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Totally Positive Wronskian Matrices and Symmetric Functions

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Abstract: The elements of the bidiagonal decomposition (BD) of a totally positive (TP) collocation matrix can be expressed in terms of symmetric functions of the nodes. Making use of this result, and studying the relation between Wronskian and collocation matrices of a given TP basis of functions, we can express the entries of the BD of Wronskian matrices as the values of certain symmetric functions evaluated at a single node. Moreover, in the case of polynomial bases, we obtain compact formulae for the entries of the BD of their Wronskian matrices. Interesting examples illustrate the applications of the obtained formulae.

Keywords: Wronskian matrices; collocation matrices; bidiagonal decompositions; Schur functions; totally positive matrices

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1. Introduction

The class of totally positive matrices possesses rich mathematical properties and numerous applications. It has been extensively studied (cf. [1–6]), and attracts significant interest across various fields of mathematics, including approximation theory, combinatorics, computer-aided geometric design, and economics. Let us recall that a matrix is TP if all its minors are nonnegative, and strictly totally positive (STP) if all its minors are positive.

Different characterizations of the total positivity property of a matrix exist in terms of the sign of certain collections of its minors. They imply a reduction in the number of determinants that need to be analyzed to determine whether a matrix is TP or STP. For example, a matrix is STP if and only if all its minors with consecutive rows and columns are positive (see [7,8]). This characterization may be refined, as pointed out in [4], where it is shown that only minors with consecutive rows and columns that include either the first row or the first column need to be checked. These minors are usually referred to as initial minors. The above observations significantly reduce the complexity of the tests for STP matrices. For later use in this paper, we quote this fundamental result in the next theorem.

Theorem 1 (Theorem 4.1 of [4])**.** *Let A be a real $l \times n$ matrix. Then, A is STP if and only if every initial minor of A is positive.*

Considering the Cauchy–Binet formula for determinants, it was established that the product of the TP matrices results in another TP matrix (see Theorem 3.1 of [1]). This property opened the possibility of factorizing TP matrices into products of simpler TP matrices. This endeavor has been the focus of extensive literature [9–14]. In this regard, the most definite fact is that a nonsingular TP matrix always admits a bidiagonal decomposition that can be fully determined by its initial minors [6].

In [5,6], it is shown that a nonsingular TP matrix $A \in \mathbb{R}^{n\times n}$ can be written as

$$A = F_{n-1} \cdots F_1 D G_1 G_2 \cdots G_{n-1}, \tag{1}$$

where D is a diagonal matrix $D = \operatorname{diag}(p_{i,i})_{1\le i\le n}$ whose diagonal entries are called pivots, and $F_i \in \mathbb{R}^{n\times n}$ (respectively, $G_i \in \mathbb{R}^{n\times n}$), $i = 1,\ldots,n-1$, are the TP, lower (respectively, upper) triangular bidiagonal matrices of the following form:

$$F_i = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & m_{i+1,1} & 1 & & \\ & & & \ddots & \ddots & \\ & & & & m_{n,n-i} & 1 \end{pmatrix}, \quad G_i^T = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & \widetilde{m}_{i+1,1} & 1 & & \\ & & & \ddots & \ddots & \\ & & & & \widetilde{m}_{n,n-i} & 1 \end{pmatrix}.$$

The off-diagonal entries $m_{i,j}$ are usually called multipliers, and satisfy $m_{i,j} = p_{i,j}/p_{i-1,j}$, with

$$p_{i,1} = a_{i,1},\ 1\le i\le n, \quad p_{i,j} = \frac{\det A[i-j+1,\ldots,i \mid 1,\ldots,j]}{\det A[i-j+1,\ldots,i-1 \mid 1,\ldots,j-1]},\ 1 < j \le i \le n, \quad (2)$$

where $A[i_1,\ldots,i_r|j_1,\ldots,j_s]$ denotes the submatrix from A formed with rows $i_1,\ldots,i_r$ and columns $j_1,\ldots,j_s$. In the following, we shall denote $A[i_1,\ldots,i_r] := A[i_1,\ldots,i_r|i_1,\ldots,i_r]$. Similarly, $\widetilde{m}_{i,j} = \widetilde{p}_{i,j}/\widetilde{p}_{i-1,j}$, where $\widetilde{p}_{i,1} = a_{1,i}$ and $\widetilde{p}_{i,j}$ can be obtained by means of the quotient of minors in (2) for the transposition A^t of the matrix A.

The bidiagonal decomposition (BD) (1) of nonsingular TP matrices is important for theoretical reasons and practical numerical applications, ensuring great accuracy in linear algebra computations involving matrices whose BD can be computed to high relative accuracy (HRA). There is rich literature delivering algorithms for resolving algebraic problems related to TP matrices to HRA through their BD (cf. [9–15]).

Given a basis $F = (f_1, f_2, \ldots, f_n)$ of functions defined on an interval $I \subseteq \mathbb{R}$ and a set of parameters $X = \{x_1, x_2, \ldots, x_n\}$ with $x_1 < x_2 < \cdots < x_n$ within I, the collocation matrix of the basis F at X is defined as:

$$M_F(X) := \big(f_j(x_i)\big)_{1\le i,j\le n}. \quad (3)$$

The basis F is TP (STP) if, for any $X = \{x_1, x_2, \ldots, x_n\}$ with $x_1 < x_2 < \cdots < x_n$, the matrix $M_F(X)$ is TP (STP).

