



Frölicher spectral sequence of compact complex manifolds with special Hermitian metrics

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Abstract

In this paper we focus on the interplay between the behaviour of the Frölicher spectral sequence and the existence of special Hermitian metrics on the manifold, such as balanced, SKT or generalized Gauduchon. The study of balanced metrics on nilmanifolds endowed with strongly non-nilpotent complex structures allows us to provide infinite families of compact balanced manifolds with Frölicher spectral sequence not degenerating at the second page. Moreover, this result is extended to non-degeneration at any arbitrary page. Similar results are obtained for the Frölicher spectral sequence of compact generalized Gauduchon manifolds. We also find a compact SKT manifold whose Frölicher spectral sequence does not degenerate at the second page, thus providing a counterexample to a conjecture by Popovici.

Keywords Complex manifold · Frölicher spectral sequence · Balanced metric · Pluriclosed metric

Mathematics Subject Classification 53C55 · 32J27 · 53C15

1 Introduction

Let X be a complex manifold. Frölicher introduced in [15] a spectral sequence $\{E_r^{*,*}(X)\}_{r \geq 1}$ associated to the double complex $(\Omega^{*,*}(X), \partial, \bar{\partial})$, where $\partial + \bar{\partial} = d$. We will refer to it as the Frölicher spectral sequence (FSS for short) of X . This sequence is also known as the Hodge-de Rham spectral sequence, as its first page is given by the Dolbeault cohomology $H_{\bar{\partial}}^{*,*}(X)$ of X and it converges to the de Rham cohomology $H_{dR}^*(X, \mathbb{C})$. Hence, the spaces $E_r^{*,*}(X)$ provide (possibly new) complex invariants of the manifold.

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An important question is to understand the interplay between the behaviour of the Frölicher spectral sequence and the existence of special Hermitian metrics on the manifold. It is well-known that the FSS of any compact Kähler manifold degenerates at the first page. In the context of non-Kähler Hermitian geometry, well-known relevant classes of metrics arise, as for instance *balanced* or *generalized Gauduchon* (in particular, *pluriclosed*) metrics. The aim of this paper is to construct compact complex manifolds endowed with these types of metrics and having Frölicher spectral sequence not degenerating at different pages.

Recall that a Hermitian metric F on a compact complex manifold X with $\dim_{\mathbb{C}} X = n$ is called *balanced* if the form F^{n-1} is closed [29]. The Iwasawa manifold is an example of a compact balanced manifold with FSS degenerating at the second page, with $E_1 \neq E_2$. More generally, any compact quotient X of a nilpotent complex Lie group G by a lattice is balanced (indeed, any left-invariant Hermitian metric on G is balanced) and satisfies $E_2(X) = E_{\infty}(X)$ [39].

The Iwasawa manifold, as well as the compact quotients of nilpotent complex Lie groups, are special concrete examples of *nilmanifolds*. There are some balanced nilmanifolds in the literature with FSS satisfying $E_2 \neq E_{\infty}$. For instance, Cordero, Fernández and Gray obtained in [6] a 6-dimensional complex nilmanifold with FSS not degenerating at the third page (for the existence of balanced metrics on it, see Sect. 3 below). Furthermore, for every $k \geq 2$, Bigalke and Rollenske constructed in [4] a $(4k - 2)$ -dimensional complex nilmanifold with $E_k \neq E_{\infty}$, which turns out to be balanced by a recent result by Sferruzza and Tardini [42].

Each of the nilmanifolds above occurs as a concrete example in a certain complex dimension. In fact, as far as we know, there are no infinite families of manifolds in the literature, in the sense of having infinitely many different complex homotopy types, all living in the same complex dimension with $E_k \neq E_{\infty}$ for some k . The main goal of this paper is to construct such families. The starting point of our construction will be the so-called *strongly non-nilpotent* complex structures on nilmanifolds, recently studied in [25] and classified in [27] in four complex dimensions.

Apart from the important role played by nilmanifolds in non-Kähler Hermitian geometry, there are some other reasons motivating the study of complex nilmanifolds in relation to the FSS. For instance, Kasuya proves in [22] that, in the larger class of solvmanifolds, if one considers those constructed from a semi-direct product of \mathbb{C}^n by a nilpotent Lie group, then the page at which their FSS degenerates cannot be greater than that of the nilmanifold. So, in this sense, nilmanifolds constitute a preferred class to search for compact complex manifolds with non-degenerate FSS. In addition, complex nilmanifolds, and in particular strongly non-nilpotent complex structures in four dimensions, have a remarkable role in relation to the problem of finding manifolds realizing certain generators of the universal ring of cohomological invariants recently studied by Stelzig in [44].

We recall that the FSS of any complex nilmanifold X with $\dim_{\mathbb{C}} X = 3$ is studied in [5]; in particular, it is proved that the existence of a balanced metric on X implies that $E_2(X) = E_{\infty}(X)$. Note that this is indeed a restriction, as there exist complex 3-dimensional nilmanifolds X with $E_2(X) \neq E_3(X)$ (see [5, Theorem 4.1]). Therefore, complex dimension four is the lowest possible dimension for a balanced nilmanifold to have FSS not degenerating at the second page. In Theorem 3.4 we prove that there are infinitely many nilmanifolds satisfying these properties and with different complex (hence, real or rational) homotopy types. Moreover, this result is extended in Theorem 3.8 to non-degeneration at any arbitrary page.

In this paper we also deal with compact generalized Gauduchon manifolds, which were introduced and studied by Fu, Wang and Wu in [16]. We recall that a Hermitian metric F on a compact complex manifold X with $\dim_{\mathbb{C}} X = n$ is called *k-th Gauduchon*, for some

$1 \leq k \leq n - 1$, if it satisfies the condition $\partial\bar{\partial}F^k \wedge F^{n-k-1} = 0$. Observe that the value $k = n - 1$ corresponds to the *standard* (also known as *Gauduchon*) metrics [18]. Note also that any pluriclosed (or SKT) metric is in particular 1-st Gauduchon, as $\partial\bar{\partial}F = 0$.

We prove in Theorem 4.4 that there are infinitely many generalized Gauduchon nilmanifolds with different complex homotopy type whose Frölicher spectral sequence can be arbitrarily non-degenerate. Regarding pluriclosed metrics, Popovici proved in [34, 35] that the existence of a Hermitian metric on X with ‘small torsion’ implies $E_2(X) = E_\infty(X)$, and furthermore, he conjectured that any compact complex manifold X admitting an SKT metric has FSS degenerating at the second page [34, Conjecture 1.3].

In Proposition 4.5 we give a counterexample to this conjecture, based on the complex geometry of compact Lie groups. More concretely, we consider the compact semisimple Lie group $SO(9)$ equipped with a left-invariant complex structure J found by Pittie in [32, 33], which is compatible with a bi-invariant metric g . Recall that any such compact Lie group is Bismut flat and its fundamental form F is dd^c -harmonic, as stated by Alexandrov and Ivanov in [1].

The paper is structured as follows. In Sect. 2 we study the Frölicher spectral sequence of 8-dimensional nilmanifolds endowed with strongly non-nilpotent complex structures. A general study of the existence of balanced metrics on such complex nilmanifolds is given in Sect. 3, from which we arrive at the results in Theorems 3.4 and 3.8 mentioned above. Finally, Sect. 4 is devoted to the FSS of compact pluriclosed and generalized Gauduchon manifolds.

2 Complex nilmanifolds with 1-dimensional center and non-degenerate Frölicher spectral sequence

In this section we study the FSS on 8-dimensional nilmanifolds with one-dimensional center endowed with invariant complex structures. Infinite families of complex nilmanifolds with $E_2 \neq E_\infty$ are obtained in complex dimension 4.

Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$. We recall that the Frölicher spectral sequence of X is the spectral sequence associated to the double complex $(\Omega^{*,*}(X), \partial, \bar{\partial})$, where $\partial + \bar{\partial} = d$ is the usual decomposition of the exterior differential d on X . This spectral sequence was first introduced in [15], in terms of a certain filtration, and it can be described as a collection of complexes

$$\dots \xrightarrow{d_r} E_r^{p-r, q+r-1}(X) \xrightarrow{d_r} E_r^{p, q}(X) \xrightarrow{d_r} E_r^{p+r, q-r+1}(X) \xrightarrow{d_r} \dots$$

that are canonically associated with the complex structure of X , for every $r \geq 1$. The 1 -st page consists of the Dolbeault cohomology groups of X , i.e. $E_1^{p, q}(X) = H_{\bar{\partial}}^{p, q}(X)$, while the differentials d_1 are induced by ∂ as $d_1([\alpha]) = [\partial\alpha]$, for every Dolbeault class $[\alpha] \in H_{\bar{\partial}}^{p, q}(X)$. For an arbitrary r , the differentials d_r on the r -th page are of type $(r, -r+1)$ but they are still induced by ∂ acting on a certain $(p+r-1, q-r+1)$ -form associated to any representative of every class in $E_r^{p, q}(X)$ (see the description below). It turns out that $d_r \circ d_r = 0$, and the $(r+1)$ -th page is induced from the previous r -th page as the kernel of d_r over the image of the incoming differential d_r .

There exists a positive integer r from which all the differentials vanish identically, namely, $d_s = 0$ for all $s \geq r$. This is equivalent to having $E_r^{p, q}(X) = E_{r+k}^{p, q}(X)$ for every $k \geq 1$ and any $0 \leq p, q \leq n$. This space $E_r^{p, q}(X)$ is denoted by $E_\infty^{p, q}(X)$ and the FSS is said to be *degenerated at the r -th page*, then writing $E_r(X) = E_\infty(X)$.

The Frölicher spectral sequence gives a link between the complex structure of X and its differential structure. Indeed, it converges to the de Rham cohomology of X in the sense that there are isomorphisms $H_{dR}^k(X, \mathbb{C}) \simeq \bigoplus_{p+q=k} E_{\infty}^{p,q}(X)$, for every $k \in \{0, \dots, 2n\}$.

It is worthy to recall that a Hodge theory is introduced in [36] through the construction of elliptic pseudo-differential operators, associated with any given Hermitian metric on X , whose kernels are isomorphic to the spaces $E_r^{p,q}(X)$ in every bidegree (p, q) . This extended to any arbitrary positive integer r a previous construction in [34] for $r = 2$. We also remind that Serre duality for $E_r^{p,q}$ is proved by Stelzig in [43] and by Milivojević in [30]. This duality is also obtained as a consequence of Hodge theory (see [36] for more details).

The following general description of the terms in the Frölicher spectral sequence was given in [7] and it will be useful for our purposes. For every $r \geq 1$ and any $0 \leq p, q \leq n$, the space $E_r^{p,q}(X)$ is isomorphic to the quotient \mathbb{C} -vector space

$$E_r^{p,q}(X) = \frac{\mathcal{X}_r^{p,q}(X)}{\mathcal{Y}_r^{p,q}(X)}, \tag{1}$$

where

$$\mathcal{X}_1^{p,q}(X) = \{\alpha_{p,q} \in \Omega^{p,q}(X) \mid \bar{\partial}\alpha_{p,q} = 0\}, \quad \mathcal{Y}_1^{p,q}(X) = \bar{\partial}(\Omega^{p,q-1}(X)),$$

and for every $r \geq 2$

$$\begin{aligned} \mathcal{X}_r^{p,q}(X) = \{ & \alpha_{p,q} \in \Omega^{p,q}(X) \mid \bar{\partial}\alpha_{p,q} = 0, \text{ and there exist } r-1 \text{ forms } \alpha_{p+1,q-1}, \dots \\ & \dots, \alpha_{p+r-2,q-r+2}, \alpha_{p+r-1,q-r+1} \text{ satisfying} \tag{2} \\ 0 = & \partial\alpha_{p,q} + \bar{\partial}\alpha_{p+1,q-1} = \dots = \partial\alpha_{p+r-2,q-r+2} + \bar{\partial}\alpha_{p+r-1,q-r+1}\}, \end{aligned}$$

and

$$\begin{aligned} \mathcal{Y}_r^{p,q}(X) = \{ & \bar{\partial}\beta_{p,q-1} + \partial\beta_{p-1,q} \in \Omega^{p,q}(X) \mid \text{there exist } r-2 \text{ forms } \beta_{p-2,q+1}, \dots \\ & \dots, \beta_{p-r+2,q+r-3}, \beta_{p-r+1,q+r-2} \text{ satisfying} \tag{3} \\ 0 = & \bar{\partial}\beta_{p-1,q} + \partial\beta_{p-2,q+1} = \dots = \bar{\partial}\beta_{p-r+2,q+r-3} + \partial\beta_{p-r+1,q+r-2} \\ & = \bar{\partial}\beta_{p-r+1,q+r-2}\}. \end{aligned}$$

Furthermore, the differentials $d_r : E_r^{p,q}(X) \longrightarrow E_r^{p+r,q-r+1}(X)$ are explicitly given by

$$d_r([\alpha_{p,q}]) = [\partial\alpha_{p+r-1,q-r+1}], \tag{4}$$

for any $[\alpha_{p,q}] \in E_r^{p,q}(X)$.

Let G be a simply connected real Lie group endowed with a left-invariant complex structure J , and suppose that G admits a discrete subgroup Γ so that the quotient space $\Gamma \backslash G$ is compact. Let us denote by X the latter manifold endowed with the (naturally induced) complex structure J .

Consider \mathfrak{g} , the Lie algebra of G , endowed with the (linear integrable) complex structure J . Then, we can define the corresponding sequence $E_r(\mathfrak{g}, J)$ associated to the pair (\mathfrak{g}, J) . The description (1) together with (2) and (3), and so the homomorphisms (4), apply to this sequence.

Proposition 2.1 *Let $X = (\Gamma \backslash G, J)$ be a compact quotient of a simply connected Lie group G by a lattice Γ , endowed with a complex structure naturally induced by a left-invariant complex structure J on G . Let \mathfrak{g} be the Lie algebra of G . Fix an integer $r \geq 1$, and suppose that the homomorphism $d_r : E_r^{p,q}(\mathfrak{g}, J) \longrightarrow E_r^{p+r,q-r+1}(\mathfrak{g}, J)$ is non-zero for some p, q . Then, the Frölicher spectral sequence of X does not degenerate at the r -th page.*

Proof Let $[\alpha_{p,q}]$ be an element in $E_r^{p,q}(\mathfrak{g}, J)$ such that $d_r([\alpha_{p,q}]) \neq 0$ in $E_r^{p+r, q-r+1}(\mathfrak{g}, J)$. Since $\alpha_{p,q} \in \mathcal{X}_r^{p,q}(\mathfrak{g}, J)$ and there is a natural inclusion $\iota: \mathcal{X}_r^{p,q}(\mathfrak{g}, J) \hookrightarrow \mathcal{X}_r^{p,q}(X)$, the form $\alpha_{p,q}$ defines an element $[\alpha_{p,q}]$ in $E_r^{p,q}(X)$. Notice that we can choose the $r - 1$ forms $\alpha_{p+1, q-1}, \dots, \alpha_{p+r-2, q-r+2}, \alpha_{p+r-1, q-r+1}$ in (2) to be left-invariant.

