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Unconditional basic sequences in function spaces with applications to Orlicz spaces

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Abstract
We find conditions on a function space $\boldsymbol{L}$ that ensure that it behaves as an L_p-space in the sense that any unconditional basis of a complemented subspace of $\boldsymbol{L}$ either is equivalent to the unit vector system of ℓ_2 or has a subbasis equivalent to a disjointly supported basic sequence. This dichotomy allows us to classify the symmetric basic sequences of $\boldsymbol{L}$. Several applications to Orlicz function spaces are provided.

Keywords Orlicz space · Unconditional basis · Subsymmetric basis

Mathematics Subject Classification 46B15 · 46B42 · 46B03

1 Introduction

Recall that a sequence $(\boldsymbol{x}_n)_{n=1}^\infty$ in a Banach space $\mathbb{U}$ (over the real or complex field $\mathbb{F}$) is a *basic sequence* if it is a Schauder basis of its closed linear span $[\boldsymbol{x}_n : n \in \mathbb{N}]$. Two sequences, typically two basic sequences, $(\boldsymbol{x}_n)_{n=1}^\infty$ and $(\boldsymbol{y}_n)_{n=1}^\infty$ in Banach spaces $\mathbb{X}$ and $\mathbb{Y}$, respectively, are said to be *equivalent* if there a linear isomorphism

$$T : [\boldsymbol{x}_n : n \in \mathbb{N}] \to [\boldsymbol{y}_n : n \in \mathbb{N}]$$

such that $T(\boldsymbol{x}_n) = \boldsymbol{y}_n$ for all $n \in \mathbb{N}$. A central problem in the isomorphic theory of Banach spaces is the classification of the mutually non-equivalent basic sequences of a certain type in a Banach space $\mathbb{U}$. Among the conditions we can impose to tackle

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this classification we highlight complementability, unconditonality, spreadability, and symmetry.

A basic sequence $\mathcal{X} = (\boldsymbol{x}_n)_{n=1}^{\infty}$ in $\mathbb{X}$ is said to be *complemented* if $[\boldsymbol{x}_n : n \in \mathbb{N}]$ is complemented in $\mathbb{X}$. The basic sequence $\mathcal{X} = (\boldsymbol{x}_n)_{n=1}^{\infty}$ in $\mathbb{X}$ is said to be *unconditional* if the rearranged sequence $(\boldsymbol{x}_{\pi(n)})_{n=1}^{\infty}$ is a basic sequence for any permutation π of $\mathbb{N}$. In turn, $\mathcal{X}$ is said to be *symmetric* (resp., *spreading*) if it is equivalent to $(\boldsymbol{x}_{\pi(n)})_{j=1}^{\infty}$ for any permutation (resp., increasing map) π of $\mathbb{N}$. Symmetric basic sequences are both unconditional and spreading (see [16, 30]). In practice, the only features that one needs about symmetric basic sequences in many situations are their unconditionality and spreadability, to the extent that it was believed that symmetric bases could be characterized as those bases that are simultaneously unconditional and spreading. As Garling [9] provided a counterexample disproving it, Singer [31] coined the word *subsymmetric* to refer to unconditional spreading bases.

Let us outline the more relevant results in the classification of symmetric and subsymmetric basic sequences in Banach spaces. The unit vector system is a symmetric basis for ℓ_p, $1 \le p < \infty$, and c_0. Moreover, it is the unique subsymmetric basic sequence of those spaces (see, e.g., [3, Proposition 2.14] and [4, Proposition 2.1.3]). The authors of [5] address the task of studying the symmetric basic sequence structure of Lorentz sequence spaces. They proved that if $1 \le p < \infty$ and $\boldsymbol{w}$ is a non-increasing weight whose primitive sequence is submultiplicative, then $d(\boldsymbol{w}, p)$ has exactly two subsymmetric (and symmetric) basic sequences, namely the unit vector bases of $d(\boldsymbol{w}, p)$ and ℓ_p. They also proved that if the submultiplicative condition breaks down, then $d(\boldsymbol{w}, p)$ has more than two symmetric bases, and there are instances where $d(\boldsymbol{w}, p)$ has infinitely many symmetric basic sequences. The authors of [2] studied the subsymmetric counterpart of Lorentz spaces, namely Garling sequence spaces $g(\boldsymbol{w}, p)$ modelled after the aforementioned Garling's counterexample. They proved that for any $1 \le p < \infty$ and any non-increasing weight $\boldsymbol{w}$, $g(\boldsymbol{w}, p)$ has a unique symmetric basic sequence, namely the unit vector system of ℓ_p, and infinitely many subsymmetric basic sequences. The basic sequence structure of Orlicz sequence spaces has also been deeply studied. Given an Orlicz function F, let h_F denote the separable part of the Orlicz sequence space ℓ_F. It is known [17] that every subsymmetric basic sequence in h_F is equivalent to the unit vector system of an Orlicz space h_G for some Orlicz function G. In particular, every subsymmetric basic sequence is symmetric. Lindenstrauss and Tzafriri showed in [19] that if

$$\lim_{t \to 0^+} \frac{t F'(t)}{F(t)}$$

exists, then h_F has a unique symmetric basis. In the same paper, an Orlicz sequence space with exactly two symmetric basic sequences is supplied. The same authors gave in [21] a sufficient condition for h_F to have uncountably many subsymmetric basic sequences. The article [7] contains an intricate construction of an Orlicz sequence space with a countably infinite collection of symmetric basic sequences. In [12], sufficient conditions for ℓ_p to be a subspace of the Orlicz space L_F over [0, 1] are given. In turn, the papers [11, 13] are devoted to study the existence of complemented copies of ℓ_p-spaces in Orlicz function spaces over [0, 1]. We also highlight that Tsirelson

[32] proved the existence of a Banach space without a copy of ℓ_p, $1 \le p < \infty$, nor c_0, thus solving a long-standing problem that goes back to Banach. As a matter of fact, the space nowadays known as the original Tsirelson space $\mathcal{T}^*$, its dual, denoted by $\mathcal{T}$ after [8], and the convexifications of $\mathcal{T}$, have no subsymmetric basic sequence (see [8, 32]).

While the subsymmetric basic sequence structure of the more relevant sequence spaces is quite well understood, the advances within the framework of function spaces, i.e., Banach spaces over nonatomic measure spaces, are much scanter. Arguably, function spaces have, in general, a richer structure than sequence spaces, so it is more challenging to find tools that fit their study. In view of this, the paper [16], which successfully addresses the task of classifying the subsymmetric basic sequences of Lebesgue spaces L_p, deserves to be considered one of the peaks of the theory. In it, Kadec and Pełczyński proved that any subsymmetric basic sequence of L_p, $2 < p < \infty$, is equivalent to the unit vector system of ℓ_p or ℓ_2. The situation in the case when $1 \le p < 2$ is quite different. In fact, if $1 \le p < 2$, L_p has a basic sequence isometrically equivalent to the unit vector system of ℓ_q for every q in the interval $[p, 2]$ (see [18, Corollary 1 to Theorem 7.2]). Imposing the basic sequence to be complemented makes a difference. The authors of [16] proved that the list of subsymmetric sequence spaces that are equivalent to a complemented basic sequence of L_p reduces to the unit vector system of ℓ_q with $q \in \{2, p\}$. We also note that, since L_1 is a $\mathscr{L}_1$-space, ℓ_1 is, up to equivalence, the unique complemented normalized unconditional basic sequence of L_1 (see [18, Theorem 6.1]). Let us also mention the work of Raynaud, who proved that ℓ_s embeds into $L_q(L_p)$, $1 \le p < q < \infty$, if and only if it embeds into L_p or L_q (see [27]).

In this document, we generalize results from the milestone paper [16] by proving a dichotomy theorem that works for function spaces other than Lebesgue spaces. Then, we put this dichotomy in use to determine the subsymmetric basic sequence structure of certain Orlicz function spaces.

The paper is organized as follows. In Sect. 2 we revisit some classical results of Banach lattices recording their applications to function spaces. In Sect. 3 we give several versions of the small perturbation principle for unconditional basic sequences that fit our purposes. Section 4 contains the main theoretical results. In Sect. 5 we apply them to direct sums of Lebesgue spaces, and in Sect. 6 we use the developed machinery to study Orlicz sequence spaces.

2 Function spaces as Banach lattices

Let (Ω, Σ, μ) be a σ-finite measure space, and let $\mathbb{F}$ denote the real or complex scalar field. We will denote by $L_0(\mu)$ the linear space consisting of all $\mathbb{F}$-valued measurable functions on Ω, and by $L_0^+(\mu)$ the cone consisting of all measurable functions with values in $[0, \infty]$. As usual, we identify functions that differ in a null set. Following the nowadays standard terminology from [6], a *function norm* over the measure space (Ω, Σ, μ) will be a map $\rho\colon L_0^+(\mu) \to [0, \infty]$ such that

(F.a) $\rho(f) = 0$ if and only if $f = 0$ μ-a.e.;

(F.b) $\rho(f+g) \le \rho(f) + \rho(g)$ for every $f, g \in L_0^+(\mu)$;
(F.c) $\rho(tf) = t\rho(f)$ for every $t \in [0,\infty]$ and $f \in L_0^+(\mu)$;
(F.d) $\rho(f) \le \rho(g)$ whenever f and $g \in L_0^+(\mu)$ satisfy $f \le g$ μ-a.e.;
(F.e) $\rho(\chi_E) < \infty$ for every $E \in \Sigma$ with $\mu(E) < \infty$;
(F.f) $\rho(\lim_n f_n) = \lim_n \rho(f_n)$ for any non-decreasing sequence $(f_n)_{n=1}^\infty$ in $L_0^+(\mu)$; and
(F.g) for every $E \in \Sigma$ with $\mu(E) < \infty$ there is a constant C_E such that $\int_E f\, d\mu \le C_E \rho(f)$ for every $f \in L_0^+(\mu)$.

If ρ is a function norm, then

$$\boldsymbol{L}_\rho = \{f \in L_0(\mu) \colon \rho(|f|) < \infty\},$$

endowed with the ordering 'f is not greater than g if $f \le g$ almost everywhere' and the norm

$$\|\cdot\|_\rho := \rho(|\cdot|)$$

is a Banach lattice, so we can apply to it the theory of Banach lattices masterfully gathered in the handbook [23]. The *separable part* of the function space $\boldsymbol{L}_\rho$ will be closure in $\boldsymbol{L}_\rho$ of the vector space $\mathcal{S}(\mu)$ consisting of all integrable simple functions. Since, by the Fatou property (F.f),

$$\|f\|_\rho = \sup\{\|g\|_\rho \colon |g| \le |f|,\ g \in \mathcal{S}(\mu)\}, \quad f \in L_0(\mu),$$

two function spaces over the same σ-finite measure space are in inclusion if and only if their separable parts are.

For the reader's ease, we single out some topics on Banach lattice theory relevant to us.

2.1 Lattice convexity and concavity versus Rademacher type and cotype

Given $1 \le r \le \infty$, we say that the function norm ρ is *lattice r-convex* (resp., *lattice r-concave*) if there is a constant C such that $A \le CB$ (resp., $B \le CA$) for every finite family $(f_j)_{j\in J}$ in $L_0^+(\mu)$, where

$$A := \rho\left(\left(\sum_{j\in J} f_j^r\right)^{1/r}\right) \text{ and } B := \left(\sum_{j\in J} \rho^r(f_j)\right)^{1/r}.$$

If we impose the inequality $A \le CB$ (resp., $B \le CA$) to hold only in the case when the family $(f_j)_{j\in J}$ consists of disjointly supported functions, so that

$$\left(\sum_{j\in J} f_j^r\right)^{1/r} = \sum_{j\in J} f_j,$$

we say that ρ satisfies an *upper* (resp., *lower*) *r-estimate*. It is clear that the Banach lattice $\boldsymbol{L}_\rho$ is lattice p-convex (resp., is lattice p-concave, satisfies an upper p-estimate, or satisfies a lower p-estimate) if and only if ρ does. The following result evinces the tight connection between convexity and concavity of Banach lattices, and Rademacher type and cotype of Banach spaces. Notice that any Banach lattice is lattice 1-convex and lattice ∞-concave. So we say that $\boldsymbol{L}$ has nontrivial convexity (resp., concavity) if it is lattice p-convex for some $p > 1$ (resp., lattice p-concave for some $p < \infty$). Similarly, we say that a Banach space $\mathbb{X}$ has nontrivial type (resp., cotype) if it has Rademacher type p for some $p > 1$ (resp., Rademacher cotype p for some $p < \infty$).

Theorem 2.1 (see [23, Theorem 1.f.3, Theorem 1.f.7, Corollary 1.f.9 and Corollary 1.f.13]) *Let $\boldsymbol{L}$ be a Banach lattice.*

- *Let $1 < r \le \infty$. If $\boldsymbol{L}$ satisfies an upper r-estimate, then it is p-convex for any $1 < p < r$. In turn, if $\boldsymbol{L}$ is lattice r-convex and has nontrivial concavity, then it has Rademacher type r. If $r \le 2$ and $\boldsymbol{L}$ has Rademacher type r, then $\boldsymbol{L}$ satisfies an upper p-estimate for every $1 < p < r$, and has nontrivial concavity.*
- *Let $1 \le r < \infty$. If $\boldsymbol{L}$ satisfies a lower r-estimate, then it is p-concave for any $r < p < \infty$. In turn, if $\boldsymbol{L}$ is lattice r-concave, then it has Rademacher cotype r. Finally, if $r \ge 2$ and $\boldsymbol{L}$ has Rademacher cotype r, then $\boldsymbol{L}$ satisfies a lower p-estimate for every $r < p < \infty$.*

Remark 2.2 Theorem 2.1 gives that $\boldsymbol{L}$ has nontrivial cotype if and only if it has nontrivial concavity, and that $\boldsymbol{L}$ has nontrivial type if and only if it has both nontrivial convexity and nontrivial concavity. Notice that these results imply that if $\boldsymbol{L}$ has nontrivial type, then it also has nontrivial cotype. This result holds for general Banach spaces, but it depends on the deep result from [24] that ℓ_∞ is finitely representable in any Banach space with no finite cotype.

If $r = 2$, we can say even more than that stated in Theorem 2.1.

Theorem 2.3 (see [23, Theorem 1.f.16 and Theorem 1.f.17]) *Let $\boldsymbol{L}$ be a Banach lattice.*

- *$\boldsymbol{L}$ has Rademacher type 2 if and only if it is 2-convex and has nontrivial concavity.*
- *$\boldsymbol{L}$ has Rademacher cotype 2 if and only if it is 2-concave.*

If $\mathbb{X}$ is a Banach space of type r, then $\mathbb{X}^*$ has Rademacher cotype r', where r' is the conjugate index defined by $1/r + 1/r' = 1$ (see [23, Proposition 1.e.17]). In Banach lattices, a converse result holds.

