mathematics

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A Bilevel Approach to the Facility Location Problem with Customer Preferences Under a Mill Pricing Policy

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Abstract: This paper addresses the facility location problem under a mill pricing policy, integrating customers' behavior through the concept of preferences. The problem is modeled as a bilevel optimization problem, where the existence of ties in customers' preferences can lead to an ill-posed bilevel problem due to the possible existence of multiple optima to the lower-level problem. As the commonly employed optimistic and pessimistic strategies are inadequate for this problem, a specific approach is proposed bearing in mind the customers' rational behavior. In this work, we propose a novel formulation of the problem as a bilevel model in which each customer faces a lexicographic biobjective problem in which the preference is maximized and the total cost of accessing the selected facility is minimized. This allows for a more accurate representation of customer preferences and the resulting decisions regarding facility location and pricing. To address the complexities of this model, we apply duality theory to the lower-level problems and, ultimately, reformulate the bilevel problem as a single-level mixed-integer optimization problem. This reformulation incorporates big-M constants, for which we provide valid bounds to ensure computational tractability and solution quality. The computational study conducted allows us to assess, on the one hand, the effectiveness of the proposed reformulation to address the bilevel model and, on the other hand, the impact of the length of the customer preference lists and fixed opening cost for facilities on the computational time and the optimal solution.

Keywords: facility location; mill pricing; preferences; bilevel optimization; lexicographic biobjective

MSC: 90C27; 90C46; 90C90

Citation: Calvete, H.I.; Galé, C.; Hernández, A.; Iranzo, J.A. A Bilevel Approach to the Facility Location Problem with Customer Preferences Under a Mill Pricing Policy. *Mathematics* **2024**, *12*, 3459. https://doi.org/10.3390/math12223459

Academic Editor: Andrea Scozzari

Received: 11 October 2024
Revised: 28 October 2024
Accepted: 2 November 2024
Published: 5 November 2024

1. Introduction

Facility location problems are a fundamental area of study in operations research, with applications ranging from the placement of production plants and warehouses in supply chain management to the location of schools, fire stations, or hospitals, which are typically managed by public administrations. Hence, these problems play a critical role in decision-making processes across both the public and private sectors. The essence of facility location problems is to determine the optimal placement of facilities to best serve a set of customers, guided by specific optimality criteria. Effective facility location decisions can increase customer satisfaction, improve supply chain efficiency, and provide a competitive pricing advantage in the marketplace. Without aiming to be exhaustive, refs. [1–7] and the references therein offer a thorough survey of the field.

This paper focuses on mill pricing in facility location problems where a company seeks to maximize profit by determining both the optimal locations for a set of facilities and the mill price for each facility, while customers are responsible for covering the travel costs to the facility they choose to visit [8]. In addition, it is assumed that each customer has established preferences over locations, and it is this preference function that drives his/her decision on the selected facility. This problem exhibits an implicit hierarchical structure, where two interrelated subproblems can be identified: the location and pricing problems, concerning the facilities at the upper level of the hierarchy, and the facility selection problem, at the lower level, which pertains to each customer choosing his/her facility based on the highest preference among the affordable opened facilities. This hierarchical structure leads to a bilevel formulation of the problem, where the upper-level decision-maker maximizes net profit, calculated as the difference between the revenue generated from customer payments for accessing the facilities and the fixed costs associated with keeping those facilities open. At the lower level, each customer seeks to maximize the preference value obtained from accessing one of the facilities. Ties in preferences can lead to an ill-posed bilevel optimization model due to the possible existence of multiple optima to the lower-level problem. The characteristics of the problem addressed in this paper render both the optimistic and pessimistic approaches, commonly used to handle this issue, inadequate. Instead, it seems more rational for customers to choose the least costly facility among those tied in terms of preference value. Taking this into account, we propose reformulating the lower-level problem as a lexicographic biobjective problem, where the first objective is to maximize preferences and the second objective is to minimize the amount paid by the customer.

The key contributions of this paper to the literature can be summarized as follows:

- Addressing customer preferences for the first time in the context of pricing in facility location problems and formulating the problem as a bilevel optimization problem.
- Proposing a new methodology that addresses the existence of multiple optima to the lower-level problem by including a lexicographic biobjective function, as the optimistic and pessimistic approaches have proven to be inadequate.
- Reformulating the bilevel problem as a single-level mixed-integer optimization problem, leveraging the properties of the lexicographic biobjective problem involved in the lower-level problem.
- Showing the effectiveness of the formulation based on the computational experiments conducted.

The remainder of this paper is structured as follows. Section 2 reviews the relevant literature pertinent to our study, highlighting key research gaps. Section 3 precisely defines the problem, introduces the first bilevel formulation, and discusses the challenges that arise when ties in preferences are involved. Section 4 proposes a bilevel reformulation of the problem to address the possible existence of multiple optima at the lower-level problem, which can arise due to ties in preferences. Section 5 demonstrates that prices can be discretized, as there exists an optimal solution in which they take values related to the customers' budgets and facility reach costs. Section 6 reformulates the problem as a single-level optimization model using duality theory applied to each follower's optimization problem. Section 7 provides valid bounds for the big-M parameters involved in the single-level formulation. Section 8 outlines the modifications required for addressing the problem in the absence of ties in preferences. Section 9 presents the results of the computational experiments conducted. Finally, Section 10 closes the paper with some concluding remarks.

2. Literature Review

Although the importance of simultaneously considering both the location and pricing problems has been recognized as early as the work of Greenhut [9], the volume of studies that involve both decisions is not very large. Hanjoul et al. [8] discuss the role of pricing in location decisions, differentiating between cases where customers are responsible for travel costs (mill pricing policy) and those where they are not (delivered pricing policy). They also present mathematical formulations for the respective problems. Additionally, they propose an algorithm to solve the uniform mill pricing problem, in which the commodity is offered at the same mill price at each open facility, while customers cover their own transportation costs. The algorithm is based on a binary search over the range of admissible mill price values. Kochetov et al. [10] recognize the bilevel structure of the problem and develop two hybrid algorithms based on local search: a variable neighborhood descent and a genetic local search. They compare the performance of these algorithms with CPLEX software and demonstrate their competitiveness. Panin and Plyasunov [11] study the computational complexity of the bilevel problem and Panin and Plyasunov [12] review various problems related to facility location and pricing processes that they have worked on, presenting the most significant results they have obtained. Lin and Tian [13] focus on the mill pricing policy and study a version of the facility location problem with mill pricing in which prices are assumed to be discrete. The authors justify this assumption by arguing that setting prices with an arbitrary number of decimal places is impractical. Therefore, they define a set of candidate price levels for each facility. They also identify the inherent hierarchical structure of the problem, formulating it as a bilevel optimization problem. In order to solve it, they propose two resolution methods. The first method involves reformulating the bilevel problem as a single-level mixed-integer optimization problem using the so-called Closed Assignment Constraints. This reformulation allows the problem to be solved using commercial solvers. The second method is an exact algorithm based on branch-and-cut procedures that incorporates specialized bilevel feasibility cuts. After conducting computational experiments, they conclude that the second method outperforms the first in terms of efficiency.

Regarding preferences, the concept of allowing customers to choose the facility they patronize was first introduced by Hanjoul and Peeters [14]. In their approach, they assumed that each customer has a preference hierarchy based on personal characteristics, as well as factors related to the facilities and the trips to those locations. They proposed the simple plant location problem with order model (SPLPO), where each customer selects the facility he/she prefers most from the available options. This model also assumes that the facility locator, the decision-maker responsible for selecting which facilities to open, is aware of the customers' preference rankings and incorporates this information into the decision-making process. These preference rankings are incorporated into the mathematical model as a set of constraints, which are added to the simple plant location problem formulation. To solve the model, the authors devised a heuristic algorithm and applied it to small case studies. Cánovas et al. [15] further examined the SPLPO, enhancing its formulation by introducing valid inequalities and implementing several preprocessing techniques. Hanjoul and Peeters [14], in their groundbreaking work, also acknowledged the existence of an implicit hierarchical structure in the SPLPO. This structure reflects the two interconnected subproblems: the facility location problem at the upper level, which involves choosing which facilities to open, and the customer allocation problem at the lower level, which assigns customers to these facilities. Recognizing that bilevel optimization models are well-suited to handle such hierarchical decision processes, Hansen et al. [16], Vasilyev and Klimentova [17], Vasilyev et al. [18] reformulated the SPLPO as a bilevel model. In this model, the upper-level decision-maker aims to minimize total costs by selecting facilities, while the lower-level problem focuses on allocating customers to minimize the aggregate of their preferences (with smaller values indicating stronger preferences). Those authors assume strict preference orderings, meaning each customer unequivocally prefers one facility over another. Under this assumption, the bilevel model can be reduced to a single-level formulation. Hansen et al. [16] introduced a reformulation of this single-level problem that outperformed previous versions in terms of the efficiency of their linear programming relaxations. Vasilyev et al. [18] improved upon this by incorporating a new class of valid inequalities, rather than expanding the number of variables. Building on this earlier work, Vasilyev and Klimentova [17] developed a branch-and-cut method to find optimal solutions. Finally, Camacho-Vallejo et al. [19] proposed an evolutionary algorithm to solve the bilevel model. On the other hand, Lin et al. [20] introduce preferences into the p-median problem and develop two exact branch-and-cut solution approaches. The first approach leverages the bilevel structure of the problem to derive an effective bilevel feasibility cut, while the second approach utilizes Benders decomposition, which is further accelerated through analytical Benders separation and heuristic separation.

A key feature of solving the SPLPO is the absence of a capacity constraint for each facility. This ensures that once the facilities to be opened are selected, each customer can be assigned to their most preferred facility, as there is sufficient capacity in each facility to accommodate them. Calvete et al. [21] generalize the SPLPO model by introducing capacity constraints. The model seeks to minimize the total costs of opening facilities and assigning customers while accounting for both customer preferences and these capacity constraints on the facilities. Two approaches are proposed to address this problem, extending the single-level and bilevel formulations of the SPLPO, where customers are free to select their most preferred open facility. After analyzing the implications of these two approaches, the paper adopts the hierarchical approach, resulting in the formulation of a bilevel optimization problem. They develop an effective metaheuristic algorithm, that uses the general framework inspired by evolutionary algorithms and leverages the bilevel structure of the model. Capacity constraints and customer preferences have also been addressed by Casas-Ramírez et al. [22]. In their work, each customer has a demand, and the capacity of a facility refers to the total demand it can accommodate. Due to the complexity of the corresponding lower-level problem, they propose to consider semi-feasible bilevel solutions. Kang et al. [23] introduce a generalization of the classic maximum cover problem, focusing on the construction of finite-capacity facilities under budgetary constraints. Their objective is to serve the maximum number of customers, thereby maximizing the benefits derived from serving customers, while also considering customer preferences. To tackle this problem, they developed a heuristic algorithm based on maximum flow techniques.

The work presented in this paper bridges the gap between the existing facility location studies that focus on pricing strategies and those that emphasize customer preferences, by integrating both aspects into a unified framework that allows for accounting for individual preferences in the decision-making process of setting facility prices.

