Quantum α-Fractal Approximation

N. Vijender [a], A. K. B. Chand[b], M. A. Navascués[c] and M. V. Sebastián[d]

[a] Department of Mathematics, Vellore Institute of Technology Chennai, Chennai-600 127, India; [b] Department of Mathematics, Indian Institute of Technology Madras, Chennai - 600036, India; [c] Departamento de Matemática Aplicada, Escuela de Ingeniería y Arquitectura, Universidad de Zaragoza-50018, Spain; [d] Centro Universitario de la Defensa, Academia General Militar, Zaragoza-50090, Spain

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ABSTRACT
Fractal approximation is a well studied concept, but the convergence of all the existing fractal approximants towards the original function follows usually if the magnitude of the corresponding scaling factors approaches zero. In this article, for a given function $f \in \mathcal{C}(I)$, by exploiting fractal approximation theory and considering the classical q-Bernstein polynomials as *base functions*, we construct a sequence $\{f_n^{(q,\alpha)}(x)\}_{n=1}^{\infty}$ of (q,α)-fractal functions that converges uniformly to f even if the norm/magnitude of the scaling functions/scaling factors does not go to zero. The convergence of the sequence $\{f_n^{(q,\alpha)}(x)\}_{n=1}^{\infty}$ of (q,α)-fractal functions towards f follows from the convergence of the sequence of q-Bernstein polynomials of f towards f. For a given sequence $\{f_m(x)\}_{m=1}^{\infty}$ of positive functions on a compact real interval that converges uniformly to a function f, we develop a double sequence $\left\{\{f_{m,n}^{(q,\alpha)}(x)\}_{n=1}^{\infty}\right\}_{m=1}^{\infty}$ of (q,α)-fractal functions that converges uniformly to f.

KEYWORDS
q-Bernstein polynomials; fractal functions; constrained approximation; fractals; interpolation

1. Introduction

Lupas [12] first introduced the q-analogue of Bernstein polynomials and this created a new extension of approximation theory, namely, q-approximation theory. In the last twenty five years, several researchers [7–10, 20–22] have proposed the q-extension of various results of classical approximation theory. However, classical approximation theory and q-approximation theory deal with the approximation of functions using smooth functions or infinitely differentiable functions. But, the classical smooth functions may not provide good representatives of irregular functions, for instance, Weierstrass function, and real-world sampled signals such as financial series, seismic data, speech signals, bioelectric recordings, etc. Fractal functions provide a constructive approximation theory for non-differentiable functions. Fractal functions concern mainly at data/function which present details at different scales or some degree of self-similarity.

CONTACT N. Vijender. Email: vijendernallapu@gmail.com

By exploiting the theory of iterated function system (IFS) [11], Barnsley [2] introduced the concept of a fractal interpolation function (FIF) to provide a mathematical representation of a data set that is generated from irregularity and/or self-affine structure. Later, fractal interpolation has been developed both in theory and applications by many authors, see for example [3–6]. Furthermore, Barnsley [2] has extended the idea of fractal interpolation to approximate a continuous function f defined on a real compact interval I, and this led to the concept of fractal approximation or α-fractal function f^α of f [16, 18, 19]. Generally speaking, (i) α-fractal functions are non-differentiable, (ii) the graph of f^α is a union of transformed copies of itself, and (iii) fractal dimension of graph of α-fractal function is non-integer. Owing to these fractal characteristics, f^α may be treated as the fractal approximant of f. In this way, every continuous function can be approximated with fractal functions.

Navascués et. al [14–19] have studied various properties of the α-fractal function f^α of f including approximation properties, among various desirable properties of a good approximant. Navascués et. al [14–19] have proved that the α-fractal function f^α of f converges towards f provided the magnitude of the scaling factors of f^α goes to zero. In this paper, using the theory of fractal functions and classical q-approximation, for a given function $f \in \mathcal{C}(I)$, we propose a sequence $\{f_n^{(q,\alpha)}(x)\}_{n=1}^{\infty}$ of (q,α)-fractal functions that converges to f even if the magnitude/norm of the corresponding scaling factors/functions does not go to zero. In the construction of the sequence $\{f_n^{(q,\alpha)}(x)\}_{n=1}^{\infty}$ of (q,α)-fractal functions, we use the sequence $\{B_{n,q}(f,x)\}_{n=1}^{\infty}$ of q-Bernstein polynomials of f as *base functions*. Owing to this reason, the convergence of the sequence $\{f_n^{(q,\alpha)}(x)\}_{n=1}^{\infty}$ of (q,α)-fractal functions towards the function f follows from the convergence of the q-Bernstein polynomials towards f. The shape of the (q,α)-fractal functions depends on the choice of $q \in (0,1)$ and the scaling functions. When $q \to 1$, the q-Bernstein polynomial coincides with the classical Bernstein polynomial, and in this case we call (q,α)-fractal functions simply α-fractal functions. Further, the convergence of these α-fractal functions towards f follows from the convergence of the q-Bernstein polynomials of f towards f. For a given sequence $\{f_m(x)\}_{m=1}^{\infty}$ of positive functions on a compact real interval that converges uniformly to a function f, by taking $B_{n,q}(f_m,x), n,m \in \mathbb{N}$ as base functions and identifying suitable conditions on corresponding scaling functions, we obtain a double sequence $\big\{\{f_{m,n}^{(q,\alpha)}(x)\}_{n=1}^{\infty}\big\}_{m=1}^{\infty}$ of positive (q,α)-fractal functions that converges uniformly to f.

For the given continuous functions f and g such that $f \geq g$, how to obtain approximants of f and g such that approximant of f be greater than approximant of g ? Such type of approximation is called a *constrained approximation problem*. We consider fractal constrained approximation problem in this paper, i.e., for given continuous functions f and g on a real compact interval I such that $f(x) \geq g(x)$ for all $x \in I$, we construct (i) a sequence $\{f_n^{(q,\alpha)}(x)\}_{n=1}^{\infty}$ of (q,α)-fractal functions that converges to f, and (ii) a sequence $\{g_n^{(q,\alpha)}(x)\}_{n=1}^{\infty}$ of (q,α)-fractal functions that converges to g such that $f_n^{(q,\alpha)}(x) \geq g_n^{(q,\alpha)}(x)$ for each $x \in I$ and $n \in \mathbb{N}$. The constrained approximation problem considered in this manuscript provides a methodology to approximate a positive function $f \in \mathcal{C}(I)$, by means of positive maps.

2. Background and Preliminaries

In this section we shall reintroduce the basics of fractal interpolation and α-fractal functions.