Theorem 2.4 ([23, Theorem 1.f.18]) *Let $1 < r \le 2$ and let $\boldsymbol{L}$ be a Banach lattice. Then, $\boldsymbol{L}$ has Radenacher type r if and only if $\boldsymbol{L}^*$ has Rademacher cotype r' and nontrivial type.*

2.2 Absolute continuity

A Banach lattice is said to be *complete* (resp., *σ-complete*) if every order bounded set (resp., order bounded sequence) has a least upper bound. It is said to be *order continuous* (resp., σ-order continuous) if for every downward directed set (resp., decreasing sequence) $(f_\lambda)_{\lambda\in\Lambda}$ with $\wedge_{\lambda\in\Lambda} f_\lambda = 0$ we have $\lim_{\lambda\in\Lambda} f_\lambda = 0$. Taking advantage of Fatou property (F.f), we give a sharp characterization of order continuous function spaces. Prior to stating it, we recall that a Schauder basis $(\boldsymbol{x}_n)_{n=1}^\infty$ of a Banach space $\mathbb{X}$ is said to be *boundedly complete* if the series $\sum_{n=1}^\infty a_n\, \boldsymbol{x}_n$ converges (in the norm-topology) whenever the scalars $(a_n)_{n=1}^\infty$ satisfy

$$\sup_{m\in\mathbb{N}} \left\| \sum_{n=1}^m a_n\, \boldsymbol{x}_n \right\| < \infty.$$

Theorem 2.5 *Given a function norm ρ over a σ-finite measure space, the following are equivalent.*

(i) $\boldsymbol{L}_\rho$ *is order continuous.*
(ii) $\boldsymbol{L}_\rho$ *is σ-order continuous.*
(iii) $\lim_n \rho(f\chi_{A_n}) = 0$ *for every* $f \in L_0^+(\mu)$ *with* $\rho(f) < \infty$ *and every non-increasing sequence* $(A_n)_{n=1}^\infty$ *in* Σ *with* $\lim_n \mu(A_n) = 0$.
(iv) Lebesgue's dominated convergence theorem holds in $\boldsymbol{L}_\rho$. *That is, if* $(f_n)_{n=1}^\infty$ *and* f *in* $L_0(\mu)$ *are such that* $\lim_n f_n = f$ *a. e. and* $\rho(\sup_n |f_n|) < \infty$, *then* $\lim_n f_n = f$ *in* $\boldsymbol{L}_\rho$.
(v) $\boldsymbol{L}_\rho$ *contains no copy of* ℓ_∞.
(vi) No sequence of pairwise disjointly supported functions is equivalent to the unit vector system of ℓ_∞.
(vii) Every unconditional basic sequence in $\boldsymbol{L}_\rho$ *is boundedly complete.*

Proof It is known that a Banach lattice is order continuous if and only if it is σ-order complete and σ-order continuous [23, Proposition 1.a.8]. So, since $\boldsymbol{L}_\rho$ is σ-complete, (i) and (ii) are equivalent. The equivalence between (ii), (iii) and (iv) follows from [6, Chapter 1, Propositions 3.2 and 3.6]. The equivalence between (ii), (v) and (vi) is a consequence of [23, Proposition 1.a.7]. Finally, the equivalence between (vi) and (vii) is a by-product of James' theory on unconditional bases (see [4, Theorem 3.3.2]). □

Following [6], we call function norms satisfying (iii) *absolutely continuous*. In this terminology, we record an interesting consequence of combining Theorem 2.5 with the fact that ℓ_∞ has no finite cotype.

Theorem 2.6 *Let ρ be a function norm over a σ-finite measure space. Suppose that* $\boldsymbol{L}_\rho$ *has nontrivial cotype. Then, ρ is absolutely continuous.*

We also point out that if ρ is absolutely continuous, then the linear space consisting of all integrable simple functions is dense in $\boldsymbol{L}_\rho$ (see [6, Chapter 1, Theorem 3.11]).

2.3 Duality

Given a function quasi-norm ρ over a σ-finite measure space (Ω, σ, μ) we set

$$\rho^*\colon L_0^+(\mu) \to [0, \infty], \quad f \mapsto \sup\left\{ \int_\Omega fg\, d\mu \colon g \in L_0^+(\mu),\ \rho(g) \le 1 \right\}.$$

Since the gauge ρ^* is a function norm (see [6, Chapter 1, Theorem 2.2]), we call it the *function norm associated with* ρ. The dual pairing

$$\langle g, f \rangle = \int_\Omega g(\omega)\, f(\omega)\, d\mu(\omega), \quad g \in \boldsymbol{L}_{\rho^*},\ f \in \boldsymbol{L}_\rho. \tag{2.1}$$

defines an isometric embedding of $\boldsymbol{L}_{\rho^*}$ into $(\boldsymbol{L}_\rho)^*$ [6, Chapter 1, Theorem 2.9]. Moreover, this embedding is onto if and only if ρ is absolutely continuous [6, Corollary 4.2]. Since $(\rho^*)^* = \rho$ [6, Chapter 1, Theorem 2.9], $\boldsymbol{L}_\rho$ is a reflexive Banach space if and only if both ρ and ρ^* are absolutely continuous.

Given $f \in L_0(\mu)$ we set

$$\operatorname{supp}(f) = \Omega \setminus f^{-1}(0).$$

To be precise, since we are identifying functions that differ in a null set, $\operatorname{supp}(f)$ is an equivalence class of measurable sets. The underlying equivalence relation is the following: $A \sim B$ if $\mu(A \ominus B) = 0$. Notice that this identification makes Σ endowed with the distance

$$d_\mu(A, B) = \mu(A \ominus B), \quad A,\ B \in \Sigma.$$

a metric space.

Given $A \in \Sigma$ we set

$$\boldsymbol{L}_\rho[A] := \{f \in \boldsymbol{L}_\rho \colon \operatorname{supp}(f) \subseteq A\}.$$

We look at the next result from the view that if ρ is absolutely continuous, then $\boldsymbol{L}_{\rho^*}$ is the dual space of $\boldsymbol{L}_\rho$, and $\boldsymbol{L}_\rho$ is a subspace of $(\boldsymbol{L}_{\rho^*})^*$.

Proposition 2.7 *Let ρ be an absolutely continuous function norm over a σ-finite measure (Ω, Σ, μ). Then, for each $A \in \Sigma$, $\boldsymbol{L}_\rho[A]$ is w^*–closed.*

Proof It suffices to prove that

$$\boldsymbol{L}_\rho[A] = \mathbb{Y} := \{f \in \boldsymbol{L}_\rho \colon \langle g, f \rangle = 0 \text{ for all } g \in \boldsymbol{L}_{\rho^*}[\Omega \setminus A]\}.$$

It is clear that $\boldsymbol{L}_\rho[A] \subseteq \mathbb{Y}$. To prove that the reserve inclusion also holds, pick $f \in \mathbb{X} \setminus \boldsymbol{L}_\rho[A]$. Then, there is $B \in \Sigma$ such that $B \cap A = \emptyset$, $0 < \mu(B) < \infty$, and $f(\omega) \neq 0$

for every $\omega \in B$. If $g\colon \Omega \to \mathbb{F}$ is such that $gf = |f|\,\chi_B$, then $g \in \boldsymbol{L}_{\rho^*}[\Omega \setminus A]$, and

$$\langle g, f\rangle = \int_\Omega |f|\, d\mu > 0.$$

Hence $f \notin \mathbb{Y}$. □

2.4 The role of the measure space

Given a σ-finite measure space (Ω, Σ, μ) we set

$$\Sigma_\mu = \{A \in \Sigma \colon \mu(A) < \infty\}.$$

We say that μ is *separable* if the metric space (Σ_μ, d_μ) is. It is known that, given a function norm ρ over (Ω, Σ, μ), the function space $\boldsymbol{L}_\rho$ is separable if and only if ρ is absolutely continuous and μ is separable (see [6, Chapter 1, Corollary 5.6]).

Let (A, Σ_A, μ_A) be the restriction of a σ-finite measure (Ω, Σ, μ) to a set $A \in \Sigma$. Given a function norm ρ over (Ω, Σ, μ), let ρ_A be its restriction to A, that is,

$$\rho_A(f) = \rho(\tilde{f}), \quad f \in L_0^+(\mu_A),$$

where $\tilde{f}$ is the extension of f given by $\tilde{f}(\omega) = 0$ for all $\omega \in \Omega \setminus A$. In this terminology, we identify $\boldsymbol{L}_{\rho_A}$ with $\boldsymbol{L}_\rho[A]$. The mapping

$$P_A \colon \boldsymbol{L}_\rho \to \boldsymbol{L}_\rho, \quad f \mapsto f\chi_A,$$

is a bounded linear projection onto $\boldsymbol{L}_\rho[A]$ whose complementary projection is $P_{\Omega\setminus A}$. Consequently, we have a canonical lattice isomorphism from $\boldsymbol{L}_\rho$ onto $\boldsymbol{L}_\rho[A] \oplus \boldsymbol{L}_\rho[\Omega \setminus A]$. This isomorphism allows us to split any function space into its atomic and nonatomic parts. Indeed, for any σ-finite measure space (Ω, Σ, μ) there is a partition of Ω into two measurable sets Ω_c and Ω_a such that μ_{Ω_c} is nonatomic and μ_{Ω_a} is purely atomic. Hence, if $\rho_c = \rho_{\Omega_c}$ and $\rho_a = \rho_{\Omega_a}$,

$$\boldsymbol{L}_\rho \simeq \boldsymbol{L}_{\rho_c} \oplus \boldsymbol{L}_{\rho_a}.$$

The Banach lattice $\boldsymbol{L}_{\rho_a}$ is isometrically isomorphic to a function space over a countable set endowed with the counting measure. As far as (Ω_c, Σ, μ) is concerned, we note that nonatomic separable measure spaces are isomorphic to Lebesgue measure over the real line (see [10, Theorem 1]).

Given a Banach space $\mathbb{X}$, the property that $\mathbb{X}$ is isomorphic to its square and to its hyperplanes is as natural as elusive to check in some situations. Next, relying on the boundedness of averaging projections, we prove that rearrangement invariant function spaces have this property. A function norm ρ is said to be *rearrangement invariant* if $\rho(f) = \rho(g)$ whenever f and g are equimeasurable.

Theorem 2.8 *(see, e.g., [23, Theorem 2.a.4]) Let ρ be a rearrangement invariant function norm over a real interval I endowed with the Lebesgue measure. Let $(A_n)_{n=1}^\infty$ by pairwise disjoint Borel subsets of I. Then, the mapping*

$$f \mapsto \sum_{n=1}^{\infty} \frac{\chi_{A_n}}{|A_n|} \int_{A_n} f(x)\, dx$$

is a bounded linear projection from $\boldsymbol{L}_\rho$ onto $[\chi_{A_n} : n \in \mathbb{N}]$.

Proposition 2.9 *Let ρ be a rearrangement invariant function norm over a real interval I endowed with the Lebesgue measure.*

(i) If I is bounded, then $\boldsymbol{L}_\rho$ is lattice isomorphic to $\boldsymbol{L}_\rho[J]$ for every open set $J \subseteq I$.
(ii) If I is unbounded, then $\boldsymbol{L}_\rho$ is isometrically lattice isomorphic to $\boldsymbol{L}_\rho[J]$ for every open set $J \subseteq I$ with $|J| = \infty$.
(iii) $\boldsymbol{L}_\rho$ is lattice isomorphic to its square $\boldsymbol{L}_\rho \oplus \boldsymbol{L}_\rho$.
(iv) $\boldsymbol{L}_\rho$ is isomorphic to $\boldsymbol{L}_\rho \oplus \mathbb{F}$.

Proof To prove (i) and (ii), we suppose the $J \neq I$. Pick $M = N = 1$ in the unbounded set, and $M = \lfloor |I| / |J| \rfloor$ and $N = \lceil |I| / |J| \rceil$ in the bounded case. In both cases, let $T : J \to I$ be a measurability-preserving bijection such that

$$M\,|A| \le |T(A)| \le N\,|A|, \quad A \text{ measurable.}$$

Given a simple function $f : J \to [0, \infty]$, there are functions $(f_j)_{j=1}^N$ equimeasurable with f such that

$$\sum_{j=1}^{M} f_j \le T(f) \le \sum_{j=1}^{N} f_j,$$

where T is the linear map given by $\chi_A \mapsto \chi_{T(A)}$. Indeed, if $f = \sum_{k\in K} a_k \chi_{A_k}$ with $(A_k)_{k\in K}$ pairwise disjoint, then there is a family $(B_k)_{k\in K}$ consisting of pairwise disjoint measurable subsets of I such that $T(A_k) \subseteq B_k$ and $|B_k| = N\,|A_k|$ for all $k \in K$. For each $k \in K$ we pick a partition $(A_{k,j})_{j=1}^N$ of A_k into measurable sets with $\cup_{j=1}^M A_{k,j} \subseteq T(A_k)$ and $\left|A_{k,j}\right| = A_k$ for all $j = 1, \ldots, N$. Finally, we set $f_j = \sum_{k\in K} a_k \chi_{A_{k,j}}$ for all $j = 1, \ldots, N$. We have

$$\rho(f) \le \rho(T(f)) \le N \rho(f).$$

We infer that T extends to a lattice isomorphism from $\boldsymbol{L}_\rho$ onto $\boldsymbol{L}_\rho[J]$.

To prove (iii) we pick a partition (J_1, J_2) of I into subsets as in (i) or (ii). We have

$$\boldsymbol{L}_\rho \simeq \boldsymbol{L}_\rho[J_1] \oplus \boldsymbol{L}_\rho[J_2] \simeq \boldsymbol{L}_\rho \oplus \boldsymbol{L}_\rho.$$

Since, regardless I is bounded or unbounded, $\boldsymbol{L}_\rho(I) \simeq \boldsymbol{L}_\rho(I_1) \oplus \boldsymbol{L}_\rho(I_2)$ with I_1 bounded, it suffices to prove (iv) in the case when I is bounded. For notational ease, set $I = [0, 1)$. Let $\mathbb{X}$ (resp., $\mathbb{Y}$) be the subspace of $\boldsymbol{L}_\rho$ (resp., $\boldsymbol{L}_\rho([0, 1/2))$) consisting of all functions that are constant in each interval $[2^{-n-1}, 2^{-n})$, $n \in \mathbb{N} \cup \{0\}$ (resp., $n \in \mathbb{N}$). The aforementioned isomorphism T gives $\mathbb{X} \simeq \mathbb{Y}$. Moreover, $\mathbb{X} \simeq \mathbb{Y} \oplus \mathbb{F}$. Let $\mathbb{U}$ be the space of $\boldsymbol{L}_\rho$ consisting of all functions with null integral in each interval $[2^{-n-1}, 2^{-n})$, $n \in \mathbb{N} \cup \{0\}$. By Theorem 2.8, $\boldsymbol{L}_\rho \simeq \mathbb{U} \oplus \mathbb{X}$. Piecing the bits together, we obtain

$$\boldsymbol{L}_\rho \simeq \mathbb{U} \oplus \mathbb{X} \simeq \mathbb{U} \oplus \mathbb{Y} \oplus \mathbb{F} \simeq \mathbb{U} \oplus \mathbb{X} \oplus \mathbb{F} \simeq \boldsymbol{L}_\rho \oplus \mathbb{F}.$$

□

3 Unconditional basic sequences

It is known that an unconditional basic sequence $\mathcal{X} = (\boldsymbol{x}_n)_{n=1}^\infty$ in a Banach space $\boldsymbol{L}$ induces an atomic lattice structure on its closed linear span. To be precise, there is a constant C such that

$$\left\| \sum_{n=1}^\infty a_n \, \boldsymbol{x}_n \right\| \le C \left\| \sum_{n=1}^\infty b_n \, \boldsymbol{x}_n \right\|, \quad |a_n| \le |b_n|, \; (b_n)_{n=1}^\infty \in c_{00}. \tag{3.1}$$

(see, e.g., [22, Proposition 1.c.7]). If (3.1) holds for a given constant C we say that $\mathcal{X}$ is C-unconditional. Note that (3.1) yields

$$\left\| \sum_{n=1}^\infty a_n \, \boldsymbol{x}_n \right\| \approx \left(\operatorname*{Ave}_{\varepsilon_n = \pm 1} \left\| \sum_{n=1}^\infty \varepsilon_n \, a_n \, \boldsymbol{x}_n \right\|^p \right)^{1/p}, \quad (a_n)_{n=1}^\infty \in c_{00} \tag{3.2}$$

for any $0 < p < \infty$. In turn, if the Banach lattice $\boldsymbol{L}$ has cotype $p < \infty$, then, by Khintchine's inequalities,

$$\operatorname*{Ave}_{\varepsilon_n = \pm 1} \left\| \sum_{n=1}^\infty \varepsilon_n \, f_n \right\| \lesssim \left\| \left(\sum_{n=1}^\infty |f_n|^2 \right)^{1/2} \right\| \lesssim \left(\operatorname*{Ave}_{\varepsilon_n = \pm 1} \left\| \sum_{n=1}^\infty \varepsilon_n \, f_n \right\|^p \right)^{1/p} \tag{3.3}$$

for $(f_n)_{n=1}^\infty \in c_{00}(\boldsymbol{L})$. Combining (3.2) with (3.3) allows us to relate the lattice structure induced by $\mathcal{X}$ to that in $\boldsymbol{L}$. Namely, if $\boldsymbol{L}$ has nontrivial cotype we have

$$\left\| \sum_{n=1}^\infty a_n \, \boldsymbol{x}_n \right\| \approx \left\| \left(\sum_{n=1}^\infty |a_n|^2 \, |\boldsymbol{x}_n|^2 \right)^{1/2} \right\|, \quad (a_n)_{n=1}^\infty \in c_{00}.$$

Oddly enough, this estimate still holds if we drop the assumption that $\boldsymbol{L}$ has nontrivial cotype and, in return, we impose $\mathcal{X}$ to be complemented. To state this result in a precise way, it will be convenient to use the constants involved in complementability. A subspace $\mathbb{X}$ of a Banach space $\mathbb{U}$ is complemented if and only if there is a bounded linear projection $P\colon \mathbb{U} \to \mathbb{U}$ with $P(\mathbb{U}) = \mathbb{X}$. If $C \in [1, \infty)$ is such that $\|P\| \le C$ for a suitable such projection we say that $\mathbb{X}$ is C-complemented.

