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Quantum Bernstein Fractal Functions

Abstract

In this article, taking the quantum Bernstein functions as base functions, we have proposed the class of quantum Bernstein fractal functions. When $f \in C(I)$, the base function is taken as the classical q-Bernstein polynomials, we propose the class of quantum fractal functions through a multivalued quantum fractal operator. When $f \in \mathcal{L}^p(I), 1 \leq p \leq \infty$, the base function is assumed as q-Kantorovich-Bernstein polynomial to construct the sequence of (q, α)-Kantorovich-Bernstein fractal functions that converges uniformly to f. Some approximation properties of these new class of quantum fractal interpolants have been studied.

KEYWORDS:
q-Bernstein polynomials; fractal functions; constrained approximation; fractals; interpolation.
28A80, 26C15, 26A48, 26A51, 65D05

1 | INTRODUCTION

Quantum calculus (in short q-calculus) is in the homework of the classical infinitesimal calculus without the notion of limit. It works as a bridge between mathematics and physics for the last five decades. One can find its application in different mathematical areas such as number theory, combinatorics, orthogonal polynomials, basic hypergeometric functions apart from quantum theory, mechanics and theory of relativity in physics. In 1912, using polynomials, Bernstein gave an alternative proof of the Weierstrass theorem: Every continuous function on $[a, b]$ can be uniformly approximated by a sequence of polynomial functions. Since Bernstein polynomials play an important role in approximation theory, many researchers have studied this polynomial and its different generalizations, see for instance [1, 2, 3]. Lupas [4] first introduced the q-analogue of Bernstein polynomials that brought into the existence of a new research area called q-approximation theory. Numerous authors [5, 6, 7, 8, 9, 10, 11, 12] have investigated and proposed the q-extension of various results of classical approximation theory.

However, classical approximation theory and q-approximation theory deal with the approximation of functions using smooth functions or infinitely differentiable functions. But, the classical smooth functions may not provide good representatives of irregular functions, for instance, Weierstrass function, and real-world sampled signals such as financial series, seismic data, speech signals, bioelectric recordings, etc. Fractal functions provide a constructive approximation theory for non-differentiable functions. Fractal functions concern mainly at data/function which present details at different scales or some degree of self-similarity.

By exploiting the theory of iterated function system (IFS) [13], Barnsley [14] introduced the concept of fractal interpolation function (FIF) to provide a mathematical representation of a data set that is generated from irregularity and/or self-affine structure. The calculus of fractal functions was investigated in [15, 16] and this research provided a methodology for the construction of C^r-fractal splines. Very recently, shape preserving fractal interpolation was studied in the references [17, 18, 19, 20]. In these articles various types of fractal splines that preserve the fundamental shapes of the interpolation data were developed. Shape preserving fractal surfaces, and their convergence and stability aspects were investigated in [21, 22, 23]. Furthermore, Barnsley [14] has extended the idea of fractal interpolation to approximate a continuous function f defined on a real compact interval I, and this led to the concept of fractal approximation or α-fractal function f^α of f [24, 25, 26]. In general, (i) α-fractal functions are non-differentiable (ii) the graph of f^α is a union of transformed copies of itself (iii) fractal dimension of graph of α-fractal function is non-integer. Owing to these fractal characteristics, f^α may be treated as the fractal approximant of f. In this way, every continuous function can be approximated by means of fractal functions. Further, shape preserving fractal approximation was investigated in [27]. Akhtar et al. [28] calculated the box dimension of the graph of α-fractal functions by assuming suitable conditions on the original function f and base function.

Navascués et al. [24, 25, 26, 29, 30, 31] studied various properties of the α-fractal function f^α of f including approximation properties, among various desirable properties of a good approximant. Navascués et al. [24, 25, 26, 29, 30, 31] proved that the α-fractal function f^α of f converges towards f provided the magnitude of the scaling factors of f^α goes to zero. In this paper, using the theory of fractal functions and classical q-approximation, for a given function $f \in C(I)$, we propose a sequence $\{f_n^{(q,\alpha)}\}_{n=1}^{\infty}$ of quantum fractal functions that converges to f even if the magnitude/norm of the corresponding scaling factors/functions does not go to zero. In the construction of the sequence $\{f_n^{(q,\alpha)}\}_{n=1}^{\infty}$ of quantum fractal functions, we use the sequence $\{B_{n,q}(f,\cdot)\}_{n=1}^{\infty}$ of q-Bernstein polynomials of f as *base functions*. Consequently, the convergence of the sequence $\{f_n^{(q,\alpha)}\}_{n=1}^{\infty}$ of quantum fractal functions towards the function f follows from the convergence of the q-Bernstein polynomials towards f. The shape of the quantum fractal functions depends on the choice of $q \in (0,1)$ and the scaling functions. When $q \to 1$, the q-Bernstein polynomial coincides with the classical Bernstein polynomial, and in this case we call quantum fractal functions simply α-fractal functions. Further, the convergence of these α-fractal functions towards f follows from the convergence of the q-Bernstein polynomials of f towards f. The procedure of getting a sequence $\{f_n^{(q,\alpha)}\}_{n=1}^{\infty}$ of quantum fractal functions that converges uniformly to $f \in C(I)$ determines an operator, termed the multivalued quantum fractal operator: $\mathcal{F}^{(q,\alpha)} : C(I) \rightrightarrows C(I)$, $f \to \{f_n^{(q,\alpha)}\}_{n=1}^{\infty}$. We study some basic properties of $\mathcal{F}^{(q,\alpha)}$.

Navascués and Chand [29] extended the notion of α-fractal function to $\mathcal{L}^p$-spaces and derived some approximation results under the assumption that the norm of the scaling functions tends to zero. In this manuscript, we develop (q,α)-Kantorovich-Bernstein fractal functions in $\mathcal{L}^p$-spaces without any condition on the scaling functions for convergence. Further, we study the approximation properties of (q,α)-Kantorovich-Bernstein fractal functions and quantum fractal versions of Müntz theorems in $\mathcal{L}^p$-spaces.

2 | BACKGROUND AND PRELIMINARIES

In this section we endeavor to expose the reader to the requisite preliminaries on fractal interpolation functions and its generalization through α-fractal functions.

