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One-way Markov process approach to repeat times of large earthquakes in faults

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Abstract One of the uses of Markov Chains is the simulation of the seismic cycle in a fault, i.e. as a renewal model for the repetition of its characteristic earthquakes. This representation is consistent with Reid's elastic rebound theory. We propose a general one-way Markovian model in which the waiting time distribution, its first moments, coefficient of variation, and functions of error and alarm (related to the predictability of the model) can be obtained analytically. The fact that in *any* one-way Markov cycle the coefficient of variation of the corresponding distribution of cycle lengths is always lower than one concurs with observations of large earthquakes in seismic faults. The waiting time distribution of one of the limits of this model is the negative binomial distribution; as an application, we use it to fit the Parkfield earthquake series in the San Andreas fault, California.

1 Introduction

The elastic-rebound model is the canonical "macroscopic" theory of great earthquakes [24,26]. It states that a great earthquake will occur where large elastic strains have accumulated in the crust. The earthquake itself will relieve most of the strain which will then accumulate slowly again by a steady input of tectonic stress until the elastic strain becomes sufficiently large for another

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earthquake to ensue. The duration of this "earthquake cycle" is the ratio of the strain released during an earthquake to the rate of input of tectonic strain by plate motion. The concept of cycle in this context is a fundamentally geologic one that bears little resemblance to other cycles encountered in physics, and is related to the changing strain state of a fault plane as stress steadily accumulates via tectonic plate motion [15, 19]. In the concept of earthquake cycle it is implicit that the stress and strain state of the fault plane after the kth large earthquake is statistically indistinguishable from the state after the $(k-1)$th or any previous large earthquake in the same fault.

An outcome of the elastic-rebound model is the idea of characteristic earthquakes [27, 35, 36, 1]. Although a specific seismic fault or fault segment can produce small earthquakes as well as large ones, an overwhelming part of the stored elastic energy is released by the large ones, which rupture the entire area of the fault (or fault segment) in a repetitive, cyclic manner. As the magnitude of an earthquake is related to the broken area of the fault [14], each fault (or fault segment) tends to produce large earthquakes of the same magnitude; and because these earthquakes release most of the stored elastic energy, their repetition defines the duration of the earthquake cycle. The concept of characteristic earthquake has slowly changed since its definition by [27], but the idea of a series of large repetitive earthquake rupturing periocically an entire fault remain, although not all seismologists adhere to it (e.g., [17]) due to its phenomenological definition.

Because the Earth's crust is heterogeneous and faults are not isolated from each other but communicate through long-range stress-transfer mechanisms [28, 9, 31, 3, 4], the earthquake cycle is not periodic. So, although the elastic-rebound model is in essence deterministic, its application to a heterogeneous and interacting crust implies its translation into a probabilistic framework. Only in this way can it be used for earthquake forecasting purposes.

Several authors have proposed probabilistic versions of the elastic-rebound model, in the shape of probability distribution functions (pdfs) for the duration of the earthquake cycle [25, 32, 18, 33, 20, 12]. The rationale of these pdfs ranges from purely statistical (e.g. Utsu [32]) to physically-motivated (e.g. Vázquez-Prada et al. [33]). However, due to the scarcity of registered large earthquakes in a specific fault (usually 4 to 10 earthquakes), the statistics upon which the selection of a specific pdf is based are poor. This means that different pdfs can fit the empirical distribution function.

The variability of the duration of the earthquake cycle can be appropriatelly defined in the context of a pdf by means of the coefficient of variation, α, the ratio of the standard deviation σ to the mean μ of the pdf.

$$\alpha = \frac{\sigma}{\mu} \tag{1}$$

In the seismological literature the coefficient of variation is also known as the *aperiodicity*, a very descriptive name when applied to the duration of the earthquake cycle: when $\alpha = 0$ the earthquake cycle is perfectly periodic, when $0 < \alpha < 1$ the earthquake cycle is quasiperiodic, and when $\alpha > 1$ the earthquake cycle is said to have a *clustering* of events. The case $\alpha = 1$ is particularly important because the exponential distribution has this property, and the exponential distribution is the pdf of an earthquake cycle where large earthquakes occur in time following a Poisson distribution (i.e, they are random in time).

