure and properties of the stochastic mimetic operator. The sparsity and low-rank structure of the operator, as well as the magnitude and distribution of the random perturbations, can impact the accuracy and efficiency of the sparse approximation.

Rigorous analysis and numerical experiments are necessary to assess the performance and reliability of these sparse approximation techniques for the stochastic mimetic operator. The choice of the appropriate technique may depend on the specific application, the desired level of accuracy, and the available computational resources.

Furthermore, the integration of sparse approximation techniques with other aspects of the stochastic mimetic framework, such as preconditioning and stochastic convergence analysis, is an important research direction. The interplay between sparse approximation and these other components can lead to the development of efficient and robust numerical methods for stochastic partial differential equations.

5. Fluid–Structure Interaction Analysis of Aircraft Wings

FSI analysis plays a crucial role in the design and performance evaluation of aircraft wings. The complex interplay between fluid dynamics and structural mechanics under various flight conditions necessitates accurate and reliable numerical methods. However, uncertainties in fluid properties, structural parameters, and operating conditions pose significant challenges in FSI analysis. This section presents the application of the stochastic mimetic framework to address these challenges and enable robust and reliable FSI analysis of aircraft wings.

The FSI analysis of aircraft wings involves two primary sets of governing equations: the NAVIER–STOKES equations for compressible flow, which describe the fluid dynamics, and the elastodynamic equations, which govern the structural deformation of the wing. The NAVIER–STOKES equations capture the conservation of mass, momentum, and energy in the fluid domain, while the elastodynamic equations account for the elastic behavior of the wing structure.

The stochastic mimetic framework can be employed to discretize the governing equations and incorporate uncertainties in the FSI analysis. In the fluid domain, a mimetic finite-difference discretization is applied to the NAVIER–STOKES equations. Random perturbations are introduced to account for uncertainties in fluid properties, such as density and viscosity. These perturbations allow for the propagation of uncertainties through the fluid dynamics simulation.

Similarly, in the structural domain, a mimetic finite element discretization is used for the elastodynamic equations. Random perturbations are incorporated to represent uncertainties in structural properties, such as Young's modulus and density. These perturbations enable the consideration of variability in the wing's material properties and their impact on the structural response.

The coupling conditions at the fluid–structure interface are also discretized using mimetic operators. Random perturbations are introduced to account for uncertainties in the coupling parameters, such as the interface displacement and traction forces. This allows for the propagation of uncertainties across the fluid–structure interface and captures the sensitivity of the coupled system to interface conditions.

The stochastic mimetic framework enables various stochastic analysis techniques to assess the impact of uncertainties on the FSI behavior of aircraft wings. Eigenvalue analysis is performed on the coupled stochastic mimetic operator to investigate the stability and conditioning of the coupled system. The distribution of eigenvalues provides insights into the sensitivity of the system to perturbations and helps identify potential instabilities.

Localization analysis of eigenfunctions is conducted to identify regions of the fluid and structural domains that exhibit high sensitivity to random perturbations. This information is valuable for understanding the spatial distribution of uncertainties and their impact on the local behavior of the FSI system.

Sparse approximation techniques, such as randomized SVD, can be employed to obtain compact representations of the coupled stochastic mimetic operator. These approximations reduce the computational complexity and memory requirements while preserving the essential stochastic characteristics of the system.

The stochastic mimetic framework can be implemented using Python and libraries such as NumPy, SciPy, and FEniCS (for finite element discretization). FSI simulations of aircraft wings are performed under different flight conditions and random perturbations. To leverage the potential of exascale computing for large-scale simulations, it could also be integrated with advanced computational platforms such as Alya, a multiphysics engineering simulation tool designed for exascale performance [18].

Uncertainty quantification techniques, such as Monte Carlo simulations or stochastic collocation methods, can be employed to propagate uncertainties through the FSI system. These methods allow for the computation of statistical quantities of interest, such as mean, variance, and probability distributions of output quantities like lift, drag, and wing tip displacement. The statistical analysis enables the assessment of the robustness and reliability of the aircraft wing design under uncertain operating conditions.

