Highlights

Optimal properties of tensor product of B-bases[1]

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- Minimal conditioning for the ∞-norm of collocation matrices of the tensor product of normalized B-bases is shown.
- The maximality of the minimal eigenvalue and singular value of collocation matrices of the tensor product of normalized B-bases is shown.

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Abstract

It is proved the optimal conditioning for the ∞-norm of collocation matrices of the tensor product of normalized B-bases among the tensor product of all normalized totally positive bases of the corresponding space of functions. Bounds for the minimal eigenvalue and singular value and illustrative numerical examples are also included.

Keywords: tensor product, B-basis, totally positive basis, conditioning
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1. Introduction and main results

Given a system of functions $u = (u_0, \dots, u_n)$ defined on $I \subseteq \mathbb{R}$, the *collocation matrix* of u at $t_0 < \cdots < t_m$ in I is given by $(u_j(t_i))_{i=0,\dots,m}^{j=0,\dots,n}$. If $\sum_{i=0}^{n} u_i(t) = 1$ for all $t \in I$, then we say that the system is *normalized*. If all collocation matrices of u have all their minors nonnegative, then we say that the system is *totally positive* (TP). Normalized totally positive (NTP) systems play a crucial role in Computer Aided Geometric Design because they lead to shape preserving representations. Among all NTP bases of a space, the basis with optimal shape preserving properties is the *normalized B-basis* ([1, 2]). The Bernstein basis of polynomials and the B-spline basis are examples of normalized B-bases of their corresponding spaces. In this

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paper we extend some optimal properties of normalized B-bases given in [2] to their corresponding tensor products. Recall that, given two systems $u^1 = (u_0^1, \dots, u_m^1)$ and $u^2 = (u_0^2, \dots, u_n^2)$ of functions defined on $[a, b]$ and $[c, d]$, respectively, the system $u^1 \otimes u^2 := (u_i^1(x) \cdot u_j^2(y))_{i=0,\dots,m}^{j=0,\dots,n}$ is called a tensor product system and generates a tensor product surface. The *Kronecker product* of two square matrices $A = (a_{ij})_{1\le i,j\le m}$ and $B = (b_{ij})_{1\le i,j\le n}$, $A \otimes B$, is defined to be the $mn \times mn$ block matrix

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1m}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mm}B \end{pmatrix}.$$

Given the collocation matrices $B_1 := (u_j^1(x_i))_{0\le i,j\le m}$ and $B_2 := (u_j^2(y_i))_{0\le i,j\le n}$ of u^1 and u^2, $B_1 \otimes B_2$ is the collocation matrix of $u^1 \otimes u^2$ at $((x_i, y_j)_{j=0,\dots,n})_{i=0,\dots,m}$.

Given two square real matrices $A = (a_{ij})_{1\le i,j\le n}$ and $B = (b_{ij})_{1\le i,j\le n}$, $A \le B$ denotes that $a_{ij} \le b_{ij}$ for all i, j. Given a complex matrix $C = (c_{ij})_{1\le i,j\le n}$, A is said to *dominate* C if $|c_{ij}| \le a_{ij}$ for all i, j. If matrices A and B are nonsingular, by Corollary 4.2.11 of [4] we have that $A \otimes B$ is nonsingular and

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}. \tag{1}$$

The next result shows the optimal properties of a collocation matrix of the tensor product of normalized B-bases among all the corresponding collocation matrices of the tensor product of NTP bases of the spaces.