3. Statement and Formulation of the Problem

This section focuses on the detailed description and mathematical formulation of the *Facility Location problem with customer Preferences under a Mill Pricing policy*, hereafter referred to as the FLPMP problem. This problem considers a set of potential facilities to be opened and a set of customers interested in accessing any of them. Each facility is associated with a fixed opening cost as well as a usage price. Moreover, there are travel costs associated with each paired customer–facility. To keep things simple and reflect what is most realistic in practice, closed facilities are required to have a price set to zero, as it is unnecessary to price a facility that no one will attend. In addition, each customer has a fixed budget and assigns a preference value to each facility he/she is interested in, resulting in a ranked list of facilities for each customer based on his/her preferences. The higher the value assigned to a facility, the greater the customer's preference for it. Note that since the mill pricing policy is adopted, i.e., travel costs are borne by customers [8], the sum of the price of the chosen facility and the travel costs to reach it cannot exceed the customer's budget. Furthermore, as a rational assumption, we consider that customers are not interested in facilities for which their budget does not even cover the cost of reaching them.

As mentioned in Section 1, this problem has a hierarchical structure with two levels of decision-making, leading to a bilevel formulation of the problem. The upper-level decision-maker, or leader, represents the entity or institution responsible for deciding which potential facilities to open and setting the price for each one, aiming to maximize the net profit. At the lower level, there is a set of multiple independent decision-makers, or followers, representing the customers, who decide which open facility to access based on their preferences and budget, aiming to maximize the preference value.

The main notation required for the mathematical formulation of the problem is provided in Table 1. Additional notation will be introduced as needed. As mentioned above, each customer $i \in I$ only considers those facilities $j \in J$ for which the associated travel cost does not exceed his/her budget, i.e., $b_i - c_{ij} \geqslant 0$. Among these facilities, the customer selects a set of preferred ones and ranks them from the most to the least preferred. Once ranked, the customer assigns a preference value equal to $|J|$ to the most preferred facilities, equal to $|J|-1$ to the second most preferred, and so on, until all preferred facilities have been assigned a preference value that is greater than zero, where $|\cdot|$ refers to the cardinal of the corresponding set. Then, for each customer $i \in I$, the set $J_i = \{j \in J \colon s_{ij} > 0\}$ contains the facilities he/she is interested in (and, of course, can reach). Similarly, for each facility $j \in J$, the set $I_j = \{i \in I \colon s_{ij} > 0\}$ contains the customers who are interested in that facility (and, of course, whose budgets allow them to reach it). Hence, variables x_{ij} are only defined for $i \in I, j \in J_i$. On the other hand, when the indices of the variables do not need to be explicitly specified, we use $\boldsymbol{p}$, $\boldsymbol{y}$, $\boldsymbol{x}$ and $\boldsymbol{x}_i$ to denote the set of variables $\{p_j\}_{j\in J}$, $\{y_j\}_{j\in J}$, $\{x_{ij}\}_{i\in I, j\in J_i}$ and $\{x_{ij}\}_{j\in J_i}$, respectively.

Table 1. Notation.

Sets and Indices	**Description**
$I = \{1, \ldots, \lvert I\rvert\}$	Set of customers. Indexed by i.
$J = \{1, \ldots, \lvert J\rvert\}$	Set of facilities. Indexed by j.
Variables	**Description**
$p_j \geqslant 0$	Price set to facility $j \in J$.
$y_j \in \{0,1\}$	1 if facility $j \in J$ is opened, 0 otherwise.
$x_{ij} \in \{0,1\}$	1 if customer $i \in I$ accesses facility $j \in J_i$, 0 otherwise.
Parameters	**Description**
$f_j \geqslant 0$	Fixed opening cost of facility $j \in J$.
$c_{ij} \geqslant 0$	Travel cost from customer $i \in I$ to facility $j \in J$.
$s_{ij} > 0$	Preference value assigned by customer $i \in I$ to facility $j \in J_i$.
$b_i \geqslant 0$	Budget of customer $i \in I$.

The FLPMP problem can be formulated as the following bilinear–linear bilevel mixed-integer optimization problem with multiple independent followers:

$$\text{“}\max_{\boldsymbol{p},\boldsymbol{y}}\text{”} \quad \sum_{i\in I}\sum_{k\in J_i} p_k x_{ik} - \sum_{k\in J} f_k y_k \tag{1a}$$

s.t.:

$$p_j \leqslant M y_j \qquad j \in J \tag{1b}$$

$$p_j \geqslant 0 \qquad j \in J \tag{1c}$$

$$y_j \in \{0,1\} \qquad j \in J \tag{1d}$$

where for each customer $i \in I$, $\{x_{ij}\}_{j\in J_i}$ solve

$$\max_{\boldsymbol{x}_i} \quad \sum_{k\in J_i} s_{ik} x_{ik} \tag{1e}$$

s.t.:

$$\sum_{k\in J_i} x_{ik} \leqslant 1 \tag{1f}$$

$$\sum_{k\in J_i} (c_{ik} + p_k) x_{ik} \leqslant b_i \tag{1g}$$

$$x_{ij} \leqslant y_j \qquad j \in J_i \tag{1h}$$

$$x_{ij} \in \{0,1\} \qquad j \in J_i \tag{1i}$$

Expression (1a), the upper-level objective function, maximizes the net profit as the difference between the revenue from customer pricing and the total cost incurred by opening facilities. Constraints (1b) ensure that the price of a closed facility is zero, where M is a sufficiently large upper bound for the prices. Constraints (1c) and (1d) define the continuous and binary nature of the variables p_j and y_j, respectively. For each customer

$i \in I$, expression (1e), the lower-level objective function, maximizes the total preference of the customer. Constraint (1f) ensures that the customer accesses at most one facility. Constraint (1g) guarantees that the total costs incurred by the customer remains within his/her budget. Constraints (1h) prevent the customer from accessing facilities that are not open. Finally, constraints (1i) determine the binary character of the variables x_{ij}.

Quotation marks in the upper-level objective function refer to the ambiguity in formulating the bilevel optimization problem when the lower-level problem has multiple optimal solutions. Key insights regarding the optimality of bilevel problems are detailed in [24–29] and the references therein. In the context of the FLPMP problem, when customers rank the facilities in a strict order, i.e., no ties in preferences are allowed, the optimal solution to each customer's lower-level problem is unique. This solution entails choosing the most preferred facility among those that are affordable, if such facilities exist, or opting not to attend any facility if none is affordable or if all preferred facilities are closed. In this case, the quotation marks can be dropped since it is guaranteed that the bilevel problem is well posed. On the contrary, when customers exhibit indifference among several facilities in terms of preference, i.e., there exist ties in their preference lists, it is no longer guaranteed that each customer's lower-level problem will yield a unique optimal solution. In such cases, it becomes essential to establish a decision rule that enables the selection of one optimal solution from the set of multiple optima.

Various strategies have been proposed in the literature to tackle this challenge in general bilevel problems. The most widely adopted method is the optimistic or weak approach, which involves choosing the optimal solution to the lower-level problem that yields the best possible value of the upper-level objective function. This approach is popular due to its relative simplicity and tractability [26]. Conversely, the pessimistic or strong approach focuses on selecting the optimal solution to the lower-level problem that results in the worst possible value of the upper-level objective function. Pessimistic bilevel problems tend to be more complex to handle; nonetheless, various solutions have been proposed in the literature to tackle these issues [30,31]. In addition to optimistic and pessimistic approaches, some authors have suggested alternative approaches that are neither entirely optimistic nor pessimistic but rather fall into an intermediate or moderate category [32].

In conclusion, selecting an appropriate approach when multiple optima exist in the lower-level problem is a critical challenge and largely hinges on the specific characteristics of the problem being addressed. The following section focuses on this issue concerning the FLPMP with ties and proposes a reformulation of the problem to tackle it effectively. The case in which ties do not exist in customer preference lists is addressed in Section 8.

4. Bilevel Reformulation of the FLPMP with Ties

In the context of the FLPMP, customers must cover both travel costs and the price of the chosen facility. When a customer is indifferent among several facilities, i.e., they have the same preference value, the rational choice would be selecting the most economical one, which in this case is not the one with the lowest price but rather the one that leads to the lowest total cost. This rational behavior aligns with neither the optimistic approach, which selects the facility with the highest price, nor the pessimistic approach, which selects the one with the lowest price. For the sake of capturing the rational behavior of each follower, we propose to reformulate the lower-level problem of each customer, (1e)–(1i), as a lexicographic biobjective optimization problem. The first objective function maximizes the preference value derived from accessing one of the open facilities, while the second objective function minimizes the incurred costs.

Bearing in mind all of the aforementioned statements, the FLPMP with ties in customer preference lists can be formulated as the following bilinear–linear bilevel mixed-integer optimization problem with multiple independent followers and a lexicographic biobjective optimization problem at the lower level.

$$\max_{p,y,x} \quad \sum_{i \in I} \sum_{k \in J_i} p_k x_{ik} - \sum_{k \in J} f_k y_k \tag{2a}$$

s.t.:

$$p_j \leqslant M y_j \qquad j \in J \tag{2b}$$

$$p_j \geqslant 0 \qquad j \in J \tag{2c}$$

$$y_j \in \{0, 1\} \qquad j \in J \tag{2d}$$

where for each customer $i \in I$, $\{x_{ij}\}_{j \in J_i}$ solve

$$\operatorname*{lexmax}_{x_i} \quad \left[\sum_{k \in J_i} s_{ik} x_{ik}, \quad - \sum_{k \in J_i} (c_{ik} + p_k) x_{ik} \right] \tag{2e}$$

s.t.:

$$\sum_{k \in J_i} x_{ik} \leqslant 1 \tag{2f}$$

$$\sum_{k \in J_i} (c_{ik} + p_k) x_{ik} \leqslant b_i \tag{2g}$$

$$x_{ij} \leqslant y_j \qquad j \in J_i \tag{2h}$$

$$x_{ij} \in \{0, 1\} \qquad j \in J_i \tag{2i}$$

Even when the lower-level problem is reformulated as a lexicographic biobjective optimization problem, there may still be multiple optimal solutions. This can happen when facilities with the same preference value also incur the same total costs (either because they have identical prices and travel costs, or because different prices and travel costs result in the same overall cost). In such cases, since customers are only aware of their preferences and budgets, all multiple optima are equivalent from the lower-level perspective in terms of the objective function values. Thus, it makes sense to let the upper-level decision-maker choose the optimal solution that benefits him/her most, which is the one with the highest price. This solution is also justifiable from the customer's perspective, as it results in the lowest travel costs, which can be interpreted as a measure of proximity to the facility. This reasoning explains why the upper level also optimizes over the variables $\boldsymbol{x}$ and why quotation marks in (2a) are no longer necessary.

