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Communication

Conversion of Arbitrary Three-Dimensional Polarization States to Regular States via Spin Cancellation

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Abstract: The present work is motivated by the necessity of handling and controlling three-dimensional polarization states, whose appropriate preparation has increasing interest in areas like nanotechnologies, quantum computing and near-field phenomena. By virtue of the so-called characteristic decomposition, any polarization state of light can be represented as an incoherent superposition of a pure state, a fully unpolarized state and a discriminating state. The discriminating component has nonzero spin in general, in which case the state is said to be nonregular. A simple procedure to transform an arbitrary nonregular state to a regular one through its incoherent composition with a pure state is described, resulting in a state that lacks a discriminating component. In addition, a method to suppress the spin vector of any given polarization state through its incoherent combination with a circularly polarized pure state is presented. Both approaches allow for the configuration of polarization states with simple features.

Keywords: polarized light; nonregularity; spin; density matrix; qutrit

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1. Introduction

The preparation and control of polarization states is a topic of growing interest in near-field phenomena, nanophotonics and quantum technologies [1–12], as well as in multiple applications of laser-driven polarization distributions like processing and structuring of material surfaces, sharp focusing, capture and manipulation of microparticles, and microscopy [13–18].

Thus, many interesting approaches involving either spatial distributions of polarization states or physical situations where the three components of the electric field of the electromagnetic wave at a given point in space should be considered have been dealt with from different points of view [19–24].

While some of the above-mentioned physical situations involve coherent, or deterministic, optical fields whose electric vector evolves within a fixed plane for each point in space, the present work deals with polychromatic random light [25], whose polarization state at each particular point in space may require general three-dimensional (3D) treatment. In fact, any polarization state of random stationary light can be represented as an incoherent superposition of a pure state (pure referring to totally polarized), a discriminating state and a fully unpolarized state [26,27]. States whose discriminating component has nonzero spin are called nonregular, while regular states exhibit the particularly simple structure where the discriminating component is a two-dimensional unpolarized state [28,29]. Since a genuine feature of nonregular states is that the spin vector of their discriminating components has transversal character [29], nonregularity appears as a fundamental property of 3D polarization states whose control deserves attention.

The fact that nonregularity involves intricate polarization properties which influence light–matter interactions makes it desirable to develop methods to transform nonregular states into regular ones in order to improve the control of the polarimetric anisotropies of the interacting polarization states.

The main aim of this work is to introduce, for the first time, a general procedure to convert a nonregular state to a regular one (thus exhibiting much simpler characteristic structure) by means of the incoherent composition of the original state with a pure state which is determined from the original nonregular state itself. In other words, this communication focuses on the formulation of the theoretical framework for the removal of nonregularity from an arbitrary 3D polarization state.

The problem is solved by means of the characteristic decomposition of the polarization matrix of the state, which is represented with respect to the intrinsic reference frame of the discriminating component. This allows for the identification of a pure state (in general elliptically polarized) whose incoherent composition with the original state produces a regular state lacking a discriminating component.

In addition, the issue of cancelling the spin vector of a generic polarization state is addressed through its incoherent superposition with a circularly polarized state whose spin vector is opposite to that of the original state.

Both procedures can be applied to the configuration and control of general three-dimensional polarization states and, due to the formal similarity of polarization density matrices and quantum 3×3 density matrices, also provide deeper insights into the structure of density matrices representing three-level quantum states as, for instance, qutrits [5].

2. Theoretical Framework

The polarization matrix, which contains all of the second-order measurable information about the state of polarization (including intensity) of an electromagnetic wave, is defined as the following 3×3 Hermitian matrix [26]:

$$\mathbf{R} = \langle \boldsymbol{\varepsilon}(t) \otimes \boldsymbol{\varepsilon}^{\dagger}(t) \rangle = \begin{pmatrix} \langle \varepsilon_x(t)\, \varepsilon_x^*(t) \rangle & \langle \varepsilon_x(t)\, \varepsilon_y^*(t) \rangle & \langle \varepsilon_x(t)\, \varepsilon_z^*(t) \rangle \\ \langle \varepsilon_y(t)\, \varepsilon_x^*(t) \rangle & \langle \varepsilon_y(t)\, \varepsilon_y^*(t) \rangle & \langle \varepsilon_y(t)\, \varepsilon_z^*(t) \rangle \\ \langle \varepsilon_z(t)\, \varepsilon_x^*(t) \rangle & \langle \varepsilon_z(t)\, \varepsilon_y^*(t) \rangle & \langle \varepsilon_z(t)\, \varepsilon_z^*(t) \rangle \end{pmatrix}, \tag{1}$$

where the elements are the second-order moments of the zero-mean analytic signals $\varepsilon_i(t)$ $(i = x, y, z)$ (complex random processes) [25,30] associated with the three Cartesian components of the electric field vector at point $\mathbf{r}$ in space with respect to the given Cartesian reference frame XYZ [31]. Superscript † denotes the conjugate transpose, $\otimes$ stands for the Kronecker Product, and the brackets $\langle \ldots \rangle$ indicate time averaging. In the case of stationary and ergodic fields, the brackets can also be interpreted as ensemble averaging over the ensemble of sample realizations [25,32]. The convention $\mathbf{R} = \langle \boldsymbol{\varepsilon}(t) \otimes \boldsymbol{\varepsilon}^{\dagger}(t) \rangle$, which is common in polarization optics, is used in place of the convention $\mathbf{R} = \langle \boldsymbol{\varepsilon}^*(t) \otimes \boldsymbol{\varepsilon}^{\mathrm{T}}(t) \rangle$, which is frequently used in optical coherence theory.