Suppose that the FSS of X degenerates at the r -th page. Then, $d_r([\alpha_{p,q}]) = 0$ in $E_r^{p+r, q-r+1}(X)$. By (2) and (4), together with (1), this means that $\partial\alpha_{p+r-1, q-r+1} \in \mathcal{Y}_r^{p+r, q-r+1}(X)$. From the description (3), there exist r forms

$$\beta_{p+r, q-r}, \beta_{p+r-1, q-r+1}, \beta_{p+r-2, q-r+2}, \dots, \beta_{p+2, q-2}, \beta_{p+1, q-1}$$

on the complex manifold X satisfying

$$\partial\alpha_{p+r-1, q-r+1} = \bar{\partial}\beta_{p+r, q-r} + \partial\beta_{p+r-1, q-r+1},$$

and

$$\begin{aligned} \bar{\partial}\beta_{p+r-1, q-r+1} + \partial\beta_{p+r-2, q-r+2} &= 0, \dots \\ \bar{\partial}\beta_{p+2, q-2} + \partial\beta_{p+1, q-1} &= 0, \quad \bar{\partial}\beta_{p+1, q-1} = 0. \end{aligned}$$

As the Lie group G has a lattice, G is unimodular. In particular, G admits a bi-invariant volume form, so we can apply the well-known symmetrization process (see for instance [5] and the references therein for details). Given any form β on X , we denote by $\tilde{\beta}$ the left-invariant form on G given by the symmetrization of β . Recall that J being left-invariant, the bidegree of the forms is preserved, and one has $\widetilde{\partial\beta} = \partial\tilde{\beta}$ and $\widetilde{\bar{\partial}\beta} = \bar{\partial}\tilde{\beta}$.

Note that $\partial\alpha_{p+r-1, q-r+1}$ coincides with its symmetrization because it is left-invariant. Therefore, from the equalities above, we get r left-invariant forms

$$\tilde{\beta}_{p+r, q-r}, \tilde{\beta}_{p+r-1, q-r+1}, \tilde{\beta}_{p+r-2, q-r+2}, \dots, \tilde{\beta}_{p+2, q-2}, \tilde{\beta}_{p+1, q-1}$$

satisfying

$$\partial\alpha_{p+r-1, q-r+1} = \bar{\partial}\tilde{\beta}_{p+r, q-r} + \partial\tilde{\beta}_{p+r-1, q-r+1},$$

and

$$\begin{aligned} \bar{\partial}\tilde{\beta}_{p+r-1, q-r+1} + \partial\tilde{\beta}_{p+r-2, q-r+2} &= 0, \dots \\ \bar{\partial}\tilde{\beta}_{p+2, q-2} + \partial\tilde{\beta}_{p+1, q-1} &= 0, \quad \bar{\partial}\tilde{\beta}_{p+1, q-1} = 0. \end{aligned}$$

But this implies $\partial\alpha_{p+r-1, q-r+1} \in \mathcal{Y}_r^{p+r, q-r+1}(\mathfrak{g}, J)$, which is a contradiction to the hypothesis that $d_r([\alpha_{p,q}])$ is non-zero in $E_r^{p+r, q-r+1}(\mathfrak{g}, J)$.

In conclusion, $d_r([\alpha_{p,q}]) \neq 0$ in $E_r^{p+r, q-r+1}(X)$, and the FSS of X does not degenerate at the r -th page. □

From now on in this section we will consider that G is a nilpotent Lie group, and thus $X = (\Gamma \backslash G, J)$ is a complex nilmanifold. It should be noticed that if the natural map $\iota: \Lambda^{*,*}(\mathfrak{g}, J) \hookrightarrow \Omega^{*,*}(X)$ induces an isomorphism in Dolbeault cohomology, then one has also an isomorphism $E_r^{p,q}(X) \cong E_r^{p,q}(\mathfrak{g}, J)$ for every $k \geq 1$ and any $0 \leq p, q \leq n$ (see [43] for general results about E_r -isomorphisms of bounded double complexes). This is indeed the case in complex dimension three [14, 37], or in arbitrary dimension for any nilpotent complex structure [38].

However, the complex structures J on the nilmanifolds X that we will study in this section are (strongly) non-nilpotent, so Proposition 2.1 will be applied to derive the non-degeneration of the FSS of X by means of the non-degeneration of the sequence $E_r(\mathfrak{g}, J)$.

Observe that the proof of Proposition 2.1 implies that the natural inclusion ι induces an injection $E_r^{p,q}(\mathfrak{g}, J) \hookrightarrow E_r^{p,q}(X)$ that commutes with the differentials d_r .

Let \mathfrak{g} be a nilpotent Lie algebra (NLA for short) endowed with a complex structure J . Since \mathfrak{g} is nilpotent, its center is always non-trivial. If we now assume that the center has dimension 1, then the only J -invariant subspace in it is the trivial one. Complex structures having the latter property are known as *strongly non-nilpotent* (SnN for short), and they are studied in [25]. We recall that, up to real dimension 8, the conditions “ J being SnN” and “ \mathfrak{g} having 1-dimensional center” are equivalent. Furthermore, it is proved in [27, Proposition 3.1] that when \mathfrak{g} has dimension 8 and J is of the previous type, the dimension of the space $E_1^{0,1}(\mathfrak{g}, J)$ is either 2 or 3. This provides a partition of the space of SnN complex structures J into two families.

Definition 2.2 Let \mathfrak{g} be an 8-dimensional NLA endowed with an SnN complex structure J . We say that J belongs to *Family I* (resp. *Family II*) if $E_1^{0,1}(\mathfrak{g}, J)$ has maximal dimension (resp. minimal dimension).

The following classification of SnN complex structures is available in [27]:

Proposition 2.3 [27, Theorem 3.3] *Let J be a complex structure on an 8-dimensional NLA \mathfrak{g} with one dimensional center. Then, there exists a basis of $(1, 0)$ -forms $\{\omega^k\}_{k=1}^4$ in terms of which the complex structure equations of (\mathfrak{g}, J) are one (and only one) of the following:*

(i) *if J belongs to **Family I**, then*

$$\begin{cases} d\omega^1 = 0, \\ d\omega^2 = \varepsilon \omega^{1\bar{1}}, \\ d\omega^3 = \omega^{14} + \omega^{1\bar{4}} + a \omega^{2\bar{1}} + i \delta \varepsilon b \omega^{1\bar{2}}, \\ d\omega^4 = i v \omega^{1\bar{1}} + b \omega^{2\bar{2}} + i \delta (\omega^{1\bar{3}} - \omega^{3\bar{1}}), \end{cases} \tag{5}$$

where $\delta = \pm 1$, $(a, b) \in \mathbb{R}^2 - \{(0, 0)\}$ with $a \geq 0$, and (ε, v, a, b) being one of the following: $(0, 0, 0, 1)$, $(0, 0, 1, 0)$, $(0, 0, 1, 1)$, $(0, 1, 0, \pm 1)$, $(0, 1, 1, b)$, $(1, 0, 0, 1)$, $(1, 0, 1, |b|)$ or $(1, 1, a, b)$.

(ii) *if J belongs to **Family II**, then*

$$\begin{cases} d\omega^1 = 0, \\ d\omega^2 = \omega^{14} + \omega^{1\bar{4}}, \\ d\omega^3 = a \omega^{1\bar{1}} + \varepsilon (\omega^{12} + \omega^{1\bar{2}} - \omega^{2\bar{1}}) + i \mu (\omega^{24} + \omega^{2\bar{4}}), \\ d\omega^4 = i v \omega^{1\bar{1}} - \mu \omega^{2\bar{2}} + i b (\omega^{1\bar{2}} - \omega^{2\bar{1}}) + i (\omega^{1\bar{3}} - \omega^{3\bar{1}}), \end{cases} \tag{6}$$

where $a, b \in \mathbb{R}$, and the tuple $(\varepsilon, \mu, v, a, b)$ takes the following values: $(1, 1, 0, a, b)$, $(1, 0, 1, a, b)$, $(1, 0, 0, 0, b)$, $(1, 0, 0, 1, b)$, $(0, 1, 0, 0, 0)$ or $(0, 1, 0, 1, 0)$.

Our first goal is to provide (the first known) examples of nilmanifolds endowed with SnN complex structures such that the differential $d_2 \neq 0$. We recall that up to real dimension 6 all such complex nilmanifolds have FSS degenerating at the second page (see [5, Theorem 4.1] for the NLAs \mathfrak{h}_{19}^- and \mathfrak{h}_{26}^+).

Next we will study the spaces $E_r^{0,2}(\mathfrak{g}, J)$ for every SnN J on any 8-dimensional NLA \mathfrak{g} . One reason for focusing on the bidegree $(p, q) = (0, 2)$ is motivated by the recent paper by Stelzig [44], where this bidegree plays an important role in complex dimension 4 in

relation to the problem of finding manifolds realizing certain generators of the universal ring of cohomological invariants in degree 4 (see [44, Problem 11.2]).

Moreover, note that one can take advantage of the explicit description (1–4) when particularized to any bidegree (p, q) with $p = 0$, since then $\mathcal{Y}_r^{0,q} = \{\bar{\partial}\beta_{0,q-1}\}$ for every $r \geq 1$. In particular, for $q = 2$ the corresponding terms for the pair (\mathfrak{g}, J) can be described as follows:

$$E_r^{0,2}(\mathfrak{g}, J) = \frac{\mathcal{X}_r^{0,2}(\mathfrak{g}, J)}{\bar{\partial}(\Lambda^{0,1}(\mathfrak{g}, J))}, \tag{7}$$

where $\mathcal{X}_1^{0,2}(\mathfrak{g}, J) = \{\alpha_{0,2} \in \Lambda^{0,2}(\mathfrak{g}, J) \mid \bar{\partial}\alpha_{0,2} = 0\}$, and

$$\begin{aligned} \mathcal{X}_2^{0,2}(\mathfrak{g}, J) &= \{\alpha_{0,2} \in \mathcal{X}_1^{0,2}(\mathfrak{g}, J) \mid \partial\alpha_{0,2} + \bar{\partial}\alpha_{1,1} = 0 \text{ for some } \alpha_{1,1}\}, \\ \mathcal{X}_3^{0,2}(\mathfrak{g}, J) &= \{\alpha_{0,2} \in \mathcal{X}_1^{0,2}(\mathfrak{g}, J) \mid \partial\alpha_{0,2} + \bar{\partial}\alpha_{1,1} = \partial\alpha_{1,1} + \bar{\partial}\alpha_{2,0} = 0 \\ &\quad \text{for some } \alpha_{1,1} \text{ and } \alpha_{2,0}\}, \end{aligned} \tag{8}$$

whereas $\mathcal{X}_r^{0,2}(\mathfrak{g}, J) = \mathcal{X}_4^{0,2}(\mathfrak{g}, J)$ for any $r \geq 4$, with

$$\mathcal{X}_4^{0,2}(\mathfrak{g}, J) = \{\alpha_{0,2} \in \mathcal{X}_1^{0,2}(\mathfrak{g}, J) \mid \partial\alpha_{0,2} + \bar{\partial}\alpha_{1,1} = \partial\alpha_{1,1} + \bar{\partial}\alpha_{2,0} = \partial\alpha_{2,0} = 0 \text{ for some } \alpha_{1,1} \text{ and } \alpha_{2,0}\}.$$

Moreover, $d_r \equiv 0$ for $r \geq 4$, and for $1 \leq r \leq 3$ we have that $d_r : E_r^{0,2}(\mathfrak{g}, J) \rightarrow E_r^{r,3-r}(\mathfrak{g}, J)$ is defined by

$$d_r([\alpha_{0,2}]) = [\partial\alpha_{r-1,3-r}],$$

for any $[\alpha_{0,2}] \in E_r^{0,2}(\mathfrak{g}, J)$. Note that $E_{r+1}^{0,2}(\mathfrak{g}, J) = \ker d_r$ because the incoming d_r is identically zero by bidegree reasons.

In the following result we study in detail the terms $E_r^{0,2}$ in the FSS of an interesting subclass of complex structures in the Family I.

Proposition 2.4 *Let J be an SnN complex structure in Family I defined by $\varepsilon = 1$ and $ab \neq 0$. Let $\Theta(\delta, v, a, b) = ((a - b)^2 - 2\delta vb)((a + b)^2 - 2\delta vb)$. Then,*

$$E_2^{0,2}(\mathfrak{g}, J) \neq E_3^{0,2}(\mathfrak{g}, J) \iff \Theta(\delta, v, a, b) \neq 0.$$

Moreover, in this case we have:

$$\begin{aligned} E_1^{0,2}(\mathfrak{g}, J) &= E_2^{0,2}(\mathfrak{g}, J) = \langle [\omega^{\bar{1}2}], [\omega^{\bar{1}3}], [\omega^{\bar{2}4}], [\omega^{\bar{3}4}] \rangle, \\ E_r^{0,2}(\mathfrak{g}, J) &= \langle [\omega^{\bar{1}2}], [\omega^{\bar{1}3}], [\omega^{\bar{2}4}] \rangle, \text{ for } r \geq 3. \end{aligned}$$

Proof We use the description given in (7) and (8) to compute the terms $E_r^{0,2}(\mathfrak{g}, J)$. From the complex structure equations (5) it follows that the space of $\bar{\partial}$ -closed $(0, 2)$ -forms is given by $\mathcal{X}_1^{0,2}(\mathfrak{g}, J) = \langle \omega^{\bar{1}2}, \omega^{\bar{1}3}, \omega^{\bar{1}4}, \omega^{\bar{2}4}, \omega^{\bar{3}4} \rangle$. Since $\bar{\partial}(\Lambda^{0,1}(\mathfrak{g}, J)) = \langle \omega^{\bar{1}4} \rangle$, we obtain the space $E_1^{0,2}(\mathfrak{g}, J)$ given in the statement above.