In this paper, we will show that the collocation matrices $M_F(X)$ connect the realm of total positivity and the field of symmetric functions. This relation happens because the initial minors of a collocation matrix

$$\det M_F(X)[i-j+1,\ldots,i \mid 1,\ldots,j], \quad \det M_F(X)^t[i-j+1,\ldots,i \mid 1,\ldots,j] \quad (4)$$

are antisymmetric functions of the variables $x_1, x_2, \ldots, x_n$ and, in turn, can be expressed as the product of a Vandermonde determinant and a symmetric function of $x_1,\ldots,x_n$. Consequently, each initial minor of an $n\times n$ collocation matrix can be computed by evaluating a symmetric function. An interesting problem is finding the n^2 symmetric functions that encode the BD of the collocation matrices (3).

As a precedent, it was found in [16] that, surprisingly, the elements of the BD decomposition of the Cauchy-polynomial–Vandermonde matrices could be expressed in terms of Schur functions. Later, a systematic line of research was initiated in [15], where explicit formulas in terms of Schur polynomials were provided for the initial minors of collocation matrices of arbitrary polynomial bases. As a result, the straightforward computation of the BD for any collocation matrix of a polynomial basis can be achieved by evaluating the Schur polynomials at the nodes. The formulas provided in [15] were also used to determine the maximal interval, for which the polynomial bases are STP. Moreover, the techniques developed in [15] can be extended to any non-polynomial basis of functions, provided that the corresponding symmetric functions associated with the initial minors of the collocation matrices can be identified.

Intriguingly, the connection between TP bases and symmetric functions may be extended to various mathematical objects, provided that they are related to collocation matrices through certain limits. They include Wronskian matrices, which are the main focus of this work. In this paper, the initial minors of Wronskian matrices will be expressed as the limit of finite differences of certain collocation matrices. This observation allows us to apply the findings of [15], and derive a concise formula for the initial minors of Wronskian matrices.

The paper is organized as follows. Section 2 introduces the necessary concepts to make the article as self-contained as possible. Section 3 shows that any minor with consecutive rows of a Wronskian matrix can be expressed in terms of the limits of determinants of collocation matrices at equally spaced nodes. As a consequence, conditions guaranteeing the total positivity of Wronskian matrices are derived. Moreover, their minors with consecutive rows and columns are expressed in terms of symmetric functions. In particular, for the polynomial case, these minors are written in terms of Schur polynomials in Section 4. The applicability of the achieved formula is illustrated in Section 5 for Bernstein bases and recursive polynomial bases, such as Jacobi, Laguerre, Hermite, and Bessel bases. We conclude with an appendix containing the pseudocode of an algorithm for the computation of the minors (4) in the case of polynomial bases.

2. Preliminary Results

Let us consider the monomial polynomial basis $(m_1, \dots, m_n)$ with

$$m_i(x) := x^{i-1}, \quad i = 1, \dots, n.$$

The corresponding collocation matrix at $X = \{x_1, x_2, \dots, x_n\}$ is the well-known Vandermonde matrix

$$V_X = V_{x_1,\dots,x_n} := \begin{pmatrix} 1 & x_1 & \dots & x_1^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & x_n & \dots & x_n^{n-1} \end{pmatrix},$$

and its determinant satisfies

$$\det V_X = \prod_{i<j} (x_j - x_i).$$

Given $h > 0$, and the equally spaced sequence

$$x_k := x + (k-1)h, \quad k = 1, \dots, n,$$

the Vandermonde determinant satisfies

$$\det V_X = h^{\frac{n(n-1)}{2}} \prod_{r=1}^{n-1} r!.$$

This result can be extended to initial minors of V_X as

$$\det V_{x_k, x_{k+1}, \dots, x_l} = h^{\frac{(l-k)(l-k+1)}{2}} \prod_{r=1}^{l-k} r!. \tag{5}$$

Vandermonde matrices are STP if $0 < x_1 < \cdots < x_n$. Thus, we can say that the monomial polynomial basis is STP on the interval $(0, \infty)$.

Now, let us recall that $f(x_1, x_2, \dots, x_n)$ is a symmetric function if

$$f(x_1, x_2, \dots, x_n) = f(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)}),$$

for any permutation σ of the indices $\{1,2,\ldots,n\}$. On the other hand, $f(x_1,x_2,\ldots,x_n)$ is antisymmetric if

$$f(x_{\sigma(1)},x_{\sigma(2)},\ldots,x_{\sigma(n)}) = \mathrm{sgn}(\sigma)f(x_1,x_2,\ldots,x_n),$$

where $\mathrm{sgn}(\sigma)$ is the signature of σ, taking the value $+1$ if σ is even and -1 if σ is odd.

Note that any minor of the collocation matrix $M_F(X)$ can be considered as an antisymmetric function of the nodes involved. Moreover, since the Vandermonde determinant is nonzero for different values of the nodes, it is always possible to express any minor of the collocation matrix as the product of a Vandermonde determinant and a symmetric function.

Given a partition $\lambda := (\lambda_1,\lambda_2,\ldots,\lambda_p)$ of size $|\lambda| := \lambda_1+\cdots+\lambda_p$ and length $l(\lambda) := p$, such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p > 0$, the Jacobi definition of the corresponding Schur polynomial in n variables is, via the Weyl's formula,

$$s_\lambda(x_1,x_2,\ldots,x_n) := \frac{\det(x_i^{\lambda_j+n-j})_{1\leq i,j\leq n}}{\det(x_i^{n-j})_{1\leq i,j\leq n}},$$

and, by convention, $s_\emptyset(x_1,x_2,\ldots,x_n) := 1$ for the empty partition $\emptyset$.