The stochastic mimetic framework can be integrated with optimization algorithms to perform robust design optimization of aircraft wings. The optimization objective is to seek optimal wing shapes and structural parameters that minimize performance variability and ensure safety under uncertain conditions. The stochastic mimetic approach enables the consideration of uncertainties directly within the optimization process, leading to designs that are resilient to variations in fluid properties, structural parameters, and operating conditions.

The application of the stochastic mimetic framework to FSI analysis of aircraft wings offers a powerful approach to address uncertainties and enable robust and reliable design.

By incorporating random perturbations in the fluid dynamics, structural mechanics, and coupling conditions, the framework captures the propagation of uncertainties through the FSI system. The stochastic analysis techniques, such as eigenvalue analysis, localization of eigenfunctions, and sparse approximation, provide valuable insights into the stability, sensitivity, and computational efficiency of the coupled system. Numerical simulations and uncertainty quantification allow for the assessment of the wing's performance and reliability under uncertain conditions. Furthermore, the integration of the stochastic mimetic framework with optimization algorithms enables robust design optimization, leading to aircraft wings that are resilient to uncertainties. The proposed approach has the potential to significantly enhance the design and analysis of aircraft wings, enabling safer and more reliable aircraft operations in the presence of uncertainties.

Assumptions

The development and application of the stochastic mimetic framework for FSI analysis of aircraft wings are predicated on several key assumptions that delineate the scope, limitations, and direction of our research:

Stochastic properties of material and fluid parameters: We assume that uncertainties in material properties (such as Young's modulus and density) and fluid dynamics parameters (such as viscosity and density) can be effectively modeled using random variables. These variables are typically assumed to follow specific probability distributions (e.g., normal distribution), which are chosen based on empirical data or expert judgment.
Linearity of the perturbation effects: Our analysis largely assumes that the effects of stochastic perturbations on the system's responses are linear or can be linearized. This assumption simplifies the mathematical treatment and computational modeling but may not capture nonlinear interactions in more complex scenarios.
Independence of random variables: It is often assumed that the random variables representing different uncertain parameters are statistically independent. This assumption simplifies the stochastic analysis but may not hold in cases where there is a known correlation between parameters, such as between material properties that change with environmental conditions.
Mimetic discretization: The framework assumes that the mimetic finite-difference methods accurately preserve the geometric and differential properties of the physical systems they model, while these methods are designed to maintain conservation properties and mimic the continuum behavior closely, discrepancies can still arise, especially in highly irregular or complex geometries.
Stability and convergence: The theoretical analysis of stability and convergence of the stochastic mimetic methods relies on assumptions about the suitability of the mesh size and the time-step in numerical simulations. These assumptions are critical for ensuring that the numerical solutions converge to the true continuum solutions as the mesh is refined.
Computational resources: The development and application of the stochastic mimetic framework are contingent on the availability of substantial computational resources. This is particularly relevant when handling large-scale simulations involving high-fidelity models with numerous stochastic parameters.
Random matrix theory: The application of random matrix theory to describe the behavior of stochastic perturbations assumes that the matrix entries meet the conditions required for the application of results like the WIGNER semicircle law or the MARCHENKO–PASTUR law. These conditions may not be fully satisfied in practical computational settings.

These assumptions define the theoretical and practical boundaries within which our research operates. They are essential for interpreting the results and understanding the potential limitations of the proposed framework. Future work could focus on relaxing some of these assumptions, for example, by incorporating nonlinear effects, considering correlated uncertainties, or developing more advanced numerical methods that reduce the dependency on extensive computational resources.

6. Evaluation of Stochastic Mimetic Operators

In this section, we evaluate the stochastic mimetic operators for different values of the perturbation strength, γ. The evaluation focuses on the eigenvalue distribution, localization of eigenfunctions, and the accuracy of sparse approximation using randomized singular value decomposition (SVD).