Note that $\omega^{\bar{1}2}$ is a d -closed form, so $[\omega^{\bar{1}2}] \in E_r^{0,2}(\mathfrak{g}, J)$ for every $r \geq 2$. Moreover, the classes $[\omega^{\bar{1}3}]$ and $[\omega^{\bar{2}4}]$ also belong to $E_r^{0,2}(\mathfrak{g}, J)$ for every $r \geq 2$, due to the following relations:

$$\begin{aligned} \partial\omega^{\bar{1}3} + \bar{\partial}(a\omega^{\bar{2}2}) &= 0, & \partial\omega^{\bar{2}4} + \bar{\partial}(2i v \omega^{\bar{2}2} + \omega^{\bar{2}4} - \omega^{\bar{4}2}) &= 0, \\ \partial(a\omega^{\bar{2}2}) + \bar{\partial}(-\omega^{\bar{1}3}) &= 0, & \partial(2i v \omega^{\bar{2}2} + \omega^{\bar{2}4} - \omega^{\bar{4}2}) + \bar{\partial}(\omega^{\bar{2}4}) &= 0, \\ & & \partial(-\omega^{\bar{1}3}) &= 0; & \partial(\omega^{\bar{2}4}) &= 0. \end{aligned}$$

Let us now focus on the $(0, 2)$ -form $\omega^{\bar{3}4}$. Since $ab \neq 0$, one can verify that $\partial\omega^{\bar{3}4} + \bar{\partial}\gamma = 0$ for the $(1, 1)$ -form γ given by

$$\gamma = \frac{i\delta(-a^2 + b^2 + 2\delta vb)}{2b}\omega^{2\bar{3}} - \frac{a^2 - b^2 + 2\delta vb}{2ab}(b\omega^{3\bar{2}} + a\omega^{4\bar{1}}) - \frac{i\delta(a^2 + b^2 - 2\delta vb)}{2ab}\omega^{3\bar{4}} - \omega^{4\bar{3}}.$$

Therefore, $[\omega^{\bar{3}4}] \in E_2^{0,2}(\mathfrak{g}, J)$, and $E_1^{0,2}(\mathfrak{g}, J) = E_2^{0,2}(\mathfrak{g}, J)$.

Notice that any $(1, 1)$ -form $\alpha_{1,1}$ satisfying the condition $\partial\omega^{\bar{3}4} + \bar{\partial}\alpha_{1,1} = 0$ can be written as $\alpha_{1,1} = \gamma + \sigma$, with $\sigma \in \mathcal{X}_1^{1,1}(\mathfrak{g}, J)$, i.e. σ is any $\bar{\partial}$ -closed $(1,1)$ -form. Since we have to study the solutions of the equation $\partial\alpha_{1,1} + \bar{\partial}\alpha_{2,0} = 0$, for some form $\alpha_{2,0}$ of bidegree $(2, 0)$, it is enough to consider the $(1,1)$ -forms σ in the quotient space

$$\mathcal{V}(\mathfrak{g}, J) = \mathcal{X}_1^{1,1}(\mathfrak{g}, J) / \{d\text{-closed } (1, 1)\text{-forms}\}.$$

Using (5) it is not difficult to see that $\mathcal{V}(\mathfrak{g}, J) = \langle \omega^{1\bar{3}} + \omega^{2\bar{4}}, \omega^{1\bar{4}} \rangle$, so we can express σ as

$$\sigma = c_1(\omega^{1\bar{3}} + \omega^{2\bar{4}}) + c_2\omega^{1\bar{4}}, \quad \text{where } c_1, c_2 \in \mathbb{C}.$$

Then, we get

$$\begin{aligned} \partial(\gamma + \sigma) = & i(v - \delta b)c_1\omega^{12\bar{1}} + \frac{i\delta a(a^2 - b^2 - 2\delta vb) + 2b^2c_2}{2b}\omega^{12\bar{2}} - i\delta c_1\omega^{12\bar{3}} \\ & + \frac{a^2(b + \delta v) - b(b - 2\delta v)(b - \delta v) - 2i\delta abc_2}{2ab}\omega^{13\bar{1}} - \frac{a^2 + b^2 - 2\delta vb}{2ab}\omega^{13\bar{3}} \\ & + c_1\omega^{14\bar{1}} + \frac{a^2 + b^2 - 2\delta vb}{2a}\omega^{14\bar{2}} - \frac{i\delta(a^2 + b^2 - 2\delta vb)}{2ab}\omega^{14\bar{4}} \\ & - i\delta c_1\omega^{23\bar{1}} + \frac{i\delta(a^2 + b^2 - 2\delta vb)}{2a}\omega^{23\bar{2}} - \frac{i\delta(a^2 + b^2 - 2\delta vb)}{2b}\omega^{24\bar{1}}. \end{aligned}$$

Now, observe that any $(2, 0)$ -form $\alpha_{2,0}$ can be written as $\alpha_{2,0} = \sum_{1 \leq r < s \leq 4} \lambda_{rs}\omega^{rs}$, with $\lambda_{rs} \in \mathbb{C}$. A direct calculation shows

$$\begin{aligned} \bar{\partial}\alpha_{2,0} = & (-a\lambda_{13} + i v \lambda_{24})\omega^{12\bar{1}} - b(\lambda_{14} - i \delta \lambda_{23})\omega^{12\bar{2}} + i \delta \lambda_{24}\omega^{12\bar{3}} + \lambda_{23}\omega^{12\bar{4}} \\ & + i(\delta \lambda_{14} + i \lambda_{23} + v \lambda_{34})\omega^{13\bar{1}} + i \delta \lambda_{34}\omega^{13\bar{3}} - \lambda_{24}\omega^{14\bar{1}} - i \delta b \lambda_{34}\omega^{14\bar{2}} \\ & - \lambda_{34}\omega^{14\bar{4}} + i \delta \lambda_{24}\omega^{23\bar{1}} + b \lambda_{34}\omega^{23\bar{2}} - a \lambda_{34}\omega^{24\bar{1}}. \end{aligned}$$

We have to study the equation $0 = \partial(\gamma + \sigma) - \bar{\partial}\alpha_{2,0} = \sum A_{rs\bar{k}}\omega^{rs\bar{k}}$, where the coefficients $A_{rs\bar{k}}$ can be directly obtained from the previous expressions. In particular, we will obtain some relations among c_1, c_2, λ_{rs} and the parameters defining the complex structure J that ensure $A_{rs\bar{k}} = 0$ for all r, s, k . First, observe that from $A_{12\bar{4}} = A_{14\bar{1}} = A_{14\bar{4}} = 0$ one gets

$$\lambda_{23} = 0, \quad \lambda_{24} = -c_1, \quad \lambda_{34} = \frac{i\delta(a^2 + b^2 - 2\delta vb)}{2ab}.$$

These values force most of the coefficients $A_{rs\bar{k}}$ to vanish, with the exception of $A_{12\bar{1}}, A_{12\bar{2}}$ and $A_{13\bar{1}}$. From the vanishing of the first and second aforementioned coefficients, one obtains

$$\lambda_{13} = \frac{ic_1(\delta b - 2v)}{a}, \quad \lambda_{14} = -\frac{i\delta a(a^2 - b^2 - 2\delta vb) + 2b^2c_2}{2b^2}.$$

This immediately gives that $A_{1\bar{3}\bar{1}} = 0$ if and only if

$$(a - b)^2 - 2\delta vb)(a + b)^2 - 2\delta vb) = 0.$$

Consequently, if $\Theta(\delta, v, a, b) \neq 0$ then $[\omega^{\bar{3}\bar{4}}] \in E_2^{0,2}(\mathfrak{g}, J)$, but $[\omega^{\bar{3}\bar{4}}] \notin E_3^{0,2}(\mathfrak{g}, J)$. However, if $\Theta(\delta, v, a, b) = 0$ then $[\omega^{\bar{3}\bar{4}}] \in E_r^{0,2}(\mathfrak{g}, J)$ for every $r \geq 3$, because $\lambda_{23} = 0$ implies that $\partial\alpha_{2,0} = 0$. This completes the proof of the proposition. \square

In the following result an interesting subclass of structures in the Family II is studied.

Proposition 2.5 *Let J be an SnN complex structure in Family II defined by $\varepsilon = \mu = 1$ (hence $v = 0$). Then, $E_2^{0,2}(\mathfrak{g}, J) \neq E_3^{0,2}(\mathfrak{g}, J)$. Moreover,*

$$\begin{aligned} E_1^{0,2}(\mathfrak{g}, J) &= E_2^{0,2}(\mathfrak{g}, J) = \langle [i\omega^{\bar{1}\bar{2}} - \omega^{\bar{2}\bar{4}}], [\omega^{\bar{1}\bar{3}} - i\omega^{\bar{3}\bar{4}}] \rangle, \\ E_r^{0,2}(\mathfrak{g}, J) &= \langle [i\omega^{\bar{1}\bar{2}} - \omega^{\bar{2}\bar{4}}] \rangle, \text{ for } r \geq 3. \end{aligned}$$

Proof We use again the description given in (7) and (8) to compute the desired terms of the FSS. From the complex structure equations (6) it follows that the space of $\bar{\partial}$ -closed $(0, 2)$ -forms is given by $\mathcal{X}_1^{0,2}(\mathfrak{g}, J) = \langle \omega^{\bar{1}\bar{2}}, \omega^{\bar{1}\bar{4}}, \omega^{\bar{2}\bar{4}}, \omega^{\bar{1}\bar{3}} - i\omega^{\bar{3}\bar{4}} \rangle$. Since $\bar{\partial}(\Lambda^{0,1}(\mathfrak{g}, J)) = \langle \omega^{\bar{1}\bar{4}}, \omega^{\bar{1}\bar{2}} - i\omega^{\bar{2}\bar{4}} \rangle$, we obtain the space $E_1^{0,2}(\mathfrak{g}, J)$ in the statement.

Now, since

$$\begin{aligned} \partial(i\omega^{\bar{1}\bar{2}} - \omega^{\bar{2}\bar{4}}) + \bar{\partial}\omega^{\bar{4}\bar{2}} &= 0, \\ \partial\omega^{\bar{4}\bar{2}} &= 0, \end{aligned}$$

we conclude that the $(0, 2)$ -form $i\omega^{\bar{1}\bar{2}} - \omega^{\bar{2}\bar{4}}$ defines a non-zero class in $E_r^{0,2}(\mathfrak{g}, J)$ for every r .

Let us now consider the form $\omega^{\bar{1}\bar{3}} - i\omega^{\bar{3}\bar{4}}$. One can check that $\partial(\omega^{\bar{1}\bar{3}} - i\omega^{\bar{3}\bar{4}}) + \bar{\partial}\gamma = 0$, where

$$\gamma = \frac{3ia}{2}\omega^{\bar{1}\bar{2}} + \frac{1}{2}\omega^{\bar{1}\bar{3}} + \frac{1}{2}\omega^{\bar{3}\bar{1}} + \frac{i}{2}\omega^{\bar{3}\bar{4}} + i\omega^{\bar{4}\bar{3}},$$

therefore $E_2^{0,2}(\mathfrak{g}, J) = E_1^{0,2}(\mathfrak{g}, J)$. To finish the proof we will show that the form $\omega^{\bar{1}\bar{3}} - i\omega^{\bar{3}\bar{4}}$ does not belong to the space $\mathcal{X}_3^{0,2}(\mathfrak{g}, J)$. As in the proof of Proposition 2.4, this is equivalent to proving that $\partial(\gamma + \sigma) \notin \bar{\partial}(\Lambda^{2,0}(\mathfrak{g}, J))$ for every $(1, 1)$ -form σ representing a class in the quotient space

$$\mathcal{V}(\mathfrak{g}, J) = \mathcal{X}_1^{1,1}(\mathfrak{g}, J) / \{d\text{-closed } (1, 1)\text{-forms}\}.$$

In other words, next we will prove that $\partial(\gamma + \sigma) + \bar{\partial}\alpha_{2,0} \neq 0$ for every such σ and every form $\alpha_{2,0}$ of bidegree $(2, 0)$.

Using (6) one can see that $\mathcal{V}(\mathfrak{g}, J) = \langle \omega^{\bar{1}\bar{4}}, \omega^{\bar{2}\bar{4}} \rangle$, so we can express σ as $\sigma = c_1\omega^{\bar{1}\bar{4}} + c_2\omega^{\bar{2}\bar{4}}$, where $c_1, c_2 \in \mathbb{C}$. A direct computation shows that

$$\begin{aligned} \partial(\gamma + \sigma) &= (1 - ibc_1)\omega^{1\bar{2}\bar{1}} - (c_1 + ibc_2)\omega^{1\bar{2}\bar{2}} - ic_2\omega^{1\bar{2}\bar{3}} + \frac{i}{2}\omega^{1\bar{2}\bar{4}} \\ &\quad - ic_1\omega^{1\bar{3}\bar{1}} + \frac{b}{2}\omega^{1\bar{3}\bar{2}} + \frac{1}{2}\omega^{1\bar{3}\bar{3}} + \frac{ia}{2}\omega^{1\bar{4}\bar{1}} + \frac{i}{2}\omega^{1\bar{4}\bar{2}} \\ &\quad + c_2\omega^{1\bar{4}\bar{4}} - \frac{b + 2ic_2}{2}\omega^{2\bar{3}\bar{1}} + \frac{i}{2}\omega^{2\bar{3}\bar{2}} - \frac{i}{2}\omega^{2\bar{4}\bar{1}} - \frac{1}{2}\omega^{2\bar{4}\bar{4}}. \end{aligned}$$

Now, writing any $(2,0)$ -form $\alpha_{2,0}$ as $\alpha_{2,0} = \sum_{1 \leq r < s \leq 4} \lambda_{rs} \omega^{rs}$, with $\lambda_{rs} \in \mathbb{C}$, from (6) we get

$$\begin{aligned} \bar{\partial}\alpha_{2,0} = & (\lambda_{13} + ib\lambda_{14} + a\lambda_{23})\omega^{12\bar{1}} + (\lambda_{14} + \lambda_{23} + ib\lambda_{24})\omega^{12\bar{2}} + i\lambda_{24}\omega^{12\bar{3}} - i\lambda_{13}\omega^{12\bar{4}} \\ & + i\lambda_{14}\omega^{13\bar{1}} + ib\lambda_{34}\omega^{13\bar{2}} + i\lambda_{34}\omega^{13\bar{3}} - \lambda_{23}\omega^{13\bar{4}} - a\lambda_{34}\omega^{14\bar{1}} - \lambda_{34}\omega^{14\bar{2}} \\ & - \lambda_{34}\omega^{14\bar{4}} + i(\lambda_{24} - b\lambda_{34})\omega^{23\bar{1}} - \lambda_{34}\omega^{23\bar{2}} + \lambda_{34}\omega^{24\bar{1}} - i\lambda_{34}\omega^{24\bar{4}}. \end{aligned}$$