2.1. Fractal interpolation

Let $x_1 < x_2 < \cdots < x_{N-1} < x_N$ $(N > 2)$ be a partition of the closed interval $I = [x_1, x_N]$, and $y_1, y_2, \ldots, y_N$ be a collection of real numbers. Let $L_i, i \in \mathbb{N}_{N-1}$, be a set of homeomorphic mappings from I to $I_i = [x_i, x_{i+1}]$ satisfying

$$L_i(x_1) = x_i, L_i(x_N) = x_{i+1}. \tag{1}$$

Let F_i be a function from $I \times K$ to K (K is suitable compact subset of $\mathbb{R}$), which is continuous in the x-direction and contractive in the y-direction (with contractive factor $|\alpha_i| \leq \kappa < 1$) such that

$$F_i(x_1, y_1) = y_i, \quad F_i(x_N, y_N) = y_{i+1}, \; i \in \mathbb{N}_{N-1}. \tag{2}$$

Let us consider $\mathcal{G} = \{g : I \to \mathbb{R} \mid g \text{ is continuous, } g(x_1) = y_1 \text{ and } g(x_N) = y_N\}$. We define a metric on $\mathcal{G}$ by $\rho(h, g) = \max\Big\{|h(x) - g(x)| \;:\; x \in I\Big\}$ for $h, g \in \mathcal{G}$. Then $(\mathcal{G}, \rho)$ is a complete metric space. Define the Read-Bajraktarević operator T on $(\mathcal{G}, \rho)$ by

$$Tg(x) = F_i(L_i^{-1}(x), g \circ L_i^{-1}(x)), \; x \in I_i. \tag{3}$$

Using the properties of L_i and (1)-(2), Tg is continuous on the interval I_i ; $i \in \mathbb{N}_{N-1}$, and at each of the points $x_2, \ldots, x_{N-1}$. Also,

$$\rho(Tg, Th) \leq |\alpha|_\infty \rho(g, h),$$

where $|\alpha|_\infty = \max\{|\alpha_i| : i \in \mathbb{N}_{N-1}\} < 1$. Hence, T is a contraction map on the complete metric space $(\mathcal{G}, \rho)$. Therefore, by the Banach fixed point theorem, T possesses a unique fixed point (say) f^* on $\mathcal{G}$, i.e., $(Tf^*)(x) = f^*(x)$ for all $x \in I$. According to (3), the function f^* satisfies the functional equation: $f^*(x) = F_i(L_i^{-1}(x), f^* \circ L_i^{-1}(x))$, $x \in I_i$. Further, using (1)-(2), it is easy to verify that $f^*(x_i) = y_i$, $i \in \mathbb{N}_N$. Defining a mapping $w_i : I \times K \to I_i \times K$ as $w_i(x, y) = (L_i(x), F_i(x, y)), (x, y) \in I \times K, \; i \in \mathbb{N}_{N-1}$, the graph $G(f^*)$ of f^* satisfies:

$$G(f^*) = \bigcup_{i \in \mathbb{N}_{N-1}} w_i(G(f^*)),$$

and hence f^* is called fractal interpolation function (FIF) corresponding to the IFS $\mathcal{I} = \{I \times K, w_i(x, y) = (L_i(x), F_i(x, y)), i \in \mathbb{N}_{N-1}\}$.

Barnsley and Navascués [2, 16, 18] observed that the concept of FIFs can be used to define a class of fractal functions associated with a given real-valued continuous function f on a compact interval I.
For a given $f \in \mathcal{C}(I)$, consider a partition $\Delta = \{x_1, x_2, \ldots, x_N\}$ of $[x_1, x_N]$ satisfying $x_1 < x_2 < \cdots < x_N$, a continuous function $b : I \to \mathbb{R}$ that fulfills the conditions

$b(x_1) = f(x_1)$, $b(x_N) = f(x_N)$ and $b \neq f$, and $N-1$ real numbers $\alpha_i, i \in \mathbb{N}_{N-1}$ satisfying $|\alpha_i| < 1$. Define an IFS through the maps

$$L_i(x) = a_i x + b_i, F_i(x,y) = \alpha_i y + f(L_i(x)) - \alpha_i b(x), i \in \mathbb{N}_{N-1}.$$

The corresponding FIF denoted by $f^\alpha_{\Delta,b} = f^\alpha$ is referred to as α-fractal function for f (fractal approximation of f) with respect to a scaling vector $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{N-1})$, base function b, and partition Δ. Here the set of data points is $\{(x_i, f(x_i)) : i \in \mathbb{N}_N\}$. The function f^α is the fixed point of the Read-Bajraktarević (RB) operator $T^\alpha : \mathcal{C}_f(I) \to \mathcal{C}_f(I)$ defined by

$$(T^\alpha g)x = \alpha_i g(L_i^{-1}(x)) + f(x) - \alpha_i b(L_i^{-1}(x)), x \in I_i, i \in \mathbb{N}_{N-1},$$

where $\mathcal{C}_f(I) = \Big\{g \in \mathcal{C}(I) \ : \ g(x_1) = f(x_1), \ g(x_N) = f(x_N)\Big\}$. Consequently, the α-fractal function f^α corresponding to f satisfies the self-referential equation

$$f^\alpha(x) = \alpha_i f^\alpha(L_i^{-1}(x)) + f(x) - \alpha_i b(L_i^{-1}(x)), \ x \in I_i, \ i \in \mathbb{N}_{N-1}. \tag{4}$$

The fractal dimension (box dimension or Hausdorff dimension) of f^α depends on the choice of the scaling vector α. For instance, Nasim Akhtar et al. [13] calculated box dimension of graph of α-fractal functions by assuming suitable conditions on the original function f and base function b. The following proposition provides the details of it.

Proposition 2.1. *Let $f \in \mathcal{C}(I)$ and $b : I \to \mathbb{R}$ be Lipschitz functions with $b(x_1) = f(x_1)$, $b(x_N) = f(x_N)$. Let $\Delta = \{x_1, x_2, \dots, x_N\}$ be a partition of I satisfying $x_1 < x_2 < \cdots < x_N$. and $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{N-1})$. If the data points $(x_i, f(x_i)), i \in \mathbb{N}_N$ are not collinear, then graph G of the α-fractal function f^α has the box dimension*

$$dim_B(G) = \begin{cases} D & \text{if } \sum_{i=1}^{N-1} |\alpha_i| > 1, \\ 1 & \text{otherwise}, \end{cases}$$

where D is solution of $\sum_{i=1}^{N-1} |\alpha_i| a_i^{D-1} = 1$.