2.1 | Fractal interpolation

Let $\mathbb{N}_k$ denote the first k natural numbers, $I = [x_1, x_N]$ be a closed and bounded interval of $\mathbb{R}$, and $C(I)$ be the Banach space of all real-valued continuous functions on I equipped with the supremum norm. Consider the interpolation data $\{(x_i, y_i) : i \in \mathbb{N}_N\}$ with strictly abscissae and $N > 2$. Let $L_i, i \in \mathbb{N}_{N-1}$, be a set of homeomorphic mappings from I to $I_i = [x_i, x_{i+1}]$ satisfying

$$L_i(x_1) = x_i, L_i(x_N) = x_{i+1}. \tag{1}$$

Let F_i be a function from $I \times K$ to K (K is suitable compact subset of $\mathbb{R}$), which is continuous in the x-direction and contractive in the y-direction (with contractive factor $|\alpha_i| \le \kappa < 1$) such that

$$F_i(x_1, y_1) = y_i, \quad F_i(x_N, y_N) = y_{i+1}, \ i \in \mathbb{N}_{N-1}. \tag{2}$$

Let us consider $\mathcal{G} = \{g \in C(I) \mid g(x_1) = y_1 \text{ and } g(x_N) = y_N\}$. We define a metric on $\mathcal{G}$ by $\rho(h,g) = \max\Big\{|h(x) - g(x)| : x \in I\Big\}$ for $h, g \in \mathcal{G}$. Then $(\mathcal{G}, \rho)$ is a complete metric space. Define the Read-Bajraktarević operator T on $(\mathcal{G}, \rho)$ by

$$Tg(x) = F_i(L_i^{-1}(x), g \circ L_i^{-1}(x)), \ x \in I_i. \tag{3}$$

Using the properties of L_i and (1)-(2), Tg is continuous on the interval I_i ; $i \in \mathbb{N}_{N-1}$, and at each of the points $x_2, \dots, x_{N-1}$. Also,

$$\rho(Tg, Th) \le |\alpha|_\infty \rho(g,h),$$

where $|\alpha|_\infty = \max\{|\alpha_i| : i \in \mathbb{N}_{N-1}\} < 1$. Hence, T is a contraction map on the complete metric space $(\mathcal{G}, \rho)$. Therefore, by the Banach fixed point theorem, T possesses a unique fixed point (say) f^* on $\mathcal{G}$, i.e., $(Tf^*)(x) = f^*(x)$ for all $x \in I$. According to

(3), the function f^* satisfies the functional equation: $f^*(x) = F_i(L_i^{-1}(x), f^* \circ L_i^{-1}(x))$, $x \in I_i$. Further, using (1)-(2), it is easy to verify that $f^*(x_i) = y_i$, $i \in \mathbb{N}_N$. Defining a mapping $w_i : I \times K \to I_i \times K$ as $w_i(x, y) = (L_i(x), F_i(x, y)), (x, y) \in I \times K,\ i \in \mathbb{N}_{N-1}$, the graph $G(f^*)$ of f^* satisfies:

$$G(f^*) = \bigcup_{i \in \mathbb{N}_{N-1}} w_i(G(f^*)), \tag{4}$$

and hence f^* is called fractal interpolation function (FIF) corresponding to the IFS $\mathcal{I} = \{I \times K, w_i(x, y) = (L_i(x), F_i(x, y)), i \in \mathbb{N}_{N-1}\}$.

Barnsley and Navascués [14, 24, 25] observed that the concept of FIFs can be used to define a class of fractal functions associated with a given function $f \in \mathcal{C}(I)$.

For a given $f \in \mathcal{C}(I)$, consider a partition $\Delta = \{x_1, x_2, \dots, x_N\}$ of $[x_1, x_N]$ satisfying $x_1 < x_2 < \cdots < x_N$, a continuous function $b : I \to \mathbb{R}$ that fulfills the conditions $b(x_1) = f(x_1)$, $b(x_N) = f(x_N)$ and $b \neq f$, and $N-1$ real numbers $\alpha_i, i \in \mathbb{N}_{N-1}$ satisfying $|\alpha_i| < 1$. Define an IFS through the maps

$$L_i(x) = a_i x + b_i, F_i(x, y) = \alpha_i y + f(L_i(x)) - \alpha_i b(x),\ i \in \mathbb{N}_{N-1}.$$

The corresponding FIF denoted by $f^\alpha_{\Delta,b} = f^\alpha$ is referred to as α-fractal function for f (fractal approximation of f) with respect to a scaling vector $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{N-1})$, base function b, and partition Δ. Here the set of data points is $\{(x_i, f(x_i)) : i \in \mathbb{N}_N\}$. The function f^α is the fixed point of the Read-Bajraktarević (RB) operator $T : \mathcal{C}_f(I) \to \mathcal{C}_f(I)$ defined by

$$(Tg)x = \alpha_i g(L_i^{-1}(x)) + f(x) - \alpha_i b(L_i^{-1}(x)), x \in I_i, i \in \mathbb{N}_{N-1},$$

where $\mathcal{C}_f(I) = \Big\{ g \in \mathcal{C}(I) \ : \ g(x_1) = f(x_1),\ g(x_N) = f(x_N) \Big\}$. Consequently, the α-fractal function f^α corresponding to f satisfies the self-referential equation

$$f^\alpha(x) = \alpha_i f^\alpha(L_i^{-1}(x)) + f(x) - \alpha_i b(L_i^{-1}(x)),\ x \in I_i,\ i \in \mathbb{N}_{N-1}. \tag{5}$$

The fractal dimension (box dimension or Hausdorff dimension) of f^α depends on the choice of the scaling vector α. For instance, Nasim Akhtar et. al [32] calculated box dimension of graph of α-fractal functions by assuming suitable conditions on the original function f and base function b. The following proposition provides the details of it.

Proposition 1. Let $f \in \mathcal{C}(I)$ and $b : I \to \mathbb{R}$ be Lipschitz functions with $b(x_1) = f(x_1)$, $b(x_N) = f(x_N)$. Let $\Delta = \{x_1, x_2, \dots, x_N\}$ be a partition of I satisfying $x_1 < x_2 < \cdots < x_N$. and $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{N-1})$. If the data points $(x_i, f(x_i)), i \in \mathbb{N}_N$ are not collinear, then graph G of the α-fractal function f^α has the box dimension

$$dim_B(G) = \begin{cases} D & \text{if } \sum_{i=1}^{N-1} |\alpha_i| > 1, \\ 1 & \text{otherwise,} \end{cases}$$

where D is solution of $\sum_{i=1}^{N-1} |\alpha_i| a_i^{D-1} = 1$.