Thus, $\mathbf{R}$ is characterized by nine quantities which are measurable through the corresponding 3D Stokes parameters [33–41]. Obviously, $\mathbf{R}$ takes different specific forms depending on the Cartesian reference frame XYZ considered.

Let us now bring out the unitary similarity transformation that diagonalizes $\mathbf{R}$

$$\mathbf{R} = \mathbf{U}\mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)\mathbf{U}^{\dagger}, \quad [\lambda_1 \geq \lambda_2 \geq \lambda_3], \tag{2}$$

where $\mathbf{U}$ is a unitary matrix, and $(\lambda_1, \lambda_2, \lambda_3)$ are the real eigenvalues of $\mathbf{R}$, which are necessarily non-negative due to the fact that $\mathbf{R}$ has the mathematical structure of a covariance matrix of the three zero-mean random signals $\varepsilon_i(t)$ [32]. The eigenvalues have been taken in decreasing order $(\lambda_1 \geq \lambda_2 \geq \lambda_3)$ without loss of generality. Let us also observe that $\mathrm{tr}\mathbf{R} = \lambda_1 + \lambda_2 + \lambda_3$ represents the intensity I of the state. For certain purposes, it is appropriate to use the polarization density matrix $\hat{\mathbf{R}} = \mathbf{R}/I$, whose eigenvalues are denoted as $\hat{\lambda}_i = \lambda_i / I$ $(i = 1, 2, 3)$, with $\hat{\lambda}_1 + \hat{\lambda}_2 + \hat{\lambda}_3 = 1$.

The above diagonalization of $\mathbf{R}$ leads directly to the so-called spectral decomposition

$$\begin{gathered} \mathbf{R} = \hat{\lambda}_1 I\, \hat{\mathbf{R}}_{p1} + \hat{\lambda}_2 I\, \hat{\mathbf{R}}_{p2} + \hat{\lambda}_3 I\, \hat{\mathbf{R}}_{p3}\,, \\ \left[\hat{\mathbf{R}}_{p1} = \mathbf{U}\mathrm{diag}(1,0,0)\mathbf{U}^{\dagger},\ \hat{\mathbf{R}}_{p2} = \mathbf{U}\mathrm{diag}(0,1,0)\mathbf{U}^{\dagger},\ \hat{\mathbf{R}}_{p3} = \mathbf{U}\mathrm{diag}(0,0,1)\mathbf{U}^{\dagger}\right], \end{gathered} \tag{3}$$

which shows that $\mathbf{R}$ can be interpreted as the incoherent superposition of three pure states whose associated analytic signal vectors are mutually orthogonal [32,42].

The above interpretation relies on the concept of superposition of pure polarization states, which refers to the coincidence, at a given point in space, of a number of pure states, for instance belonging to different random stationary light beams. Depending on the single-point correlations between the fluctuating components of the analytic signal vectors of the superimposed (also composed or combined) states, the resulting state can be pure (coherent composition) or partially polarized (partially coherent or incoherent composition). In fact, any partially polarized state can always be considered a partially coherent superposition of a number n of pure states, represented by the respective analytic signal vectors $\boldsymbol{\varepsilon}_i$ $(i = 1, \ldots, n)$ and described as follows ([32], Section 1.12.3) (the time dependence is omitted for brevity):

$$\begin{aligned}
\mathbf{R} &= \langle \boldsymbol{\varepsilon} \otimes \boldsymbol{\varepsilon}^{\dagger} \rangle = \left\langle \left(\sum_{i=1}^{n} \boldsymbol{\varepsilon}_i \right) \otimes \left(\sum_{i=1}^{n} \boldsymbol{\varepsilon}_i \right)^{\dagger} \right\rangle = \mathbf{X} + \mathbf{Y} \\
\mathbf{X} &= \left\langle \sum_{i=1}^{n} \boldsymbol{\varepsilon}_i \otimes \boldsymbol{\varepsilon}_i^{\dagger} \right\rangle = \sum_{i=1}^{n} \langle \boldsymbol{\varepsilon}_i \otimes \boldsymbol{\varepsilon}_i^{\dagger} \rangle = \sum_{i=1}^{n} \mathbf{R}_i \\
\mathbf{Y} &= \left\langle \sum_{i,j=1, i \neq j}^{n} \left[\left(\boldsymbol{\varepsilon}_i \otimes \boldsymbol{\varepsilon}_j^{\dagger} \right) + \left(\boldsymbol{\varepsilon}_j \otimes \boldsymbol{\varepsilon}_i^{\dagger} \right) \right] \right\rangle = \sum_{i,j=1, i \neq j}^{n} \left[\left\langle \boldsymbol{\varepsilon}_i \otimes \boldsymbol{\varepsilon}_j^{\dagger} \right\rangle + \langle \boldsymbol{\varepsilon}_j \otimes \boldsymbol{\varepsilon}_i^{\dagger} \rangle \right]
\end{aligned} \tag{4}$$

where the matrix $\mathbf{X}$ formally corresponds to the polarization matrix describing the incoherent composition of the pure states $\boldsymbol{\varepsilon}_i$, whereas the term $\mathbf{Y} = \mathbf{R} - \mathbf{X}$ is a Hermitian matrix that is not positive semidefinite and consequently does not correspond to any polarization state. When the superposed states $\boldsymbol{\varepsilon}_i$ are fully uncorrelated, the average in the expression of $\mathbf{Y}$ becomes the zero matrix ($\mathbf{Y}$=$\mathbf{0}$), so that $\mathbf{R}$=$\mathbf{X}$ (incoherent composition).