The following proposition explains about the sufficient condition for the α-fractal function f^α to be irregular (non-differentiable) on I. It can be proved using Lemma 5.1 and Theorem 5.2 of [16] and hence the proof is omitted.

Proposition 2.2. *Let $f \in \mathcal{C}^1(I)$. Let $\Delta = \{x_1, x_2, \dots, x_N\}$ be a partition of $I = [x_1, x_N]$ satisfying $x_1 < x_2 < \cdots < x_N$ and $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{N-1})$. If*

$$\sum_{i=1}^{N-1} |\alpha_i| > 1,$$

then the set of points of non-differentiability of f^α is dense in I.

To obtain fractal functions with more flexibility, iterated function system wherein scaling factors are replaced by scaling functions received attention in the recent literature [23] on fractal functions. That is, one may consider the IFS with maps

$$L_i(x) = a_i x + b_i, F_i(x,y) = \alpha_i(x)y + f(L_i(x)) - \alpha_i b(x), i \in \mathbb{N}_{N-1},$$

where $\alpha_i, i \in \mathbb{N}_{N-1}$ are continuous functions on I satisfying $\max\{\|\alpha_i\|_\infty : \imath \in \mathbb{N}_{N-1}\} < 1$. The corresponding α-fractal function is the fixed point of the RB-operator

$$(T^\alpha g)x = \alpha_i(L_i^{-1}(x))g(L_i^{-1}(x)) + f(x) - \alpha_i(L_i^{-1}(x))b(L_i^{-1}(x)), x \in I_i, i \in \mathbb{N}_{N-1}.$$

Consequently, the α-fractal function f^α corresponding to f satisfies the self-referential equation

$$f^\alpha(x) = \alpha_i(L_i^{-1}(x))f^\alpha(L_i^{-1}(x)) + f(x) - \alpha_i(L_i^{-1}(x))b(L_i^{-1}(x)), x \in I_i, i \in \mathbb{N}_{N-1}. \quad (5)$$

3. (q, α)-fractal approximation

From (5), we get the following inequality:

$$\|f^\alpha - f\|_\infty \leq \frac{\|\alpha\|_\infty}{1 - \|\alpha\|_\infty}\|f - b\|_\infty, \quad (6)$$

where $\|\alpha\|_\infty = \max\{\|\alpha_i\|_\infty : i \in \mathbb{N}_{N-1}\}$. For a fixed base function b, the α-fractal function f^α converges uniformly to $f \in \mathcal{C}(I)$ if $\|\alpha\|_\infty \to 0$. To get the convergence of the α-fractal function f^α towards f for all the scaling functions, we choose the base function b as q-Bernstein polynomial $B_{n,q}(f,x)$ of f, i.e., $b = B_{n,q}(f,x)$, where from [1]

$$B_{n,q}(f,x) = \frac{1}{(x_N - x_1)^n} \sum_{k=0}^{n} \binom{n}{k}_q (x - x_1)^k \prod_{s=0}^{n-k-1} (x_N - x_1 - q^s x) f\left(x_1 + (x_N - x_1)\frac{[k]_q}{[n]_q}\right), \quad (7)$$

$x \in I, q \in (0,1), n \in \mathbb{N}$, $[k]_q = \frac{1-q^k}{1-q}$,

$$[k]_q! = \begin{cases} [k]_q[k-1]_q[k-2]_q \dots [2]_q[1]_q, & \text{if } k \neq 0, \\ 1, & \text{if } k = 0, \end{cases}$$

$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$, $f \in \mathcal{C}(I)$, $B_{n,q}(f,x_1) = f(x_1)$, $B_{n,q}(f,x_N) = f(x_N)$. When $q \to 1$, $B_{n,q}(f,x)$ coincides with the classical n-th Bernstein polynomial. If we take as base function $b = B_{n,q}(f,x)$ in (7), then the corresponding α-fractal function $f^\alpha = f_n^{(q,\alpha)}$ is called (q,α)- fractal function corresponding to $f \in \mathcal{C}(I)$, and

$$f_n^{(q,\alpha)}(x) = \alpha_i(L_i^{-1}(x))f_n^{(q,\alpha)}(L_i^{-1}(x)) + f(x) - \alpha_i(L_i^{-1}(x))B_{n,q}(f, L_i^{-1}(x)), \quad (8)$$

$x \in I_i,\ i \in \mathbb{N}_{N-1}, n \in \mathbb{N}$. From (7), we can notice that the properties of the q-Bernstein polynomial $B_{n,q}(f,x)$ depends on $q \in (0,1)$. Therefore, from (8), it is easy to notice that shape of the (q,α)- fractal function $f_n^{(q,\alpha)}$ depends on the choice of $q \in (0,1)$ apart from the choice of scaling functions. Furthermore, from the construction of fractal functions (see previous section), it follows that (q,α)- fractal function $f_n^{(q,\alpha)}, n \in \mathbb{N}$, of

$f \in \mathcal{C}(I)$ is obtained via the IFS defined by

$$\mathcal{I}_n = \{I \times \mathbb{R}, (L_i(x), F_{n,i}(x,y)) : i \in \mathbb{N}_{N-1}\}, n \in \mathbb{N}, \tag{9}$$

where $F_{n,i}(x,y) = \alpha_i(x)y + f(L_i(x)) - \alpha_i(x)B_{n,q}(f,x)$.

Theorem 3.1. *Let $f \in \mathcal{C}(I)$. There exists a sequence $\{f_n^{(q,\alpha)}(x)\}_{n=1}^{\infty}$ of (q,α)-fractal functions that converges uniformly to f on I. Further, $f_n^{(q,\alpha)}, n \in \mathbb{N}$, satisfies the following inequalities:*

$$\frac{1-\|\alpha\|_\infty 2^n}{1+\|\alpha\|_\infty}\|f\|_\infty \le \|f_n^{(q,\alpha)}\| \le \frac{1+\|\alpha\|_\infty 2^n}{1-\|\alpha\|_\infty}\|f\|_\infty. \tag{10}$$

Proof. Let $f_n^{(q,\alpha)}, n \in \mathbb{N}$, be the (q,α)- fractal function corresponding to f. Then, from (8), it is easy to deduce that

$$\begin{aligned}\|f_n^{(q,\alpha)} - f\|_\infty \le& \|\alpha\|_\infty \|f_n^{(q,\alpha)} - B_{n,q}(f,.)\|_\infty,\\ \le& \|\alpha\|_\infty[\|f_n^{(q,\alpha)} - f\|_\infty + \|f - B_{n,q}(f,.)\|_\infty].\end{aligned}$$