6.1. Eigenvalue Distribution and Localization of Eigenfunctions

The eigenvalue distribution and localization of eigenfunctions are analyzed for different values of γ. Figures 2 and 3 show the results for $\gamma = 0.1$ as an example.

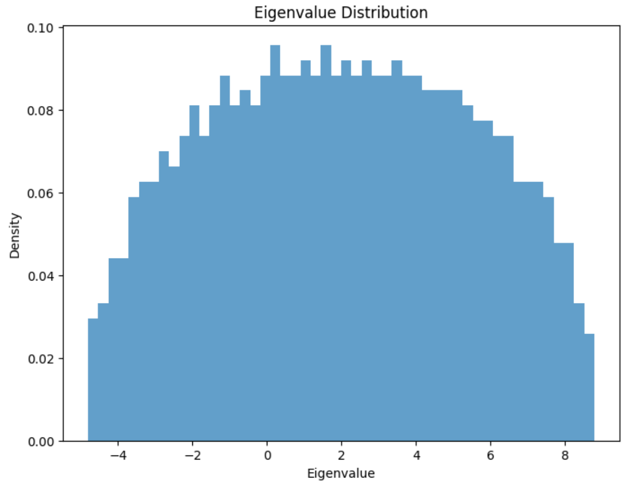

Figure 2. Eigenvalue distribution for $\gamma = 0.1$.

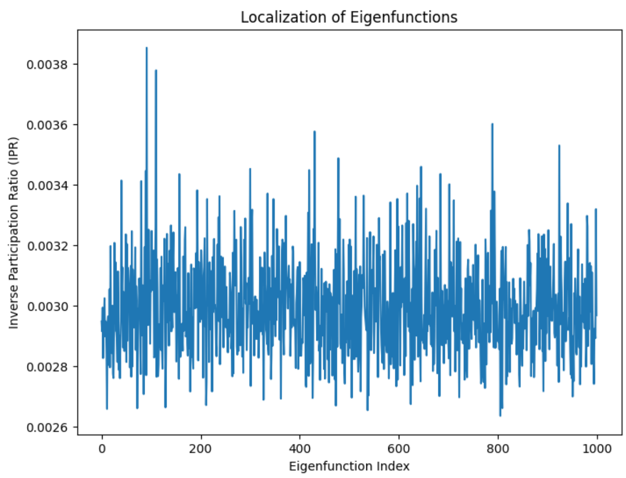

Figure 3. Localization of eigenfunctions (inverse participation ratio) for $\gamma = 0.1$.

The sparse approximation error is computed using randomized SVD for different values of γ. The results indicate that the approximation error remains relatively stable across different values of γ, with a value of approximately 0.9004 for $\gamma = 0.1$.

Table 1 presents a summary of the statistics for the eigenvalue distribution, localization of eigenfunctions, and approximation error for different values of γ.

Table 1. Summary statistics for different values of γ.

γ	Min Eigenvalue	Max Eigenvalue	Mean Eigenvalue	Std Dev Eigenvalue	Mean IPR	Max IPR	Approx Error
0.01	−0.2458	4.2513	1.9998	1.4492	0.0029	0.0048	0.9002
0.10	−4.7895	8.8878	2.0015	3.4637	0.0030	0.0037	0.9003
0.50	−29.5431	33.4538	1.9968	15.8916	0.0030	0.0036	0.9006

The table shows that, as γ increases, the range of eigenvalues expands, indicating a wider spread in the eigenvalue distribution. The mean inverse participation ratio (IPR) remains relatively constant, suggesting that the overall localization of eigenfunctions does not change significantly with γ. The approximation error is also stable across different values of γ, indicating that the RSVD method provides a consistent level of accuracy in approximating the stochastic mimetic operator.

The evaluation of stochastic mimetic operators demonstrates their robustness in handling uncertainties characterized by different values of γ. The eigenvalue distribution and localization of eigenfunctions provide insights into the stability and sensitivity of the system. The consistent approximation error achieved by RSVD highlights the effectiveness of sparse approximation techniques in reducing computational complexity while maintaining accuracy.