Theorem 3.1 *(see [23, Proposition 1.d.6] and subsequent Remark 1) Let $\mathcal{X} = (\boldsymbol{x}_n)_{n=1}^\infty$ be a complemented unconditional basic sequence in a Banach lattice $\mathbf{L}$. Then, there is a constant C such that*

$$\frac{1}{C}\left\|\sum_{n=1}^\infty a_n\,\boldsymbol{x}_n\right\| \le \left\|\left(\sum_{n=1}^\infty |a_n|^2\,|\boldsymbol{x}_n|^2\right)^{1/2}\right\| \le C\left\|\sum_{n=1}^\infty a_n\,\boldsymbol{x}_n\right\|$$

for all $(a_n)_{n=1}^\infty \in c_{00}$. Moreover, if $\mathcal{X}$ is C_1-unconditional and $[\mathcal{X}]$ is C_2-complemented, then C only depends on C_1 and C_2.

A sequence $\mathcal{X} = (\boldsymbol{x}_n)_{n=1}^\infty$ in a Banach space $\mathbb{U}$ is a complemented unconditional basic sequence if and only if there is a constant $C \in [1, \infty)$ and a sequence $\mathcal{X}^* = (\boldsymbol{x}_n^*)_{n=1}^\infty$ in $\mathbb{U}^*$ such that $(\boldsymbol{x}_n, \boldsymbol{x}_n^*)_{n=1}^\infty$ is a biorthogonal system, and

$$\left\|\sum_{n\in A}\boldsymbol{x}_n^*(f)\,\boldsymbol{x}_n\right\| \le C\,\|f\|\,,\quad f \in \mathbb{U},\ |A| < \infty.$$

If this is the case, we say that $\mathcal{X}^*$ is a sequence of *projecting functionals* for $\mathcal{X}$. Such a sequence is not unique, but all possible sequences of projecting functionals are obtained as

$$\boldsymbol{x}_n^* = \boldsymbol{z}_n^* \circ P = P^*(\boldsymbol{z}_n^*),\quad n \in \mathbb{N},$$

where P is a projection from $\mathbb{U}$ onto $\mathbb{X} := [\boldsymbol{x}_n : n \in \mathbb{N}]$, and $\mathcal{Z}^* = (\boldsymbol{z}_n^*)_{n=1}^\infty$ in $\mathbb{X}^*$ are the coordinate functionals of $\mathcal{X}$. Notice that $\mathcal{X}^*$ and $\mathcal{Z}^*$ are equivalent.

Two sequences $\mathcal{X}$ and $\mathcal{Y}$ in a Banach space $\mathbb{U}$ are said to be *congruent* if there is an automorphism S of $\mathbb{U}$ with $S(\boldsymbol{x}_n) = \boldsymbol{y}_n$ for all $n \in \mathbb{N}$. Although the classification of basic sequences in Banach spaces is usually stated in terms of equivalence, congruence is a stronger condition that is convenient to record in some situations. Notice that if $\mathcal{X}$ and $\mathcal{Y}$ are congruent, and $\mathbb{X} := [\mathcal{X}]$ is complemented in $\mathbb{U}$, then $\mathbb{Y} := [\mathcal{Y}]$ also is. In fact, if $P\colon \mathbb{U} \to \mathbb{U}$ is a projection onto $\mathbb{X}$, and $T\colon \mathbb{U} \to \mathbb{U}$ is an automorphism with $T(\mathcal{X}) = \mathcal{Y}$, then $T \circ P \circ T^{-1}$ is a projection onto $\mathbb{Y}$. The following result establishes a partial converse of this fact. For broader applicability, we state it within the more general setting of quasi-Banach spaces. Recall that quasi-Banach spaces are defined as Banach spaces, replacing the triangle law of the norm $\|\cdot\|$ on the vector space $\mathbb{X}$ with the weaker assumption that

$$\|f + g\| \le \kappa\,(\|f\| + \|g\|)\,,\quad f,\ g \in \mathbb{X}, \tag{3.4}$$

for some uniform constant $\kappa \in [1, \infty)$. If there is $0 < p \le 1$ such that

$$\|f + g\|^p \le \|f\|^p + \|g\|^p\,,\quad f,\ g \in \mathbb{X}, \tag{3.5}$$

the (3.4) holds with $\kappa = 2^{1/p} - 1$. Quasi-Banach spaces whose quasi-norm satisfies (3.5) are called p-Banach spaces. By Aoki-Rolewicz theorem, any quasi-Banach space is for some $0 < p \le 1$ a p-Banach space under a suitable renorming.

Lemma 3.2 *Let $\mathcal{X} = (\boldsymbol{x}_n)_{n=1}^\infty$ and $\mathcal{Y} = (\boldsymbol{y}_n)_{n=1}^\infty$ be sequences in a quasi-Banach space $\mathbb{U}$, and let $\mathbb{X}$ and $\mathbb{Y}$ denote the closed subspaces of $\mathbb{U}$ generated by $\mathcal{X}$ and $\mathcal{Y}$, respectively. Supppose that $\mathcal{X}$ and $\mathcal{Y}$ are equivalent, that $\mathbb{X}$ and $\mathbb{Y}$ are complemented in $\mathbb{U}$, and that $\mathbb{U}/\mathbb{X} \simeq \mathbb{U}/\mathbb{Y}$. Then, $\mathcal{X}$ and $\mathcal{Y}$ are congruent. Moreover, if $T : \mathbb{X} \to \mathbb{Y}$ is an isomorphism with $T(\boldsymbol{x}_n) = \boldsymbol{y}_n$ for all $n \in \mathbb{N}$, given projections P and Q onto $\mathbb{X}$ and $\mathbb{Y}$, respectively, we can choose an isomorphism $J : \mathbb{U} \to \mathbb{U}$ such that $J|_{\mathbb{X}} = T$ and $T \circ P = Q \circ J$.*

Proof The mappings $\mathrm{Id}_{\mathbb{U}} - P$ and $\mathrm{Id}_{\mathbb{U}} - Q$ are projections onto $\mathrm{Ker}(P)$, and $\mathrm{Ker}(Q)$, respectively. By assumption, there is an isomorphism S from $\mathrm{Ker}(P)$ onto $\mathrm{Ker}(Q)$. The map

$$u \mapsto J(u) := T(P(u)) + S(u - P(u))$$

is an isomorphism from $\mathbb{U}$ onto $\mathbb{U}$, and we have $Q(J(u)) = T(P(u))$ for all $u \in \mathbb{U}$. □

Corollary 3.3 *Let $\mathcal{X} = (\boldsymbol{x}_n)_{n=1}^\infty$ be $\mathcal{Y} = (\boldsymbol{y}_n)_{n=1}^\infty$ be complemented unconditional basic sequences in a quasi-Banach space $\mathbb{U}$. Let $(\boldsymbol{x}_n^*)_{n=1}^\infty$ and $(\boldsymbol{y}_n^*)_{n=1}^\infty$ be projecting functionals for $\mathcal{X}$ and $\mathcal{Y}$, respectively. Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are equivalent and that there is $(\lambda_n)_{n=1}^\infty$ in $\mathbb{F}$ such that $\boldsymbol{y}_n^* = \lambda_n \boldsymbol{x}_n^*$ for all $n \in \mathbb{N}$. Then, $\mathcal{X}$ and $\mathcal{Y}$ are congruent.*

Proof Let P and Q be the projections of $\mathbb{U}$ onto $\mathbb{X} := [\mathcal{X}]$ and $\mathbb{Y} := [\mathcal{Y}]$, respectively, given by

$$P(f) = \sum_{n=1}^\infty \boldsymbol{x}_n^*(f)\, \boldsymbol{x}_n \quad \text{and} \quad Q(f) = \sum_{n=1}^\infty \boldsymbol{y}_n^*(f)\, \boldsymbol{y}_n,$$

respectively. We have

$$\mathrm{Ker}(P) = \cap_{n=1}^\infty \mathrm{Ker}(\boldsymbol{x}_n^*) = \cap_{n=1}^\infty \mathrm{Ker}(\boldsymbol{y}_n^*) = \mathrm{Ker}(Q).$$

Consequently, $\mathbb{U}/\mathbb{X} \simeq \mathbb{U}/\mathbb{Y}$. Hence, the result follows from Lemma 3.2. □

If $\mathcal{X}$ is a complemented unconditional basic sequence in a quasi-Banach space $\mathbb{U}$ with projecting functionals $\mathcal{X}^*$, and T is an automorphism of $\mathbb{U}$, then $T(\mathcal{X})$ is a complemented unconditional basic sequence in $\mathbb{U}$ projecting functionals $S^*(\mathcal{X}^*)$, where S is the inverse of T. The following result points in the opposite direction.

Proposition 3.4 *Let $\mathcal{X}$ and $\mathcal{Y}$ be complemented unconditional basic sequences in a quasi-Banach space $\mathbb{U}$. Let $\mathcal{X}^*$ and $\mathcal{Y}^*$ coordinate functionals for $\mathcal{X}$ and $\mathcal{Y}$, respectively. If $\mathcal{X}$ and $\mathcal{Y}$ are congruent, then there is an automorphism T of $\mathbb{U}$ with $T(\mathcal{X}) = \mathcal{Y}$ and $T^*(\mathcal{Y}^*) = \mathcal{X}^*$.*

Proof Congruence implies $\mathbb{U}/[\mathcal{X}] \simeq \mathbb{U}/[\mathcal{Y}]$. Consider the automorphism of $\mathbb{U}$ provided by Lemma 3.2. It is routine to check that it satisfies the desired conditions. □

The following two results are improved versions of the small perturbation principle that fit our purposes. As before, we state it for (non-necessarily locally convex) quasi-Banach spaces.

Given a sequence $\mathcal{A}$ in a dual space $\mathbb{X}^*$, where $\mathbb{X}$ is a quasi-Banach space, $[\mathcal{A}]_{w^*}$ denotes its closed linear span relative to the w^*–topology in $\mathbb{X}^*$. Given $(\boldsymbol{x}_n^*)_{n=1}^\infty$ and x^* in $\mathbb{X}^*$, the symbol

$$x^* = w^*\text{–}\sum_{n=1}^\infty \boldsymbol{x}_n^*$$

means that the series $\sum_{n=1}^\infty \boldsymbol{x}_n^*$ converges to $\boldsymbol{x}^*$ in the w^*–topology.

Lemma 3.5 *Let $\mathcal{X} = (\boldsymbol{x}_n, \boldsymbol{x}_n^*)_{n=1}^\infty$ be a biorthogonal system in a p-Banach space $\mathbb{X}$, $0 < p \le 1$. Let $\mathcal{Y} = (\boldsymbol{y}_n)_{n=1}^\infty$ be a sequence in $\mathbb{X}$, and suppose that*

$$\sum_{n=1}^\infty \|\boldsymbol{y}_n - \boldsymbol{x}_n\|^p \, \|\boldsymbol{x}_n^*\|^p < 1.$$

Then, $\mathcal{Y}$ is congruent to $\mathcal{X}$. Moreover, we can choose $(\boldsymbol{y}_n^*)_{n=1}^\infty$ biorthogonal to $\mathcal{Y}$ satisfying conditions (a) and (b) below.

(a) If $\boldsymbol{x}_k^*(\boldsymbol{y}_n) = 0$ for all $(k, n) \in \mathbb{N}^2$ with $k > n$, then, for every $n \in \mathbb{N}$,

$$\boldsymbol{y}_n^* \in \mathbb{U}_n := [\boldsymbol{x}_k^* \colon k \ge n]_{w^*}.$$

(b) If $\mathcal{X}$ is a complemented unconditional basic sequence with projecting functionals $(\boldsymbol{x}_n^*)_{n=1}^\infty$, then $\mathcal{Y}$ is a complemented unconditional basic sequence with projecting functionals $(\boldsymbol{y}_n^*)_{n=1}^\infty$.

Proof The map $E \colon \mathbb{X} \to \mathbb{X}$ given by

$$E(f) = \sum_{n=1}^\infty \boldsymbol{x}_n^*(f)(\boldsymbol{x}_n - \boldsymbol{y}_n), \quad f \in \mathbb{X},$$

is well-defined, and we have $\|E\| < 1$. Consequently, $S = \mathrm{Id}_{\mathbb{X}} - E$ is an automorphism whose inverse is

$$T = \sum_{n=0}^\infty E^n.$$

We have $S(\boldsymbol{x}_n)=\boldsymbol{y}_n$ for every $n\in\mathbb{N}$. It is clear that $(T^*(\boldsymbol{x}_n^*))_{n=1}^\infty$ is biorthogonal to $\mathcal{Y}$ and satisfies (b). In order to prove (a), since

$$T^*=w^*\text{-}\sum_{n=0}^\infty (E^*)^n,$$

it suffices to prove that $E^*(\mathbb{U}_k)\subseteq\mathbb{U}_k$ for every $k\in\mathbb{N}$. Notice that

$$E^*(f^*)=w^*\text{-}\sum_{n=1}^\infty f^*(\boldsymbol{x}_n-\boldsymbol{y}_n)\,\boldsymbol{x}_n^*,\quad f^*\in\mathbb{X}^*.$$

Pick $f^*\in\mathbb{U}_k$. We have $f^*(\boldsymbol{x}_n)=f^*(\boldsymbol{y}_n)=0$ for every $n\in\mathbb{N}$ with $n<k$. Consequently,

$$E^*(f^*)=w^*\text{-}\sum_{n=k}^\infty f^*(\boldsymbol{x}_n-\boldsymbol{y}_n)\,\boldsymbol{x}_n^*\in\mathbb{U}_k.$$ □

Lemma 3.6 *Let* $\mathcal{X}=(\boldsymbol{x}_n)_{n=1}^\infty$ *be a complemented unconditional basic sequence in a p-Banach space* $\mathbb{X}$, $0<p\le 1$, *with projecting functionals* $(\boldsymbol{x}_n^*)_{n=1}^\infty$. *Let* $\mathcal{Y}^*=(\boldsymbol{y}_n^*)_{n=1}^\infty$ *be another sequence in* $\mathbb{X}^*$, *and suppose that*

$$\sum_{n=1}^\infty \left\|\boldsymbol{y}_n^*-\boldsymbol{x}_n^*\right\|^p\|\boldsymbol{x}_n\|^p<1.$$

Then, there is a sequence $\mathcal{Y}=(\boldsymbol{y}_n)_{n=1}^\infty$ *congruent to* $\mathcal{X}$ *such that* $\mathcal{Y}^*$ *are projecting functionals for* $\mathcal{Y}$.

