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ODEFTC: Optimal Distributed Estimation based on Fixed-Time Consensus ☆

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ABSTRACT

Distributed state estimation has been a significant research topic in recent years due to its applications for multi-robot and large-scale systems. Several approaches have been proposed in the context of continuous-time systems with stochastic noise, with limitations regarding observability, assumptions on the noise bounds, or requirements to pre-compute auxiliary global information offline. Moreover, many of these approaches are suboptimal with respect to a centralized implementation, and optimal proposals only apply to time-invariant systems. The present work proposes the ODEFTC algorithm for distributed state estimation based on fixed-time consensus. The proposal computes state estimates and corresponding covariance matrices online, making it suitable for time-variant systems. We verify the stability of the proposal through formal analysis, and we show that the optimal centralized solution, given by the Kalman-Bucy filter, can be recovered asymptotically. Additionally, we provide numerical results and an in-depth statistical and numerical discussion to show the advantages of our proposal against other approaches in the literature.

1. Introduction

1.1. Motivation

Distributed problems are currently a relevant research topic due to their many potential applications, such as multi-robot cooperation or control of large-scale networked systems [1]. In particular, for state estimation, the distributed case consists of a network of sensing nodes that collectively observe a system using their local measurement information and communication with neighboring nodes.

There are several reasons why the distributed approach can be beneficial for state estimation. The redundancy of sensors can decrease the uncertainty of the estimates while also improving the robustness of the ensemble in the case of single-point failure [2]. The benefits of redundancy can be exploited to deal with missed detections or false positives when measurements from different sensors are available [3]. In addition, sharing information ensures that each node can obtain an estimate of the full state of the plant, even if it is not completely observable using its local measurement alone. Thus, several works in the literature address this problem to provide distributed implementations of popular filters.

In the context of systems affected by stochastic noise, many proposals in the literature address the discrete-time case to develop distributed implementations of the Kalman filter [4]. See, for example, the review in [5], which summarizes different approaches and evaluates their optimality in recovering the optimal centralized solution. In particular, consensus-based solutions for discrete-time filtering typically rely on the application of static average consensus protocols to fuse the information available to the nodes at a given sampling instant, requiring several exchanges of information per sample to reach agreement, recovering optimal results as the number of iterations per sample tends to infinity. Additionally, other issues have been studied in the discrete-time context, e.g., cyber-attacks [6], model uncertainties [7], missing data [8,9], transmission delays [10] or communication constraints [11,12].

Several approaches have been proposed for continuous-time systems affected by stochastic noise to solve the problem of distributed state estimation with some limitations. In [13], a distributed implementation of the Kalman-Bucy filter is provided under a rather restrictive assumption of observability of the system from each node in the network. Another approach is given in [14], with the restriction of having full state availability, i.e., the measurement matrix of the nodes being equal to the identity. Moreover, no proof of the stability of this filter is provided. These restrictive observability assumptions are relaxed in the following works to collective observability of the network.

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The proposal in [15], which builds on the ideas from [16], provides a filter with proven stability in the mean of the estimation error. However, it assumes convergence of the estimated covariance matrices to the asymptotic value reached in the centralized implementation of the filter, which is difficult to verify in practice. A proposal to extend the filter in [14] to work for measurement matrices different from the identity is given in [17]. Still, the formal proofs of stability and optimality are shown to contain technical errors in [18], where it is shown that consensus may not always be achieved with this approach. In [2], a distributed Kalman-Bucy filter using consensus of the measured signals is proposed but requires the assumption of null or bounded measurement noise rather than Gaussian noise. Moreover, convergence to the optimal centralized solution is only achieved in the absence of noise. A similar approach is also used in [19] under event-triggered communication, with similar limitations.

In [20], an approximated distributed Kalman-Bucy filter is proposed, which exploits the asymptotic form of the filter. This work guarantees the optimality and stability of the distributed filter with respect to the centralized implementation and does not require additional assumptions on the noise present in the system. However, it requires pre-computing the asymptotic covariance matrix of the estimation error via a separate diffusion protocol. This idea was also similarly used in [21] for the case with correlated measurement noise. However, note that this approach is limited to linear time-invariant (LTI) systems, given that there is no constant asymptotic covariance matrix in time-variant systems. In addition, most of the mentioned works also address LTI systems as well, and the ones that consider linear time-variant (LTV) systems [2,13] do not achieve optimal results in the presence of stochastic noise. Thus, designing an optimal distributed state estimator for continuous-time stochastic systems is still an open problem.

1.2. Innovations

Note that the continuous-time formulation of the problem lends itself to the application of *dynamic* consensus algorithms, which allow the tracking of time-varying signals, in contrast to the static consensus approach usually taken in discrete-time setups.

In recent years, fixed-time consensus algorithms have been developed, that achieve exact agreement after a fixed amount of time, which does not depend on the initial conditions. Several proposals exist in the literature, considering different control objectives and applications, such as formation control of robots or smart grids [22]. Moreover, some fixed-time consensus protocols, such as [23], guarantee a predefined convergence time according to the choice of parameters in the consensus algorithm.

Hence, in this work, we leverage a fixed-time dynamic consensus algorithm as a tool to design our distributed state estimator. To the best of our knowledge, fixed-time consensus algorithms have not yet been exploited for this purpose.

1.3. Contributions

In this work, we propose the ODEFTC algorithm, a distributed estimator for continuous-time systems affected by stochastic noise under a general assumption of collective observability. Our proposal, based on a fixed-time dynamic consensus algorithm, computes the state estimates and the estimated covariance matrix online without requiring the pre-computation of any quantities. Thus, it is suitable for LTV systems. We show through formal analysis that our proposal can recover the performance of the centralized optimal Kalman-Bucy filter [24] asymptotically. Henceforth, it achieves similar values between the true covariance of the estimation error and the estimated covariance computed by each network node. Moreover, we include simulation experiments to validate our approach and to show the improvements made with respect to other works in the literature.

1.4. Sections

The paper is organized as follows. The preliminaries are given in Section 2, which contains the problem statement, a summary of the optimal centralized solution and auxiliary results regarding fixed-time consensus. Section 3 presents our ODEFTC algorithm and provides a convergence analysis. A discussion is included in Section 4 to compare our proposal with related works, as well as simulation experiments to validate our algorithm and its performance compared to other approaches in the literature. Finally, Section 5 summarizes the conclusions for this work.

1.5. Notation

Let $\mathbf{I}_n \in \mathbb{R}^{n\times n}$ be the identity matrix and $\mathbb{1}$ the vector of ones of appropriate dimensions. Let $\mathrm{sgn}(x) = 1$ if $x > 0, \mathrm{sgn}(x) = -1$ if $x < 0$ and $\mathrm{sgn}(0) = 0$. The trace of a matrix is $\mathrm{tr}(\bullet)$, and $\mathrm{std}(\bullet)$ represents standard deviation. Given a set of matrices $\mathbf{P}_1, \ldots, \mathbf{P}_N$, we denote with $\mathrm{diag}_{i=1}^N(\mathbf{P}_i)$ the block diagonal matrix with each $\mathbf{P}_i$ as block components. Let $\|\bullet\|$ represent the standard Euclidean norm for vector input and its induced matrix norm for matrix input, and let $\|\bullet\|_\mathrm{F}$ denote the Frobenius norm. The expectation operator is denoted as $\mathbb{E}\{\bullet\}$. The covariance is denoted by $\mathrm{cov}\{\bullet\}$. For a matrix M_i computed at node i, we use the notation m_i to refer to an arbitrary element in the proof of Lemma 2.

2. Preliminaries

2.1. Problem statement

Consider a linear system described by a state vector $\mathbf{x}(t) \in \mathbb{R}^n$ satisfying:

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{w}(t), \quad t \geq 0 \tag{1}$$

where $\mathbf{A}(t) \in \mathbb{R}^{n\times n}$, $\mathbf{B}(t) \in \mathbb{R}^{n\times n_\mathbf{w}}$ and $\mathbf{w}(t) \in \mathbb{R}^{n_\mathbf{w}}$ denotes an unknown input, typically accounting for any non-modeled dynamics or disturbances. The unknown input $\mathbf{w}(t)$ is usually modeled as an $n_\mathbf{w}$-dimensional white Gaussian noise process with known positive definite covariance matrix $\mathbf{W}(t) \in \mathbb{R}^{n_\mathbf{w}\times n_\mathbf{w}}$ [2], requiring to understand (1) as a Stochastic Differential Equation (SDE). In this case, $\mathbf{x}(t)$ follows a Gaussian distribution, with the initial condition $\mathbf{x}(0)$ having a known mean $\mathbf{x}_0$ and covariance matrix $\mathbf{P}_0$.

The system is collectively observed by N sensors comprising a communication network, with its topology described by an undirected connected graph $\mathcal{G}$. The node set composed by the sensors is denoted by $\mathcal{V} = \{1, \ldots, N\}$ for the sensors and $\mathcal{E} \subseteq \mathcal{V}\times\mathcal{V}$ is the edge set representing the communication links between neighboring nodes. The adjacency matrix $\mathbf{A}_\mathcal{G} = [a_{ij}] \in \{0, 1\}^{N\times N}$ has elements $a_{ij} = 1$ if $(i, j) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. The Laplacian matrix is denoted by $\mathbf{Q}_\mathcal{G}$. We denote the set of neighbors of node i by $\mathcal{N}_i = \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$.

Each sensor i in the network produces a local measurement

$$\mathbf{y}_i(t) = \mathbf{C}_i(t)\mathbf{x}(t) + \mathbf{v}_i(t), \forall t \geq 0 \tag{2}$$

where $\mathbf{C}_i(t) \in \mathbb{R}^{n_\mathbf{y}\times n}$ and $\mathbf{v}_i(t)$ is an $n_\mathbf{y}$-dimensional white Gaussian noise process with known positive definite covariance matrix $\mathbf{R}_i(t) \in \mathbb{R}^{n_\mathbf{y}\times n_\mathbf{y}}$.

Assumption 1. The pair $(\mathbf{A}(t), \mathbf{C}(t))$ is observable $\forall t \geq 0$, with $\mathbf{C}(t) := [\mathbf{C}_1(t)^\top, \ldots, \mathbf{C}_N(t)^\top]^\top$.

Assumption 2. There exist constants $0 \leq a, b, c, w_1 \leq w_2, r_1 \leq r_2$ such that $\|\mathbf{A}(t)\| \leq a$, $\|\mathbf{B}(t)\| \leq b$, $\|\mathbf{C}(t)\| \leq c$, $w_1 \leq \|\mathbf{W}(t)\| \leq w_2$, $r_1 \leq \|\mathbf{R}(t)\| \leq r_2$ for all $t \geq 0$.

Remark 1. Note that Assumption 1 requires the system to be observable using the collective measurements obtained by the network, but it does not pose the requirement of the system being observable from each sensor node alone. In addition, Assumption 2 is commonly required for standard Kalman filtering.

The goal for the network nodes is to collectively estimate the system's state, using their own measurements and communication with neighboring nodes.

2.2. Optimal centralized solution: Kalman-Bucy filter

The Kalman-Bucy filter obtains state estimates $\hat{\mathbf{x}}(t)$ for the state $\mathbf{x}(t)$, as well as a covariance matrix $\mathbf{P}(t) = \mathbb{E}\{(\mathbf{x}(t)-\hat{\mathbf{x}}(t))(\mathbf{x}(t)-\hat{\mathbf{x}}(t))^\top\}$ for the estimation. It is known to be the optimal filter to obtain state estimates for continuous-time systems affected by stochastic noise, in the sense that it minimizes $\mathbb{E}\{(\mathbf{x}(t)-\hat{\mathbf{x}}(t))^\top(\mathbf{x}(t)-\hat{\mathbf{x}}(t))\} = \mathrm{tr}(\mathbf{P}(t))$. Considering $\mathbf{y}(t) = [\mathbf{y}_1(t)^\top, \dots, \mathbf{y}_N(t)^\top]^\top$, $\mathbf{C}(t)$ defined in Assumption 1, and $\mathbf{R}(t) = \mathrm{diag}_{i=1}^N(\mathbf{R}_i(t))$, the equations for the centralized filter are given by [24]:

$$\mathbf{K}(t) = \mathbf{P}(t)\mathbf{C}(t)^\top\mathbf{R}(t)^{-1} \tag{3a}$$

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}(t)\hat{\mathbf{x}}(t) + \mathbf{K}(t)\left(\mathbf{y}(t) - \mathbf{C}(t)\hat{\mathbf{x}}(t)\right) \tag{3b}$$

$$\dot{\mathbf{P}}(t) = \mathbf{A}(t)\mathbf{P}(t) + \mathbf{P}(t)\mathbf{A}(t)^\top + \mathbf{B}(t)\mathbf{W}(t)\mathbf{B}(t)^\top - \mathbf{P}(t)\mathbf{C}(t)^\top\mathbf{R}(t)^{-1}\mathbf{C}(t)\mathbf{P}(t) \tag{3c}$$

Moreover, for the time-invariant case, $\mathbf{P}(t)$ is known to converge to an asymptotic solution $\mathbf{P}_\infty$, given by the following Riccati equation:

$$\mathbf{0} = \mathbf{A}\mathbf{P}_\infty + \mathbf{P}_\infty\mathbf{A}^\top + \mathbf{B}\mathbf{W}\mathbf{B}^\top - \mathbf{P}_\infty\mathbf{C}^\top\mathbf{R}^{-1}\mathbf{C}\mathbf{P}_\infty \tag{4}$$

which provides the asymptotic form of the filter with gain $\mathbf{K}_\infty = \mathbf{P}_\infty\mathbf{C}^\top\mathbf{R}^{-1}$. Note that, in order to solve (3c) or (4), knowledge of the quantity

$$\bar{\mathbf{Z}}(t) = \mathbf{C}(t)^\top\mathbf{R}(t)^{-1}\mathbf{C}(t) = \sum_{i=1}^N \mathbf{C}_i(t)^\top\mathbf{R}_i(t)^{-1}\mathbf{C}_i(t) \tag{5}$$

is required, which depends on the matrices $\mathbf{C}_i(t)$, $\mathbf{R}_i(t)$ of the whole network, making the problem of distributed estimation challenging.