6.2. Statistical Analysis of Beam Displacement under Stochastic Fluid Force

In this experiment, we perform a statistical analysis of the displacement of a beam under the influence of a stochastic fluid force. The beam is modeled as a one-dimensional elastic structure with a length of 10 m, a cross-sectional area of 0.01 square meters, and a material density of 2700 kg/m^3. The Young's modulus of the beam material is set to 70 GPa. Uncertainties in the material properties and the fluid force are introduced through random perturbations with a magnitude of 10% of the original values.

The stochastic fluid force is simplified to a uniform load with a mean value of 1000 N and a standard deviation of 100 N. The beam is discretized into 100 elements, and the structural response is solved using the finite element method. The variational problem is defined with stochastic expressions for the density and Young's modulus, and the fluid force is applied as a body load.

Monte Carlo simulations are performed to quantify the uncertainty in the displacement of the beam. A total of 100 simulations are conducted, each with a different realization of the stochastic properties and fluid force. The displacement of the beam is recorded for each simulation, and statistical measures such as the mean, standard deviation, variance, median, and interquartile range are computed.

The results of the statistical analysis are presented in Figure 4. The mean displacement shows the average response of the beam to the stochastic fluid force, while the standard deviation and interquartile range provide information about the variability of the displacement. The median displacement is also plotted to compare with the mean, indicating the central tendency of the displacement distribution.

The analysis demonstrates the capability of the stochastic mimetic framework to capture the uncertainty in the structural response of the beam due to variations in material properties and fluid force. This approach can be extended to more complex FSI problems, such as the analysis of aircraft wings under uncertain operating conditions.

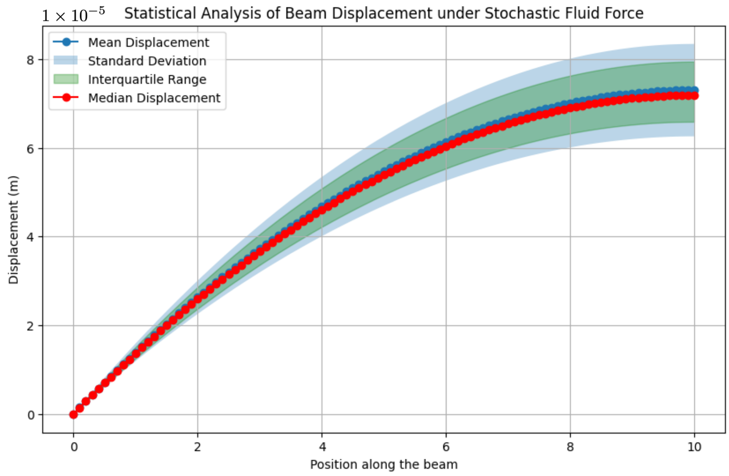

Figure 4. Statistical analysis of beam displacement under stochastic fluid force. The mean displacement is shown with the standard deviation and interquartile range represented as shaded areas. The median displacement is also plotted for comparison.

6.3. Stochastic Analysis of an Aircraft Wing

In this experiment, we extend the stochastic analysis to a more realistic model of an aircraft wing. The wing is modeled as a rectangular beam with a span of 10 m and a chord length of 1 m. The structural domain is discretized using a mesh with 50 elements along the span and 10 elements along the chord. The material properties of the wing, such as density and Young's modulus, are subject to stochastic perturbations to account for uncertainties in the structural parameters.

The aerodynamic lift force acting on the wing is also modeled as a stochastic load, varying linearly along the span of the wing. The lift force is expressed as $1000 \cdot x[0]/\text{length} + \text{perturb\_lift} \cdot \text{rand}$, where $x[0]$ is the spanwise coordinate, length is the span of the wing, perturb_lift is the uncertainty in the lift force, and rand is a random variable representing the stochastic perturbation.