Schur polynomials are symmetric functions in their arguments. In addition, we now list other well-known properties that will be used in the following sections:

(i) $s_\lambda(x_1,\ldots,x_n) > 0$ for positive values of x_i, $i=1,\ldots,n$. Additionally,

$$s_\lambda(\underbrace{1,\ldots,1}_{j}) = D_{\lambda,j}, \tag{6}$$

with

$$D_{\lambda,j} = \prod_{1\leq k<l\leq j} \frac{\lambda_k-\lambda_l+l-k}{l-k} \geq 0. \tag{7}$$

(ii) $s_\lambda(x_1,\ldots,x_n) = 0$ if $l(\lambda) > n$.

(iii) $s_\lambda(x_1,\ldots,x_n)$ is a homogeneous function of degree $|\lambda|$, i.e.,

$$s_\lambda(\alpha\, x_1,\alpha\, x_2,\ldots,\alpha\, x_n) = \alpha^{|\lambda|}s_\lambda(x_1,x_2,\ldots,x_n).$$

(iv) As running over all the partitions of size $|\lambda|$, the corresponding Schur polynomials provide a basis for the space of symmetric homogeneous polynomials of degree $|\lambda|$. When considering all partitions, Schur polynomials furnish a basis of symmetric functions.

For more details, interested readers are referred to [17].

We finish this section by defining some symmetric functions that will be used in the following:

$$\begin{aligned} g_{F,i,j}(x_{i-j+1},\ldots,x_i) &:= \tfrac{\det M_F(X)[i-j+1,\ldots,i|1,\ldots,j]}{\det V_{x_{i-j+1},\ldots,x_i}}, \quad 1\leq j\leq i\leq n,\\ g_{F,j,i}(x_1,\ldots,x_j) &:= \tfrac{\det M_F^t(X)[i-j+1,\ldots,i|1,\ldots,j]}{\det V_{x_1,\ldots,x_j}}, \quad 1\leq j< i\leq n. \end{aligned} \tag{8}$$

3. Initial Minors and Total Positivity of Wronskian Matrices

Let $F=(f_1,f_2,\ldots,f_n)$ be a system of C^{n-1} functions on I. The Wronskian matrix of F at $x\in I$ is defined as:

$$W_F(x) := \begin{pmatrix} f_1(x) & f_2(x) & \cdots & f_n(x)\\ f_1'(x) & f_2'(x) & \cdots & f_n'(x)\\ \vdots & \vdots & \ddots & \vdots\\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{pmatrix},$$

where $f'(x)$ and $f^{(k)}(x)$, $k = 2, \dots, n-1$, denote the first and k-th derivative of f at x.

For a given $F = (f_1, \dots, f_n)$, we shall denote

$$F_0 := F, \quad F_m := (f_1^{(m)}, f_2^{(m)}, \dots, f_n^{(m)}), \quad m = 1, \dots, n-1. \tag{9}$$

Note that the system F_m needs to not be a basis, since it is not guaranteed that the functions are linearly independent.

Let us recall that the forward finite-difference approximation of the derivative of a function f at $x = x_0$ is given by:

$$f'(x_0) \approx \frac{f(x_0 + h) - f(x_0)}{h},$$

where $h > 0$ is a small step size. Then, we define the forward finite-difference of f as:

$$\Delta_h f(x) := f(x + h) - f(x),$$

and recursively, for $n \in \mathbb{N}$, the higher-order difference of f:

$$\Delta_h^n f(x) := \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} f(x + ih).$$

The relationship of the higher-order differences with the respective derivatives is straightforward, and can be expressed as follows:

$$f^{(n)}(x) = \lim_{h \to 0} \frac{1}{h^n} \Delta_h^n f(x). \tag{10}$$

The next result demonstrates that the determinant of a Wronskian matrix can be expressed as a limit of collocation matrices.

Proposition 1. *Let $F = (f_1, f_2, \dots, f_n)$ be a basis of C^{n-1} functions on $I \subseteq \mathbb{R}$, and $W_F(x)$ be the Wronskian matrix of F at $x \in I$. Then, we have*

$$\det W_F(x) = \lim_{h \to 0^+} h^{n(1-n)/2} \begin{vmatrix} f_1(x_1) & f_2(x_1) & \dots & f_n(x_1) \\ f_1(x_2) & f_2(x_2) & \dots & f_n(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ f_1(x_n) & f_2(x_n) & \dots & f_n(x_n) \end{vmatrix}, \tag{11}$$

for $x_k := x + (k-1)h$, $k = 1, \dots, n$.