2.3. Fixed-time stability for consensus

In this work, we address the problem of consensus-based distributed estimation using fundamentals of fixed-time consensus. Here, we summarize some useful definitions and results that we will use in the following. Consider a dynamical system

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)) \tag{6}$$

where $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear function with initial conditions $\mathbf{x}_0 = \mathbf{x}(0) \in \mathbb{R}^n$ and the origin being an equilibrium point, i.e. $\mathbf{f}(\mathbf{0}) = \mathbf{0}$. Then, consider the following definitions pertaining fixed-time stability:

Definition 1 (*Global Finite-Time Stability [25]*). The origin of system (6) is called globally finite-time stable if it is globally asymptotically stable and any solution $\mathbf{x}(t, \mathbf{x}_0)$ of (6) reaches the equilibrium point at a finite-time moment. This is, $\mathbf{x}(t, \mathbf{x}_0) = \mathbf{0}, \forall t \geq T(\mathbf{x}_0)$, where $T : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ denotes the settling time function.

Definition 2 (*Fixed-Time Stability [26]*). The origin of system (6) is fixed-time stable if it is globally finite-time stable and has a bounded settling time function. This is, there exists $T_{\max} > 0$ such that $T(\mathbf{x}_0) \leq T_{\max}, \forall \mathbf{x}_0 \in \mathbb{R}^n$.

From Definitions 1 and 2, note that fixed-time stability ensures that the system under consideration converges to its equilibrium point in a time no larger than a fixed time $T_{\max}$, regardless of the initial conditions for the system.

Table 1
Design parameters for the ODEFTC algorithm.

Parameter	Description	Value
κ	Gain for consensus on $\hat{\mathbf{x}}_i(t)$ in (9b)	> 0
α	Gain for consensus on $\hat{\mathbf{Z}}_i(t)$ in (9e)	> 0
γ	Exponent for $\phi(\bullet, \xi, \gamma)$ in (8)–(9e)	$\in (0, 1)$
ξ	Discontinuous term gain in $\phi(\bullet, \xi, \gamma)$ in (8)–(9e)	As in Lemmas 1-2

Based on these definitions, a fixed-time consensus algorithm is a distributed algorithm that reaches agreement on a quantity of interest by all nodes of the system, in a time no larger than $T_{\max}$ for any initial conditions, by using only communication with neighboring nodes. Equivalently, the dynamics of the disagreement in the consensus protocol are fixed-time stable towards the origin. Note that agreement is maintained for all $t \geq T_{\max}$. Fixed-time consensus protocols have been studied in the literature for a range of different applications, as illustrated by [22]. In this work, we exploit the following result, which guarantees convergence of the consensus protocol before a fixed time $T_{\max} > 0$:

Lemma 1 (*Adapted from [23], Th. 6*). *Let $\mathcal{G}$ be a connected undirected graph with algebraic connectivity $\lambda_{\mathcal{G}}$, formed by N nodes and ℓ edges. Moreover, consider a consensus protocol of the form*

$$\dot{s}_i(t) = d_i(t) + \alpha \sum_{j \in \mathcal{N}_i} \phi(s_j(t) - s_i(t), \xi, \gamma) \tag{7}$$

where $s_i(t) \in \mathbb{R}$, $|d_i(t)| \leq L', \forall t \geq 0$,

$$\phi(\bullet, \xi, \gamma) = (|\bullet|^{1-\gamma} + |\bullet|^{1+\gamma} + \xi)\mathrm{sgn}(\bullet) \tag{8}$$

with the constants $\alpha > 0$ and $\gamma \in (0, 1)$ being design parameters of the consensus algorithm, and $\xi \geq L'/(\alpha\sqrt{\lambda_{\mathcal{G}}})$. Then, $s_i(t) = (1/N)\sum_{i=1}^N s_i(0)$, $\forall t \geq T_{\max}, \forall i \in \mathcal{V}$, with convergence time $T_{\max} = \ell\pi/(\alpha\gamma\lambda_{\mathcal{G}})$. This is, the dynamic average consensus disagreements $s_i(t) - (1/N)\sum_{i=1}^N s_i(0)$ are fixed-time stable towards the origin.

Note that the algorithm in Lemma 1 achieves exact consensus in fixed-time despite the time-varying variables $d_i(t)$, which enables us to design a distributed estimator that is suitable for systems with LTV dynamics.

3. The ODEFTC algorithm

In this Section, we introduce the ODEFTC algorithm, based on a fixed-time consensus protocol, to achieve optimal distributed state estimation for continuous-time stochastic systems.

The following equations give the proposed ODEFTC algorithm for an arbitrary node i:

$$\mathbf{K}_i(t) = N\mathbf{P}_i(t)\mathbf{C}_i(t)^\top\mathbf{R}_i(t)^{-1} \tag{9a}$$

$$\begin{aligned}\dot{\hat{\mathbf{x}}}_i(t) = {} & \mathbf{A}(t)\hat{\mathbf{x}}_i(t) + \mathbf{K}_i(t)\left(\mathbf{y}_i(t) - \mathbf{C}_i(t)\hat{\mathbf{x}}_i(t)\right) \\ & + \kappa\mathbf{P}_i(t)\sum_{j\in\mathcal{N}_i}(\hat{\mathbf{x}}_j(t) - \hat{\mathbf{x}}_i(t))\end{aligned} \tag{9b}$$

$$\dot{\mathbf{P}}_i(t) = \mathbf{A}(t)\mathbf{P}_i(t) + \mathbf{P}_i(t)\mathbf{A}(t)^\top + \mathbf{B}(t)\mathbf{W}(t)\mathbf{B}(t)^\top - \mathbf{P}_i(t)\hat{\mathbf{Z}}_i(t)\mathbf{P}_i(t) \tag{9c}$$

$$\hat{\mathbf{Z}}_i(t) = N\mathbf{C}_i(t)^\top\mathbf{R}_i(t)^{-1}\mathbf{C}_i(t) - \mathbf{Q}_i(t) \tag{9d}$$

$$\dot{\mathbf{Q}}_i(t) = \alpha\sum_{j\in\mathcal{N}_i}\phi(\hat{\mathbf{Z}}_i(t) - \hat{\mathbf{Z}}_j(t), \xi, \gamma) \tag{9e}$$

Here, the function ϕ from (8) is applied element-wise. Moreover, $\kappa, \alpha, \xi, \gamma > 0$ are design parameters yet to be specified. The design parameters for the algorithm are summarized in Table 1. In addition, Table 2 summarizes the variables defined so far for the reader's convenience.

The filter is initialized at $t = 0$ to $\sum_{i=1}^N \mathbf{Q}_i(0) = \mathbf{0}$, $\hat{\mathbf{x}}_i(0) = \mathbf{x}_0$, $\mathbf{P}_i(0) = \mathbf{P}_0$, with $\mathbf{P}_0$ being a $\mathbb{R}^{n\times n}$ symmetric non-negative matrix. For simplicity, we assume that the number of nodes N is a known parameter for

Table 2
List of variables used in this work.

Variables	Description
$\mathbf{x}(t)$	System state
$\mathbf{A}(t)$, $\mathbf{B}(t)$	System matrices
$\mathbf{w}(t)$, $\mathbf{W}(t)$	System disturbance and its covariance
$\mathbf{y}_i(t)$	Measurement at node i
$\mathbf{C}_i(t)$	Measurement matrix at node i
$\mathbf{v}_i(t)$, $\mathbf{R}_i(t)$	Measurement noise and its covariance at node i
$\hat{\mathbf{x}}_i(t)$, $\mathbf{P}_i(t)$	State estimate and estimated error covariance at node i using ODEFTC
$\hat{\mathbf{Z}}_i(t)$, $\mathbf{Q}_i(t)$	Estimate for the inverse covariance matrix at node i and auxiliary variable
$\mathcal{P}_i(t)$	True error covariance of the estimates from (9b) at node i
$\mathbf{K}_i(t)$	Filtering gain for ODEFTC at node i
$\hat{\mathbf{x}}(t)$, $\mathbf{P}(t)$	Estimate and error covariance from centralized Kalman-Bucy filter
$\mathbf{K}(t)$	Filtering gain for the centralized Kalman-Bucy filter
$\bar{\mathbf{Z}}(t)$	Inverse measurement covariance matrix for the network
$\mathbf{C}(t)$, $\mathbf{R}(t)$	Measurement and covariance matrices in the centralized case
$\mathbf{P}_\infty$	Asymptotic covariance matrix for the centralized LTI Kalman-Bucy filter

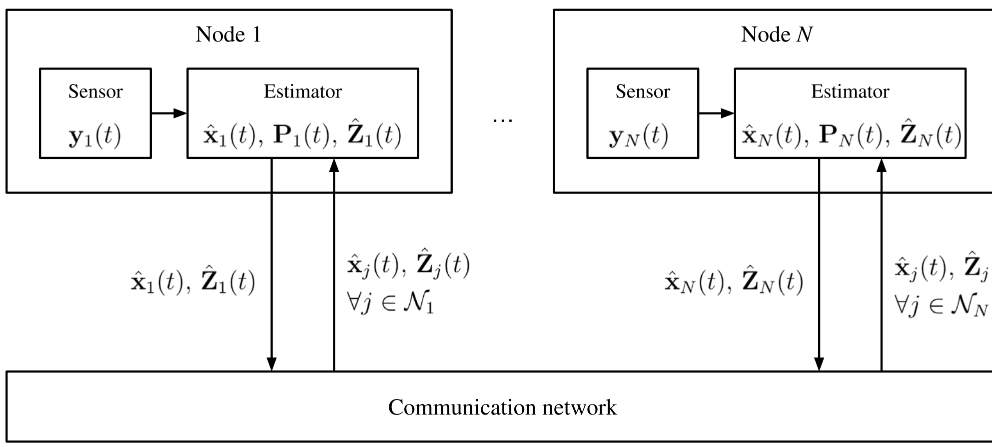

Fig. 1. Distributed state estimation setup.

the nodes, but it can also be computed in a distributed fashion as in, e.g., [27]. Due to the online computation of $\hat{\mathbf{x}}_i(t)$ and $\hat{\mathbf{Z}}_i(t)$ through consensus, both of these values need to be communicated to neighboring nodes. The proposed setup and transmitted information are summarized in Fig. 1.

Note that (9a)–(9b) is motivated by the filter proposed in [20], where a similar choice of $\mathbf{K}_i(t)$ was shown to recover the performance of the centralized filter for LTI systems, but we use a time-varying matrix $\mathbf{P}_i(t)$ to extend the proposal to LTV systems, as opposed to using $\mathbf{P}_\infty$ from (4). Then, (9c) is designed similarly to the expression for $\mathbf{P}(t)$ in the centralized filter (3c). Still, we have substituted the global information term $\bar{\mathbf{Z}}(t)$ defined in (5) with the estimates $\hat{\mathbf{Z}}_i(t)$, which are computed through the fixed-time dynamic consensus protocol (9d)–(9e).

The ODEFTC algorithm provides an expression to compute state estimates $\hat{\mathbf{x}}_i(t)$, as well as an estimated covariance $\mathbf{P}_i(t)$ of the estimation error $\mathbf{x}(t) - \hat{\mathbf{x}}_i(t)$. However, note that due to the distributed setting it is necessary to verify that the *estimated* covariance $\mathbf{P}_i(t)$ computed as (9c) recovers the *true* covariance $\mathcal{P}_i(t) = \mathrm{cov}\{\mathbf{x}(t) - \hat{\mathbf{x}}_i(t)\}$, for the estimates generated by consensus in (9b).