Let us study the condition $0 = \partial(\gamma + \sigma) - \bar{\partial}\alpha_{2,0} = \sum A_{r,s\bar{k}} \omega^{r,s\bar{k}}$, where the coefficients $A_{r,s\bar{k}}$ are obtained directly from the previous expressions. From $A_{13\bar{1}} = A_{13\bar{4}} = 0$ one gets $\lambda_{14} = -c_1$ and $\lambda_{23} = 0$. Hence, $A_{12\bar{1}} = 0$ is equivalent to $\lambda_{13} = 1$. However, $A_{12\bar{4}} = 0$ gives $\lambda_{13} = -\frac{1}{2}$, which is a contradiction. \square

We recall that, by the classification obtained in [27, Theorem 1.1], an 8-dimensional NLA \mathfrak{g} with one dimensional center admits a complex structure if and only if it is isomorphic to one (and only one) in the following list:

$$\begin{aligned} \mathfrak{g}_1^\gamma &= (0^5, 13 + 15 + 24, 14 - 23 + 25, 16 + 27 + \gamma \cdot 34), \text{ where } \gamma \in \{0, 1\}, \\ \mathfrak{g}_2^\alpha &= (0^4, 12, 13 + 15 + 24, 14 - 23 + 25, 16 + 27 + \alpha \cdot 34), \text{ where } \alpha \in \mathbb{R}, \\ \mathfrak{g}_3^\gamma &= (0^4, 12, 13 + \gamma \cdot 15 + 25, 15 + 24 + \gamma \cdot 25, 16 + 27), \text{ where } \gamma \in \{0, 1\}, \\ \mathfrak{g}_4^{\alpha, \beta} &= (0^4, 12, 15 + (\alpha + 1) \cdot 24, (\alpha - 1) \cdot 14 - 23 + (\beta - 1) \cdot 25, 16 + 27 + 34 - 2 \cdot 45), \\ &\text{ where } (\alpha, \beta) \in \mathbb{R}^* \times \mathbb{R}^+ \text{ or } \mathbb{R}^+ \times \{0\}, \\ \mathfrak{g}_5 &= (0^4, 2 \cdot 12, 14 - 23, 13 + 24, 16 + 27 + 35), \\ \mathfrak{g}_6 &= (0^4, 2 \cdot 12, 14 + 15 - 23, 13 + 24 + 25, 16 + 27 + 35), \\ \mathfrak{g}_7 &= (0^5, 15, 25, 16 + 27 + 34), \\ \mathfrak{g}_8 &= (0^4, 12, 15, 25, 16 + 27 + 34), \\ \mathfrak{g}_9^\gamma &= (0^3, 13, 23, 35, \gamma \cdot 12 - 34, 16 + 27 + 45), \text{ where } \gamma \in \{0, 1\}, \\ \mathfrak{g}_{10}^\gamma &= (0^3, 13, 23, 14 + 25, 15 + 24, 16 + \gamma \cdot 25 + 27), \text{ where } \gamma \in \{0, 1\}, \\ \mathfrak{g}_{11}^{\alpha, \beta} &= (0^3, 13, 23, 14 + 25 - 35, \alpha \cdot 12 + 15 + 24 + 34, 16 + 27 - 45 - \beta(2 \cdot 25 + 35)), \\ &\text{ where } (\alpha, \beta) = (0, 0), (1, 0) \text{ or } (\alpha, 1) \text{ with } \alpha \in [0, +\infty), \\ \mathfrak{g}_{12}^\gamma &= (0^2, 12, 13, 23, 14 + 25, 15 + 24, 16 + 27 + \gamma \cdot 25), \text{ where } \gamma \in \{0, 1\}. \end{aligned}$$

In the description of the nilpotent Lie algebras above we are using the standard abbreviated notation, where the i -th component of the tuple contains the differential of the i -th element of the basis. We recall that complex structures on the NLAs $\mathfrak{g}_1^\gamma, \dots, \mathfrak{g}_8$ belong to Family I, whereas those on $\mathfrak{g}_9^\gamma, \mathfrak{g}_{10}^\gamma, \mathfrak{g}_{11}^{\alpha, \beta}$ and \mathfrak{g}_{12}^γ belong to Family II. Moreover, the previous list is ordered according to the dimensions of the ascending central series of the algebras.

We recall that the precise relation between these 8-dimensional NLAs and the classification of complex structures showed in Proposition 2.3 can be found in [27]. Nevertheless, the essential information is gathered in the Tables 1 and 2 (columns 1, 2 and 5), where we also sum up the behaviour of the sequence $E_r^{0,2}(\mathfrak{g}, J)$ for any complex structure J in the Families I and II, respectively (columns 3 and 4). Note that this completes the results obtained in Propositions 2.4 and 2.5. We omit the details here.

In both tables we denote by $e_r^{0,2}$ the dimension of the term $E_r^{0,2}(\mathfrak{g}, J)$ in the spectral sequence. In Table 1, the parameter s stands for the sign of $b - 2\delta v$.

Remark 2.6 In [44, Problem 11.2] Stelzig asks for the construction, for every $n \geq 3$, of a compact complex manifold X with $\dim_{\mathbb{C}} X = n$ and with non-vanishing differential on page

Table 1 FSS for complex structures in Family I

Ascending type	$J \equiv (\epsilon, \nu, a, b)$	$(e_1^{0,2}, e_2^{0,2}, e_3^{0,2})$	Sequence $E_r^{0,2}(g, J)$	NLA g
(1, 3, 8)	$(0, 0, 1, b), b \in \{0, 1\}$	(4, 2, 2)	$E_1^{0,2} \neq E_2^{0,2} = E_\infty^{0,2}$	g_1^b
(1, 3, 6, 8)	$(0, 1, 1, \frac{\delta}{2})$	(4, 3, 3)	$E_1^{0,2} \neq E_2^{0,2} = E_\infty^{0,2}$	$g_2^{-4\delta b}$
	$(0, 1, 1, b), b \in \mathbb{R} - \{\frac{\delta}{2}\}$	(4, 2, 2)		
	$(1, 1, a, 0), a \in (0, 2)$	(4, 3, 3)	$E_1^{0,2} \neq E_2^{0,2} = E_\infty^{0,2}$	g_2^0
	$(1, 1, 2, 0)$			g_3^1
	$(1, 1, a, 0), a \in (2, \infty)$			g_3^0
	$(1, 0, 1, 0)$			
	$(1, 0, 1, 1)$	(4, 4, 4)	$E_1^{0,2} = E_\infty^{0,2}$	$g_4^{\frac{\nu a}{b}, \frac{ b-2\delta\nu }{a}}$
	$(1, 0, 1, b), b \in \mathbb{R}^+ - \{1\}$	(4, 4, 3)	$E_1^{0,2} = E_2^{0,2} \neq E_3^{0,2} = E_\infty^{0,2}$	
	$(1, 1, a, b), a > 0, b \in \mathbb{R}^*, \Theta(\delta, 1, a, b) = 0$	(4, 4, 4)	$E_1^{0,2} = E_\infty^{0,2}$	
	$(1, 1, a, b), a > 0, b \in \mathbb{R}^*, \Theta(\delta, 1, a, b) \neq 0$	(4, 4, 3)	$E_1^{0,2} = E_2^{0,2} \neq E_3^{0,2} = E_\infty^{0,2}$	
(1, 4, 8)	$(1, 1, 0, 2\delta)$	(4, 4, 4)	$E_1^{0,2} = E_\infty^{0,2}$	g_5
(1, 4, 6, 8)	$(1, 0, 0, 1)$	(4, 3, 3)	$E_1^{0,2} \neq E_2^{0,2} = E_\infty^{0,2}$	g_6
	$(1, 1, 0, b), b \neq 0, 2\delta$			
(1, 5, 8)	$(0, 0, 0, 1)$	(4, 4, 4)	$E_1^{0,2} = E_\infty^{0,2}$	g_7
(1, 5, 6, 8)	$(0, 1, 0, b), b \in \{-1, 1\}$	(4, 2, 2)	$E_1^{0,2} \neq E_2^{0,2} = E_\infty^{0,2}$	g_8

Table 2 FSS for complex structures in Family II

Ascending type	$J \equiv (\varepsilon, \mu, \nu, a, b)$	$(e_1^{0,2}, e_2^{0,2}, e_3^{0,2})$	Sequence $E_r^{0,2} (g, J)$	NLA \mathfrak{g}
(1, 3, 5, 8)	$(0, 1, 0, 0, 0)$	$(2, 2, 2)$	$E_1^{0,2} = E_\infty^{0,2}$	\mathfrak{g}_9^0
	$(0, 1, 0, 1, 0)$			\mathfrak{g}_9^1
	$(1, 0, 0, a, 0), a \in \{0, 1\}$	$(2, 2, 2)$	$E_1^{0,2} = E_\infty^{0,2}$	\mathfrak{g}_{10}^0
	$(1, 0, 0, a, b), a \in \{0, 1\}, b \in \mathbb{R}^*$			\mathfrak{g}_{10}^1
	$(1, 1, 0, 0, 0)$	$(2, 2, 1)$	$E_1^{0,2} = E_2^{0,2} \neq E_3^{0,2} = E_\infty^{0,2}$	$\mathfrak{g}_{11}^{0,0}$
	$(1, 1, 0, a, 0), a \in \mathbb{R}^*$			$\mathfrak{g}_{11}^{1,0}$
(1, 3, 5, 6, 8)	$(1, 1, 0, a, b), a \in \mathbb{R}, b \in \mathbb{R}^*$			$\mathfrak{g}_{11}^{\frac{2\sqrt{3} a }{ b }, 1}$
	$(1, 0, 1, a, 0), a \in \mathbb{R}$	$(2, 2, 2)$	$E_1^{0,2} = E_\infty^{0,2}$	\mathfrak{g}_{12}^0
	$(1, 0, 1, a, b), a \in \mathbb{R}, b \in \mathbb{R}^*$			\mathfrak{g}_{12}^1

$E_{n-1}(X)$ starting in bidegree $(0, n-1)$ or $(0, n-2)$. For $n = 3$, any complex nilmanifold $X = (\Gamma \backslash G, J)$ has differential $d_2: E_2^{0,1}(X) \rightarrow E_2^{2,0}(X)$ identically zero (see [44, Proposition 8.11] or [5, Theorem 4.1]). From the Tables 1 and 2 we get that $d_3: E_3^{0,2}(\mathfrak{g}, J) \rightarrow E_3^{3,0}(\mathfrak{g}, J)$ vanishes for any SnN complex structure J on an 8-dimensional NLA \mathfrak{g} . One may then ask whether the differential $d_{n-1}: E_{n-1}^{0,n-2} \rightarrow E_{n-1}^{n-1,0}$ always vanishes on complex nilmanifolds.

We can finally apply the previous results (namely, Propositions 2.1, 2.4 and 2.5) to obtain new compact complex manifolds with $d_2 \neq 0$. To our knowledge, no compact complex manifold of complex dimension 4 with $d_3 \neq 0$ is known. Let G be the simply-connected nilpotent Lie group associated to \mathfrak{g} , where \mathfrak{g} is isomorphic to any one of the NLAs in the list above. Since $\gamma \in \{0, 1\}$, all the Lie algebras are rational except for possibly $\mathfrak{g}_2^{\alpha, \beta}$ and $\mathfrak{g}_{11}^{\alpha, \beta}$. For these algebras, it follows from their structure equations that they are also rational algebras whenever $\alpha, \beta \in \mathbb{Q}$. Hence, the existence of a lattice Γ for the associated nilpotent Lie groups is guaranteed by the well-known Mal'cev theorem [28]. Therefore, many compact complex nilmanifolds $X = (\Gamma \backslash G, J)$ with $\dim_{\mathbb{C}} X = 4$ can be defined in this way.

Theorem 2.7 *Let $\Gamma \backslash G$ be a nilmanifold endowed with a complex structure J in any of the following cases:*

- J in Family I defined by $(\varepsilon, \nu, a, b) = (1, 0, 1, b)$ with $b \in \mathbb{Q}^+ - \{1\}$, or $(1, 1, a, b)$ with $(a, b) \in \mathbb{Q}^+ \times \mathbb{Q}^*$ satisfying $\Theta(\delta, 1, a, b) \neq 0$;
- J in Family II defined by $(\varepsilon, \mu, \nu, a, b) = (1, 1, 0, 0, 0)$, $(1, 1, 0, a, 0)$ with $a \in \mathbb{R}^*$, or $(1, 1, 0, a, b)$ with $(a, b) \in \mathbb{R} \times \mathbb{R}^*$ satisfying $\frac{2\sqrt{3}|a|}{|b|} \in \mathbb{Q}$.

Then, the compact complex manifold $X = (\Gamma \backslash G, J)$ has Frölicher spectral sequence not degenerating at the second page.

Proof Let J be a complex structure in Family I defined by $(\varepsilon, \nu, a, b) = (1, 0, 1, b)$, with $b \in \mathbb{Q}^+ - \{1\}$, or $(1, 1, a, b)$, with $(a, b) \in \mathbb{Q}^+ \times \mathbb{Q}^*$ satisfying $\Theta(\delta, 1, a, b) \neq 0$. In view of Table 1, in the first case the underlying Lie algebra is $\mathfrak{g}_4^{\frac{1}{b}, b}$, and in the second one we have $\mathfrak{g}_4^{\frac{sa}{b}, \frac{|b-2\delta|}{a}}$, where s denotes the sign of $b - 2\delta$. Since $\delta = \pm 1$, we always get rational Lie algebras. Note also that

$$\Theta(\delta, 1, a, b) = (a^2 - b^2)^2 - 4\delta b(a^2 + b^2) + 4b^2,$$

hence, given any $b \in \mathbb{Q}^*$ we can choose δ so that $\delta b < 0$ and thus $\Theta(\delta, 1, a, b) \neq 0$.

Now, let J be a complex structure in Family II defined by $(\varepsilon, \mu, \nu, a, b) = (1, 1, 0, 0, 0)$, $(1, 1, 0, a, 0)$ with $a \in \mathbb{R}^*$, or $(1, 1, 0, a, b)$ with $(a, b) \in \mathbb{R} \times \mathbb{R}^*$ satisfying $\frac{2\sqrt{3}|a|}{|b|} \in \mathbb{Q}$. In view of Table 2, the underlying NLAs are $\mathfrak{g}_{11}^{0,0}$, $\mathfrak{g}_{11}^{1,0}$ or $\mathfrak{g}_{11}^{q,1}$, with $q = \frac{2\sqrt{3}|a|}{|b|} \in \mathbb{Q}$ and $q \geq 0$, depending on the case, which are always rational. Note that for the last two NLAs there are one-parameter families of (non-isomorphic) complex structures.

