Nonperturbative determination of the $\mathcal{N}=1$ supersymmetric Yang-Mills gluino condensate at large N

Claudio Bonanno,[1,*] Pietro Butti,[2,†] Margarita García Pérez,[1,‡] Antonio González-Arroyo,[1,3,§] Ken-Ichi Ishikawa,[4,5,‖] and Masanori Okawa[5,¶]

[1]*Instituto de Física Teórica UAM-CSIC, c/ Nicolás Cabrera 13-15, Universidad Autónoma de Madrid, Cantoblanco, E-28049 Madrid, Spain*
[2]*Departamento de Física Téorica, Facultad de Ciencias and Centro de Astropartículas y Física de Altas Energías (CAPA), Universidad de Zaragoza, Calle Pedro Cerbuna 12, E-50009, Zaragoza, Spain*
[3]*Departamento de Física Teórica, Universidad Autónoma de Madrid, Módulo 15, Cantoblanco, E-28049 Madrid, Spain*
[4]*Core of Research for the Energetic Universe, Graduate School of Advanced Science and Engineering, Hiroshima University, Higashi-Hiroshima, Hiroshima 739-8526, Japan*
[5]*Graduate School of Advanced Science and Engineering, Hiroshima University, Higashi-Hiroshima, Hiroshima 739-8526, Japan*

(Received 24 June 2024; accepted 12 September 2024; published 10 October 2024)

We present the first nonperturbative large N calculation of the $\mathcal{N}=1$ supersymmetric SU(N) Yang-Mills gluino condensate obtained by means of numerical simulations of the lattice-discretized theory, exploiting large-N twisted volume reduction. We present two different determinations based, respectively, on the Banks-Casher formula and on the Gell-Mann–Oakes–Renner relation, both giving perfectly consistent results. By expressing the lattice results in the Novikov-Shifman-Vainshtein-Zakharov (NSVZ) scheme, we are able for the first time to compare numerical and analytic computations. Our most accurate determination of the renormalization group invariant (RGI) gluino condensate gives $\Sigma_{\mathrm{RGI}}/\Lambda^3_{\mathrm{NSVZ}} = [1.18(08)_{\mathrm{stat}}(12)_{\mathrm{syst}}]^3 = 1.64(33)_{\mathrm{stat}}(50)_{\mathrm{syst}} = 1.64(60)$, in agreement with the N dependence and the value predicted by the weak coupling instanton-based approach $\Sigma_{\mathrm{RGI}}/\Lambda^3_{\mathrm{NSVZ}} = 1$.

DOI: 10.1103/PhysRevD.110.074507

I. INTRODUCTION

The value of the gluino condensate in the $\mathcal{N}=1$ supersymmetric (SUSY) SU(N) Yang-Mills theory has been a subject of debate since the first exact instanton-based calculations were conducted in the early 1980s [1–4]. A comprehensive overview can be found in [5]. At that time, two distinct methodologies were employed to obtain the gluino condensate. In one approach, the so-called strong-coupling (SC) instanton approach [1–3], the gluino condensate was derived from the one-instanton contribution to the $2N$-point function $\langle \mathrm{Tr}\lambda^2(x_1)\cdots\mathrm{Tr}\lambda^2(x_N)\rangle$, a quantity related, assuming clustering, to the object of interest $\langle \mathrm{Tr}\lambda^2\rangle^N$. In contrast, the weak-coupling (WC) instanton approach incorporated additional matter fields and operated within a Higgs phase, enabling a controlled weak coupling calculation of the condensate [4]. By invoking holomorphicity, this approach allowed to decouple the additional matter fields, leading to the value of the condensate in the $\mathcal{N}=1$ SUSY theory.

The two methods yielded two distinct values for the renormalization group invariant (RGI) gluino condensate, exhibiting different leading N dependence in the large-N limit:

$$\Sigma_{\mathrm{RGI}} \equiv \frac{1}{(4\pi)^2 b_0 N}|\langle \mathrm{Tr}\lambda^2\rangle| = \begin{cases} 2e\Lambda^3_{\mathrm{NSVZ}}/N, & \mathrm{SC}, \\ \Lambda^3_{\mathrm{NSVZ}}, & \mathrm{WC}, \end{cases} \tag{1}$$

where Λ_{NSVZ} refers to the Λ parameter in the Novikov-Shifman-Vainshtein-Zakharov (NSVZ) scheme, given at all orders by [1,6]

$$\Lambda^3_{\mathrm{NSVZ}} = \frac{\mu^3}{b_0\lambda_{\mathrm{NSVZ}}(\mu)}\exp\left(\frac{-8\pi^2}{\lambda_{\mathrm{NSVZ}}(\mu)}\right), \tag{2}$$

*Contact author: claudio.bonanno@csic.es
†Contact author: pbutti@unizar.es
‡Contact author: margarita.garcia@csic.es
§Contact author: antonio.gonzalez-arroyo@uam.es
‖Contact author: ishikawa@theo.phys.sci.hiroshima-u.ac.jp
¶Contact author: okawa@hiroshima-u.ac.jp

with $\lambda_{\mathrm{NSVZ}}(\mu)$ denoting the renormalized 't Hooft coupling in this scheme, and $b_0 = 3/(4\pi)^2$, the first universal coefficient of the $\mathcal{N} = 1$ SUSY β function.[1]

An alternative approach to computing the gluino condensate is through the use of fractional instantons, namely, self-dual configurations with fractional topological charge $Q = 1/N$ [8]. In contrast to standard instantons, these allow for direct saturation of the value of the gluino two-point function, circumventing the need for clustering [9,10]. However, it was not until recently that the proportionality constant between the condensate and $\Lambda^3_{\mathrm{NSVZ}}$ was computed using fractional objects. The result obtained in the WC approach has been reproduced using the fractional constituents of finite-temperature calorons [11,12]. In contrast, using the fractional instanton solutions [13,14] on asymmetric tori endowed with 't Hooft's twisted boundary conditions [15] results in a value that, for gauge group SU(2), is 2 times larger than that of WC [16]. This factor of 2 is expected to become a factor of N for SU(N),[2] adding to the controversy surrounding the determination of the condensate.

Given the nonperturbative nature of the gluino condensate, numerical Monte Carlo simulations of the lattice-discretized theory offer a natural framework to compute this quantity from first principles. Despite a significant progress in the past decade concerning SUSY Yang-Mills lattice calculations [18–28] (see also Refs. [29–31] and references therein), determining the gluino condensate has proven to be a highly nontrivial numerical challenge. So far, only a few SU(2) determinations have been obtained [18,19,26,27], all lacking continuum limit, and all missing the matching to the NSVZ scheme, thus preventing comparison with analytic predictions.

In this paper we present the first nonperturbative computation of the gluino condensate at large-N using a lattice-discretized version of the supersymmetric theory. We anticipate that our result, extrapolated to the continuum and zero-gluino-mass limit, and matched to the NSVZ scheme, will be in agreement with the WC prediction, including the N dependence and the numerical coefficient of the condensate.

The paper is organized as follows. In Sec. II we examine the conventions that have led to the definition of Σ_{RGI}, as given in Eq. (1). In Sec. III we present the two different methodologies used to determine the gluino condensate on the lattice. The final results for the gluino condensate are presented in Sec. IV. All technical details concerning the lattice simulations, the determination of the condensate, and of the dynamically generated scale Λ_{NSVZ}, are discussed respectively in Appendixes A, B, and C.

II. FROM THE LATTICE TO THE NSVZ SCHEME

Before presenting our results, it is essential to establish the conventions that underpin the definition of Σ_{RGI}, as given in Eq. (1).[3] It should be noted that, although defined in the NSVZ scheme, Σ_{RGI}, or equivalently $\langle \mathrm{Tr}\lambda^2\rangle$, is a renormalization group invariant and scheme-independent quantity [6,32–34]. The relationship between the quantity $\langle \mathrm{Tr}\lambda^2\rangle$ and the renormalized gluino two-point function in the NSVZ scheme, defined in terms of canonically normalized gluino fields and dependent on the chosen scheme and scale, is given by [6,32]

$$\Sigma_{\mathrm{R}}^{(\mathrm{NSVZ})}(\mu) = \frac{N[1-\lambda_{\mathrm{NSVZ}}(\mu)/(8\pi^2)]}{\lambda_{\mathrm{NSVZ}}(\mu)}|\langle \mathrm{Tr}\lambda^2\rangle|. \tag{3}$$

To obtain a generalization of this expression for any arbitrary renormalization scheme "s", the starting point is the Callan-Symanzik equations, which define the β function and the gluino mass anomalous dimension:

$$\beta_{\mathrm{s}}(\lambda_{\mathrm{s}}) = \frac{\mathrm{d}\lambda_{\mathrm{s}}}{\mathrm{d}\log(\mu^2)}, \qquad \tau_{\mathrm{s}}(\lambda_{\mathrm{s}}) = \frac{\mathrm{d}\log\left(m_{\mathrm{R}}^{(\mathrm{s})}(\mu)\right)}{\mathrm{d}\log(\mu)}. \tag{4}$$

The equation for the mass anomalous dimension can be formally integrated. The result, when combined with the renormalization group invariance of $m_{\mathrm{R}}^{(\mathrm{s})}(\mu)\Sigma_{\mathrm{R}}^{(\mathrm{s})}(\mu)$, can be used to define a renormalization group and scheme-independent condensate [35–38]:

$$\Sigma_{\mathrm{RGI}} = \mathcal{A}\Sigma_{\mathrm{R}}^{(\mathrm{s})}(\mu)[2b_0\lambda_{\mathrm{s}}(\mu)]^{\frac{d_0}{2b_0}} \times \exp\left[\int_0^{\lambda_{\mathrm{s}}(\mu)} dx\left(\frac{\tau_{\mathrm{s}}(x)}{2\beta_{\mathrm{s}}(x)} - \frac{1}{x}\right)\right], \tag{5}$$

where the coefficients d_0, b_0, and b_1 represent the universal terms in the asymptotic expansions of the β and τ functions. For $\mathcal{N} = 1$ supersymmetric Yang-Mills theory, these coefficients take the values $(4\pi)^2 b_0 = 3$, $(4\pi)^4 b_1 = 6$, and $d_0 = 2b_0$. The normalization factor $\mathcal{A}$ can be readily determined by employing the relationship between the exact β function and the mass anomalous dimension in the NSVZ scheme [39]:

$$\frac{\tau_{\mathrm{NSVZ}}(x)}{2\beta_{\mathrm{NSVZ}}(x)} = \frac{1}{x(1-b_1 x/b_0)}. \tag{6}$$

Upon insertion of this expression into Eq. (5), a comparison to Eqs. (1) and (3) yields the result $\mathcal{A} = 8\pi^2/(9N^2)$.