In the general case of partially correlated fields, $\mathbf{Y}$ does not vanish ($\mathbf{Y} \neq \mathbf{0}$) and thus determines the structure of polarimetric purity of the state $\mathbf{R}$. In the limiting case where all superimposed fields are mutually fully correlated (coherent composition), the composed state is pure, with the associated analytic signal vector $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_1 + \boldsymbol{\varepsilon}_2 \ldots + \boldsymbol{\varepsilon}_n$ and with the associated polarization matrix $\mathbf{R}$=$\boldsymbol{\varepsilon} \otimes \boldsymbol{\varepsilon}^{\dagger}$.

The spectral decomposition (3) can be rearranged to build the corresponding characteristic decomposition [26,31,43],

$$\begin{gathered}
\mathbf{R} = P_1 I \hat{\mathbf{R}}_p + (P_2 - P_1) I \hat{\mathbf{R}}_m + (1 - P_2) I \hat{\mathbf{R}}_{u-3D}, \\
\hat{\mathbf{R}}_p = \mathbf{U}\mathrm{diag}(1,0,0)\mathbf{U}^{\dagger}, \quad \hat{\mathbf{R}}_m = \tfrac{1}{2}\mathbf{U}\mathrm{diag}(1,1,0)\mathbf{U}^{\dagger}, \quad \hat{\mathbf{R}}_{u-3D} = \tfrac{1}{3}\mathrm{diag}(1,1,1)\,,
\end{gathered} \tag{5}$$

where $\hat{\mathbf{R}}_p$ represents a pure state (which coincides with $\hat{\mathbf{R}}_{p1}$ in Equation (3)), $\hat{\mathbf{R}}_{u-3D}$ (proportional to the identity matrix) is a fully unpolarized state, and $\hat{\mathbf{R}}_m$ is called the discriminating component of $\mathbf{R}$, while the coefficients of the convex sum are regulated by the indices of polarimetric purity (IPP) defined from the eigenvalues of $\hat{\mathbf{R}}$ in the following manner [44,45]:

$$P_1 = \hat{\lambda}_1 - \hat{\lambda}_2, \quad P_2 = 1 - 3\hat{\lambda}_3. \tag{6}$$

Let us note that the convention $\lambda_1 \geq \lambda_2 \geq \lambda_3$ should be preserved to obtain the proper definition of the above IPP, and consequently, $0 \leq P_1 \leq P_2 \leq 1$ [45].

The structure of the characteristic decomposition shows that the discriminating component $\hat{\mathbf{R}}_m$ has two equal nonzero eigenvalues $\hat{\lambda}_1(\mathbf{R}_m) = \hat{\lambda}_2(\mathbf{R}_m) = 1/2$, so that, taken as isolated, it is characterized by $P_1(\mathbf{R}_m) = 0$ and $P_2(\mathbf{R}_m) = 1$ [27]. Regarding the whole state $\mathbf{R}$, it is pure if and only if $P_1 = 1$ (i.e., $P_1 = P_2 = 1$), while it is fully unpolarized if and only if $P_2 = 0$ (i.e., $P_1 = P_2 = 0$).

Thus, the characteristic decomposition reflects the structure of polarimetric purity of the polarization state [46]. While any two-dimensional (2D) polarization state (characterized by the fact that the electric field fluctuates in a fixed plane, called the polarization plane) can be represented as an incoherent superposition of a pure state and a 2D unpolarized state (the field fluctuates fully randomly within a fixed polarization plane), the characteristic decomposition of a general 3D state includes the peculiar discriminating state in addition to the pure and fully depolarized (maximally mixed) components.

Consequently, the IPP regulate the structure of polarimetric purity–randomness of **R**, while they are insensitive to the type of polarization exhibited by the spectral components. The overall polarimetric purity of a state **R** is given by the associated degree of polarimetric purity (or degree of polarization) [26],

$$P_{3D} = \sqrt{\frac{3P_1^2 + P_2^2}{4}} = \sqrt{\frac{1}{2}\left(3\mathrm{tr}\hat{\mathbf{R}}^2 - 1\right)} = \sqrt{\frac{1}{2}\left(3\sum_{i=1}^{3}\hat{\lambda}_i^2 - 1\right)}\,, \tag{7}$$

which takes values within the interval between $P_{3D} = 0$, for fully unpolarized states, and $P_{3D} = 1$, for fully polarized states.

States whose discriminating component corresponds to a 2D unpolarized state, i.e., $\hat{\mathbf{R}}_m = \mathrm{diag}\,(1/2,1/2,0)$, are called regular and represent a borderline case of general, nonregular states. Consequently, a polarization state is regular if and only if its discriminating component lacks spin, while maximal nonregularity is achieved by the so-called perfect nonregular states characterized by the fact that they can be represented by an equiprobable incoherent mixture of a circularly polarized state and a linearly polarized state whose electric field fluctuates along a direction normal to the polarization plane of the circular component [27,28]. The degree of nonregularity provides a measure of the distance of the state to a regular one and is defined as [28]

$$P_N(\mathbf{R}) = 4\,(P_2 - P_1)\,\hat{m}_3, \quad [0 \le P_N(\mathbf{R}) \le 1], \tag{8}$$

where $\hat{m}_3$ is the smallest eigenvalue of $\mathrm{Re}\,(\hat{\mathbf{R}}_m)$, with $0 \le \hat{m}_3 \le 1/4$.