To illustrate the problem addressed in this paper, Table 2 provides the parameters and the optimal solution for an instance of the FLPMP involving 9 customers and 5 facilities. The inner left side of the table shows the preferences assigned by each customer to each facility. The symbol "–" means that the customer is not interested in the facility, either because he/she does not wish to access it or cannot reach it and therefore does not assign it a preference value. On the other hand, the inner right side of the table shows the travel costs associated with each customer–facility pair. The budget of each customer is shown in the last column. Finally, the last row of the table displays the fixed opening costs for facilities (on the left) and the optimal prices (on the right, in red). The facilities accessed by each customer in the optimal solution along with their preferences are highlighted in red in the table. Moreover, the optimal solution is depicted in Figure 1. Customers and facilities are displayed in blue and red, respectively, along with their budgets, fixed opening costs, prices, and the customer's choices represented by black arrows. The upper-level optimal objective function value is 47.

This instance provides an opportunity to highlight some interesting issues. Customer i_9 is equally interested in facilities j_4 and j_5 and the travel costs associated with these facilities are also the same. Therefore, customer i_9 chooses facility j_4 because it offers a lower price and incurs lower total costs. In contrast, customer i_6 is equally interested in facilities j_4 and j_5. However, although the price of j_4 is lower than that of j_5, customer i_9 chooses facility j_5 since the overall costs associated with both facilities are the same, and choosing facility j_5 benefits the upper-level decision-maker. Finally, customer i_4 does not access any facility, as j_1 is closed and the other facilities of interest (j_2, j_3 and j_4) have prices that exceed his/her budget after accounting for travel costs.

Table 2. An instance of the FLPMP with 9 customers and 5 facilities.

	Preferences					Travel Costs					
Customers	j_1	j_2	j_3	j_4	j_5	j_1	j_2	j_3	j_4	j_5	**Budget**
i_1	4	**5**	–	–	–	11	**3**	19	17	19	16
i_2	4	–	**5**	4	–	9	19	**7**	15	23	17
i_3	–	5	–	4	**5**	21	11	29	15	**11**	20
i_4	4	3	5	4	–	8	10	8	8	14	11
i_5	4	5	–	**5**	4	10	8	18	**6**	8	12
i_6	3	4	–	5	**5**	14	10	22	8	**6**	14
i_7	–	–	5	**4**	–	16	14	8	**6**	14	13
i_8	–	5	**5**	5	–	17	15	**3**	11	19	15
i_9	–	–	–	**5**	5	21	19	17	**5**	5	16
	Fixed opening costs					Optimal prices					
	5	5	5	5	5	**0**	**13**	**10**	**6**	**8**	

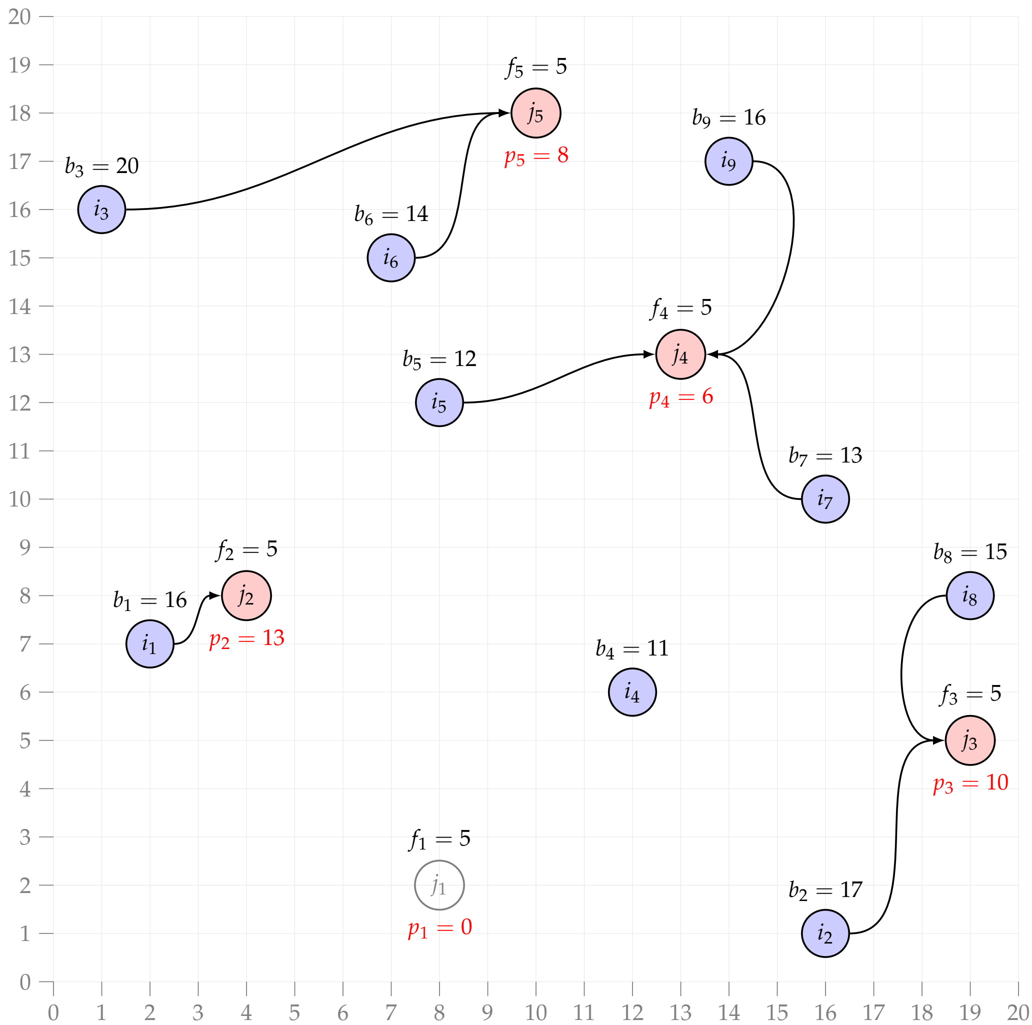

Figure 1. An optimal solution to the instance of the FLPMP detailed in Table 2.

5. Prices Can Be Discretized

The purpose of this section is to prove that prices, p_j, $j \in J$, can be discretized, as they only take values in predefined sets which depend on the budgets and travel costs.

For each facility $j \in J$, let us define $Q_j = \{b_i - c_{ij} : i \in I_j\}$. The elements of Q_j represent the remaining budget, after accounting for travel costs of customers who are interested in facility j. Note that Q_j contains each number only once, even in the case of repeated values. We assume that the elements in Q_j are sorted in increasing order. Let $L_j = \{1, \ldots, |L_j|\}$ be the set of indices that identify the elements in Q_j. Let q_j^l be the l-th element in the sorted set Q_j. In addition, for each facility $j \in J$, we define the function $\sigma_j \colon I_j \to L_j$ as $\sigma_j(i) = l$ if $b_i - c_{ij} = q_j^l$, i.e., σ_j maps the customer $i \in I_j$ to the index $l \in L_j$ corresponding to the position that the value $b_i - c_{ij}$ occupies in the ordered set Q_j.

Proposition 1. *There exists an optimal solution to the problem (2a)–(2i) such that the price p_j, for $j \in J$, takes a value from the set Q_j if facility j is open and is equal to zero otherwise.*

Proof. Suppose that an optimal solution $(\overline{\boldsymbol{p}}, \overline{\boldsymbol{y}}, \overline{\boldsymbol{x}})$ exists for which $\overline{p}_{j_0} \notin Q_{j_0}$ for some $j_0 \in J$. The aim of the proof is to construct another optimal solution $(\widetilde{\boldsymbol{p}}, \widetilde{\boldsymbol{y}}, \widetilde{\boldsymbol{x}})$ such that $\widetilde{p}_{j_0} \in Q_{j_0}$ if facility j_0 is open, and $\widetilde{p}_{j_0} = 0$ if facility j_0 is not open.

If facility j_0 is chosen by a customer in the optimal solution $(\overline{\boldsymbol{p}}, \overline{\boldsymbol{y}}, \overline{\boldsymbol{x}})$, then $\overline{p}_{j_0} \leqslant q_{j_0}^{|L_{j_0}|}$. Therefore, $q_{j_0}^l < \overline{p}_{j_0} < q_{j_0}^{l+1}$, for any $l \in L_{j_0}$. On the other hand, if customer i chooses facility j_0, then $\sigma_{j_0}(i) \geqslant l + 1$. Therefore, it is possible to construct the feasible solution $(\widetilde{\boldsymbol{p}}, \widetilde{\boldsymbol{y}}, \widetilde{\boldsymbol{x}})$ such that $\widetilde{p}_j = \overline{p}_j$ for every $j \neq j_0$, $\widetilde{p}_{j_0} = q_{j_0}^{l+1}$ and $\widetilde{\boldsymbol{y}} = \overline{\boldsymbol{y}}$, $\widetilde{\boldsymbol{x}} = \overline{\boldsymbol{x}}$. This would yield a strictly greater objective function value since the decisions regarding the open facilities and the customers' choices remain the same, but the customers choosing facility j_0 would be paying more. Thus, we arrive at a contradiction to the optimality of $(\overline{\boldsymbol{p}}, \overline{\boldsymbol{y}}, \overline{\boldsymbol{x}})$.

As a result, nobody chooses facility j_0 in the optimal solution $(\overline{\boldsymbol{p}}, \overline{\boldsymbol{y}}, \overline{\boldsymbol{x}})$. This may be due to the fact that they cannot afford it, or because they are choosing a more preferred facility even if they can afford it. Whatever the case, the solution that modifies the previous one by simply not opening facility j_0 and setting its price $\overline{p}_{j_0}$ to zero is also optimal as it yields the same objective function value. □

As a consequence of the preceding result, the FLPMP with ties admits a formulation in which the continuous variables p_j are replaced with a new set of binary variables. For each $j \in J$ and each $l \in L_j$, let v_j^l be a binary variable that takes value 1 if the price of facility j is set to $q_j^l \in Q_j$. Otherwise, it takes value 0. Using this set of variables, the price p_j of the facility $j \in J$ can be expressed as:

$$p_j = \sum_{l=1}^{|L_j|} q_j^l v_j^l \tag{3}$$

as long as the following set of constraints is added:

$$\sum_{l=1}^{|L_j|} v_j^l \leqslant y_j \qquad j \in J \tag{4}$$

Moreover, whether a customer can afford to access a certain facility or not is modeled by imposing the following set of constraints for each customer $i \in I$:

$$x_{ij} \leqslant \sum_{l=1}^{\sigma_j(i)} v_j^l \qquad j \in J_i \tag{5}$$

Based on the previous statements, the FLPMP with ties can be formulated as the following bilevel optimization problem with binary variables:

$$\max_{\boldsymbol{v}, \boldsymbol{y}, \boldsymbol{x}} \qquad \sum_{i \in I} \sum_{k \in J_i} \left(\sum_{l=1}^{\sigma_k(i)} q_k^l v_k^l \right) x_{ik} - \sum_{k \in J} f_k y_k \tag{6a}$$

s.t.:

$$\sum_{l=1}^{|L_j|} v_j^l \leqslant y_j \qquad j \in J \tag{6b}$$

$$v_j^l \in \{0,1\} \qquad j \in J \quad l \in L_j \tag{6c}$$

$$y_j \in \{0,1\} \qquad j \in J \tag{6d}$$

where for each customer $i \in I$, $\{x_{ij}\}_{j \in J_i}$ solve

$$\operatorname*{lexmax}_{x_i} \quad \left[\sum_{j \in J_i} s_{ik} x_{ik}, \quad -\sum_{k \in J_i} \left(c_{ik} + \left(\sum_{l=1}^{\sigma_k(i)} q_k^l v_k^l \right) \right) x_{ik} \right] \tag{6e}$$

s.t.:

$$\sum_{k \in J_i} x_{ij} \leqslant 1 \tag{6f}$$

$$x_{ij} \leqslant \sum_{l=1}^{\sigma_j(i)} v_j^l \qquad j \in J_i \tag{6g}$$

$$x_{ij} \leqslant y_j \qquad j \in J_i \tag{6h}$$

$$x_{ij} \in \{0,1\} \qquad j \in J_i \tag{6i}$$

To exactly solve the problem (6a)–(6i), it is reformulated in the following section as a single-level optimization problem by characterizing the set of optimal solutions to the lower-level problems corresponding to each customer.