Theorem 1. *Let $u^1 = (u_0^1, \dots, u_m^1)$ be an NTP basis on $[a, b]$ of a space of functions $\mathcal{U}_1$, $u^2 = (u_0^2, \dots, u_n^2)$ be an NTP basis on $[c, d]$ of a space of functions $\mathcal{U}_2$ and let $v^1 = (v_0^1, \dots, v_m^1)$ and $v^2 = (v_0^2, \dots, v_n^2)$ be the normalized B-bases of $\mathcal{U}_1$ and $\mathcal{U}_2$, respectively. Given the increasing sequences of nodes $\boldsymbol{t} = (t_i)_{i=0}^m$ on $[a, b]$ and $\boldsymbol{r} = (r_i)_{i=0}^n$ on $[c, d]$, the nonsingular collocation matrices A_1 and M_1 of the bases u^1 and v^1, respectively, at $\boldsymbol{t}$, and A_2 and M_2 of the bases u^2 and v^2, respectively, at $\boldsymbol{r}$, the following properties hold:*

i) *The matrix $|(A_1 \otimes A_2)^{-1}|$ dominates $(M_1 \otimes M_2)^{-1}$.*

ii) *The minimal eigenvalue (resp., singular value) of $A_1 \otimes A_2$ is bounded above by the minimal eigenvalue (resp., singular value) of $M_1 \otimes M_2$.*

iii) *$\kappa_\infty(M_1 \otimes M_2) \le \kappa_\infty(A_1 \otimes A_2)$.*

Proof. i) By Corollary 1 of [2], $|A_1^{-1}|$ dominates $|M_1^{-1}|$ and $|A_2^{-1}|$ dominates $|M_2^{-1}|$. Hence, $|A_1^{-1}| \otimes |A_2^{-1}|$ dominates $|M_1^{-1}| \otimes |M_2^{-1}|$, and, since $|(A_1 \otimes A_2)^{-1}| = |A_1^{-1} \otimes A_2^{-1}| = |A_1^{-1}| \otimes |A_2^{-1}|$ by (1), $|(A_1 \otimes A_2)^{-1}|$ dominates $(M_1 \otimes M_2)^{-1}$.

ii) Let B_1 be an $n \times n$ matrix and B_2 an $m \times m$ matrix. If λ is an eigenvalue of B_1 and μ is an eigenvalue of B_2, then $\lambda\mu$ is an eigenvalue of $B_1 \otimes B_2$ and every eigenvalue of $B_1 \otimes B_2$ arises as such a product of eigenvalues of B_1 and B_2 (see Theorem 4.2.12 of [4]). By Corollary 2 of [2], we have that $\lambda_{\min}(A_1) \leq \lambda_{\min}(M_1)$ and that $\lambda_{\min}(A_2) \leq \lambda_{\min}(M_2)$. Hence,

$$\lambda_{\min}(M_1 \otimes M_2) = \lambda_{\min}(M_1)\lambda_{\min}(M_2) \geq \lambda_{\min}(A_1)\lambda_{\min}(A_2) = \lambda_{\min}(A_1 \otimes A_2).$$

The case of singular values is analogous to that of eigenvalues recalling that every nonzero singular value of $B_1 \otimes B_2$ is the product of a singular value of B_1 and a singular value of B_2 (see Theorem 4.2.15 of [4]).

iii) First, let us see that the infinity norm of the Kronecker product of two matrices $A = (a_{ij})_{1 \leq i,j \leq m}$ and $B = (b_{ij})_{1 \leq i,j \leq n}$ satisfies that $||A \otimes B||_\infty = ||A||_\infty ||B||_\infty$:

$$||A \otimes B||_\infty = \max_{0 \leq i \leq nm-1} \sum_{j=1}^{m} |a_{t+1,j}| (\sum_{k=1}^{n} |b_{r+1,k}|), \text{ where } t = \left[\frac{i}{n}\right], r = i - tn. \quad (2)$$

Denoting $R_t := \sum_{j=1}^{m} |a_{tj}|$ and $S_r = \sum_{k=1}^{n} |b_{rk}|$ we can rewrite (2) as

$$||A \otimes B||_\infty = \max_{0 \leq tn+r \leq nm-1} R_{t+1} S_{r+1} = \max_{1 \leq t \leq m} R_t \max_{1 \leq r \leq n} S_r = ||A||_\infty ||B||_\infty.$$