Additional complementary descriptors can be defined through the intrinsic representation of **R**, which is determined by means of the diagonalization of the real part Re**R** of **R** [47],

$$\mathrm{Re}\mathbf{R} = \mathbf{Q}_O \mathrm{diag}(a_1, a_2, a_3)\mathbf{Q}_O^{\mathrm{T}}\,, \quad [a_1 \ge a_2 \ge a_3], \tag{9}$$

where $\mathbf{Q}_O$ is a proper orthogonal matrix ($\mathbf{Q}_O^{\mathrm{T}} = \mathbf{Q}_O^{-1}$, $\det\mathbf{Q}_O = +1$), superscript T indicates the transpose matrix and the non-negative diagonal elements (a_1, a_2, a_3) (taken in decreasing order without loss of generality) are called the principal intensities of **R**. The intrinsic polarization matrix $\mathbf{R}_O$ (representing the same state as **R**, but referenced with respect to the so-called intrinsic reference frame $X_O Y_O Z_O$ instead of the generic original one XYZ) is defined by $\mathbf{R}_O = \mathbf{Q}_O^{\mathrm{T}}\mathbf{R}\mathbf{Q}_O$ and can be expressed in the simple form [31,47]

$$\mathbf{R}_O \equiv \begin{pmatrix} a_1 & -in_{O3}/2 & in_{O2}/2 \\ in_{O3}/2 & a_2 & -in_{O1}/2 \\ -in_{O2}/2 & in_{O1}/2 & a_3 \end{pmatrix} = I \begin{pmatrix} \hat{a}_1 & -i\hat{n}_{O3}/2 & i\hat{n}_{O2}/2 \\ i\hat{n}_{O3}/2 & \hat{a}_2 & -i\hat{n}_{O1}/2 \\ -i\hat{n}_{O2}/2 & i\hat{n}_{O1}/2 & \hat{a}_3 \end{pmatrix}, \tag{10}$$
$$[a_1 \ge a_2 \ge a_3,\ \ I = a_1 + a_2 + a_3,\ \ \hat{a}_i = a_i/I,\ \ \hat{n}_{Oi} = n_{Oi}/I \quad (i = 1,2,3)],$$

where the off-diagonal elements are fully determined by the spin vector in its intrinsic representation $\mathbf{n}_O \equiv (n_{O1}, n_{O2}, n_{O3})^{\mathrm{T}}$ [31,47]. Thus, the information held by the polarization matrix of any polarization state can be parametrized in terms of nine scalar descriptors, namely the three principal intensities (a_1, a_2, a_3), the three components (n_{O1}, n_{O2}, n_{O3}) of the spin vector along the respective intrinsic axes $X_O Y_O Z_O$ and the three angles determining the rotation associated with $\mathbf{Q}_O$ [31,48]. Consequently, regardless of the spatial orientation of the principal intensities of the polarization state, the intrinsic polarization properties are determined by the polarization object constituted by the polarization ellipsoid defined by (a_1, a_2, a_3) and the spin vector [47,48] (see Figure 1).

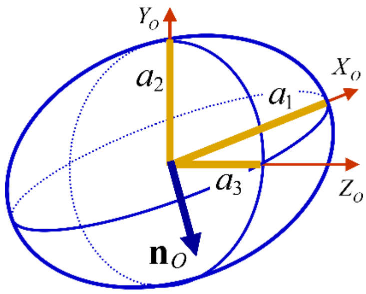

Figure 1. The polarization object encompasses the intrinsic information of a polarization state by means of the intensity ellipsoid, whose semiaxes a_1, a_2, a_3 are the eigenvalues of the real part of the polarization matrix (taken in decreasing order, $a_1 \geq a_2 \geq a_3$), together with the intrinsic representation $\mathbf{n}_O$ of the spin vector of the state.

The principal intensities determine three meaningful quantities, namely the intensity $I = a_1 + a_2 + a_3$, the degree of linear polarization $P_l = \hat{a}_1 - \hat{a}_2$ and the degree of directionality $P_d = 1 - 3\hat{a}_3$. Other additional descriptors are the degree of circular polarization $P_c = |\mathbf{n}|/I$, given by the intensity-normalized absolute value of the spin vector, and the degree of elliptical purity $P_e = \sqrt{P_l^2 + P_c^2}$ [49]. The set P_l, P_c, P_d constitutes the so-called components of purity (CP) of the polarization state [50], while the set $(I, P_l, P_d, \hat{n}_{O1}, \hat{n}_{O2}, \hat{n}_{O3})$ constitutes, precisely, the six nonzero intrinsic Stokes parameters [40].

Complementary to the IPP, the CP carry qualitative information on the type of polarization exhibited by the state $\mathbf{R}$ in such a manner that the contributions of the CP as sources of the overall purity of $\mathbf{R}$ are evidenced by the relation [50]

$$P_{3D} = \sqrt{\frac{3\left(P_l^2 + P_c^2\right) + P_d^2}{4}}, \tag{11}$$

which establishes an essential link between the IPP and the CP via Equation (7).

3. Regularizing Procedure

The nonregularity of a polarization state is governed by the properties of its discriminating component. In fact, a polarization state is regular if and only if its discriminating component exhibits nonzero spin or, equivalently, $\mathbf{R}_m$ is a real matrix, in which case, as said above, it must take the form of a 2D unpolarized state.