Proof. By applying elementary properties of determinants, we can express

$$\begin{vmatrix} f_1(x_1) & \dots & f_n(x_1) \\ f_1(x_2) & \dots & f_n(x_2) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \dots & f_n(x_n) \end{vmatrix} = \begin{vmatrix} f_1(x_1) & \dots & f_n(x_1) \\ \Delta_h f_1(x_1) & \dots & \Delta_h f_n(x_1) \\ \vdots & \ddots & \vdots \\ \Delta_h^{n-1} f_1(x_1) & \dots & \Delta_h^{n-1} f_n(x_1) \end{vmatrix},$$

and then,

$$\begin{vmatrix} f_1(x_1) & \dots & f_n(x_1) \\ \frac{1}{h}\Delta_h f_1(x_1) & \dots & \frac{1}{h}\Delta_h f_n(x_1) \\ \vdots & \ddots & \vdots \\ \frac{1}{h^{n-1}}\Delta_h^{n-1} f_1(x_1) & \dots & \frac{1}{h^{n-1}}\Delta_h^{n-1} f_n(x_1) \end{vmatrix} = h^{n(1-n)/2} \begin{vmatrix} f_1(x_1) & \dots & f_n(x_1) \\ f_1(x_2) & \dots & f_n(x_2) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \dots & f_n(x_n) \end{vmatrix}. \tag{12}$$

The Formula (11) follows as we take the limit $h \to 0$ in (12) and consider (10). □

With similar reasoning, the previous result can be extended to any minor of $W_F(x)$ with consecutive rows.

Proposition 2. *Let $F = (f_1, f_2, \dots, f_n)$ be the basis of C^{n-1} functions on $I \subseteq \mathbb{R}$, and $W_F(x)$ be the Wronskian matrix of F at $x \in I$. Then, for any $1 \le j \le i \le n$ and $1 \le \alpha_1 < \cdots < \alpha_j \le n$,*

$$\det W_F(x)[i-j+1,\dots,i|\alpha_1,\dots,\alpha_j] = \lim_{h\to 0^+} h^{j(1-j)/2} \begin{vmatrix} f_{\alpha_1}^{(i-j)}(x_1) & \dots & f_{\alpha_j}^{(i-j)}(x_1) \\ f_{\alpha_1}^{(i-j)}(x_2) & \dots & f_{\alpha_j}^{(i-j)}(x_2) \\ \vdots & \ddots & \vdots \\ f_{\alpha_1}^{(i-j)}(x_j) & \dots & f_{\alpha_j}^{(i-j)}(x_j) \end{vmatrix}, \quad (13)$$

where $x_k := x + (k-1)h$, $k = 1, \dots, n$.

Using the previous result, conditions to guarantee the total positivity property of a Wronskian matrix can be derived.

Theorem 2. *Let $F = (f_1, f_2, \dots, f_n)$ be the basis of C^{n-1} functions on (a,b) and F_k; $k = 1, \dots, n-1$; the systems are defined in (9). If the Wronskian matrix $W_F(x)$ at $x \in (a,b)$ is nonsingular, and F_k is TP on (a,b), for $k = 0, \dots, n-1$, then $W_F(x)$ is TP at $x \in (a,b)$.*

Proof. According to Theorem 2.3 of [18], we only have to check that all minors of $W_F(x)$ with consecutive rows are nonnegative. Let us consider $(x-\delta, x+\delta) \subset (a,b)$ and $h < \frac{\delta}{j-1}$. The condition that F_{i-j} is TP on (a,b) for $i - j = 0, \dots, n-1$ implies that, for any pair (i,j) such that $1 \le j \le i \le n$, the determinants in (13) at the nodes $x_k = x + (k-1)h$ are nonnegative, and remain so when $h \to 0^+$. □

Now, we shall express the initial minors of $W_F(x)$ in terms of symmetric functions.

Theorem 3. *Let $F = (f_1, f_2, \dots, f_n)$ be a basis of C^{n-1} functions on $I \subseteq \mathbb{R}$, and $W_F(x)$ be the Wronskian matrix of F at $x \in I$. Then,*

$$\det W_F(x)[i-j+1,\dots,i|1,\dots,j] = g_{F_{i-j},j,j}(x,\dots,x) \prod_{r=1}^{j-1} r!,$$
$$\det W_F^t(x)[i-j+1,\dots,i|1,\dots,j] = g_{F,j,i}(x,\dots,x) \prod_{r=1}^{j-1} r!, \quad (14)$$

where $g_{F_{i-j},j,j}$ and $g_{F,j,i}$, $1 \le j \le i \le n$, are the symmetric functions in (8).

Proof. By applying (8) in Formula (13) for the sequence $\alpha_k = k$, and taking the limit $h \to 0^+$, we deduce that

$$\begin{aligned} &\det W_F(x)[i-j+1,\dots,i|1,\dots,j] \\ &= \lim_{h\to 0^+} h^{\frac{j(1-j)}{2}} \begin{vmatrix} f_1^{(i-j)}(x_1) & \dots & f_j^{(i-j)}(x_1) \\ f_1^{(i-j)}(x_2) & \dots & f_j^{(i-j)}(x_2) \\ \vdots & \ddots & \vdots \\ f_1^{(i-j)}(x_j) & \dots & f_j^{(i-j)}(x_j) \end{vmatrix} = \lim_{h\to 0^+} h^{\frac{j(1-j)}{2}} \det M_{F_{i-j}}(X)[1,\dots,j] \\ &= \lim_{h\to 0^+} h^{\frac{j(1-j)}{2}} |V_{x_1,\dots,x_j}| g_{F_{i-j},j,j}(x_1,\dots,x_j) = \prod_{r=1}^{j-1} r! \; g_{F_{i-j},j,j}(x,\dots,x), \end{aligned}$$