6. A Single-Level Reformulation of the FLPMP with Ties

Consider the lower-level problem (6e)–(6i) corresponding to the i-th customer given the value of the upper-level variables $\left\{y_j, v_j^l \colon j \in J, l \in L_j\right\}$. These variables provide information about which facilities are open and at what price. Once their values are known, it becomes possible to determine which facilities are accessible to the i-th customer. These facilities are those that are both affordable and open. Let $J(i) \subseteq J_i$ be the set of accessible facilities to the i-th customer:

$$J(i) = \left\{ j \in J_i \colon y_j = 1 \text{ and } \sum_{l=1}^{\sigma_j(i)} v_j^l = 1 \right\} \tag{7}$$

Notice that, if $J(i) = \emptyset$, the customer cannot access any facility. Hence, from (6g) and (6h), we obtain that $x_{ij} = 0, j \in J_i$ is the optimal solution to the lower-level problem (6e)–(6h). Otherwise, i.e., if $J(i) \neq \emptyset$, $x_{ij} = 0$ when $j \in J_i \setminus J(i)$, and the lower-level problem corresponding to the i-th customer can be written as

$$\operatorname*{lexmax}_{x_i} \quad \left[\sum_{k \in J(i)} s_{ik} x_{ik}, \quad -\sum_{k \in J(i)} \left(c_{ik} + \left(\sum_{l=1}^{\sigma_k(i)} q_k^l v_k^l \right) \right) x_{ik} \right] \tag{8a}$$

s.t.:

$$\sum_{k \in J(i)} x_{ik} \leqslant 1 \tag{8b}$$

$$x_{ij} \in \{0,1\} \qquad j \in J(i) \tag{8c}$$

Next, the purpose is to characterize the optimal solutions of problem (8a)–(8c) using duality theory. Since lexicographic optimization considers one objective at a time, we focus first on the maximization problem (8a)–(8c) with respect to the first objective function:

$$\max_{\boldsymbol{x}_i} \quad \sum_{k\in J(i)} s_{ik}x_{ik} \tag{9a}$$

s.t.:

$$\sum_{k\in J(i)} x_{ik} \leqslant 1 \tag{9b}$$

$$x_{ij} \in \{0,1\} \qquad j \in J(i) \tag{9c}$$

Its optimal solution consists of selecting the facility $j \in J(i)$ with the largest preference or any of the tied facilities if there are several facilities with the same largest preference.

Taking into account the structure of constraint (9b), it is guaranteed that the linear relaxation of the problem (9a)–(9c) has an optimal solution with integer values and, thus, solves the original problem. The set of constraints (10) ensures optimality with respect to the first objective function:

$$\sum_{k\in J(i)} s_{ik}x_{ik} \geqslant s_{ij} \quad j \in J(i) \tag{10}$$

Once the first objective function has been optimized, the lexicographic approach proceeds to optimize the second objective function, which, in this case, selects the facility with the lowest total cost among those that are accessible and have the same highest preference. As pointed out above, if there are still tied facilities in terms of preference and cost, the selection is left to the upper-level decision-maker, as the optimal values of the lower-level objective function remain unchanged. In this case, the upper-level decision-maker has the opportunity to improve his/her objective function value without negatively impacting the lower-level decision-makers.

In light of the aforementioned points, the set of optimal solutions of the problem (11a)–(11d) includes an optimal solution to the problem (8a)–(8c).

$$\min_{\boldsymbol{x}_i} \quad \sum_{k\in J(i)} \left(c_{ik} + \left(\sum_{l=1}^{\sigma_k(i)} q_k^l v_k^l \right) \right) x_{ik} \tag{11a}$$

s.t.:

$$\sum_{k\in J(i)} x_{ik} \leqslant 1 \tag{11b}$$

$$\sum_{k\in J(i)} s_{ik}x_{ik} \geqslant s_{ij} \qquad j \in J(i) \tag{11c}$$

$$x_{ij} \geqslant 0 \qquad j \in J(i) \tag{11d}$$

The dual problem of (11a)–(11d) is:

$$\max_{u_i, \boldsymbol{w}_i} \quad -u_i + \sum_{k\in J(i)} s_{ik}w_{ik} \tag{12a}$$

s.t.:

$$-u_i + s_{ij} \sum_{k\in J(i)} w_{ik} \leqslant c_{ij} + \sum_{l=1}^{\sigma_j(i)} q_j^l v_j^l \qquad j \in J(i) \tag{12b}$$

$$u_i \geqslant 0 \tag{12c}$$

$$w_{ij} \geqslant 0 \qquad j \in J(i) \tag{12d}$$

where u_i is the dual variable associated with the constraint (11b) and $\{w_{ij}\}_{j\in J(i)}$ are the dual variables associated with the set of constraints (11c). As the primal problem (11a)–(11d) has an optimal solution, the dual problem (12a)–(12d) also has an optimal solution, and the optimal objective function values of both problems coincide. Hence, applying duality theory, it is guaranteed that $\{x_{ij}\}_{j\in J(i)}$ and $\{u_i, w_{ij}\}_{j\in J(i)}$ are optimal solutions to their respective problems if, and only if

$$\sum_{k \in J(i)} \left(c_{ik} + \left(\sum_{l=1}^{\sigma_k(i)} q_k^l v_k^l \right) \right) x_{ik} = -u_i + \sum_{k \in J(i)} s_{ik} w_{ik} \tag{13a}$$

$$\sum_{k \in J(i)} x_{ik} \leqslant 1 \tag{13b}$$

$$\sum_{k \in J(i)} s_{ik} x_{ik} \geqslant s_{ij} \qquad j \in J(i) \tag{13c}$$

$$-u_i + s_{ij} \sum_{k \in J(i)} w_{ik} \leqslant c_{ij} + \left(\sum_{l=1}^{\sigma_j(i)} q_j^l v_j^l \right) \qquad j \in J(i) \tag{13d}$$

$$x_{ij} \geqslant 0 \qquad j \in J(i) \tag{13e}$$

$$u_i \geqslant 0 \tag{13f}$$

$$w_{ij} \geqslant 0 \qquad j \in J(i) \tag{13g}$$

Isolating variable u_i from (13a) and replacing it at (13d) and (13f), the following set of constraints is obtained:

$$\sum_{k \in J(i)} x_{ik} \leqslant 1 \tag{14a}$$

$$\sum_{k \in J(i)} s_{ik} x_{ik} \geqslant s_{ij} \qquad j \in J(i) \tag{14b}$$

$$\sum_{k \in J(i)} \left(c_{ik} + \left(\sum_{l=1}^{\sigma_k(i)} q_k^l v_k^l \right) \right) x_{ik} - \sum_{k \in J(i)} s_{ik} w_{ik} + s_{ij} \sum_{k \in J(i)} w_{ik}$$
$$\leqslant c_{ij} + \left(\sum_{l=1}^{\sigma_j(i)} q_j^l v_j^l \right) \qquad j \in J(i) \tag{14c}$$

$$\sum_{k \in J(i)} s_{ik} w_{ik} - \sum_{k \in J(i)} \left(c_{ik} + \left(\sum_{l=1}^{\sigma_k(i)} q_k^l v_k^l \right) \right) x_{ik} \geqslant 0 \tag{14d}$$

$$x_{ij} \geqslant 0 \qquad j \in J(i) \tag{14e}$$

$$w_{ij} \geqslant 0 \qquad j \in J(i) \tag{14f}$$

The set of constraints (14a)–(14f) has been derived under the assumption that the upper-level decision variables are known, and thus, only the lower-level decision variables corresponding to facilities $j \in J(i)$ are needed, as the remaining variables are set to zero by constraints (6g) and (6h). Therefore, in order to replace the lower-level problem of the i-th customer by constraints (14a)–(14f), it is necessary to extend them over the entire set of facilities J_i, ensuring that the constraints apply when $j \in J(i)$ and impose no additional conditions when $j \in J_i \setminus J(i)$. Below, we explain how each constraint is extended.

- Variables w_{ij} were defined only for $j \in J(i)$. Now, variables w_{ij} for $j \in J_i \setminus J(i)$ are introduced and set to zero. Thus, constraints (14f) are replaced by

$$w_{ij} \leqslant M_{ij} \sum_{l=1}^{\sigma_j(i)} v_j^l \qquad j \in J_i \tag{15}$$

$$w_{ij} \geqslant 0 \qquad j \in J_i \tag{16}$$

  where M_{ij} is a big enough constant that guarantees these constraints are only restrictive when $j \in J_i \setminus J(i)$
- Constraints (14e) are replaced by $x_{ij} \geqslant 0, j \in J_i$, as long as constraints (6g) and (6h) are added. Note that these constraints set variables x_{ij} to zero for $j \in J_i \setminus J(i)$.

- Summation indices are extended from $J(i)$ to J_i in constraints (14a)–(14d), as only terms equal to zero are included.
- Constraints (14b) are replaced by

$$\sum_{k\in J_i} s_{ik}x_{ik} \geqslant s_{ij} \sum_{l=1}^{\sigma_j(i)} v_j^l \qquad j \in J_i \tag{17}$$

  Note that when $j \in J(i)$, these correspond to constraints (14b). On the other hand, when $j \in J_i \setminus J(i)$, $\sum_{l=1}^{\sigma_j(i)} v_j^l = 0$, and the constraints reduce to $\sum_{k\in J_i} s_{ik}x_{ik} \geqslant 0$, which always hold.
- Finally, constraints (14c) are replaced by

$$\begin{aligned}&\sum_{k\in J_i}\left(c_{ik} + \left(\sum_{l=1}^{\sigma_k(i)} q_k^l v_k^l\right)\right)x_{ik} - \sum_{k\in J_i} s_{ik}w_{ik} + s_{ij}\sum_{k\in J_i} w_{ik}\\ &\leqslant c_{ij} + \left(\sum_{l=1}^{\sigma_j(i)} q_j^l v_j^l\right) + \widetilde{M}_{ij}\left(1 - \sum_{l=1}^{\sigma_j(i)} v_j^l\right) \qquad j \in J_i\end{aligned} \tag{18}$$

  where $\widetilde{M}_{ij}$ is a big enough constant that ensures these constraints are only restrictive when $j \in J(i)$.