It has been shown that, when referenced to its intrinsic reference frame $X_{mO}Y_{mO}Z_{mO}$, the discriminating component adopts the simple form [8,28]

$$\mathbf{R}_{mO} = \frac{1}{2} I \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos^2\chi & -i\cos\chi\sin\chi \\ 0 & i\cos\chi\sin\chi & \sin^2\chi \end{pmatrix}, \quad [-\pi/4 \leq \chi \leq \pi/4]. \tag{12}$$

Let us now consider the representation of a nonregular $\mathbf{R}$ state (i.e., $\chi \neq 0$) and its characteristic components with respect to the reference frame $X_{mO}Y_{mO}Z_{mO}$. This transformation is achieved through $\mathbf{R}' = \mathbf{Q}_{mO}\,\mathbf{R}\,\mathbf{Q}_{mO}^{\mathrm{T}}$, where the proper orthogonal matrix $\mathbf{Q}_{mO}$ is what carries out the change to the intrinsic representation of the discriminating component $\mathbf{R}_{mO} = \mathbf{Q}_{mO}\,\mathbf{R}_m\,\mathbf{Q}_{mO}^{\mathrm{T}}$. Therefore, the characteristic decomposition of the new polarization matrix $\mathbf{R}'$ representing the state $\mathbf{R}$, but referenced with respect to the reference frame $X_{mO}Y_{mO}Z_{mO}$, adopts the form

$$\mathbf{R}' = IP_1\hat{\mathbf{R}}'_p + I(P_2 - P_1)\hat{\mathbf{R}}_{mO} + I(1-P_2)\hat{\mathbf{R}}_{u-3D}, \quad \begin{bmatrix} \hat{\mathbf{R}}'_p = \mathbf{Q}_{mO}\mathbf{U}\mathrm{diag}(1,0,0)\mathbf{U}^{\dagger}\mathbf{Q}^{\mathrm{T}}_{mO}, \\ \hat{\mathbf{R}}_{mO} = \frac{1}{2}\mathbf{Q}_{mO}\mathbf{U}\mathrm{diag}(1,1,0)\mathbf{U}^{\dagger}\mathbf{Q}^{\mathrm{T}}_{mO}, \\ \hat{\mathbf{R}}_{u-3D} = \frac{1}{3}\mathbf{Q}_{mO}\mathbf{U}\mathbf{I}\mathbf{U}^{\dagger}\mathbf{Q}^{\mathrm{T}}_{mO} = \frac{I}{3}\mathbf{I}, \end{bmatrix} \tag{13}$$

where $\mathbf{I}$ stands for the 3×3 identity matrix.

Recall that, from Equation (12), the discriminating component in its intrinsic representation $\hat{\mathbf{R}}_{mO}$ can be interpreted through the decomposition [28]

$$\begin{aligned} &\hat{\mathbf{R}}_{mO} = \frac{I}{2}\,\hat{\mathbf{R}}_{l-x} + \frac{I}{2}\,\hat{\mathbf{R}}_{e-x}, \\ &\hat{\mathbf{R}}_{l-x} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ &\hat{\mathbf{R}}_{e-x} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & c_\chi^2 & -ic_\chi s_\chi \\ 0 & ic_\chi s_\chi & s_\chi^2 \end{pmatrix}, \quad \begin{bmatrix} s_\chi = \sin\chi,\ c_\chi = \cos\chi \\ -\pi/4 \le \chi \le \pi/4 \end{bmatrix}. \end{aligned} \tag{14}$$

Here, $\hat{\mathbf{R}}_{l-x}$ and $\hat{\mathbf{R}}_{e-x}$ are the polarization matrices of a linearly polarized pure state whose electric field fluctuates along the axis X_{mO} and an elliptically polarized pure state whose polarization ellipse lies on plane $Y_{mO}Z_{mO}$ (i.e., its spin vector lies along the X_{mO} axis).

To achieve the cancellation of the spin vector of the discriminating component $\mathbf{R}_{mO}$ (as required for the transformation of $\mathbf{R}'$ to a regular state) while preserving the orientation of the spin vector of the pure component, it is sufficient to combine it incoherently with a state $(I/2)\,\hat{\mathbf{R}}_{\perp e-x}$ orthonormal to $(I/2)\,\hat{\mathbf{R}}_{e-x}$ (see Figure 2) so that the polarization matrix of the resulting state is given by

$$\begin{aligned} &\hat{\mathbf{R}}_{mO} + \frac{I}{2}\hat{\mathbf{R}}_{\perp e-x} = \frac{I}{2}\,\hat{\mathbf{R}}_{l-x} + \frac{I}{2}\,\hat{\mathbf{R}}_{e-x} + \frac{I}{2}\hat{\mathbf{R}}_{\perp e-x} \\ &= \frac{I}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{I}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & c_\chi^2 & -ic_\chi s_\chi \\ 0 & ic_\chi s_\chi & s_\chi^2 \end{pmatrix} + \frac{I}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & s_\chi^2 & ic_\chi s_\chi \\ 0 & -ic_\chi s_\chi & c_\chi^2 \end{pmatrix} \\ &= \frac{I}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \underbrace{\frac{I}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{I\,\hat{\mathbf{R}}_{u-2D}} = \underbrace{\frac{I}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\frac{3I}{2}\,\hat{\mathbf{R}}_{u-3D}}. \end{aligned} \tag{15}$$