[1]The Λ parameter is defined following the standard QCD convention, trivially mapped to the usual ones in SUSY instanton calculations, see, e.g., Ref. [7].

[2]Recently, a new study [17] from the same authors of [16] appeared, where a solution to this apparent discrepancy was found, resulting in an N dependence in agreement with ours.

[3]A discussion along the lines presented here, but restricted to leading order perturbation theory, was given in Ref. [7].

The final ingredient necessary to match the lattice- and instanton-based determinations of the condensate is the RGI Λ parameter in a conventionally defined scheme such as $\overline{\rm MS}$, which can be reliably determined on the lattice. This dynamically generated scale is defined by formally integrating the Callan-Symanzik equation defining the β function, leading to

$$\Lambda_{\rm s} = \mu[b_0\lambda_{\rm s}(\mu)]^{\frac{-b_1}{2b_0^2}} \exp\left(\frac{-1}{2b_0\lambda_{\rm s}(\mu)}\right) \times \exp\left[-\int_0^{\lambda_{\rm s}(\mu)} dx\left(\frac{1}{2\beta_s(x)}+\frac{1}{2b_0x^2}-\frac{b_1}{2b_0^2x}\right)\right]. \quad (7)$$

A one-loop calculation enables the matching of the NSVZ and $\overline{\rm MS}$ schemes, as expressed by the equation: $\Lambda_{\rm NSVZ} = e^{-1/18}\Lambda_{\overline{\rm MS}}$ [40].

When all the elements are combined, the result, at next-to-leading order, is (see Appendix B)

$$\Sigma_{\rm RGI} = \mathcal{A}2b_0\lambda_{\overline{\rm MS}}(\mu)\left[1+\frac{d_1^{(\overline{\rm MS})}-2b_1}{2b_0}\lambda_{\overline{\rm MS}}(\mu)\right]\Sigma_{\rm R}^{(\overline{\rm MS})}(\mu), \quad (8)$$

where one can rely on the two-loop relation between the renormalized coupling and the Λ parameter, i.e.,

$$2b_0\lambda_{\overline{\rm MS}}(\mu) = -\frac{1}{\log(y)}-\frac{b_1}{2b_0^2}\frac{\log\left[\log(1/y^2)\right]}{\log^2(y)}, \quad (9)$$

with $y \equiv \Lambda_{\overline{\rm MS}}/\mu$. Using these expressions, we will be able to determine the RGI condensate in terms of the lattice determination of $\Sigma_{\rm R}^{(\overline{\rm MS})}(\mu)$.

III. METHODS

Our lattice-based extraction of the renormalized gluino condensate employs two distinct methodologies. The first is a Banks-Casher-based approach, which relates the spectral density of the Dirac operator to the condensate. The second is based on a Gell-Mann–Oakes–Renner-like relation that is applicable in SUSY Yang-Mills theory when endowed with a gluino mass SUSY-breaking term (see below). The two aforementioned methods, employed in SUSY Yang-Mills for the first time in this study, have recently been successfully employed by the authors to determine the quark condensate in large-N Yang-Mills theories [41]. It should be also noted that the spectral method has been extensively used in the QCD literature [41–49].

The Banks-Casher (BC) equation establishes a relationship between the condensate in the massless-gluino limit and the value of the spectral density at the origin. In their seminal paper, Leutwyler and Smilga proposed the following expression for the specific case of $\mathcal{N}=1$ supersymmetric Yang-Mills theory [50]:

$$\frac{\Sigma}{2\pi} = \lim_{\lambda\to 0}\lim_{m\to 0}\lim_{V\to\infty}\rho(\lambda,m), \quad (10)$$

where $i\lambda + m$ stands for a generic eigenvalue of the massive Dirac operator $\not{D}+m$. The additional factor of 2 with respect to the usual BC relation has to do with the Majorana nature of gluinos [50].

Giusti and Lüscher demonstrated that a more convenient quantity for determining the condensate on the lattice is the mode number [42]. This is defined in relation to the spectral density, as follows:

$$\langle\nu(M,m)\rangle = V\int_{-\lambda_M}^{-\lambda_M} d\lambda\rho(\lambda,m), \quad (11)$$

with $\lambda_M = \sqrt{M^2-m^2}$. The integration of Eq. (10) between $-\lambda_M$ and λ_M is straightforward, leading in the chiral and thermodynamic limit to

$$\Sigma \equiv \lim_{m\to 0}\lim_{V\to\infty}\frac{\pi\nu(M,m)}{4V\lambda_M}. \quad (12)$$

The advantage of this formulation is that the mode number is a renormalization group invariant quantity [42], which means that the renormalization properties of Σ are fully dictated by those of λ_M. Further technical details about the practical implementation of the Giusti-Lüscher method and its related lattice renormalization procedure can be found in Appendix B.

The gluino condensate can also be determined from a Gell-Mann–Oakes–Renner-like (GMOR) relation. As is well known, the lattice formulation explicitly breaks supersymmetry. However, Kaplan [51] and Curci and Veneziano [52] demonstrated that the $\mathcal{N}=1$ SUSY Yang-Mills theory could be recovered as the continuum limit of a lattice Yang-Mills theory coupled to a massless adjoint Majorana fermion. In the continuum and chiral limits of such a theory, supersymmetry is recovered as an accidental symmetry. The chiral behavior of the nonsinglet adjoint pion, an unphysical particle not present in the SUSY spectrum, can be used to determine the point where SUSY gets restored. The mass of this adjoint pion is related, at leading order in the chiral expansion, to the gluino mass by a relation analogous to the Gell-Mann–Oakes–Renner one in QCD [20,21]:

$$m_\pi^2 = 2\frac{\Sigma_{\rm R}^{(\rm s)}}{F_\pi^2}m_{\rm R}^{(\rm s)}, \quad (13)$$

where by F_π we denote the adjoint-pion decay constant in the massless gluino limit. By employing this relationship, one can derive a determination of the renormalized gluino condensate, provided that the mass of the adjoint pion and its decay constant, as well as the renormalized gluino mass,

are known on the lattice. In Appendix B we provide all the details concerning the lattice computation of F_π and $m_{\mathrm{R}}^{(\mathrm{s})}$.

It should be noted that the methods described above allow for the determination of $\Sigma_{\mathrm{R}}^{(\mathrm{s})}/Z_{\mathrm{S}}^{(\mathrm{s})}$ and $\Sigma_{\mathrm{R}}^{(\mathrm{s})}/Z_{\mathrm{P}}^{(\mathrm{s})}$. A nonperturbative determination of Z_{S} and Z_{P} represents a highly nontrivial challenge on its own, which falls beyond the scope of this paper. Consequently, we have opted to rely on a nonperturbative determination of $Z_{\mathrm{P}}/Z_{\mathrm{S}}$ based on the spectral methods described in Refs. [41–43,47–49], combined with the two-loop determination of $Z_{\mathrm{S}}^{(\overline{\mathrm{MS}})}(\mu)$ provided in Ref. [53]. In order to keep into account the perturbative renormalization and the two-loop truncation of Eqs. (5) and (7), a 30% systematic error has been added to our final results for the condensate. The size of the systematic error has been chosen on the basis of the typically observed mismatch between two-loop and nonperturbative results for renormalization constants in Yang-Mills theories [41,54–56] (for explicit examples, see Table 6 of [55], where a ∼10% between perturbative and nonperturbative determinations of Z_{A} is reported, and Tables 6 and 7 of [54], where ∼16% and ∼37% are reported for, respectively, Z_{S} and Z_{P} at $N = 3$, $b = 0.334$). This will be our dominant source of uncertainty.