The structural response of the wing under the stochastic aerodynamic lift is solved using the finite element method in FEniCS. The displacement of the wing is computed and visualized in 3D in Figure 5 to illustrate the deformation of the wing under the stochastic load.

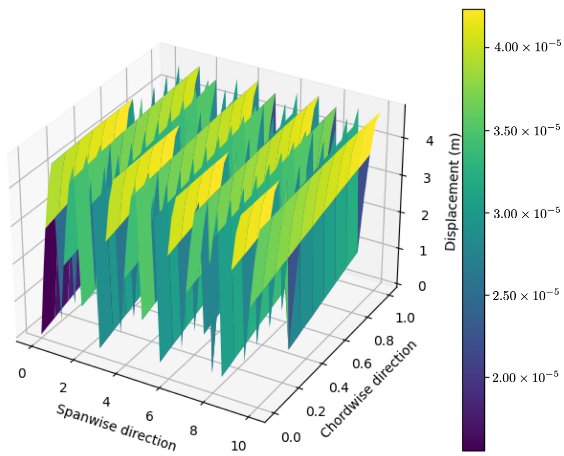

Figure 5. Displacement of an aircraft wing under stochastic aerodynamic lift.

To quantify the uncertainty in the structural response, Monte Carlo simulations are performed. A total of 100 simulations are conducted, with each simulation using a new set of random values for the stochastic properties. The mean and standard deviation of the displacement are computed across all simulations to provide statistical measures of the wing's displacement, as shown in Figure 6.

The results of the Monte Carlo simulations provide valuable insights into the variability of the wing's structural response due to uncertainties in material properties and aerodynamic loads. This stochastic analysis is crucial for designing aircraft wings that are robust and reliable under uncertain operating conditions.

In addition to the uncertainties in material properties and aerodynamic loads, we further enhance the stochastic analysis by introducing randomness in the mimetic operator, which represents the discretization process in the finite element method. This approach captures the variability in the numerical representation of the physical domain, providing a more comprehensive understanding of the uncertainties in the structural response of the wing.

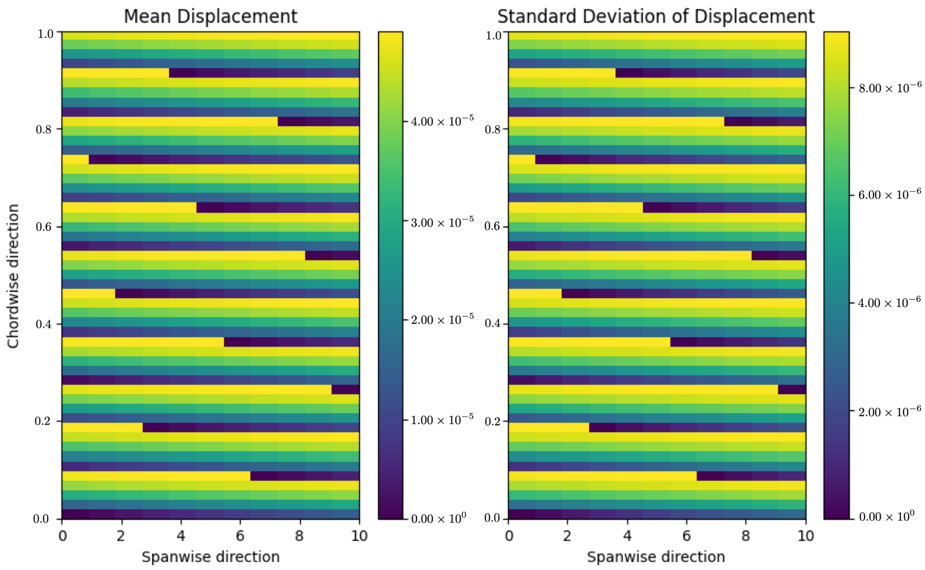

Figure 6. Statistical analysis of the displacement of an aircraft wing under stochastic aerodynamic lift. (**left**) Mean displacement. (**right**) Standard deviation of displacement.