To get the desired result it suffices to observe that for any lattice Γ in the Lie group G associated to any of these rational Lie algebras, the FSS satisfies $E_2(\Gamma \backslash G, J) \neq E_3(\Gamma \backslash G, J)$ by Propositions 2.1, 2.4 and 2.5. □

3 The Frölicher spectral sequence of balanced manifolds

In this section we study the existence of balanced metrics on 8-dimensional nilmanifolds M endowed with an SnN complex structure J . As an application, we get infinitely many

compact balanced manifolds with complex dimension 4 and different homotopy types whose FSS does not degenerate at the second page. This is also extended to non-degeneration at any arbitrary page.

A compact complex manifold X of complex dimension n is said to be *balanced* if there is a Hermitian metric F on X satisfying $dF^{n-1} = 0$. These metrics were first studied in [29] and play an important role in geometry and many aspects of theoretical and mathematical physics. Notice that for any $n \geq 3$ one has

$$dF^{n-1} = (n - 1) dF \wedge F^{n-2} = (n - 1) (\partial F \wedge F^{n-2} + \bar{\partial} F \wedge F^{n-2}).$$

Since F is a real form, the two summands in the last expression above are conjugate to each other, and the balanced condition is equivalent to

$$\partial F \wedge F^{n-2} = 0. \tag{9}$$

We recall that when $X = (M, J)$ is a nilmanifold M endowed with an invariant complex structure J , by the symmetrization process the existence of balanced metric on X implies the existence of an invariant one, i.e. a balanced metric on the underlying Lie algebra (\mathfrak{g}, J) .

Let us consider a basis of invariant $(1, 0)$ -forms $\{\omega^k\}_{k=1}^4$ on a complex nilmanifold $X = (M, J)$, where M is 8-dimensional. Then, in terms of this basis, any invariant Hermitian metric F on X is given by

$$F = \sum_{k=1}^4 i x_{k\bar{k}} \omega^{k\bar{k}} + \sum_{1 \leq k < l \leq 4} (x_{k\bar{l}} \omega^{k\bar{l}} - \bar{x}_{k\bar{l}} \omega^{l\bar{k}}), \tag{10}$$

for some coefficients $x_{k\bar{k}} \in \mathbb{R}$ and $x_{k\bar{l}} \in \mathbb{C}$. Associated to F , we consider the 4×4 matrix

$$H = \begin{pmatrix} x_{1\bar{1}} & -i x_{1\bar{2}} & -i x_{1\bar{3}} & -i x_{1\bar{4}} \\ i \bar{x}_{1\bar{2}} & x_{2\bar{2}} & -i x_{2\bar{3}} & -i x_{2\bar{4}} \\ i \bar{x}_{1\bar{3}} & i \bar{x}_{2\bar{3}} & x_{3\bar{3}} & -i x_{3\bar{4}} \\ i \bar{x}_{1\bar{4}} & i \bar{x}_{2\bar{4}} & i \bar{x}_{3\bar{4}} & x_{4\bar{4}} \end{pmatrix}. \tag{11}$$

Let us denote by H_{rs} the determinant of the 3×3 submatrix obtained by removing the r -th row and the s -th column from H . Note that $H_{sr} = \bar{H}_{rs}$. The metric F being positive implies, in particular, $H_{rr} > 0$ for every $1 \leq r \leq 4$.

Lemma 3.1 *Let M be an 8-dimensional nilmanifold endowed with an SnN complex structure J . Let F be any Hermitian metric on (M, J) defined by (10) in a $(1, 0)$ -frame $\{\omega^k\}_{k=1}^4$ satisfying the Eqs. (5) or (6). We have:*

(i) *If J belongs to Family I, then F is balanced if and only if*

$$\varepsilon = \nu = 0, \quad a H_{12} + \bar{H}_{14} = 0, \quad b H_{22} - 2\delta \Im(H_{13}) = 0.$$

(ii) *If J belongs to Family II, then F is balanced if and only if*

$$\begin{aligned} \nu = 0, \quad H_{14} = 0, \quad 2\varepsilon \Im(H_{12}) - \mu \bar{H}_{24} - ia H_{11} &= 0, \\ \mu H_{22} - 2b \Im(H_{12}) + 2 \Im(H_{13}) &= 0. \end{aligned}$$

Proof To prove (i) we calculate (9) for $n = 4$ taking into account the complex equations (5). We get

$$0 = \frac{1}{2} \partial F \wedge F^2 = A_I \omega^{1234\bar{1}\bar{2}\bar{3}} + B_I \omega^{1234\bar{1}\bar{2}\bar{4}} + C_I \omega^{1234\bar{1}\bar{3}\bar{4}},$$

where

$$\begin{aligned} A_I &= -\nu H_{11} - i b H_{22} + 2 i \delta \Im(H_{13}), \\ B_I &= -\delta \varepsilon b \bar{H}_{12} - i a H_{12} - i \bar{H}_{14}, \\ C_I &= -i \varepsilon H_{11}. \end{aligned}$$

Since $H_{11} > 0$, the condition $C_I = 0$ implies $\varepsilon = 0$. Similarly, since $\Re(A_I) = -\nu H_{11} = 0$ we get $\nu = 0$. Now, taking $\varepsilon = \nu = 0$, the remaining conditions are $\Im(A_I) = 0$ and $B_I = 0$, and the statement in the part (i) of the lemma follows.

For the proof of (ii) we use the complex equations (6) to compute (9) for $n = 4$. One has

$$0 = \frac{1}{2} \partial F \wedge F^2 = A_{II} \omega^{1234\bar{1}\bar{2}\bar{3}} + B_{II} \omega^{1234\bar{1}\bar{2}\bar{4}} + C_{II} \omega^{1234\bar{1}\bar{3}\bar{4}},$$

where

$$\begin{aligned} A_{II} &= -\nu H_{11} + i (\mu H_{22} - 2 b \Im(H_{12}) + 2 \Im(H_{13})), \\ B_{II} &= -2 \varepsilon \Im(H_{12}) + \mu \bar{H}_{24} + i a H_{11}, \\ C_{II} &= i \bar{H}_{14}. \end{aligned}$$

From $\Re(A_{II}) = -\nu H_{11} = 0$ and $H_{11} > 0$, it follows that $\nu = 0$. The remaining conditions in the statement come directly from $\Im(A_{II}) = 0$, $B_{II} = 0$ and $C_{II} = 0$. □

The following result provides a classification of the SnN complex structures J in eight dimensions admitting balanced metrics.

Proposition 3.2 *Let M be an 8-dimensional nilmanifold endowed with an SnN complex structure J . Then, (M, J) admits a balanced metric if and only if J is equivalent to a complex structure defined by, either (5) with the tuple (ε, ν, a, b) being one of the following*

$$(0, 0, 0, 1), (0, 0, 1, 0), (0, 0, 1, 1),$$

or (6) with tuple $(\varepsilon, \mu, \nu, a, b)$ one of the following

$$(1, 1, 0, a, b), (1, 0, 0, 0, b), (0, 1, 0, 0, 0), (0, 1, 0, 1, 0).$$

Proof When J belongs to Family I, it follows from Lemma 3.1 (i) that the possible values for the tuple (ε, ν, a, b) are $(0, 0, 0, 1)$, $(0, 0, 1, 0)$ or $(0, 0, 1, 1)$. We will find a balanced metric F for each case. Observe that $a, b \in \{0, 1\}$. We distinguish two cases according to the value of b :

- If $b = 0$, then $a = 1$ and the equations in Lemma 3.1 (i) reduce to $H_{12} + \bar{H}_{14} = 0$ and $\Im(H_{13}) = 0$. To obtain a solution it suffices to consider the metric F defined by taking H in (11) as the identity matrix.
- If $b = 1$, we consider the metric F defined by

$$\begin{aligned} x_{1\bar{1}} = x_{3\bar{3}} = x_{4\bar{4}} = 1, \quad x_{2\bar{2}} = 4, \quad x_{1\bar{2}} = i, \quad x_{1\bar{3}} = x_{1\bar{4}} = x_{3\bar{4}} = 0, \\ x_{2\bar{3}} = \frac{\delta}{2}, \quad x_{2\bar{4}} = i a. \end{aligned}$$

It can be checked that this choice indeed defines a positive-definite metric for which one has $H_{12} = 1$, $H_{13} = \frac{\delta i}{2}$, $H_{14} = -a$, and $H_{22} = 1$. Since $\delta = \pm 1$, all the conditions in Lemma 3.1 (i) are satisfied and the metric F is balanced.

Suppose now that J belongs to Family II. Lemma 3.1 (ii) implies that the possible values for the tuple $(\varepsilon, \mu, \nu, a, b)$ are $(1, 1, 0, a, b)$, $(1, 0, 0, 0, b)$, $(1, 0, 0, 1, b)$, $(0, 1, 0, 0, 0)$ or $(0, 1, 0, 1, 0)$.

Let us distinguish two cases depending on the value of the parameter μ :

- If $\mu = 0$, then we have $(\varepsilon, \mu, \nu, a, b) = (1, 0, 0, 0, b)$ or $(1, 0, 0, 1, b)$. Observe that one of the equations in Lemma 3.1 (ii) becomes $2\Im(H_{12}) - iaH_{11} = 0$, which is equivalent to $\Im(H_{12}) = 0$ and $aH_{11} = 0$. Since $H_{11} > 0$, we conclude that $a = 0$ in order that J admits a balanced metric. Therefore, there are no balanced metrics for the tuple $(1, 0, 0, 1, b)$. However, for the case $(1, 0, 0, 0, b)$ one can check that the metric F defined by taking H in (11) as the identity matrix is balanced.
- If $\mu = 1$, we take F defined by

$$x_{1\bar{1}} = x_{3\bar{3}} = x_{4\bar{4}} = 1, \quad x_{2\bar{2}} = a^2 + \frac{3}{4}, \quad x_{1\bar{2}} = x_{1\bar{4}} = x_{2\bar{3}} = x_{3\bar{4}} = 0,$$

$$x_{1\bar{3}} = \frac{1}{2}, \quad x_{2\bar{4}} = -a.$$

It can be checked that it defines a positive-definite metric. Moreover, $H_{11} = \frac{3}{4}$, $H_{12} = 0$, $H_{13} = -\frac{3i}{8}$, $H_{14} = 0$, $H_{22} = \frac{3}{4}$, and $H_{24} = \frac{3ai}{4}$. Since all the conditions in Lemma 3.1 (ii) are satisfied, the metric F is balanced.

□

The following result is a consequence of Proposition 3.2 and the values given in the second and fifth columns of Tables 1 and 2. Here we are also using that the existence of a balanced metric implies the existence of an invariant one.

Theorem 3.3 *If an 8-dimensional nilmanifold endowed with an S_nN complex structure admits a balanced metric, then its underlying Lie algebra is isomorphic to \mathfrak{g}_1^γ , \mathfrak{g}_7 , \mathfrak{g}_9^γ , \mathfrak{g}_{10}^γ or $\mathfrak{g}_{11}^{\alpha,\beta}$.*

This result has a certain converse which is useful in the construction of balanced nilmanifolds satisfying additional properties. Indeed, by a similar argument as in Sect. 2, starting with the rational Lie algebras \mathfrak{g}_1^γ , \mathfrak{g}_7 or \mathfrak{g}_9^γ we get compact balanced nilmanifolds. Similarly, when we start with \mathfrak{g}_{10}^γ endowed with any complex structure corresponding to a tuple $(\varepsilon, \mu, \nu, a, b) = (1, 0, 0, 0, b)$. Finally, compact balanced nilmanifolds can also be constructed starting with $\mathfrak{g}_{11}^{\alpha,\beta}$ for $(\alpha, \beta) = (0, 0)$, $(1, 0)$ or $(q, 1)$ with rational $q \geq 0$, since from Table 2 in the latter case we can always choose complex structures defined by tuples $(\varepsilon, \mu, \nu, a, b) = (1, 1, 0, a, b)$, with $b \neq 0$, such that $\frac{2\sqrt{3}|a|}{|b|} = q$.

We observe that all the compact balanced nilmanifolds X with underlying Lie algebra isomorphic to $\mathfrak{g}_{11}^{\alpha,\beta}$ satisfy $E_2(X) \neq E_\infty(X)$, due to Proposition 2.5. Moreover, non-isomorphic Lie algebras $\mathfrak{g}_{11}^{\alpha,\beta}$ and $\mathfrak{g}_{11}^{\alpha',\beta'}$ give rise to nilmanifolds X and X' with different minimal model by Hasegawa theorem [20], hence with different real homotopy type (see [3, 8, 10, 19, 45] for results in homotopy theory). Furthermore, their complex homotopy types are also different:

Theorem 3.4 *There are infinitely many complex (hence, real or rational) homotopy types of compact balanced manifolds of complex dimension 4 with Frölicher spectral sequence not degenerating at the second page.*

Proof By the discussion above, it is enough to consider a family of compact balanced nilmanifolds Y_α^4 with underlying Lie algebra isomorphic to $\mathfrak{g}_{11}^{\alpha,\beta}$ with $\alpha = q \geq 0$, $q \in \mathbb{Q}$, and $\beta = 1$. To complete the proof it only remains to prove that the Lie algebras $\mathfrak{g}_{11}^{\alpha,1}$ and $\mathfrak{g}_{11}^{\alpha',1}$ are

$\partial(\tau^j \wedge \overline{\tau^j}) = 0$ for $1 \leq j \leq 3n - 2$, and

$$\begin{aligned} \partial(\tau^{3n-1} \wedge \overline{\tau^{3n-1}}) &= \tau^1 \wedge \tau^{n+1} \wedge \overline{\tau^{3n-1}} + \tau^{2n} \wedge \overline{\tau^n} \wedge \tau^{3n-1}, \\ &\vdots \\ \partial(\tau^{4n-3} \wedge \overline{\tau^{4n-3}}) &= \tau^{n-1} \wedge \tau^{2n-1} \wedge \overline{\tau^{4n-3}} + \tau^{3n-2} \wedge \overline{\tau^{2n-2}} \wedge \tau^{4n-3}, \\ \partial(\tau^{4n-2} \wedge \overline{\tau^{4n-2}}) &= \tau^n \wedge \overline{\tau^{2n}} \wedge \tau^{4n-2}. \end{aligned}$$

These equalities clearly imply that $\xi \wedge \partial(\tau^j \wedge \overline{\tau^j}) = 0$ for every $\xi \in \bigwedge^{3n-3} \langle \tau^1 \wedge \overline{\tau^1}, \dots, \tau^{3n-2} \wedge \overline{\tau^{3n-2}} \rangle$ and for any $1 \leq j \leq 4n - 2$, so (14) vanishes. Thus, any diagonal metric F_ρ is balanced.