In the following, we establish the main properties of ODEFTC. Particularly, we show that the filter is stable and that the estimation accuracy with respect to the optimal centralized solution can be made arbitrarily high as κ is increased, both in terms of the true covariance $\mathcal{P}_i(t)$ of the error, as well as in the estimated covariance matrices $\mathbf{P}_i(t)$.

3.1. Convergence of the covariance matrices

To analyze the convergence of $\mathbf{P}_i(t)$ obtained with our ODEFTC algorithm, first note that (9c) depends on the value of $\hat{\mathbf{Z}}_i(t)$, which is computed through a fixed-time consensus algorithm to recover the matrix $\bar{\mathbf{Z}}(t)$ of the network, as defined in (5). Thus, we start by showing the fixed-time stability of $\hat{\mathbf{Z}}_i(t)$ towards $\bar{\mathbf{Z}}(t)$ in the following result.

Assumption 3. Define $\mathbf{Z}_i(t) = N\mathbf{C}_i(t)^\top \mathbf{R}_i(t)^{-1}\mathbf{C}_i(t)$. Then, $\|\dot{\mathbf{Z}}_i(t)\| \leq L$, $\forall t \geq 0$ for some known $L \geq 0$.

Note that Assumption 3 is trivially fulfilled by LTI systems. For LTV systems, it can still be fulfilled in many cases, such as when the matrices $\mathbf{C}_i(t)$, $\mathbf{R}_i(t)$ contain sinusoidal terms. Recalling that these matrices represent the measurement model and its noise covariance, assuming bounds on them is reasonable in practice. In addition, similar assumptions are adopted in related works, e.g. [2]. Now, we can state our result.

Lemma 2. *Let Assumption 3 hold as well as the assumptions and notation of Lemma 1 for the graph $\mathcal{G}$. For given design parameters $\alpha > 0$ and $\gamma \in (0,1)$, let $L' = 2L$ and set ξ and $T_{\max}$ as defined in Lemma 1. Then, the protocol (9d)–(9e) complies with $\hat{\mathbf{Z}}_i(t) = \bar{\mathbf{Z}}(t) = \mathbf{C}(t)^\top \mathbf{R}(t)^{-1}\mathbf{C}(t)$, $\forall t \geq T_{\max}$.*

Proof. Given that the $\phi(\bullet, \xi, \gamma)$ operator is applied element-wise, in the following we analyze convergence of an arbitrary element $\hat{z}_i(t)$ of the matrix $\hat{\mathbf{Z}}_i(t)$. Let $q_i(t)$ be the corresponding element of $\mathbf{Q}_i(t)$ and $z_i(t)$ the corresponding element for the matrix $\mathbf{Z}_i(t)$. Then, the dynamics for $\hat{z}_i(t)$ are given by

$$\dot{\hat{z}}_i(t) = \dot{z}_i(t) - \dot{q}_i(t) = \dot{z}_i(t) - \alpha \sum_{j\in\mathcal{N}_i} \phi(\hat{z}_i(t) - \hat{z}_j(t), \xi, \gamma) \tag{10}$$

where $|\dot{z}_i(t)| \leq L$ by assumption. Define $\bar{z}(t) = (1/N)\sum_{i=1}^{N} z_i(t)$, i.e. the corresponding element of the matrix $\bar{\mathbf{Z}}(t)$, and let $s_i(t) = \hat{z}_i(t) - \bar{z}(t)$. Noting that $\hat{z}_i(t) - \hat{z}_j(t) = -(s_j(t) - s_i(t))$, it follows that

$$\dot{s}_i(t) = (\dot{z}_i(t) - \dot{\bar{z}}(t)) + \alpha \sum_{j\in\mathcal{N}_i} \phi(s_j(t) - s_i(t), \xi, \gamma) \tag{11}$$

where $\sum_{i=1}^{N} s_i(0) = 0$, recalling that $\sum_{i=1}^{N} q_i(0) = 0$. Using the result from Lemma 1 with $|d_i(t)| = |\dot{\bar{z}}(t) - \dot{z}_i(t)| \leq 2L = L'$, we obtain that $s_i(t) = \sum_{i=1}^{N} s_i(0) = 0,\ \forall t \geq T_{\max}$. This leads to $\hat{z}_i(t) = \bar{z}(t),\ \forall t \geq T_{\max}$, implying the same for all elements of the matrix $\hat{\mathbf{Z}}_i(t)$ so that $\hat{\mathbf{Z}}_i(t) = \bar{\mathbf{Z}}(t),\ \forall t \geq T_{\max}$. □

Lemma 2 ensures that each node can recover the value of the inverse covariance matrix $\bar{\mathbf{Z}}(t)$ of the network in a fixed time $T_{\max}$. Now, we can state the following theorem showing the stability and convergence of $\mathbf{P}_i(t)$.

Theorem 1. *Let Assumptions 1, 2 and 3 hold as well as the conditions in Lemma 2. The estimated covariance $\mathbf{P}_i(t)$ computed as (9c) fulfills the following properties:*

1. *$\mathbf{P}_i(t)$ is uniformly bounded $\forall t \geq 0$.*
2. *After a fixed time $T_{\max}$, $\mathbf{P}_i(t)$ evolves as (3c) from the optimal centralized filter.*
3. *$\lim_{t\to\infty}(\mathbf{P}_i(t) - \mathbf{P}(t)) = \mathbf{0}$ with $\mathbf{P}(t)$ from the optimal centralized filter (3c).*
4. *If the system is LTI, $\lim_{t\to\infty} \mathbf{P}_i(t) = \mathbf{P}_\infty$, with $\mathbf{P}_\infty$ being the solution to the Riccati Eq. (4).*

Proof. Recalling the fixed convergence time $T_{\max}$ of $\hat{\mathbf{Z}}_i(t)$ to $\bar{\mathbf{Z}}(t)$, we first prove that the solution for the Riccati differential equation for $\mathbf{P}_i(t)$ given in (9c) exists for $t \in [0, T_{\max}]$, i.e. a finite-time escape towards infinity does not occur while the matrices $\hat{\mathbf{Z}}_i(t)$ converge towards $\bar{\mathbf{Z}}(t)$. This is ensured by [24, Theorem 1], which points out that the solution to this Riccati equation is uniquely determined for $t \geq 0$. Its proof follows from the existence theorem presented in [28], where existence of solutions for the Riccati equation of the "dual" optimal control problem is shown, starting at any symmetric non-negative initial condition, and noticing that the variance of $\mathbf{x}(t)$ must be finite in any finite time interval. Note that the existence of solution holds regardless of the value taken by $\hat{\mathbf{Z}}_i(t)$ during $t \in [0, T_{\max}]$, which is always a symmetric non-negative matrix by construction and bounded due to the convergence towards $\bar{\mathbf{Z}}(t)$ as per Lemma 2, where $\bar{\mathbf{Z}}(t)$ is also bounded due to Assumption 2. Moreover, note that $\mathbf{P}_i(t)$ is also always symmetric and non-negative.

For $t \geq T_{\max}$, we have that $\hat{\mathbf{Z}}_i(t) = \bar{\mathbf{Z}}(t)$. By substituting the value of $\bar{\mathbf{Z}}(t)$ in (9c) it follows that, for $t \geq T_{\max}$,

$$\dot{\mathbf{P}}_i(t) = \mathbf{A}(t)\mathbf{P}_i(t) + \mathbf{P}_i(t)\mathbf{A}(t)^\top + \mathbf{B}(t)\mathbf{W}(t)\mathbf{B}(t)^\top - \mathbf{P}_i(t)\mathbf{C}(t)^\top\mathbf{R}(t)^{-1}\mathbf{C}(t)\mathbf{P}_i(t) \tag{12}$$

which precisely matches the expression for $\mathbf{P}(t)$ in the centralized implementation (3c). Therefore, $\mathbf{P}_i(t)$ is ruled by the same dynamics as the centralized solution $\mathbf{P}(t)$ after time $T_{\max}$, proving item 2. Moreover, [24, Theorem 4] ensures that every solution for $t \geq T_{\max}$ starting at a symmetric non-negative matrix $\mathbf{P}_i(T_{\max})$ converges asymptotically towards the condition $\lim_{t\to\infty}(\mathbf{P}_i(t) - \mathbf{P}(t)) = \mathbf{0}$ for arbitrary $\mathbf{P}(0)$, concluding the proof of items 1 and 3. Finally, for an LTI system, [24, Theorem 5] states that $\lim_{t\to\infty} \mathbf{P}(t) = \mathbf{P}_\infty$, which is the solution for the Riccati Eq. (4), showing item 4. □

3.2. Convergence of the state estimates

Now, we analyze the performance of ODEFTC in terms of the state estimates $\hat{\mathbf{x}}_i(t)$. Before stating the main theorem of this work, we introduce an auxiliary Lemma to aid its proof.

Lemma 3. *Define*

$$\mathbf{A}^*(\kappa, t) = diag_{i=1}^{N}\left(\mathbf{A}(t) - \mathbf{K}_i(t)\mathbf{C}_i(t)\right) - \kappa\, diag_{i=1}^{N}(\mathbf{P}_i(t))\left(\mathbf{Q}_\mathcal{G} \otimes \mathbf{I}_n\right) \tag{13}$$

Then, there exist $\kappa_0 > 0$ such that $\forall \kappa > \kappa_0$ the origin of the system $\dot{\mathbf{e}}(t) = \mathbf{A}^*(\kappa, t)\mathbf{e}(t)$ is asymptotically stable.

Proof. The proof is given in the Appendix. □

Finally, we can state the following result for our ODEFTC algorithm, showing that the estimates obtained with it can achieve the same performance as with the centralized implementation.

Theorem 2. *Let κ_0 as in Lemma 3. For $\kappa > \kappa_0$, the ODEFTC algorithm (9) complies the following properties for the estimates $\hat{\mathbf{x}}_i(t)$ and the true covariance of the estimation error, $\boldsymbol{\mathcal{P}}_i(t) = \mathrm{cov}\{\mathbf{x}(t) - \hat{\mathbf{x}}_i(t)\}$:*

1. *The estimates $\hat{\mathbf{x}}_i(t)$ are unbiased.*
2. *$\boldsymbol{\mathcal{P}}_i(t)$ is uniformly bounded $\forall t \geq 0$ and sufficiently large $\kappa > 0$.*
3. *For the LTI case, $\lim_{t\to\infty} \boldsymbol{\mathcal{P}}_i(t) = \mathbf{P}_\infty$ as $\kappa \to \infty$, $t \to \infty$.*

Proof. Let $\mathbf{e}_i(t) = \mathbf{x}(t) - \hat{\mathbf{x}}_i(t)$ be the estimation error at node i. According to the system dynamics (1) and the estimation dynamics defined in (9b), we can write the error dynamics as follows:

$$\begin{aligned}
\dot{\mathbf{e}}_i(t) &= \dot{\mathbf{x}}(t) - \dot{\hat{\mathbf{x}}}_i(t) \\
&= \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{w}(t) - \mathbf{A}(t)\hat{\mathbf{x}}_i(t) - \mathbf{K}_i(t)\left(\mathbf{y}_i(t) - \mathbf{C}_i(t)\hat{\mathbf{x}}_i(t)\right) \\
&\quad - \kappa\mathbf{P}_i(t)\sum_{j\in\mathcal{N}_i}\left(\hat{\mathbf{x}}_j(t) - \hat{\mathbf{x}}_i(t)\right) \\
&= \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{w}(t) - \mathbf{A}(t)\hat{\mathbf{x}}_i(t) - \mathbf{K}_i(t)\mathbf{C}_i(t)\mathbf{x}(t) - \mathbf{K}_i(t)\mathbf{v}_i(t) \\
&\quad + \mathbf{K}_i(t)\mathbf{C}_i(t)\hat{\mathbf{x}}_i(t) - \kappa\mathbf{P}_i(t)\sum_{j\in\mathcal{N}_i}\left(\hat{\mathbf{x}}_j(t) - \hat{\mathbf{x}}_i(t)\right) \\
&= \left(\mathbf{A}(t) - \mathbf{K}_i(t)\mathbf{C}_i(t)\right)\mathbf{e}_i(t) - \kappa\mathbf{P}_i(t)\sum_{j\in\mathcal{N}_i}\left(\mathbf{e}_i(t) - \mathbf{e}_j(t)\right) + \mathbf{n}_i(t)
\end{aligned} \tag{14}$$

where we have defined $\mathbf{n}_i(t) = \mathbf{B}(t)\mathbf{w}(t) - \mathbf{K}_i(t)\mathbf{v}_i(t)$.