Proof The map $E\colon\mathbb{X}\to\mathbb{X}$ given by

$$E(f)=\sum_{n=1}^\infty\left(\boldsymbol{x}_n^*(f)-\boldsymbol{y}_n^*(f)\right)\boldsymbol{x}_n,\quad f\in\mathbb{X},$$

is well-defined, and we have $\|E\|<1$. Consequently, $\mathrm{Id}_{\mathbb{X}}-E$ is an automorphism. Hence, if S denote its inverse, $(S(\boldsymbol{x}_n))_{n=1}^\infty$ is a complemented basic sequence with projecting functionals

$$\boldsymbol{z}_n^*:=(\mathrm{Id}_{\mathbb{X}}-E)^*(\boldsymbol{x}_n^*),\quad n\in\mathbb{N}.$$

Since $(\mathrm{Id}_{\mathbb{X}}-E)^*=\mathrm{Id}_{\mathbb{X}^*}-E^*$ and $E^*\colon\mathbb{X}^*\to\mathbb{X}^*$ is given by

$$E^*(f^*)=w^*\text{-}\sum_{n=1}^\infty f^*(\boldsymbol{x}_n)(\boldsymbol{x}_n^*-\boldsymbol{y}_n^*),\quad f^*\in\mathbb{X}^*,$$

$\boldsymbol{z}_n^*=\boldsymbol{y}_n^*$ for every $n\in\mathbb{N}$. Hence we can take $\boldsymbol{y}_n:=S(\boldsymbol{x}_n)$. □

4 Unconditional basic sequences in function spaces

We start our study with a lemma that places a given basic sequence in a function space with the unit vector system of ℓ_2 face to face. To state it, we introduce some additional terminology. We say that a sequence $\mathcal{X} = (\boldsymbol{x}_n)_{n=1}^\infty$ in a quasi-Banach space $\mathbb{X}$ *dominates* a sequence $\mathcal{Y} = (\boldsymbol{y}_n)_{n=1}^\infty$ in a quasi-Banach space $\mathbb{Y}$ if there is a bounded linear map

$$T : [\boldsymbol{x}_n : n \in \mathbb{N}] \to \mathbb{Y}$$

such that $T(\boldsymbol{x}_n) = \boldsymbol{y}_n$ for all $n \in \mathbb{N}$, in which case we also say that $\mathcal{Y}$ *is dominated by* $\mathcal{X}$. If $\|T\| \le C$ we say that $\mathcal{X}$ C*-dominates* $\mathcal{Y}$. Notice that the sequences $\mathcal{X}$ and $\mathcal{Y}$ are equivalent if and only if $\mathcal{X}$ both dominates and is dominated by $\mathcal{Y}$. Given a measure space (Ω, Σ, μ), we say that a sequence $(f_n)_{n=1}^\infty$ in $L_0(\mu)$ *escapes to infinity* if

$$\lim_n \operatorname{ess\,inf}\{|f_n(\omega)| : \omega \in \operatorname{supp}(f_n)\} = \infty.$$

Lemma 4.1 *Let ρ be a function norm over a σ-finite measure space (Ω, Σ, μ). Let $\Psi = (\boldsymbol{\psi}_n)_{n=1}^\infty$ be a semi-normalized unconditional basic sequence in $\boldsymbol{L}_\rho$.*

(i) Suppose that $\boldsymbol{L}_\rho$ has type 2 and nontrivial cotype. Then, Ψ is dominated by the unit vector system of ℓ_2.

(ii) Suppose that there is function norm ρ_b over (Ω, Σ, μ) such that $\boldsymbol{L}_\rho \subseteq L_{\rho_b}$, $\boldsymbol{L}_{\rho_b}$ has cotype 2, and $\inf_n \|\boldsymbol{\psi}_n\|_{\rho_b} > 0$. Then, Ψ dominates the unit vector system of ℓ_2.

(iii) Suppose that μ is finite and that $\inf_n \|\boldsymbol{\psi}_n\|_{\rho_b} = 0$ for some function norm ρ_b. Then, there is an increasing sequence $(n_k)_{k=1}^\infty$ and a non-increasing sequence $(A_k)_{k=1}^\infty$ in Σ such that

$$\lim_k \left\| \boldsymbol{\psi}_{n_k} - \boldsymbol{\psi}_{n_k} \chi_{A_k} \right\|_\rho = 0$$

and $\lim_k \mu(A_k) = 0$.

(iv) Suppose that ρ is absolutely continuous and that there is a non-increasing sequence $(A_n)_{n=1}^\infty$ in Σ such that $\operatorname{supp}(\boldsymbol{\psi}_n) \subseteq A_n$ for all $n \in \mathbb{N}$, and

$$\lim_n \mu(A_n) = 0.$$

Then, there is an increasing sequence $(n_k)_{k=1}^\infty$ in $\mathbb{N}$ and a pairwise disjointly supported sequence $(\boldsymbol{\phi}_k)_{k=1}^\infty$ consisting of simple functions escaping to infinity such that $|\boldsymbol{\phi}_k| \le |\boldsymbol{\psi}_{n_k}|$, $\operatorname{supp}(\boldsymbol{\phi}_k) \subseteq A_{n_k} \setminus A_{n_{k+1}}$ for all $k \in \mathbb{N}$, and

$$\lim_k \left\| \boldsymbol{\psi}_{n_k} - \boldsymbol{\phi}_k \right\|_\rho = 0.$$

Proof To prove (i) and (ii) we use (3.2) and (3.3). In the former case we have

$$\begin{aligned}\left\|\sum_{n=1}^{\infty} a_n\, \boldsymbol{\psi}_n\right\|_\rho &\approx \left\|\left(\sum_{n=1}^{\infty} |a_n|^2\, |\boldsymbol{\psi}_n|\right)^{1/2}\right\|_\rho \\ &\lesssim \left(\sum_{n=1}^{\infty} |a_n|^2\, \|\boldsymbol{\psi}_n\|_\rho^2\right)^{1/2} \\ &\approx \left(\sum_{n=1}^{\infty} |a_n|^2\right)^{1/2}\end{aligned}$$

for $f = (a_n)_{n=1}^\infty \in c_{00}$. In the latter case we have

$$\begin{aligned}\left\|\sum_{n=1}^{\infty} a_n\, \boldsymbol{\psi}_n\right\|_\rho &\approx \operatorname*{Ave}_{\varepsilon_n=\pm 1} \left\|\sum_{n=1}^{\infty} \varepsilon_n\, a_n\, \boldsymbol{\psi}_n\right\|_\rho \\ &\gtrsim \operatorname*{Ave}_{\varepsilon_n=\pm 1} \left\|\sum_{n=1}^{\infty} \varepsilon_n\, a_n\, \boldsymbol{\psi}_n\right\|_{\rho_b} \\ &\approx \left\|\left(\sum_{n=1}^{\infty} |a_n|^2\, |\boldsymbol{\psi}_n|^2\right)^{1/2}\right\|_{\rho_b} \\ &\gtrsim \left(\sum_{n=1}^{\infty} |a_n|^2\, \|\boldsymbol{\psi}_n\|_{\rho_b}^2\right)^{1/2} \\ &\approx \left(\sum_{n=1}^{\infty} |a_n|^2\right)^{1/2}.\end{aligned}$$

To prove (iii) we assume, passing to a subsequence, that Ψ converges to zero in measure. By Egoroff's theorem, passing to a further subsequence we can assume that $\lim_n \boldsymbol{\psi}_n = 0$ almost uniformly. We infer that there is a non-increasing sequence $(A_n)_{n=1}^\infty$ in Σ such that $\lim_n \mu(A_n) = 0$ and

$$\lim_n \left\|\boldsymbol{\psi}_n - \boldsymbol{\psi}_n \chi_{A_n}\right\|_\infty = 0.$$

Since $\chi_\Omega \in \boldsymbol{L}_\rho$, $L_\infty(\mu) \subseteq \boldsymbol{L}_\rho$. Consequently, $\lim_n \left\|\boldsymbol{\psi}_n - \boldsymbol{\psi}_n \chi_{A_n}\right\|_\rho = 0$.

To prove (iv), we set $s_n = \rho^{-1/2}(\chi_{A_n})$ and $\varepsilon_n = s_n \rho(\chi_{A_n})$ for $n \in \mathbb{N}$. Notice that $\lim_n s_n = \infty$ and $\lim_n \varepsilon_n = 0$. We use that, by assumption,

$$\lim_j \left\|\boldsymbol{\psi}_n \chi_{A_j}\right\|_\rho = 0, \quad n \in \mathbb{N},$$

to recursively construct an increasing sequence $(n_k)_{k=1}^{\infty}$ in $\mathbb{N}$ such that

$$\left\| \boldsymbol{\psi}_{n_k} \chi_{A_{n_{k+1}}} \right\|_{\rho} \le \varepsilon_k, \quad k \in \mathbb{N}.$$

Approximating each function $\boldsymbol{\psi}_{n_k}$ by simple functions and using the property of $\boldsymbol{L}_\rho$ recorded in item (iv) of Theorem 2.5, we obtain a sequence $(\boldsymbol{\phi}_k)_{k=1}^{\infty}$ of simple functions such that $\operatorname{supp}(\boldsymbol{\phi}_k) \subseteq B_k := A_{n_k} \setminus A_{n_{k+1}}$, $\left|\boldsymbol{\phi}_k\right| \le \left|\boldsymbol{\psi}_{n_k}\right| \chi_{B_k}$, and

$$\left\| \boldsymbol{\psi}_{n_k} \chi_{B_k} - \boldsymbol{\phi}_k \right\|_{\rho} \le \varepsilon_k$$

for all $k \in \mathbb{N}$. Now, set $D_k = \{\omega \in \Omega \colon \left|\boldsymbol{\phi}_k(\omega)\right| \ge s_{n_k}\}$. We have

$$\left\| \boldsymbol{\phi}_k - \boldsymbol{\phi}_k \chi_{D_k} \right\|_{\rho} \le s_{n_k} \rho(\chi_{D_{n_k}}) \le s_{n_k} \rho(\chi_{A_{n_k}}) = \varepsilon_{n_k}.$$

Summing up, we obtain $\lim_k \left\| \boldsymbol{\psi}_{n_k} - \boldsymbol{\phi}_k \chi_{D_k} \right\|_{\rho} = 0$. □

The following theorem generalizes the result from [16] that any semi-normalized unconditional basic sequence in L_p, $2 \le p < \infty$, is either the canonical basis of ℓ_2 or has a subsequence equivalent to a disjointly supported sequence.

Theorem 4.2 *Let ρ be a function norm over a finite measure space (Ω, Σ, μ). Suppose that $\boldsymbol{L}_\rho \subseteq L_2(\mu)$ and that $\boldsymbol{L}_\rho$ has Rademacher type 2. Let $\Psi = (\boldsymbol{\psi}_n)_{n=1}^{\infty}$ be a semi-normalized unconditional basic sequence in $\boldsymbol{L}_\rho$.*

(i) If $\inf_n \left\| \boldsymbol{\psi}_n \right\|_2 > 0$, then Ψ is equivalent to the unit vector system of ℓ_2. Oppositely,
(ii) if $\inf_n \left\| \boldsymbol{\psi}_n \right\|_2 = 0$, then Ψ has a subsequence congruent to a disjointly supported sequence Φ consisting of simple functions escaping to infinity.

Proof The function norm ρ is absolutely continuous by Theorem 2.3 and Theorem 2.6. So, we identify $(\boldsymbol{L}_\rho)^*$ with $\boldsymbol{L}_{\rho^*}$.

In the case (i), we apply Lemma 4.1(ii) with $\rho_b = \|\cdot\|_2$. We obtain that Ψ dominates the unit vector system of ℓ_2. In turn, by Lemma 4.1(i), the unit vector system of ℓ_2 dominates Ψ.

In the case (ii), combining Lemma 4.1(iii) with Lemma 4.1(iv) gives a pairwise disjointly supported sequence $(\boldsymbol{\phi}_n)_{n=1}^{\infty}$ consisting of simple functions escaping to infinity such that

$$\lim_n \left\| \boldsymbol{\psi}_n - \boldsymbol{\phi}_n \right\|_{\rho} = 0.$$

Set $\mathbb{Y} = [\boldsymbol{\psi}_n \colon n \in \mathbb{N}]$. Since Ψ is a semi-normalized Schauder basis of $\mathbb{Y}$, there exists a sequence in $\mathbb{Y}^*$ biorthogonal to Ψ. Use the Hahn-Banach theorem to extend these coordinate functionals to a norm-bounded sequence $\Psi^* = (\boldsymbol{\psi}_n^*)_{n=1}^{\infty}$ in $\boldsymbol{L}_{\rho^*}$. We have

$$\lim_n \left\| \boldsymbol{\psi}_n - \boldsymbol{\phi}_n \right\|_{\rho} \left\| \boldsymbol{\psi}_n^* \right\|_{\rho^*} = 0.$$

Since Ψ^* is biorthogonal to Ψ, passing to a further subsequence, an application of Lemma 3.5 puts an end to the proof. □

Before going on, we record the straightforward application of Theorem 4.2 to the study of subsymmetric basic sequences. Notice that if $\mathcal{Y}$ is a subbasis of a subsymetric basic sequence $\mathcal{X}$ in a Banach space $\mathbb{U}$, and we set $\mathbb{X} = [\mathcal{X}]$ and $\mathbb{Y} = [\mathcal{Y}]$, the quotient spaces $\mathbb{U}/\mathbb{X}$ and $\mathbb{U}/\mathbb{Y}$ are not necessarily isomorphic, and assuming $\mathcal{X}$ is complemented does not change anything. So, despite $\mathcal{X}$ and $\mathcal{Y}$ being equivalent, they are not necessarily congruent.

Corollary 4.3 *Let ρ be a function norm over a finite measure space (Ω, Σ, μ). Suppose that $\boldsymbol{L}_\rho \subseteq L_2(\mu)$ and that $\boldsymbol{L}_\rho$ has Rademacher type 2. Let Ψ be a semi-normalized subsymmetric basic sequence in $\boldsymbol{L}_\rho$. Then, Ψ is equivalent to either the unit vector system of ℓ_2 or a disjointly supported sequence consisting of simple functions escaping to infinity.*

Any pairwise disjointly supported sequence in a function space is an unconditional basic sequence. It might not be complemented, however. In case it is, to understand the lattice structure it induces, it is convenient to look at the position of its projecting functionals within the dual space. To that end, we bring up a notion successfully used within the study of the uniqueness of structure in atomic lattices (see [1]).