The predictability of a time series whose events follow a specific pdf is related to its aperiodicity [22,37,30]. Applied to the earthquake cycle this means that the predictability of the next large (characteristic) earthquake in a series is related to the aperiodicity of the pdf describing the duration of the cycles: aperiodicities close to zero imply greater predictability than aperiodicities close to one. Sykes and Menke [29] have calculated the aperiodicity of the earthquake cycle of several seismic faults. All the studied faults have aperiodicities smaller than 0.6, meaning that the earthquake cycle is quasiperiodic. Ellsworth et al. [8] also studied the aperiodicity of the earthquake cycle in several fault segments and concluded that all of them are between 0.11 and 0.97. Recently, Abaimov et al. [1,2] studied the creeping section of the San Andreas Fault, where instead of through medium or large earthquakes, elastic strain is released in an almost continuous way via small slip events 20 to 100 days apart. The aperiodicity of these slip series is in the range $0.473 < \alpha < 0.677$ [1]. It seems, thus, that $\alpha < 1$ is a property of the earthquake cycle in seismic faults and that this behaviour spans cycles with durations from days to hundreds of years. Can this be reproduced by simple models of single-fault seismicity?

In this paper we propose a general *one-way* Markovian model of the earthquake cycle with aperiodicities lower than 1. The term one-way in this family of models refers to the fact that after a time step, the state of strain in a fault can remain either stationary or grow by a finite amount. In other words, in this model a decrease in the strain, such as could take place in a random walk type model, is forbidden. Time increases in discrete steps and strain is also added in finite units. The N positions of the model correspond to states of the system with progressive growing strain. The scheme of this model is shown in figure 1, for $N = 6$.

The relaxation of the system through a sudden and complete loss of strain, which simulates the occurrence of an earthquake, occurs when the N^{th} position of chain is reached. In Fig. 1, the relaxation is represented by the wavy line.

This article is organized as follows: Section 2 contains the general form of the stochastic matrix of one-way Markov cycles together with the specialization to the case of the Box-Model and the case where all the parameters are equal. This second case is nothing but a Negative Binomial Process. Sections 3 and 4 contain the distribution function for the cycle length and the two first moments of that distribution, respectively. The distribution function and first moments of the two particular cases mentioned above are also included. In Section 5, the so called fraction of error and fraction of alarm time are calculated. In Section 6, using a Negative Binomial Distribution we fit the data of the Parkfield earthquake series. Finally, in Section 7 we write the conclusions. Additionally, we have considered it of interest to explicitly present, for a non trivial case such as $N = 3$, how the distribution function in the case where all the parameters are equal tends to the Negative Binomial Distribution. This proof is written in the Appendix.

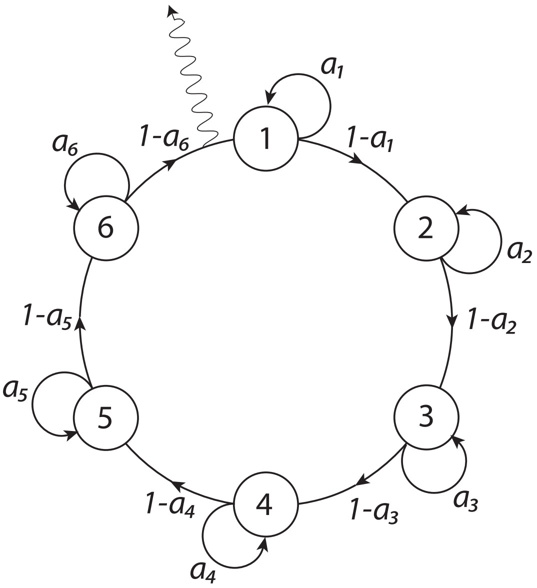

Fig. 1 Scheme of a one-way Markov cycle with $N = 6$. The probability of staying in state i is a_i and the probability of jumping from state i to state $i+1$ is $(1-a_i)$. The wavy line between states 6 and 1 means that at the end of the cycle all the stored energy is released.