Hence we obtain

$$\|f_n^{(q,\alpha)} - f\|_\infty \le \frac{\|\alpha\|_\infty}{1-\|\alpha\|_\infty}\|f - B_{n,q}(f,.)\|_\infty. \tag{11}$$

From [1], we have

$$\|B_{n,q}(f,.) - f\|_\infty \to 0 \text{ as } n \to \infty. \tag{12}$$

Using (12) in (11), we conclude that the sequence $\{f_{n,q}(x)\}_{n=1}^{\infty}$ of (q,α)-fractal functions converges uniformly to f. Again from [1], we have

$$\|B_{n,q}(.,.)\|_\infty = 2^n. \tag{13}$$

We can rewrite (11) as

$$\|f_n^{(q,\alpha)}\|_\infty - \|f\|_\infty \le \|f_n^{(q,\alpha)} - f\|_\infty \le \frac{\|\alpha\|_\infty}{1-\|\alpha\|_\infty}\big\{\|f\|_\infty + \|B_{n,q}(f,.)\|_\infty\big\}. \tag{14}$$

Using (13) in (14), we get the right side inequality of (10). Next, from (8), we obtain

$$|f_n^{(q,\alpha)}(x) - f(x)| \le \|\alpha_i\|_\infty\big\{\|f_n^{(q,\alpha)}\|_\infty + \|B_{n,q}(f,.)\|_\infty\big\},\ x \in I_i,\ \ i \in \mathbb{N}_{N-1}, n \in \mathbb{N},$$

which implies that

$$\|f\|_\infty - \|f_n^{(q,\alpha)}\|_\infty \le \|f_n^{(q,\alpha)} - f\|_\infty \le \|\alpha\|_\infty\big\{\|f_n^{(q,\alpha)}\|_\infty + \|B_{n,q}(f,.)\|_\infty\big\}.$$

Using (13) in the above inequality, we get the left side inequality of (10). □

The proof of the following theorem follows from Proposition 2.2 and Theorem 3.1.

Theorem 3.2. *Let $f \in \mathcal{C}^1(I)$. Let $\Delta = \{x_1, x_2, \dots, x_N\}$ be a partition of $I = [x_1, x_N]$ satisfying $x_1 < x_2 < \cdots < x_N$ and $\alpha_i(x) = \lambda_i$ for all $i \in \mathbb{N}_{N-1}$ and $x \in I$. Suppose that $f'(x)$ and $B'_{n,q}(f,x)$ respectively do not agree with $f(x_N) - f(x_1)$ and $N(B_{n,q}(f,x_N) - B_{n,q}(f,x_1))$ in a nonempty open subinterval of I. If all the (q,α)-fractal functions in the sequence $\{f_n^{(q,\alpha)}(x)\}_{n=1}^{\infty}$ are obtained with same choice of scaling factors $\lambda_i, i \in \mathbb{N}_{N-1}$, which satisfy the condition $\sum_{i=1}^{N-1} |\lambda_i| > 1$, then all the (q,α)-fractal functions are non-differentiable in a dense subset of I and converge uniformly to f when $n \to \infty$.*

Remark 1. For a prescribed continuous function f, the proposed (q,α)-fractal approximation provides a methodology to construct a sequence of (q,α)-fractal functions with specified box dimension converging to f.

Theorem 3.3. *Let $f \in \mathcal{C}(I)$. Let $\Delta = \{x_1, x_2, \dots, x_N\}$ be a partition of I satisfying $x_1 < x_2 < \cdots < x_N$. Suppose that $\alpha_i(x) = \lambda_i$ for all $x \in I$ and $i \in \mathbb{N}_{N-1}$, where λ_i, $i \in \mathbb{N}_{N-1}$, are real numbers such that $|\lambda_i| < 1$. For each $n \in \mathbb{N}$, let $f_n^{(q,\alpha)}$ be the (q,α)-fractal function of f. Then*

$$\int_{x_1}^{x_N} f_n^{(q,\alpha)}(x)dx = \frac{1}{1 - \sum\limits_{i=1}^{N-1} a_i \lambda_i} \left[\int_{x_1}^{x_N} f(x)dx - \sum_{i=1}^{N-1} a_i \lambda_i \int_{x_1}^{x_N} B_{n,q}(f,x)dx \right]$$

Proof. Using (8), we get

$$\begin{aligned}
\int_{x_1}^{x_N} f_n^{(q,\alpha)}(x)dx &= \sum_{i=1}^{N-1} \int_{x_i}^{x_{i+1}} f_n^{(q,\alpha)}(x)dx \\
&= \sum_{i=1}^{N-1} \int_{x_i}^{x_{i+1}} \lambda_i f_n^{(q,\alpha)}(L_i^{-1}(x))dx + \sum_{i=1}^{N-1} \int_{x_i}^{x_{i+1}} f(x)dx \\
&\quad - \sum_{i=1}^{N-1} \int_{x_i}^{x_{i+1}} \lambda_i B_{n,q}(f, L_i^{-1}(x))dx
\end{aligned}$$

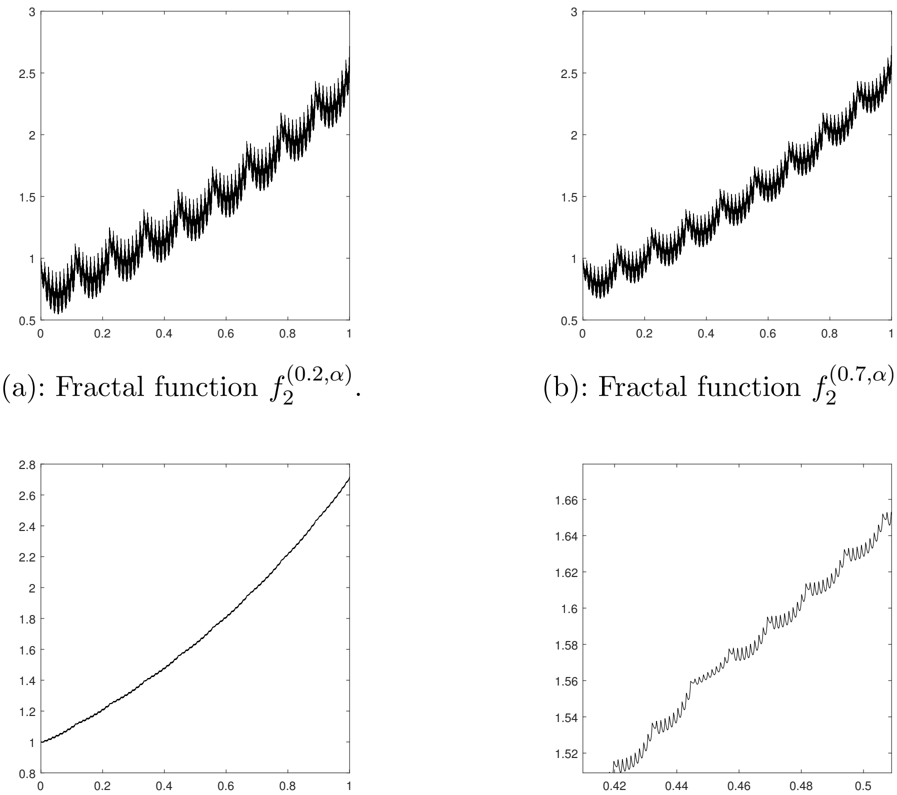
Let $L_i^{-1}(x) = \tau$. Then, $dx = a_i d\tau$. Hence, from the above equation, we obtain