As a result, the FLPMP with ties can be reformulated as the following single-level mixed-integer bilinear optimization problem:

$$\max_{v,w,x,y} \quad \sum_{i\in I}\sum_{k\in J_i}\left(\sum_{l=1}^{\sigma_k(i)} q_k^l v_k^l\right)x_{ik} - \sum_{k\in J} f_k y_k \tag{19a}$$

s.t.:

$$\sum_{l=1}^{|L_j|} v_j^l \leqslant y_j \qquad j \in J \tag{19b}$$

$$\sum_{k\in J_i} x_{ik} \leqslant 1 \qquad i \in I \tag{19c}$$

$$x_{ij} \leqslant y_j \qquad i \in I \quad j \in J_i \tag{19d}$$

$$x_{ij} \leqslant \sum_{l=1}^{\sigma_j(i)} v_j^l \qquad i \in I \quad j \in J_i \tag{19e}$$

$$w_{ij} \leqslant M_{ij} \sum_{l=1}^{\sigma_j(i)} v_j^l \qquad i \in I \quad j \in J_i \tag{19f}$$

$$\sum_{k\in J_i} s_{ik}x_{ik} \geqslant s_{ij} \sum_{l=1}^{\sigma_j(i)} v_j^l \qquad i \in I \quad j \in J_i \tag{19g}$$

$$\begin{aligned}&\sum_{k\in J_i}\left(c_{ik} + \left(\sum_{l=1}^{\sigma_k(i)} q_k^l v_k^l\right)\right)x_{ik} - \sum_{k\in J_i} s_{ik}w_{ik} + s_{ij}\sum_{k\in J_i} w_{ik}\\ &\leqslant c_{ij} + \left(\sum_{l=1}^{\sigma_j(i)} q_j^l v_j^l\right) + \widetilde{M}_{ij}\left(1 - \sum_{l=1}^{\sigma_j(i)} v_j^l\right) \qquad i \in I \quad j \in J_i\end{aligned} \tag{19h}$$

$$\sum_{k\in J_i} s_{ik}w_{ik} - \sum_{k\in J_i}\left(c_{ik} + \left(\sum_{l=1}^{\sigma_k(i)} q_k^l v_k^l\right)\right)x_{ik} \geqslant 0 \qquad i \in I \tag{19i}$$

$$v_j^l \in \{0,1\} \qquad j \in J \quad l \in L_j \tag{19j}$$

$$y_j \in \{0,1\} \qquad j \in J \tag{19k}$$
$$x_{ij} \in \{0,1\} \qquad i \in I \quad j \in J_i \tag{19l}$$
$$w_{ij} \geqslant 0 \qquad i \in I \quad j \in J_i \tag{19m}$$

In order to linearize the above problem, we introduce a new set of non-negative variables $\{z_{ij} : i \in I, j \in J_i\}$. These variables are defined for each customer $i \in I$ and each facility $j \in J_i$ as the profit obtained from customer i accessing facility j. That is to say,

$$z_{ij} = \left(\sum_{l=1}^{\sigma_j(i)} q_j^l v_j^l \right) x_{ij} \tag{20}$$

In order to guarantee this equality, the following three sets of additional constraints must be added:

$$z_{ij} \leqslant \sum_{l=1}^{\sigma_j(i)} q_j^l v_j^l \qquad i \in I \quad j \in J_i \tag{21a}$$
$$z_{ij} \leqslant \overline{M}_{ij} x_{ij} \qquad i \in I \quad j \in J_i \tag{21b}$$
$$z_{ij} \geqslant \sum_{l=1}^{\sigma_j(i)} q_j^l v_j^l - \overline{M}_{ij}(1 - x_{ij}) \qquad i \in I \quad j \in J_i \tag{21c}$$

The first and third constraints together ensure that the profit obtained from a customer accessing a certain facility is equal to the price set for that facility when the customer chooses to access it. The second constraint imposes that the profit obtained from a customer must be zero if the customer does not access that facility.

Therefore, the FLPMP with ties can finally be formulated as the following single-level linear mixed-integer optimization problem:

$$\max_{\boldsymbol{v},\boldsymbol{w},\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}} \quad \sum_{i \in I} \sum_{k \in J_i} z_{ik} - \sum_{k \in J} f_k y_k \tag{22a}$$

s.t.:

$$\sum_{l=1}^{|L_j|} v_j^l \leqslant y_j \qquad j \in J \tag{22b}$$
$$\sum_{k \in J_i} x_{ik} \leqslant 1 \qquad i \in I \tag{22c}$$
$$x_{ij} \leqslant y_j \qquad i \in I \quad j \in J_i \tag{22d}$$
$$x_{ij} \leqslant \sum_{l=1}^{\sigma_j(i)} v_j^l \qquad i \in I \quad j \in J_i \tag{22e}$$
$$w_{ij} \leqslant M_{ij} \sum_{l=1}^{\sigma_j(i)} v_j^l \qquad i \in I \quad j \in J_i \tag{22f}$$
$$\sum_{k \in J_i} s_{ik} x_{ik} \geqslant s_{ij} \sum_{l=1}^{\sigma_j(i)} v_j^l \qquad i \in I \quad j \in J_i \tag{22g}$$
$$\begin{aligned} &\sum_{k \in J_i} c_{ik} x_{ik} + \sum_{k \in J_i} z_{ik} - \sum_{k \in J_i} s_{ik} w_{ik} + s_{ij} \sum_{k \in J_i} w_{ik} \\ &\leqslant c_{ij} + \left(\sum_{l=1}^{\sigma_j(i)} q_j^l v_j^l \right) + \widetilde{M}_{ij} \left(1 - \sum_{l=1}^{\sigma_j(i)} v_j^l \right) \qquad i \in I \quad j \in J_i \end{aligned} \tag{22h}$$
$$\sum_{k \in J_i} s_{ik} w_{ik} - \sum_{k \in J_i} c_{ik} x_{ik} - \sum_{k \in J_i} z_{ik} \geqslant 0 \qquad i \in I \tag{22i}$$

$$z_{ij} \leqslant \sum_{l=1}^{\sigma_j(i)} q_j^l v_j^l \qquad i \in I \quad j \in J_i \tag{22j}$$

$$z_{ij} \leqslant \overline{M}_{ij} x_{ij} \qquad i \in I \quad j \in J_i \tag{22k}$$

$$z_{ij} \geqslant \sum_{l=1}^{\sigma_j(i)} q_j^l v_j^l - \overline{M}_{ij}(1 - x_{ij}) \qquad i \in I \quad j \in J_i \tag{22l}$$

$$v_j^l \in \{0,1\} \qquad j \in J \quad l \in L_j \tag{22m}$$

$$y_j \in \{0,1\} \qquad j \in J \tag{22n}$$

$$z_{ij} \geqslant 0 \qquad i \in I \quad j \in J_i \tag{22o}$$

$$x_{ij} \in \{0,1\} \qquad i \in I \quad j \in J_i \tag{22p}$$

$$w_{ij} \geqslant 0 \qquad i \in I \quad j \in J_i \tag{22q}$$

Note that the final model has $|J| + \sum_{j \in J} |L_j| + \sum_{i \in I} |J_i|$ binary variables, $2\sum_{i \in I} |J_i|$ continuous variables and $|J| + 2|I| + 8\sum_{i \in I} |J_i|$ constraints.

Given the significant role that determining appropriate big-M values plays in reformulating the bilevel optimization problem using duality properties [33], the next section explores this issue in depth. The goal is to compute suitable constants for the model (22a)–(22q) by leveraging its unique characteristics.

7. Deriving Valid Values for the Big-Ms

Proposition 2. *For every customer $i \in I$ and every facility $j \in J_i$, $\overline{M}_{ij} = b_i - c_{ij}$ is a valid constant in constraints (22k) and (22l).*

Proof. The profit obtained from a customer $i \in I$ accessing a facility $j \in J$, represented by the variable z_{ij}, is always less than or equal to $b_i - c_{ij}$. Thus, $\overline{M}_{ij} = b_i - c_{ij}$ is a valid constant in constraints (22k). On the other hand,

$$\sum_{l=1}^{\sigma_j(i)} q_j^l v_j^j \leqslant q_j^{\sigma_j(i)} = b_i - c_{ij}. \tag{23}$$

Thus, $\overline{M}_{ij} = b_i - c_{ij}$ is also a valid constant in constraints (22l). □

To obtain valid bounds for M_{ij} and $\widetilde{M}_{ij}$, we assume that we are given the values of the upper-level variables $\{p_j, y_j\}_{j \in J}$ and that $J(i) \neq \emptyset$ for $i \in I$. Otherwise, as mentioned before, $x_{ij} = w_{ij} = z_{ij} = 0$, $j \in J_i$, and constraints (22f) and (22h) hold regardless of the values of these big-Ms. The following additional notation is introduced:

$$s_i^{\max} = \max\{s_{ij} \colon j \in J(i)\} \tag{24}$$

$$J(i)^+ = \{j \in J(i) \colon s_{ij} = s_i^{\max}\} \tag{25}$$

$$a_i^{\min} = \min\{c_{ij} + p_j \colon j \in J(i)^+\} \tag{26}$$

Note that customer $i \in I$ accesses a facility $h \in J(i)^+$ such that $c_{ih} + p_h = a_i^{\min}$, given that $J(i) \neq \emptyset$.

Proposition 3. *Given the values of the upper-level variables $\{p_j, y_j\}_{j \in J}$, for each customer $i \in I$ such that $J(i) \neq \emptyset$, let $h \in J(i)^+$ be the facility such that $c_{ih} + p_h = a_i^{min}$. Then,*

$$u_i = -a_i^{min} + s_{ih} w_{ih} \tag{27a}$$
$$w_{ih} = \max\left\{ \max_{j \in J(i) \setminus J(i)^+} \left\{ \frac{a_i^{min} - (c_{ij} + p_j)}{s_{ih} - s_{ij}} \right\}, \frac{a_i^{min}}{s_{ih}} \right\} \tag{27b}$$
$$w_{ij} = 0 \qquad \textit{for } j \in J(i) \textit{ and } j \neq h \tag{27c}$$

is an optimal solution to the dual problem (12a)–(12d).

Proof. The proof is constructive in the sense that it derives the solution (27a)–(27c) by selecting a feasible solution to the dual problem (12a)–(12d), which yields an objective function value equal to the optimal objective function value of the corresponding primal problem. Note that the solution to the lower-level primal problem considered is $x_{ih} = 1$, $x_{ij} = 0, j \in J_i, j \neq h$.