2 One-way Markov cycles: Two particular cases

This model can be viewed as an array of N sites, or cells. These N sites are ordered by the index i, $i = 1, 2, \ldots, N$. As in genuine cellular automata, time increases in discrete steps. At the beginning of each cycle ($i = 1$), the array is empty of particles. In the first time step one particle is thrown to hit site 1, the probability of success being $(1 - a_1)$ and, in consequence, the probability of failure is a_1. In each failed attempt, the particle than misses the site is lost. Typically, after several failures a particles hits site 1 and is incorporated to the array at this position. Once the first site is occupied, the succesive throws of particles are aimed to hit site 2. All is identical to the first case except that now the probability of success is $(1 - a_2)$ and that of failure is a_2.

The occupation of site 1 is the first transit of the one-way Markov cycle ant the occupation of site 2 is the second transit. And then is the turn of sites 3, 4, ..., N. When site N is occupied, the cycle ends because all the particles accumulated in the array are released in a global relaxation. These simple rules are graphycally illustrated in Fig. 1 and materialized in the Markov matrix $[M]$, that for a cycle of size N has the form

$$[M] = \begin{pmatrix} a_1 & 1-a_1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & a_2 & 1-a_2 & \cdots & 0 & 0 & 0 \\ 0 & 0 & a_3 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{N-2} & 1-a_{N-2} & 0 \\ 0 & 0 & 0 & \cdots & 0 & a_{N-1} & 1-a_{N-1} \\ 1-a_N & 0 & 0 & \cdots & 0 & 0 & a_N \end{pmatrix}, \quad (2)$$

where the N parameters a_i, $i = 1, 2, 3, .., , N$, are $0 < a_i < 1$. Note that, including the size N, the number of free paramenters in the model is $N + 1$. From the point of view of the earthquake cycle defined in Section 1, if we conceptualize a fault plane as a 2D array of cells and each cell can be either stressed or un-stressed, the probability a_i of remaining in the same position of the cycle means no increase in stress during time step i, while jumping to the next position in the cycle, which occurs with probability $(1 - a_i)$, is associated with an increase in stress on the fault plane (i.e., a change of one cell from the un-stressed to the stressed state), thus approaching the final state when all the fault plane is stressed and the large earthquake that terminates the cycle takes place.

Denoting by $[M]^T$ the transpose of the Markov matrix, the components of its eigenvector with eigenvalue unity, c_i , are:

$$\begin{aligned} c_i &= \frac{1}{C} \prod_{j(\neq i)=1}^{N} (1 - a_j) \\ C &= \sum_{i=1}^{N} \prod_{j(\neq i)=1}^{N} (1 - a_j) \end{aligned} \quad (3)$$

where C is the normalization factor. The component c_i is the probability, statistically speaking, of finding the system in the position i of the cycle.

A particular case of this general scheme is that of the Box Model (BM) [11]. This is a cellular automaton where the stochastic filling of a box represents the increase of elastic energy in a fault during the seismic cycle. To visualize the model, consider an array of N cells. The position of the cells is irrelevant, but we can assume that they are arranged in the shape of a box. At the beginning of each cycle, the box is completely empty. At each time step, one ball is thrown, at random, to one of the cells in the box. That is, each cell has equal probability, $1/N$, of receiving the ball. If the cell that is chosen is empty, it will become occupied. If it was already occupied, the thrown ball is lost. Thus, each cell can be either occupied by a ball (stressed) or empty (un-stressed). When a new throw completes the occupation of the N cells of the box, it topples, becoming completely empty, and a new cycle starts. The emptying of the box after it is full is analogous to the generation of a characterisitic earthquake. In this model, the values of the N parameters are:

$$a_i = \frac{i-1}{N} \tag{4}$$

Another significant particular case corresponds to the case when

$$a_i = a \qquad \forall i \tag{5}$$

That is, all the parameters are identical. In such a Markov process, the distribution of the cycle length is that of a Negative Binomial Distribution (NBD) [7] where the probability of success is $(1-a)$ and N successes are required. In other words, the negative binomial distribution is a discrete probability distribution of the number of Bernoulli trials before a specified (non-random) number of successes (denoted N) occur. For example, if one throws a die repeatedly until the third time '1' appears ($N = 3$), then the probability distribution of the number of trials (i.e. the sum of '1's and non-'1's) that have been needed will be negative binomial.