Hence, the condition number satisfies by (1) that

$$\begin{aligned} \kappa_\infty(B_1 \otimes B_2) &= ||B_1 \otimes B_2||_\infty ||(B_1 \otimes B_2)^{-1}||_\infty \\ &= ||B_1||_\infty ||B_2||_\infty ||B_1^{-1}||_\infty ||B_2^{-1}||_\infty = \kappa_\infty(B_1)\kappa_\infty(B_2). \end{aligned}$$

By Corollary 2 of [2], we have that $\kappa_\infty(M_1) \leq \kappa_\infty(A_1)$ and $\kappa_\infty(M_2) \leq \kappa_\infty(A_2)$. So, we conclude that $\kappa_\infty(M_1 \otimes M_2) \leq \kappa_\infty(A_1 \otimes A_2)$.

□

2. Numerical tests

In this section two numerical examples illustrating the theoretical results will be presented. The first example will be constructed by performing the tensor product of three different NTP bases $u^n = (u_0^n, \ldots, u_n^n)$ of the space $\mathcal{P}_n([0,1])$ of polynomials of degree not greater than n, which were used in [2]. A second example will be presented considering the tensor product of rational bases $r^n = (r_0^n, \dots, r_n^n)$ constructed from the three NTP bases considered in the first example with positive weights and the tensor product of rational monomial bases (the monomial basis is TP in $[0,1]$) also with positive weights. In fact, if u^n is a TP basis, it can be checked that the rational basis $(r_0^n, \dots, r_n^n)$, $r_i^n(x) = w_i u_i^n(x)/(\sum_{j=0}^n w_j u_j^n(x))$, with weights $w_i^n > 0$, is NTP. The basis $u^n = (b_0^n, \dots, b_n^n)$ formed by the Bernstein polynomials of degree n (see Example 6 a) in [2]) is the normalized B-basis of $\mathcal{P}_n([0,1])$ and the corresponding rational Bernstein basis r_B^n defined by $r_i^n(x) = w_i b_i^n(x)/(\sum_{j=0}^n w_j b_j^n(x))$ with $w_i > 0$, $i = 0, \dots, n$, is the normalized B-basis of its spanned space $\langle r_B^n \rangle$.

We will also consider the Said-Ball basis $s^n = (s_0^n, \dots, s_n^n)$ and the DP basis $c^n = (c_0^n, \dots, c_n^n)$, which are both NTP basis. The Said-Ball basis (see [3]) is defined by

$$s_i^n(x) = \binom{\lfloor n/2 \rfloor + i}{i} x^i (1-x)^{\lfloor n/2 \rfloor + 1}, \quad 0 \le i \le \lfloor (n-1)/2 \rfloor,$$

$s_i^n(x) = s_{n-i}^n(1-x)$, $\lfloor n/2 \rfloor + 1 \le i \le n$, and, if n is even

$$s_{n/2}^n(x) = \binom{n}{n/2} x^{n/2} (1-x)^{n/2},$$

where $\lfloor m \rfloor$ is the greatest integer less than or equal to m. The DP basis is given by $c_0^n(x) = (1-x)^n$, $c_n^n(x) = x^n$, $c_i^n(x) = x(1-x)^{n-i}$, $1 \le i \le \lfloor n/2 \rfloor - 1$, $c_i^n(x) = x^i(1-x)$, $\lfloor (n+1)/2 \rfloor + 1 \le i \le n-1$, and, if n is even $c_{\frac{n}{2}}^n(x) = 1 - x^{\frac{n}{2}+1} - (1-x)^{\frac{n}{2}+1}$, and, if n is odd,

$$c_{\frac{n-1}{2}}(x) = x(1-x)^{\frac{n+1}{2}} + \frac{1}{2}\left[1 - x^{\frac{n+1}{2}+1} - (1-x)^{\frac{n+1}{2}+1}\right], \; c_{\frac{n+1}{2}}(x) = c_{\frac{n-1}{2}}(1-x).$$