The randomized mimetic operator is implemented by adding a random perturbation term to the stiffness matrix in the variational problem, expressed as (young_modulus_stochastic ×thickness + perturb_operator × random_normal()) × dot(grad(u), grad(v)) × dx. This term introduces variability in the discretization process, allowing us to investigate the sensitivity of the structural response to changes in the numerical representation of the wing.

The inclusion of randomness in the mimetic operator leads to a greater variability in the displacement results across different simulations. The standard deviation of the displacement provides a measure of this variability, indicating the sensitivity of the structural response to the discretization process. The Monte Carlo simulations with the randomized mimetic operator yield a distribution of displacement results that account for uncertainties in both the physical properties of the wing and the numerical representation of the structural domain.

By incorporating randomness in the mimetic operator, we gain insights into the robustness of the aircraft wing design under a wider range of uncertainties. This approach provides a more comprehensive assessment of the wing's performance and reliability, ensuring that the design is resilient to variations in both physical properties and numerical discretization. The results highlight the importance of considering uncertainties in the discretization process in addition to material properties and external loads for a more accurate and reliable analysis of aircraft wings.

6.4. Analysis

We developed a stochastic operator through the addition of a random perturbation matrix to a deterministic mimetic Laplacian. This approach facilitates the exploration of uncertainty effects on eigenvalue distributions and the localization of eigenfunctions, which are crucial for understanding the stability and robustness of FSI simulations under uncertain conditions.

The experiments employed a matrix size of $n = 1000$ with various levels of stochastic perturbation (γ), assessing eigenvalue distribution, localization via the IPR, and efficiency of RSVD for matrix approximation.

The most important takeaways of this are as follows:

1. Eigenvalue distribution and stability: Our stochastic framework allows for the visualization and analysis of eigenvalue distributions, providing insights into the stability of the system under different perturbation strengths.
2. IPR analysis: We quantified the localization of eigenfunctions, a factor critical in understanding the energy concentration and potential weak spots in structural behavior under stochastic influences.
3. Computational efficiency: By implementing RSVD, we demonstrated that our framework achieves significant computational efficiency, reducing the computational overhead associated with large-scale eigenvalue problems in stochastic settings.

The main differentiating features of our methodology include the following:

- Enhanced computational efficiency: Unlike traditional FSI simulations that scale poorly with increased stochastic dimensions, our method maintains computational feasibility through efficient matrix approximation techniques.
- Robustness in uncertainty quantification: Our approach provides a more detailed and robust quantification of uncertainty impacts on system behavior, surpassing traditional methods that often neglect or oversimplify these aspects.

Experimental Methodology

We introduced a stochastic component to the standard mimetic Laplacian matrix, which is conventionally deterministic. This was performed to model the uncertainty inherent in practical applications. The stochastic operator $L_{\text{stochastic}}$ is generated as follows:

$$L_{\text{stochastic}} = L + \gamma R,$$

where L is the deterministic mimetic Laplacian, R represents a matrix of random perturbations drawn from a standard normal distribution, and γ is the perturbation strength parameter. This experiment was conducted using a matrix size of $n = 1000$ and $\gamma = 0.1$.

The eigenvalues of $L_{\text{stochastic}}$ were computed to assess the impact of stochasticity on the spectral properties of the system. The distribution of eigenvalues gives insight into the stability and dynamical behavior of the system under stochastic influences.

The IPR was calculated to investigate the localization properties of the eigenfunctions, which are indicative of how eigenmodes are affected by the random perturbations. The IPR for each eigenfunction is defined as:

$$\text{IPR}(o) = \frac{\sum_{k=1}^{n} |\psi_o(k)|^4}{\left(\sum_{k=1}^{n} |\psi_o(k)|^2\right)^2},$$

where $\psi_o(k)$ denotes the k-th component of the o-th normalized eigenfunction.

To further analyze the stochastic operator, we utilized RSVD for matrix approximation, which is crucial for dimensionality reduction in high-dimensional data analysis. The approximation quality was evaluated using the relative FROBENIUS norm of the error.