IV. RESULTS

The gluino condensate has been determined using the Monte Carlo ensembles generated in Ref. [57]. The reader is referred to this paper and to Appendix A for all the technical details concerning the simulations. These were performed using the so-called large-N twisted volume reduction [58,59], which was generalized to include the case of adjoint fermions in Ref. [60]. The relationship between space-time and color degrees of freedom that emerges in the large-N limit due to Eguchi-Kawai equivalence [61] enables the large-N theory to be simulated as a model of SU(N) matrices on a lattice with a single space-time point and twisted boundary conditions [15]. This equivalence has been successfully employed to investigate numerous instances of Yang-Mills theories in the large-N limit [41,55,62,63], as well as several non-supersymmetric theories with Dirac fermions in the adjoint representation [64]. This technique was applied in Ref. [57] to simulate $\mathcal{N} = 1$ supersymmetric Yang-Mills theory, generating configurations with dynamical massive gluinos at several values of the inverse bare 't Hooft coupling ($b = 1/\lambda_{\mathrm{L}} = 0.34,\ 0.345,\ 0.35$ are used here), and for several values of N (169, 289, and 361). In accordance with the strategy proposed by the DESY-Jena-Regensburg-Münster collaboration [23–25], we employed Wilson fermions and adopted the well-known reweighting method to address the computation of the Pfaffian, which is not positive definite for this particular fermion discretization. This entailed incorporating the Pfaffian sign in the observable and sampling the Monte Carlo configurations with its modulus. In [57] we explicitly computed the sign of the Pfaffian on the ensembles used for this study, and found it was always positive, thus no reweighting was needed. In [57] we also determined the lattice spacing in physical units, a prerequisite for relating any lattice-based measurement to a common physical standard. To this end, we employed the hadronic reference scale $\sqrt{8t_0}$, based on the renormalization group flow of the gauge action density, which has emerged as the standard for lattice computations [65], and which will be used subsequently in the estimation of the Λ parameter.

Our determination of the dynamically generated scale of the theory, Λ, relies on the use of asymptotic scaling, which has proven to be highly reliable in the case at hand. By making use of the defining equation (7), we determined the value of the Λ parameter relative to the hadronic scale $\sqrt{8t_0}$ through the two-loop expression:

$$\sqrt{8t_0}\Lambda_{\mathrm{s}} = \lim_{a_\chi \to 0} \frac{\sqrt{8t_0}}{a_\chi} \exp\{-f(\lambda_{\mathrm{s}})\},$$
$$f(x) = \frac{1}{2b_0}\left[\frac{1}{x} + \frac{b_1}{b_0}\log(b_0 x)\right], \tag{14}$$

where the momentum scale has been set to $\mu = 1/a_\chi$, with a_χ the lattice spacing in the massless-gluino limit obtained in [57]. It is widely acknowledged that the use of the naive bare coupling, $\lambda_{\mathrm{L}} = 1/b$, is not expected to exhibit good scaling properties. However, it is possible to accelerate the convergence of the perturbative expansion by defining appropriate improved couplings. In the literature, there are several standard choices based on the use of the average plaquette P. In this study, we considered three possibilities: $\lambda_t^{(\mathrm{I})} = 1/(bP)$, $\lambda_t^{(\mathrm{E})} = 8(1-P)$, and $\lambda_t^{(\mathrm{E}')} = -8\log(P)$. Each of these definitions introduces a different lattice scheme, with an associated determination of the Λ parameter. The matching of all these definitions to the $\overline{\mathrm{MS}}$ scheme is known; details on the matching prescriptions are provided in Appendix C. Figure 1 shows the continuum extrapolation leading to the determination of $\sqrt{8t_0}\Lambda_{\mathrm{NSVZ}}$. The values of $a_\chi/\sqrt{8t_0}$ shown in the figure correspond to the chiral (zero-gluino-mass) limit, as obtained in Ref. [57]. As previously noted, asymptotic scaling works remarkably well and leads to

$$\sqrt{8t_0}\Lambda_{\mathrm{NSVZ}} = 0.376(25). \tag{15}$$

For the sake of completeness we also quote $\sqrt{8t_0}\Lambda_{\overline{\mathrm{MS}}} = 0.397(26)$.

We now proceed to the calculation of the gluino condensate. In this section, we will merely present the final results. Further details on the extraction of the condensate using either the BC or GMOR methods can be found in Appendix B. A summary of all of the intermediate numerical results used to extract the condensate in the supersymmetric

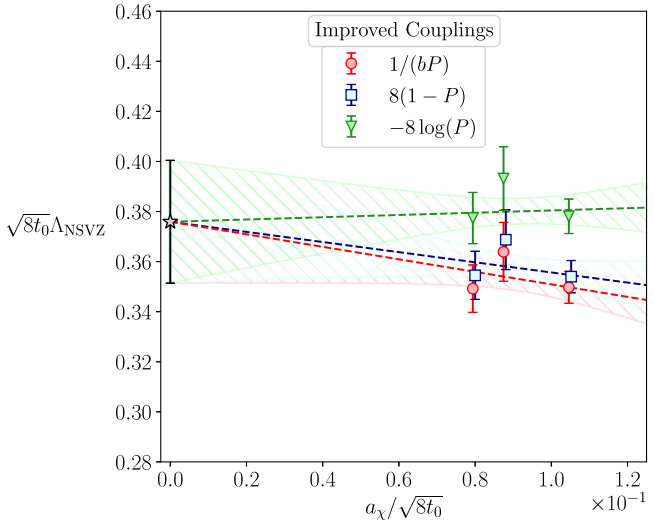

FIG. 1. Determination of $\sqrt{8t_0}\Lambda_{\rm NSVZ}$ (star) from asymptotic scaling with different improved couplings assuming $\mathcal{O}(a_\chi)$ artifacts. Dashed lines and shaded areas represent the joint continuum extrapolation best fit obtained imposing a common continuum limit.

limit is provided in Table II in Appendix B. It should be noted that in order to determine the condensate using the GMOR method, the value of the adjoint-pion decay constant in the SUSY (chiral-continuum) limit was required. This was calculated, resulting in a value

$$\frac{F_\pi}{N\Lambda_{\rm NSVZ}} = 0.092(14). \tag{16}$$

A significant outcome of our analysis is the leading N dependence of the condensate, when written in units of $\Lambda_{\rm NSVZ}$, cf. Eq. (1). The three available values of N, namely $N = 169$, 289, and 361, were analyzed using both the BC and the GMOR methods. In the BC approach, we were able to reliably determine the condensate only for volumes satisfying $m_\pi \ell_{\rm eff} = m_\pi(a\sqrt{N}) \gtrsim 10$. The outcomes at finite gluino mass are presented as functions of the pion mass squared in Fig. 2. There is no observed dependence on the number of colors. This result, in full accordance with the WC instanton-based approach, is clearly in contradiction with the $1/N$ dependence predicted by the SC determination.

Given the excellent agreement among determinations for different values of N, the final numbers for the gluino condensate were obtained through a joint chiral and continuum extrapolation of the results at $N = 361$, collected in Table I and obtained at finite gluino mass and lattice spacing as described in Appendixes B 1 and B 2. The scale is set by fixing $\sqrt{8t_0}\Lambda_{\rm NSVZ}$ to the value given in Eq. (15).

In Fig. 3, displaying the values reported in Table I of $\Sigma^{1/3}_{\rm RGI}/\Lambda_{\rm NSVZ}$ as a function of the lattice spacing in units of $\sqrt{8t_0}$, we perform a simultaneous chiral and continuum

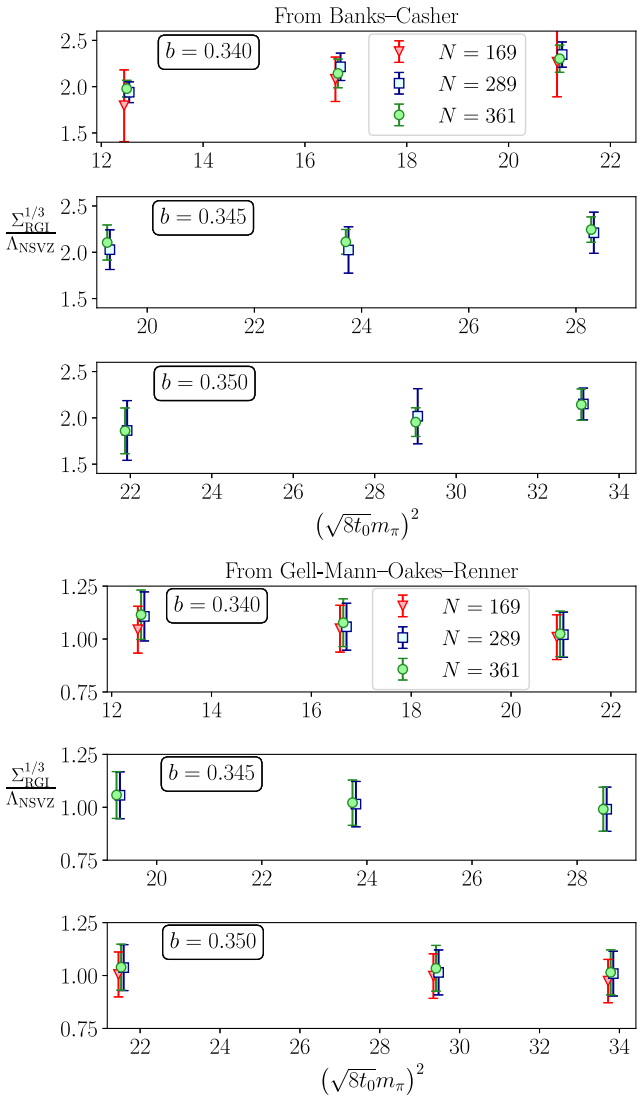

FIG. 2. Behavior of the gluino condensate $\Sigma^{1/3}_{\rm RGI}/\Lambda_{\rm NSVZ}$ as a function of N for several values of the adjoint-pion mass m_π and the bare 't Hooft coupling $1/b$. The x axis refers to the values of $\sqrt{8t_0}m_\pi$ determined for $N = 361$. Points are slightly shifted horizontally for readability.