Let $(t_i^n)_{i=1}^{n+1}$ be the sequence of points given by $t_i = i/(n+2)$ for $i = 1, \dots, n+1$. Let us consider the Kronecker products of the collocation matrices of the Bernstein, Said-Ball and DP bases of $\mathcal{P}_n([0,1])$ for $n = 3, 4, 5$ at $(t_i^n)_{i=1}^{n+1}$ by itself: $M^n \otimes M^n$, $B_1^n \otimes B_1^n$ and $B_2^n \otimes B_2^n$, respectively. Then, the computation of the eigenvalues and the singular values of these matrices have been carried out with Mathematica using a precision of 100 digits. We can see the corresponding minimal eigenvalues and singular values in Table 1. It can be observed that the minimal eigenvalue, resp. singular value, of $M^n \otimes M^n$ is higher than the minimal eigenvalue, resp. singular value, of $B_1^n \otimes B_1^n$ and $B_2^n \otimes B_2^n$ as Theorem 1 has stated.

n	$M^n \otimes M^n$		$B_1^n \otimes B_1^n$		$B_2^n \otimes B_2^n$	
	λ_{min}	σ_{min}	λ_{min}	σ_{min}	λ_{min}	σ_{min}
3	$2.30e-03$	$2.19e-03$	$8.28e-04$	$8.28e-04$	$3.23e-04$	$3.20e-04$
4	$3.43e-04$	$3.23e-04$	$2.17e-04$	$1.97e-04$	$1.92e-05$	$1.11e-05$
5	$5.10e-05$	$4.78e-05$	$1.04e-05$	$1.03e-05$	$3.54e-07$	$2.77e-07$

Table 1: The minimal eigenvalue and singular value of $M^n \otimes M^n$, $B_1^n \otimes B_1^n$ and $B_2^n \otimes B_2^n$

We have also computed $k_\infty(M^n \otimes M^n)$, $k_\infty(B_1^n \otimes B_1^n)$ and $k_\infty(B_2^n \otimes B_2^n)$ for $n = 3, 4, 5$. Table 2 shows the results. It can be observed that $k_\infty(M^n \otimes M^n) \leq k_\infty(B_i^n \otimes B_i^n)$ for $i = 1, 2$, as it has been shown in Theorem 1.

n	$k_\infty(M^n \otimes M^n)$	$k_\infty(B_1^n \otimes B_1^n)$	$k_\infty(B_2^n \otimes B_2^n)$
3	$5.1883e+02$	$1.7361e+03$	$7.1797e+03$
4	$3.9690e+03$	$6.5610e+03$	$1.6080e+05$
5	$2.5264e+04$	$1.3949e+05$	$6.0028e+06$

Table 2: Infinity condition number k_∞ of $M^n \otimes M^n$, $B_1^n \otimes B_1^n$ and $B_2^n \otimes B_2^n$

As it has been said before, the rational Said-Ball, DP and monomial bases with positive weights are NTP. Taking a sequence of positive weights $(w_i^n)_{i=0}^n$ and taking into account that $\sum_{j=0}^n w_j^n b_j^n(x) \in \mathcal{P}_n([0,1])$ and that s^n, c^n and $m^n = (1, x, \ldots, x^n)$ are bases of $\mathcal{P}_n([0,1])$, then there exists three sequence of weights $(\overline{w}_i^n)_{i=0}^n$, $(\tilde{w}_i^n)_{i=0}^n$ and $(\hat{w}_i^n)_{i=0}^n$ satisfying

$$\sum_{j=0}^n w_j^n b_j^n(x) = \sum_{j=0}^n \overline{w}_j^n s_j^n(x) = \sum_{j=0}^n \tilde{w}_j^n b_j^n(x) = \sum_{j=0}^n \hat{w}_j^n c_j^n(x), \quad x \in [0,1]. \quad (3)$$