The eigenvalue analysis highlighted an increase in the spectral radius due to stochastic perturbations, suggesting a higher system reactivity to inputs. The IPR results indicated increased localization of modes, primarily at the edges of the spectrum. The RSVD demonstrated an effective low-rank approximation with a relative error quantified by:

$$\text{Error} = \frac{\|L_{\text{stochastic}} - L_{\text{approx}}\|_F}{\|L_{\text{stochastic}}\|_F}, \tag{26}$$

where L_{approx} is the low-rank approximation of $L_{\text{stochastic}}$.

We performed a Monte Carlo simulation to study the variability of the eigenvalues and IPR across different realizations of the stochastic operator. This provided a robust statistical framework to understand the effects of random perturbations on the system dynamics.

The experimental results confirm that the stochastic perturbations significantly affect both the spectral properties and the spatial localization of the system. This study lays the groundwork for further investigations into the robust control and stability analysis of stochastic dynamical systems.

7. Discussion

The development and application of the stochastic mimetic framework for FSI analysis of aircraft wings have yielded promising results, demonstrating its potential to enhance the reliability and robustness of aircraft designs under uncertain conditions. In this section, we discuss the key findings, implications, and future directions arising from this study.

7.1. Key Findings

1. Spectral analysis: The investigation of the eigenvalue behavior of the mimetic Laplacian operator under stochastic perturbations has provided valuable insights into the stability of the coupled fluid–structure system. The preservation of key mathematical properties by mimetic operators, even in a stochastic setting, underscores their suitability for FSI analysis.
2. Localization of eigenfunctions: The analysis of eigenfunction localization has revealed the sensitivity of the system to random perturbations. Understanding the spatial distribution of uncertainties through localization properties is crucial for identifying regions of the wing that are most susceptible to variations in fluid and structural parameters.
3. Sparse approximation: The application of random matrix techniques for sparse approximation of the stochastic mimetic operator has demonstrated the potential for computational efficiency gains. This approach enables the handling of large-scale problems by reducing the dimensionality of the operator while retaining its essential characteristics.

The findings of this study have significant implications for the design and analysis of aircraft wings:

1. Robust design: The stochastic mimetic framework allows for the direct integration of uncertainties into the design process, leading to more robust and reliable wing designs that can withstand a wide range of operating conditions.
2. Uncertainty quantification: The framework provides a systematic approach for quantifying uncertainties in FSI analysis, enabling the assessment of the impact of variability in material properties, fluid dynamics, and structural behavior on the overall performance of the wing.
3. Computational efficiency: The use of sparse approximation techniques enhances the computational efficiency of simulations, making it feasible to perform extensive uncertainty analyses and optimizations within reasonable time frames.

Several avenues for future research emerge from this study:

1. Extension to other applications: The stochastic mimetic framework can be extended to other FSI problems beyond aircraft wings, such as wind turbine blades, marine structures, and bio-inspired systems, where uncertainties play a crucial role in their performance.

2. Integration with machine learning: Machine learning techniques can be combined with the stochastic mimetic framework to develop data-driven models for predicting the behavior of fluid–structure systems under uncertainty, further enhancing the efficiency and accuracy of simulations.
3. Advanced preconditioning strategies: The development of more sophisticated preconditioning methods tailored for the stochastic mimetic operator can improve the convergence of iterative solvers, especially for highly ill-conditioned systems.
4. Experimental validation: Experimental studies can be conducted to validate the predictions of the stochastic mimetic framework, providing a benchmark for its accuracy and reliability in real-world applications.

7.2. Novelty and Significance of the Stochastic Mimetic Framework

The proposed stochastic mimetic framework represents a significant advancement in the field of computational fluid dynamics and structural mechanics, particularly in the context of FSI analysis for aerospace applications. This subsection highlights the novel aspects of this research and delineates how it extends beyond existing methods documented in the literature.