extrapolation of the gluino condensate assuming $\mathcal{O}(8t_0 m_\pi^2)$ and $\mathcal{O}(a/\sqrt{8t_0})$ corrections, as expected from chiral perturbation theory for Wilson fermions. The dashed lines and shaded areas represent the best fit with the coefficient of the $\mathcal{O}(8t_0 m_\pi^2)$ correction set to zero. The results are compared to those predicted by the WC instanton-based approach, represented by the horizontal line indicating $\Sigma^{1/3}_{\rm RGI}/\Lambda_{\rm NSVZ} = 1$. As can be observed, the BC and the GMOR methods yield results which are perfectly in accordance when extrapolated to the supersymmetric limit:

$$\begin{aligned}\frac{\Sigma_{\rm RGI}}{\Lambda^3_{\rm NSVZ}} &= [1.18(08)_{\rm stat}\,(12)_{\rm syst}]^3 \\ &= 1.64(33)_{\rm stat}(50)_{\rm syst}\end{aligned} \qquad \text{(GMOR)} \tag{17}$$

TABLE I. Summary of the numerical results reported in Fig. 3, and used to extrapolate the gluino condensate towards the SUSY (i.e., chiral-continuum) limit.

b	κ	$\frac{a}{\sqrt{8t_0}}$	$m_\pi\sqrt{8t_0}$	$\Sigma^{1/3}_{\rm RGI}/\Lambda_{\rm NSVZ}$ (BC)	$\Sigma^{1/3}_{\rm RGI}/\Lambda_{\rm NSVZ}$ (GMOR)
0.340	0.1850	0.2132	4.58	2.24(14)	1.02(11)
	0.1890	0.1762	4.08	2.09(15)	1.08(11)
	0.1910	0.1521	3.55	1.93(8)	1.11(12)
	0.1930	0.1172	2.24	⋯	1.20(13)
0.345	0.1800	0.1954	5.34	2.19(13)	0.99(10)
	0.1840	0.1686	4.87	2.06(13)	1.02(11)
	0.1868	0.1438	4.39	2.06(18)	1.06(11)
	0.1896	0.1095	3.23	⋯	1.11(12)
0.350	0.1800	0.1535	5.81	2.09(16)	1.01(11)
	0.1825	0.1361	5.42	1.91(15)	1.03(11)
	0.1850	0.1155	4.64	1.82(24)	1.04(11)
	0.1875	0.0934	3.07	⋯	1.08(11)

$$\frac{\Sigma_{\rm RGI}}{\Lambda^3_{\rm NSVZ}} = [1.24(19)_{\rm stat}(12)_{\rm syst}]^3 = 1.89(87)_{\rm stat}(55)_{\rm syst}. \quad \text{(BC)} \qquad (18)$$

Furthermore, these numbers are in good agreement with the WC determination. Although the errors in our calculation are significant, it is crucial to highlight that this is the first time a numerical determination of this kind has been carried out in the existing literature. Previous calculations of the condensate in four-dimensional supersymmetric Yang-Mills theory [18,19,26,27] were limited to a single lattice spacing and gauge group SU(2), and were never matched to the NSVZ scheme.

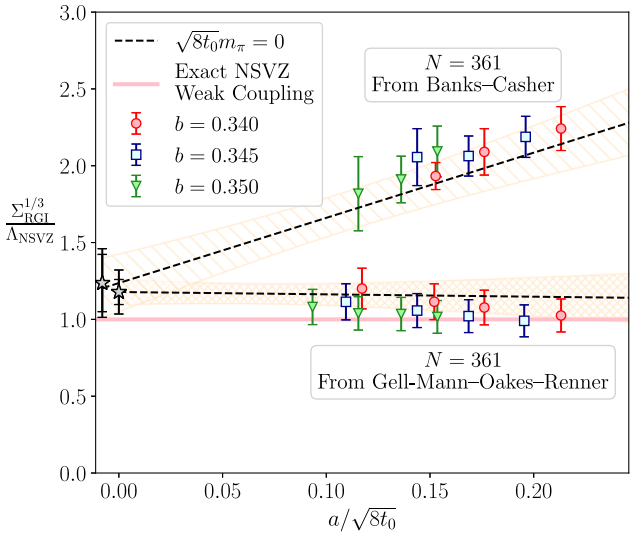

FIG. 3. Chiral-continuum extrapolations of the gluino condensate (stars) obtained from the Banks-Casher and from the Gell-Mann–Oakes–Renner relations, assuming $\mathcal{O}(a)$ and $\mathcal{O}(m_\pi^2)$ corrections. Dashed lines and shaded areas represent the best fit with the coefficient of the $\mathcal{O}(m_\pi^2)$ correction set to zero.

V. CONCLUSIONS

In this paper we have presented the first nonperturbative lattice calculation of the gluino condensate in $\mathcal{N} = 1$ supersymmetric Yang-Mills theory in the large-N limit. We have exploited large-N volume independence, and adopted two independent strategies, giving perfectly agreeing results. After extrapolation to the supersymmetric limit and matching with the NSVZ scheme, our results confirm the N dependence predicted by semiclassical instanton methods based on the so-called weak coupling (WC) approach [4,11,12]. Our most accurate determination of the condensate in units of the dynamically generated scale of the theory is $\Sigma_{\rm RGI}/\Lambda^3_{\rm NSVZ} = [1.18(08)_{\rm stat}(12)_{\rm syst}]^3 = 1.64(33)_{\rm stat}(50)_{\rm syst} = 1.64(60)$, consistent with the expected result from WC. The error budget in our result is largely influenced by the systematic error resulting from the absence of a nonperturbative determination of the renormalization constant Z_S. While addressing this issue would be a valuable avenue for future research, it would also necessitate the commitment to a long-term project that is beyond the scope of the current study.

ACKNOWLEDGMENTS

It is a pleasure to thank José L. F. Barbón, Georg Bergner, Gregorio Herdoíza, Nikolai Husung, Carlos Pena, and Erich Poppitz for useful discussions. This work is partially supported by the Spanish Research Agency (Agencia Estatal de Investigación) through the grant IFT Centro de Excelencia Severo Ochoa CEX2020-001007-S, funded by MCIN/AEI/10.13039/501100011033, and by Grant No. PID2021-127526NB-I00, funded by MCIN/AEI/10.13039/501100011033 and by "ERDF A way of making Europe." We also acknowledge support from the Project No. H2020-MSCAITN-2018-813942 (EuroPLEx) and the EU Horizon 2020 research and innovation program, STRONG-2020 project, under Grant Agreement No. 824093. P. B. is supported by Grant No. PGC2022-126078NB-C21 funded by MCIN/AEI/10.13039/501100011033 and "ERDF A way of making Europe." P. B. also acknowledges support by Grant Diputación General de Aragón-Fondo Social Europeo (DGA-FSE) Grant No. 2020-E21-17R Aragon Government and the European Union—NextGenerationEU Recovery and Resilience Program on "Astrofísica y Física de Altas Energías" CEFCA-CAPA-ITAINNOVA. K. I. I. is supported in part by MEXT as "Feasibility studies for the next-generation computing infrastructure." M. O. is supported by JSPS KAKENHI Grant No. 21K03576. Numerical calculations have been performed on the FINISTERRAE III cluster at CESGA (Centro de Supercomputación de Galicia). We have also used computational resources of Oakbridge-CX at the University of Tokyo

through the HPCI System Research Project (Project ID: hp230021 and hp220011).

APPENDIX A: THE TEK LATTICE FORMULATION OF $\mathcal{N}=1$ LARGE-N SU(N) SUSY YANG-MILLS

Our entire lattice setup, including the gauge configuration ensembles, comes from the one explained and adopted in Ref. [57], which we briefly summarize here.

The twisted Eguchi-Kawai (TEK) model is a $d=4$-matrix model with partition function given by

$$\mathcal{Z}_{\mathrm{TEK}}=\int [dU]\mathrm{Pf}\{CD_{\mathrm{W}}^{(\mathrm{TEK})}[U]\}e^{-S_{\mathrm{TEK}}[U]}, \quad \text{(A1)}$$

with the charge-conjugation operator C satisfying $\gamma_\mu^{\mathrm{t}}C=-\gamma_\mu C$ and $C^{\mathrm{t}}=-C$.

The gauge action, expressed in terms of the gauge link $U_\mu\in\mathrm{SU}(N)$ in the fundamental representation, reads

$$S_{\mathrm{TEK}}[U]=bN\sum_{\mu\neq\nu}\mathrm{Tr}\{\mathrm{I}-z_{\mu\nu}^{*}U_\mu U_\nu U_\mu^{\dagger}U_\nu^{\dagger}\}, \quad \text{(A2)}$$

where $z_{\mu\nu}=e^{i2\pi n_{\mu\nu}/N}$ is the twist factor, with $\sqrt{N}\in\mathbb{N}$, $n_{\mu\nu}=\sqrt{N}k(N)$ for $\nu>\mu$ and $n_{\mu\nu}=-n_{\nu\mu}$ (we used $k=5$, 5, 7 for $N=169$, 289, 361). Instead, the adjoint Wilson-Dirac operator reads

$$D_{\mathrm{W}}^{(\mathrm{TEK})}[U]=\frac{1}{2\kappa}\mathrm{I}-\frac{1}{2}\sum_{\mu}\left[(\mathrm{I}-\gamma_\mu)U_\mu^{(\mathrm{adj})}+(\mathrm{I}+\gamma_\mu)(U_\mu^{(\mathrm{adj})})^{\dagger}\right], \quad \text{(A3)}$$

with κ the Wilson hopping parameter, and $U_\mu^{(\mathrm{adj})}$ denoting the $d=4$ SU(N) gauge fields in the adjoint representation.