3 Distribution Function for the cycle lengths

The distribution function of the cycle length for a one-way Markov cycle of N states, $P_N(n)$, can be obtained from the Markov matrix of the system, $\mathbf{M}$, by the application of the following three steps: i) The element of the last row and first column of $\mathbf{M}$ is changed by a 0. After this pruning, the matrix will be called $\mathbf{M}'$. ii) The new matrix $\mathbf{M}'$ is multiplied by itself $n-1$ times to obtain $\mathbf{M}'^{(n-1)}$ and the element of the first row, last column of this matrix is identified. iii) $P_N(n)$ is the product of this selected matrix element times $1/N$. The whys of this recipe are explained, for example, in [34]. Using this procedure, one obtains

$$P_N(n) = \left[\prod_{i=1}^{N}(1-a_i)\right]\left[\sum_{i=1}^{N}\frac{a_i^{n-1}}{\prod_{j(\neq i)=1}^{N}(a_i-a_j)}\right] \qquad n = N, N+1, \ldots, \infty \tag{6}$$

where n represents the length of the cycle expressed in time steps of the model.

This is the general form of the discrete distribution function in any one-way Markov cycle. This formula has been obtained from systematics. That is, after having explicitely calculated $P_2(n)$, $P_3(n)$, etc., one deduces that the form of $P_N(n)$ is what is written in Eq. 6.

When these models are applied in seismicity, the fact that until time step $n = N$ the probability of completing a cycle is null is called a stress shadow, i.e., a time period during which no earthquake can occurr in the fault due to the fact that the previous one has release all of the stored energy.

In the two particular cases mentioned above, the Box Model and the Negative Binomial Distribution, one obtains:

$$P_N(n) = \sum_{i=1}^{N-1} (-1)^{i+1} \binom{N-1}{i-1} (1 - \frac{i}{N})^{n-1} \qquad n = N, N+1, \ldots, \infty \tag{7}$$

for the BM, and

$$P_N(n) = (1-a)^N a^{n-N} \binom{n-1}{N-1} \qquad n = N, N+1, \ldots, \infty \tag{8}$$

for the NBD.

4 The two first moments

In a geometric process where the probability of success is $(1 - a)$, the mean and variance of the distribution are:

$$\mu = \frac{1}{1-a}, \tag{9}$$

and

$$\sigma^2 = \frac{a}{(1-a)^2}. \tag{10}$$

Thus, the coefficient of variation, or aperiodicity, of a geometric process is

$$\alpha = \frac{\sigma}{\mu} = a^{1/2} < 1 \tag{11}$$

Therefore, as a one-way Markov cycle is nothing more than a succession of N independent geometric processes, the mean and variance can be written as:

$$\mu = \frac{1}{1-a_1} + \frac{1}{1-a_2} + \ldots + \frac{1}{1-a_N}, \tag{12}$$

and

$$\sigma^2 = \frac{a_1}{(1-a_1)^2} + \frac{a_2}{(1-a_2)^2} + \ldots + \frac{a_N}{(1-a_N)^2}. \quad (13)$$

Thus, the aperiodicity, α, is given by:

$$\alpha = \frac{\left[\frac{a_1}{(1-a_1)^2} + \frac{a_2}{(1-a_2)^2} + \ldots + \frac{a_N}{(1-a_N)^2}\right]^{1/2}}{\frac{1}{1-a_1} + \frac{1}{1-a_2} + \ldots + \frac{1}{1-a_N}}. \quad (14)$$

This aperiodicity is rigorously lower than 1 because the different subprocesses which build the one-way Markov cycle are geometric and independent.

In the particular case of the NBD, we have:

$$\mu = \frac{N}{1-a}, \quad (15)$$

$$\sigma^2 = \frac{Na}{(1-a)^2}, \quad (16)$$

and

$$\alpha = \frac{\sigma}{\mu} = \sqrt{\frac{a}{N}}. \quad (17)$$

5 Fraction of error and fraction of alarm

A convenient way to assess the predictability of a time series is by trying to forecast its events by declaring *alarms* at particular times. The aim is to declare alarms before all the events in order not to miss any, but to declare them just before the events in order to minimize the total alarm time. Many strategies can be devised to declare the alarms but there is a reference strategy to which all others can be compared [23,16,34]. This strategy consists of setting the alarm a fixed time interval after each event (waiting time) and maintaining it until the occurrence of the event. If the following event in the time series occurs before the alarm is raised, it is counted as a prediction error; if the following event in the time series occurs after the alarm is raised, it is counted as a prediction success and the alarm is then cancelled.