To obtain fractal functions with more flexibility, iterated function system wherein scaling factors are replaced by scaling functions received attention in the recent literature [33] on fractal functions. That is, one may consider the IFS with maps

$$L_i(x) = a_i x + b_i, F_i(x, y) = \alpha_i(x) y + f(L_i(x)) - \alpha_i b(x), i \in \mathbb{N}_{N-1},$$

where $\alpha_i, i \in \mathbb{N}_{N-1}$ are continuous functions on I satisfying $\max\{\|\alpha_i\|_\infty : \pi \in \mathbb{N}_{N-1}\} < 1$. The corresponding α-fractal function is the fixed point of the RB-operator

$$(Tg)x = \alpha_i(L_i^{-1}(x)) g(L_i^{-1}(x)) + f(x) - \alpha_i(L_i^{-1}(x)) b(L_i^{-1}(x)), x \in I_i, i \in \mathbb{N}_{N-1}. \tag{6}$$

Consequently, the α-fractal function f^α corresponding to f satisfies the self-referential equation

$$f^\alpha(x) = \alpha_i(L_i^{-1}(x)) f^\alpha(L_i^{-1}(x)) + f(x) - \alpha_i(L_i^{-1}(x)) b(L_i^{-1}(x)), x \in I_i, i \in \mathbb{N}_{N-1}. \tag{7}$$

3 | QUANTUM FRACTAL APPROXIMATION

From (7), we get the following inequality:

$$\|f^\alpha - f\|_\infty \leq \frac{\|\alpha\|_\infty}{1 - \|\alpha\|_\infty} \|f - b\|_\infty, \tag{8}$$

where $\|\alpha\|_\infty = \max\{\|\alpha_i\|_\infty : i \in \mathbb{N}_{N-1}\}$. For a fixed base function b, the α-fractal function f^α converges uniformly to $f \in \mathcal{C}(I)$ if $\|\alpha\|_\infty \to 0$. To get the convergence of the α-fractal function f^α towards f without altering the scaling functions, we choose the base function b as q-Bernstein polynomial $B_{n,q}(f,x)$ of f, i.e., $b = B_{n,q}(f,x)$ (see for instance [34]),

$$B_{n,q}(f,x) = \frac{1}{(x_N - x_1)^n} \sum_{k=0}^{n} \binom{n}{k}_q (x-x_1)^k f\Big(x_1 + (x_N - x_1)\frac{[k]_q}{[n]_q}\Big) \prod_{s=0}^{n-k-1} \big(x_N - x_1 - q^s x\big), x \in I, \tag{9}$$

where $q \in (0,1), n \in \mathbb{N}$, $[k]_q = \frac{1-q^k}{1-q}$,

$$[k]_q! = \begin{cases} [k]_q[k-1]_q[k-2]_q \dots [2]_q[1]_q, & \text{if } k \neq 0, \\ 1, & \text{if } k = 0, \end{cases}$$

$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$, $f \in \mathcal{C}(I)$, $B_{n,q}(f,x_1) = f(x_1)$, $B_{n,q}(f,x_N) = f(x_N)$. When $q \to 1$, $B_{n,q}(f,x)$ coincides with the classical n-th Bernstein polynomial. If we take the base function as $b = B_{n,q}(f,x)$ in (9), then the corresponding fractal function $\mathcal{F}^{q,\alpha}_{\Delta,B_n}(f) = f_n^{(q,\alpha)}$ is called a quantum Bernstein fractal function associated with $f \in \mathcal{C}(I)$, and

$$f_n^{(q,\alpha)}(x) = f(x) + \alpha_i(L_i^{-1}(x))[f_n^{(q,\alpha)}(L_i^{-1}(x)) - B_{n,q}(f, L_i^{-1}(x))], \; x \in I_i, \; i \in \mathbb{N}_{N-1}, n \in \mathbb{N}. \tag{10}$$

Therefore, from (10), it is easy to notice that shape and properties of the quantum fractal function $f_n^{(q,\alpha)}$ depends on the choice of $q \in (0,1)$ apart from the choice of scaling functions. Note that the quantum fractal function $f_n^{(q,\alpha)}, n \in \mathbb{N}$, of $f \in \mathcal{C}(I)$ is obtained via the IFS defined by

$$\mathcal{I}_n = \{I \times \mathbb{R}, (L_i(x), F_{n,i}(x,y)) : i \in \mathbb{N}_{N-1}\}, n \in \mathbb{N}, \tag{11}$$

where $F_{n,i}(x,y) = f(L_i(x)) - \alpha_i(x)(y - B_{n,q}(f,x))$.

Theorem 1. Let $f \in \mathcal{C}(I)$. There exists a sequence of quantum Bernstein fractal functions $\{f_n^{(q,\alpha)}(x)\}_{n=1}^\infty$ that converges uniformly to f on I. Further, $f_n^{(q,\alpha)}, n \in \mathbb{N}$, satisfies the following inequalities:

$$\frac{1 - \|\alpha\|_\infty}{1 + \|\alpha\|_\infty}\|f\|_\infty \leq \|f_n^{(q,\alpha)}\| \leq \frac{1 + \|\alpha\|_\infty}{1 - \|\alpha\|_\infty}\|f\|_\infty. \tag{12}$$

Proof. Let $f_n^{(q,\alpha)}, n \in \mathbb{N}$, be the quantum fractal function corresponding to f. Then, from (10), it is easy to deduce that

$$\begin{aligned} \|f_n^{(q,\alpha)} - f\|_\infty \leq & \|\alpha\|_\infty \|f_n^{(q,\alpha)} - B_{n,q}(f,.)\|_\infty, \\ \leq & \|\alpha\|_\infty [\|f_n^{(q,\alpha)} - f\|_\infty + \|f - B_{n,q}(f,.)\|_\infty]. \end{aligned}$$

Hence we obtain

$$\|f_n^{(q,\alpha)} - f\|_\infty \leq \frac{\|\alpha\|_\infty}{1 - \|\alpha\|_\infty}\|f - B_{n,q}(f,.)\|_\infty. \tag{13}$$