Proposition 3.6 [42, Theorem 3.3] *For every $n \geq 2$, the nilmanifold X^{4n-2} is balanced.*

In a recent paper Stelzig proves that the Künneth formula is compatible with the Hodge filtration and its conjugate, so the well-known Künneth formula for the Dolbeault cohomology implies a Künneth formula for all higher pages of the Frölicher spectral sequence (see [44, Section 6] and the proof of [44, Proposition 4.8] for more details).

Proposition 3.7 [44] *Let X and Y be compact complex manifolds, and let $Z = X \times Y$. For every $r \geq 1$ and every (p, q) with $0 \leq p, q \leq \dim_{\mathbb{C}} Z$, the following equality holds*

$$e_r^{p,q}(Z) = \sum_{\substack{p_1+p_2=p \\ q_1+q_2=q}} e_r^{p_1,q_1}(X) \cdot e_r^{p_2,q_2}(Y),$$

where $e_r^{p,q}(\cdot)$ denotes the dimension of $E_r^{p,q}(\cdot)$, i.e. $e_r^{p,q}(\cdot) = \dim E_r^{p,q}(\cdot)$.

We will apply this formula together with the results above to prove the following

Theorem 3.8 *For every integer $n \geq 3$, there exist infinitely many compact balanced manifolds Z with $\dim_{\mathbb{C}} Z = 4n + 2$ and different complex (hence, real or rational) homotopy types such that $d_n \neq 0$ (hence, the Frölicher spectral sequence $E_r(Z)$ does not degenerate at the n -th page).*

Similar results hold for $d_1 \neq 0$ and $d_2 \neq 0$ when complex dimensions 7 and 4 are respectively considered.

Proof First, we prove the theorem for each $n \geq 3$. Let X^{4n-2} be the nilmanifold in the Bigalke-Rollenske family, which is balanced by Proposition 3.5. Let us also consider any of the complex balanced manifolds Y_α^4 in the infinite family given in Theorem 3.4. Then, by [29, Proposition 1.9] the compact complex manifold $Z_\alpha^{4n+2} = X^{4n-2} \times Y_\alpha^4$ is balanced too.

Now observe that both X^{4n-2} and Y_α^4 are nilmanifolds, and recall that Y_α^4 is constructed from a nilpotent Lie algebra $\mathfrak{g}_{11}^{\alpha,1}$, with $\alpha \geq 0$ and $\alpha \in \mathbb{Q}$. By [20] the (\mathbb{Q}) -minimal model of Z_α^{4n+2} is isomorphic to the commutative differential graded algebra (CDGA) defined by the exterior algebra of the rational Lie algebra $\mathfrak{h} \oplus \mathfrak{g}_{11}^{\alpha,1}$, where \mathfrak{h} denotes the nilpotent Lie algebra underlying the Bigalke-Rollenske nilmanifold X^{4n-2} . Hence, for every non-negative $\alpha, \alpha' \in \mathbb{Q}$ with $\alpha \neq \alpha'$, the minimal models of Z_α^{4n+2} and $Z_{\alpha'}^{4n+2}$ are isomorphic to $\mathfrak{g}^\alpha = \mathfrak{h} \oplus \mathfrak{g}_{11}^{\alpha,1}$ and $\mathfrak{g}^{\alpha'} = \mathfrak{h} \oplus \mathfrak{g}_{11}^{\alpha',1}$, respectively. Let us denote their complexifications by $\mathfrak{g}_{\mathbb{C}}^\alpha$ and $\mathfrak{g}_{\mathbb{C}}^{\alpha'}$. By [41, Lemma 1], if a finite-dimensional Lie algebra over \mathbb{C} is decomposed as a direct sum of indecomposable ideals, then the isomorphism classes of these ideals are unique. Hence, if we assume that $\mathfrak{h}_{\mathbb{C}} \oplus (\mathfrak{g}_{11}^{\alpha,1})_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}}^\alpha \cong \mathfrak{g}_{\mathbb{C}}^{\alpha'} = \mathfrak{h}_{\mathbb{C}} \oplus (\mathfrak{g}_{11}^{\alpha',1})_{\mathbb{C}}$, then $(\mathfrak{g}_{11}^{\alpha,1})_{\mathbb{C}} \cong (\mathfrak{g}_{11}^{\alpha',1})_{\mathbb{C}}$

and the nilpotent Lie algebras $\mathfrak{g}_{11}^{\alpha,1}$ and $\mathfrak{g}_{11}^{\alpha',1}$ would be isomorphic over \mathbb{C} . However, this is a contradiction (see the proof of Theorem 3.4). Consequently, the (nil)manifolds Z_α^{4n+2} and $Z_{\alpha'}^{4n+2}$ have different complex homotopy type for $\alpha \neq \alpha'$, thus also different real or rational homotopy type.

Let us now apply the Künneth formula in Proposition 3.7 to $Z_\alpha^{4n+2} = X^{4n-2} \times Y_\alpha^4$ with $p = 0, q = n - 1$ and $r \in \{n, n + 1\}$. Then, we get:

$$\begin{aligned} e_n^{0,n-1}(Z_\alpha^{4n+2}) &= e_n^{0,n-1}(X^{4n-2}) \cdot e_n^{0,0}(Y_\alpha^4) + \sum_{l=1}^4 e_n^{0,n-1-l}(X^{4n-2}) \cdot e_n^{0,l}(Y_\alpha^4) \\ &> e_{n+1}^{0,n-1}(X^{4n-2}) \cdot e_{n+1}^{0,0}(Y_\alpha^4) + \sum_{l=1}^4 e_{n+1}^{0,n-1-l}(X^{4n-2}) \cdot e_{n+1}^{0,l}(Y_\alpha^4) \\ &= e_{n+1}^{0,n-1}(Z_\alpha^{4n+2}). \end{aligned}$$

Here we have used that $e_n^{0,n-1}(X^{4n-2}) > e_{n+1}^{0,n-1}(X^{4n-2})$ by Proposition 3.5, together with the well-known Frölicher inequalities $e_r^{p,q}(\cdot) \geq e_{r+1}^{p,q}(\cdot)$, valid for every compact complex manifold and any r, p, q . In conclusion, for every integer $n \geq 2$, the infinite family of compact balanced manifolds Z_α^{4n+2} have differential $d_n \neq 0$.

The result for the second page comes directly from Theorem 3.4. Therefore, we next focus on the non-degeneration at the first page. Note that we can no longer make use of the Bigalke-Rollenske nilmanifolds since they satisfy $d_n \neq 0$ with $n \geq 2$. Moreover, we do not know if the nilmanifolds Y_α^4 have non-zero differential d_1 . So, we will proceed as follows.

Let X^3 be any compact balanced nilmanifold with Frölicher spectral sequence not degenerating at the first page. We can consider, for instance, the Iwasawa manifold or any of those studied in [5]. Let $Z_\alpha^7 = X^3 \times Y_\alpha^4$, with $\alpha \geq 0$ and $\alpha \in \mathbb{Q}$. A similar argument as above allows us to conclude that Z_α^7 and $Z_{\alpha'}^7$ have different complex homotopy type whenever $\alpha \neq \alpha'$, they are balanced and their Frölicher spectral sequence $\{E_r(Z_\alpha^7)\}_{r \geq 1}$ is not degenerating at the 1-st page (neither at the 2-nd page). \square

It is worthy to remark that, in the result concerning the second page, the dimension of Z is optimal in the class of complex nilmanifolds; indeed, Bazzoni and Muñoz prove in [3] that the number of real homotopy types of 6-dimensional nilmanifolds is finite.

However, for other pages the dimension of the balanced manifolds Z in the infinite families seems to be far from being optimal, even in the class of complex nilmanifolds. For instance, for the third page we can consider the (3-step) nilmanifold of real dimension 12 endowed with a nilpotent complex structure given in [6]. Let us denote by X^6 this complex nilmanifold, whose complex structure equations are

$$d\omega^1 = d\omega^2 = d\omega^3 = 0, \quad d\omega^4 = \omega^{12} + \omega^{1\bar{2}}, \quad d\omega^5 = -\omega^{2\bar{1}}, \quad d\omega^6 = \omega^{14} + \omega^{1\bar{3}}.$$

It is proved in [6] that $E_3(X^6) \not\cong E_4(X^6)$. For any $\rho = (\rho_1, \dots, \rho_6) \in (\mathbb{R}^+)^6$, one can check that the “diagonal” invariant Hermitian metrics on X^6 defined by $F_\rho = \frac{i}{2} \sum_{k=1}^6 \rho_k \omega^{k\bar{k}}$ are balanced. Now, a similar argument as the one given in the proof of Theorem 3.8 shows that the infinite family of balanced manifolds $Z_\alpha^{10} = X^6 \times Y_\alpha^4$ satisfies the required properties for $n = 3$ in complex dimension 10.

Finally, it is still unclear whether there are obstructions in the Frölicher spectral sequence under the existence of a balanced metric on a compact complex manifold. We recall that in the class of balanced nilmanifolds of complex dimension 3 the Frölicher spectral sequence always degenerates at the second page [5]. We believe that the following general question could have a positive answer:

Question. Let X be a compact balanced manifold with $\dim_{\mathbb{C}} X = n$. Does the Frölicher spectral sequence of X degenerate at a k -th page for some $k \leq n - 1$?

4 Frölicher spectral sequence of SKT and generalized Gauduchon manifolds

In this section we construct compact complex manifolds endowed with SKT and, more generally, generalized Gauduchon metrics having non-degenerate Frölicher spectral sequence.

Let us first recall the definition of generalized Gauduchon metrics, introduced and studied by Fu, Wang and Wu in [16].

Definition 4.1 [16] Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$, and let $1 \leq k \leq n - 1$ be an integer. A Hermitian metric F on X is called k -th Gauduchon if it satisfies the condition

$$\partial\bar{\partial}F^k \wedge F^{n-k-1} = 0.$$

From the definition, it is clear that the value $k = n - 1$ corresponds to the classical *standard* (also known as *Gauduchon*) metrics [18]. Moreover, observe that any SKT metric ($\partial\bar{\partial}F = 0$) is in particular 1-st Gauduchon.

We recall that by [11, Theorem 2.5], any conformally balanced 1-st Gauduchon metric F on a compact complex manifold X with $\dim_{\mathbb{C}} X = n \geq 3$ whose Bismut connection has (restricted) holonomy contained in $SU(n)$ is necessarily Kähler. In [21] Ivanov and Papadopoulos extend this result to any generalized k -th Gauduchon metric, for any $k \neq n - 1$.

Very little is known about the relation between the existence of k -th Gauduchon metrics, for some $1 \leq k \leq n - 2$, on a compact complex manifold X and the degeneration of its FSS. A first consequence of the results in the preceding sections is the following one:

Proposition 4.2 *There are infinitely many complex homotopy types of compact 1-st Gauduchon manifolds Z with $\dim_{\mathbb{C}} Z = 4$ and Frölicher spectral sequence not degenerating at the second page.*

Proof It is enough to consider the nilmanifolds Y_{α}^4 with underlying Lie algebra isomorphic to $\mathfrak{g}_{11}^{q,1}$ with $q \geq 0$ and $q \in \mathbb{Q}$, given in the proof of Theorem 3.4. In fact, these nilmanifolds admit generalized Gauduchon metrics by [26, Theorem 5.2]. □

Next, we extend this result to obtain non-degeneration at an arbitrarily large page. Let X be a complex nilmanifold with $\dim_{\mathbb{C}} X = n \geq 2$. For any invariant Hermitian metric F on X , the real (n, n) -form $\frac{i}{2} \partial\bar{\partial}F \wedge F^{n-2}$ is proportional to the volume form F^n . Therefore,

$$\frac{i}{2} \partial\bar{\partial}F \wedge F^{n-2} = c_1(F) F^n, \tag{15}$$

for some constant $c_1(F) \in \mathbb{R}$. By [16, Proposition 11] the sign of the constant $c_1(F)$ is an invariant of the conformal class of F . Observe that F is 1-st Gauduchon if and only if $c_1(F) = 0$. Note that when $n = 2$ the constant $c_1(F) = 0$ because any invariant metric is standard.

Proposition 4.3 [24, Proposition 2.3 and Corollary 2.4] *Let X and X' be complex nilmanifolds endowed with invariant Hermitian metrics F and F' , respectively.*

- (i) *For any real positive number $\lambda > 0$, we have $c_1(\lambda F) = \lambda^{-1}c_1(F)$.*
- (ii) *The product Hermitian metric $F + F'$ on the nilmanifold $X \times X'$ satisfies*

$$c_1(F + F') = \frac{n(n-1)}{(n+n')(n+n'-1)} c_1(F) + \frac{n'(n'-1)}{(n+n')(n+n'-1)} c_1(F'),$$

where $n = \dim_{\mathbb{C}} X$ and $n' = \dim_{\mathbb{C}} X'$. In particular, if $c_1(F) > 0$ and $c_1(F') < 0$, then $X \times X'$ has a 1-st Gauduchon metric.

It is worth to recall that for complex nilmanifolds of complex dimension 3, the existence of an invariant Hermitian metric F with $c_1(F) < 0$ implies the existence of a 1-st Gauduchon metric (possibly non-invariant and non-SKT) on the nilmanifold (see [11, Theorem 3.6] for details). In [24, Proposition 2.9] a classification of complex structures admitting Hermitian metrics with $c_1 < 0$ is given. Using such classification together with [5, Theorem 4.1], one has that the FSS of a complex nilmanifold X of complex dimension 3 degenerates at the second page whenever an invariant Hermitian metric with $c_1 \leq 0$ exists on X .

We use the results obtained in Sect.3 together with the proposition above to extend Proposition 4.2 to arbitrarily high pages.

Theorem 4.4 *For every integer $n \geq 3$, there exist infinitely many compact 1-st Gauduchon manifolds W with $\dim_{\mathbb{C}} W = 4n + 5$ and different complex (hence, real or rational) homotopy types such that $d_n \neq 0$ (hence, the Frölicher spectral sequence $E_r(W)$ is not degenerating at the n -th page).*

Similar results hold for $d_1 \neq 0$ and $d_2 \neq 0$ when complex dimensions 7 and 4 are respectively considered.

Furthermore, all the previous compact complex manifolds are k -th Gauduchon for every k .