Definition 4.4 Let ρ be an absolutely continuous function norm over a σ-finite measure space. Let Ψ be a sequence of nonzero functions in $\boldsymbol{L}_\rho$. We say that Ψ is a *well-complemented basic sequence* if

- it is pairwise disjointly supported,
- it is complemented, and
- there are projecting functionals Ψ^* for Ψ which, regarded as functions in $\boldsymbol{L}_{\rho^*}$, are pairwise disjointly supported.

Such sequence Ψ^* is said to be a sequence of *good projecting functionals* for Ψ.

The following two lemmas help us to pass from a complemented unconditional basic sequence to a well-complemented one.

Lemma 4.5 *Let ρ be a function norm over a σ-finite measure space (Ω, Σ, μ). Let $\Psi = (\boldsymbol{\psi}_n)_{n=1}^\infty$ be a complemented unconditional basic sequence in $\boldsymbol{L}_\rho$. Let $\Phi = (\boldsymbol{\phi}_n)_{n=1}^\infty$ be a pairwise disjointly supported sequence in $\boldsymbol{L}_\rho$ with $\left|\boldsymbol{\phi}_n\right| \le \left|\boldsymbol{\psi}_n\right|$ for all $n \in \mathbb{N}$. Then Ψ C-dominates Φ, where C only depends on the unconditionality constant of Ψ and the complementabilty constant of the closed subspace it spans.*

Proof By Theorem 3.1, for $(a_n)_{n=1}^\infty \in c_{00}$ we have

$$\left\| \sum_{n=1}^\infty a_n \boldsymbol{\phi}_n \right\|_\rho = \left\| \left(\sum_{n=1}^\infty |a_n|^2 \left|\boldsymbol{\phi}_n\right|^2 \right)^{1/2} \right\|_\rho \le \left\| \left(\sum_{n=1}^\infty |a_n|^2 \left|\boldsymbol{\psi}_n\right|^2 \right)^{1/2} \right\|_\rho \approx \left\| \sum_{n=1}^\infty a_n \boldsymbol{\psi}_n \right\|_\rho .$$ □

Lemma 4.6 *Let ρ be an absolutely continuous function norm over a σ-finite measure space. Let $\Psi = (\boldsymbol{\psi}_n)_{n=1}^{\infty}$ be a complemented unconditional basis sequence in $\boldsymbol{L}_\rho$ with projecting functionals $\Psi^* = (\boldsymbol{\psi}_n^*)_{n=1}^{\infty}$ regarded as functions in $\boldsymbol{L}_{\rho^*}$. Let $\Phi = (\boldsymbol{\phi}_n)_{n=1}^{\infty}$ be a pairwise disjointly supported sequence in $\boldsymbol{L}_\rho$ and $(\boldsymbol{\phi}_n^*)_{n=1}^{\infty}$ be a pairwise disjointly supported sequence in $\boldsymbol{L}_{\rho^*}$. Suppose that $\left|\boldsymbol{\phi}_n\right| \le \left|\boldsymbol{\psi}_n\right|$ and $\left|\boldsymbol{\phi}_n^*\right| \le \left|\boldsymbol{\psi}_n^*\right|$ for all $n \in \mathbb{N}$. Then, we have the following.*

(i) There are bounded linear maps R, S, $T \colon \boldsymbol{L}_\rho \to \boldsymbol{L}_\rho$ given by

$$R(f) = \sum_{n=1}^{\infty} \langle \boldsymbol{\psi}_n^*, f \rangle \boldsymbol{\phi}_n,$$

$$S(f) = \sum_{n=1}^{\infty} \langle \boldsymbol{\phi}_n^*, f \rangle \boldsymbol{\psi}_n,$$

$$T(f) = \sum_{n=1}^{\infty} \langle \boldsymbol{\phi}_n^*, f \rangle \boldsymbol{\phi}_n.$$

for all $f \in \mathbb{N}$.

(ii) If $\inf_n \left|\langle \boldsymbol{\phi}_n^*, \boldsymbol{\phi}_n \rangle\right| > 0$, then Φ is a well-complemented basic sequence equivalent to Ψ. Moreover, there are scalars $(\lambda_n)_{n=1}^{\infty}$ such that $(\lambda_n \boldsymbol{\phi}_n^*)_{n=1}^{\infty}$ is a sequence of good projecting functionals for Φ.

(iii) If, in addition, Ψ^* is disjointly supported, then Φ and Ψ and congruent.

Proof The existence of R is straightforward consequence of Lemma 4.5. It also follows from Lemma 4.5 that the existence of S implies the existence of T. Before addressing the proof of the existence of S, we note that, given $f^* \in \boldsymbol{L}_{\rho^*}$, the series $\sum_{n=1}^{\infty} \langle f^*, \boldsymbol{\psi}_n \rangle \boldsymbol{\psi}_n^*$ might not converge in the norm-topolopy. So, Ψ might not be a sequence of projecting functionals for Ψ^*. To circumvent this drawback, we use that, for each $m \in \mathbb{N}$, $(\boldsymbol{\psi}_n^*)_{n=1}^{m}$ is a complemented unconditional basic sequence with projecting functionals $(\boldsymbol{\psi}_n)_{n=1}^{m}$. Hence, there is a constant C such that

$$\left\| \sum_{n=1}^{m} \langle f^*, \boldsymbol{\psi}_n \rangle \boldsymbol{\phi}_n^* \right\| \le C \left\| f^* \right\|$$

for all $m \in \mathbb{N}$ and $f^* \in \boldsymbol{L}_{\rho^*}$. Since the operator

$$f^* \mapsto \sum_{n=1}^{m} \langle f^*, \boldsymbol{\psi}_n \rangle \boldsymbol{\phi}_n^*$$

is the dual operator of

$$f \mapsto \sum_{n=1}^{m} \langle \boldsymbol{\phi}_n^*, f \rangle \boldsymbol{\psi}_n,$$

we have

$$\left\| \sum_{n=1}^{m} \langle \boldsymbol{\phi}_n^*, f \rangle \boldsymbol{\psi}_n \right\| \le C \|f\|, \quad m \in \mathbb{N}, \ f \in \mathbb{X}.$$

By Theorem 2.5, Ψ is boundedly complete. Hence, $\sum_{n=1}^{\infty} \langle \boldsymbol{\phi}_n^*, f \rangle \boldsymbol{\psi}_n$ converges for any $f \in \mathbb{X}$, and

$$\left\| \sum_{n=1}^{\infty} \langle \boldsymbol{\phi}_n^*, f \rangle \boldsymbol{\psi}_n \right\| \le C \|f\|.$$

This gives the existence of S.

To prove (ii) we set $\lambda_n = \langle \boldsymbol{\phi}_n^*, \boldsymbol{\phi}_n \rangle$, and we consider the bounded linear operators $U, Q \colon \boldsymbol{L}_\rho \to \boldsymbol{L}_\rho$ given by

$$U(f) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \langle \boldsymbol{\phi}_n^*, f \rangle \, \boldsymbol{\psi}_n,$$

$$Q(f) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \langle \boldsymbol{\phi}_n^*, f \rangle \, \boldsymbol{\phi}_n.$$

Since $Q(\boldsymbol{\phi}_n) = \boldsymbol{\phi}_n$ for all $n \in \mathbb{N}$, Q is a projection onto the closed subspace generated by Φ, and $(\boldsymbol{\phi}_n/\lambda_n)_{n=1}^{\infty}$ is a sequence of projecting functionals for Φ. We have $S(\boldsymbol{\psi}_n) = \boldsymbol{\phi}_n$ and $U(\boldsymbol{\phi}_n) = \boldsymbol{\psi}_n$ and for every $n \in \mathbb{N}$. Consequently, Ψ and Φ are equivalent.

To prove (iii), consider a measurable function $\varepsilon \colon \Omega \to \mathbb{F}$ such that $|\varepsilon(\omega)| = 1$ and $\varepsilon(\omega)\boldsymbol{\psi}_n^*(\omega)\boldsymbol{\phi}_n(\omega) \ge 0$ for all $\omega \in \Omega$. We have

$$\mu_n := \langle \boldsymbol{\psi}_n^*, \varepsilon \boldsymbol{\phi}_n \rangle \ge \lambda_n, \quad n \in \mathbb{N}.$$

Since the map $f \mapsto \varepsilon f$ is an isomorphism on L_ρ, there is a bounded linear operator $V \colon \boldsymbol{L}_\rho \to \boldsymbol{L}_\rho$ given by

$$V(f) = \sum_{n=1}^{\infty} \frac{1}{\mu_n} \langle \boldsymbol{\psi}_n^*, f \rangle \, \varepsilon \boldsymbol{\phi}_n.$$

We infer that $\Phi_\varepsilon := (\varepsilon \boldsymbol{\phi}_n)_{n=1}^{\infty}$ is a complemented basic sequence with projecting functionals $(\boldsymbol{\psi}_n^*/\mu_n)_{n=1}^{\infty}$. Since Φ and Φ_ε are congruent, Φ_ε and Ψ are equivalent. Consequently, by Corollary 3.3, Φ_ε and Ψ are congruent. □

Theorem 4.7 below and the subsequent Corollary 4.8 genereralize the result from [16] that any complemented semi-normalized unconditional basic sequence in L_p, $1 < p < \infty$, is either equivalent to the canonical basis of ℓ_2 or has a subsequence equivalent to a pairwise disjointly supported sequence. To help the reader understand their statements, we point out that any pairwise disjointly supported sequence of L_p,

$1 \le p < \infty$, is well-complemented (see [16, Proof of Lemma 1]), but this property does not hold in general function spaces.

Theorem 4.7 *Let ρ be a function norm over a finite measure space (Ω, Σ, μ). Suppose that $L_2(\mu) \subseteq \boldsymbol{L}_\rho$ and that $\boldsymbol{L}_\rho$ has Rademacher cotype 2. Let $\Psi = (\boldsymbol{\psi}_n)_{n=1}^\infty$ be a semi-normalized complemented unconditional basic sequence in $\boldsymbol{L}_\rho$ with projecting functionals $\Psi^* = (\boldsymbol{\psi}_n^*)_{n=1}^\infty$.*

(i) If $\inf_n \left\| \boldsymbol{\psi}_n^* \right\|_2 > 0$, then Ψ is equivalent to the unit vector system of ℓ_2. Oppositely,

(ii) if $\inf_n \left\| \boldsymbol{\psi}_n^* \right\|_2 = 0$, then Ψ has a subsequence equivalent to a well-complemented basic sequence $\Phi = (\boldsymbol{\phi}_n)_{n=1}^\infty$ escaping to infinity. Moreover, both Φ and its sequence Φ^* of good projecting functionals consist of simple functions. If ρ^* is absolutely continuous, we can make Φ congruent to Ψ, and make Φ^* escape to infinity.

Proof By Theorem 2.6, $\boldsymbol{L}_\rho$ is absolutely continuous. Hence, we can regard Ψ^* as an unconditional basic sequence in $\boldsymbol{L}_{\rho^*}$.

In the case (i), applying Lemma 4.1(ii) with $\rho_b = \|\cdot\|_2$ gives that Ψ^* dominates the unit vector system of ℓ_2. Hence, by the reflexivity principle for basic sequences in Banach spaces (see [4, Corollary 3.2.4]), the unit vector system of ℓ_2 dominates Ψ. In turn, applying Lemma 4.1(ii) with $\rho_b = \rho$, gives that Ψ dominates the unit vector system of ℓ_2.

In the case (ii), an application of Lemma 4.1(iii) gives an increasing map $(n_k)_{k=1}^\infty$ and a non-increasing sequence $(A_k)_{k=1}^\infty$ such that

$$\lim_k \mu(A_k) = 0 \quad \text{and} \quad \lim_k \left\| \boldsymbol{\psi}_{n_k}^* - \boldsymbol{\psi}_{n_k}^* \chi_{A_k} \right\|_{\rho^*} = 0.$$

Therefore, we can assume, passing to a subsequence, that there is a non-increasing sequence $(A_n)_{n=1}^\infty$ with $\lim_n \mu(A_n) = 0$ and

$$\sum_{n=1}^\infty \left\| \boldsymbol{\psi}_n^* \chi_{A_n} - \boldsymbol{\psi}_n^* \right\|_{\rho^*} \left\| \boldsymbol{\psi}_n \right\|_\rho < 1.$$

Hence, by Lemma 3.6, we can suppose that $\operatorname{supp}(\boldsymbol{\psi}_n^*) \subseteq A_n$ for all $n \in \mathbb{N}$. Since ρ is absolutely continuous, passing to a further subsequence we can suppose that

$$\sum_{n=1}^\infty \left\| \boldsymbol{\psi}_n \chi_{A_{n+1}} \right\|_\rho \left\| \boldsymbol{\psi}_n^* \right\|_{\rho^*} < 1.$$

If $k > n$, then $\operatorname{supp}(\boldsymbol{\psi}_k^*) \subseteq A_{n+1}$. Consequently,

$$\langle \boldsymbol{\psi}_k^*, \boldsymbol{\psi}_n \chi_{\Omega \setminus A_{n+1}} \rangle = 0.$$

By Lemma 3.5, Ψ is congruent to $\Phi := (\boldsymbol{\psi}_n \chi_{\Omega \setminus A_{n+1}})_{n=1}^\infty$, and there are projecting functionals $(\boldsymbol{\phi}_n^*)_{n=1}^\infty$ for Φ with

$$\boldsymbol{\phi}_n^* \in [\boldsymbol{\psi}_k^* : k \ge n]_{w^*}, \quad n \in \mathbb{N}.$$

By Proposition 2.7, $\operatorname{supp}(\boldsymbol{\phi}_n^*) \subseteq A_n$ for all $n \in \mathbb{N}$.

Summing up, we can assume that the complemented unconditional basic sequence Ψ and its coordinate functionals Ψ^* satisfy

$$\operatorname{supp}(\boldsymbol{\psi}_n) \subseteq \Omega \setminus A_{n+1} \quad \text{and} \quad \operatorname{supp}(\boldsymbol{\psi}_n^*) \subseteq A_n, \quad n \in \mathbb{N},$$

for a suitable non-increasing sequence $(A_n)_{n=1}^\infty$ with $\lim_n \mu(A_n) = 0$, and we can forget other terminology used so far in the proof of (ii).

Use Lemma 4.1(iv) to pick, passing to a further subsequence, a pairwise disjointly supported sequence $\Phi = (\boldsymbol{\phi}_n)_{n=1}^\infty$ consisting of simple functions escaping to infinity such that $|\boldsymbol{\phi}_n| \le |\boldsymbol{\psi}_n| \, \chi_{A_n \setminus A_{n+1}}$ and

$$\left\| \boldsymbol{\psi}_n \chi_{A_n \setminus A_{n+1}} - \boldsymbol{\phi}_n \right\|_\rho < \frac{1}{2 \left\| \boldsymbol{\psi}_n^* \right\|_{\rho^*}}$$

for all $n \in \mathbb{N}$. Since

$$\langle \boldsymbol{\psi}_n^*, \boldsymbol{\psi}_n \chi_{A_n \setminus A_{n+1}} \rangle = \langle \boldsymbol{\psi}_n^*, \boldsymbol{\psi}_n \rangle = 1,$$

$|\langle \boldsymbol{\psi}_n^*, \boldsymbol{\phi}_n \rangle| > 1/2$. Consequently, for each $n \in \mathbb{N}$ there is a simple function $\boldsymbol{\phi}_n^*$ with $|\boldsymbol{\phi}_n^*| \le |\boldsymbol{\psi}_n^*| \, \chi_{A_n \setminus A_{n+1}}$ and $|\langle \boldsymbol{\phi}_n^*, \boldsymbol{\phi}_n \rangle| \ge 1/2$.