Let $\mathbf{e}(t) = [\mathbf{e}_1(t)^\top, \dots, \mathbf{e}_N(t)^\top]^\top$. Then, we can write

$$\begin{aligned}
\dot{\mathbf{e}}(t) = &\ \mathrm{diag}_{i=1}^{N}\left(\mathbf{A}(t) - \mathbf{K}_i(t)\mathbf{C}_i(t)\right)\mathbf{e}(t) \\
&- \kappa\,\mathrm{diag}_{i=1}^{N}(\mathbf{P}_i(t))\left(\mathbf{Q}_\mathcal{G} \otimes \mathbf{I}_n\right)\mathbf{e}(t) + \mathbf{n}(t) \\
= &\ \mathbf{A}^*(\kappa, t)\mathbf{e}(t) + \mathbf{n}(t)
\end{aligned} \tag{15}$$

with $\mathbf{A}^*(\kappa, t)$ defined in Lemma 3 and denoting $\mathbf{n}(t) = [\mathbf{n}_1(t)^\top, \dots, \mathbf{n}_N(t)^\top]^\top$.

Item 1: Recall that the estimates are initialized as $\hat{\mathbf{x}}_i(0) = \mathbf{x}_0$, which corresponds to the mean of the initial state $\mathbf{x}(0)$. Then, $\mathbb{E}\{\mathbf{e}_i(0)\} = \mathbb{E}\{\mathbf{x}(0) - \hat{\mathbf{x}}_i(0)\} = \mathbf{0}$ and $\mathbb{E}\{\mathbf{e}(0)\} = \mathbf{0}$ as well. The evolution of the error is given by (15). Note that for the elements of the term $\mathbf{n}(t)$ we have $\mathbb{E}\{\mathbf{n}_i(t)\} = \mathbb{E}\{\mathbf{B}(t)\mathbf{w}(t) - \mathbf{K}_i(t)\mathbf{v}_i(t)\} = \mathbf{0}$ due to $\mathbb{E}\{\mathbf{w}(t)\} = \mathbf{0}$ and $\mathbb{E}\{\mathbf{v}_i(t)\} = \mathbf{0}$. Then, given the dynamics (15), it follows that $\mathbb{E}\{\mathbf{e}(t)\} = \mathbf{0},\ \forall t \geq 0$, showing that the estimates are unbiased.

Item 2: To show boundedness of $\boldsymbol{\mathcal{P}}_i(t)$, we prove that the dynamics induced by $\mathbf{A}^*(\kappa, t)$ are stable for some sufficiently large κ, and that the error term $\mathbf{n}(t)$ related to the stochastic noise has bounded covariance. Note that the noiseless error system $\dot{\mathbf{e}}(t) = \mathbf{A}^*(\kappa, t)\mathbf{e}(t)$ is already shown to be stable in Lemma 3 for a large enough κ. Moreover, the covariance of $\mathbf{n}(t)$ is given by

$$\mathrm{cov}\{\mathbf{n}(t)\} = \mathbb{1}\mathbb{1}^\top \otimes \mathbf{B}(t)\mathbf{W}(t)\mathbf{B}(t)^\top + \mathrm{diag}_{i=1}^{N}(\mathbf{K}_i(t)\mathbf{R}_i(t)\mathbf{K}_i(t)^\top) \tag{16}$$

where we have used $\mathrm{cov}\{\mathbf{w}(t)\} = \mathbf{W}(t)$, $\mathrm{cov}\{\mathbf{v}_i(t)\} = \mathbf{R}_i(t)$. Recalling that $\mathbf{K}_i(t) = N\mathbf{P}_i(t)\mathbf{C}_i(t)^\top\mathbf{R}_i(t)^{-1}$, its value is bounded due to Assumption 2 and $\mathbf{P}_i(t)$ being uniformly bounded, as shown in Theorem 1. Thus, $\mathbf{n}(t)$ has uniformly bounded covariance. As a result, the same conclusion applies to the covariance of (15).

Item 3: Note that, if we consider the LTI case with constant matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}_i, \mathbf{W}, \mathbf{R}_i$ and we substitute $\mathbf{P}_i(t)$ with $\mathbf{P}_\infty$, the asymptotic filter proposed in [20] is recovered. For that filter, it is shown in [20,

Table 3
Summary of related methods for distributed state estimation of continuous-time systems with stochastic noise.

Filter	LTV system	Observability	Communication	Computation of $\mathbf{P}_i(t)$	Stability	Optimality
[13]	Yes	Local	$\hat{\mathbf{x}}_i(t)$	Online, non optimal	Convergence of estimates to true state in absence of noise	–
[14]	No	Full state	$\hat{\mathbf{x}}_i(t)$, $\mathbf{P}_i(t)$	Online, non optimal	–	–
[15]	No	Collective	$\hat{\mathbf{x}}_i(t)$, $\mathbf{P}_i(t)$	Online, non optimal	Convergence of estimates to true state in absence of noise, assuming $\mathbf{P}_i(t) \to \mathbf{P}_\infty$	–
[17]	No	Collective	$\hat{\mathbf{x}}_i(t)$, $\mathbf{P}_i(t)$	Online, non optimal	Contains technical errors (see [18])	Contains technical errors (see [18])
[2]	Yes	Collective	$\hat{\mathbf{z}}_i(t)$, $\hat{\mathbf{Z}}_i(t)$	Online, convergence to optimal	Bounded error w.r.t. centralized assuming bounded noise	Convergence to optimal only in absence of measurement noise
[19]	No	Collective	$\hat{\mathbf{z}}_i(t)$, $\hat{\mathbf{Z}}_i(t)$, event-triggered	Online, convergence to optimal	No analysis given	No analysis given
[20]	No	Collective	$\hat{\mathbf{x}}_i(t)$	$\mathbf{P}_\infty$ computed beforehand	Boundedness of covariance of the estimation error	Covariance converges to optimal centralized solution
ODEFTC	Yes	Collective	$\hat{\mathbf{x}}_i(t)$, $\hat{\mathbf{Z}}_i(t)$	Online, fixed-time algorithm, convergence to optimal	Boundedness of covariance of the estimation error	Covariance converges to optimal centralized solution

Theorem 3] that, when $\kappa \to \infty$, the true covariance of the estimation error tends to the covariance $\mathbf{P}_\infty$ in the centralized implementation as $t \to \infty$.

Recall that we have shown that $\mathbf{P}_i(t) \to \mathbf{P}_\infty$ as $t \to \infty$ in Theorem 1. Thus, considering the filter from [20] as a "nominal" system, we can analyze the behavior of the ODEFTC algorithm as a "perturbed" system with respect to the nominal one, with a vanishing perturbation due to the difference between $\mathbf{P}_i(t)$ and $\mathbf{P}_\infty$.

For ODEFTC, the estimation error dynamics are given by (15). Therefore, the evolution for the true covariance $\mathcal{P}(t) = \text{cov}\{\mathbf{e}(t)\}$ is given by

$$\begin{aligned}\dot{\mathcal{P}}(t) &= \mathbf{A}^*(\kappa,t)\mathcal{P}(t) + \mathcal{P}(t)\mathbf{A}^*(\kappa,t)^\top \\ &\quad + \mathbb{1}\mathbb{1}^\top \otimes \mathbf{B}\mathbf{W}\mathbf{B}^\top + \text{diag}_{i=1}^N(\mathbf{K}_i(t)\mathbf{R}_i\mathbf{K}_i(t)^\top)\end{aligned} \tag{17}$$

For the nominal system from [20] we have that

$$\dot{\mathbf{e}}'(t) = \mathbf{A}'(\kappa)\mathbf{e}'(t) + \mathbf{n}'(t) \tag{18}$$

where $\mathbf{n}'(t)$ is defined similarly to $\mathbf{n}(t)$ in (15) with $\mathbf{K}_i' = N\mathbf{P}_\infty\mathbf{C}_i^\top\mathbf{R}_i^{-1}$ and

$$\mathbf{A}'(\kappa) = \text{diag}_{i=1}^N(\mathbf{A} - \mathbf{K}_i'\mathbf{C}_i) - \kappa(\mathbf{Q}_\mathcal{G} \otimes \mathbf{P}_\infty) \tag{19}$$

Then, its true covariance $\mathcal{P}'(t)$ evolves as

$$\begin{aligned}\dot{\mathcal{P}}'(t) &= \mathbf{A}'(\kappa)\mathcal{P}'(t) + \mathcal{P}'(t)\mathbf{A}'^\top(\kappa) \\ &\quad + \mathbb{1}\mathbb{1}^\top \otimes \mathbf{B}\mathbf{W}\mathbf{B}^\top + \text{diag}_{i=1}^N(\mathbf{K}_i'\mathbf{R}_i\mathbf{K}_i'^\top)\end{aligned} \tag{20}$$

Comparing both systems, we can write the evolution for our covariance as that of the nominal system with an added perturbation $\mathbf{G}(t,\mathcal{P}(t))$

$$\begin{aligned}\dot{\mathcal{P}}(t) &= \mathbf{A}'(\kappa)\mathcal{P}'(t) + \mathcal{P}'(t)\mathbf{A}'^\top(\kappa) + \mathbb{1}\mathbb{1}^\top \otimes \mathbf{B}\mathbf{W}\mathbf{B}^\top \\ &\quad + \text{diag}_{i=1}^N(\mathbf{K}_i'\mathbf{R}_i\mathbf{K}_i'^\top) + \mathbf{G}(t,\mathcal{P}(t))\end{aligned} \tag{21}$$

where the perturbation is given by

$$\begin{aligned}\mathbf{G}(t,\mathcal{P}(t)) &= (\mathbf{A}^*(\kappa,t) - \mathbf{A}'(\kappa))\mathcal{P}(t) + \mathcal{P}(t)(\mathbf{A}^*(\kappa,t) - \mathbf{A}'(\kappa))^\top \\ &\quad + \text{diag}_{i=1}^N(\mathbf{K}_i(t)\mathbf{R}_i\mathbf{K}_i(t)^\top) - \text{diag}_{i=1}^N(\mathbf{K}_i'\mathbf{R}_i\mathbf{K}_i'^\top)\end{aligned} \tag{22}$$

Note that

$$\begin{aligned}&\mathbf{A}^*(\kappa,t) - \mathbf{A}'(\kappa) = \\ &= \text{diag}_{i=1}^N(\mathbf{A} - N\mathbf{P}_i(t)\mathbf{C}_i^\top\mathbf{R}_i^{-1}\mathbf{C}_i) - \kappa\,\text{diag}_{i=1}^N(\mathbf{P}_i(t))\left(\mathbf{Q}_\mathcal{G} \otimes \mathbf{I}_n\right) \\ &\quad - \text{diag}_{i=1}^N(\mathbf{A} - N\mathbf{P}_\infty\mathbf{C}_i^\top\mathbf{R}_i^{-1}\mathbf{C}_i) - \kappa(\mathbf{Q}_\mathcal{G} \otimes \mathbf{P}_\infty) \\ &= \text{diag}_{i=1}^N(N(\mathbf{P}_\infty - \mathbf{P}_i(t))\mathbf{C}_i^\top\mathbf{R}_i^{-1}\mathbf{C}_i) + \kappa\,\text{diag}_{i=1}^N(\mathbf{P}_\infty - \mathbf{P}_i(t))(\mathbf{Q}_\mathcal{G} \otimes \mathbf{I}_n)\end{aligned} \tag{23}$$

and

$$\begin{aligned}&\text{diag}_{i=1}^N(\mathbf{K}_i(t)\mathbf{R}_i\mathbf{K}_i(t)^\top) - \text{diag}_{i=1}^N(\mathbf{K}_i'\mathbf{R}_i\mathbf{K}_i'^\top) \\ &= \text{diag}_{i=1}^N(N\mathbf{P}_i(t)\mathbf{C}_i^\top\mathbf{R}_i^{-1}\mathbf{C}_i\mathbf{P}_i(t)^\top) - \text{diag}_{i=1}^N(N\mathbf{P}_\infty\mathbf{C}_i^\top\mathbf{R}_i^{-1}\mathbf{C}_i\mathbf{P}_\infty) \\ &= \text{diag}_{i=1}^N(N(\mathbf{P}_i(t) - \mathbf{P}_\infty)\mathbf{C}_i^\top\mathbf{R}_i^{-1}\mathbf{C}_i(\mathbf{P}_i(t) - \mathbf{P}_\infty))\end{aligned} \tag{24}$$

Defining the following terms

$$\begin{aligned}\eta(t) &= 2\|\mathbf{A}^*(\kappa,t) - \mathbf{A}'(\kappa)\| \\ \varepsilon(t) &= \|\text{diag}_{i=1}^N(\mathbf{K}_i(t)\mathbf{R}_i\mathbf{K}_i(t)^\top) - \text{diag}_{i=1}^N(\mathbf{K}_i'\mathbf{R}_i\mathbf{K}_i'^\top)\|\end{aligned} \tag{25}$$

we can write

$$\|\mathbf{G}(t,\mathcal{P}(t))\| \le \eta(t)\|\mathcal{P}(t)\| + \varepsilon(t) \tag{26}$$

Moreover, note that

$$\lim_{t\to\infty} \eta(t) = 0, \qquad \lim_{t\to\infty} \varepsilon(t) = 0 \tag{27}$$

due to $\lim_{t\to\infty} \mathbf{P}_i(t) = \mathbf{P}_\infty$. Therefore, given that $\|\mathcal{P}(t)\|$ in (26) is also bounded as a result of item 2 of Theorem 2, the perturbation $\mathbf{G}(t,\mathcal{P}(t))$ vanishes as $t \to \infty$. Moreover, $\mathbf{A}'(\kappa)$ being Hurwitz as a consequence of [20, Theorem 1] implies exponential convergence of the nominal system as well as the existence of an appropriate Lyapunov function such that an argument of continuity of solutions in the infinite interval can be applied [29]. This is, for any $\delta, \delta' > 0$ there exist sufficiently large $T > 0$ such that $\|\mathbf{G}(t,\mathcal{P}(t))\| \le \delta$ from (26) and $\|\mathcal{P}(t) - \mathcal{P}'(t)\| \le \delta'$ from [29, Theorem 9.1]. Henceforth, $\lim_{t\to\infty} \|\mathcal{P}(t) - \mathcal{P}'(t)\| = 0$, and as a consequence $\lim_{t\to\infty} \mathcal{P}(t) = \lim_{t\to\infty} \mathcal{P}'(t) = \mathbb{1}\mathbb{1}^\top \otimes \mathbf{P}_\infty$. Note that the result in [29, Theorem 9.1] is given for a system with an equilibrium point at the origin, but it can be straightforwardly applied without loss of generality to our case with an equilibrium at $\mathbb{1}\mathbb{1}^\top \otimes \mathbf{P}_\infty$. □

Remark 2. The main results of this work, namely Theorems 1 and 2, show that the ODEFTC algorithm (9) can recover the performance of the optimal centralized filter (3), both in terms of the true covariance $\mathcal{P}_i(t)$ of the estimation error as well as the estimated covariance $\mathbf{P}_i(t)$. In particular, for the LTI case, both $\mathbf{P}_i(t)$ and $\mathcal{P}_i(t)$ converge to the covariance $\mathbf{P}_\infty$ from the centralized setting. Thus, (9c) asymptotically recovers the covariance of the estimates from (9b).