- Integration of stochastic analysis with mimetic discretization methods:

  Novelty: The primary novelty of this research lies in its integration of stochastic elements directly into mimetic discretization methods, while mimetic methods are well regarded for their ability to preserve the geometric and physical properties of differential operators, their combination with stochastic analysis to handle uncertainties in material properties and operational conditions is largely unexplored.
  Beyond the literature: Traditional approaches in FSI typically involve deterministic simulations that may not adequately account for the inherent uncertainties present in real-world scenarios. Previous research has focused separately on enhancing the accuracy of deterministic mimetic methods or on applying stochastic methods to conventional finite element or finite volume frameworks. This research bridges this gap by embedding stochastic perturbations within the mimetic framework, providing a more robust and realistic modeling approach.

- Robustness and safety in aerospace design:

  Significance: The framework significantly enhances the robustness and safety of designs by enabling the simulation of FSI under a variety of uncertain operational conditions. This is crucial for the aerospace industry where safety and performance are paramount.
  Beyond the literature: Previous efforts often required extensive safety margins to accommodate uncertainties, leading to potentially overly conservative designs. The stochastic mimetic framework allows for more nuanced, tailored safety margins based on probabilistic analyses, leading to optimal material use and weight reductions without compromising safety.

- Advanced uncertainty quantification (UQ) techniques:

  Novelty: The application of advanced UQ techniques in the context of mimetic discretization is novel, particularly the use of stochastic GALERKIN methods and polynomial chaos expansions adapted to the mimetic framework.
  Beyond the literature: While UQ techniques are well-established in computational science, their application in mimetic methods is not widely documented. This research not only adapts these techniques to the mimetic framework but also tailors them to the specific challenges and structures of FSI problems in aerospace engineering.

- Computational efficiency and large-scale applications:

  Significance: The framework addresses computational efficiency through the development of novel sparse approximation techniques and preconditioning strategies specifically designed for stochastic systems characterized by high variability.
  Beyond the literature: Existing studies have often struggled with the computational cost associated with stochastic simulations, especially in high dimensions typical of FSI problems. The proposed sparse techniques and preconditioners are designed to effectively handle these large-scale systems, making the simulations more feasible and faster.

In conclusion, the stochastic mimetic framework provides a substantial leap forward in the simulation of fluid–structure interactions under uncertainty. It pushes the boundaries of current computational capabilities in aerospace engineering, offering a methodologically rich, robust, and efficient tool for tackling the complex challenges of designing the next generation of aerospace structures.

8. Conclusions

In this study, we have presented a stochastic mimetic framework for the fluid–structure interaction (FSI) analysis of aircraft wings, aimed at addressing the challenges posed by uncertainties in operating conditions, material properties, and fluid dynamics. The framework leverages the spectral properties of mimetic finite-difference operators and integrates stochastic analysis techniques to provide a comprehensive tool for uncertainty quantification and robust design optimization.

Our findings demonstrate the potential of the stochastic mimetic approach in enhancing the reliability and robustness of aircraft wing designs. The spectral and localization analysis of the stochastic mimetic operator offers insights into the stability and sensitivity of the coupled system, while sparse approximation techniques contribute to computational efficiency.

Future research directions include extending the framework to other FSI applications, integrating machine learning for data-driven modeling, developing advanced preconditioning strategies, and conducting experimental validation. The stochastic mimetic framework holds promise for advancing aerospace engineering and other fields where uncertainty plays a critical role in the design and analysis of complex systems.

Overall, this study contributes to the development of rigorous and scalable numerical methods for FSI analysis, paving the way for safer and more efficient aircraft in the presence of uncertainties.

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Abbreviations

The following abbreviations are used in this manuscript:

fluid–structure interaction	FSI
machine learning	ML
inverse participation ratio	IPR
interpolative decomposition	ID
singular value decomposition	SVD
orthogonal matching pursuit	OMP
conjugate gradient	CG
generalized minimal residual	GMRES
uncertainty quantification	UQ

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