Sequences of positive weights $(w_i^n)_{i=0}^n$ have been randomly generated for $n = 3, 4, 5$, where each w_i^n is an integer in the interval $[1, 1000]$, until we have obtained a sequence such that there exists positive sequences $(\overline{w}_i^n)_{i=0}^n$, $(\tilde{w}_i^n)_{i=0}^n$ and $(\hat{w}_i^n)_{i=0}^n$ satisfying (3). Then we have the normalized B-basis r_B, and the NTP rational bases of $\langle r_B \rangle$ corresponding to the Said-Ball basis, the DP basis and the monomial basis. So, in the second example we have considered the Kronecker products of the collocation matrices of the generated rational Bernstein, Said-Ball, DP and monomial bases for $n = 3, 4, 5$ at $(t_i^n)_{i=1}^{n+1}$ by itself: $M_T^n = MR^n \otimes MR^n$, $B_{1,T}^n = BR_1^n \otimes BR_1^n$, $B_{2,T}^n = BR_2^n \otimes BR_2^n$ and $B_{3,T}^n = BR_3^n \otimes BR_3^n$, respectively. Then, the computation of the eigenvalues and the singular values of these matrices have been carried out with

Mathematica using a precision of 100 digits. We can see the corresponding minimal eigenvalues and singular values of M_T^n, $B_{1,T}^n$ and $B_{3,T}^n$ in Table 3. It can be observed that the minimal eigenvalue, resp. singular value, of M_T^n is higher than the minimal eigenvalue, resp. singular value, of $B_{1,T}^n$ and $B_{3,T}^n$ as Theorem 1 has proved.

n	M_T^n		$B_{1,T}^n$		$B_{3,T}^n$	
	λ_{min}	σ_{min}	λ_{min}	σ_{min}	λ_{min}	σ_{min}
3	$1.95e-03$	$1.74e-03$	$4.06e-04$	$3.78e-04$	$4.39e-06$	$3.82e-6$
4	$2.57e-04$	$2.05e-04$	$1.30e-04$	$1.09e-04$	$8.86e-08$	$2.35e-08$
5	$4.75e-05$	$4.36e-05$	$8.83e-06$	$8.66e-06$	$2.60e-10$	$1.63e-10$

Table 3: The minimal eigenvalue and singular value of M_T^n, $B_{1,T}^n$ and $B_{3,T}^n$

We have also computed $k_\infty(M_T^n)$, $k_\infty(B_{1,T}^n)$, $k_\infty(B_{2,T}^n)$ and $k_\infty(B_{3,T}^n)$ for $n = 3, 4, 5$ with Mathematica. The results can be seen in Table 4. It can be observed that $k_\infty(M_T^n) \leq k_\infty(B_{i,T}^n)$ for $i = 1, 2, 3$ (see Theorem 1).

n	$k_\infty(M_T^n)$	$k_\infty(B_{1,T}^n)$	$k_\infty(B_{2,T}^n)$	$k_\infty(B_{3,T}^n)$
3	$8.1049e+02$	$5.6308e+03$	$3.5425e+04$	$5.8525e+05$
4	$7.1105e+03$	$1.3484e+04$	$2.0327e+06$	$1.3229e+08$
5	$3.1318e+04$	$1.6543e+05$	$4.0614e+07$	$1.7440e+10$

Table 4: Infinity condition number k_∞ of M_T^n, $B_{1,T}^n$, $B_{2,T}^n$ and $B_{3,T}^n$

References

[1] Carnicer, J. M., Peña J. M. Totally positive bases for shape preserving curve design and optimality of B-splines. Comput. Aided Geom. Des. 11 (1994), 633–654.

[2] Delgado J., Peña J. M. Extremal and optimal properties of B-bases collocation matrices. Numer. Math. 146 (2020), 105–118.

[3] Goodman T. N. T., Said H. B. Shape preserving properties of the generalised Ball basis. Comput. Aided Geom. Des. 8 (1991), 115–121

[4] Horn R. A., Johnson C. R.: Topics in Matrix Analysis. Cambridge University Press, Cambridge, (1991)