By applying this result to the composition $\mathbf{R}' + (I/2)(P_2 - P_1)\,\hat{\mathbf{R}}_{\perp e-x} \equiv \mathbf{R}_\Sigma$ of the complete state $\mathbf{R}'$ and the added pure state $(I/2)(P_2 - P_1)\,\hat{\mathbf{R}}_{\perp e-x}$, the polarization matrix $\mathbf{R}_\Sigma$ of the composed state is given by

$$\begin{aligned} \mathbf{R}_\Sigma &= \mathbf{R}' + (I/2)(P_2 - P_1)\hat{\mathbf{R}}_{\perp e-x} \\ &= IP_1\hat{\mathbf{R}}'_p + I(P_2 - P_1)\left[\hat{\mathbf{R}}_{mO} + (1/2)\,\hat{\mathbf{R}}_{\perp e-x}\right] + I(1-P_2)\mathbf{R}_{u-3D} \\ &= IP_1\hat{\mathbf{R}}'_p + I(P_2 - P_1)\left[\tfrac{3}{2}\hat{\mathbf{R}}_{u-3D}\right] + (1-P_2)\hat{\mathbf{R}}_{u-3D} \\ &= IP_1\hat{\mathbf{R}}'_p + I\left(1 + \tfrac{1}{2}P_2 - \tfrac{3}{2}P_1\right)\hat{\mathbf{R}}_{u-3D}, \qquad \left[1 + \tfrac{1}{2}P_2 - \tfrac{3}{2}P_1 = \hat{\lambda}_2\right], \end{aligned} \tag{16}$$

which can be rearranged as

$$\mathbf{R}' + (I/2)(P_2 - P_1)\hat{\mathbf{R}}_{\perp e-x} = I'P_1'\hat{\mathbf{R}}_p' + I'(1 - P_1')\hat{\mathbf{R}}_{u-3D}, \\ \left[I' = I\left(1 + \frac{P_2 - P_1}{2}\right), \quad P_1' = \frac{P_1}{1 + \frac{P_2 - P_1}{2}} \right], \tag{17}$$

thus showing that the combined state lacks a discriminating component; so, it is simply equivalent to a superposition of the pure state $\mathbf{R}_p'$ and a 3D unpolarized state $\mathbf{R}_{u-3D}$. As indicated in Equation (17), the IPP of the resulting state are given by $P_1' = P_2'$ in terms of the IPP P_1 and P_2 of $\mathbf{R}$.

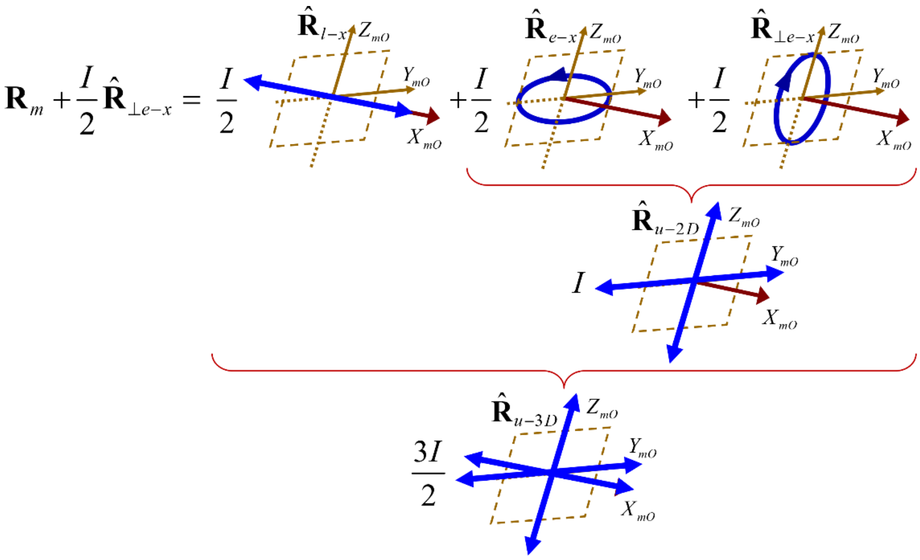

Figure 2. The incoherent composition of the discriminating component $\mathbf{R}_m$ with a specific pure state determined univocally from $\mathbf{R}_m$ itself results in a fully unpolarized state. As the first step, $\mathbf{R}_m$ is decomposed into an incoherent superposition of a linearly polarized state $\mathbf{R}_{l-x}$ and an elliptically polarized state $\mathbf{R}_{e-x}$ whose polarization plane is orthogonal to the direction along which the linear component fluctuates. Through the appropriate composition of $\mathbf{R}_m$ and a pure state $\mathbf{R}_{\perp e-x}$ whose 3D Jones vector is orthogonal to that of $\mathbf{R}_{e-x}$, the resulting state is completely unpolarized.