4. Discussion and validation

4.1. Comparison to related work

In Table 3, we summarize the comparison of different approaches in the literature and our ODEFTC algorithm.

First, note that only [2,13] and our ODEFTC apply to LTV systems, while other works only address the LTI case. However, for [13,14], restrictive assumptions on the system's observability are required while the rest of the proposals in the literature need the system to be collectively observable using the aggregated information of the whole sensor network.

Regarding the computation of $\mathbf{P}_i(t)$, the information that needs to be communicated in each approach varies. Generally, if $\mathbf{P}_i(t)$ is computed online, this requires the transmission of either the matrix $\mathbf{P}_i(t)$ itself or the estimate $\hat{\mathbf{Z}}_i(t)$ for the inverse covariance matrix $\bar{\mathbf{Z}}(t)$ of the network. The exception is [13], which computes $\mathbf{P}_i(t)$ using only local information, requiring stronger observability assumptions and leading to non-optimal results. Moreover, [20] pre-computes the asymptotic matrix $\mathbf{P}_\infty$ through a separate distributed protocol. In addition, all approaches communicate the state estimates $\hat{\mathbf{x}}_i(t)$, except [2,19] which communicate their estimate $\hat{\mathbf{z}}_i(t)$ of the average measurement of the network (in information form) given by $\mathbf{C}(t)^\top \mathbf{R}(t)^{-1}\mathbf{y}(t)$.

In terms of the stability of the state estimates, [13,15] show convergence to the true state in absence of noise, with [15] requiring an assumption that the estimated covariance matrices $\mathbf{P}_i(t)$ computed by the nodes converge to the asymptotic centralized solution $\mathbf{P}_\infty$, which may not be fulfilled in practice. In [2], stability is only shown under the assumption of null or bounded measurement noise, rather than Gaussian noise, reaching a bounded error with respect to the centralized implementation. For [20] and our ODEFTC, it is shown that the true covariance $\boldsymbol{\mathcal{P}}_i(t)$ of the estimates is bounded, without the need to consider an absence of noise in the proof. As for the other works, [14,19] do not provide proof of stability for the state estimates, and the one in [17] contains technical errors (also in the proof of optimality), which are critical in some cases according to [18].

Regarding the optimality of the estimates, both [20] and ODEFTC are shown to recover the performance of the optimal centralized filter. The approach in [2] would only recover similar estimates as the centralized filter in absence of measurement noise, i.e. if $\mathbf{v}_i(t) = \mathbf{0}$. However, note that in Kalman filtering it is assumed that measurement noise with a given covariance $\mathbf{R}_i(t) \succ \mathbf{0}$ is present, so having null noise is contradictory with this assumption. The other works either do not analyze this aspect, or the proofs contain errors.

In regards to the computation of the estimated covariance matrices $\mathbf{P}_i(t)$, the works [13–15,17] compute them online, but obtain matrices that do not necessarily correspond to the true covariance $\boldsymbol{\mathcal{P}}_i(t)$ of the estimation error. This means that the matrices $\mathbf{P}_i(t)$ in these works may not be useful to evaluate the confidence on their state estimates. The proposals in [2,19] recover the same covariance matrix $\mathbf{P}_i(t)$ as in the centralized approach, but this estimated covariance does not necessarily match the *true* covariance of the state estimates produced by the distributed filter (in [2], they match only in absence of measurement noise, while no analysis is given in [19]).

A different approach is taken in [20], where the asymptotic matrix $\mathbf{P}_\infty$ of the centralized solution is pre-computed by the nodes of the network before running the filter to generate state estimates. Then, since the estimates recover the performance of the centralized solution as $t \to \infty$, their true covariance $\boldsymbol{\mathcal{P}}_i(t)$ also tends to $\mathbf{P}_\infty$. However, the approximation of using $\mathbf{P}_\infty$ from the start may cause overconfidence during the transient period.

In contrast, with ODEFTC we compute the estimated covariance matrices online, through a fixed-time consensus algorithm. After the fixed convergence time $T_{\max}$, the estimated covariance $\mathbf{P}_i(t)$ computed at each node evolves through the same dynamics as $\mathbf{P}(t)$ in the optimal centralized filter, and converges to the value $\mathbf{P}_\infty$ as well, as $t \to \infty$. Additionally, note that computing the matrices $\mathbf{P}_i(t)$ online makes our algorithm applicable for LTV systems, as opposed to [20].

In summary, our approach improves the results from [20] by extending the optimal proposal to the LTV case, and provides an advantage with respect to previous works addressing LTV systems [2,13] by reconstructing the covariance value $\mathbf{P}(t)$ of the centralized filter using only local communication and providing stable estimates even in the presence of Gaussian noise, without assuming noise bounds.

Remark 3. In terms of the computational complexity of the different approaches, note that the main bottleneck for our ODEFTC and for the related works lies on the matrix multiplications, with complexity $\mathcal{O}(n^2)$ per iteration where n is the state dimension, and matrix inversions, which can be performed offline beforehand in the LTI case. Concerning the number of neighbors, all approaches, including our own, have a comparable complexity $\mathcal{O}(|\mathcal{N}_i|)$ per iteration.

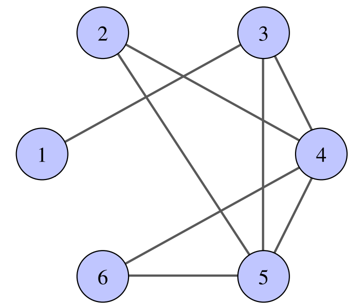

Fig. 2. Sensor network used in the simulation experiments.

4.2. Simulation experiments

We now provide simulation examples to validate our proposal and compare its results with previous approaches. The experiments have been executed on Matlab R2022a, on a computer equipped with an Intel Core i7-8700 CPU.

In all experiments we consider a sensor network consisting of $N = 6$ sensing nodes, represented in Fig. 2. The pseudo-code for the implementation of ODEFTC is given in Algorithm 1, where we omit the time dependence for simplicity and use the superindex $\bullet^-$ to indicate the value of a variable in the previous simulation step. We have set the simulation step to $h = 10^{-4}$ s.

Algorithm 1: ODEFTC

Initialize estimates $\hat{\mathbf{x}}_i \leftarrow \mathbf{x}_0$, $\mathbf{P}_i \leftarrow \mathbf{P}_0$, $\mathbf{K}_i \leftarrow N\mathbf{P}_i\mathbf{C}_i^\top\mathbf{R}_i^{-1}$
Initialize auxiliary variables $\mathbf{Q}_i = \mathbf{0}$ (which comply $\sum_{i=1}^N \mathbf{Q}_i = \mathbf{0}$)
for *each time step* **do**
  for *each node* $i \in \{1, \dots, N\}$ **do**
    Get measurement $\mathbf{y}_i^-$
    $\hat{\mathbf{x}}_i \leftarrow \hat{\mathbf{x}}_i^- + h\left(\mathbf{A}^-\hat{\mathbf{x}}_i^- + \mathbf{K}_i^-\left(\mathbf{y}_i^- - \mathbf{C}_i^-\hat{\mathbf{x}}_i^-\right) + \kappa\mathbf{P}_i^-\sum_{j\in\mathcal{N}_i}(\hat{\mathbf{x}}_j^- - \hat{\mathbf{x}}_i^-)\right)$
    $\mathbf{P}_i \leftarrow \mathbf{P}_i^- + h\left(\mathbf{A}^-\mathbf{P}_i^- + \mathbf{P}_i^-\mathbf{A}^{-\top} + \mathbf{B}^-\mathbf{W}^-\mathbf{B}^{-\top} - \mathbf{P}_i^-\hat{\mathbf{Z}}_i^-\mathbf{P}_i^-\right)$
    $\mathbf{Q}_i \leftarrow \mathbf{Q}_i^- + h\left(\alpha\sum_{j\in\mathcal{N}_i}\phi(\hat{\mathbf{Z}}_i^- - \hat{\mathbf{Z}}_j^-, \xi, \gamma)\right)$
    $\hat{\mathbf{Z}}_i \leftarrow N\mathbf{C}_i^\top\mathbf{R}_i^{-1}\mathbf{C}_i - \mathbf{Q}_i$
    $\mathbf{K}_i \leftarrow N\mathbf{P}_i\mathbf{C}_i^\top\mathbf{R}_i^{-1}$
    Broadcast $\hat{\mathbf{x}}_i$ to neighbors $j \in \mathcal{N}_i$
  end
end

In the following, we test our approach on both LTI and LTV systems of the form (1) to validate our theoretical results. We explore the effect of the tuning parameter κ on the performance of the proposal, particularly in recovering the optimal solution as $\kappa \to \infty$. In addition, we compare the performance of our approach with the centralized Kalman-Bucy filter, as well as with other distributed approaches for LTI and LTV systems.

For the comparison with other approaches in the literature, we run simulations with a length of T_f, for each of the considered approaches.

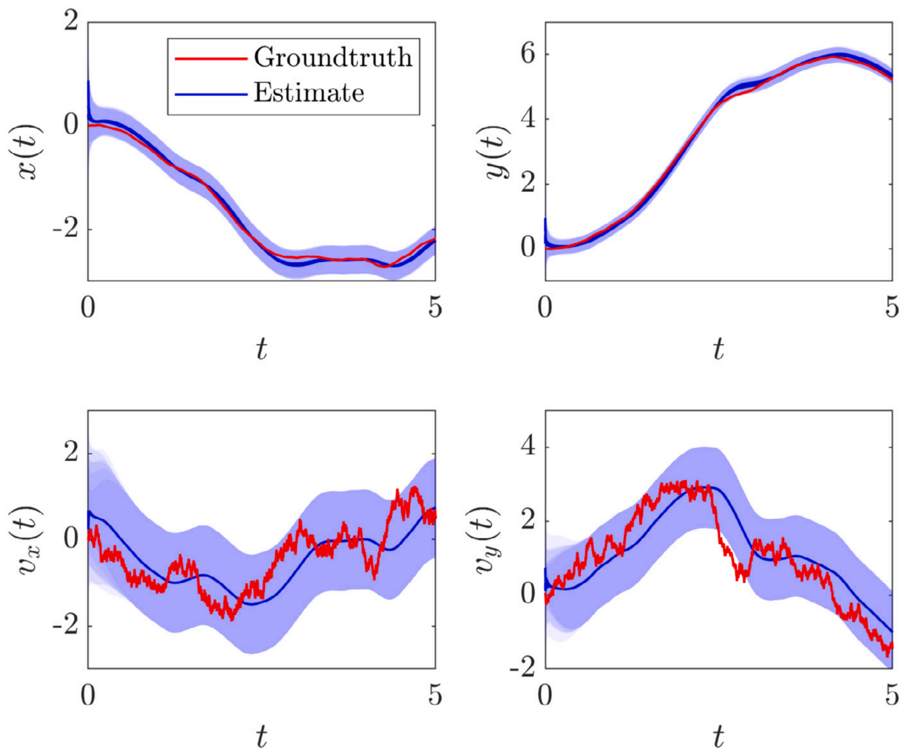

Fig. 3. Estimation example for a LTI system with our ODEFTC algorithm. The shaded blue area represents the confidence interval, according to the estimated covariance.