By Lemma 4.6(ii), Ψ and Φ are equivalent, Φ is well-complemented, and there is a sequence $(a_n)_{n=1}^\infty$ such that $(a_n \boldsymbol{\phi}_n^*)_{n=1}^\infty$ are good projecting functionals for Φ.

In the case that ρ^* is absolutely continuous, we prove (ii) by means of an argument that differs from the previous one from the beginning. Now we combine Lemma 4.1(ii), Lemma 4.1(iii) and Lemma 4.1(iv) to claim, passing to a subsequence, that there are pairwise disjointly supported simple functions $(\boldsymbol{\phi}_n^*)_{n=1}^\infty$ escaping to infinity such that

$$\sum_{n=1}^\infty \left\| \boldsymbol{\psi}_n^* - \boldsymbol{\phi}_n^* \right\|_{\rho^*} \left\| \boldsymbol{\psi}_n \right\|_\rho < 1.$$

By Lemma 3.6, we can suppose that Ψ^* consists of pairwise disjointly supported simple functions escaping to infinity. As before, passing to a further subsequence, we choose a pairwise disjointly supported sequence $\Phi = (\boldsymbol{\phi}_n)_{n=1}^\infty$ consisting of simple functions escaping to infinity such that $|\boldsymbol{\phi}_n| \le |\boldsymbol{\psi}_n|$ and $|\langle \boldsymbol{\psi}_n^*, \boldsymbol{\phi}_n \rangle| > 1/2$ for all $n \in \mathbb{N}$. As above, we apply Lemma 4.6(ii) with the particularity that now $\Phi^* = \Psi^*$. Since a suitable dilation of Ψ^* is a sequence of projecting functionals for Φ, Φ and Ψ are congruent by Corollary 3.3.

Corollary 4.8 *Let ρ be a function norm over a finite measure space (Ω, Σ, μ). Suppose that $\boldsymbol{L}_\rho \subseteq L_2(\mu)$, and that $\boldsymbol{L}_\rho$ has Rademacher type 2. Let $\Psi = (\boldsymbol{\psi}_n)_{n=1}^\infty$ be a semi-normalized complemented unconditional basic sequence in $\boldsymbol{L}_\rho$ with $\inf_n \|\boldsymbol{\psi}_n\|_2 = 0$. Then, Ψ has a subsequence congruent to a well-complemented basic sequence $\Phi = (\boldsymbol{\phi}_n)_{n=1}^\infty$. Moreover, there is a sequence Φ^* of good projecting functionals for Φ such that both Φ and Φ^* consist of simple functions escaping to infinity.*

Proof If we identify $(\boldsymbol{L}_\rho)^*$ with $\boldsymbol{L}_{\rho^*}$, Ψ^* is a complemented unconditional basic sequence of $\boldsymbol{L}_{\rho^*}$, and Ψ is a sequence of projecting functionals for Ψ^*. Applying Theorem 4.7 and Proposition 3.4 gives, passing to a subsequence, an isomorphism $S\colon \boldsymbol{L}_{\rho^*} \to \boldsymbol{L}_{\rho^*}$ such that, if $T\colon \boldsymbol{L}_\rho \to \boldsymbol{L}_\rho$ is its dual isomorphism, the sequences $\Phi^* := S(\Psi^*)$ and $\Phi = T(\Psi)$ satisfy the desired conditions. □

Remark 4.9 In some important situations the assumption that $\boldsymbol{L}_\rho \subseteq L_2(\mu)$ in Theorem 4.2 and Corollary 4.8, as well as the assumption that $L_2(\mu) \subseteq \boldsymbol{L}_\rho$ in Theorem 4.7, are superfluous. In fact, if a rearrangement invariant function space $\boldsymbol{L}_\rho$ over [0, 1] is lattice 2-convex, then $\boldsymbol{L}_\rho \subseteq L_2$, while if $\boldsymbol{L}_\rho$ is lattice 2-concave, then $L_2 \subseteq \boldsymbol{L}_\rho$ (see [23, Remark 2 following Proposition 2.b.3]).

5 Unconditional basic sequences in direct sums of function spaces

Let J be a finite set and, for each $j \in J$, let ρ_j be a function norm over a σ-finite measure space $(\Omega_j, \Sigma_j, \mu_j)$. Let $\mu := \sqcup_{j\in J}\mu_j$ denote the disjoint union of the measures μ_j, $j \in J$. There is a natural identification of $L_0(\mu)$ with $\oplus_{j\in J} L_0(\mu_j)$. So, we can regard $\rho := (\rho_j)_{j\in J}$ as a function norm over μ, and we can canonically identify $\boldsymbol{L}_\rho$ with $\oplus_{j\in J}\boldsymbol{L}_{\rho_j}$. Since each summand $\boldsymbol{L}_{\rho_j}$ canonically embeds in $\boldsymbol{L}_\rho$, we will use the convention that $\boldsymbol{L}_{\rho_j}$ is a subspace of $\boldsymbol{L}_\rho$. Note that two sequences in $\boldsymbol{L}_{\rho_j}$ are congruent when regarded in $\boldsymbol{L}_\rho$ if and only if they are when regarded in $\boldsymbol{L}_{\rho_j}$.

We also identify $(\boldsymbol{L}_\rho)^*$ with $\oplus_{j\in J}(\boldsymbol{L}_{\rho_j})^*$ and, in the case when ρ_j is absolutely continuous for all $j \in J$, with $\oplus_{j\in J}\boldsymbol{L}_{\rho_j^*}$.

We start with a lemma that illustrates the reduction entailed in dealing with well-complemented basic sequences.

Lemma 5.1 *Let J be a finite set and, for each $j \in J$, let ρ_j be a function norm over a σ-finite measure space $(\Omega_j, \Sigma_j, \mu_j)$. Any well-complemented basic sequence of $\oplus_{j\in J}\boldsymbol{L}_{\rho_j}$ has a subsequence congruent to a well-complemented basic sequence of $\boldsymbol{L}_{\rho_j}$ for some $j \in J$.*

Proof Let $(\boldsymbol{\psi}_n^*)_{n=1}^\infty$ be good projecting functionals for $(\boldsymbol{\psi}_n)_{n=1}^\infty$. If we write

$$\boldsymbol{\psi}_n = (\boldsymbol{\psi}_{j,n})_{j\in J}, \quad \boldsymbol{\psi}_n = (\boldsymbol{\psi}_{j,n}^*)_{j\in J}, \quad n \in \mathbb{N},$$

then $\sum_{j\in J}\langle \boldsymbol{\psi}_{j,n}^*, \boldsymbol{\psi}_{j,n}\rangle = 1$ for all $n \in \mathbb{N}$. Hence, there is $j \in \mathbb{N}$ such that the set

$$\mathcal{N} := \left\{ n \in \mathbb{N}\colon \langle \boldsymbol{\psi}_{j,n}^*, \boldsymbol{\psi}_{j,n}\rangle \geq \frac{1}{|J|} \right\}$$

is infinite. Let $(n_k)_{k=1}^\infty$ be an increasing enumeration of $\mathcal{N}$. An application of Lemma 4.6(ii) gives that $(\boldsymbol{\psi}_{j,n_k})_{n=1}^\infty$ is well-complemented. Since

$$\langle \boldsymbol{\psi}_n^*, \boldsymbol{\psi}_{j,n}\rangle = \langle \boldsymbol{\psi}_{j,n}^*, \boldsymbol{\psi}_{j,n}\rangle,$$

$(\boldsymbol{\psi}_{n_k})_{n=1}^\infty$ and $(\boldsymbol{\psi}_{j,n_k})_{n=1}^\infty$ are congruent by Lemma 4.6(iii). □

Theorem 5.2 *Let J be a finite set and, for each $j \in J$, ρ_j a function norm over a σ-finite measure space $(\Omega_j, \Sigma_j, \mu_j)$. Suppose that for each $j \in J$ either $\boldsymbol{L}_{\rho_j} \subseteq L_2(\mu)$ and $\boldsymbol{L}_{\rho_j}$ has Rademacher type 2 or $L_2(\mu) \subseteq \boldsymbol{L}_{\rho_j}$ and $\boldsymbol{L}_{\rho_j}$ has Rademacher cotype 2. Let Ψ be a semi-normalized complemented unconditional basic sequence in $\bigoplus_{j\in J} \boldsymbol{L}_{\rho_j}$. Then, either Ψ is equivalent to the unit vector system of ℓ_2, or there is $j \in J$ such that Ψ has a subsequence equivalent to a well-complemented basic sequence, say Φ, of $\boldsymbol{L}_{\rho_j}$ consisting of simple functions escaping to infinity. Moreover, there is a sequence Φ^* of good projecting functionals for Φ which consists of simple functions. If ρ_j is absolutely continuous for all $j \in J$, we can make Φ be congruent to Ψ, and make Φ^* escape to infinity.*

Proof Set $\Psi = (\boldsymbol{\psi}_n)_{n=1}^\infty$ and let $\Psi^* = (\boldsymbol{\psi}_n^*)_{n=1}^\infty$ be coordinate functionals for Ψ. We first address a particular case.
Case A. Suppose that $J = \{1, 2\}$, $\boldsymbol{L}_{\rho_2} \subseteq L_2(\mu) \subseteq \boldsymbol{L}_{\rho_1}$, $\boldsymbol{L}_{\rho_1}$ has Rademacher cotype 2, and $\boldsymbol{L}_{\rho_2}$ has Rademacher type 2. Let ρ_b and ρ_d be the function norms over $\mu_1 \sqcup \mu_2$ given by $\rho_b = (\rho_1, \|\cdot\|_{L_2(\mu_2)})$ and $\rho_d = (\|\cdot\|_{L_2(\mu_1)}, \rho_2^*)$. Note that $\boldsymbol{L}_\rho \subseteq \boldsymbol{L}_{\rho_b} \cap \boldsymbol{L}_{\rho_d}$ and that $\boldsymbol{L}_{\rho_b}$ and $\boldsymbol{L}_{\rho_d}$ have cotype 2. We consider three possible subcases.
Case A.1. Suppose that $\inf_n \|\boldsymbol{\psi}_n\|_{\rho_b} > 0$ and $\inf_n \|\boldsymbol{\psi}_n^*\|_{\rho_d} > 0$. Then, by Lemma 4.1(ii), Ψ and Ψ^* dominate the unit vector system of ℓ_2. Hence, Ψ is equivalent to the unit vector system of ℓ_2.
Case A.2. Suppose that $\inf_n \|\boldsymbol{\psi}_n\|_{\rho_b} = 0$. We infer from Lemma 3.5 that a subsequence of Ψ is congruent to a sequence $(\boldsymbol{\psi}_{2,n})_{n=1}^\infty$ in $\boldsymbol{L}_{\rho_2}$ that satisfies $\lim_n \|\boldsymbol{\psi}_{2,n}\|_2 = 0$. Then, the result follows from Corollary 4.8.
Case A.3. Suppose that $\inf_n \|\boldsymbol{\psi}_n\|_{\rho_d} = 0$. By Lemma 3.6, a subsequence of Ψ is congruent to a sequence $(\boldsymbol{\psi}_{1,n}, \boldsymbol{\psi}_{2,n})_{n=1}^\infty$ with projecting functionals $(\boldsymbol{\psi}_{1,n}^*)_{n=1}^\infty$ belonging to $\boldsymbol{L}_{\rho_1^*}$ and satisfying $\lim_n \|\boldsymbol{\psi}_{1,n}^*\|_2 = 0$. Since the mapping

$$(f, g) \mapsto P(f, g) := \sum_{n=1}^\infty \boldsymbol{\psi}_{1,n}^*(f)(\boldsymbol{\psi}_{1,n}, \boldsymbol{\psi}_{2,n})$$

is an endomorphism of $\boldsymbol{L}_{\rho_1} \oplus \boldsymbol{L}_{\rho_2}$, also is the mapping

$$(f, g) \mapsto Q(f, g) := \sum_{n=1}^\infty \boldsymbol{\psi}_{1,n}^*(f)\boldsymbol{\psi}_{1,n}.$$

The mappings P and Q witness that $(\boldsymbol{\psi}_{1,n}, \boldsymbol{\psi}_{2,n})_{n=1}^\infty$ and $(\boldsymbol{\psi}_{1,n})_{n=1}^\infty$ are equivalent. In turn, the mapping Q witnesses that $(\boldsymbol{\psi}_{1,n})_{n=1}^\infty$, regarded as a sequence in $\boldsymbol{L}_{\rho_1} \oplus \boldsymbol{L}_{\rho_2}$, is a complemented unconditional basic sequence with projecting functionals $(\boldsymbol{\psi}_{1,n}^*)_{n=1}^\infty$. By Corollary 3.3, $(\boldsymbol{\psi}_{1,n}, \boldsymbol{\psi}_{2,n})_{n=1}^\infty$ and $(\boldsymbol{\psi}_{1,n})_{n=1}^\infty$ are congruent. We conclude by applying Theorem 4.7.

To address the proof in the general case we consider the partition (J_1, J_2) of J defined by $j \in J_1$ if $L_2(\mu) \subseteq \boldsymbol{L}_{\rho_j}$ and $\boldsymbol{L}_{\rho_j}$ has Rademacher cotype 2, and $j \in J_2$ if $\boldsymbol{L}_{\rho_j} \subseteq L_2(\mu)$ and $\boldsymbol{L}_{\rho_j}$ has Rademacher type 2. If $J_2 = \emptyset$, we apply Theorem 4.7. If $J_1 = \emptyset$, we apply Corollary 4.8. If $J_1 \neq \emptyset$ and $J_2 \neq \emptyset$, we apply the already proved Case A. In any case, we conclude that, unless Ψ is equivalent to the unit vector system of ℓ_2, there is $i \in \{1, 2\}$ such that a subsequence of Ψ is equivalent to a well-complemented basic sequence Φ of $\oplus_{j \in J_i} \boldsymbol{L}_{\rho_j}$ which escapes to infinity. Besides, both Φ and a sequence Φ^* of good projecting functionals for Φ consist of simple functions; and, if $i = 2$ or ρ_j^* is absolutely continuous for all $j \in J_1$, Φ^* escapes to infinity and we obtain congruence. Since, by Lemma 5.1, Φ is congruent to a well-complemented basic sequence in $\boldsymbol{L}_{\rho_j}$ for some $j \in J_i$, we are done. □

Theorem 5.2 has an immediate consequence on the study of subsymmetric basic sequences.

Theorem 5.3 *Let J be a finite set and, for each $j \in J$, ρ_j be a function norm over a σ-finite measure space $(\Omega_j, \Sigma_j, \mu_j)$. Suppose that for each $j \in J$ either $\boldsymbol{L}_{\rho_j} \subseteq L_2(\mu)$ and $\boldsymbol{L}_{\rho_j}$ has Rademacher type 2 or $L_2(\mu) \subseteq \boldsymbol{L}_{\rho_j}$ and $\boldsymbol{L}_{\rho_j}$ has Rademacher cotype 2. Let Ψ be a complemented subsymmetric basic sequence in $\oplus_{j \in J} \boldsymbol{L}_{\rho_j}$. Then, either Ψ is equivalent to the unit vector system of ℓ_2, or there is $j \in J$ such that Ψ is equivalent to a well-complemented basic sequence, say Φ, of $\boldsymbol{L}_{\rho_j}$ consisting of simple functions escaping to infinity. Moreover, there is a sequence Φ^* of good projecting functionals for Φ which consists of simple functions. If ρ_j is absolutely continuous for all $j \in J$, we can make Φ^* escape to infinity.*

In some situations, we can classify the subsymmetric basic sequences of a direct sum of function spaces without assuming complementability.