As a consequence of the rotational invariance of $\mathbf{R}_{u-3D}$ $(\mathbf{Q}_{mO}^{\mathrm{T}}\mathbf{R}_{u-3D}\mathbf{Q}_{mO} = \mathbf{R}_{u-3D})$, the polarization matrix of the combined state can be represented with respect to the original reference frame XYZ in the form (Figure 3)

$$\mathbf{R}_{\Sigma-XYZ} = \mathbf{Q}_{mO}^{\mathrm{T}}\mathbf{R}'\mathbf{Q}_{mO} = I'P_1'\hat{\mathbf{R}}_p + I'(1 - P_1')\hat{\mathbf{R}}_{u-3D}. \tag{18}$$

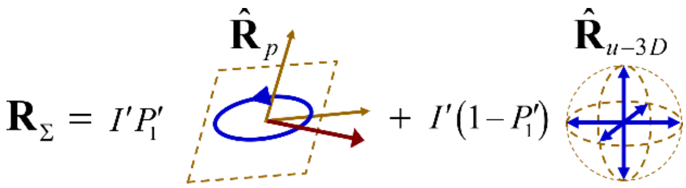

Figure 3. The incoherent composition of the state $\mathbf{R}$ with a specific pure state determined univocally from the discriminating component bends up being equivalent to a composition of the pure component $\mathbf{R}_p$ of $\mathbf{R}$ and a fully unpolarized state.

Due to the simple structure of $\mathbf{R}_{\Sigma-XYZ}$, its polarization descriptors take the values

$$P'_l = P'_1 = P'_d = P'_2 = P'_{3D} = \frac{P_1}{1+\frac{P_2-P_1}{2}},\;\; P'_c = \frac{1}{1+\frac{P_2-P_1}{2}}P_c,\;\; P'_N = 0. \tag{19}$$

Let us observe that, since the original state has been incoherently mixed with the added pure component, the purity is reduced $(P'_1 < P_1, P'_2 < P_2, P'_{3D} < P_{3D})$.

It should also be noted that the state $\mathbf{R}$ can alternatively be transformed to a regular state through the composition $\mathbf{R} + (P_2 - P_1)\mathbf{R}_k$, with $\mathbf{R}_k$ being any pure state satisfying the condition $\mathbf{n}\,(\mathbf{R}_k) = -\mathbf{n}\,(\mathbf{R}_m)$; that is, the spin vector of the added state $\mathbf{R}_k$ is opposite to the spin vector of the discriminating component $\mathbf{R}_m$ of $\mathbf{R}$. In particular, when $\mathbf{R}_k$ represents a circularly polarized state, the intensity of $\mathbf{R}_k$ is the smallest among the other infinite states $\mathbf{R}_k$ that perform the cancellation of the spin vector of $\mathbf{R}_m$. However, the resulting state $\mathbf{R} + (P_2 - P_1)\mathbf{R}_k$ has a nonvanishing discriminating state except for the particular case $\mathbf{R}_k = \mathbf{Q}^{\mathrm{T}}_{mO}[(I/2)\,\hat{\mathbf{R}}_{\perp e-x}]\,\mathbf{Q}_{mO}$ described above.

In summary, given a nonregular polarization state, it can always be incoherently combined with a pure state in such a way that the resulting state is regular and lacks a discriminating component; so, it is equivalent to an incoherent superposition of the pure component of the original state and a fully unpolarized state.

4. Spin Cancellation Procedure

The polarization matrix $\mathbf{R}$ of a given state can always be expressed as

$$\mathbf{R} = \mathrm{Re}(\mathbf{R}) + i\mathrm{Im}(\mathbf{R}). \tag{20}$$

Since $\mathbf{R}$ is a positive semidefinite Hermitian matrix, $\mathrm{Re}\,(\mathbf{R})$ is symmetric and has non-negative diagonal elements, while $\mathrm{Im}\,(\mathbf{R})$ is an antisymmetric matrix of the form

$$\mathrm{Im}(\mathbf{R}) = \begin{pmatrix} 0 & -n_3/2 & n_2/2 \\ n_3/2 & 0 & -n_1/2 \\ -n_2/2 & n_1/2 & 0 \end{pmatrix}, \tag{21}$$

where $n_i (i = 1,2,3)$ are the components of the spin vector $\mathbf{n}$ of the state represented by $\mathbf{R}$. This means that the imaginary parts of the off-diagonal elements of $\mathbf{R}$ determine the components of $\mathbf{n}$ referenced with respect to the Cartesian reference frame XYZ taken to represent the fluctuating components of the electric field [31,48].

To further simplify the mathematical expressions, it is worth noting that it is always possible to rotate the laboratory reference frame XYZ to the new axes $X_nY_nZ_n$ so the direction of Z_n coincides with the direction of the spin vector $\mathbf{n}$. Let us note also that $X_nY_nZ_n$ is not unique because it leaves free the orientations of axes X_nY_n, provided they are orthogonal to Z_n. Mathematically, $\mathbf{R}_n = \mathbf{Q}_n\mathbf{R}\mathbf{Q}_n^{\mathrm{T}}$, where $\mathbf{Q}_n$ is any proper orthogonal matrix that performs the required rotation transformation of the polarization matrix [31]. Thus, the transformed polarization matrix $\mathbf{R}_n$ representing the state takes the form

$$\begin{aligned} &\mathbf{R}_n = \mathrm{Re}(\mathbf{R}_n) + i\mathrm{Im}(\mathbf{R}_n), \\ &\mathrm{Re}(\mathbf{R}_n) = \mathbf{Q}_n\,\mathrm{Re}(\mathbf{R})\;\mathbf{Q}_n^{\mathrm{T}} \equiv \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{12} & c_{22} & c_{23} \\ c_{13} & c_{23} & c_{33} \end{pmatrix}, \\ &\mathrm{Im}(\mathbf{R}_n) = \mathbf{Q}_n\,\mathrm{Im}(\mathbf{R})\;\mathbf{Q}_n^{\mathrm{T}} = \begin{pmatrix} 0 & -n/2 & 0 \\ n/2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (n \geq 0). \end{aligned} \tag{22}$$