The fraction of errors f_e (number of missed events divided by the total number of events) and the fraction of alarm time f_a (total alarm time divided by the total duration of the time series) can be computed as a function of the waiting time n and the purpose is to find the optimum waiting time. This optimum waiting time depends on the relative importance that failing to predict an event has compared to keeping the alarm on. An objective function, called loss function, L, can be defined that incorporates this trade-off in each particular case. One very simple option is $L = f_e + f_a$, where both, failure to predict an earthquake and a long alarm time, are equally penalized. Obviously the aim is to find the waiting time $n = n^*$ that minimizes $L(n)$. Depending on the context where an alarm-based prediction strategy is applied, the loss function can be tailored to specific needs [21,22].

For any thinkable strategy based on the use of alarms, if an earthquake takes place when the alarm is on, the prediction is considered to be a success. If the earthquake takes place when the alarm is off, then it is labelled as a prediction failure.In our general one-way Markov model, and using the above-mentioned strategy, the fraction of error function adopts the form

$$f_e(n) = \sum_{n'=N}^{n} P(n') = 1 - \sum_{n'=n+1}^{\infty} P(n'). \quad (18)$$

Note that f_e is the accumulated distribution of Eq. (6). Performing the sum over the n' index, the result is:

$$f_e(n) = 1 - \left[\prod_{i=1}^{N}(1-a_i)\right] \sum_{i=1}^{N} \left[\frac{a_i^n}{(1-a_i)} \frac{1}{\prod_{j(\neq i)=1}^{N}(a_i - a_j)}\right]. \quad (19)$$

Regarding the fraction of alarm time function, f_a, its general form is

$$f_a(n) = \frac{\sum_{n'=n}^{\infty} P(n')(n'-n)}{\sum_{n'=n}^{\infty} P(n')n'} = \frac{\sum_{n'=n}^{\infty} P(n')(n'-n)}{\mu}, \quad (20)$$

and, for the particular case of a one-way Markov cycle, we have:

$$\mu f_a(n) = \sum_{n'=n}^{\infty} n' P(n') - n \sum_{n'=n}^{\infty} P(n') =$$

$$= \left[\prod_{i=1}^{N}(1-a_i)\right] \sum_{i=1}^{N} \left[\frac{a_n^i}{\prod_{j(\neq i)=1}^{N}(a_i - a_j)} \left(\frac{n}{a_i(1-a_i)} + \frac{1}{(1-a_i)^2}\right)\right]$$
$$-n\left(1 + P(n) - f_e(n)\right). \quad (21)$$

6 Application of the model to the Parkfield series

As said in Section 2, two particular cases of the models included in the family of one-way Markov cycles are the BM and the NBD. The waiting time distribution of the BM was used in [11] to fit the series of earthquakes occurred at the Parkfield segment of the San Andreas fault in California. Here we will apply a NBD to the same series because it constitutes the best studied sequence of characteristic earthquakes in the world. The mean and aperiodicity of the Parkfield earthquake series are 24.5 years and 0.378 respectively. During the mid 1980s a prediction experiment was set up in this fault segment in order to predict the time of the next earthquake in the series [6]. Finally the earthquake did occur in 2004, but outside the prediction window of the 1985 experiment, demonstrating that a reliable short-term earthquake prediction is still not achievable [5].

A negative binomial process is a particular case where all the probabilities of advancing in the one-way Markov process are equal. Thus, in principle, in this model one would have to deal with two parameters, N and a. But this can be simplified if N and a are related. One interesting possibilty is

$$1 - a = \frac{1}{N}, \tag{22}$$

which corresponds to an statistical process of filling the N sites of an array *in an ordered way* (note the difference with the Box Model, where the filling process is not ordered). Thus, in this particular case the occupation of any site is a geometric process with a probability of success equal to $1/N$ while $(N-1)/N$ is the probability of failure. Inserting Eq. (22) into Eq. (17), we obtain the aperiodicity of this concrete model:

$$\alpha = \frac{\sqrt{N-1}}{N}. \tag{23}$$

The aperiodicity has a maximum value of 0.5 for $N = 2$ and then decays monotonously to 0 as N tends to ∞.