We select, $w_{ij} = 0$ for $j \in J(i)$ and $j \neq h$ and $u_i = -a_i^{\min} + s_{ih} w_{ih}$. Moreover, to guarantee that $u_i \geqslant 0$, it must be ensured that $w_{ih} \geqslant \frac{a_i^{\min}}{s_{ih}}$, which, as a consequence, will result in $w_{ih} \geqslant 0$. Therefore, constraint (12b) can be written as

$$-u_i + s_{ij} \sum_{k \in J(i)} w_{ik} = -u_i + s_{ij} w_{ih} \leqslant c_{ij} + \sum_{l=1}^{\sigma_j(i)} q_j^l v_j^l = c_{ij} + p_j \qquad j \in J(i) \tag{28}$$

Constraints (28) are trivially satisfied for $j \in J(i)^+$, as $s_{ij} = s_{ih}$ and $a_i^{\min} \leqslant c_{ij} + p_j$ when $j \in J(i)^+$. For those $j \notin J(i)^+$, to guarantee these constraints, it must be ensured that

$$w_{ih} \geqslant \frac{a_i^{\min} - (c_{ij} + p_j)}{s_{ih} - s_{ij}} \qquad j \in J(i) \setminus J(i)^+$$

In particular, taking w_{ih} as

$$w_{ih} = \max\left\{ \max_{j \in J(i) \setminus J(i)^+} \left\{ \frac{a_i^{\min} - (c_{ij} + p_j)}{s_{ih} - s_{ij}} \right\}, \frac{a_i^{\min}}{s_{ih}} \right\}$$

it is ensured that (27a)–(27c) is a dual feasible solution. Furthermore, for both solutions, it is verified that

$$\sum_{k \in J(i)} \left(c_{ik} + \left(\sum_{l=1}^{\sigma_k(i)} q_k^l v_k^l \right) \right) x_{ik} = c_{ih} + p_h = a_i^{\min} = -u_i + s_{ij} w_{ih} = -u_i + \sum_{j \in J(i)} s_{ij} w_{ij} \tag{29}$$

Thus, the objective function values of the primal and dual problems coincide and both solutions are optimal for their respective problems. In particular, (27a)–(27c) is an optimal solution to the dual problem (12a)–(12d). □

Proposition 4. *For every customer $i \in I$ and every facility $j \in J_i$,*

$$M_{ij} = \max\left\{ b_i - \min_{k \in J_i}\{c_{ik}\}, \frac{b_i}{s_{ij}} \right\} \tag{30}$$
$$\widetilde{M}_{ij} = s_{ij} \max_{k \in J_i}\{M_{ik}\} \tag{31}$$

are valid constants in constraints (22f) and (22h), respectively.

Proof. Note that, according to constraints (22f), M_{ij} must be an upper bound of the value of the dual variable w_{ij} for each $i \in I$ and $j \in J_i$. Moreover, taking into account constraints (22i), it follows that the left side of constraints (22h) satisfies the following inequality:

$$\sum_{k\in J_i} c_{ik}x_{ik} + \sum_{k\in J_i} z_{ik} - \sum_{k\in J_i} s_{ik}w_{ik} + s_{ij}\sum_{k\in J_i} w_{ik} \leqslant s_{ij}\sum_{k\in J_i} w_{ik} \quad i \in I \quad j \in J_i \tag{32}$$

Therefore, the task of determining a valid value for $\widetilde{M}_{ij}$ becomes finding an upper bound of the sum $\sum_{k\in J_i} w_{ik}$ for each customer $i \in I$.

Note also that constants $\left\{M_{ij}, \widetilde{M}_{ij}\right\}_{i\in I,\, j\in J_i}$ are introduced when the set of constraints (14a)–(14f) is extended from $J(i)$ to J_i. This set of constraints ensures the optimality of the lower-level problems and is derived based on the fact that, for each customer $i \in I$, $\{u_i, w_{ij}\}_{j\in J(i)}$ is an optimal solution to the dual problem (12a)–(12d). Therefore, constants $\left\{M_{ij}, \widetilde{M}_{ij}\right\}_{i\in I,\, j\in J_i}$ must be chosen so that $\{u_i, w_{ij}\}_{j\in J(i)}$ can take values that enable them to be optimal. According to Proposition 3, (27a)–(27c) is an optimal solution to the dual problem (12a)–(12d). Therefore, it is sufficient to compute constants $\left\{M_{ij}, \widetilde{M}_{ij}\right\}_{i\in I,\, j\in J_i}$ that are valid for this solution.

Given a customer $i \in I$, it follows that $a_i^{\min} \leqslant b_i$ by the definition of $a_i^{\min}$. Furthermore, keeping in mind that $p_j \geqslant 0$ for any arbitrary $j \in J$, and that $s_{ih} > s_{ij}$ for $j \in J(i) \setminus J(i)^+$, the following inequality holds:

$$\frac{a_i^{\min} - (c_{ij} + p_j)}{s_{ih} - s_{ij}} \leqslant b_i - c_{ij} \qquad j \in J(i) \setminus J(i)^+ \tag{33}$$

Hence,

$$\max_{j\in J(i)\setminus J(i)^+}\left\{\frac{a_i^{\min} - (c_{ij} + p_j)}{s_{ih} - s_{ij}}\right\} \leqslant \max_{j\in J(i)\setminus J(i)^+}\{b_i - c_{ij}\} \leqslant \leqslant \max_{j\in J_i}\{b_i - c_{ij}\} = b_i - \min_{j\in J_i}\{c_{ij}\} \tag{34}$$

On the other hand,

$$\frac{a_i^{\min}}{s_{ih}} \leqslant \frac{b_i}{s_{ih}} \tag{35}$$

Therefore,

$$w_{ih} \leqslant \max\left\{b_i - \min_{k\in J_i}\{c_{ik}\}, \frac{b_i}{s_{ih}}\right\} \tag{36}$$

and

$$M_{ih} = \max\left\{b_i - \min_{k\in J_i}\{c_{ik}\}, \frac{b_i}{s_{ih}}\right\} \tag{37}$$

is an upper bound of w_{ih}, which is a valid constant.

Based on the previous statements and considering that (27a)–(27c) is an optimal solution to the dual problem (12a)–(12d) where all the variables w_{ij} are zero except for one,

$$M_{ij} = \max\left\{b_i - \min_{k\in J_i}\{c_{ik}\}, \frac{b_i}{s_{ij}}\right\} \qquad i \in I \quad j \in J_i \tag{38}$$

are valid constants in constraints (22f).

In addition, in the dual optimal solution (27a)–(27c), $\sum_{j\in J_i} w_{ij}$ equals the value of the dual variable corresponding to the facility that incurs the lowest costs among those providing the highest preference value to customer $i \in I$. Therefore, it is less than or equal to $\max_{k\in J_i} M_{ik}$. Thus,

$$\widetilde{M}_{ij} = s_{ij}\max_{k\in J_i}\{M_{ik}\} \tag{39}$$

is a valid constant in constraints (22h). □

8. The FLPMP Without Ties

As mentioned above, when there are no ties in preferences, problem (1a)–(1i) is well-posed and quotation marks are not needed in the upper-level objective function. Moreover, its reformulation as a single-level problem is more straightforward, as there is no need to consider the lexicographic approach. Therefore, the FLPMP without ties can be formulated as the following single-level linear mixed-integer optimization problem:

$$\max_{v,w,x,y,z} \quad \sum_{i\in I}\sum_{j\in J_i} z_{ij} - \sum_{j\in J} f_j y_j \tag{40a}$$

s.t.:

$$\sum_{l=1}^{|L_j|} v_j^l \leqslant y_j \qquad j\in J \tag{40b}$$

$$\sum_{j\in J_i} x_{ij} \leqslant 1 \qquad i\in I \tag{40c}$$

$$x_{ij} \leqslant y_j \qquad i\in I \quad j\in J_i \tag{40d}$$

$$x_{ij} \leqslant \sum_{l=1}^{\sigma_j(i)} v_j^l \qquad i\in I \quad j\in J_i \tag{40e}$$

$$\sum_{k\in J_i} s_{ik}x_{ik} \geqslant s_{ij}\sum_{l=1}^{\sigma_j(i)} v_j^l \qquad i\in I \quad j\in J_i \tag{40f}$$

$$z_{ij} \leqslant \sum_{l=1}^{\sigma_j(i)} q_j^l v_j^l \qquad i\in I \quad j\in J_i \tag{40g}$$

$$z_{ij} \leqslant \overline{M}_{ij}x_{ij} \qquad i\in I \quad j\in J_i \tag{40h}$$

$$z_{ij} \geqslant \sum_{l=1}^{\sigma_j(i)} q_j^l v_j^l - \overline{M}_{ij}(1-x_{ij}) \qquad i\in I \quad j\in J_i \tag{40i}$$

$$v_j^l \in \{0,1\} \qquad j\in J \quad l\in L_j \tag{40j}$$

$$y_j \in \{0,1\} \qquad j\in J \tag{40k}$$

$$z_{ij} \geqslant 0 \qquad i\in I \quad j\in J_i \tag{40l}$$

$$x_{ij} \in \{0,1\} \qquad i\in I \quad j\in J_i \tag{40m}$$

$$w_{ij} \geqslant 0 \qquad i\in I \quad j\in J_i \tag{40n}$$

Note that Proposition 2 can also be applied to this problem, and therefore, $\overline{M}_{ij} = b_i - c_{ij}$, $i \in I$, $j \in J_i$ are valid constants in constraints (40h) and (40i).

9. Computational Study

In this section, the results of the computational experiments conducted to analyze the performance of the model and the effectiveness of the single-level reformulation are presented and discussed. The computational experiments were performed on a PC equipped with a 13th Gen Intel Core i9-13900F processor. The system has 64 GB of RAM and runs Windows 11 64-bit as operating system. All computations were implemented using Python 3.10 and Gurobi 10.0.3.

The performance of the model was tested on the set of benchmark instances for the *Facility Location and Pricing Problem (FLPr)* from the *Discrete Location Problems Library*. These instances were modified to be used in the evaluation of the performance of the FLPMP model (22a)–(22q). The original set of instances, provided by the authors of [13], consists of 20 instances without fixed costs for opening facilities, divided into two groups: 10 instances with 100 customers and 40 facilities and 10 instances with 100 customers and 100 facilities.

The modifications consist of defining customer preferences for each reachable location, as well as introducing a new scenario that includes fixed opening costs for the facilities.

Customer preferences are randomly generated so that each customer $i \in I$ assigns a preference value to all his/her reachable facilities. Preference values are assigned starting from the highest possible value, $|J|$, followed by $|J| - 1$, and continuing until all reachable facilities have been assigned a preference value. To determine which facility or facilities receive the incumbent preference value, first, the number of facilities receiving it is randomly selected from the set $\{1, 2, 3, 4\}$ with probabilities $\{0.5, 0.3, 0.1, 0.1\}$, respectively. Subsequently, specific facilities are randomly chosen from the set of reachable facilities that have not yet been assigned a preference value.