Proof First, we prove the theorem for every $n \geq 3$. For each $\alpha \geq 0$ and $\alpha \in \mathbb{Q}$, let Z_α^{4n+2} be a compact balanced manifold in the family given in Theorem 3.8. Recall that Z_α is a complex nilmanifold and it has a (invariant) balanced metric F . By [21, Lemma 3.7], one has that the constant $c_1(F) > 0$.

Let us now consider any complex nilmanifold X' of complex dimension 3 endowed with an invariant Hermitian metric F' with $c_1(F') < 0$. Note that these nilmanifolds are classified in [24, Proposition 2.9], and three is the lowest possible dimension where this can occur.

Let $W_\alpha = Z_\alpha \times X'$. It follows from Proposition 4.3 that W_α has a 1-st Gauduchon metric. Furthermore, a similar argument as in the proof of Theorem 3.8 implies that for any non-negative $\alpha, \alpha' \in \mathbb{Q}$ with $\alpha \neq \alpha'$, the (nil)manifolds $W_\alpha = Z_\alpha \times X'$ and $W_{\alpha'} = Z_{\alpha'} \times X'$ have different complex homotopy types because their minimal models are not isomorphic (over \mathbb{C}). Applying the Künneth formula to $W_\alpha = Z_\alpha \times X'$ we conclude that $d_n \neq 0$ for all the manifolds W_α . This gives the first part of the statement.

The result for the second page comes directly from Proposition 4.2, so we focus on the non-degeneration at the first page. Let X' be a 3-dimensional complex nilmanifold with Frölicher spectral sequence not degenerating at the first page and having an invariant Hermitian metric F with $c_1(F) < 0$. For example, one can consider X' defined by the following complex structure equations

$$d\omega^1 = d\omega^2 = 0, \quad d\omega^3 = \omega^{1\bar{1}} + (1 + i)\omega^{2\bar{2}},$$

and then the Hermitian metric $F = \frac{i}{2}(\omega^{1\bar{1}} + \omega^{2\bar{2}} + \omega^{3\bar{3}})$ satisfies $\frac{i}{2} \partial \bar{\partial} F \wedge F = \frac{-1}{12} F^3$, for instance. Let $Z_\alpha^7 = X' \times Y_\alpha^4$, with $\alpha \geq 0$ and $\alpha \in \mathbb{Q}$, where Y_α^4 is any of the complex balanced manifolds in the infinite family given in Theorem 3.4. A similar argument as above allows us to conclude the result for the first page in seven complex dimensions.

For the final statement, we just apply [23, Proposition 2.2], where it is proved that if an invariant Hermitian metric F on a complex nilmanifold W of complex dimension $n \geq 4$ is k_0 -th Gauduchon for some k_0 with $1 \leq k_0 \leq n - 2$, then F is k -th Gauduchon for every k with $1 \leq k \leq n - 1$. Alternatively, one can also apply [31, Proposition 3.1]. \square

Observe that the generalized Gauduchon metrics on the manifolds given in the previous result are not SKT. Indeed, Arroyo and Nicolini prove in [2, Theorem 1.2] that any complex

nilmanifold admitting an invariant SKT metric is either a torus or 2-step nilpotent. Since any nilmanifold with S_nN complex structure has nilpotency step $s \geq 3$ (see [25, Corollary 3.6]), in particular the manifolds constructed in Theorems 3.8 and 4.4 do not admit any SKT metric. Note also that the Bigalke-Rollenske nilmanifolds X are 2-step but they do not admit any SKT metric by [42, Proposition 3.5]. The latter also follows from the fact that the equations (13) for X imply that the underlying Lie algebra \mathfrak{g} satisfies that $[\mathfrak{g}, \mathfrak{g}] + J[\mathfrak{g}, \mathfrak{g}]$ is abelian and, under this condition, Fino and Vezzoni proved (see [12, Theorem 1.1] and [13, Theorem A]) that if X carries an SKT metric and a balanced metric, then X is necessarily a complex torus.

In [34, Conjecture 1.3], it is conjectured that *any compact complex manifold X admitting an SKT metric has Frölicher spectral sequence degenerating at the second page*. Some evidence for this conjecture is provided in [34, 35], where Popovici obtains sufficient metric conditions for the E_2 degeneration of the Frölicher spectral sequence. The main idea is that the existence of a Hermitian metric on X with *small torsion* (in the sense that we recall below) implies that $E_2(X) = E_\infty(X)$.

Let F be any Hermitian metric on a compact complex manifold X . Recall that the torsion operator of order zero and bidegree $(1, 0)$ associated with F (see [9, VII §1]) is given by $\tau := [\Lambda, \partial F \wedge \cdot]$, where Λ is the formal adjoint of the Lefschetz operator $L := F \wedge \cdot$ with respect to the L^2 -inner product $\langle \cdot, \cdot \rangle$ induced by F on differential forms. Define $C_F^{p,q} := \sup_{u \in \Omega^{p,q}(X), \|u\|=1} \langle ([\tau, \tau^*] + [\partial F \wedge \cdot, (\partial F \wedge \cdot)^*])u, u \rangle$. Now, let $\Delta', \Delta'' : \Omega^{p,q}(X) \rightarrow \Omega^{p,q}(X)$ be the usual Laplace-Beltrami operators, i.e. $\Delta' = \partial\bar{\partial}^* + \partial^*\bar{\partial}$ and $\Delta'' = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$. The non-negative self-adjoint differential operator $\Delta' + \Delta''$ is elliptic and, since X is compact, it has a discrete spectrum contained in $[0, +\infty)$ with $+\infty$ as its only accumulation point. Denote by $\rho_F^{p,q} := \min(\text{Spec}(\Delta' + \Delta'')^{p,q} \cap (0, +\infty))$ its smallest positive eigenvalue. Thus, $\rho_F^{p,q}$ is the size of the spectral gap of $\Delta' + \Delta''$ acting on (p, q) -forms.

Popovici proves in [34, Theorem 5.4] that if a compact complex n -dimensional manifold X carries an SKT metric F whose torsion satisfies the condition $C_F^{p,q} \leq \frac{1}{3}\rho_F^{p,q}$ for all $p, q \in \{0, \dots, n\}$, then the Frölicher spectral sequence of X degenerates at $E_2(X)$. Furthermore, in [35, Theorem 1.2] it is proved that the degeneration at E_2 occurs whenever the manifold admits a Hermitian metric F satisfying $\ker \Delta'' \subset \ker [\tau, \tau^*]$, that is, the torsion operator and its adjoint vanish on Δ'' -harmonic forms.

We next provide a counterexample to the previous conjecture, so the torsion of SKT metrics may not be small in general. The construction is based on the complex geometry of compact Lie groups.¹

Let G be a connected Lie group with Lie algebra \mathfrak{g} , and denote by $\mathfrak{g}_\mathbb{C}$ the complexification of \mathfrak{g} . Giving a left-invariant almost complex structure J on G is equivalent to the choice of a subspace $\mathfrak{s} \subset \mathfrak{g}_\mathbb{C}$ such that

$$\mathfrak{s} \cap \mathfrak{g} = \{0\}, \quad \mathfrak{g}_\mathbb{C} = \mathfrak{s} \oplus \bar{\mathfrak{s}}.$$

Hence \mathfrak{s} is the subspace of $(1, 0)$ -elements. Now, J is integrable if and only if \mathfrak{s} is a subalgebra of $\mathfrak{g}_\mathbb{C}$, i.e. $[\mathfrak{s}, \mathfrak{s}] \subset \mathfrak{s}$.

For an even-dimensional compact Lie group G , Samelson provided in [40] a construction of left-invariant complex structures on G that we briefly recall here. Let T be a maximal torus in G with Lie algebra \mathfrak{t} , and suppose that $\alpha_1, \dots, \alpha_r \in \mathfrak{t}^*$ is a set of positive roots. Here $2r$ is the rank of G . Then we have the $\text{ad}(T)$ -invariant decomposition

¹ We were informed by Jonas Stelzig of the existence of SKT nilmanifolds with arbitrarily non-degenerate FSS.

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \sum_{j=1}^r \mathfrak{s}_{\alpha_j} \oplus \sum_{j=1}^r \mathfrak{s}_{-\alpha_j},$$

where $\mathfrak{t}_{\mathbb{C}}$ is the complexification of \mathfrak{t} , and \mathfrak{s}_{α} is given by

$$\mathfrak{s}_{\alpha} := \{Z \in \mathfrak{g}_{\mathbb{C}} \mid [x, Z] = 2\pi i\alpha(x)Z \quad \forall x \in \mathfrak{t}\}.$$

So, if we choose a subspace $\mathfrak{a} \subset \mathfrak{t}_{\mathbb{C}}$ such that $\mathfrak{a} \cap \mathfrak{t} = \{0\}$ and $\mathfrak{a} \oplus \bar{\mathfrak{a}} = \mathfrak{t}_{\mathbb{C}}$, then we get a left-invariant complex structure J on G defined by

$$\mathfrak{s} = \mathfrak{a} \oplus \sum_{j=1}^r \mathfrak{s}_{\alpha_j}.$$

Now, let g be a bi-invariant metric on G , and denote by $\langle \cdot \rangle$ its \mathbb{C} -linear extension. The compatibility of the complex structure J with g is equivalent to \mathfrak{s} being isotropic, i.e. $\langle \mathfrak{s}, \mathfrak{s} \rangle = 0$. Alexandrov and Ivanov state in [1] that any compact Lie group equipped with a bi-invariant metric g and a left-invariant complex structure J compatible with g is Bismut flat and its fundamental form F is dd^c -harmonic. The latter condition means that F is SKT (since $dd^c F = 0$) and standard (since $(dd^c)^* F = 0$ is equivalent to $\partial\bar{\partial} F^{n-1} = 0$, where $2n = \dim_{\mathbb{R}} G$). This result is used in [1] to compute the Hodge numbers $h^{0,q}$ of compact Lie groups with such a Hermitian structure.

Bismut flat manifolds play a relevant role in relation to the pluriclosed flow. In [17], Garcia-Fernandez, Jordan and Streets prove global existence and convergence of the pluriclosed flow and the generalized Kähler Ricci flow on compact Bismut flat manifolds. Note that by [47] it turns out that, up to taking universal covers, all such manifolds are given by the above Samelson spaces.

In what follows, we consider the compact semisimple Lie group $SO(9)$ equipped with the bi-invariant metric g given by minus the Killing form and with a left-invariant complex structure J compatible with g found by Pittie in [32, 33], as we next recall. The Lie group $G = SO(9)$ has rank four, so we choose a maximal 4-torus T and a basis for its Lie algebra \mathfrak{t} given by

$$\mathfrak{t} = \langle e_1, e_2, e_3, e_4 \rangle$$

so that $\{e_k\}_{k=1}^4$ is orthonormal for the metric g . Consider the subspace $\mathfrak{a} \subset \mathfrak{t}_{\mathbb{C}}$ defined by

$$\mathfrak{a} = \langle e_1 + e_2 + i\sqrt{2}e_3, e_1 - e_2 + i\sqrt{2}e_4 \rangle.$$

It is clear that $\mathfrak{a} \cap \mathfrak{t} = \{0\}$ and $\mathfrak{a} \oplus \bar{\mathfrak{a}} = \mathfrak{t}_{\mathbb{C}}$, then one has a left-invariant complex structure J on G . Moreover, \mathfrak{a} is isotropic, so also is \mathfrak{s} , and thus J is compatible with the metric g .

Proposition 4.5 *The 18-dimensional compact complex manifold $X = (SO(9), J)$ satisfies $E_2(X) \neq E_{\infty}(X)$ and it has an SKT metric, which is in addition k -th Gauduchon for every $1 \leq k \leq 17$.*

Proof Let us consider the space

$$\Lambda V = \Lambda(w_{2,1}, w_{4,3}, w_{6,5}, w_{8,7}) \otimes \Lambda(v_1, v_2) \otimes \Lambda(u_1, u_2),$$

where the generators w 's have the bidegree indicated by the subindices, v_1, v_2 have bidegree $(1, 1)$ and u_1, u_2 have bidegree $(0, 1)$. Recall that, since the total degree of v_j is even, the space $\Lambda(v_1, v_2)$ is a polynomial algebra.

Let us consider $\bar{\partial}$ defined on generators by

$$\bar{\partial}u_1 = \bar{\partial}u_2 = \bar{\partial}v_1 = \bar{\partial}v_2 = \bar{\partial}w_{2,1} = 0, \quad \bar{\partial}w_{4,3} = f^2, \quad \bar{\partial}w_{6,5} = fg, \quad \bar{\partial}w_{8,7} = g^2,$$

where $f = v_1^2 + v_2^2$ and $g = v_1^2v_2^2$, and let ∂ be given by

$$\partial u_1 = v_1, \quad \partial u_2 = v_2, \quad \partial v_1 = \partial v_2 = \partial w_{2,1} = \partial w_{4,3} = \partial w_{6,5} = \partial w_{8,7} = 0.$$

Pittie proved (see also [46]) that $(\Lambda V, \bar{\partial})$ is the Dolbeault minimal model of X , and used it to show that the FSS of X does not degenerate at E_2 . For the sake of completeness, we show here that the map

$$d_2: E_2^{6,8} \longrightarrow E_2^{8,7}$$

is not identically zero. Let $\xi := u_1v_1 + u_2v_2$ and $\eta := u_1v_1v_2^2$. We have $\partial\xi = f$, $\partial\eta = g$. The element $f\xi\eta$ has bidegree $(6, 8)$, it is clearly $\bar{\partial}$ -closed and $\partial(f\xi\eta) = f^2\eta - fg\xi$. Since

$$\partial(f\xi\eta) + \bar{\partial}(w_{6,5}\xi - w_{4,3}\eta) = 0,$$

by the description (1–4) we have that $f\xi\eta \in E_2^{6,8}$ and

$$d_2(f\xi\eta) = \partial(w_{6,5}\xi - w_{4,3}\eta) = w_{4,3}g - w_{6,5}f \in E_2^{8,7}.$$

Using (3), it is a direct calculation to check that $w_{4,3}g - w_{6,5}f \notin \mathcal{Y}_2^{8,7}$, so d_2 is non-zero and $E_2^{6,8} \neq E_3^{6,8}$.

Alternatively, Tanré showed that $w_{4,3}g - w_{6,5}f$ is a non-zero Dolbeault-Massey product and, by [46, Théorème 9], the degeneration of the FSS at E_2 is equivalent to the Dolbeault formality, for every even-dimensional compact connected Lie group G such that $T \rightarrow G \rightarrow G/T$ is a principal holomorphic fiber bundle, where T is a maximal torus in G .