As performance metrics for comparison, we compute the following quantities:

$$\begin{aligned}
\mathcal{E}^i_{\hat{\mathbf{x}}} &= \frac{1}{T_f}\int_0^{T_f} \|\mathbf{x}(t) - \hat{\mathbf{x}}_i(t)\|^2 \mathrm{d}t \\
\mathcal{E}_{\hat{\mathbf{x}}} &= \frac{1}{N}\sum_{i=1}^N \mathcal{E}^i_{\hat{\mathbf{x}}}, \qquad \mathcal{D}_{\hat{\mathbf{x}}} = \mathrm{std}(\mathcal{E}^i_{\hat{\mathbf{x}}}) \\
\mathcal{E}^i_{\mathbf{P}} &= \frac{1}{T_f}\int_0^{T_f} \|\mathbf{P}^*(t) - \mathbf{P}_i(t)\|_{\mathrm{F}}\, \mathrm{d}t \\
\mathcal{E}_{\mathbf{P}} &= \frac{1}{N}\sum_{i=1}^N \mathcal{E}^i_{\mathbf{P}}, \qquad \mathcal{D}_{\mathbf{P}} = \mathrm{std}(\mathcal{E}^i_{\mathbf{P}})
\end{aligned} \tag{28}$$

where we use $\mathbf{P}^*(t) = \mathbf{P}_\infty$ for the LTI case and $\mathbf{P}^*(t) = \mathbf{P}(t)$ from the centralized filter in the LTV case. Note that $\mathcal{E}_{\hat{\mathbf{x}}}$ represents the mean square error of the estimates of a node with respect to the true state, $\mathcal{E}_{\mathbf{P}}$ represents the error between the estimated covariance matrix and the desired optimal solution, and $\mathcal{D}_{\hat{\mathbf{x}}}$, $\mathcal{D}_{\mathbf{P}}$ represent the disagreement between nodes in terms of the standard deviation between error values of the nodes. Small values of these metrics are desirable.

4.2.1. LTI systems

Let the system be described by (1) with

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \ \mathbf{W} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{29}$$

where the state is given by $\mathbf{x}(t) = [x(t), y(t), v_x(t), v_y(t)]^\top$. The sensors obtain measurements in the form of (2) with

$$\begin{aligned}
\mathbf{C}_1 = \mathbf{C}_5 &= \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \\
\mathbf{C}_2 = \mathbf{C}_4 &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \\
\mathbf{C}_3 = \mathbf{C}_6 &= \begin{bmatrix} 0 & 2 & 0 & 0 \end{bmatrix}
\end{aligned} \tag{30}$$

and the noise covariances for the measurement noises $\mathbf{v}_i(t)$ are given by $\mathbf{R}_1 = 0.01$, $\mathbf{R}_3 = \mathbf{R}_6 = 0.03$, $\mathbf{R}_5 = 0.05$ and

$$\mathbf{R}_2 = \mathbf{R}_4 = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix} \tag{31}$$

Note that for the LTI case, the computation of $\hat{\mathbf{Z}}_i(t)$ from (9d)–(9e) reduces to

$$\dot{\hat{\mathbf{Z}}}_i(t) = -\alpha \sum_{j\in\mathcal{N}_i} \phi(\hat{\mathbf{Z}}_i(t) - \hat{\mathbf{Z}}_j(t), \xi, \gamma) \tag{32}$$

with $\hat{\mathbf{Z}}_i(0) = \mathbf{Z}_i$, due to $\mathbf{Z}_i = N\mathbf{C}_i^\top \mathbf{R}_i^{-1}\mathbf{C}_i$ being constant. Thus, since $\|\dot{\mathbf{Z}}_i\| = 0$, we can set $\xi = 0$ as per Lemma 2. For the other parameters, we have set $\kappa = 100$, $\alpha = 10$, $\gamma = 0.5$.

We have initialized the system state at $t = 0$ to $\mathbf{x}(0) = [0, 0, 0, 0]^\top$. For the estimates, we have initialized them at random values for $\hat{\mathbf{x}}(0)$ and random symmetric covariance matrices $\mathbf{P}(0)$, in order to verify that convergence is achieved even if the initial conditions $\mathbf{x}_0, \mathbf{P}_0$ are unknown in practice. Figs. 3 and 4 show the resulting estimates and the convergence of the covariance matrices to the desired asymptotic value, respectively. The shaded area in Fig. 3 represents the confidence interval for the estimates, given by twice the estimated standard deviation (as given by the estimated covariance matrix $\mathbf{P}_i(t)$ for each node). The estimates for all nodes are plotted in an overlapping manner. Note that, despite the random initialization, convergence to the agreement values is quickly achieved.

To further investigate the effect of κ on the steady state true covariance $\mathcal{P}_i(t)$ of the estimation error, we have run simulations for a range of values of κ. The results are represented in Fig. 5, where we denote as $\mathcal{P}_{i,\infty}$ the true covariance obtained in steady state (i.e. excluding the initial transitory period while variables converge). In addition, recall that $\mathrm{tr}(\mathcal{P}_{i,\infty})$ is equivalent to the mean squared error (MSE) of the state estimates in steady state, so this figure can also be read as the estimation accuracy of our ODEFTC for LTI systems in this sense. Note that the system is locally observable from nodes 2 and 4, so these nodes achieve a small covariance even for small values of κ. Moreover, for $\kappa = 1$ the true covariance is very high, which might mean that $\kappa \not> \kappa_0$,

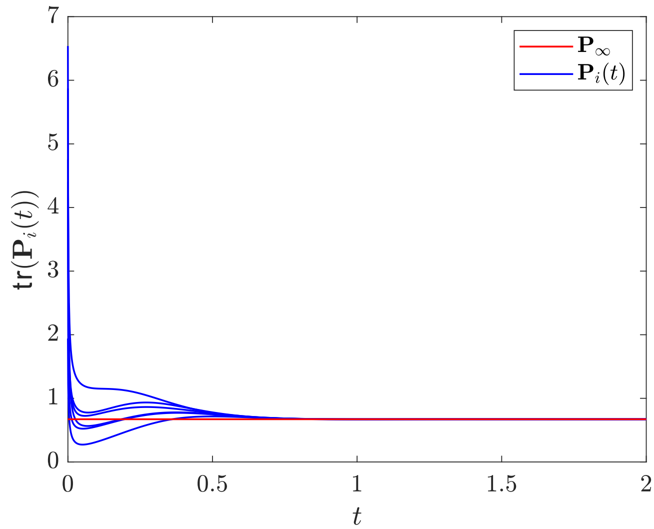

Fig. 4. Convergence of $\mathbf{P}_i(t)$ for all nodes $i \in \mathcal{V}$ towards the asymptotic value of the optimal centralized filter $\mathbf{P}_\infty$ in the LTI case.

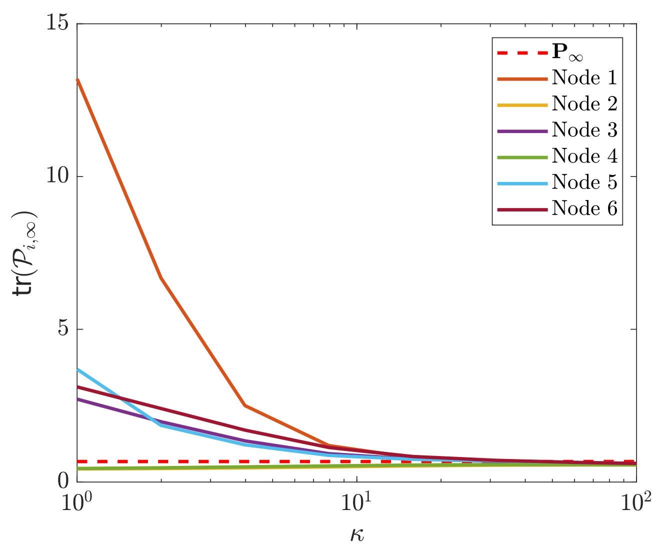

Fig. 5. Convergence of the steady state true covariance of the estimation error, $\mathcal{P}_{i,\infty}$, to the optimal centralized solution $\mathbf{P}_\infty$ as the consensus gain κ is increased.

recalling κ_0 from Lemma 3 as the minimum gain that ensures stability of the algorithm. However, as expected from the theoretical analysis, in steady state the true covariance of the estimates $\mathcal{P}_i(t)$ tends to the optimal centralized solution $\mathbf{P}_\infty$ as κ is increased, without needing $\kappa \to \infty$ in practice to achieve acceptable results.

For the comparison with other approaches in the literature, we computed the performance metrics (28) on 20 simulations with T_f = 100 s each, for the LTI filters in Table 3 that can be applied to the case of a network with collective observability, as well as the optimal solution obtained with the centralized Kalman-Bucy filter. We have initialized all simulations at $\hat{\mathbf{x}}(0) = \mathbf{x}_0 = [0,0,0,0]^\top$ and $\mathbf{P}_i(t) = \mathbf{I}_n$. We have maintained $\kappa = 100$, $\alpha = 10$, $\gamma = 0.5$ as the parameters for ODEFTC and $\kappa = 100$ as the consensus gain for [20].

The results are summarized in Table 4, where the average value of each metric over the 20 simulations is shown. In terms of the mean square error $\mathcal{E}_{\hat{\mathbf{x}}}$, our proposal as well as [17,20] obtain comparable results. However, the disagreement between nodes is smaller for our proposal and [20]. In terms of the estimated covariance matrix $\mathbf{P}_i(t)$, our proposal has similar performance to the centralized filter, with all nodes recovering $\mathbf{P}_\infty$ asymptotically. The proposals in [15,17] do not converge to the optimal value $\mathbf{P}_\infty$. Moreover, the nodes do not achieve consensus for $\mathbf{P}_i(t)$. Similar values with respect to [20] are to be expected, since our proposal recovers the asymptotic filter from [20] in steady state. However, in the following we show that our proposal can also be applied to LTV systems, as opposed to [20].

Table 4
Comparison of performance for LTI filters, in terms of the error in the state estimates $\mathcal{E}_{\hat{\mathbf{x}}}$ and estimated covariance $\mathcal{E}_{\mathbf{P}}$, as well as the corresponding disagreement between nodes, $\mathcal{D}_{\hat{\mathbf{x}}}$ and $\mathcal{D}_{\mathbf{P}}$.

Filter	$\mathcal{E}_{\hat{\mathbf{x}}}$	$\mathcal{D}_{\hat{\mathbf{x}}}$	$\mathcal{E}_{\mathbf{P}}$	$\mathcal{D}_{\mathbf{P}}$
Centralized filter	0.4871	–	0.0035	–
ODEFTC	0.5530	0.0242	0.0035	3×10^{-5}
[15]	0.8726	0.1316	0.4253	0.1508
[17]	0.5402	0.1103	4.3264	1.8493
[20]	0.5535	0.0242	–	–

4.2.2. LTV systems

Consider similar dynamics as for the previous case, but with a time-varying matrix

$$\mathbf{A}(t) = \begin{bmatrix} 0 & 0 & \sin(t) & 0 \\ 0 & 0 & 0 & \sin(t) \\ 0 & 0 & \cos(t) & 0 \\ 0 & 0 & 0 & \cos(t) \end{bmatrix} \tag{33}$$

For the measurements, we consider the same matrices $\mathbf{C}_i(t)$ as in the previous case, but we include the time-varying covariance matrices $\mathbf{R}_1(t) = 0.05 + 0.01\sin(0.1t)$, $\mathbf{R}_3(t) = 0.03 + 0.01\cos(0.1t)$. The rest of the matrices are kept the same as in the LTI example. We have set $\kappa = 1000$, $\alpha = 24$, $\gamma = 0.7$, $\xi = 100$. Fig. 6 shows the estimation result with our ODEFTC. Fig. 7 shows the recovery of the time-varying matrix $\mathbf{P}(t)$ from the centralized filter, with the Table 5 providing a numerical comparison of the variances for each state variable computed at each node. The values are shown for different instants in the simulation, to show that agreement is reached and maintained despite the time-variant dynamics.

Furthermore, Fig. 8 shows the recovery of the matrix $\bar{\mathbf{Z}}(t)$ in a time smaller than the theoretical convergence time $T_{\max}$. As seen in the Figure, the convergence time is approximately 0.015s. Considering the choice of parameters α, γ, $\ell = 8$ edges and algebraic connectivity $\lambda_{\mathcal{G}} = 0.7639$, along with the relation given in Lemmas 1–2, the theoretical guaranteed convergence time is $T_{\max} = 1.9584$ s, providing a conservative upper bound.