Since the Pfaffian of the lattice Wilson-Dirac operator is not guaranteed to be positive, we follow the approach proposed by the DESY-Jena-Regensburg-Münster collaboration [23–25], namely, we resort to sign-quenched simulations, meaning that the sign of the Pfaffian is moved to the observable via standard reweighting:

$$\langle\mathcal{O}\rangle=\frac{\langle\mathcal{O}\mathrm{sign}[\mathrm{Pf}\{CD_{\mathrm{W}}^{(\mathrm{TEK})}[U]\}]\rangle_{\mathrm{sq}}}{\langle\mathrm{sign}[\mathrm{Pf}\{CD_{\mathrm{W}}^{(\mathrm{TEK})}[U]\}]\rangle_{\mathrm{sq}}}. \quad \text{(A4)}$$

The sign-quenched (sq) expectation value is calculated sampling the following functional integral using a standard rational hybrid Monte Carlo algorithm (see Ref. [57]):

$$\mathcal{Z}_{\mathrm{TEK}}^{(\mathrm{sq})}=\int [dU]|\mathrm{Pf}\{CD_{\mathrm{W}}^{(\mathrm{TEK})}[U]\}|e^{-S_{\mathrm{TEK}}[U]}. \quad \text{(A5)}$$

The sign of the Pfaffian on the ensembles used for this study was subjected to explicit analysis in the study presented in Ref. [57]. In light of the results presented in Refs. [66–70], the sign in question can be determined by counting the number of negative real eigenvalues of D_{W}. It is anticipated that, for heavy gluino masses, there will be no flips in the sign of the Pfaffian. Consequently, the analysis focused on the lightest adjoint fermion masses at each value of b and N. No negative-real eigenvalues were identified in the spectrum. Furthermore, this absence was confirmed for a number of heavier fermion masses. Therefore, we concluded that the sign of the Pfaffian is consistently positive for the employed model parameters, thereby simplifying the reweighting method by validating the distribution using the absolute value of the Pfaffian.

APPENDIX B: EXTRACTING THE GLUINO CONDENSATE FROM THE LATTICE

In this Appendix we describe the extraction of the gluino condensate from lattice gauge configurations using the two methodologies described in Sec. III: the spectral method and the GMOR relation. We also describe how to relate the bare quantity obtained from these methods to the RGI condensate. A summary of all intermediate numerical results for the largest value of $N=361$, along with the results of [57] employed in this study, is presented in Table II.

1. Spectral method (Banks-Casher)

The Giusti-Lüscher method is a well-established numerical strategy to compute fermion condensates from lattice simulations based on the low-lying spectrum of the Dirac operator and on the Banks-Casher relation [41–43,47–49]. We summarize it in the following.

We numerically solved the Hermitian eigenvalue problem,

$$\left(\gamma_5 D_{\mathrm{W}}^{(\mathrm{TEK})}\right)u_\lambda=\lambda u_\lambda, \qquad \lambda\in\mathbb{R}, \quad \text{(B1)}$$

for the first 100 low-lying eigenvalues using the ARPACK library in order to compute the mode number $\langle\nu(M)\rangle=\langle\#|\lambda|\leq M\rangle$ as a function of the threshold mass M.

Given that M renormalizes as an eigenvalue ($\lambda=Z_{\mathrm{P}}\lambda_{\mathrm{R}}$), the Giusti-Lüscher method allows to determine the renormalized condensate, up to the renormalization constant Z_{P}, as follows:

$$\frac{\Sigma_{\mathrm{R}}}{Z_{\mathrm{P}}}=\frac{\pi}{4V}\sqrt{1-\left(\frac{m_{\mathrm{R}}}{M_{\mathrm{R}}}\right)^2}S, \quad \text{(B2)}$$

with $V=a^4$ the one-point lattice volume and a the lattice spacing, and with

$$S=\frac{\partial\langle\nu(M)\rangle}{\partial M} \quad \text{(B3)}$$

the slope of the mode number as a function of the threshold mass M, calculated for M close to $M=\langle|\lambda_{\mathrm{min}}|\rangle$ (the

TABLE II. Summary of obtained numerical results for $N = 361$, with the only exceptions of Z_P/Z_S, which was instead calculated for $N = 289$. The quantity t_1 is another hadronic reference scale which was used to set the scale in intermediate calculation. It is related to the more commonly employed reference scale t_0, which we used to display all of our final results in the main text, by the relation $\sqrt{t_0/t_1} \simeq 1.627$ [57]. The results for the critical κ_c, the scale setting, the pion and partially conserved axial current (PCAC) masses, and the plaquette, come from Ref. [57].

b	κ	am_π	am_{PCAC}	$a^3\Sigma_R/(Z_P N^2)$ (Banks-Casher)	$\sqrt{8t_1}/a$	Z_P/Z_S	P (plaquette)	$aF_\pi/(Z_A N)$	κ_c	$\sqrt{8t_1}/a_\chi$
0.340	0.1850	0.977(4)	0.2018(14)	0.00378(69)	2.883(58)	0.642(2)	0.541414(99)	0.1758(16)	0.19359(5)	5.88(11)
	0.1890	0.719(4)	0.1083(7)	0.00213(43)	3.488(88)	0.560(5)	0.549479(88)	0.1317(10)		
	0.1910	0.540(5)	0.0609(9)	0.00129(17)	4.024(58)	0.500(5)	0.55419(10)	0.1006(17)		
	0.1930	0.263(13)	0.0138(6)	⋯	5.244(175)	0.410(12)	0.560145(74)	0.0422(13)		
0.345	0.1800	1.043(6)	0.2352(32)	0.00266(43)	3.134(93)	0.683(4)	0.55072(10)	0.16841(92)	0.19095(6)	7.03(23)
	0.1840	0.821(5)	0.1507(11)	0.00163(28)	3.645(98)	0.625(6)	0.556355(81)	0.13412(87)		
	0.1868	0.631(6)	0.0913(18)	0.00118(31)	4.274(55)	0.565(8)	0.561253(71)	0.1067(13)		
	0.1896	0.353(5)	0.0289(5)	⋯	5.614(201)	0.464(10)	0.567106(91)	0.0581(12)		
0.350	0.1800	0.883(7)	0.1826(16)	0.00126(28)	4.003(109)	0.663(5)	0.564610(70)	0.13560(94)	0.18857(5)	7.74(21)
	0.1825	0.733(6)	0.1242(15)	0.00073(17)	4.516(64)	0.629(8)	0.568020(55)	0.11299(95)		
	0.1850	0.540(6)	0.0712(10)	0.00046(18)	5.323(65)	0.552(8)	0.571769(42)	0.0839(13)		
	0.1875	0.293(6)	0.0195(4)	⋯	6.582(287)	0.472(10)	0.576334(91)	0.0422(11)		

average minimum eigenvalue extrapolated to the infinite N limit). Note that Σ_R/Z_P is RGI, thus we do not need to specify any renormalization scheme/renormalization scale. Examples of mode number best fits to extract the slope are shown in Fig. 4.

Given that the Banks-Casher relation just gives the leading order term of the chiral expansion of the spectral density in powers of m and λ, the linearity of the mode number as a function of M is only valid up to $O(M^2)$ corrections. To avoid contaminations from higher-order terms, we chose the fit range according to the same criteria suggested in the original paper [42], which were also applied in other works addressing the spectral determination of fermion condensates, see, e.g., Refs. [41,49,71].

The lower bound of the fit range was chosen so as to stay close to the threshold $M/\langle|\lambda_{\mathrm{min}}|\rangle = 1$. For Wilson fermions, the lower end of the spectrum is known to deviate from the expected continuum form [42], therefore we actually started the fit range in the linear region a bit above the threshold. Then, we tried several linear best fits increasing the upper bound of the fit range until a good reduced chi square (p value larger than 5%) could be obtained. We also verified that including/excluding one further point from the lower/upper bound of the fit range did not change significantly the obtained slope, i.e., that the contamination coming from higher-order terms in M is negligible. Thus, systematic errors related to the choice of the fit range are negligible compared to our statistical ones.

Given the renormalization property of the so-called "subtracted definition" of the bare gluino mass,

$$am_{\mathrm{sub}} = \frac{1}{2\kappa} - \frac{1}{2\kappa_c} = aZ_S m_R, \tag{B4}$$

with κ_c the critical κ obtained in [57], in order to calculate the factor $m_R/M_R = (Z_P/Z_S)(m_{\mathrm{sub}}/M)$ appearing in Eq. (B2) we need the RGI ratio Z_P/Z_S.

This quantity can be computed from spectral methods using the eigenvectors obtained from the same eigenvalue problem in Eq. (B1) [41–49] as follows:

$$\left(\frac{Z_P}{Z_S}\right)^2 = \frac{\langle s_P(M)\rangle}{\langle \nu(M)\rangle}, \tag{B5}$$

$$s_P(M) \equiv \sum_{|\lambda|,|\lambda'|\leq M} |u_\lambda^\dagger \gamma_5 u_{\lambda'}|^2, \tag{B6}$$

where the ratio in Eq. (B5) is expected to exhibit a plateau as a function of M for sufficiently large values of M.