where in the last equality, we have used (5). Similarly,

$$
\begin{aligned}
&\det W_F^t(x)[i-j+1,\ldots,i|1,\ldots,j] \\
&= \begin{vmatrix} f_{i-j+1}(x) & \ldots & f_{i-j+1}^{(j-1)}(x) \\ f_{i-j+2}(x) & \ldots & f_{i-j+2}^{(j-1)}(x) \\ \vdots & \ddots & \vdots \\ f_i(x) & \ldots & f_i^{(j-1)}(x) \end{vmatrix} = \lim_{h\to 0^+} h^{\frac{j(1-j)}{2}} \begin{vmatrix} f_{i-j+1}(x_1) & \ldots & f_{i-j+1}(x_j) \\ f_{i-j+2}(x_1) & \ldots & f_{i-j+2}(x_j) \\ \vdots & \ddots & \vdots \\ f_i(x_1) & \ldots & f_i(x_j) \end{vmatrix} \\
&= \lim_{h\to 0^+} h^{\frac{j(1-j)}{2}} \det M_F^t(X)[i-j+1,\ldots,i|1,\ldots,j] = \prod_{r=1}^{j-1} r!\, g_{F,j,i}(x,\ldots,x).
\end{aligned}
$$

□

Starting from this point, the discussion will be narrowed to the polynomial basis.

4. Initial Minors and Total Positivity of Wronskian Matrices for Polynomial Bases

Let $\mathbf{P}^n(I)$ be the space of polynomials of a degree not greater than n, defined on $I \subseteq \mathbb{R}$, and $P = (p_1,\ldots,p_n)$ is a basis of $\mathbf{P}^{n-1}(I)$, such that

$$p_i(x) = \sum_{j=1}^{n} a_{i,j}\, x^{j-1}, \quad x \in I, \quad i = 1,\ldots,n.$$

Let $A = (a_{i,j})_{1\le i,j\le n}$ be the change of the basis matrix from the monomial basis $M = (m_1,\ldots,m_n)$, with $m_i(x) = x^{i-1}$; $i = 1,\ldots,n$, satisfying

$$(p_1,p_2,\ldots,p_n)^T = A(m_1,m_2,\ldots,m_n)^T.$$

The first derivatives of P form a system $P_1 = (p_1',p_2',\ldots,p_n')$ of polynomials of a degree not greater than $n-2$ are given by

$$p_i'(x) = \sum_{j=1}^{n-1} (A_1)_{i,j}\, x^{j-1}, \quad x \in I, \quad i = 1,\ldots,n,$$

where

$$(A_1)_{i,j} := \begin{cases} j\, a_{i,j+1}, & 1 \le i \le n, \quad 1 \le j \le n-1, \\ 0, & 1 \le i \le n, \quad j = n. \end{cases} \tag{15}$$

Higher derivative sets $P_m = (p_1^{(m)}, p_2^{(m)},\ldots,p_n^{(m)})$ can be defined in a similar fashion from the matrix A,

$$p_i^{(m)}(x) = \sum_{j=1}^{n-m} \frac{(j+m-1)!}{(j-1)!}\, a_{i,j+m}\, x^{j-1} = \sum_{j=1}^{n-m} (A_m)_{i,j}\, x^{j-1}, \quad x \in I, \quad i = 1,\ldots,n,$$

with

$$(A_m)_{i,j} := \begin{cases} \frac{(j+m-1)!}{(j-1)!}\, a_{i,j+m}, & 1 \le i \le n, \quad 1 \le j \le n-m, \\ 0, & 1 \le i \le n, \quad 1 \le j = n-m+1,\ldots,n. \end{cases} \tag{16}$$

In [15], it was shown how to express the BD of the collocation matrix $M_P(x)$ in terms of Schur polynomials and some minors of A. Specifically,

$$
\begin{aligned}
&\det M_P(x)[i-j+1,\ldots,i\,|\,1,\ldots,j] \\
&= |V_{x_{i-j+1},\ldots,x_i}| \sum_{l_1>\cdots>l_j} \det A[1,\ldots,j\,|\,l_j,\ldots,l_1]\, s_{(l_1-j,\ldots,l_j-1)}(x_{i-j+1},\ldots,x_i).
\end{aligned} \tag{17}
$$

Schur polynomials are naturally labeled by partitions, and their product rule and other properties are easily stated in terms of them. Thus, for operativeness, it is convenient to write the linear combination appearing in (17) in terms of partitions. We shall consider the partitions $\lambda = (\lambda_1, \ldots, \lambda_j)$, where $\lambda_r = l_r + r - j - 1$, for $r = 1, \ldots, j$. Note that, since $l_1 > \cdots > l_j$, λ is a well-defined partition. So, for the minors of A, we shall use the notation

$$A_{[i,\lambda]} := \det A[i-j+1, \ldots, i \,|\, l_j, \ldots, l_1] = \det A[i-j+1, \ldots, i \,|\, \lambda_j + 1, \ldots, \lambda_1 + j]. \quad (18)$$