$$\int_{x_1}^{x_N} f_n^{(q,\alpha)}(x)dx = \sum_{i=1}^{N-1} \lambda_i a_i \int_{x_1}^{x_N} f_n^{(q,\alpha)}(\tau)d\tau + \int_{x_1}^{x_N} f(x)dx - \sum_{i=1}^{N-1} \lambda_i a_i \int_{x_1}^{x_N} B_{n,q}(f,\tau)d\tau.$$

By simplifying the above equation, we get the required result. □

3.1. Examples

In this section, we provide numerical examples to corroborate our findings. For this purpose, let $f(x) = e^x, x \in [0,1]$. The (q,α)-fractal functions in Figures 1(a)-(c) are generated with respect to the partition $\Delta = \{0,\ 0.1111,\ 0.2222,\ 0.3333,\ 0.4444,\ 0.5556,\ 0.6667,\ 0.7778, 0.8889, 1\}$ of $[0,1]$. The (q,α)-fractal functions $f_2^{(0.2,\alpha)}$, $f_2^{(0.7,\alpha)}$, and $f_{31}^{(0.7,\alpha)}$ are generated at the third iteration respectively in Figures 1(a)-(c) with the choice of the scaling functions $\alpha_i(x) = \frac{1}{1+e^{-10x}}, x \in [0,1], i \in \mathbb{N}_9$. By comparing the



(a): Fractal function $f_2^{(0.2,\alpha)}$. (b): Fractal function $f_2^{(0.7,\alpha)}$.

(c): Fractal function $f_{31}^{(0.7,\alpha)}$. (d): A part of $f_{31}^{(0.7,\alpha)}$ under magnification.

Figure 1. (q,α)-fractal approximants of $e^x, x \in [0,1]$.

(q,α)-fractal functions $f_2^{(0.2,\alpha)}$ and $f_2^{(0.7,\alpha)}$, one can visualize the effects of q in the shape of (q,α)-fractal function. According to Theorem 3.1, the (q,α)-fractal function $f_{31}^{(0.7,\alpha)}$ provides a better approximation for $e^x, x \in [0,1]$ than that obtained by $f_2^{(0.7,\alpha)}$. By observing Figures 1(b)-(c), one can ask why the fractal functions $f_2^{(0.7,\alpha)}$ and $f_{31}^{(0.7,\alpha)}$ are not having the same sort of irregularity even if their scaling functions are same. This is due to the following reason: The fractal function $f_2^{(0.7,\alpha)}$ exhibit irregularity on all scales whereas the fractal function $f_{31}^{(0.7,\alpha)}$ exhibits irregularity on small scales. Further, small scales of irregularity of the fractal function $f_{31}^{(0.7,\alpha)}$ can be observed from Figure 1(d) which is a part of $f_{31}^{(0.7,\alpha)}$ under magnification.

4. Constrained (q,α)-fractal approximation

In this section, we establish the constrained approximation by (q,α)-fractal functions.

Theorem 4.1. *Let $f \in \mathcal{C}(I)$ and $f(x) \geq 0$ for all $x \in I$. Let $\Delta = \{x_1, x_2, \dots, x_N\}$ be a partition of I satisfying $x_1 < x_2 < \dots < x_N$. Then, the sequence $\{\mathcal{I}_n\}_{n=1}^{\infty}$ of IFSs determine a sequence $\{f_n^{(q,\alpha)}\}_{n=1}^{\infty}$ of positive (q,α)-fractal functions that converges to*

f if the scaling functions are chosen as $\|\alpha_i\|_\infty < 1$ and

$$\max\left\{\frac{-\phi(f,i)}{K_n-\phi_n}, -\frac{K_n-\Phi(f,i)}{\Phi_n}\right\} \le \alpha_i(x) \le \min\left\{\frac{\phi(f,i)}{\Phi_n}, \frac{K_n-\Phi(f,i)}{K_n-\phi_n}\right\}, i \in \mathbb{N}_{N-1}, \tag{15}$$

where $\phi(f,i) = \min_{x\in I} f(L_i(x))$, $\Phi(f,i) = \max_{x\in I} f(L_i(x))$, $\phi_n = \min_{x\in I} B_{n,q}(f,x)$, $\Phi_n = \max_{x\in I} B_{n,q}(f,x)$, and K_n is a positive real number strictly greater than both ϕ_n and $\|f\|_\infty$.

Proof. Theorem 3.1 ensures that there exists a sequence $\{f_n^{(q,\alpha)}\}_{n=1}^\infty$ of (q,α)-fractal functions that converges to f. Therefore, in what follows, it is enough to ensure that $f_n^{(q,\alpha)}, n \in \mathbb{N}$ is positive if the scaling functions are elected according to (15). Since $f \in \mathcal{C}(I)$ and $f(x) \ge 0$ for all $x \in I$, from the q-approximation theory [1], it follows that $B_{n,q}(f,x) \ge 0$ for all $x \in I$. This ensures that $\Phi_n > 0$. Since the (q,α)-fractal function $f_n^{(q,\alpha)}$ obeys the functional equation:

$$\begin{aligned} f_n^{(q,\alpha)}(L_i(x)) =& \alpha_i(x) f_n^{(q,\alpha)}(x) + q_{n,i}, q_{n,i} = f(L_i(x)) - \alpha_i(x) B_{n,q}(f,x)) \\ =& F_{n,i}(x, f_n^{(q,\alpha)}(x)) = f(L_i(x)) + \alpha_i(x)(f_n^{(q,\alpha)}(x) - B_{n,q}(f,x)),\ x \in I, \end{aligned} \tag{16}$$