We will fit the Parkfield series of earthquakes to this model using the method of moments. First, we will choose the value of N for which the aperiodocity is nearest to that of the Parkfield series, $\alpha = 0.378$. The result is $N = 6$, for which $\alpha = 0.373$. From Eq. (15), the mean value of n in this model is N^2 and thus for $N = 6$ the mean is equal to 36 time steps. Because the actual mean of the Parkfield series is 24.62 years, one step of the model corresponds to $24.62/36 = 0.68$ years, or around 8 months.

As the last earthquake of the series occurred on September 28, 2004 and the period of stress shadow is $6 \times 8 = 48$ months, it ended in September 2008. Therefore the occurrence of the next event now (2012) is not forbidden by this model, although the probability is very low.

Now the parameters are already fixed and using Eq. (8), the pdf for the fit is

$$P_6(n) = \left(\frac{1}{6}\right)^6 \left(\frac{5}{6}\right)^{n-6} \binom{n-1}{5}, n = 6, 7, \ldots, \infty. \tag{24}$$

In Figure 2 we have superimposed the cumulative histogram (empirical distribution function) of the Parkfield series to the cumulative distributions of the NBD and other five models used in the literature [10,12]. It is quite obvious from the figure that the performance of all six models is good and very similar, including the NBD. Indeed, the residuals for the NBD evaluated at the midpoints of the horizontal segments of the empirical distribution function are the lowest of the six tested models.

The hazard rate corresponding to $P_N(n)$ is defined as:

$$h_N(n) = \frac{P_N(n)}{\sum_{i=n}^{\infty} P_N(i)}. \tag{25}$$

The hazard rate is the probability for an earthquake to occurr at time step n on the condition that it has not occurred until time step $n-1$. However, in the seismological literature is customary to express the likelihood of an earthquake using the yearly conditional probability of earthquake occurrence,

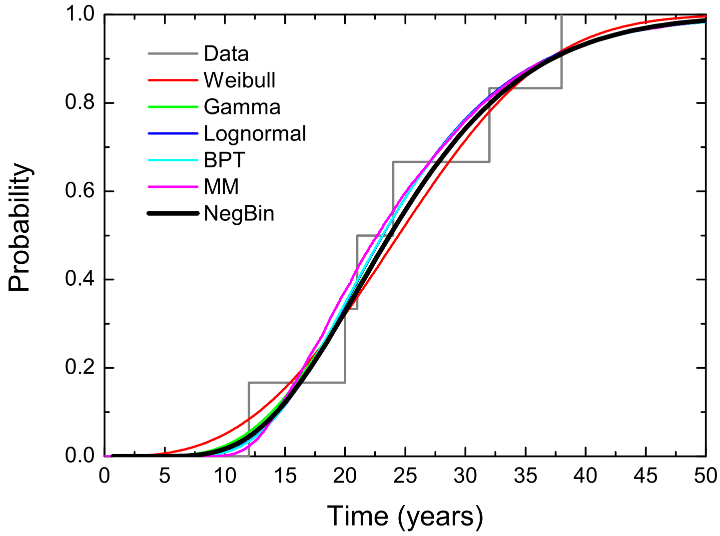

Fig. 2 Fit of the Negative Binomial model to the Parkfield series and comparison with other statistical models used in the literature.

$P(n|\tau = 1 \text{ year})$, instead of the hazard rate. This function gives the probability of having an earthquake during the next year provided it has not occurred before:

$$P_N(n|\tau = 1 \quad \text{year}) = \frac{S_N(n+\tau) - S_N(n)}{1 - S_N(n-1)}, \tag{26}$$

where $S_N(n) = \sum_{i=N}^{n} P_N(n)$ is the cummulative distribution function.

Both the hazard rate and the yearly conditional probability functions for the NBD reach a constant value for large times. Inserting Eq. (24) into Eq. (25) one easily obtains that, for long times,

$$\lim_{n\to\infty} h_N(n) = \frac{1}{N}. \tag{27}$$

As an example, the asymptotic (large time) hazard rate for the Parkfield series is $h_6(\infty) = 1/6 = 0.1667$, while the present hazard rate (for the end of the year 2012) is 0.0033, a 2% of the maximum hazard rate.