For each customer, the aforementioned process assigns a preference value to each of his/her reachable facilities. As noted when introducing the model, customers typically show interest in only a subset of these facilities. To capture this aspect, each customer $i \in I$ is associated with a parameter $\alpha_i \in [0, 1]$, which indicates that customer $i \in I$ is actually interested in at least the $\alpha_i \cdot 100\%$ of his/her most preferred reachable facilities. In other words, up to $(1 - \alpha_i) \cdot 100\%$ of his/her least preferred reachable facilities are excluded from consideration. To determine which facilities are excluded, the percentile P_{α_i} of the preference values for each customer $i \in I$ is computed, and any facility with a preference value less than or equal to $P_{\alpha_i} - 1$ is removed from the set of preferred facilities.

In the computational study, we analyzed the influence of two factors, the length of the customer preference lists and the existence or not of fixed opening costs associated with the facilities. The levels of length are *long* (meaning that the preference list contains every reachable facility) and *short* (meaning that the parameter $\alpha_i = 0.5$, $i \in I$, is applied). The levels of the fixed costs were set to $f_j = 0$, $j \in J$ (as in the original set of instances), and $f_j = 20$, $j \in J$. As a result, 80 test instances were solved. Finally, the stopping criterion has been set to a time limit of 3600 s.

Tables 3 and 4 display the results. They are organized in a similar way. The first column provides the identification of the instance and it follows the structure # number of customers_# of facilities_# of the instance. The second column refers to the length of the customer preference list. The third column stands for the value of the fixed cost of opening the facilities. The fourth column refers to the best objective function value provided by Gurobi. The fifth and sixth columns give the number of served customers and the number of open facilities in the best solution provided by Gurobi, respectively. The seventh column T provides the computational time (in seconds). The eighth column refers to the MIPGap (in percentage) provided by Gurobi when the instance is not solved to optimality.

Table 3. Results for each instance of size $|I| = 100$ and $|J| = 40$. ID refers to the identification of the instance and it follows the structure # number of customers_# of facilities_# of the instance. Length indicates whether the preference lists of each customer are complete (long) or not (short). Column f_j stands for the value of the fixed cost for opening the facilities. Column Z refers to the best objective function value provided by Gurobi. Columns m and n give the number of served customers and the number of open facilities in the best solution provided by Gurobi, respectively. Column T means computational time (in seconds) and column $\%Gap$ refers to the MIPGap (in percentage) provided by Gurobi when the instance reaches the stopping criterion.

ID	Length	f_j	Z	m	n	T	$\%Gap$
100_40_1	short	0	3067	62	33	124.81	–
		20	2480	61	23	265.06	–
	long	0	3293	69	35	577.28	–
		20	2754	68	20	1332.04	–
100_40_2	short	0	3140	59	33	87.77	–
		20	2592	60	21	129.43	–
	long	0	3347	62	33	252.22	–
		20	2761	62	25	817.54	–

Table 3. *Cont.*

ID	Length	f_j	Z	m	n	T	$\%Gap$
100_40_3	short	0	2833	66	32	66.22	–
		20	2274	68	25	68.09	–
	long	0	3080	67	34	372.39	–
		20	2496	66	24	1581.27	–
100_40_4	short	0	2352	58	30	11.72	–
		20	1811	57	23	9.15	–
	long	0	2476	58	31	48.18	–
		20	1923	57	22	307.65	–
100_40_5	short	0	3111	61	35	447.91	–
		20	2560	54	22	380.91	–
	long	0	3332	68	35	2909.20	–
		20	2815	62	22	3600.00	2.64
100_40_6	short	0	2747	59	31	19.39	–
		20	2227	56	23	32.78	–
	long	0	2822	61	31	290.30	–
		20	2321	59	19	503.66	–
100_40_7	short	0	2913	50	31	17.06	–
		20	2351	50	25	52.02	–
	long	0	3071	54	33	438.25	–
		20	2503	49	21	903.51	–
100_40_8	short	0	2793	59	33	187.36	–
		20	2256	58	21	166.43	–
	long	0	2998	61	32	531.50	–
		20	2460	56	18	2845.82	–
100_40_9	short	0	2630	55	29	35.93	–
		20	2139	51	19	31.51	–
	long	0	2777	52	30	323.67	–
		20	2277	54	21	711.22	–
100_40_10	short	0	2882	61	33	109.07	–
		20	2371	50	21	35.26	–
	long	0	3077	68	33	593.56	–
		20	2545	56	20	2373.59	–

Table 4. Results for each instance of size $|I| = 100$ and $|J| = 100$. ID refers to the identification of the instance and it follows the structure # number of customers_# of facilities_# of the instance. Length indicates whether the preference lists of each customer are complete (long) or not (short). Column f_j stands for the value of the fixed cost for opening the facilities. Column Z refers to the best objective function value provided by Gurobi. Columns m and n give the number of served customers and the number of open facilities in the best solution provided by Gurobi, respectively. Column T means computational time (in seconds) and column $\%Gap$ refers to the MIPGap (in percentage) provided by Gurobi when the instance reaches the stopping criterion.

ID	Length	f_j	Z	m	n	T	$\%Gap$
100_100_1	short	0	3532	74	57	454.91	–
		20	2679	67	32	1676.48	–
	long	0	3583	74	59	3600.00	4.27
		20	2760	64	25	3600.00	7.62
100_100_2	short	0	3696	76	57	1201.48	–
		20	2782	71	35	3600.00	0.97
	long	0	3836	77	58	3600.00	1.8
		20	2989	72	32	3600.00	5.58
100_100_3	short	0	3348	64	53	250.20	–
		20	2505	62	29	363.05	–
	long	0	3410	62	48	1483.22	–
		20	2640	60	28	3600.00	2.58

Table 4. *Cont.*

ID	Length	f_j	Z	m	n	T	$\%Gap$
100_100_4	short	0	3340	70	59	3600.00	1.38
		20	2453	59	29	3600.00	3.13
	long	0	3443	66	49	3600.00	3.92
		20	2597	57	27	3600.00	11.16
100_100_5	short	0	3377	73	60	1366.63	–
		20	2523	60	31	3395.74	–
	long	0	3520	67	54	3600.00	1.62
		20	2703	67	32	3600.00	4.79
100_100_6	short	0	3664	67	57	412.00	–
		20	2770	64	32	1071.62	–
	long	0	3710	64	53	3600.00	2.89
		20	2825	62	27	3600.00	7.64
100_100_7	short	0	3135	66	57	511.43	–
		20	2309	63	33	1033.28	–
	long	0	3210	64	53	3600.00	2.13
		20	2416	58	26	3600.00	5.65
100_100_8	short	0	3131	67	51	824.40	–
		20	2279	54	31	1432.25	–
	long	0	3294	77	59	3600.00	2.01
		20	2395	56	23	3600.00	7.97
100_100_9	short	0	3160	70	52	314.37	–
		20	2309	62	34	523.26	–
	long	0	3243	69	52	3600.00	0.32
		20	2401	62	30	3600.00	4.7
100_100_10	short	0	3827	77	57	2833.13	–
		20	2956	75	32	1273.62	–
	long	0	3993	81	59	3600.00	2.21
		20	3074	75	31	3600.00	8.39

Looking at Table 3, which corresponds to the instances with 100 customers and 40 facilities, an optimal solution is achieved in 19 out of 20 instances, with the unsolved instance having a gap of 2.64%. For instances with 100 customers and 100 facilities, shown in Table 4, fewer problems are solved to optimality, as expected. The time limit is reached in 22 out of 40 instances. In the short-length setting, 17 out of 20 instances are solved, with the remaining three showing a gap between 0.97% and 3.13%. In the long-length setting, only one instance is solved, with the remaining 19 showing gaps between 0.32% and 11.20%. Figure 2 displays the computational time required to solve each instance. Lines have been broken to distinguish between the two blocks of instances. It can be observed that the computational time required is primarily influenced by the problem size, followed by the length of the preference lists, with the existence of fixed opening costs having a smaller impact.

Next, we analyze the solutions obtained by focusing on the customers served, facilities opened, and revenues (objective function value plus total opening facility costs). Figure 3 shows the number of open facilities (with lines broken to distinguish between the two blocks of instances). As expected, more facilities are open when the number of facilities is 100 and when the fixed opening cost is 0. Furthermore, the effect of the fixed opening cost on the number of open facilities, which increases as the fixed opening cost decreases from 20 to 0, is greater when the number of potential facilities is 100. The length factor has minimal effect.

Figure 4 displays the number of customers accessing any facility. In this case, the effect of both factors is less straightforward than before. The most important effect arises from the fixed opening cost, with a great number of customers served when this cost is 0 compared with when it is 20. Finally, Figure 5 displays the revenue obtained. In this case, it is clear the effect of the fixed opening cost factor. The revenue obtained is smaller when the fixed opening cost is 20 compared with when it is 0. The effect of the length factor is minimal.

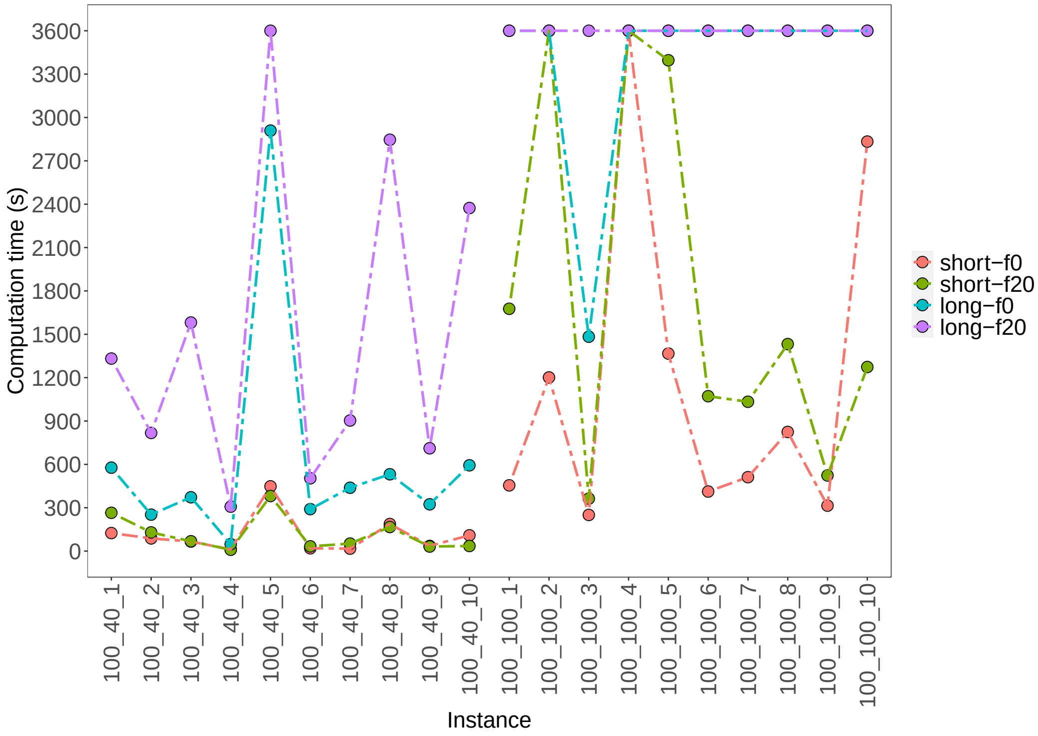

Figure 2. Computational time for each instance across the four combinations of factors.