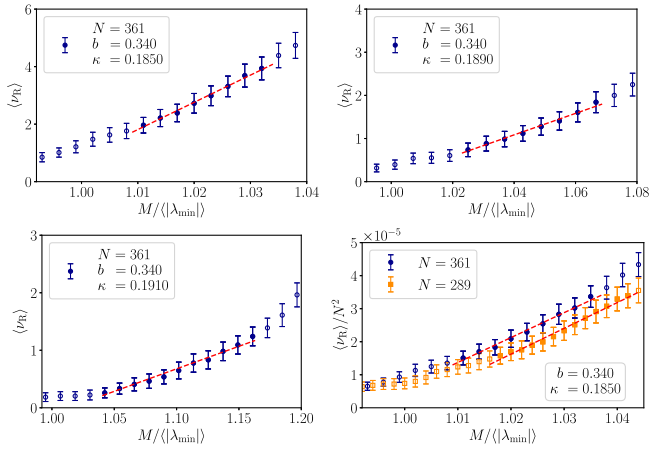

FIG. 4. Examples of calculation of mode number slope for $b = 0.34$, $N = 361$, and three values of the Wilson hopping parameter $\kappa = 0.185, 0.189, 0.191$. In the bottom right plot we compare two different values of N for $b = 0.34$ and $\kappa = 0.185$.

The expected behavior is verified from actual lattice data, see Fig. 5. Since Z_P/Z_S is a ratio of spectral sums, and since it is determined by the plateau at larger values of M, while finite-size effects mostly affect the smallest eigenvalues, finite-volume (i.e., finite N) corrections for this quantity are expected to be largely negligible.

Table II collects the values of $a^3\Sigma_R(a,m_\pi)/(Z_P N^2)$, obtained applying this method for $N=361$, and the values of $Z_P/Z_S(a,m_\pi)$, determined nonperturbatively for $N=289$. These are converted in physical units through

$$\frac{\Sigma_R(a,m_\pi)}{Z_S N^2 \Lambda_{NSVZ}^3} = \left(\frac{\sqrt{8t_0}}{a}\right)^3 \times \frac{a^3\Sigma_R(a,m_\pi)}{Z_P N^2} \times \frac{Z_P}{Z_S}(a,m_\pi) \times \left(\sqrt{8t_0}\Lambda_{NSVZ}\right)^{-3}. \tag{B7}$$

After using the perturbatively determined value of Z_S, determined as described in Appendix B 3, and the procedure to pass from the renormalized to the RGI condensate, again described in Appendix B 3, the determinations reported in Table I are obtained.

2. Pion mass method (Gell-Mann–Oakes–Renner)

Although softly broken SUSY Yang-Mills has just a single massive fermion flavor as its dynamical content, it is possible to consider, in the rigorous framework of partially quenched chiral perturbation theory, the chiral behavior of the adjoint pion, an unphysical particle whose mass is related to the gluino one by a Gell-Mann–Oakes–Renner-like relation, just as in QCD [20,21]. In this context, one finds

$$m_\pi^2 = 2\frac{\Sigma_R}{F_\pi^2} m_R, \tag{B8}$$

where F_π refers to the decay constant of the unphysical adjoint pion, while Σ_R is the physical gluino condensate we wish to calculate. Clearly, both quantities must be understood as computed in the chiral (i.e., SUSY restoring) limit $m_R \to 0$.

From the adjoint-pion mass correlators computed in [57] it is possible to extract both m_π and F_π/Z_A. Once one gets rid of Z_A and isolates F_π, from the knowledge of m_π as a function of the bare gluino mass it is possible to obtain the renormalized gluino condensate, up to the renormalization constant Z_S, as

$$\frac{\Sigma_R}{Z_S} = \frac{m_\pi^2 F_\pi^2}{2m_{sub}}. \tag{B9}$$

To obtain F_π/Z_A, we used standard techniques relying on the calculation of the pion-vacuum matrix element of the temporal component of the axial vector current:

$$\frac{F_\pi}{NZ_A} = \frac{1}{\sqrt{2}Nm_\pi}\langle 0|A_4(x=0)|\pi(\vec{p}=0)\rangle, \tag{B10}$$

where the normalization was chosen on the basis of standard counting arguments, leading to $F_\pi \sim \mathcal{O}(N)$ for adjoint fermions. The details of the generalized eigenvalue problem solved to find the largest-overlapping interpolating pion operator for twisted-reduced models can be found in the Appendixes of Refs. [55,57]. Table II collects the values of $aF_\pi/(NZ_A)$ obtained for $N=361$.

Since F_π needs renormalization, it is useful to recall that from the axial Ward identity it is possible to extract a different definition of the bare quark mass, the so-called PCAC (partially conserved axial current) mass:

$$m_{PCAC} = \frac{Z_P}{Z_A} m_R. \tag{B11}$$

Therefore, using the spectral determination of Z_P/Z_S, and the PCAC mass determinations of [57], the adjoint-pion decay constant, for each value of N, b, and gluino mass, is obtained via

$$\frac{F_\pi}{N} = \left(\frac{F_\pi}{NZ_A}\right) \times \left(\frac{m_{sub}}{m_{PCAC}}\right) \times \left(\frac{Z_P}{Z_S}\right), \tag{B12}$$

as in this combination the factors of $Z_P/(Z_S Z_A)$ exactly cancel out.

The values of the decay constant obtained in this way (normalized to the value of Λ_{NSVZ}) are displayed as a function of the adjoint-pion mass squared, in units of $8t_0$, in Fig. 6. A joint chiral and continuum extrapolation of all the results obtained for the largest $N=361$ is then performed. Assuming $\mathcal{O}(a)$ and $\mathcal{O}(m_\pi^2)$ corrections, we fit to a functional form,

$$\frac{\sqrt{8t_0}F_\pi(a,m_\pi)}{N\sqrt{8t_0}\Lambda_{NSVZ}} = \frac{F_\pi}{N\Lambda_{NSVZ}} + c_1\frac{a}{\sqrt{8t_0}} + c_2 8t_0 m_\pi^2, \tag{B13}$$

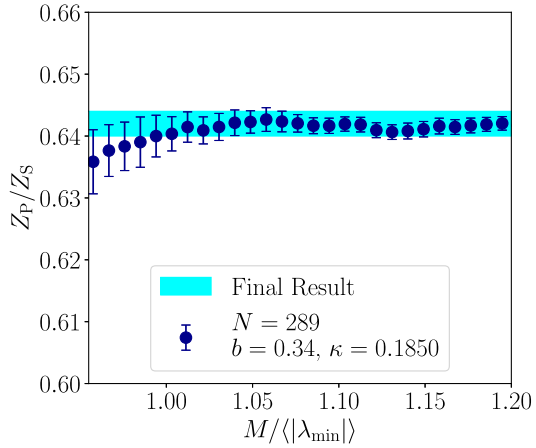

FIG. 5. Example of nonperturbative determination of Z_P/Z_S using the spectral method, Eq. (B5), for $N=289$, $b=0.34$, and $\kappa=0.185$. The shaded area represents our final result for the ratio Z_P/Z_S.

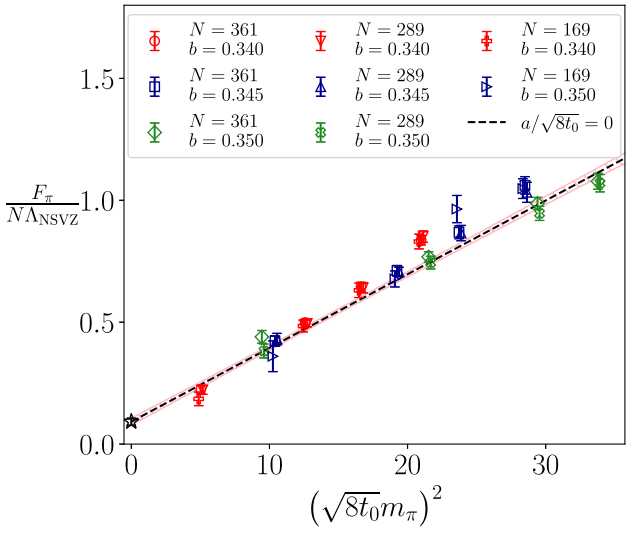

FIG. 6. Chiral-continuum extrapolation (star) of the $N = 361$ results for the adjoint-pion decay constant $F_\pi/(N\Lambda_{\rm NSVZ})$. For comparison we also show results obtained for $N = 169$ and 289. The dashed line and the shaded area represent the chiral extrapolation best fit at the zero-lattice-spacing point. Best fit yields a reduced $\tilde{\chi}^2 \simeq 14.4/9$.

where $\sqrt{8t_0}\Lambda_{\rm NSVZ}$ is set to the value in the SUSY limit, given by Eq. (15), and the values of $\sqrt{8t_0}F_\pi(a, m_\pi)$ can be determined in a straightforward way from the data given in Table II. Higher-order corrections in the chiral expansion appear to be negligible within our numerical precision, as adding a $O(m_\pi^4)$ power to the fit function does not change the obtained result for F_π in the SUSY limit. The dashed line and the shaded area in Fig. 6 represent the pion mass dependence obtained by setting $c_1 = 0$. We obtain a value for the adjoint-pion decay constant in the chiral and continuum limit given by $F_\pi/(N\Lambda_{\rm NSVZ}) = 0.092(14)$.