From [34], we have

$$\|B_{n,q}(f,.) - f\|_\infty \to 0 \text{ as } n \to \infty. \tag{14}$$

Using (14) in (13), we conclude that the sequence $\{f_{n,q}(x)\}_{n=1}^\infty$ of quantum fractal functions converges uniformly to f. Again from [35], we have

$$\|B_{n,q}(.,.)\|_\infty = 1, q \in (0,1]. \tag{15}$$

We can rewrite (13) as

$$\|f_n^{(q,\alpha)}\|_\infty - \|f\|_\infty \leq \|f_n^{(q,\alpha)} - f\|_\infty \leq \frac{\|\alpha\|_\infty}{1 - \|\alpha\|_\infty}\big\{\|f\|_\infty + \|B_{n,q}(f,.)\|_\infty\big\}. \tag{16}$$

Using (15) in (16), we get the right side inequality of (12). Next, from (10), we obtain

$$|f_n^{(q,\alpha)}(x) - f(x)| \leq \|\alpha_i\|_\infty\big\{\|f_n^{(q,\alpha)}\|_\infty + \|B_{n,q}(f,.)\|_\infty\big\}, \; x \in I_i, \; i \in \mathbb{N}_{N-1}, n \in \mathbb{N},$$

which implies that

$$\|f\|_\infty - \|f_n^{(q,\alpha)}\|_\infty \leq \|f_n^{(q,\alpha)} - f\|_\infty \leq \|\alpha\|_\infty\big\{\|f_n^{(q,\alpha)}\|_\infty + \|B_{n,q}(f,.)\|_\infty\big\}.$$

Using (15) in the above inequality, we get the left side inequality of (12). □

Proposition 2. If we consider $\mathcal{L}^p$-norm $\|f\|_{\mathcal{L}^p} = \left(\int_I |f(t)|^p dt\right)^{1/p}$, $1 < p < \infty$, for $f \in C(I)$, the following inequality holds.

$$\|f_n^{(q,\alpha)} - f\|_{\mathcal{L}^p} \leq \frac{||\alpha||_\infty}{1 - ||\alpha||_\infty} \|f - B_{n,q}(f)\|_{\mathcal{L}^p} \tag{17}$$

Proof. From (10), we have

$$\begin{aligned}
\|f_n^{(q,\alpha)} - f\|_{\mathcal{L}^p}^p &= \int_I |(f_n^{(q,\alpha)} - f)(x)|^p dx \\
&= \sum_{i=1}^{N-1} \int_{x_i}^{x_{i+1}} |\alpha_i(L_i^{-1}(x)|^p |(f_n^{(q,\alpha)} - B_{n,q}(f)) \circ L_i^{-1}(x)|^p dx \\
&= \sum_{i=1}^{N-1} \int_I a_i |\alpha_i(\tilde{x})|^p |(f_n^{(q,\alpha)} - B_{n,q}(f))(\tilde{x})|^p d\tilde{x} \\
&\leq \sum_{i=1}^{N-1} a_i \|\alpha\|_\infty^p \int_I |(f_n^{(q,\alpha)} - B_{n,q} f)(x)|^p dx \\
&= \|\alpha\|_\infty^p \|f_n^{(q,\alpha)} - B_{n,q}(f)\|_{\mathcal{L}^p}^p.
\end{aligned}$$

In the above computation, we have used the change of variable $\tilde{x} = L_i^{-1}(x)$ at the 3rd step and $\sum_{i=1}^{N-1} a_i = 1$ at the final step. From the above estimation, we have

$$\begin{aligned}
\|f_n^{(q,\alpha)} - f\|_{\mathcal{L}^p} &\leq \|\alpha\|_\infty \|f_n^{(q,\alpha)} - B_{n,q}(f)\|_{\mathcal{L}^p} \\
&\leq \|\alpha\|_\infty (\|f_n^{(q,\alpha)} - f\|_{\mathcal{L}^p} + \|f - B_{n,q}(f)\|_{\mathcal{L}^p}).
\end{aligned}$$

Further simplification of the above inequality gives the desired estimation in (17). □

Examples. Now, we want to see some examples of q-fractal functions for a given function $f(x) = x^{1/4}, x \in [0, 1]$. The quantum fractal functions in Figures 1(a)-(c) are generated with respect to the partition $\Delta = \{0, 0.25, 0.5, 1\}$ of $[0, 1]$. The quantum fractal functions $f_2^{(0.2,\alpha)}$, $f_2^{(0.7,\alpha)}$, and $f_{98}^{(0.7,\alpha)}$ are generated at the sixth iteration respectively in Figures 1(a)-(c) with the choice of the scaling functions $\alpha_i(x) = \frac{1}{1+e^{-10x}}, x \in [0, 1], i \in \mathbb{N}_3$. By comparing the quantum fractal functions $f_2^{(0.2,\alpha)}$ and $f_2^{(0.7,\alpha)}$, one can observe the effects of q in the shape of the quantum fractal function. According to Theorem 1, the quantum fractal function $f_{31}^{(0.7,\alpha)}$ provides a better approximation for $x^{1/4}, x \in [0, 1]$ than that obtained by $f_2^{(0.7,\alpha)}$. By observing Figures 1(b)-(c), one can ask why the fractal functions $f_2^{(0.7,\alpha)}$ and $f_{98}^{(0.7,\alpha)}$ don't have the same sort of irregularity even if their scaling functions are same. This is due to the following reason: The fractal function $f_2^{(0.7,\alpha)}$ exhibits irregularity on all scales whereas the fractal function $f_{31}^{(0.7,\alpha)}$ exhibits irregularity on small scales. Further, small scales of irregularity of the fractal function $f_{98}^{(0.7,\alpha)}$ can be observed from Figure 1(d) which is a part of $f_{98}^{(0.7,\alpha)}$ under magnification.

4 | MULTIVALUED QUANTUM FRACTAL OPERATOR

The definition of quantum α-fractal function $\mathcal{F}_{\Delta,B_n}^{(q,\alpha)}(f) = f_{\Delta,B_n}^{(q,\alpha)} = f_n^{(q,\alpha)}$ corresponding to each $f \in C(I)$ yields a multivalued quantum fractal operator $\mathcal{F}^{(q,\alpha)} : C(I) \rightrightarrows C(I)$ defined by

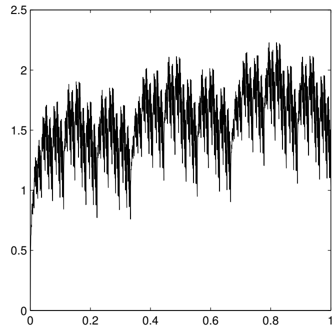
$$\mathcal{F}^{(q,\alpha)}(f) = \{\mathcal{F}_n^{(q,\alpha)}(f)\}_{n=1}^\infty = \{f_n^{(q,\alpha)}\}_{n=1}^\infty.$$

Let us record some definitions which are needed for our further investigations.