To illustrate the true estimation accuracy of our filter in the time-variant case, Fig. 9 shows a comparison of the mean squared error (MSE) of our ODEFTC with the theoretical expected value, recalling that it is given by $\text{tr}(\mathbf{P}_i(t))$ in the context of Kalman filtering. Note that, due to the time-varying matrices, the expected MSE varies over time. For this experiment, we have run 100 simulations in order to have different noise realizations, and we compute the MSE over the realizations. We include the results obtained for our ODEFTC algorithm with different choices of the gain κ, and for the centralized Kalman-Bucy filter, over the same 100 realizations. We have initialized the estimates and covariance matrices of the nodes to random values, to account for the fact that the initial conditions $\mathbf{x}_0$, $\mathbf{P}_0$ may not be known in practice. In addition, we have computed the values for each node separately and plotted them in the figure. While each node will have a different noise realization for each simulation, they behave similarly in MSE, producing estimates that are consistent with the value given by $\text{tr}(\mathbf{P}_i(t))$. In addition, increasing κ leads to better agreement between nodes and it improves the estimation accuracy, achieving results closer to the centralized filter.

Similarly, Fig. 10 shows the estimation error for the nodes in comparison to the estimation error from the centralized filter for one

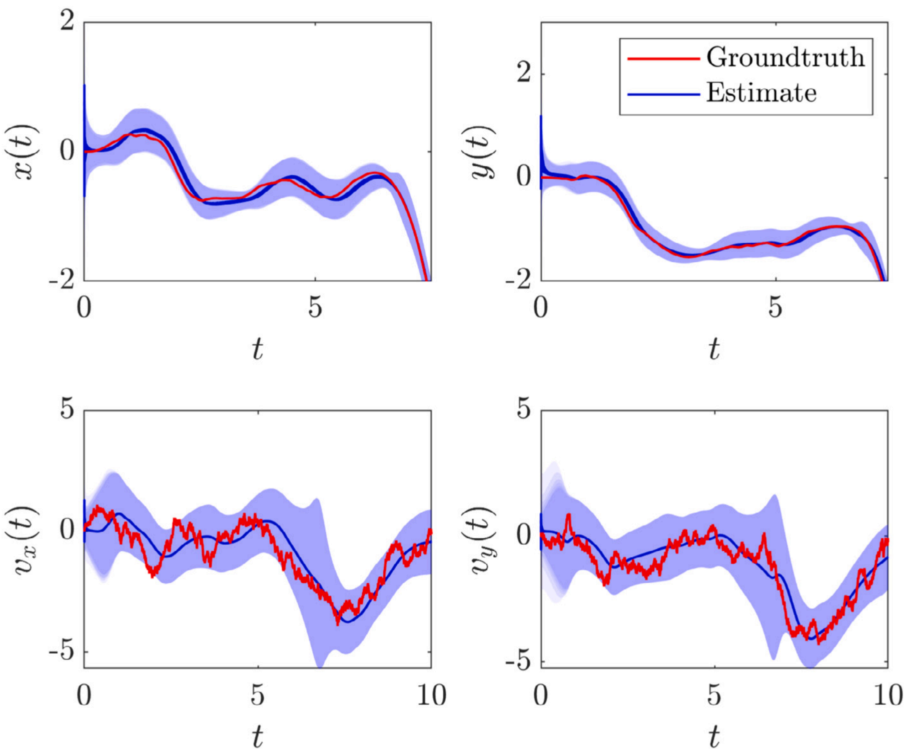

Fig. 6. Estimation example for a LTV system with our ODEFTC algorithm. The shaded blue area represents the confidence interval of the state estimates, according to their estimated covariance.

Table 5
Numerical comparison of variances for each state variable computed by each node at several times. Agreement is reached and maintained despite the LTV dynamics.

		Centralized filter	ODEFTC					
			$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$
$t=2$ s	$x(t)$	0.0231	0.0231	0.0231	0.0231	0.0231	0.0231	0.0231
	$y(t)$	0.0177	0.0177	0.0177	0.0177	0.0177	0.0177	0.0177
	$v_x(t)$	0.3352	0.3352	0.3351	0.3351	0.3351	0.3351	0.3352
	$v_y(t)$	0.3092	0.3092	0.3092	0.3092	0.3092	0.3092	0.3092
$t=5$ s	$x(t)$	0.0245	0.0245	0.0245	0.0245	0.0245	0.0245	0.0245
	$y(t)$	0.0187	0.0187	0.0187	0.0187	0.0187	0.0187	0.0187
	$v_x(t)$	0.3888	0.3888	0.3888	0.3888	0.3888	0.3888	0.3888
	$v_y(t)$	0.3552	0.3552	0.3553	0.3552	0.3552	0.3552	0.3553
$t=7$ s	$x(t)$	0.0346	0.0346	0.0346	0.0346	0.0346	0.0346	0.0346
	$y(t)$	0.0251	0.0251	0.0251	0.0251	0.0251	0.0251	0.0251
	$v_x(t)$	1.4734	1.4736	1.4734	1.4735	1.4733	1.4733	1.4734
	$v_y(t)$	1.1174	1.1173	1.1175	1.1173	1.1174	1.1174	1.1175

noise realization, confirming that comparable results to the centralized filter can be obtained even in the time-variant case.

For the comparison with other approaches, we take into account [2, 13], which consider LTV systems. We compute the metrics (28), setting the consensus gains to $\kappa = 1000$ for all algorithms. The results are summarized in Table 6. Note that the table shows results including the transitory period while convergence is not yet achieved. Our ODEFTC achieves similar results to the centralized filter, with a high level of agreement between nodes. In contrast, [13] produces a higher estimation error with increased disagreement between nodes. Moreover, this last method produces diverging covariance matrices in this case, with no agreement between nodes, since the system is not locally observable for each agent. This can be seen in Fig. 11, where the same simulation example from Fig. 6 is replicated with this algorithm. While our ODEFTC achieves agreement in both estimate $\hat{\mathbf{x}}_i(t)$ and covariance matrix $\mathbf{P}_i(t)$, here the nodes produce different covariance matrices, which increase in magnitude for nodes that cannot completely observe the system using their local measurement. Therefore, the covariance matrices produced by [13] in this context do not accurately represent the expected error of the state estimates, in contrast to our proposal. For [2], acceptable results are achieved on average, but our algorithm still outperforms this approach, achieving a closer performance to the centralized implementation. In addition, recall the guarantees for the proposal in [2] are given in absence of measurement noise. For bounded noise, a bounded estimation error compared to the estimates from the centralized filter is expected, but the assumption in Kalman filtering is that the noise is Gaussian, and therefore has an unbounded distribution. As shown in Fig. 12, the performance of the filter degrades significantly in the interval where the noise covariances are higher and the measurement noise takes larger values, producing a loss of agreement on the estimates from the nodes. In comparison, recalling Fig. 6 and our theoretical results, our proposal is able to handle stochastic noise without assuming a bounded noise distribution.

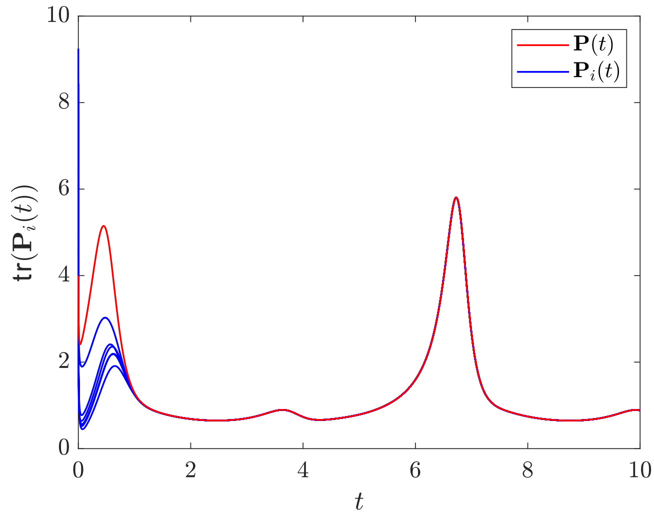

Fig. 7. Convergence of $\mathbf{P}_i(t)$ for all nodes $i \in \mathcal{V}$ to the matrix $\mathbf{P}(t)$ from the centralized filter in the LTV case.

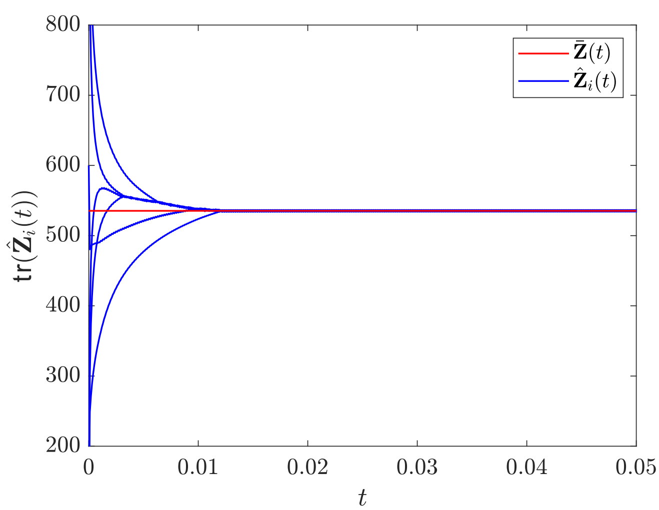

Fig. 8. Convergence of $\hat{\mathbf{Z}}_i(t)$ for all nodes $i \in \mathcal{V}$ to $\bar{\mathbf{Z}}(t)$. The theoretical guaranteed convergence time is $T_{\max} = 1.9584$ s, but agreement is reached before that in practice, in approximately 0.015 s.

Table 6
Comparison of performance for LTV filters, in terms of the error in the state estimates $\mathcal{E}_{\hat{\mathbf{x}}}$ and estimated covariance $\mathcal{E}_{\mathbf{P}}$, as well as the corresponding disagreement between nodes, $\mathcal{D}_{\hat{\mathbf{x}}}$ and $\mathcal{D}_{\mathbf{P}}$.

Filter	$\mathcal{E}_{\hat{\mathbf{x}}}$	$\mathcal{D}_{\hat{\mathbf{x}}}$	$\mathcal{E}_{\mathbf{P}}$	$\mathcal{D}_{\mathbf{P}}$
Centralized filter	0.6918	–	0	–
ODEFTC	0.7141	0.0024	0.0012	8×10^{-4}
[13]	1.0518	0.0322	63.4073	48.8591
[2]	0.8693	0.0743	0.0256	0.0082

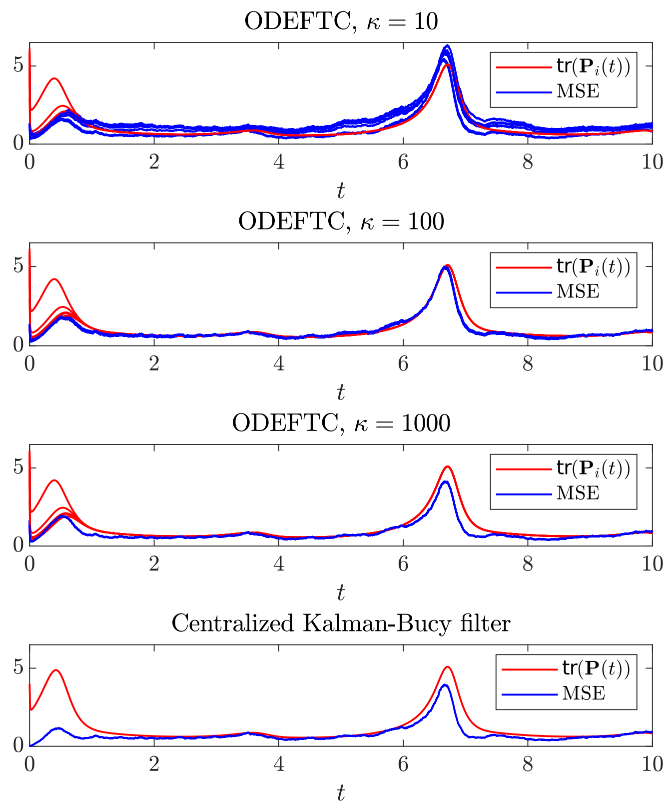

Fig. 9. Mean square error of ODEFTC compared to its expected value, given by $\text{tr}(\mathbf{P}_i(t))$, computed over 100 noise realizations. The values for all nodes are plotted. As the gain κ is increased, better agreement between nodes and improved estimation accuracy is achieved, closer to that of the centralized filter.