Be aware that as the dummy variables satisfy $l_1 > \cdots > l_j$ and $l_k \le n$, for $1 \le k \le n$, the partitions they correspond to will have j parts of maximal length $n-j$ each. In other words, the sum in (17) must be over all Young diagrams that fit in a $j \times (n-j)$ rectangular box. This fact can be expressed as

$$l(\lambda) \le j, \qquad \lambda_1 \le n-j.$$

Taking into account this notation, we can write

$$\sum_{l_j < \cdots < l_1} \det A[1, \ldots, j \,|\, l_j, \ldots, l_1] s_{(l_1-j, \ldots, l_j-1)}(x_{i-j+1}, \ldots, x_i) = \sum_{\substack{l(\lambda) \le j \\ \lambda_1 \le n-j}} A_{[j,\lambda]} s_\lambda(x_{i-j+1}, \ldots, x_i),$$

and

$$\det M_P(x)[i-j+1, \ldots, i \,|\, 1, \ldots, j] = |V_{x_{i-j+1}, \ldots, x_i}| \sum_{\substack{l(\lambda) \le j \\ \lambda_1 \le n-j}} A_{[j,\lambda]} s_\lambda(x_{i-j+1}, \ldots, x_i).$$

For polynomial functions, the symmetric functions $g_{P_{i-j},j,j}$ and $g_{P,j,i}$ are

$$\begin{aligned} g_{P_{i-j},j,j}(x_1, \ldots, x_j) &= \sum_{\substack{l(\lambda) \le j \\ \lambda_1 \le n-j}} A_{i-j\,[j,\lambda]} s_\lambda(x_1, \ldots, x_j), \\ g_{P,j,i}(x_1, \ldots, x_j) &= \sum_{\substack{l(\lambda) \le j \\ \lambda_1 \le n-j}} A_{[i,\lambda]} s_\lambda(x_1, \ldots, x_j) \end{aligned} \quad (19)$$

Using (19), it is possible to find compact formulae for the initial minors of the Wronskian matrix $W_P(x)$. They are shown in the next result.

Theorem 4. *For $1 \le j \le i \le n$, we have*

$$\det W_P(x)[i-j+1, \ldots, i|1, \ldots, j] = \prod_{r=1}^{j-1} r! \sum_{\substack{l(\lambda) \le j \\ \lambda_1 \le n-j}} A_{i-j\,[j,\lambda]} D_{\lambda,j}\, x^{|\lambda|}, \quad (20)$$

$$\det W_P^t(x)[i-j+1, \ldots, i|1, \ldots, j] = \prod_{r=1}^{j-1} r! \sum_{\substack{l(\lambda) \le j \\ \lambda_1 \le n-j}} A_{[i,\lambda]} D_{\lambda,j}\, x^{|\lambda|}, \quad (21)$$

where $D_{\lambda,j}$ are the values defined in (7).

Proof. First, we have to evaluate (19) at a single node. The value of a Schur function at a single node can be found by the use of

$$s_\lambda(\underbrace{x, \ldots, x}_{j}) = x^{|\lambda|} s_\lambda(\underbrace{1, \ldots, 1}_{j}) = x^{|\lambda|} D_{\lambda,j}.$$

The symmetric functions in (19), evaluated at a single node, take the values

$$g_{P_{i-j},j,j}(x,\ldots,x) = \sum_{\substack{l(\lambda)\le j\\ \lambda_1\le n-j}} A_{i-j\,[j,\lambda]} x^{|\lambda|} D_{\lambda,j},$$
$$g_{P,j,i}(x,\ldots,x) = \sum_{\substack{l(\lambda)\le j\\ \lambda_1\le n-j}} A_{[i,\lambda]} x^{|\lambda|} D_{\lambda,j}. \tag{22}$$

Finally, by inserting the above functions into (14), we obtain the expressions for the minors of the Wronskian matrices. □

Using (20) and (21), we can find sufficient conditions for the total positivity of the Wronskian matrix $W_P(x)$.

Theorem 5. *Let A be the matrix of change of the basis from the monomial basis of a polynomial basis P and $J = (J_{i,j})_{1\le i,j\le n}$, with $J_{i,j} := (-1)^{j-1}\delta_{i,j}$.*

(i) If A is TP then $W_P(x)$ is TP for $x \in (0,\infty)$.
(ii) If JAJ is TP then $JW_P(x)J$ is TP for $x \in (-\infty,0)$.

Proof. (i) If A is TP, the matrices A_m in (16) are also TP. Now, since all the minors of A_m and the quantities $D_{\lambda,j}$ in (7) are non-negative, from (20) and (21), the initial minors of $W_P(x)$ are non-negative for $x \in (0,\infty)$. (ii) Let us note that for any matrix M and partition $\lambda = (\lambda_1,\ldots,\lambda_j)$, we have

$$\det JMJ[i-j+1,\ldots,i|1+\lambda_j,\ldots,j+\lambda_1]$$
$$= (-1)^{j(i-1)+|\lambda|} \det M[i-j+1,\ldots,i|1+\lambda_j,\ldots,j+\lambda_1].$$

If JAJ is TP, then for $x \in (0,\infty)$, we have

$$\begin{aligned} 0 &\le \prod_{r=1}^{j-1} r! \sum_{\substack{l(\lambda)\le j\\ \lambda_1\le n-j}} JAJ_{[i,\lambda]} D_{\lambda,j}\, x^{|\lambda|} = \prod_{r=1}^{j-1} r! \sum_{\substack{l(\lambda)\le j\\ \lambda_1\le n-j}} (-1)^{j(i-1)+|\lambda|} A_{[i,\lambda]} D_{\lambda,j}\, x^{|\lambda|} \\ &= (-1)^{j(i-1)} \det W_P^t(-x)[i-j+1,\ldots,i|1,\ldots,j] \\ &= \det JW_PJ^t(-x)[i-j+1,\ldots,i|1,\ldots,j]. \end{aligned}$$

Similarly, since $JA_{i-j}J$ is TP, we have

$$0 \le \prod_{r=1}^{j-1} r! \sum_{\substack{l(\lambda)\le j\\ \lambda_1\le n-j}} JA_{i-j}J_{[j,\lambda]} D_{\lambda,j}\, x^{|\lambda|} = \det JW_AJ(-x)[i-j+1,\ldots,i|1,\ldots,j].$$

So, $JW_P(x)J$ is TP at $x \in (-\infty,0)$. □

In Algorithm A1 (see Appendix A), the implementation of the code of Formula (20) is displayed. Note that an algorithm for the implementation of the code of Formula (21) can be obtained similarly.