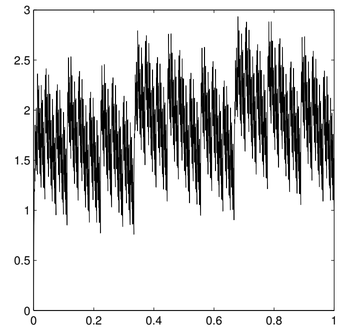
Definition 1 ([36])**.** Let X and Y be two real normed linear spaces over $\mathbb{R}$. For a multi-valued map $T : X \to Y$, the domain of T is defined by $Dom(T) = \{x \in X : T(x) \neq \emptyset\}$. Then $T : X \rightrightarrows Y$ is

- convex if for all $x_1, x_2 \in Dom(T)$ and for all $\lambda \in [0, 1]$,

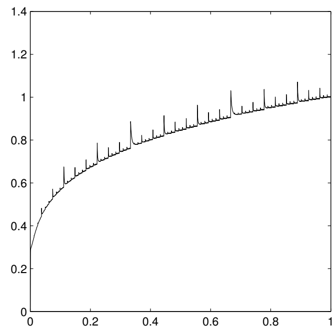
$$\lambda T(x_1) + (1 - \lambda) T(x_2) \subseteq T(\lambda x_1 + (1 - \lambda) x_2).$$



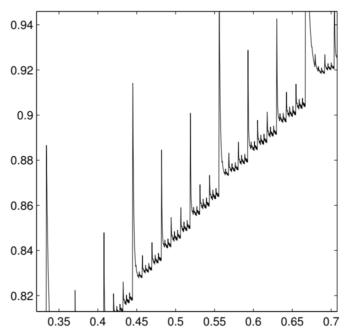
(a) The quantum fractal function $f_2^{(0.2,\alpha)}$.



(b) The quantum fractal function $f_2^{(0.7,\alpha)}$.



(c) The quantum fractal function $f_{98}^{(0.7,\alpha)}$.



(d) A part of $f_{98}^{(0.7,\alpha)}$ under magnification.

FIGURE 1 The quantum fractal approximants of $x^{1/4}, x \in [0,1]$.

- process if for all $x \in Dom(T)$ and for all $\lambda > 0$,
$$T(\lambda x) = \lambda T(x) \text{ and } 0 \in T(0).$$
- linear if for all $x_1, x_2 \in Dom(T)$ and for all $\beta, \gamma \in \mathbb{R}$,
$$\beta T(x_1) + \gamma T(x_2) \subseteq T(\beta x_1 + \gamma x_2).$$
- Lipschitz if there exists a constant $\nu > 0$ such that for all $x_1, x_2 \in Dom(T)$
$$T(x_1) \subseteq T(x_2) + \nu \|x_1 - x_2\| U_Y,$$
where U_Y is the closed unit ball in Y.

Theorem 2. [Corollary 1.4, [37]] Let X and Y be real vector spaces and $P_0(Y)$ be the collection of all nonempty subsets of Y. A multivalued map $T : X \to P_0(Y)$ is linear and $T(0) = \{0\}$ if and only if T is single-valued map.

Theorem 3. [Corollary 1.4, [37]] Let X and Y be real vector spaces and $P_0(Y)$ be the collection of all nonempty subsets of Y. If a multivalued map $T : X \to P_0(Y)$ is such that $T(x_0)$ is a singleton for some $x_0 \in X$, then $T : X \to P_0(Y)$ is convex if and only if T is single-valued and affine.

Theorem 4. The multivalued quantum fractal operator $\mathcal{F}^{(q,\alpha)} : \mathcal{C}(I) \rightrightarrows \mathcal{C}(I)$ defined by $\mathcal{F}^{(q,\alpha)}(f) = \{f_n^{(q,\alpha)}\}_{n=1}^{\infty}$ is not linear.

Proof. Clearly $\mathcal{F}^{(q,\alpha)}$ is multivalued. Also, from definition $\mathcal{F}^{(q,\alpha)}(0) = \{0\}$. Hence, by Theorem 2, $\mathcal{F}^{(q,\alpha)}$ is not linear. □

Remark 1. Note that $B_{n,q} : (\mathcal{L}^p, \|\cdot\|_p) \to (\mathcal{L}^p, \|\cdot\|_p)$ is not bounded on $(\mathcal{L}^p(I), \|\cdot\|_p)$. Thus, we can not use the standard density argument to extend the continuous quantum Bernstein fractal functions to $(\mathcal{L}^p(I), \|\cdot\|_p)$. Therefore, we will construct $\mathcal{L}^p$-quantum fractal Bernstein fractal functions by using Kantorovich-Bernstein polynomials in the following.

5 | KANTOROVICH-BERNSTEIN FRACTAL FUNCTIONS IN $\mathcal{L}^P$ SPACES

In this section, for a given function $f \in \mathcal{L}^p(I)$, $1 \le p \le \infty$, using q-Kantorovich-Bernstein operator $\Phi_{q,n}$ [34] as base function, we develop (q,α)-Kantorovich-Bernstein fractal functions in the following:
It is known [34] that for $f \in \mathcal{L}^p(I)$, $1 \le p \le \infty$, $\|f - \Phi_{q,n}(f)\|_p \to 0$ as $n \to \infty$, where

$$\Phi_{q,n}(f;x) = \frac{1}{(x_N - x_1)^n} \sum_{k=0}^{n} \binom{n}{k}_q (x - x_1)^k (x_N - x)^{n-k} [n+1]_q \int_{x_1 + \frac{k(x_N - x_1)}{[n+1]_q}}^{x_1 + \frac{(k+1)(x_N - x_1)}{[n+1]_q}} f(t) d_q t,$$

where $d_q t$ denotes the q-integration [38].

The proof of the following theorem can be obtained using the arguments similar to those used in [39].