for proving $f_n^{(q,\alpha)}(x) \in [0, K_n]$ for all $x \in I$ it is enough to verify that $F_{n,i}(x,y) \in [0,K_n]$, $i \in \mathbb{N}_{N-1}$ for all $(x,y) \in I \times [0,K_n]$. Assume $(x,y) \in I \times [0,K_n]$. Consider the scale function $\alpha_i(x)$, $i \in \mathbb{N}_{N-1}$ such that $|\alpha_i(x)| < 1$, $x \in I$.
Case-I: Without loss of generality, let $0 \le \alpha_i(x) < 1$, for all $x \in I$. Then $0 \le y \le K_n$ implies $q_{n,i}(x) \le \alpha_i(x)y + q_{n,i}(x) \le K_n\alpha_i(x) + q_{n,i}(x)$. Therefore, $0 \le F_{n,i}(x,y) = \alpha_i(x)y + q_{n,i}(x) \le K_n$, $i \in \mathbb{N}_{N-1}$ for all $(x,y) \in I \times [0,K_n]$ is true if

$$f(L_i(x)) - \alpha_i(x)B_{n,q}(f,x) \ge 0, f(L_i(x)) - \alpha_i(x)B_{n,q}(f,x) \le K_n(1-\alpha_i(x)), x \in I. \tag{17}$$

Since $f(L_i(x)) \ge \phi(f,i)$ and $B_{n,q}(f,x) \le \Phi_n$; $f(L_i(x)) - \alpha_i(x)B_{n,q}(f,x) \ge 0$ if and only if $\phi(f,i) - \alpha_i(x)\Phi_n \ge 0$. Hence $0 \le \alpha_i(x) \le \frac{\phi(f,i)}{\Phi_n}$. Next, since $f(L_i(x)) \le \Phi(f,i)$ and $B_{n,q}(f,x) \ge \phi_n$, second inequality in (17) is true if $0 \le \alpha_i(x) \le \frac{K_n - \Phi(f,i)}{K_n - \phi_n}$. In this case, $F_{n,i}(x,z) \in [0,K_n]$, $i \in \mathbb{N}_{N-1}$ for all $(x,y) \in I \times [0,K_n]$ if $0 \le \alpha_i(x) \le \min\left\{\frac{\phi(f,i)}{\Phi_n}, \frac{K_n-\Phi(f,i)}{K_n-\phi_n}\right\}$.
Case-II: Let $-1 < \alpha_i(x) \le 0$, $i \in \mathbb{N}_{N-1}$, for all $x \in I$. In this case, $0 \le y \le K_n$ implies $K_n\alpha_i(x) + q_{n,i}(x) \le \alpha_i(x)y + q_{n,i}(x) \le q_{n,i}(x)$. Consequently, $0 \le F_{n,i}(x,y) = \alpha_i(x)y + q_{n,i}(x) \le K_n$, $i \in \mathbb{N}_{N-1}$ for all $(x,y) \in I \times [0,K_n]$ holds if:

$$f(L_i(x)) - \alpha_i(x)B_{n,q}(f,x) \le K_n, K_n\alpha_i(x) + f(L_i(x)) - \alpha_i(x)B_{n,q}(f,x) \ge 0, x \in I. \tag{18}$$

Since $f(L_i(x)) \le \Phi(f,i)$ and $B_{n,q}(f,x) \ge \Phi_n$, from the first inequality of (18), we have $f(L_i(x)) - \alpha_i(x)B_{n,q}(f,x) \le \Phi(f,i) - \alpha_i(x)\Phi_n \le K$. Hence, $\alpha_i(x) \ge -\frac{K_n-\Phi(f,i)}{\Phi_n}$. Next, since $f(L_i(x)) \ge \phi(f,i)$ and $B_{n,q}(f,x) \ge \phi_n$, from the routine calculations, we observe that $\alpha_i(x) \ge \frac{-\phi(f,i)}{K_n-\phi_n}$ gives second inequality in (18). Hence, in this case, $F_{n,i}(x,z) \in [0,K_n]$, $i \in \mathbb{N}_{N-1}$ for all $(x,y) \in I \times [0,K_n]$ if $\max\left\{\frac{-\phi(f,i)}{K_n-\phi_n}, -\frac{K_n-\Phi(f,i)}{\Phi_n}\right\} \le \alpha_i(x) \le 0$.
Combining Case-I and Case-II, we obtain (15). □

The previous theorem guarantees the existence of positive (q, α)-fractal approximation for any positive function f defined on $[0, 1]$. Next theorem discusses the existence of a double sequence of positive (q, α)-fractal functions that converges uniformly to $f \in \mathcal{C}(I)$ whenever there is a sequence of positive continuous functions that converges uniformly to $f \in \mathcal{C}(I)$.

Theorem 4.2. *Let $\{f_m(x)\}_{m=1}^{\infty}$ be a sequence of positive functions in $\mathcal{C}(I)$ that converges uniformly to a function f. Let $\Delta = \{x_1, x_2, \dots, x_N\}$ be a partition of $I = [x_1, x_N]$ satisfying $x_1 < x_2 < \dots < x_N$. Suppose that $L_i : I \to I_i$, $i \in \mathbb{N}_{N-1}$ are affine maps $L_i(x) = a_i x + b_i$ satisfying $L_i(x_1) = x_i$ and $L_i(x_N) = x_{i+1}$, and $F^{\dagger}_{m,n,i}(x, y) = \alpha_i(x)y + f_m(L_i(x)) - \alpha_i(x)B_{n,q}(f_m, x)$, $i \in \mathbb{N}_{N-1}$. Let $f^{(q,\alpha)}_{m,n}$ be α-fractal function determined by the IFS $\mathcal{I}^{\dagger}_{m,n} = \{I \times K; \big(L_i(x), F^{\dagger}_{m,n,i}(x, y)\big),\ i \in \mathbb{N}_{N-1}\}$. Then, the double sequence $\big\{\{\mathcal{I}^{\dagger}_{m,n}\}_{n=1}^{\infty}\big\}_{m=1}^{\infty}$ of IFSs determines a double sequence $\big\{\{f^{(q,\alpha)}_{m,n}(x)\}_{n=1}^{\infty}\big\}_{m=1}^{\infty}$ of positive (q, α)-fractal functions that converges uniformly to f if the scaling functions of each $f^{(q,\alpha)}_{m,n}$ obey $\|\alpha_i\|_\infty < 1$ and for $i \in \mathbb{N}_{N-1}$,*