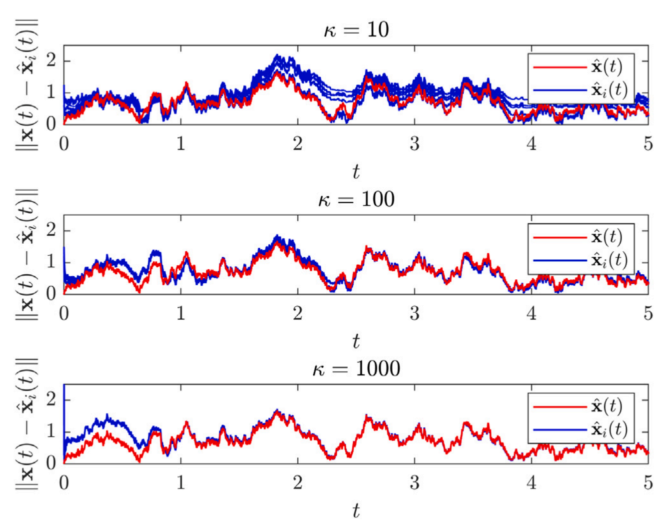

Fig. 10. Estimation error for the nodes in the distributed LTV case, compared to the error of the centralized Kalman-Bucy filter, for different values of consensus gain κ. Increasing κ leads to better agreement and to recovering the solution of the centralized filter.

5. Conclusions

In this work, we have proposed the ODEFTC algorithm to achieve distributed state estimation of continuous-time systems with stochastic noise. The proposal, which is based on fixed-time consensus, can recover the optimal centralized solution, with the covariance of the estimation error tending to that of the centralized Kalman-Bucy filter. Moreover, each node computes an estimated covariance matrix online, which also recovers the value for the centralized implementation, thus serving as means to evaluate the confidence on the estimates. Due to this online computation, our proposal can be applied to time-variant systems, in contrast to previous approaches. We have validated our proposal through several experiments and shown its effectiveness against the approaches from the state of the art.

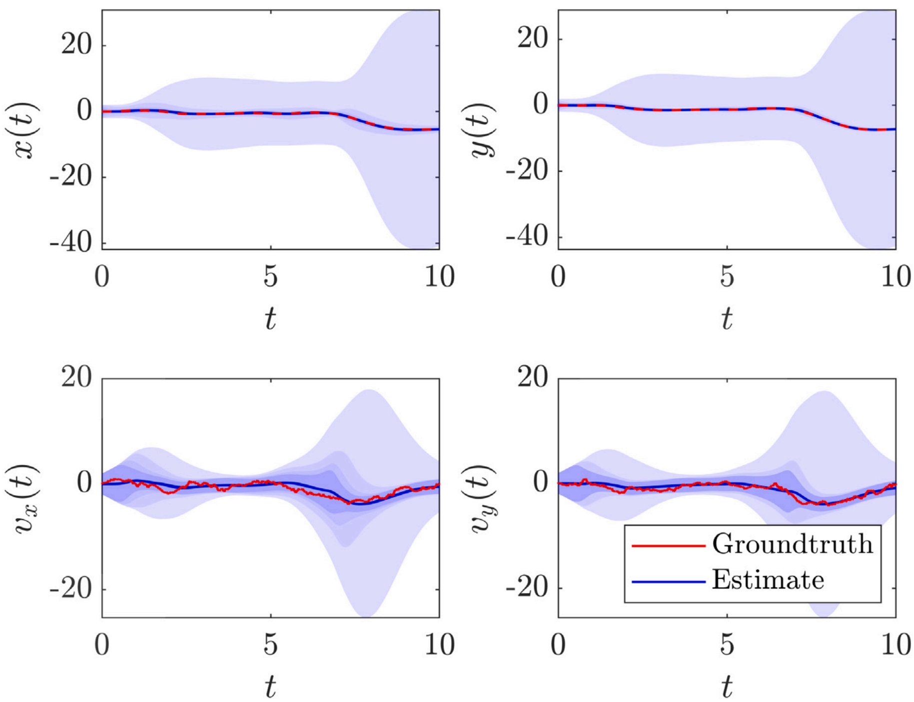

Fig. 11. Estimation example for a LTV system with the algorithm from [13]. The covariance matrices $\mathbf{P}_i(t)$ do not reach agreement and might diverge towards infinity for systems that are not locally observable from each node.

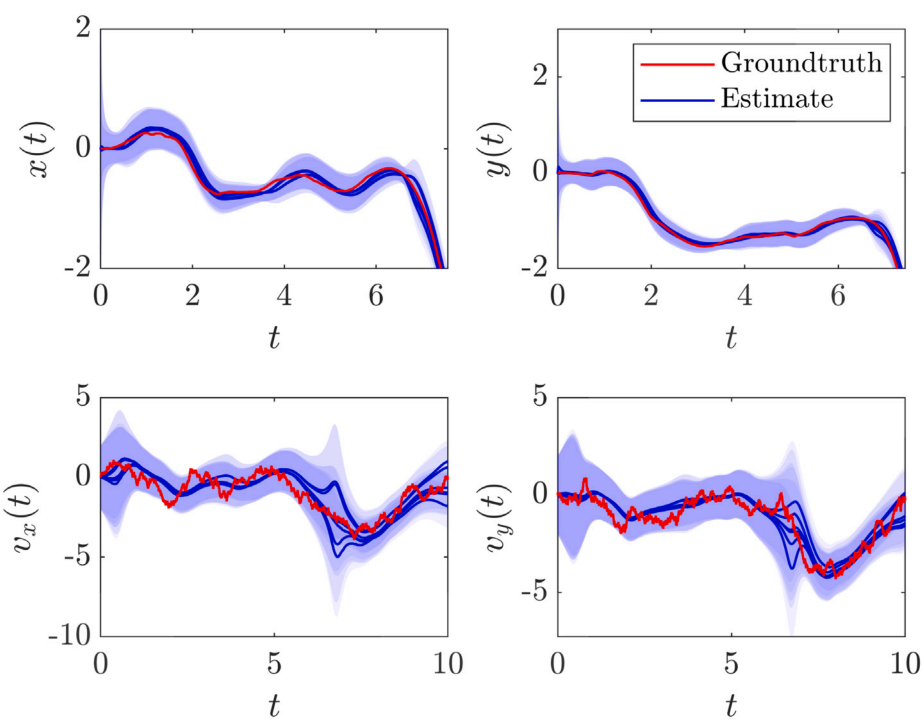

Fig. 12. Estimation example for a LTV system with the algorithm from [2]. The algorithm converges to the solution from the centralized filter only if $\mathbf{v}_i(t) = \mathbf{0}$ but, when larger measurement noise is present, its performance degrades.

CRediT authorship contribution statement

Irene Perez-Salesa: Writing – original draft, Validation, Software, Investigation, Formal analysis, Data curation, Conceptualization. **Rodrigo Aldana-López:** Writing – review & editing, Investigation, Formal analysis, Conceptualization. **Carlos Sagüés:** Writing – review & editing, Supervision, Resources, Project administration, Funding acquisition.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix. Proof of Lemma 3

Take the Lyapunov function

$$V(\mathbf{e}(t)) = \mathbf{e}(t)^\top \mathrm{diag}_{i=1}^N(\mathbf{P}_i(t)^{-1})\mathbf{e}(t) \tag{A.1}$$

recalling that $\mathbf{P}_i(t)$ is uniformly bounded as a result of Theorem 1, positive definite and symmetric. Then, using $\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{P}_i(t)^{-1}) = -\mathbf{P}_i(t)^{-1}\dot{\mathbf{P}}_i(t)\mathbf{P}_i(t)^{-1}$, we have that

$$\begin{aligned}\dot{V}(\mathbf{e}(t)) =& 2\mathbf{e}(t)^\top \mathrm{diag}_{i=1}^N(\mathbf{P}_i(t)^{-1})\dot{\mathbf{e}}(t) + \mathbf{e}(t)^\top \mathrm{diag}_{i=1}^N(\dot{\mathbf{P}}_i(t)^{-1})\mathbf{e}(t)\\ =& 2\mathbf{e}(t)^\top \mathrm{diag}_{i=1}^N(\mathbf{P}_i(t)^{-1})\mathbf{A}^*(\kappa,t)\mathbf{e}(t)\\ &- \mathbf{e}(t)^\top \mathrm{diag}_{i=1}^N(\mathbf{P}_i(t)^{-1}\dot{\mathbf{P}}_i(t)\mathbf{P}_i(t)^{-1})\mathbf{e}(t)\end{aligned} \tag{A.2}$$

Note that

$$\begin{aligned}&\mathrm{diag}_{i=1}^N(\mathbf{P}_i(t)^{-1})\mathbf{A}^*(\kappa,t)\\ &= \mathrm{diag}_{i=1}^N(\mathbf{P}_i(t)^{-1})(\mathrm{diag}_{i=1}^N\left(\mathbf{A}(t) - \mathbf{K}_i(t)\mathbf{C}_i(t)\right) - \kappa\,\mathrm{diag}_{i=1}^N(\mathbf{P}_i(t))\left(\mathbf{Q}_\mathcal{G}\otimes \mathbf{I}_n\right))\\ &= \mathrm{diag}_{i=1}^N(\mathbf{P}_i(t)^{-1}\mathbf{A}(t) - N\mathbf{C}_i(t)^\top\mathbf{R}_i(t)^{-1}\mathbf{C}_i(t)) - \kappa\left(\mathbf{Q}_\mathcal{G}\otimes\mathbf{I}_n\right)\end{aligned} \tag{A.3}$$

and

$$\begin{aligned}&\mathbf{P}_i(t)^{-1}\dot{\mathbf{P}}_i(t)\mathbf{P}_i(t)^{-1}\\ &= \mathbf{P}_i(t)^{-1}(\mathbf{A}(t)\mathbf{P}_i(t) + \mathbf{P}_i(t)\mathbf{A}(t)^\top\\ &\quad + \mathbf{B}(t)\mathbf{W}(t)\mathbf{B}(t)^\top - \mathbf{P}_i(t)\hat{\mathbf{Z}}_i(t)\mathbf{P}_i(t))\mathbf{P}_i(t)^{-1}\\ &= \mathbf{P}_i(t)^{-1}\mathbf{A}(t) + \mathbf{A}(t)^\top\mathbf{P}_i(t)^{-1} + \mathbf{P}_i(t)^{-1}\mathbf{B}(t)\mathbf{W}(t)\mathbf{B}(t)^\top\mathbf{P}_i(t)^{-1} - \hat{\mathbf{Z}}_i(t)\end{aligned} \tag{A.4}$$

Denote the following matrices as

$$\begin{aligned}\mathbf{M}_1(t) &= \mathrm{diag}_{i=1}^N(\mathbf{P}_i(t)^{-1}\mathbf{A}(t) - N\mathbf{C}_i(t)^\top\mathbf{R}_i(t)^{-1}\mathbf{C}_i(t))\\ \mathbf{M}_2(t) &= \mathrm{diag}_{i=1}^N(\mathbf{P}_i(t)^{-1}\mathbf{A}(t) + \mathbf{A}(t)^\top\mathbf{P}_i(t)^{-1}\\ &\quad + \mathbf{P}_i(t)^{-1}\mathbf{B}(t)\mathbf{W}(t)\mathbf{B}(t)^\top\mathbf{P}_i(t)^{-1} - \hat{\mathbf{Z}}_i(t))\end{aligned} \tag{A.5}$$

and note that both matrices are uniformly bounded, given that $\mathbf{A}(t)$, $\mathbf{B}(t)$, $\mathbf{W}(t)$, $\mathbf{C}_i(t)$, $\mathbf{R}_i(t)$ are bounded due to Assumption 2, $\mathbf{P}_i(t) \succ \mathbf{0}$ uniformly bounded according to Theorem 1, and $\hat{\mathbf{Z}}_i(t)$ is bounded due to the convergence to $\bar{\mathbf{Z}}(t)$ shown in Lemma 2, where $\bar{\mathbf{Z}}(t)$ is also bounded as a consequence of Assumption 2.

Considering $\mathbf{M}_1(t)$ and $\mathbf{M}_2(t)$, we have

$$\begin{aligned}\dot{V}(\mathbf{e}(t)) &= 2\mathbf{e}(t)^\top\mathbf{M}_1(t)\mathbf{e}(t) - 2\kappa\mathbf{e}(t)^\top(\mathbf{Q}_\mathcal{G}\otimes\mathbf{I}_n)\mathbf{e}(t) - \mathbf{e}(t)^\top\mathbf{M}_2(t)\mathbf{e}(t)\\ &\le 2\lambda_1\|\mathbf{e}(t)\|^2 + \lambda_2\|\mathbf{e}(t)\|^2 - 2\kappa\lambda_\mathcal{G}\|\mathbf{e}(t)\|^2\end{aligned} \tag{A.6}$$

where $\lambda_\mathcal{G}$ is the smallest non-zero eigenvalue of $\mathbf{Q}_\mathcal{G}$, i.e. the algebraic connectivity of $\mathcal{G}$, and $\lambda_1 = \sup_{t\ge0}\|\mathbf{M}_1(t)\|$, $\lambda_2 = \sup_{t\ge0}\|\mathbf{M}_2(t)\|$. Therefore, if

$$\kappa > \frac{2\lambda_1 + \lambda_2}{2\lambda_\mathcal{G}} = \kappa_0 \tag{A.7}$$

then, all trajectories $\mathbf{e}(t)$ asymptotically converge to the origin.

Data availability

Data will be made available on request.

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