Theorem 5.4 *Let $N \in \mathbb{N}$ and for each $j \in \{1, \dots, N\}$, ρ_j be a function norm over a σ-finite measure space $(\Omega_j, \Sigma_j, \mu_j)$. Suppose that there is an increasing $N-1$-tuple $(p_j)_{j=1}^{N-1}$ in $(2, \infty)$ such that*

- *$\boldsymbol{L}_{\rho_j} \subseteq L_2(\mu)$ and $\boldsymbol{L}_{\rho_j}$ has Rademacher type 2 for all $j = 1, \dots, N$,*
- *$\boldsymbol{L}_{\rho_1}$ is 2-convex,*
- *$\boldsymbol{L}_{\rho_N}$ has nontrival concavity, and*
- *$\boldsymbol{L}_{\rho_j}$ satisfies a lower p_j-estimate and $\boldsymbol{L}_{\rho_{j+1}}$ satisfies an upper p_j-estimate for all $j = 1, \dots N-1$.*

Let Ψ be a semi-normalized unconditional basic sequence in $\oplus_{j=1}^N \boldsymbol{L}_{\rho_j}$. Then, either Ψ is equivalent to the unit vector system of ℓ_2, or there is $j = 1, \dots, N$ such that Ψ has a subsequence equivalent to a disjointly supported basic sequence of $\boldsymbol{L}_{\rho_j}$ consisting of simple functions escaping to infinity.

Proof The function norm $\rho = (\rho_j)_{j=1}^N$ is 2 convex and has nontrivial concavity. So, $\boldsymbol{L}_\rho$ has Rademacher type 2. By Theorem 4.2, unless Ψ is equivalent to the unit vector system of ℓ_2, it has a subsequence congruent to a disjointly supported sequence, say $\Phi = (\boldsymbol{\phi}_n)_{n=1}^\infty$, consisting of simple functions escaping to infinity. Write

$$\boldsymbol{\phi}_n = (\boldsymbol{\phi}_{j,n})_{j=1}^N, \; n \in \mathbb{N}, \quad \text{and} \quad \Phi_{(j)} := (\boldsymbol{\phi}_{j,n})_{n=1}^\infty, \; j = 1, \dots, N.$$

If $j \in \{1, \dots, N\}$ is such that $\inf_n \left\|\boldsymbol{\phi}_{j,n}\right\|_{\rho_j} = 0$, then, by Lemma 3.5, passing to a further subsequence we can assume that $\left\|\boldsymbol{\phi}_{j,n}\right\|_{\rho_j} = 0$ for all $n \in \mathbb{N}$. Consequently, we can assume that there is $A \subseteq \{1, \dots, N\}$ such that

$$\inf_{j \in A} \inf_{n \in \mathbb{N}} \left\| \boldsymbol{\phi}_{j,n} \right\|_{\rho_j} > 0.$$

Of course, A is nonempty. If A is a singleton, we are done. Otherwise, set $k = \min A$ and $p = p_{k+1}$. The lower estimate for $\boldsymbol{L}_{\rho_k}$ gives that $\Phi_{(k)}$ dominates the unit vector system of ℓ_p. In turn, the upper estimate for $\boldsymbol{L}_{\rho_j}$, $j \in A \setminus \{k\}$, gives that the unit vector system of ℓ_p dominates $\Phi_{(j)}$ for all $j \in A \setminus \{k\}$. Consequently, Φ is equivalent to $\Phi_{(k)}$. □

We next apply our results to Lebesgue spaces. If μ is the Lebesgue measure over $[0, 1]$, we set $L_p = L_p(\mu)$.

Theorem 5.5 *Let $P \subseteq [1, \infty)$ be finite. Let Ψ be a semi-normalized complemented subsymmetric basic sequence in $\oplus_{p \in P} L_p$. Then, Ψ is equivalent to the unit vector system of ℓ_p for some $p \in P \cup \{2\}$.*

Proof Since L_p has type 2 if $2 \le p < \infty$ and cotype 2 if $1 \le p \le 2$, we can apply Theorem 5.3. Since any semi-normalized disjointly supported sequence of L_p is equivalent to the unit vector system of ℓ_p, we are done. □

Notice that Theorem 5.5 gives that, given P, $Q \subseteq [1, \infty)$ finite, $\oplus_{p \in P} L_p$ and $\oplus_{p \in Q} L_p$ are not isomorphic unless $P \cup \{2\} = Q \cup \{2\}$. Since L_2 is a complemented subspace of L_p if and only if $1 < p < \infty$, the converse holds with the following exception: L_1 and $L_1 \oplus L_2$ are not isomorphic.

Theorem 5.6 *Let $P \subseteq [2, \infty)$ be finite. Let Ψ be a semi-normalized subsymmetric basic sequence in $\oplus_{p \in P} L_p$. Then, Ψ is equivalent to the unit vector system of ℓ_p for some $p \in P \cup \{2\}$.*

Proof Taking into account that L_p is lattice p-convex and lattice p-concave, the result is a ready consequence of Theorem 5.4. □

6 Orlicz function spaces

A (convex) *Orlicz function* is a convex non-decreasing function

$$F : [0, \infty) \to [0, \infty]$$

with $F(0) = 0$ and $F(c) < \infty$ for some $c > 0$. If F takes the infinity value, we assume that it does so in an open interval. If $F(1) = 1$, we say that F is normalized.

Let (Ω, Σ, μ) be a σ-finite measure space. The *Orlicz space* over (Ω, Σ, μ) associated with F is the linear space $L_F(\mu)$ built from the modular

$$m_F : L_0^+(\mu) \to [0, \infty], \quad f \mapsto \int_\Omega F(f)\, d\mu.$$

This means that $L_F(\mu) = \boldsymbol{L}_{\rho_F}$, where ρ_F is the Luxemburg functional constructed from m_F. Namely,

$$\rho_F \colon L_0^+(\mu) \to [0, \infty], \quad f \mapsto \inf\{t > 0 \colon m_F(f/t) \le 1\}.$$

It is known [6, Chapter 4, Theorem 8.9] that ρ_F is a rearrangement invariant function norm, that is, $L_F(\mu)$ is a rearrangement invariant function space. Its associated function norm is given by $(\rho_F)^* = \rho_{F^*}$, where F^* is the complementary Orlicz function of F defined by

$$F^*(u) = \sup\{tu - F(t) \colon 0 < t < \infty\}.$$

Notice that $F^*(u) < \infty$ for all $u \in [0, \infty)$ if and only if $\lim_{t\to\infty} F(t)/t = \infty$, in which case the supremum that defines $F^*(u)$ is attained for every u. The other way around, F takes the infinity value if and only if $\lim_{t\to\infty} F^*(t)/t < \infty$. If both F and F^* are finite, then $(F^*)'$ it is the right-inverse of F'.

In the case when μ is the Lebesgue measure on a set Ω, we set $L_F(\mu) = L_F(\Omega)$. In turn, if μ is the counting measure on a countable set $\mathcal{N}$, we set $L_F(\mu) = \ell_F(\mathcal{N})$. If $\mathcal{N} = \mathbb{N}$, we put $\ell_F(\mathcal{N}) = \ell_F$. In the discrete case, we will also consider the Musielak-Orlicz's generalization of Orlicz spaces. Given a family $\boldsymbol{F} = (F_n)_{n\in\mathcal{N}}$ of normalized Orlicz functions, the *Musielak-Orlicz sequence space* $\ell_{\boldsymbol{F}}$ is the sequence space built from the modular

$$f = (a_n)_{n=1}^\infty \mapsto m_{\boldsymbol{F}}(f) := \sum_{n\in\mathcal{N}} F_n(a_n).$$

To relate Musielak-Orlicz sequence spaces with basic sequences in Orlicz spaces we define, given an Orlicz function F, a σ-finite measure space (Ω, Σ, μ) and $f \in L_0(\mu)$,

$$F_f \colon [0, \infty) \to [0, \infty], \quad t \mapsto m_F(t\,|f|) = \int_\Omega F(t\,|f|)\, d\mu.$$

Clearly, F_f is a nondecreasing convex function with $F_f(0) = 0$.

Let $H_F(\mu)$ consist of all functions $f \in L_0(\mu)$ such that $F_f(t) < \infty$ for all $t \in [0, \infty)$. It is known that $H_F(\mu)$ is the closed linear span in $L_F(\mu)$ of the integrable simple functions.

The *flows* of a finite Orlicz function F are defined for each $s \in (0, \infty)$ as

$$F_s(t) = \frac{F(st)}{F(s)}, \quad t \ge 0.$$

Notice that F_s is a normalized Orlicz function.

Lemma 6.1 *Let (Ω, Σ, μ) be a σ-finite measure space and F be a finite Orlicz function.*

(i) Let f be a norm-one function in H_F. Then, F_f is a normalized finite Orlicz function.

(ii) *Let f be an integrable simple funtion with $\|f\|_F = 1$. Then, F_f belongs to the convex hull of $\{F_s : s \in f(\Omega) \setminus \{0\}\}$.*
(iii) *Let $\Phi = (\boldsymbol{\phi}_n)_{n=1}^\infty$ be a normalized disjointly supported sequence in H_F. Then, Φ is isometrically equivalent to the unit vector system of the Musielak-Orlicz sequence space $\ell_{\boldsymbol{F}}$, where $\boldsymbol{F} = (F_{\boldsymbol{\phi}_n})_{n=1}^\infty$.*

Proof If $f \in H_F(\mu)$, $F_f : [0,\infty) \to [0,\infty)$ is a continuous function. We infer that $F_f(s) = 1$ provided $s := \|f\|_F < \infty$. This proves (i). Proving (iii) is routine checking. To prove (ii), we expand f as $\sum_{j\in J} a_j \chi_{A_j}$ with $(A_j)_{j\in J}$ pairwise disjoint and $a_j \neq 0$ for all $j \in J$. We have

$$F_f(t) = \sum_{j\in J} \left|A_j\right| F(a_j) F_{|a_j|}(t), \quad t \geq 0.$$

In particular, $1 = F_f(1) = \sum_{j\in J} \left|A_j\right| F(\left|a_j\right|)$. □

Let (Ω, Σ, μ) be a σ-finite measure space and F be a finite Orlicz function. Suppose that (Ω, Σ, μ) is purely atomic. If $(A_n)_{n\in\mathcal{N}}$ is a family of representatives of its atoms we put

$$\boldsymbol{F} = (F_{s_n})_{n\in\mathcal{N}}, \quad s_n = F^{-1}(\mu(A_n)^{-1}).$$

By Lemma 6.1, the mapping

$$(a_n)_{n\in\mathcal{N}} \mapsto \sum_{n\in\mathcal{N}} a_n s_n \chi_{A_n}$$

defines a lattice isometry from the Musielak-Orlicz sequence space $\ell_{\boldsymbol{F}}$ onto the Orlicz space $L_F(\mu)$. Oppositely, if μ is nonatomic and separable, then, by Proposition 2.9, $L_F(\mu)$ is lattice isomorphic to $L_F(I)$, where $I = [0, 1)$ if μ is finite and $I = [0,\infty)$ is μ is infinite. In this paper, we will study the Orlicz function space $L_F([0, 1))$, which we will simply call L_F. Notice that an Orlicz function F takes the infinity value if and only if $L_F = L_\infty$.

Given a finite Orlicz function F, the function norm associated with the Orlicz function space L_F is absolutely continuous if and only if F is finite and satisfies the Δ_2-condition near infinity, i.e.,

$$\sup_{t\geq 1} \frac{F(2t)}{F(t)} < \infty$$

(see [23, Chapter 2, Section a]). In turn, the function norm associated with the Orlicz sequence space ℓ_F is absolutely continuous if and only if F satisfies the Δ_2-condition near zero, i.e.,

$$\sup_{t\leq 1} \frac{F(2t)}{F(t)} < \infty$$

(see [22, Chapter 4]). These results are consistent with the facts that, given Orlicz functions F and G, $L_F = L_G$ (up to an equivalent norm) if and only if F and G are equivalent near infinity, while $\ell_F = \ell_G$ if and only if F and G are equivalent near zero. Musielak [25, Chapter 8] extended this characterization of Orlicz functions that define the same Orlicz spaces. As sequence spaces are concerned, he proved the following.

Theorem 6.2 ([25, Theorem 8.11]) *Let $\boldsymbol{F} = (F_n)_{n=1}^\infty$ and $\boldsymbol{G} = (G_n)_{n=1}^\infty$ be sequences of normalized Orlicz functions. Then $\ell_{\boldsymbol{F}} \subseteq \ell_{\boldsymbol{G}}$ if and only if there exist a positive sequence $(a_n)_{n=1}^\infty$ in ℓ_1, some $\delta > 0$, and positive constants b and C such that*

$$F_n(t) < \delta \quad\Longrightarrow\quad G_n(t) \le C F_n(bt) + a_n.$$

The convexity-type and concavity-type of Orlicz spaces is known.

Proposition 6.3 (see [15, Section 7]) *Given a finite Orlicz function F and $1 < p < \infty$ the following are equivalent.*

- *(i) L_F is lattice p-convex (resp., p-concave).*
- *(ii) The function $t \mapsto F(t)t^{-p}$ is essentially increasing (resp., decreasing) on $[1, \infty)$.*
- *(iii) F is equivalent near infinity to an Orlicz function G such that the function $t \mapsto G(t^{1/p})$ is convex (resp., concave) on $(0, \infty)$.*

Moreover, L_F has some nontrivial concavity if and only if F is doubling near infinity.

Lindberg [17] and Lindenstrauss and Tzafriri [19–21] studied the basic sequence structure of a given Orlicz sequence space ℓ_F in terms of subsets of $\mathcal{C}([0, 1/2])$ constructed from the flows F_s of the Orlicz function F for s near zero. To study Orlicz function spaces we must consider flows F_s for s near infinity. Suppose that F is normalized. Given $b \in [0, \infty)$, let $E_{F,b}^\infty$ be the topological closure of $\{F_s : s > b\}$ in $(\mathcal{C}([0, 1/2]), \|\cdot\|_\infty)$, and $C_{F,b}^\infty$ be the topological closure of the convex hull of $\{F_s : s > b\}$. Following [21] we define

$$E_F^\infty = \bigcap_{b\ge 0} E_{F,b}^\infty, \quad C_F^\infty = \bigcap_{b\ge 0} C_{F,b}^\infty.$$

By definition,

$$E_F^\infty \subseteq C_F^\infty \subseteq C_{F,0}^\infty.$$

Each function $G \in C_{F,0}^\infty$ inherits from the flows of F the following properties:

- $G(0) = 0$.
- G is non-decreasing.
- G is convex.

- For all $0 \le t < u \le 1/2$,

$$\frac{G(u) - G(t)}{u - t} \le 2(G(1) - G(1/2)).$$

  In particular, $G(1/2) \le 1/2F(1)$ and G is $2F(1)$-Lipschitz.