Theorem 5. Let $f \in \mathcal{L}^p(I)$, $1 \le p \le \infty$. Suppose $\Delta = \{x_1, x_2, \ldots, x_N\}$ be a partition of I satisfying $x_1 < x_2 < \ldots < x_N$, $I_i := [x_i, x_{i+1})$, $i \in \mathbb{N}_{N-2}$, $I_{N-1} = [x_{N-1}, x_N]$. Let $L_i(x) = a_i x + b_i$ satisfy (1). If $\alpha_i \in \mathcal{L}^\infty(I)$ for all $i \in \mathbb{N}_{N-1}$ and $b(x) = \Phi_{q,n}(f;x) \in \mathcal{L}^p(I)$, then the RB-operator given in (6) maps $\mathcal{L}^p(I)$ onto itself. Further, if the scaling function satisfies the condition

$$\begin{cases} \left[\sum\limits_{i \in \mathbb{N}_{N-1}} a_i \|\alpha_i\|_\infty^p \right]^{\frac{1}{p}} < 1 & \text{if } 1 \le p < \infty, \\ \|\alpha\|_\infty < 1 & \text{if } p = \infty, \end{cases}$$

then T is a contraction on $\mathcal{L}^p$, and gives a fixed point $f_n^{(q,\alpha)} \in \mathcal{L}^p(I)$ for each $n \in \mathbb{N}$, which satisfies the self-referential equation (4).

From here we will assume that these conditions on the scaling functions are satisfied.
We define a (q,α)-Kantorovich-Bernstein fractal function as the solution of the fixed point equation:

$$f_n^{(q,\alpha)}(x) = f(x) + (f_n^{(q,\alpha)}(L_i^{-1}(x)) - \Phi_{q,n}(f; L_i^{-1}(x)))\alpha_i(L_i^{-1}(x)) \; \forall x \in I_i, \; n \in \mathbb{N}, \; i \in N_{N-1}. \tag{18}$$

Theorem 6. For $f \in \mathcal{L}^p(I)$ and the scaling functions satisfying the conditions given in Theorem 5, there exists a sequence $\{f_n^{(q,\alpha)}(x)\}_{n=1}^\infty$ of (q,α)-Kantorovich-Bernstein fractal functions that converges uniformly to f on I.

Proof. From (18) for $1 \le p < \infty$, we obtain

$$\begin{aligned} \|f_n^{(q,\alpha)} - f\|_p^p &= \int_I |(f_n^{(q,\alpha)} - f)(x)|^p dx \\ &= \sum_{i \in \mathbb{N}_{N-1}} \int_{I_i} \left| (f_n^{(q,\alpha)}(L_i^{-1}(x)) - \Phi_{q,n}(f; L_i^{-1}(x)))\alpha_i(L_i^{-1}(x)) \right|^p dx \\ &= \sum_{i \in \mathbb{N}_{N-1}} a_i \int_I \left| (f_n^{(q,\alpha)}(t) - \Phi_{q,n}(f;t))\alpha_i(t) \right|^p dt \\ &\le \sum_{i \in \mathbb{N}_{N-1}} a_i \|\alpha_i\|_\infty^p \int_I \left| (f_n^{(q,\alpha)}(t) - \Phi_{q,n}(f;t)) \right|^p dt \\ &= \sum_{i \in \mathbb{N}_{N-1}} a_i \|\alpha_i\|_\infty^p \|f_n^{(q,\alpha)} - \Phi_{q,n}(f)\|_p^p. \end{aligned}$$

Taking pth root in both sides, we have

$$\begin{aligned} \|f_n^{(q,\alpha)} - f\|_p &\le \left[\sum_{i \in \mathbb{N}_{N-1}} a_i \|\alpha_i\|_\infty^p \right]^{\frac{1}{p}} \|f_n^{(q,\alpha)} - \Phi_{q,n}(f)\|_p, \\ &\le \left[\sum_{i \in \mathbb{N}_{N-1}} a_i \|\alpha_i\|_\infty^p \right]^{\frac{1}{p}} \left[\|f_n^{(q,\alpha)} - f\|_p + \|f - \Phi_{q,n}(f)\|_p \right], \end{aligned}$$

and further implication gives

$$\|f_n^{(q,\alpha)} - f\|_p \le \frac{\left[\sum_{i\in\mathbb{N}_{N-1}} a_i\|\alpha_i\|_\infty^p\right]^{\frac{1}{p}}}{1-\left[\sum_{i\in\mathbb{N}_{N-1}} a_i\|\alpha_i\|_\infty^p\right]^{\frac{1}{p}}}\|f-\Phi_{q,n}(f)\|_p. \tag{19}$$

A similar calculation as in Proposition 2, we obtain

$$\|f_n^{(q,\alpha)} - f\|_\infty \le \frac{\|\alpha\|_\infty}{1-\|\alpha\|_\infty}\|f-\Phi_{q,n}(f)\|_\infty. \tag{20}$$

From the last two inequalities, we get the desired result. □

Theorem 7. The (q,α)-Kantorovich-Bernstein fractal operator $\mathcal{F}^{(q,\alpha)}_{\Delta,\Phi_{q,n}} : \mathcal{L}^p(I) \mapsto \mathcal{L}^p(I), 1 \le p \le \infty, n \in \mathbb{N}$ defined by $\mathcal{F}^{(q,\alpha)}_{\Delta,\Phi_{q,n}}(f) = f_n^{(q,\alpha)}$ is linear and bounded.

Proof. Let f and g be in $\mathcal{L}^p(I)$ and λ_1, λ_2 be real scalars. The functional equations for the corresponding (q,α)-Kantorovich-Bernstein fractal functions are given by

$$f_n^{(q,\alpha)}(x) = f(x) + \alpha_i(L_i^{-1}(x))(f_n^{(q,\alpha)}(L_i^{-1}(x)) - \Phi_{q,n}(f;L_i^{-1}(x))),$$

$$g_n^{(q,\alpha)}(x) = g(x) + \alpha_i(L_i^{-1}(x))(g_n^{(q,\alpha)}(L_i^{-1}(x)) - \Phi_{q,n}(g;L_i^{-1}(x)))\ \forall x \in I_i,\ i \in \mathbb{N}_{N-1}.$$

Thus, we can write

$$\begin{aligned}(\lambda_1 f_n^{(q,\alpha)} + \lambda_2 g_n^{(q,\alpha)})(x) =&(\lambda_1 f + \lambda_2 g)(x) + \alpha_i(L_i^{-1}(x))[(\lambda_1 f_n^{(q,\alpha)} + \lambda_2 g_n^{(q,\alpha)})(L_i^{-1}(x)) \\ &- \Phi_{q,n}(\lambda_1 f + \lambda_2 g; L_i^{-1}(x))]\end{aligned} \tag{21}$$

from which we obtain that $\lambda_1 f_n^{(q,\alpha)} + \lambda_2 g_n^{(q,\alpha)}$ is a fixed point of the operator

$$(Th)(x) = (\lambda_1 f + \lambda_2 g)(x) + \alpha_i(L_i^{-1}(x))(h - \Phi_{q,n}(\lambda_1 f + \lambda_2 g; L_i^{-1}(x)).$$