The yearly conditional probability function for the Parkfield series is illustrated in Fig. 3. Again, as in Fig. 2, the NBD and five other models are compared. The present yearly probability of earthquake occurrence is 0.004, i.e., there is a 0.4% probability of having an earthquake in the following 12 months. Obviously this probability is low because the earthquake cycle is in its early stages. When the cycle is at its average duration, 24.62 years, the yearly probability is 6% (Fig. 3).

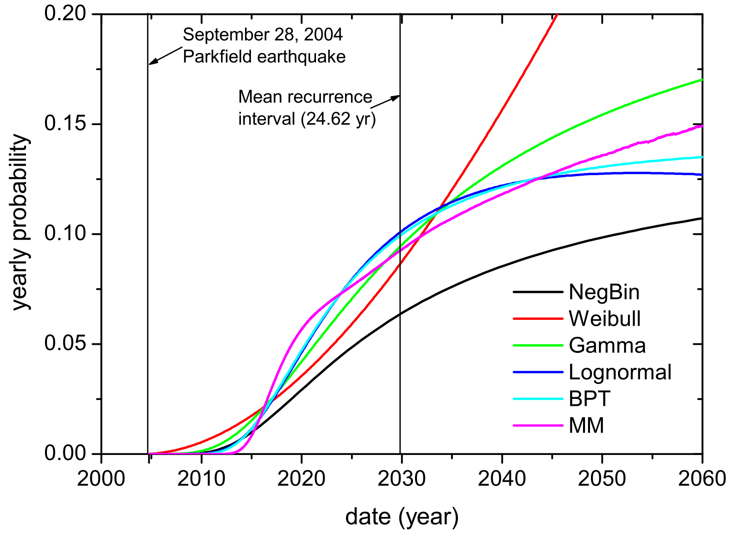

Fig. 3 Yearly conditional probability for the Parkfield series as predicted by the Negative Binomial model compared to other statistical models used in the literature.

7 Conclusions

We have introduced a family of models, one-way Markov cycles, for the description of the repetitive occurrence of earthquakes in faults. We have calculated the form of the distribution function for the cycle length. The number of independent parameters, N, coincides with the number of positions in the Markov cycle, each one corresponding to a transition probability $(1 - a_i)$ to the next state in the cycle $(i = 1, 2, \ldots N)$. The first moments of this distribution are easily calculated bearing in mind that a one-way Markov cycle is nothing more than a succession of N independent geometric processes. Thus, these moments are written as the sum of the mean, or variance, of the N stages of the cycle. Most properties of this family of models can be obtained analytically, an interesting result in itself. Two of such properties are the fraction of error to predict and the fraction of alarm time, basic funtions to assess the predictability of earthquake renewal models [13].

The above enumerated properties of the model nicely match Reid's theoretical vision of the mechanism of how earthquakes are generated [15]. As commented on in the Introduction, data on the recurrence of large earthquakes in well documented seismic faults indicate that their aperiodicity is always lower than unity [29, 8, 1, 2]. Because the aperiodicity (coefficient of variation) of the distribution of cycle lengths in any one-way Markov cycle is also lower than unity, this family of models can be used as general renewal models of earthquake recurrence.

Two limit cases of one-way Markov cycles are the Box Model (BM) and the Negative Binomial Model (NBM). The first was already used by the authors to evaluate the predictability of the Parkfield, California, series of earthquakes [11]. Here we have applied the NBM to the same earthquake series and shown that it gives competitive results in comparison to several other renewal models used in the literature. However, while renewal models using known distributions such as gamma, Weibull, log-normal, etc. are pure fits to earthquake data, our model at least provides a nave view of the process of loading and relaxation of a fault. This relationship between the model and the physics of a fault, together with the conclusion that one-way Markov cycles always have aperiodicities lower than one (in agreement with observations), are the main results of our study.