$$\max\left\{\frac{-\phi(f_m, i)}{K^*_{n,m} - \phi_{n,m}(f_m)}, -\frac{K^*_{n,m} - \Phi(f_m, i)}{\Phi_{n,m}(f_m)}\right\} \le \alpha_i(x) \le \min\left\{\frac{\phi(f_m, i)}{\Phi_{n,m}(f_m)}, \frac{K^*_{n,m} - \Phi(f_m, i)}{K^*_{n,m} - \phi_{n,m}(f_m)}\right\}, \tag{19}$$

where $\phi(f_m, i) = \min\limits_{x\in I} f_m(L_i(x))$, $\Phi(f_m, i) = \max\limits_{x\in I} f_m(L_i(x))$, $\phi_{n,m}(f_m) = \min\limits_{x\in I} B_{n,q}(f_m, x)$, $\Phi_{n,m}(f_m) = \max\limits_{x\in I} B_{n,q}(f_m, x)$, and $K^*_{n,m}$ is a positive real number strictly greater than both $\phi_{n,m}(f_m)$ and $\|f_m\|_\infty$.

Proof. Since $\{f_m(x)\}_{m=1}^{\infty}$ be a sequence of positive functions in $\mathcal{C}(I)$ that converges uniformly to a function f, for given ϵ there exists a natural numbers N_1 such that

$$\|f_m - f\|_\infty < \frac{\epsilon}{2}\ \forall m\ \ge N_1. \tag{20}$$

From the approximation theory, we can see that for each $m \in \mathbb{N}$, there exits a sequence $\{B_{n,q}(f_m, x)\}_{n=1}^{\infty}$ of q-Bernstein polynomials of f_m that converges uniformly to f_m. Therefore, for given $\epsilon > 0$ there exists a natural number N_2 such that

$$\|B_{n,q}(f_m, .) - f_m\|_\infty < \frac{\epsilon(1 - \|\alpha\|_\infty)}{2\|\alpha\|_\infty}\ \ \forall\, n \ge N_2. \tag{21}$$

Since $f^{(q,\alpha)}_{m,n}$ is the α-fractal function determined by the IFS $\mathcal{I}^{\dagger}_{m,n}$, the fractal function $f^{\alpha}_{m,n}$ enjoys the following functional equation:

$$f^{(q,\alpha)}_{m,n}(x) = \alpha_i(L_i^{-1}(x))f^{(q,\alpha)}_{m,n}(L_i^{-1}(x)) + f_m(x) - \alpha_i(L_i^{-1}(x))B_{n,q}(f_m, L_i^{-1}(x)), \tag{22}$$

$x \in I_i, i \in \mathbb{N}_{N-1}$. Also, it is easy to verify that $f^{(q,\alpha)}_{m,n}$ satisfies the following inequality:

$$\|f^{\alpha}_{m,n} - f_m\|_\infty \le \frac{\|\alpha\|_\infty}{1 - \|\alpha\|_\infty}\|f_m - B_{n,q}(f_m, .)\|_\infty. \tag{23}$$

Using (21) in (23), we obtain

$$\|f^{(q,\alpha)}_{m,n} - f_m\|_\infty < \frac{\epsilon}{2}\ \ \forall\, n \ge N_2. \tag{24}$$

Using (20) and (24) with the triangular inequality

$$\|f_{m,n}^{(q,\alpha)} - f\|_\infty \le \|f_{m,n}^{(q,\alpha)} - f_m\|_\infty + \|f_m - f\|_\infty,$$

for given $\epsilon > 0$, we get

$$\|f_{m,n}^{(q,\alpha)} - f\|_\infty < \epsilon \ \ \forall\, m,n \ge N = \max\{N_1, N_2\}. \tag{25}$$

The above inequality provides the desired double sequence. Finally, from Theorem 4.1, it is plain to verify that the functions $f_{m,n}^{(q,\alpha)}, m,n \in \mathbb{N}$ are positive on I if the scaling functions obey (19). □

Theorem 4.3. *Let f and g continuous functions on I such that $f(x) \ge g(x)$ for all $x \in I$. Let $\Delta = \{x_1, x_2, \ldots, x_N\}$ be a partition of I satisfying $x_1 < x_2 < \cdots < x_N$. For $n \in \mathbb{N}$, let $f_n^{(q,\alpha)}$ be the (q,α)-fractal function of f corresponding to the IFS $\mathcal{I}_n$. Then, the sequence $\{\mathcal{I}_n\}_{n=1}^\infty$ of IFSs determines a sequence $\{f_n^{(q,\alpha)}\}_{n=1}^\infty$ of (q,α)-fractal functions such that $f_n^{(q,\alpha)}(x) \ge g(x)$ for all $x \in I$ and $n \in \mathbb{N}$, and that converges uniformly to $f \in \mathcal{C}(I)$ if the scaling functions are chosen as prescribed in the following:*

$$0 \le \alpha_i(x) < \min\left\{\frac{\phi(f-g,i)}{\Phi_n(f) - \phi(g)}, 1\right\}, i \in \mathbb{N}_{N-1}, \tag{26}$$

where $\Phi_n(f) = \max_{x\in I} B_{n,q}(f,x)$, $\phi(f-g,i) = \min_{x\in I}(f-g)(L_i(x))$, $\phi(g) = \min_{x\in I} g(x)$.

Proof. From the construction of the (q,α)-fractal function $f^{(q,\alpha)}$, we can notice that $f^{(q,\alpha)}$ obeys the following functional equation:

$$f_n^{(q,\alpha)}(L_i(x)) = \alpha_i(x) f_n^{(q,\alpha)}(x) + f(L_i(x)) - \alpha_i(x) B_{n,q}(f,x), x \in I, i \in \mathbb{N}_{N-1}.$$

This functional equation is a rule that computes the values of $f_n^{(q,\alpha)}$ at $(N-1)^{p+2}+1$ distinct points in I at $(p+1)$-th iteration using the values of $f_n^{(q,\alpha)}$ at $(N-1)^{p+1}+1$ distinct points in I at p-th iteration. Let us consider the nodal points x_i, for $i = 1, 2, \ldots N$, to start the iteration process, and let us prove that any p-th iterated image x of them satisfies: $f_n^{(q,\alpha)}(x) \ge g(x)$.
For the zero-th iteration, we have

$$f_n^{(q,\alpha)}(x_i) = f(x_i) \ge g(x_i),$$

since $f_n^{(q,\alpha)}$ interpolates to f at the nodes, and $f \ge g$.
Let us assume now that $f_n^{(q,\alpha)}(x) \ge g(x)$ and let us prove that $f_n^{(q,\alpha)}(L_i(x)) \ge g(L_i(x))$. Using the fixed point equation, this is equivalent to

$$\alpha_i(x) f_n^{(q,\alpha)}(x) + f(L_i(x)) - \alpha_i(x) B_{n,q}(f,x) - g(L_i(x)) \ge 0.$$