Now using the uniqueness of fixed point, we get

$$\mathcal{F}^{(q,\alpha)}_{\Delta,\Phi_{q,n}}(\lambda_1 f + \lambda_2 g) = \lambda_1 f_n^{(q,\alpha)} + \lambda_2 g_n^{(q,\alpha)} = \lambda_1 \mathcal{F}^{(q,\alpha)}_{\Delta,\Phi_{q,n}}(f) + \lambda_2 \mathcal{F}^{(q,\alpha)}_{\Delta,\Phi_{q,n}}(g).$$

Again with help of (19)-(20), we have

$$\begin{aligned}\|\mathcal{F}^{(q,\alpha)}_{\Delta,\Phi_{q,n}}(f)\|_p &= \|f_n^{(q,\alpha)}\|_p \\ &\le \|f_n^{(q,\alpha)} - f\|_p + \|f\|_p \\ &\le \frac{R}{1-R}\|f - \Phi_{q,n}(f)\|_p + \|f\|_p \\ &\le \frac{R}{1-R}\|Id - \Phi_{q,n}\|_p\|f\|_p + \|f\|_p,\end{aligned} \tag{22}$$

where

$$R = \begin{cases}\left[\sum_{i\in\mathbb{N}_{N-1}} a_i\|\alpha_i\|_\infty^p\right]^{\frac{1}{p}}, & \text{for } 1 \le p < \infty, \\ \|\alpha\|_\infty, & \text{for } p = \infty.\end{cases} \tag{23}$$

Since $\|Id - \Phi_{q,n}\|_p \to 0$ as $n \to \infty$, so for given $\epsilon = 1$, there exists $M \in \mathbb{N}$ such that

$$\|Id - \Phi_{q,n}\|_p < 1\ \ \forall\ n > M.$$

Consider $\eta = \max\{\|Id - \Phi_{q,1}\|_p, \|Id - \Phi_{q,2}\|_p, \dots, \|Id - \Phi_{q,M}\|_p, 1\}$. Then from (22) we get

$$\|\mathcal{F}^{(q,\alpha)}_{\Delta,\Phi_{q,n}}\| \le 1 + \frac{R}{1-R}\eta,$$

which implies $\mathcal{F}^{(q,\alpha)}_{\Delta,\Phi_{q,n}}$ is bounded operator for each $n \in \mathbb{N}$. □

Theorem 8. Consider a scaling function which satisfies

$$\left[\sum_{i\in\mathbb{N}_{N-1}} a_i\|\alpha_i\|_\infty^p\right]^{\frac{1}{p}} < \min\{1, \|\Phi_{q,n}\|^{-1}\}, \text{ if } 1 \le p < \infty,$$

$$\|\alpha\|_\infty < \min\{1, \|\Phi_{q,n}\|^{-1}\}, \text{ if } p = \infty.$$

Then the corresponding fractal operator is bounded below. In particular, $\mathcal{F}^{(q,\alpha)}_{\Delta,\Phi_{q,n}}$ is injective and has a closed range.

Proof. From the reverse triangle inequality and the proof of Theorem 6, we obtain

$$\begin{aligned}\|f\|_p - \|f_n^{(q,\alpha)}\|_p &\le \|f - f_n^{(q,\alpha)}\|_p \\ &\le R\|f_n^{(q,\alpha)} - \Phi_{q,n}(f)\|_p \\ &\le R\|f_n^{(q,\alpha)}\|_p + R\|\Phi_{q,n}\|\|f\|_p \\ \Rightarrow \|f\|_p &\le \frac{1+R}{1-R\|\Phi_{q,n}\|}\|f_n^{(q,\alpha)}\|_p. \end{aligned} \tag{24}$$

Since $\|\Phi_{q,n}\|^{-1} > R$, the operator $\mathcal{F}^{(q,\alpha)}_{\Delta,\Phi_{q,n}}$ is bounded below and so injective. Now to prove $\mathcal{F}^{(q,\alpha)}_{\Delta,\Phi_{q,n}}$ has a closed range, let $f_{n,m}^{(q,\alpha)}$ be a sequence in $\mathcal{F}^{(q,\alpha)}_{\Delta,\Phi_{q,n}}(\mathcal{L}^p(I))$ such that $f_{n,m}^{(q,\alpha)} \to \tilde{f}$, and thus, $f_{n,m}^{(q,\alpha)}$ is a Cauchy sequence. Now

$$\|f_m - f_r\|_p \le \frac{1+R}{1-R\|\Phi_{q,n}\|}\|f_{m,n}^{(q,\alpha)} - f_{r,n}^{(q,\alpha)}\|_p$$

which shows that $\{f_m\}$ is a Cauchy sequence in $\mathcal{L}^p(I)$. Since $\mathcal{L}^p(I)$ is a complete metric space, there exists $f \in \mathcal{L}^p(I)$ such that $f_m \to f$. Using the continuity of $\mathcal{F}^{(q,\alpha)}_{\Delta,\Phi_{q,n}}$, we have $\tilde{f} = \mathcal{F}^{(q,\alpha)}_{\Delta,\Phi_{q,n}}(f) = f_n^{(q,\alpha)}$. □

6 | APPROXIMATION BY KANTOROVICH-BERNSTEIN FRACTAL FUNCTIONS

Denote $\Lambda := \{\lambda_i\}_{i=1}^{+\infty}$, $\lambda_i \ne \lambda_j$ if $i \ne j$, $\lambda_i \in \mathbb{R}^+$, $\lambda_0 = 0$. The collection $\Lambda_m = \{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_m}\}$ is called a finite Müntz system. The linear span of Λ_m is known as Müntz space and denoted by $M_m(\Lambda)$. Let $I = [a,b], a > 0$ and $\Delta := \{x_1, \dots, x_N\}$ be a partition of I satisfying $a = x_1 < \dots < x_N = b$. Choose the scaling function $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{N-1}) \in (\mathcal{L}^\infty(I))^{N-1}$ as per the prescription given in Theorem 5. We know that $\Phi_{q,n} : \mathcal{L}^p(I) \mapsto \mathcal{L}^p(I)$ is a bounded linear map and the Müntz monomial $x^{\lambda_i} \in \mathcal{L}^p(I)$ even if $\lambda_i > \frac{-1}{p}$. Therefore, we can define the (q,α)-Kantorovich-Bernstein fractal Müntz monomial $(x^{\lambda_i})_n^{(q,\alpha)} := \mathcal{F}^{(q,\alpha)}_{\Delta,\Phi_{q,n}}(x^{\lambda_i})$.