Using this value of F_π and the values of the adjoint pion and PCAC masses determined in Ref. [57] (compiled in Table II for $N = 361$), it is straightforward to ascertain the values of $\Sigma_{\rm R}/Z_{\rm S}$ through the application of Eq. (B9). To be precise, we determine

$$\frac{\Sigma_{\rm R}(a, m_\pi)}{Z_{\rm S}N^2\Lambda_{\rm NSVZ}^3} = \left(\frac{\sqrt{8t_0}}{a}\right) \times \frac{(am_\pi)^2}{2(am_{\rm sub})} \times \left(\frac{F_\pi}{\sqrt{N}\Lambda_{\rm NSVZ}}\right)^2 \times \frac{1}{\sqrt{8t_0}\Lambda_{\rm NSVZ}}. \tag{B14}$$

The outcomes obtained for the largest $N = 361$, multiplied by the perturbatively determined value of $Z_{\rm S}$, and passed to the RGI definition, cf. Appendix B 3, are summarized in Table I.

3. Renormalization and conversion to RGI definition

The methods described so far allow to determine the bare gluino condensates $\Sigma_{\rm R}/Z_{\rm P}$ and $\Sigma_{\rm R}/Z_{\rm S}$. Given that we are also able to obtain nonperturbative determinations of $Z_{\rm P}/Z_{\rm S}$, in the end we are just left with two independent determinations of, say, $\Sigma_{\rm R}/Z_{\rm S}$. A nonperturbative determination of $Z_{\rm S}$ is not available at present, therefore we have to rely on two-loop perturbation theory to estimate $Z_{\rm S}^{(\overline{\rm MS})}(\mu)$. In terms of the renormalized coupling at scale $\mu = 1/a$, the renormalization constant $Z_{\rm S}$ in the $\overline{\rm MS}$ scheme at two-loop order reads [53]

$$Z_{\rm S}^{(\overline{\rm MS})}\left(\mu = \frac{1}{a}, \lambda_{\overline{\rm MS}}\right) = 1 - \frac{12.9524104(1)}{(4\pi)^2}\lambda_{\overline{\rm MS}} - \frac{60.68(10)}{(4\pi)^4}\lambda_{\overline{\rm MS}}^2 + \mathcal{O}(\lambda_{\overline{\rm MS}}^3). \tag{B15}$$

The renormalized coupling in the $\overline{\rm MS}$ scheme at $\mu = 1/a$ was computed using improved couplings (see Appendix C), and no significant dependence on the choice of the particular improved lattice scheme was observed in the obtained results.

With these ingredients we are able to determine, up to the systematic error involved in the perturbative determination of $Z_{\rm S}$, the value of $\Sigma_{\rm R}^{(\overline{\rm MS})}(\mu = 1/a)$. As explained in the main text, the renormalized condensate $\Sigma_{\rm R}^{(s)}(\mu)$, determined in any arbitrary scheme (s) at an arbitrary renormalization scale μ, can be converted to a scheme and scale-independent RGI quantity as follows [35–38]:

$$\Sigma_{\rm RGI} = \mathcal{A}\Sigma_{\rm R}^{(s)}(\mu)[2b_0\lambda_{\rm s}(\mu)]^{\frac{d_0}{2b_0}} \times \exp\left[\int_0^{\lambda_{\rm s}(\mu)} dx\left(\frac{\tau_{\rm s}(x)}{2\beta_{\rm s}(x)} - \frac{1}{x}\right)\right], \tag{B16}$$

with $\mathcal{A} = 8\pi^2/(9N^2)$ a normalization factor chosen to match the analytic calculation conventions of the NSVZ determination. At two-loop order, expanding β and τ, this relation simplifies as

$$\Sigma_{\rm RGI} = \mathcal{A}2b_0\Sigma_{\rm R}^{(s)}(\mu)\lambda_{\rm s}(\mu)\left[1 + \lambda_{\rm s}(\mu)\left(\frac{d_1^{(s)}}{2b_0} - \frac{b_1}{b_0}\right)\right],$$

where b_0, b_1, and $d_0 = 2b_0$ are universal, and $d_1^{(s)}$ is scheme dependent. In the $\overline{\rm MS}$ scheme at scale $\mu = 1/a$, one obtains

$$\Sigma_{\rm RGI} = \mathcal{A}2b_0\Sigma_{\rm R}^{(\overline{\rm MS})}(\mu = 1/a)\lambda_{\overline{\rm MS}}(\mu = 1/a) \times \left[1 + \frac{d_1^{(\overline{\rm MS})} - 2b_1}{2b_0}\lambda_{\overline{\rm MS}}(\mu = 1/a)\right], \tag{B17}$$

with $d_1^{(\overline{\rm MS})} = 32/(4\pi)^4$ [72].

Given the order of truncation we are working at, we rely on two-loop perturbation theory also to express the 't Hooft

coupling in terms of the dynamically generated scale of the theory [37], which we are able to reliably compute from the lattice (see Appendix C):

$$2b_0\lambda_{\overline{\rm MS}}(\mu = 1/a) = -\frac{1}{\log(a\Lambda_{\overline{\rm MS}})} - \frac{b_1}{2b_0^2}\frac{\log\left[-2\log(a\Lambda_{\overline{\rm MS}})\right]}{\log^2(a\Lambda_{\overline{\rm MS}})}. \quad \text{(B18)}$$

We have checked that truncating Eqs. (B17)–(B18) at one-loop order leads to a change in the final results for the condensate which is well below the 30% systematic error we have assigned to the use of the perturbative formula for the renormalization factor Z_S.

APPENDIX C: THE DYNAMICALLY GENERATED SCALE $\Lambda_{\rm NSVZ}$ FROM THE LATTICE

The dynamically generated scale $\Lambda_{\rm NSVZ}$ can be obtained from the one of the $\overline{\rm MS}$ scheme, which is more amenable to be computed on the lattice, via $\Lambda_{\rm NSVZ} = e^{-1/18}\Lambda_{\overline{\rm MS}}$ [40].

We computed $\Lambda_{\overline{\rm MS}}$ via asymptotic scaling:

$$\sqrt{8t_0}\Lambda_{\overline{\rm MS}} = \lim_{a_\chi\to 0}\frac{\sqrt{8t_0}}{a_\chi}\exp\{-f(\lambda_{\overline{\rm MS}})\},$$
$$f(x) = \frac{1}{2b_0}\left[\frac{1}{x} + \frac{b_1}{b_0}\log(b_0 x)\right], \quad \text{(C1)}$$

with a_χ the lattice spacing extrapolated to the massless-gluino limit, computed in [57].

Instead of matching directly the bare lattice 't Hooft coupling $\lambda_{\rm L} = 1/b$ to the $\overline{\rm MS}$ scheme, it is better first to pass through an intermediate scheme defined by the so-called improved couplings $\lambda_t^{(\rm s)}$, which are related to the bare one by

$$\lambda_{\rm L} = \lambda_t^{(\rm s)} - 2b_0\left(\lambda_t^{(\rm s)}\right)^2\log(\Lambda_{\rm s}/\Lambda_{\rm L}), \quad \text{(C2)}$$

with $\Lambda_{\rm L}$ the Λ parameter related to $\lambda_{\rm L}$.

In this work we have considered three different improved couplings: $\lambda_t^{(\rm I)} = 1/(bP)$, $\lambda_t^{(\rm E)} = 8(1-P)$, and $\lambda_t^{(\rm E')} = -8\log(P)$, where P is the averaged clover plaquette $P \equiv \frac{1}{6N}\sum_{\mu>\nu}\langle{\rm Tr\,Clover}_{\mu\nu}\rangle$. All of these improved couplings define its own renormalization scheme, which can be matched to the bare lattice one via

$$\log\left(\frac{\Lambda_{\rm L}}{\Lambda_{\rm I}}\right) = -\frac{w_1}{2b_0} = \log\left(\frac{\Lambda_{\rm L}}{\Lambda_{\overline{\rm MS}}}\times 2.7373\right), \quad \text{(C3)}$$

$$\log\left(\frac{\Lambda_{\rm L}}{\Lambda_{\rm E}}\right) = -\frac{w_2}{2b_0 w_1} = \log\left(\frac{\Lambda_{\rm L}}{\Lambda_{\overline{\rm MS}}}\times 29.005\right), \quad \text{(C4)}$$

$$\log\left(\frac{\Lambda_{\rm L}}{\Lambda_{\rm E'}}\right) = -\frac{a_1}{2b_0} = \log\left(\frac{\Lambda_{\rm L}}{\Lambda_{\overline{\rm MS}}}\times 5.60\right). \quad \text{(C5)}$$

Here w_1 and w_2 [73,74] are the first two coefficients of the perturbative expansion of the plaquette P:

$$w_1 = \frac{1}{8}, \quad \text{(C6)}$$

$$w_2 = 0.0051069297 - N_{\rm f}0.0013858405(1), \quad \text{(C7)}$$

$$a_1 = \frac{w_2}{w_1} + \frac{w_1}{2}, \quad \text{(C8)}$$

with $N_{\rm f} = 1/2$. Finally, using the known relation between $\Lambda_{\rm L}$ and $\Lambda_{\overline{\rm MS}}$ in a theory with adjoint fermions [73],

$$\frac{\Lambda_{\overline{\rm MS}}}{\Lambda_{\rm L}} = 73.46674161161081, \quad \text{(C9)}$$

and the values of the plaquette determined in Ref. [57] (cf. Table II for $N = 361$), we have all the necessary ingredients to pass from improved lattice couplings to the $\overline{\rm MS}$ one.

The result of applying this exercise to determine $\Lambda_{\rm NSVZ}$ through the three different improved couplings is presented in Fig. 1, in Sec. IV. Asymptotic scaling works remarkably well leading to a value: $\sqrt{8t_0}\Lambda_{\rm NSVZ} = 0.376(25)$.