5. Examples

In order to show the application of Formula (20) for the computation of the initial minors $\det W_P(x)[i-j+1,\ldots,i|1,\ldots,j]$, let us present two interesting examples.

5.1. Wronskian Matrices of the Bernstein Basis

The Bernstein basis of the space $P^{n-1}[0,1]$ of polynomials of degree less than or equal to $n-1$ on the interval $[0,1]$ is $B := (B_1^{n-1}, \dots, B_n^{n-1})$, with

$$B_i^{n-1}(x) := \binom{n-1}{i-1} x^{i-1}(1-x)^{n-i}, \quad x \in [0,1], \quad i = 1, \dots, n. \tag{23}$$

For $n = 4$, $(B_1^3, B_2^3, B_3^3, B_4^3)^T = A(1, x, x^2, x^3)^T$, with

$$A = \begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and the Wronskian matrix of the Bernstein basis (23) is

$$W_B(x) = \begin{pmatrix} (1-x)^3 & 3x(1-x)^2 & 3x^2(1-x) & x^3 \\ -3(1-x)^2 & 9x^2-12x+3 & -9x^2+6x & 3x^2 \\ 6(1-x) & 18x-12 & -18x+6 & 6x \\ -6 & 18 & -18 & 6 \end{pmatrix}.$$

Now, we shall focus on $\det W_B(x)[2,3,4|1,2,3]$ to illustrate our proposed procedure for obtaining the initial minors of $W_B(x)$.

For computing $\det W_B(x)[2,3,4|1,2,3]$, we have to sum in (20) over all partitions λ, such that $l(\lambda) \le 3$ and $\lambda_1 \le 1$. In general, partitions with $l(\lambda) \le j$ and $\lambda_1 \le n-j$ are in one-to-one correspondence with integer non-negative decreasing j-tuplas whose first entry does not exceed $n-j$. This observation is taken into account in the present example, and in the implementation of (20) in Algorithm A1 (see Appendix A). In our example, we must consider all non-negative decreasing three-tuplas with initial entry less or equal to 1. Thus, we will be summing over $\{(1,1,1),(1,1,0),(1,0,0),(0,0,0)\}$. Moreover, from (15) we obtain

$$A_1 = \begin{pmatrix} -3 & 6 & -3 & 0 \\ 3 & -12 & 9 & 0 \\ 0 & 6 & -9 & 0 \\ 0 & 0 & 3 & 0 \end{pmatrix}.$$

From (18), we can write $A_{1[3,\lambda]}$ as follows

$$A_{1[3,(1,1,1)]} = \det A_1[1,1,3|2,3,4] = 0, \quad A_{1[3,(1,1,0)]} = \det A[1,2,3|1,3,4] = 0,$$
$$A_{1[3,(1,0,0)]} = \det A_1[1,2,3|1,2,4] = 0, \quad A_{1[3,(0,0,0)]} = \det A_1[1,2,3|1,2,3] = -54.$$

Taking into account (7), we can derive

$$D_{(1,1,1),3} = 1, \quad D_{(1,1,0),3} = 3, \quad D_{(1,0,0),3} = 3, \quad D_{(0,0,0),3} = 1,$$

and

$$x^{|(1,1,1)|} = x^3, \quad x^{|(1,1,0)|} = x^2, \quad x^{|(1,0,0)|} = x^1, \quad x^{|(0,0,0)|} = x^0.$$

Finally, from (20), we obtain

$$\det W_B(x)[2,3,4|1,2,3] = 1!2!\big(A_{1[3,(1,1,1)]}D_{(1,1,1),3}x^{|(1,1,1)|} + A_{1[3,(1,1,0)]}D_{(1,1,0),3}x^{|(1,1,0)|}$$
$$+ A_{1[3,(1,0,0)]}D_{(1,0,0),3}x^{|(1,0,0)|} + A_{1[3,(0,0,0)]}D_{(0,0,0),3}x^{|(0,0,0)|}\big) = -108.$$

Following the same reasoning, we can obtain any other initial minor.

5.2. Polynomial Recursive Bases

The example of polynomial recursive bases shows that Equations (20) and (21) can be specially useful for the computation of the initial minors, if the structure of the corresponding matrix of the change of the basis permits a systematic computation of its minors with consecutive rows.