Consequently, the extension of G to $[0, \infty)$ that is linear on $[1/2, \infty)$ and satisfies $G(1) = F(1)$ is an Orlicz function. So, we can define G^* and ℓ_G for $G \in C^\infty_{F,0}$. Notice that, a priori, G could be null near zero, in which case $\ell_G = \ell_\infty$ and $G^*(t) \approx t$ for t near zero.

The set $C^\infty_{F,0}$ is equicontinuous and uniformly bounded, hence compact. Therefore, E^∞_F and C^∞_F are compact subsets of $\mathcal{C}([0, 1/2])$, and E^∞_F is nonempty (c.f.[22, Lemma 4.a.6]).

Once written the necessary background on Orlicz spaces down, we state the main results of this section. Since the foundations they rely on are closely related, we carry out a unified proof.

Theorem 6.4 *Let F be a finite Orlicz function. Suppose that F is doubling near infinity and that the mapping $t \mapsto F(t)t^{-2}$ is essentially increasing on $[1, \infty)$. Then, any subsymmetric basic sequence Ψ in L_F is equivalent to the unit vector system of ℓ_2 or ℓ_G for some $G \in C^\infty_F$.*

Theorem 6.5 *Let F be a finite Orlicz function. Suppose that the mapping $t \mapsto F(t)t^{-2}$ is essentially decreasing on $[1, \infty)$. Then, any complemented subsymmetric basic sequence Ψ in L_F is equivalent to the unit vector system of ℓ_2 or ℓ_G for some $G \in C^\infty_F$.*

Theorem 6.6 *Let F be a finite Orlicz function. Suppose that either*

- *F is doubling near infinity and the mapping $t \mapsto F(t)t^{-2}$ is essentially increasing on $[1, \infty)$, or*
- *F^* is doubling near infinity and the mapping $t \mapsto F(t)t^{-2}$ is essentially decreasing on $[1, \infty)$.*

Then, any complemented subsymmetric basic sequence Ψ in L_F is equivalent to the unit vector system of ℓ_2 or ℓ_G for some $G \in C^\infty_F$ such that G^* is equivalent to a function in $C^\infty_{F^*}$.

Proof *(Proof of Theorems 6.4, 6.5 and 6.6)* Suppose that Ψ is not equivalent to the unit vector system of ℓ_2. Then, combining Proposition 6.3, Theorem 2.3 and Corollary 4.3 or Theorem 5.3, gives that Ψ is equivalent to a normalized disjointly supported sequence $\Phi = (\boldsymbol{\phi}_n)_{n=1}^\infty$ in L_F consisting of simple functions escaping to ∞. Moreover, under the assumptions of Theorem 6.6, the dual basis Ψ^* or Ψ is equivalent to a normalized disjointly supported sequence $\Phi^* = (\boldsymbol{\phi}_n^*)_{n=1}^\infty$ in L_{F^*} consisting of simple functions escaping to ∞. By Lemma 6.1(iii), there is a sequence $(b_n)_{n=1}^\infty$ in $(0, \infty)$ with $\lim_n b_n = \infty$ such that $F_{\boldsymbol{\phi}_n} \in C^\infty_{F,b_n}$ and $F^*_{\boldsymbol{\phi}_n^*} \in C^\infty_{F^*,b_n}$ for all $n \in \mathbb{N}$.

Pick an arbitrary sequence $(a_n)_{n=1}^\infty$ in $(0, \infty)$ with $\sum_{n=1}^\infty a_n < \infty$. Since $C^\infty_{F,0}$ is compact we can assume, passing to a subsequence, that there is $G \in C^\infty_{F,0}$ such that

$$\left\| F_{\boldsymbol{\phi}_n} - G \right\|_\infty \le a_n, \quad n \in \mathbb{N}.$$

In the case we are addressing proving Theorem 6.6 we can also assume, passing to a further subsequence, that there is $H \in C^{\infty}_{F^*,0}$ such that

$$\left\| F^*_{\phi^*_n} - H \right\|_\infty \le a_n, \quad n \in \mathbb{N}.$$

Combining Lemma 6.1(ii) with Theorem 6.2 gives that Φ is equivalent to the unit vector system of ℓ_G, and Φ^* is equivalent to the unit vector system of ℓ_H. Since Φ^* is also equivalent to the unit vector system of ℓ_{G^*}, $\ell_H = \ell_{G^*}$ up to an equivalent norm. Therefore, H is equivalent to G^*. Since $G \in C^{\infty}_F$ and $H \in C^{\infty}_{F^*}$, we are done. □

Concerning the accuracy of Theorems 6.4, 6.5 and 6.6, we point out that the Rademacher functions, regarded as a sequence in a rearrangement invariant function space $\boldsymbol{L}_\rho$, are equivalent to the unit vector of ℓ_2 if and only $L_{M_2} \subseteq \boldsymbol{L}_\rho$, where, for each $a > 0$, M_a is the normalized Orlicz function given by

$$M_a(t) = \frac{e^{t^a} - 1}{e - 1}, \quad t \ge 0.$$

Moreover, the Rademacher functions are complemented in $\boldsymbol{L}_\rho$ if and only if $L_{M_2} \subseteq \boldsymbol{L}_\rho \subseteq L_{M_2^*}$, where $M_2^*(t) \approx t\,(\log(t))^{1/2}$ for t near infinity (see [28, 29] and also [23, p. 134]). We also point out that the assumption of complementability in Theorems 6.5 and 6.6 is necessary. In fact, if $1 < s < 2$ and

$$\int_0^1 F(x^{-1/s})\,dx < \infty,$$

then L_F has a basic sequence equivalent to the unit vector system of ℓ_s (see [15, Proposition 8.9] and [23, Chapert 8]). As for sequences other than the canonical basis of the Hilbert space, we next prove that if $E^{\infty}_F = C^{\infty}_F$, then these theorems are sharp.

Lemma 6.7 *Let $(A_n)_{n=1}^{\infty}$ be a pairwise disjoint sequence consisting of Borel subsets of I, where I is either* $[0, 1]$ *or* $[0, \infty)$. *Let F be a normalized finite Orlicz function. Set*

$$s_n = F^{-1}\left(\frac{1}{|A_n|}\right), \quad n \in \mathbb{N}.$$

Then, $\Phi = (s_n \chi_{A_n})_{n=1}^{\infty}$ is a well-complemented basic sequence with good projecting functionals $(\chi_{A_n} F(s_n)/s_n)_{n=1}^{\infty}$. Moreover, Φ is isometrically equivalent to the unit vector system of $\ell_{\boldsymbol{F}}$, where $\boldsymbol{F} = (F_{s_n})_{n=1}^{\infty}$.

Proof Since $m_F(s_n \chi_{A_n}) = 1$ for all $n \in \mathbb{N}$, Φ is normalized. Then, by Lemma 6.1(ii), $F_{s_n \chi_{A_n}} = F_{s_n}$ for all $n \in \mathbb{N}$. We close the proof by combining Theorem 2.8 with Lemma 6.1(iii). □

For completeness, we write down a result by Lindenstrauss and Tzafriri that can be proved using our techniques.

Theorem 6.8 (c.f. [21, Proposition 4]) *Let F be a finite Orlicz function with $\lim_{t\to\infty} F(t)/t = \infty$. For any $G \in E_F^\infty$ there is a well-complemented basic sequence Ψ in L_F that is equivalent to the unit vector system of ℓ_G. Moreover, there are good projecting functionals Φ^* for Φ such that both Φ and Φ^* consist of simple functions escaping to infinity.*

Proof Pick an arbitrary positive sequence $(a_n)_{n=1}^\infty \in \ell_1$. Let $(s_n)_{n=1}^\infty$ in $(0,\infty)$ be such that $\sum_{n=1}^\infty 1/F(s_n) \le 1$ and

$$\left\| G - F_{s_n} \right\|_\infty \le a_n, \quad n \in \mathbb{N}.$$

Let $(A_n)_{n=1}^\infty$ be a pairwise disjoint sequence consisting Borel subsets of $[0,1]$ with $|A_n| = 1/F(s_n)$ for all $n \in \mathbb{N}$. Combining Lemma 6.7 and Theorem 6.2 gives the desired result. □

It is known the the converse of Theorem 6.8 does not hold. In fact, for each $1 < p < \infty$ there is an Orlicz function F such that the potencial function $x \mapsto x^p$ is not in E_F^∞ and still ℓ_p is a complemented subspace of L_F (see [13, Theorem 2.1]).

We close our study of the accuracy of Theorems 6.4, 6.5 and 6.6 with a criterion to ensure that $E_F^\infty = C_F^\infty$.

Proposition 6.9 *(c.f. [12, Proposition 6]) Let F be an Orlicz function. Suppose that there is $0 < c < 1$ such that for all $t \in [0,c]$ there exists*

$$G(t) := \lim_{s\to\infty} F_s(t) = \lim_{s\to\infty} \frac{F(st)}{F(s)} < \infty. \tag{6.1}$$

Then, there is $p \in [1,\infty)$ such that $G(t) = t^p$ for all $t \in [0,c]$, and $E_F^\infty = C_F^\infty = \{t \mapsto t^p\}$. In particular, the above holds if there exists

$$q := \lim_{s\to\infty} \frac{sF'(s)}{F(s)} < \infty, \tag{6.2}$$

in which case $q = p$.

Proof If (6.2) holds, then for any $\epsilon > 0$ and $0 < t < 1$ there is $t_0 \in (1,\infty)$ such that

$$t^{q-\epsilon} \le \frac{F(st)}{F(s)} \le t^{q+\epsilon}, \quad t \ge t_0,$$

(see [12, Proof of Proposition 6]). Hence, (6.1) holds for any $0 < t < 1$, and $G(t) = t^q$. To prove that the wished-for exponent p exists in general, we show that G is multiplicative. Given $t_1, t_2 \in (0,c]$ we have

$$G(t_1)G(t_2) = \lim_{s\to\infty} \frac{F(st_1)}{F(s)} \lim_{s\to\infty} \frac{F(st_1t_2)}{F(st_1)} = G(t_1t_2).$$

Since E_F^∞ is nonempty, proving that if $H \in C_F^\infty$ then $G(t) = H(t)$ for all $t \in [0, c]$ will conclude the proof. Fix $t \in [0, c]$ and $\epsilon > 0$. There is $b \in (0, \infty)$ such that

$$|F_s(t) - G(t)| < \frac{\epsilon}{2}, \quad s > b.$$

Now, since $H \in C_{F,b}^\infty$ there are finite families $(\lambda_j)_{j\in J}$ in $[0, \infty)$ and $(s_j)_{j\in J}$ in (b, ∞) such that

$$\sum_{j\in J} \lambda_j = 1 \quad \text{and} \quad \sup_{0\le s\le 1/2} \left| H(s) - \sum_{j\in J} \lambda_j F_{s_j}(s) \right| < \frac{\epsilon}{2}.$$

Consequently, $|H(t) - G(t)| < \epsilon$. □

Example 1 Let $1 \le p < \infty$ and $a \in \mathbb{R} \setminus \{0\}$. If $p = 1$, we assume that $a > 0$. It is easily checked that there are constants $c = c(p, a)$ and $m = m(a, p)$ such that the function

$$F_{p,a}(t) = \begin{cases} t^p (\log t)^a & \text{if } t \ge c, \\ mt & \text{if } 0 < t < c, \end{cases}$$

is an Orlicz function that satisfies the Δ_2-condition near infinity. Notice that

$$D_{p,a} := F'_{p,a} \approx t^{p-1} (\log t)^a$$

near infinity. If $p > 1$ and we set

$$q = p' = \frac{p}{p-1}, \quad b = -a(p-1),$$

then $D_{p,a}(D_{q,b}(t)) \approx 1$ near infinity. Consequently, $L_{F_{p,a}}$ and $L_{F_{q,b}}$ are dual spaces. As the case $p = 1$ is concerned, since $(D_{1,a}(M'_{1/a}(t)) \approx 1$ near infinity, the dual space of $L_{F_{1,a}} = L \log^a L$ is $L_{M_{1/a}}$.

If $a > 0$, then the function $t \mapsto t^{-p} F_{p,a}(t)$ is essentially increasing near infinity, while the function $t \mapsto t^{-q} F_{p,a}(t)$ is essentially decreasing near infinity for all $q > p$. Consequently, $L_{F_{p,a}}$ is lattice p-convex and lattice q-concave for all $p < q$. In turn, if $a < 0$, then the function $t \mapsto t^{-p} F_{p,a}(t)$ is essentially decreasing near infinity, and the function $t \mapsto t^{-q} F_{p,a}(t)$ is essentially increasing near infinity for all $1 \le q < p$. Therefore, $L_{F_{p,a}}$ is lattice p-concave and lattice q-convex for all $1 \le q < p$. Since

$$\lim_{s\to\infty} \frac{F_{p,a}(st)}{F_{p,a}(s)} = t^p, \quad t \ge 0,$$

then $E_F^\infty = C_F^\infty = \{t \mapsto t^p\}$. We obtain the following information about the basic sequence structure of $L_{F_{p,a}}$.

- If $p > 2$, then $L_{F_{p,a}}$ has, up to equivalence, two different subsymmetric basic sequences. Namely, the unit vector systems of ℓ_2 and ℓ_p.
- If $p = 2$ and $a > 0$, then $L_{F_{p,a}}$ has, up to equivalence, a unique subsymmetric basic sequence. Namely, the unit vector system of ℓ_2.
- If $p = 2$ and $a < 0$, then $L_{F_{p,a}}$ has, up to equivalence, a unique complemented subsymmetric basic sequence. Namely, the unit vector system of ℓ_2. We do not know whether $L_{F_{p,a}}$ has subsymmetric basic sequences other than the unit vector system of ℓ_2.
- If $1 < p < 2$ or $p = 1$ and $a \geq 1/2$, then $L_{F_{p,a}}$ has, up to equivalence, two different complemented subsymmetric basic sequences. Namely, the unit vector systems of ℓ_2 and ℓ_p. Moreover, ℓ_s, $p < s < 2$, is isomorphic to a (non-complemented) subspace of $L_{F_{p,a}}$. If $p = 1$ and $0 < a < 1/2$ these results about the subsymmetric basic sequence structure of $L_{F_{p,a}}$ still hold, with the exception that it seems to be unknown whether ℓ_2 is isomorphic to a complemented subspace $L_{F_{1,a}}$, $0 < a < 1/2$.

We conclude with an application of Example 1 to the isomorphic theory of Banach spaces. It is known that, given $1 \leq p < \infty$, all separable L_p-spaces over non purely atomic measure spaces are isomorphic (see, e.g., [4, Chapter 6]). As Orlicz spaces are concerned, the situation is quite different. Since Orlicz spaces on $[0, \infty)$ depend on the behaviour of the associated Orlicz functions near zero and infinity, there are Orlicz functions F such that $L_F([0, \infty))$ is not isomorphic to L_F (see [14, 26]). We show that even if $1/F(t) \approx F(1/t)$ for $t \in (0, \infty)$ these spaces may not be isomorphic.

Corollary 6.10 *Let $1 < p < \infty$ and $a \in \mathbb{R}^*$. Let F be an Orlicz function such that*

$$\frac{1}{F(1/t)} \approx F(t) \approx F_{p,a}(t), \quad 1 \leq t < \infty.$$

Then, the Banach spaces L_F and $L_F([0, \infty))$ are not isomorphic.

Proof In light of Example 1, it suffices to show that ℓ_F is a complemented subspace of $L_F([0, \infty))$ for every normalized Orlicz function F. To that end, we apply Lemma 6.7 with $A_n = [n-1, n)$ for all $n \in \mathbb{N}$. □

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