Finally, by [1] the fundamental form F is dd^c -harmonic, so it is SKT and Gauduchon. The last assertion in the proposition follows from [23, Corollary 2.3] because F is left-invariant. □

Note that $E_r(X) = E_\infty(X)$ for every $r \geq 3$, so the FSS of X degenerates at the third page (see [32, 33] for details).

Appendix A. The nilmanifolds Y_α^4 have pairwise non-isomorphic \mathbb{C} -minimal models

In this section we complete the proof of Theorem 3.4 by showing that the Lie algebras $\mathfrak{g}_{11}^{\alpha,1}$, $\alpha \in [0, +\infty)$, underlying Y_α^4 are pairwise non-isomorphic over \mathbb{C} . Recall that the structure equations of $\mathfrak{g}_{11}^{\alpha,1}$ are

$$\begin{aligned} dv^1 = dv^2 = dv^3 = 0, \quad dv^4 = v^{13}, \quad dv^5 = v^{23}, \quad dv^6 = v^{14} + v^{25} - v^{35}, \\ dv^7 = \alpha v^{12} + v^{15} + v^{24} + v^{34}, \quad dv^8 = v^{16} - 2v^{25} + v^{27} - v^{35} - v^{45}. \end{aligned} \tag{16}$$

We will prove that if $f: \mathfrak{g}_{11}^{\alpha,1} \longrightarrow \mathfrak{g}_{11}^{\alpha',1}$ is an isomorphism of Lie algebras, then $\alpha = \alpha'$. Note that the dual map $f^*: (\mathfrak{g}_{11}^{\alpha',1})^* \longrightarrow (\mathfrak{g}_{11}^{\alpha,1})^*$ extends to a map $F: \bigwedge^*(\mathfrak{g}_{11}^{\alpha',1})^* \longrightarrow \bigwedge^*(\mathfrak{g}_{11}^{\alpha,1})^*$ that commutes with the differentials, i.e. $F \circ d = d \circ F$.

Let $\{v^k\}_{k=1}^8$ (resp. $\{v'^k\}_{k=1}^8$) be a basis for $(\mathfrak{g}_{11}^{\alpha,1})^*$ (resp. $(\mathfrak{g}_{11}^{\alpha',1})^*$) satisfying the equations (16) for α (resp. α'). In terms of these bases, F is determined by

$$F(v'^k) = \sum_{j=1}^8 \lambda_j^k v^j, \quad k = 1, \dots, 8,$$

where the matrix $\Lambda = (\lambda_j^k)_{k,j=1,\dots,8}$ belongs to $GL(8, \mathbb{C})$, and the condition $F \circ d = d \circ F$ reads as

$$F(dv^k) - d(Fv'^k) = 0, \quad \text{for each } 1 \leq k \leq 8. \tag{17}$$

In [26] the isomorphism problem for this family of algebras was studied in the case of the field of real numbers. We notice that [26, Lemma 3.4] is still valid over \mathbb{C} , as it is the first part of the proof of [26, Lemma 3.5]. Thus, one arrives at a complex matrix $\Lambda = (\lambda_j^k)_{k,j}$ of the form

$$\Lambda = \begin{pmatrix} \lambda_1^1 & \lambda_2^1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda_1^2 & \lambda_2^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_3^3 & 0 & 0 & 0 & 0 & 0 \\ \lambda_1^4 & \lambda_2^4 & \lambda_3^4 & \lambda_4^4 & \lambda_5^4 & 0 & 0 & 0 \\ \lambda_1^5 & \lambda_2^5 & \lambda_3^5 & \lambda_4^5 & \lambda_5^5 & 0 & 0 & 0 \\ \lambda_1^6 & \lambda_2^6 & \lambda_3^6 & \lambda_4^6 & \lambda_5^6 & \lambda_6^6 & \lambda_7^6 & 0 \\ \lambda_1^7 & \lambda_2^7 & \lambda_3^7 & \lambda_4^7 & \lambda_5^7 & \lambda_6^7 & \lambda_7^7 & 0 \\ \lambda_1^8 & \lambda_2^8 & \lambda_3^8 & \lambda_4^8 & \lambda_5^8 & \lambda_6^8 & \lambda_7^8 & \lambda_8^8 \end{pmatrix}, \tag{18}$$

where $\lambda_1^1 \lambda_2^2 - \lambda_2^1 \lambda_1^2 \neq 0, \lambda_3^3, \lambda_8^8 \neq 0$, together with

$$\begin{aligned} \lambda_4^4 &= \lambda_1^1 \lambda_3^3, & \lambda_5^5 &= \lambda_2^2 \lambda_3^3, & \lambda_6^6 &= \lambda_2^2 (\lambda_3^3)^2, & \lambda_7^7 &= -\lambda_1^1 (\lambda_3^3)^2, \\ \lambda_4^5 &= \lambda_2^1 \lambda_3^3, & \lambda_6^7 &= -\lambda_2^1 (\lambda_3^3)^2, & \lambda_7^6 &= -\lambda_1^1 (\lambda_3^3)^2, \end{aligned}$$

and

$$(\lambda_2^1)^2 + (\lambda_2^2)^2 - \lambda_2^2 \lambda_3^3 = 0, \quad \lambda_2^1 (\lambda_3^3 + 2 \lambda_2^2) = 0. \tag{19}$$

We distinguish two cases depending on the vanishing of the complex coefficient λ_2^1 .

Let us first suppose that $\lambda_2^1 = 0$. Note that in this case the second part of the proof of [26, Lemma 3.5] and [26, Lemma 3.6] are still valid for \mathbb{C} . Hence, for $\alpha \neq \alpha'$, any F with $\lambda_2^1 = 0$ is not an isomorphism over \mathbb{C} .

Consequently, we will now assume that $\lambda_2^1 \neq 0$. Then, the second expression in (19) gives $\lambda_3^3 = -2 \lambda_2^2$, and replacing it in the first one, we get

$$(\lambda_2^1)^2 + 3 (\lambda_2^2)^2 = 0. \tag{20}$$

On the other hand, the coefficients of v^{14} and v^{15} in the condition (17) for $k = 7$ are, respectively, $2 \lambda_1^1 \lambda_1^2 + \lambda_2^1 \lambda_3^3$ and $\lambda_2^1 \lambda_1^2 + \lambda_1^1 (\lambda_2^2 - \lambda_3^3)$, so using again $\lambda_3^3 = -2 \lambda_2^2$ we get

$$\lambda_1^1 \lambda_1^2 - \lambda_2^1 \lambda_2^2 = 0, \quad \lambda_2^1 \lambda_1^2 + 3 \lambda_1^1 \lambda_2^2 = 0. \tag{21}$$

The Eqs. (20 and 21) can be solved explicitly in terms of λ_2^1 . Notice that $(\lambda_1^1)^2 = (\lambda_2^2)^2$. There are four solutions:

$$\lambda_1^1 = \vartheta \xi \frac{i}{\sqrt{3}} \lambda_2^1, \quad \lambda_1^2 = \xi \lambda_2^1, \quad \lambda_2^2 = \vartheta \frac{i}{\sqrt{3}} \lambda_2^1, \quad \lambda_3^3 = -2 \vartheta \frac{i}{\sqrt{3}} \lambda_2^1, \quad \vartheta, \xi \in \{\pm 1\}. \tag{22}$$

We will use these solutions below, after taking into account first other equations coming from the conditions (17) for $k = 6, 7, 8$ (note that (17) is already fulfilled for $1 \leq k \leq 5$).

The coefficients of v^{13} and v^{23} in the condition (17) for $k = 6, 7$ give the following equations:

$$\begin{aligned} \lambda_4^6 &= \lambda_1^1 \lambda_3^4 + \lambda_3^3 \lambda_1^5 + \lambda_1^2 \lambda_3^5, & \lambda_5^6 &= \lambda_2^1 \lambda_3^4 + \lambda_3^3 \lambda_2^5 + \lambda_2^2 \lambda_3^5, \\ \lambda_4^7 &= -\lambda_3^3 \lambda_1^4 + \lambda_1^2 \lambda_3^4 + \lambda_1^1 \lambda_3^5, & \lambda_5^7 &= -\lambda_3^3 \lambda_2^4 + \lambda_2^2 \lambda_3^4 + \lambda_2^1 \lambda_3^5. \end{aligned}$$

Using this, from the coefficients of $v^{13}, v^{14}, v^{16}, v^{23}$ and v^{34} in the condition (17) for $k = 8$ we have the following equalities:

$$\begin{aligned} \lambda_4^8 &= (\lambda_3^3 + \lambda_3^4) \lambda_1^5 - (2 \lambda_1^2 + \lambda_1^4) \lambda_3^5 + \lambda_1^1 \lambda_3^6 + \lambda_1^2 \lambda_3^7, \\ \lambda_5^8 &= (\lambda_3^3 + \lambda_3^4) \lambda_2^5 - (2 \lambda_2^2 + \lambda_2^4) \lambda_3^5 + \lambda_2^1 \lambda_3^6 + \lambda_2^2 \lambda_3^7, \\ \lambda_6^8 &= -2(\lambda_1^2)^2 \lambda_3^3 - 2 \lambda_1^2 \lambda_3^3 \lambda_1^4 + ((\lambda_1^1)^2 + (\lambda_1^2)^2) \lambda_3^4 + 2 \lambda_1^1 \lambda_3^3 \lambda_1^5 + 2 \lambda_1^1 \lambda_1^2 \lambda_3^5, \\ \lambda_7^8 &= -\lambda_1^2 (\lambda_3^3)^2 - \lambda_1^2 \lambda_3^3 \lambda_3^4 + \lambda_1^1 \lambda_3^3 \lambda_3^5, \\ \lambda_8^8 &= (\lambda_1^1 \lambda_2^2 - \lambda_2^1 \lambda_1^2) (\lambda_3^3)^2. \end{aligned}$$

Now, considering these equalities together with the solutions (22), the conditions (17) for all $1 \leq k \leq 8$ are equivalent to the following system:

$$\begin{aligned} \lambda_1^4 - \sqrt{3} i \vartheta \xi \lambda_2^4 + \sqrt{3} i \vartheta \lambda_1^5 - \xi \lambda_2^5 &= -4 \xi \lambda_2^1, \\ 3 \lambda_1^4 - \sqrt{3} i \vartheta \xi \lambda_2^4 + \sqrt{3} i \vartheta \lambda_1^5 - 3 \xi \lambda_2^5 &= -4 \xi \alpha (\lambda_2^1)^2, \\ 3 \sqrt{3} i \vartheta \xi \lambda_1^4 + 3 \lambda_2^4 + 3 \xi \lambda_1^5 + 3 \sqrt{3} i \vartheta \lambda_2^5 &= -4 \lambda_2^1 (\sqrt{3} i \vartheta - 2 \xi \lambda_2^1), \\ \sqrt{3} i \vartheta \lambda_1^4 - 3 \xi \lambda_2^4 + 3 \lambda_1^5 - \sqrt{3} i \vartheta \xi \lambda_2^5 &= -4 \xi \lambda_2^1 (\alpha' - \frac{i}{\sqrt{3}} \vartheta \alpha \lambda_2^1), \\ 3 \sqrt{3} i \vartheta \xi \lambda_1^4 + 3 \xi \lambda_1^5 &= -4 \lambda_2^1 (\sqrt{3} i \vartheta + \xi \lambda_2^1), \\ 4 \xi \alpha (\lambda_2^1)^3 + 2 i \sqrt{3} \vartheta \xi \alpha (\lambda_2^1)^2 \lambda_3^4 - 2 i \sqrt{3} \vartheta \lambda_2^1 \lambda_1^5 - 3 \lambda_2^4 \lambda_1^5 + 6 \xi \lambda_2^1 \lambda_2^5 \\ + 3 \lambda_1^4 \lambda_2^5 + 2 \xi \alpha (\lambda_2^1)^2 \lambda_3^5 + 3 \lambda_2^1 \lambda_1^6 - i \sqrt{3} \vartheta \xi \lambda_2^1 \lambda_2^6 + i \sqrt{3} \vartheta \lambda_2^1 \lambda_1^7 - 3 \xi \lambda_2^1 \lambda_2^7 &= 0. \end{aligned} \tag{23}$$

We focus our attention on the first five equations in (23) and consider them as a system of five linear equations in $\lambda_1^4, \lambda_2^4, \lambda_1^5$ and λ_2^5 . Applying Gaussian elimination, one reaches the matrix

$$\left(\begin{array}{cccc|ccc} 1 & -\sqrt{3} i \vartheta \xi & \sqrt{3} i \vartheta & -\xi & & & -4 \xi \lambda_2^1 \\ 0 & 1 & -\xi & 0 & & & -\frac{2i}{\sqrt{3}} \vartheta \lambda_2^1 (3 - \alpha \lambda_2^1) \\ 0 & 0 & 1 & \sqrt{3} i \vartheta \xi & & & \frac{2}{3} \vartheta \xi \lambda_2^1 (2 \vartheta \xi \lambda_2^1 - \sqrt{3} i (1 - \alpha \lambda_2^1)) \\ 0 & 0 & 0 & 0 & & & -\frac{4}{\sqrt{3}} \vartheta \xi \lambda_2^1 (\sqrt{3} \vartheta \alpha' + 2 i (3 - 2 \alpha \lambda_2^1)) \\ 0 & 0 & 0 & 0 & & & -4 \vartheta \lambda_2^1 (2 \vartheta \xi \lambda_2^1 + \sqrt{3} i (2 - \alpha \lambda_2^1)) \end{array} \right).$$

By hypothesis $\lambda_2^1 \neq 0$, so the system can be solved if and only if

$$\begin{cases} \sqrt{3} \vartheta \alpha' + 2 i (3 - 2 \alpha \lambda_2^1) = 0, \\ 2 \vartheta \xi \lambda_2^1 + \sqrt{3} i (2 - \alpha \lambda_2^1) = 0. \end{cases}$$

If $\alpha = 0$, then α' would be an imaginary number, which is not possible. Otherwise, λ_2^1 can be solved from the first equation and substituting its value in the second one, we get

$$2 \sqrt{3} i (\alpha - \xi \alpha') - 3 \vartheta (\alpha \alpha' - 4 \xi) = 0.$$

Since $\alpha, \alpha' \in (0, \infty)$, the imaginary part of this equation gives $\alpha' = \alpha$.

Hence, we have proved that given $\alpha, \alpha' \in [0, +\infty)$, the algebras $\mathfrak{g}_{11}^{\alpha,1}$ and $\mathfrak{g}_{11}^{\alpha',1}$ are isomorphic over \mathbb{C} if and only if $\alpha = \alpha'$.

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Declarations

Conflict of interest Not applicable.

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