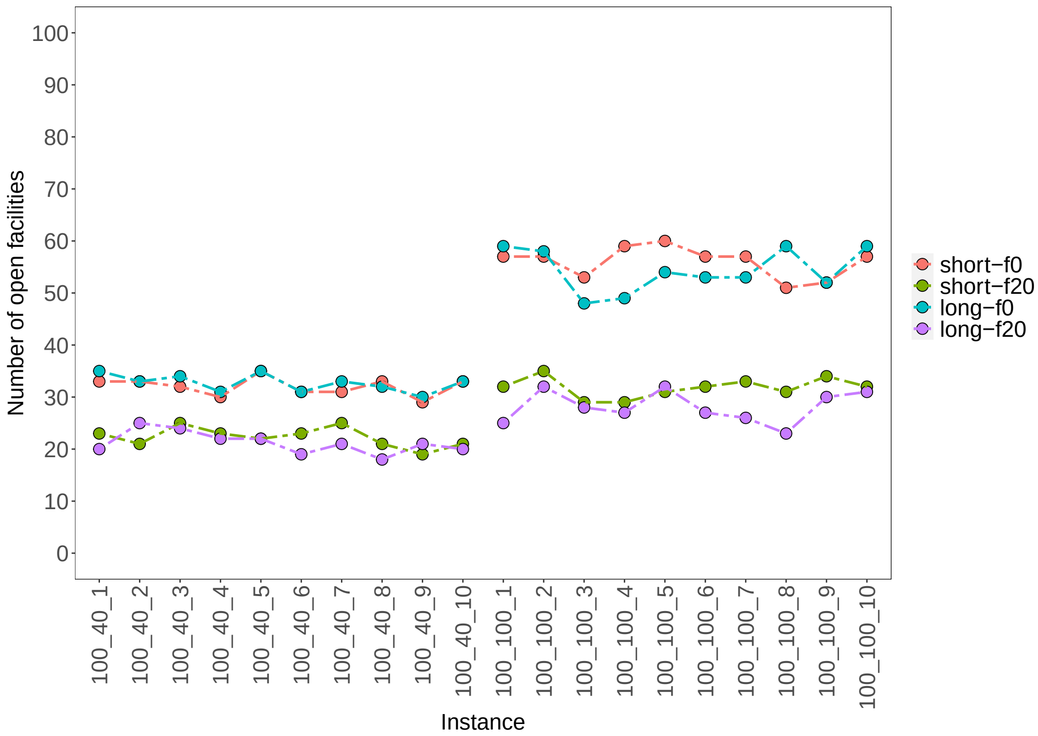

Figure 3. Number of open facilities in the best solution obtained for each instance across the four combinations of factors.

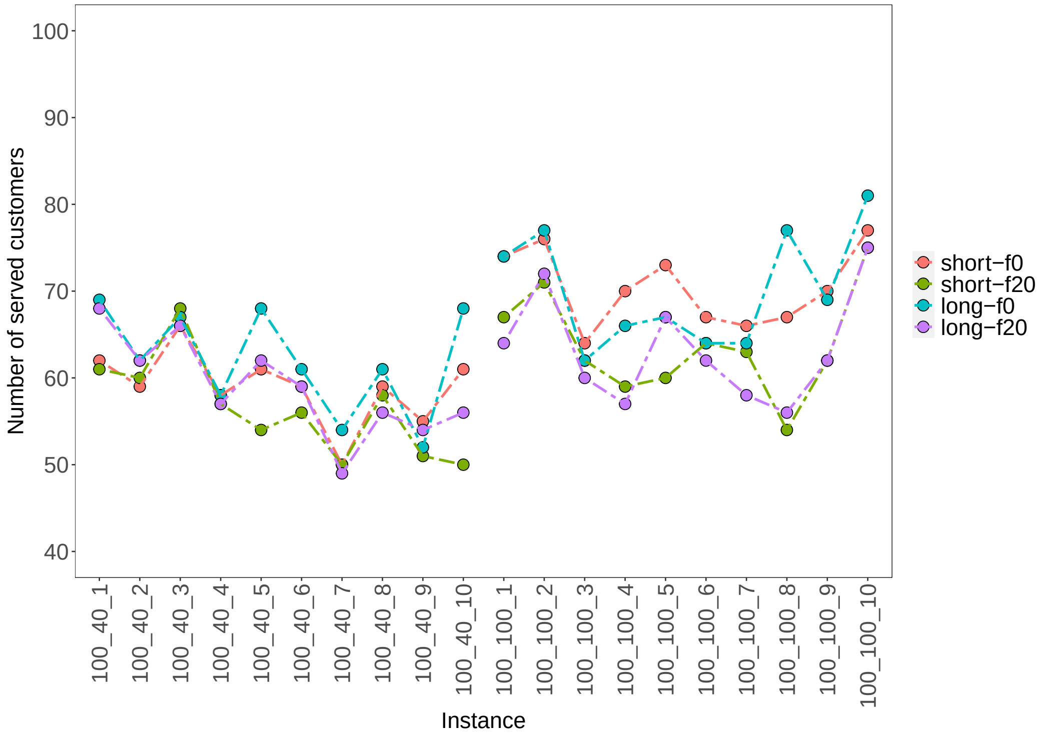

Figure 4. Number of customers that access any facility in the best solution obtained for each instance across the four combinations of factors.

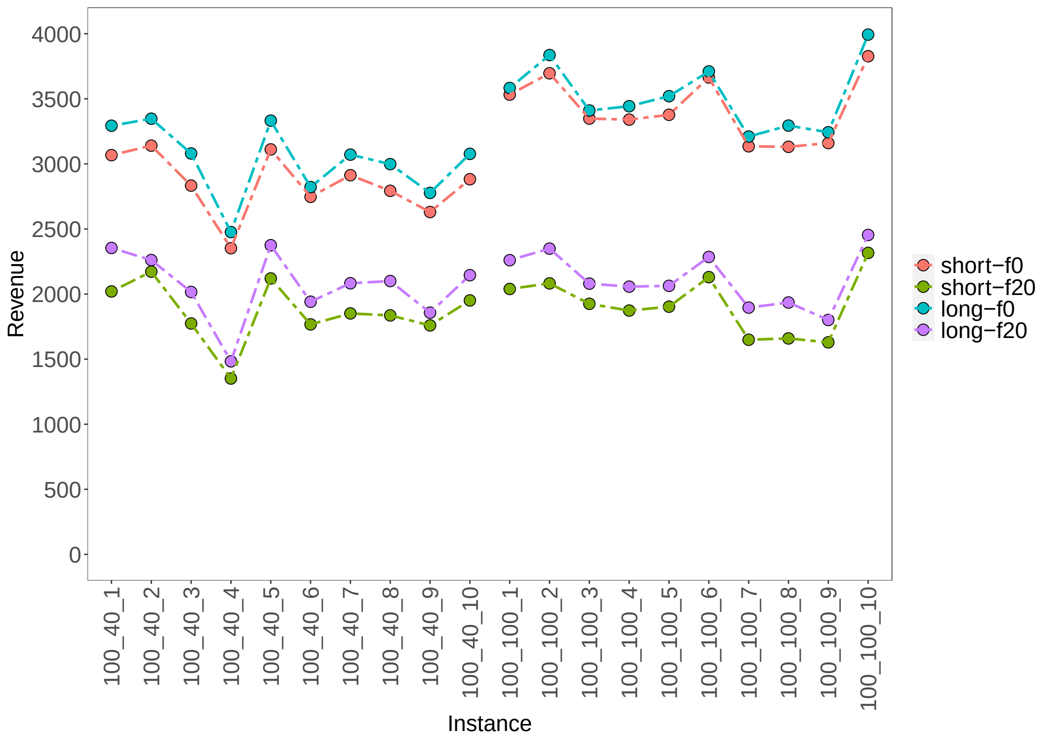

Figure 5. Revenue achieved in the best solution obtained for each instance across the four combinations of factors.

10. Conclusions and Future Research

This paper makes a significant contribution to the facility location and pricing problems by addressing customers' preferences for the first time. Assuming a mill pricing policy, that is, travel costs are borne by customers, we formulate the problem as a bilevel optimization model with multiple independent followers. The formulation specifically tackles the complexities that arise when there are ties in customer preference lists. These ties can result in an ill-posed bilevel optimization model due to the existence of multiple optima to the lower-level problem. After justifying why traditional optimistic and pessimistic approaches are inadequate for this problem, we propose a novel methodology that effectively captures customer preferences, as well as customer rational behavior in selecting facilities.

Under the proposed approach, the lower-level problem of each follower consists of solving a lexicographic biobjective problem where the first objective function maximizes the customer preference obtained by accessing a facility and the second objective function minimizes the total cost incurred in accessing that facility. The reformulation of the bilevel problem into a single-level mixed-integer optimization problem leverages the properties of the lexicographic biobjective problem of the lower level. Moreover, it is proved that there exists an optimal solution to the problem in which prices of facilities take values in a discrete set related to customer's budgets and travel costs. Then, duality theory is applied and valid constants for the big-Ms involved in the single-level reformulation of the problem are computed. The computational experiments conducted show the effectiveness of the formulation proposed in this paper and enable us to evaluate the effects of key parameters in the real system, such as the length of the customer preference lists and whether or not fixed opening costs exist for facilities. The analysis revealed that the computational time required was primarily influenced by the problem size, followed by the length of the preference lists, while the existence of fixed opening costs had a smaller impact.

Although the proposed optimization model is suitable for addressing the FLPMP, a key limitation is the increased computational time when the size of the instances grows. This challenge presents an opportunity to explore enhancements to the single-level reformulation of the problem, such as by defining appropriate cutting planes, or by employing alternative solution approaches like heuristics, metaheuristic algorithms, or exact decomposition methods. In addition, applying the model to new contexts could provide insights into its generalizability, incorporating features such as customers' demand, facility capacity constraints, dynamic customer preferences, fluctuating fixed costs for opening facilities, or uncertainty in some of the parameters of the model. These possible extensions would broaden the applicability of the model and create opportunities for developing more robust and adaptive solutions.

Author Contributions: Conceptualization, H.I.C. and A.H.; methodology, H.I.C., C.G. and A.H.; formal analysis, H.I.C., C.G., A.H. and J.A.I.; statistical analysis, C.G. and A.H. writing—original draft preparation, H.I.C. and A.H.; writing—review and editing, H.I.C., C.G., A.H. and J.A.I. All authors have read and agreed to the published version of the manuscript.

Funding: This research has been funded by AEI Spanish Ministry of Science and Innovation under grants PID2022-139543OB-C43 and TED2021-130961B-I00, and by Gobierno de Aragón under grant E41-23R.

Data Availability Statement: The original data presented in the study are openly available in Github at https://github.com/opt-uz/FLPMP (accessed on 24 October 2024).

Acknowledgments: The authors gratefully acknowledge Yun Hui Lin and Qingyun Tian [13] for providing the set of benchmarks instances of the FLPr.

Conflicts of Interest: The authors declare no conflicts of interest.

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