[1] V. A. Novikov, M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Nucl. Phys. **B229**, 407 (1983).
[2] G. C. Rossi and G. Veneziano, Phys. Lett. **138B**, 195 (1984).
[3] D. Amati, G. C. Rossi, and G. Veneziano, Nucl. Phys. **B249**, 1 (1985).
[4] V. A. Novikov, M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Nucl. Phys. **B260**, 157 (1985).
[5] T. J. Hollowood, V. V. Khoze, W.-J. Lee, and M. P. Mattis, Nucl. Phys. **B570**, 241 (2000).
[6] M. A. Shifman and A. I. Vainshtein, Nucl. Phys. **B277**, 456 (1986).
[7] A. Armoni, M. Shifman, and G. Veneziano, Phys. Lett. B **579**, 384 (2004).
[8] G. 't Hooft, Commun. Math. Phys. **81**, 267 (1981).
[9] E. Cohen and C. Gomez, Phys. Rev. Lett. **52**, 237 (1984).

[10] A. R. Zhitnitsky, Nucl. Phys. **B340**, 56 (1990).
[11] N. M. Davies, T. J. Hollowood, V. V. Khoze, and M. P. Mattis, Nucl. Phys. **B559**, 123 (1999).
[12] N. M. Davies, T. J. Hollowood, and V. V. Khoze, J. Math. Phys. (N.Y.) **44**, 3640 (2003).
[13] M. García Pérez, A. González-Arroyo, and C. Pena, J. High Energy Phys. 09 (2000) 033.
[14] A. González-Arroyo, J. High Energy Phys. 02 (2020) 137.
[15] G. 't Hooft, Nucl. Phys. **B153**, 141 (1979).
[16] M. M. Anber and E. Poppitz, J. High Energy Phys. 01 (2023) 118.
[17] M. M. Anber and E. Poppitz, arXiv:2408.16058.
[18] J. Giedt, R. Brower, S. Catterall, G. T. Fleming, and P. Vranas, Phys. Rev. D **79**, 025015 (2009).
[19] S. W. Kim, H. Fukaya, S. Hashimoto, H. Matsufuru, J. Nishimura, and T. Onogi (JLQCD Collaboration), Proc. Sci., LATTICE2011 (**2011**) 069 [arXiv:1111.2180].
[20] G. Münster and H. Stüwe, Proc. Sci., LATTICE2014 (**2014**) 310 [arXiv:1411.1540].
[21] G. Münster and H. Stüwe, J. High Energy Phys. 05 (2014) 034.
[22] G. Bergner and S. Piemonte, Phys. Rev. D **97**, 074510 (2018).
[23] S. Ali, H. Gerber, I. Montvay, G. Münster, S. Piemonte, P. Scior, and G. Bergner, Eur. Phys. J. C **78**, 404 (2018).
[24] S. Ali, G. Bergner, H. Gerber, P. Giudice, I. Montvay, G. Münster, S. Piemonte, and P. Scior, J. High Energy Phys. 03 (2018) 113.
[25] S. Ali, G. Bergner, H. Gerber, I. Montvay, G. Münster, S. Piemonte, and P. Scior, Phys. Rev. Lett. **122**, 221601 (2019).
[26] G. Bergner, C. López, and S. Piemonte, Phys. Rev. D **100**, 074501 (2019).
[27] S. Piemonte, G. Bergner, and C. López, Phys. Rev. D **102**, 014503 (2020).
[28] G. Bergner, S. Piemonte, and M. Ünsal, J. High Energy Phys. 07 (2024) 048.
[29] G. Bergner and S. Catterall, Int. J. Mod. Phys. A **31**, 1643005 (2016).
[30] G. Bergner, G. Münster, and S. Piemonte, Universe **8**, 617 (2022).
[31] D. Schaich, Eur. Phys. J. Special Topics **232**, 305 (2023).
[32] J. Hisano and M. A. Shifman, Phys. Rev. D **56**, 5475 (1997).
[33] N. Arkani-Hamed and H. Murayama, Phys. Rev. D **57**, 6638 (1998).
[34] N. Arkani-Hamed and H. Murayama, J. High Energy Phys. 06 (2000) 030.
[35] J. Gasser and H. Leutwyler, Phys. Rep. **87**, 77 (1982).
[36] S. Sint and P. Weisz (ALPHA Collaboration), Nucl. Phys. **B545**, 529 (1999).
[37] L. Giusti, F. Rapuano, M. Talevi, and A. Vladikas, Nucl. Phys. **B538**, 249 (1999).
[38] M. Della Morte, R. Hoffmann, F. Knechtli, J. Rolf, R. Sommer, I. Wetzorke, and U. Wolff (ALPHA Collaboration), Nucl. Phys. **B729**, 117 (2005).
[39] I. Jack and D. R. T. Jones, Phys. Lett. B **415**, 383 (1997).
[40] D. Finnell and P. Pouliot, Nucl. Phys. **B453**, 225 (1995).
[41] C. Bonanno, P. Butti, M. García Peréz, A. González-Arroyo, K.-I. Ishikawa, and M. Okawa, J. High Energy Phys. 12 (2023) 034.
[42] L. Giusti and M. Lüscher, J. High Energy Phys. 03 (2009) 013.
[43] M. Lüscher and F. Palombi, J. High Energy Phys. 09 (2010) 110.
[44] K. Cichy, E. Garcia-Ramos, and K. Jansen, J. High Energy Phys. 10 (2013) 175.
[45] K. Cichy, E. Garcia-Ramos, and K. Jansen (ETM Collaboration), J. High Energy Phys. 02 (2014) 119.
[46] K. Cichy, E. Garcia-Ramos, K. Jansen, K. Ottnad, and C. Urbach (ETM), J. High Energy Phys. 09 (2015) 020.
[47] C. Bonanno, G. Clemente, M. D'Elia, and F. Sanfilippo, J. High Energy Phys. 10 (**2019**) 187.
[48] A. Athenodorou, C. Bonanno, C. Bonati, G. Clemente, F. D'Angelo, M. D'Elia, L. Maio, G. Martinelli, F. Sanfilippo, and A. Todaro, J. High Energy Phys. 10 (2022) 197.
[49] C. Bonanno, F. D'Angelo, and M. D'Elia, J. High Energy Phys. 11 (2023) 013.
[50] H. Leutwyler and A. V. Smilga, Phys. Rev. D **46**, 5607 (1992).
[51] D. B. Kaplan, Phys. Lett. **136B**, 162 (1984).
[52] G. Curci and G. Veneziano, Nucl. Phys. **B292**, 555 (1987).
[53] A. Skouroupathis and H. Panagopoulos, Phys. Rev. D **76**, 094514 (2007); **78**, 119901(E) (2008).
[54] G. S. Bali, F. Bursa, L. Castagnini, S. Collins, L. Del Debbio, B. Lucini, and M. Panero, J. High Energy Phys. 06 (2013) 071.
[55] M. García Pérez, A. González-Arroyo, and M. Okawa, J. High Energy Phys. 04 (**2021**) 230.
[56] L. Castagnini, Meson spectroscopy in Large-N QCD, Ph.D. thesis, Regensburg University, 2015.
[57] P. Butti, M. García Pérez, A. González-Arroyo, K.-I. Ishikawa, and M. Okawa, J. High Energy Phys. 07 (2022) 074.
[58] A. González-Arroyo and M. Okawa, Phys. Rev. D **27**, 2397 (1983).
[59] A. González-Arroyo and M. Okawa, J. High Energy Phys. 07 (2010) 043.
[60] A. González-Arroyo and M. Okawa, Phys. Rev. D **88**, 014514 (2013).
[61] T. Eguchi and H. Kawai, Phys. Rev. Lett. **48**, 1063 (1982).
[62] A. González-Arroyo and M. Okawa, Phys. Lett. B **718**, 1524 (2013).
[63] M. García Pérez, A. González-Arroyo, L. Keegan, and M. Okawa, J. High Energy Phys. 01 (2015) 038.
[64] M. García Pérez, A. González-Arroyo, L. Keegan, and M. Okawa, J. High Energy Phys. 08 (2015) 034.
[65] M. Lüscher, J. High Energy Phys. 08 (2010) 071; 03 (2014) 92.
[66] G. Bergner, P. Giudice, G. Münster, I. Montvay, and S. Piemonte, J. High Energy Phys. 03 (2016) 080.
[67] J. Wuilloud, The Wilson-Dirac operator eigenspectrum for the theories of QCD and super Yang-Mills with one flavour, Ph.D. thesis, University of Münster, 2010.
[68] G. Bergner, G. Münster, D. Sandbrink, U. D. Özugurel, and I. Montvay, Proc. Sci. LATTICE2011 (**2011**) 055 [arXiv:1111.3012].
[69] G. Bergner and J. Wuilloud, Comput. Phys. Commun. **183**, 299 (2012).

[70] S. Piemonte, $\mathcal{N} = 1$ supersymmetric Yang-Mills theory on the lattice, Ph.D. thesis, University of Münster, 2015.
[71] G. P. Engel, L. Giusti, S. Lottini, and R. Sommer, Phys. Rev. Lett. **114**, 112001 (2015).
[72] J. A. M. Vermaseren, S. A. Larin, and T. van Ritbergen, Phys. Lett. B **405**, 327 (1997).
[73] P. Weisz, Phys. Lett. **100B**, 331 (1981).
[74] M. García Pérez, A. González-Arroyo, and M. Okawa, J. High Energy Phys. 10 (2017) 150.