A APPENDIX: The Negative Binomial Distribution as a Limit: Case $N = 3$

In this Appendix we show explicitly that, for $N = 3$, the limit of eq. 6 when the three parameters are equal is eq. 8. For simplicity in the notation, let us call $a_1 = a$, $a_2 = b$, $a_3 = c$. Eq. 6 for $N = 3$ reads as follows:

$$\frac{P_3(n)}{K} = \frac{a^{n-1}}{(a-b)(a-c)} + \frac{b^{n-1}}{(b-a)(b-c)} + \frac{c^{n-1}}{(c-a)(c-b)}$$
$$K = (1-a)(1-b)(1-c) \tag{28}$$

To carry out the limit, we introduce new variables x and y.

$$a = xc$$
$$b = yc \tag{29}$$

The limit we seek will be implemented by tending x and y to 1. Substituting the new variables into eq. 28, the result is:

$$\frac{c^{n-1}x^{n-1}}{c^2(x-y)(x-1)} + \frac{c^{n-1}y^{n-1}}{c^2(y-x)(y-1)} + \frac{c^{n-1}}{c^2(1-x)(1-y)} =$$
$$= c^{n-3}\left[\frac{x^{n-1}(y-1) - y^{n-1}(x-1) + (x-y)}{(x-y)(x-1)(y-1)}\right] \tag{30}$$

Elaborating eq.30 slightly, we obtain :

$$\frac{c^{n-3}}{(x-y)(x-1)(y-1)}\left[y(x^{n-1}-1) - x(x^{n-2}-1) - y^{n-1}(x-1)\right] \tag{31}$$

Henceforth it is convenient to use the following type of polynomials:

$$P_n(x) = x^n + x^{n-1} + x^{n-2} + \ldots + x + 1$$
$$P_n(1) = n + 1 \qquad (32)$$
$$P_n(x,y) = x^n + x^{n-1}y + x^{n-2}y^2 + \ldots + xy^{n-1} + y^n$$

These polynomials fulfil the so-called cyclotomic property, namely

$$(x^n - 1) = (x-1)P_{n-1}(x)$$
$$(x^n - y^n) = (x-y)P_{n-1}(x,y) \qquad (33)$$

So, dividing the second factor in eq. 31 by $(x-1)$ we obtain

$$\frac{c^{n-3}}{(x-y)(y-1)}\left[yP_{n-2}(x) - xP_{n-3}(x) - y^{n-1}\right] =$$
$$= \frac{c^{n-3}}{(x-y)(y-1)}\left[y(P_{n-3}(x) + x^{n-2}) - xP_{n-3}(x) - y^{n-1}\right] = \qquad (34)$$
$$= \frac{c^{n-3}}{(x-y)(y-1)}\left[(y-x)P_{n-3}(x) + y(x^{n-2} - y^{n-2})\right]$$

Now we divide the second factor of eq. 34 by $(x-y)$

$$\frac{c^{n-3}}{(y-1)}[yP_{n-3}(x,y) - P_{n-3}(x)] \qquad (35)$$
$$\frac{c^{n-3}}{(y-1)}[y(x^{n-3} + x^{n-4}y + x^{n-5}y^2 + \ldots + xy^{n-4} + y^{n-3}) - (x^{n-3} + x^{n-4} + \ldots + x + 1)] =$$
$$\frac{c^{n-3}}{(y-1)}[(y-1)x^{n-3} + (y^2-1)x^{n-4} + \ldots + (y^{n-4}-1)x^2 + (y^{n-3}-1)x + (y^{n-2}-1)] =$$
$$c^{n-3}[x^{n-3} + x^{n-4}P_1(y) + x^{n-5}P_2(y) + \ldots + x^2P_{n-5}(y) + xP_{n-4}(y) + P_{n-3}(y)]$$

Returning to eq. 28, using eq. 35, and performing the limit $x, y \to 1$, we obtain:

$$\lim_{x,y\to 1} P_3(n) = (1-c)^3 c^{n-3}[1 + 2 + 3 + \ldots + (n-3) + (n-2)] =$$
$$(1-c)^3 c^{n-3}\frac{(n-1)(n-2)}{2} \qquad (36)$$

This formula coincides with eq. 8 when $N = 3$

$$P_3(n) = (1-c)^3 c^{n-3}\binom{n-1}{2}$$
$$n = 3, 4, \ldots, \infty \qquad (37)$$

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