Given values $b_1, \dots, b_n$, such that $b_i > 0$, $i = 1, \dots, n$, let us define the polynomial recursive basis $P := (p_1, \dots, p_n)$ as

$$p_i = \sum_{j=1}^{i} b_j \, x^{j-1}, \quad i = 1, \dots, n.$$

The change of the basis matrix such that $(p_1, \dots, p_n)^T = A(m_1, \dots, m_n)^T$, with $m_i(x) := x^{i-1}$, $i = 1, \dots, n$, is a nonsingular lower triangular and TP matrix of the following form:

$$A = \begin{pmatrix} b_1 & 0 & 0 & \dots & 0 \\ b_1 & b_2 & 0 & \dots & 0 \\ b_1 & b_2 & b_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ b_1 & b_2 & b_3 & \dots & b_n \end{pmatrix}. \tag{24}$$

So, the basis $(p_1, \dots, p_n)$ is TP for $x \in [0, \infty)$. Therefore, by virtue of Theorem 5, the Wronskian matrix $W_P(x)$ is TP for $x \in [0, \infty)$.

The matrix A_1 is

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ b_2 & 0 & 0 & \dots & 0 & 0 \\ b_2 & 2b_3 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ b_2 & 2b_3 & 3b_4 & \dots & (n-1)b_n & 0 \end{pmatrix},$$

and, similarly, by using (16), we obtain A_m with $m = 1, \dots, n$.

Let us note that the only non-zero minors of A_m are

$$A_m[i-j+1, \dots, i|\, l, i-j+2-m, \dots, i-m] = C_{i,j,m} \, b_{l+m} \prod_{k=2}^{j} b_{i-j+k},$$

for $l = 1, \dots, i-j-m+1$, where $A_0 := A$ and

$$C_{i,j,m} = (l+m-1)_m \prod_{k=1}^{j-1} (i-j+k)_m, \quad i-j \geq m$$

with the Pochhammer symbol for descending factorials $(j)_m := j(j-1)\cdots(j-m+1)$. Specifically, for the case $m = i-j$, which is relevant in (20), we have

$$A_{i-j,[j,\lambda]} = \det A_{i-j}[1, \dots, j|l, 2, \dots, j], \quad l = 1.$$

So, the only partition which contributes to (20) is $\lambda = \emptyset$, and

$$\det W_P(x)[i-j+1, \dots, i|1, \dots, j] = \prod_{r=1}^{j-1} r! \prod_{k=1}^{j} b_{i-j+k},$$

which does not depend on x.

In order to compute (21), let us note that, because of (24), the only contribution to the sum in the RHS of (21) comes from the partitions $\lambda = (\lambda_1, \ldots, \lambda_j)$ with

$$\begin{aligned} \lambda_r &= i-j, \quad r = 1, \ldots, j-1, \\ \lambda_j &= 0, 1, \ldots, i-j, \end{aligned}$$

Let us call

$$\lambda_l := (\underbrace{i-j, \ldots, i-j}_{j-1}, l-1), \quad l = 1, \ldots, i-j+1. \tag{25}$$

Applied on these partitions, Formula (7) reduces to

$$D_{\lambda_l, j} = \binom{i-l}{j-1}.$$

Thus, finally, we obtain

$$\det W_P^t(x)[i-j+1, \ldots, i|1, \ldots, j] = \prod_{r=1}^{j-1} r! \prod_{k=2}^{j} b_{i-j+k} \sum_{l=1}^{i-j+1} b_l \binom{i-l}{j-1} x^{(i-j)(j-1)+l-1}.$$

6. Conclusions

The elements of the bidiagonal decomposition of a totally positive collocation matrix can be expressed in terms of symmetric functions of the nodes. By examining the relationship between the Wronskian and collocation matrices of a given TP basis of functions, we have developed a method to express the entries of the BD of Wronskian matrices. These entries can be calculated as the values of certain symmetric functions evaluated at a single node. Furthermore, in the case of polynomial bases, we have derived explicit formulas for the entries of the BD of their Wronskian matrices, facilitating a deeper understanding and simpler computation in applications involving TP matrices.

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Appendix A

In Algorithm A1, the implementation of the code of Formula (20) is displayed.

Algorithm A1: Pseudocode for the computation of Formula (20)

Require: i,j,A
Ensure: $\det W_A(x)[i-j+1, \ldots, i|1, \ldots, j]$

```
n = size(A,1);
m = i − j;
Am = Am(A,i,j);
parts = particiones(j,n);
suma = 0;
  for k = 1:size(parts,1)
     lambda = parts(k,:);
```

Algorithm A1: *Cont.*

```
        a = Alambda(Am, j, lambda);
        d = Dlambdaj(lambda,3);
        suma = suma + a*d*x^sum(lambda);
    end
prod = 1;
    for r = 1:j−1
        prod = prod * factorial(r);
    end
total = prod * suma

function part = partitionsR(from, level)
part = [];
    for value = from: −1:0
        if level > 1
            res = partitionsR(min(from,value),level-1);
            part = [part; [value .∗ ones(size(res,1),1) res]];
        else
            part = [part; value];
        end
    end

function partitions = partitions(j, n)
partitions = partitionsR(n−j,j);

function Alambda = Alambda(A, j, lambda)
rows = 1:j;
cols = flip(lambda) + (1:j);
Alambda = det(A(rows,cols));

function Am = Am(A,i,j)
m = i-j;
n = size(A,1);
r = (m+1):n;
Am = [A(:,r), zeros(n,m)];
    for j=1:n-m
        factorj = factorial(j + m−1)/factorial(j−1);
        Am(:,j) = Am(:,j).*factorj;
    end

function d = Dlambdaj(lambda,j)
d = 1;
    for kl=nchoosek(1:j,j−1)'
        k = kl(1);
        l = kl(2);
        factor = (lambda(k) − lambda(l) + l − k)/(l − k);
        d = d * factor;
    end
```

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