Now, by incoherently combining the state with a circularly polarized pure state $\mathbf{R}_{cl}$ whose spin vector is $-\mathbf{n}$, the composed state, represented with respect to the reference frame $X_n Y_n Z_n$, is given by

$$\mathbf{R}_t = \mathbf{R}_n + n\,\hat{\mathbf{R}}_{cl} = \begin{pmatrix} c_{11} & c_{12} - in/2 & c_{13} \\ c_{12} + in/2 & c_{22} & c_{23} \\ c_{13} & c_{23} & c_{33} \end{pmatrix} + n\frac{1}{2}\begin{pmatrix} 1 & i & 0 \\ -i & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} c_{11}+n/2 & c_{12} & c_{13} \\ c_{12} & c_{22}+n/2 & c_{23} \\ c_{13} & c_{23} & c_{33} \end{pmatrix} = \mathrm{Re}(\mathbf{R}_n) + n\,\hat{\mathbf{R}}_{u-2D}, \tag{23}$$
$$\left[\hat{\mathbf{R}}_{u-2D} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right].$$

Obviously, even though the reference frame $X_n Y_n Z_n$ has been taken for simplicity, the above spin suppression only requires that the spin vector of the state $\mathbf{R}_{cl}$ equals -**n**, regardless of the reference frame considered. Let us observe that the respective intrinsic reference frames of the states **R** and $\mathbf{R}_{cl}$ do not coincide in general.

Let us also note that the composition of the original state (represented either by **R** or $\mathbf{R}_n$, depending on the reference frame considered) with other pure states (not necessarily circularly polarized) exhibiting spin -**n** would also produce spin cancellation, but involving intensities larger than the intensity n of $\mathbf{R}_{cl}$.

5. Conclusions

Given a polarization state **R**, its spin vector be cancelled through its superposition with a pure state whose spin vector is opposite to that of **R**. Thus, there are infinite pure states that satisfy the cancellation condition. Among them, the one with the smallest intensity is a circularly polarized state with spin -**n**, in which case the resulting state can also be interpreted as an incoherent composition of the state represented by the real part of the polarization matrix of the original state and a 2D unpolarized state $n\hat{\mathbf{R}}_{u-2D} = n\,\mathrm{diag}(1/2, 1/2, 0)$ whose intensity equals the absolute value n of **n** (Figure 4).

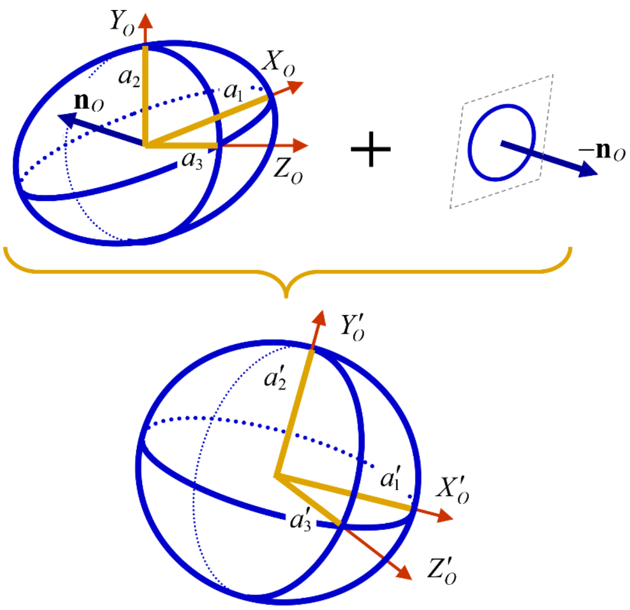

Figure 4. The incoherent composition of a given 3D polarization state with a specific circularly polarized state produces a regular state that lacks spin, and therefore, the resulting state can also be considered an incoherent composition of three linearly polarized states whose electric fields fluctuate along three mutually orthogonal directions.

In any case, the resulting state is regular and is equivalent to an incoherent superposition of three linearly polarized states whose electric fields fluctuate along mutually orthogonal directions. Furthermore, in general, the components of the characteristic decomposition of the resulting state are given by a linearly polarized state (the pure component), a 2D unpolarized state (the regular discriminating component) and a fully unpolarized state.

Although spin cancellation can always be performed through the composition procedure described above, it does not imply that any polarization state can be decomposed as an incoherent combination of a pure state and a mixed state with zero spin. In fact, the inspection of the characteristic decomposition whose discriminating component can exhibit nonzero spin shows that only regular states admit such a simple decomposition.

Even though the general spin cancellation procedure converts $\mathbf{R}$ to a regular state, there are other possible methods to regularize $\mathbf{R}$ via the cancellation of the spin vector of its discriminating component, leading, in general, to a resulting state whose characteristic components are given by the pure component $\mathbf{R}_p$ of $\mathbf{R}$ (thus, with nonzero spin in general), a 2D unpolarized state $\mathbf{R}_{u-2D}$ and a fully unpolarized state $\mathbf{R}_{u-3D}$. In particular, the combination $\mathbf{R} + (P_2 - P_1)\,\mathbf{Q}_{mO}^{\mathrm{T}}[(I/2)\,\hat{\mathbf{R}}_{\perp e-x}]\,\mathbf{Q}_{mO}$ described in Section 3 produces a regular state that lacks a discriminating component, so it is equivalent to a simple composition of the pure component $\mathbf{R}_p$ of $\mathbf{R}$ and $\mathbf{R}_{u-3D}$.

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