Assuming that α_i is positive and applying the inductive hypothesis, this inequality is implied by

$$\alpha_i(x) g(x) + f(L_i(x)) - \alpha_i(x) B_{n,q}(f,x) - g(L_i(x)) \ge 0,$$

that is to say, α_i must be chosen as

$$0 \leq \alpha_i(x) \leq \frac{\phi(f-g,i)}{\Phi_n(f)-\phi(g)}.$$

By arguments of density and continuity, we obtain the inequality proposed for any $x \in I$. □

Theorem 4.4. *Let $f, g \in \mathcal{C}(I)$ such that $f(x) \geq g(x)$ for all $x \in I$. Let $\Delta = \{x_1, x_2, \dots, x_N\}$ be a partition of I satisfying $x_1 < x_2 < \cdots < x_N$. Then, there exist the sequences $\{f_n^{\alpha}(x)\}_{n=1}^{\infty}$ and $\{g_n^{\alpha}(x)\}_{n=1}^{\infty}$ of (q,α)-fractal functions that converge uniformly to f and g respectively such that $f_n^{(q,\alpha)}(x) \geq g_n^{(q,\alpha)}(x)$ for all $x \in I$ if the scaling functions $\alpha_i(x)$, $i \in \mathbb{N}_{N-1}$ obey*

$$0 \leq \alpha_i(x) \leq \frac{\phi(f-g,i)}{\Phi_n(f-g)}, i \in \mathbb{N}_{N-1},$$

where $\Phi_n(f-g) = \max\limits_{x\in I} B_{n,q}(f_n - g_n, x)$, $\phi(f-g,i) = \min\limits_{x\in I}(f-g)(L_i(x))$.

Proof. From the construction of the (q,α)-fractal functions, it is easy to obtain that $f^{(q,\alpha)}$ and $g^{(q,\alpha)}$ obey the following functional equations:

$$f_n^{(q,\alpha)}(L_i(x)) = \alpha_i(x) f_n^{(q,\alpha)}(x) + f(L_i(x)) - \alpha_i(x) B_{n,q}(f,x), x \in I, i \in \mathbb{N}_{N-1}.$$

$$g_n^{(q,\alpha)}(L_i(x)) = \alpha_i(x) g_n^{(q,\alpha)}(x) + g(L_i(x)) - \alpha_i(x) B_{n,q}(g,x), x \in I, i \in \mathbb{N}_{N-1}.$$

By using the similar argument as in the Theorem 4.3, we can see that to prove $f_n^{(q,\alpha)}(z) \geq g_n^{(q,\alpha)}(z)$ for all $z \in I$, it is enough to show that $f_n^{(q,\alpha)}(x) \geq g_n^{(q,\alpha)}(x)$, $x \in I$ implies that $f_n^{(q,\alpha)}(L_i(x)) \geq f_n^{(q,\alpha)}(L_i(x))$, $x \in I$ and for all $i \in \mathbb{N}_{N-1}$.

Assume that $f_n^{(q,\alpha)}(x) \geq g_n^{(q,\alpha)}(x)$, $x \in I$. We have to ensure that

$$\alpha_i(x)\Big(f_n^{(q,\alpha)}(x) - g_n^{(q,\alpha)}(x)\Big) + \Big(f(L_i(x)) - g(L_i(x))\Big) - \alpha_i(x)\Big(B_{n,q}(f,x) - B_{n,q}(g,x)\Big) \geq 0, x \in I. \tag{27}$$

From the q-approximation theory [1], we can see that q-Bernstein operator $B_{n,q}$ is monotonic. Hence, $f(x) \geq g(x)$ for all $x \in I$ implies $B_{n,q}(f,x) - B_{n,q}(g,x) \geq 0$ for all $x \in I$. This ensures that $\Phi_n(f-g) \geq 0$. Now, using the inductive hypothesis, $f_n^{(q,\alpha)}(x) \geq g_n^{(q,\alpha)}(x)$, $x \in I$ and $\alpha_i(x) \geq 0$ for all $x \in I$, $i \in \mathbb{N}_{N-1}$, and definitions of $\Phi_n(f)$ and $\phi(f-g,i)$, we infer that the condition $\alpha_i(x) \leq \frac{\phi(f-g,i)}{\phi_n(f-g)}$ ensures (27). Thus we complete the proof. □

Remark 2. Owing to the implicit nature of the fractal functions, it is difficult to prove Theorem 4.3-4.4 using the condition $\alpha_i(x) \leq 0, x \in I, i \in \mathbb{N}_{N-1}$. Owing to this reason, the condition $\alpha_i(x) \leq 0, x \in I, i \in \mathbb{N}_{N-1}$ is avoided in Theorems 4.3-4.4.

5. Conclusion

In the present paper, we have introduced a new approximation method through the theory of fractal functions and q-Bernstein polynomials. For a given function

$f \in \mathcal{C}(I)$, the convergence of sequence of (q,α)-fractal functions towards f does not need any condition on the scaling functions whereas the convergence of the existing α-fractal functions towards f needs scaling functions close to zero. The shape of the proposed fractal approximants depends on the free variable $q \in (0,1)$ apart from the scaling functions. Hence, for the given continuous function f, the proposed (q,α)-fractal approximants provide a large number of approximants than that would be obtained by the existing fractal approximants. For the sequence $\{f_m(x)\}_{m=1}^{\infty}$ of positive functions that converges uniformly to a non-negative function f, we have identified the double sequence $\big\{\{\mathcal{I}^{\dagger}_{m,n}\}_{n=1}^{\infty}\big\}_{m=1}^{\infty}$ of IFSs so that corresponding double sequence $\big\{\{f^{(q,\alpha)}_{m,n}\}_{n=1}^{\infty}\big\}_{m=1}^{\infty}$ of (q,α)-fractal functions converges to f with the property that $f^{(q,\alpha)}_{m,n}(x) \geq 0$ for all $x \in I$ and $m,n \in \mathbb{N}$. By imposing suitable conditions on the scaling functions, we studied the constrained fractal approximation by the proposed (q,α)-fractal functions.

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