Definition 2. A (q,α)-Kantorovich-Bernstein fractal Müntz polynomial is a finite linear combination of the functions $(x^{\lambda_i})_n^{(q,\alpha)}$, where $\lambda_i \in \Lambda$, $i \in \mathbb{N}$, and $\alpha \in (\mathcal{L}^\infty(I))^{N-1}$ satisfies the condition of Theorem 5. In particular, when $\alpha = \mathbf{0}$, this linear combination is called quantum Bernstein Müntz polynomial.

Let $S = \{(x^{\lambda_i})_n^{(q,\alpha)} : i, n \in \mathbb{N}\}$. The set

$$M^{(q,\alpha)}(\Lambda) := Span(S)$$

is defined as the quantum Bernstein fractal Müntz space associated with Λ. We need the following definition in the sequel:

Definition 3. ([40]) A set A is fundamental in a normed linear space B if the family of linear combinations of elements of A is a dense set of B.

Theorem 9. Quantum fractal version of first Müntz theorem: Let Δ be a partition of $I = [a,b], b > 0$. If the scaling vector α is chosen according to Theorem 5, then the system S restricted to values λ_i such that $-\frac{1}{2} < \lambda_i \to \infty$ is fundamental in $\mathcal{L}^2(I)$, whenever $\sum_{\lambda_i\ne 0}\frac{1}{\lambda_i} = +\infty$.

Proof. Let $g \in \mathcal{L}^2(I)$ and $\epsilon > 0$ be given. From classical Müntz's first theorem (see for instance [40]), it is known that the set of functions $\{x^{\lambda_1}, x^{\lambda_2}, \dots\}$, where $-\frac{1}{2} < \lambda_i \to \infty$ is fundamental in the least-square norm if and only if $\sum_{\lambda_i\ne 0}\frac{1}{\lambda_i} = +\infty$.

Thus, for $\epsilon/2 > 0$, there exists a Müntz polynomial $q_m \in Span\{x^{\lambda_1}, x^{\lambda_2}, \dots\}$ such that

$$\|g - q_m\|_2 < \frac{\epsilon}{2}. \tag{25}$$

With the scaling function α, we construct the (q,α)-Kantorovich-Bernstein fractal Müntz polynomial as $(q_m)_n^{(q,\alpha)} = \mathcal{F}^{(q,\alpha)}_{\Delta,\Phi_{q,n}}(q_m)$ by using the linearity of $\mathcal{F}^{(q,\alpha)}_{\Delta,\Phi_{q,n}}$. Since $\|q_m - \Phi_{q,n}(q_m)\|_2 \to 0$ as $n \to \infty$, there exists $M_1 \in \mathbb{N}$ such that

$$\|q_m - \Phi_{q,n}(q_m)\|_2 < \frac{\epsilon\left[1 - \sqrt{\sum_{i\in\mathbb{N}_{N-1}} a_i\|\alpha_i\|_\infty^2}\right]}{2\sqrt{\sum_{i\in\mathbb{N}_{N-1}} a_i\|\alpha_i\|_\infty^2}} \quad \text{for } n > M_1. \tag{26}$$

Using (26) in (19), we obtain

$$\|(q_m)_n^{(q,\alpha)} - q_m\|_2 \le \frac{\sqrt{\sum_{i\in\mathbb{N}_{N-1}} a_i\|\alpha_i\|_\infty^2}}{1 - \sqrt{\sum_{i\in\mathbb{N}_{N-1}} a_i\|\alpha_i\|_\infty^2}}\|q_m - \Phi_{q,n}(q_m)\|_2 < \frac{\epsilon}{2} \quad \text{for } n > M_1. \tag{27}$$

Combining (25) and (27), we have

$$\|g - (q_m)_n^{(q,\alpha)}\|_2 \le \|g - q_m\|_2 + \|(q_m)_n^{(q,\alpha)} - q_m\|_2 < \epsilon \quad \text{for } n > M_1.$$

Consequently $(q_m)_n^{(q,\alpha)} \in M^{(q,\alpha)}(\Lambda)$ approximates to g in $\mathcal{L}^2$-norm and the set considered is fundamental in $\mathcal{L}^2(I)$. □

Corollary 1. The system S is complete in $\mathcal{L}^2(I)$ if the scaling vector α is chosen according to the prescription of Theorem 5, $-\frac{1}{2} < \lambda_i \to \infty$ and $\sum_{\lambda_i\neq 0} \frac{1}{\lambda_i} = +\infty$.

Proof. In the above theorem, we have proved that $\{(x^{\lambda_i})_n^{(q,\alpha)} \;:\; i,n \in \mathbb{N}\}$ where λ_i satisfy the conditions described is fundamental in the normed linear space $\mathcal{L}^2(I)$. According to Banach's theorem (see for instance [41]), the system S is complete. □

We can generalize the above results for any fundamental system of $\mathcal{L}^p(I), 1 \le p < \infty$. The proof follows similar lines and hence it is omitted.

Theorem 10. Let Δ be a partition of $I = [a,b], a > 0$ and the scaling vector α be chosen according to the prescription of Theorem 5. If the system $\{f_j \;:\; j \in \mathbb{N}\}$ is fundamental in $\mathcal{L}^p(I), 1 \le p < \infty$, then the corresponding quantum Bernstein fractal system $\{(f_j^{\lambda_i})_n^{(q,\alpha)} \;:\; i,j,n \in \mathbb{N}\}$ is also fundamental whenever $-\frac{1}{p} < \lambda_i \to \infty$ and $\sum_{\lambda_i\neq 0} \frac{1}{\lambda_i} = +\infty$.

Now, we will state the full Müntz theorem in $L^p[0,1], 1 \le p \le \infty$ for quantum fractal Bernstein Müntz polynomials. The proof follows similar steps as described in Theorem 9 with proper choice of classical Müntz polynomial, i.e., the exponents satisfy the condition prescribed by Borwein and Erdélyi [42].

Theorem 11. Let $1 \le p \le \infty$ and $\Delta := 0 = x_0 < x_1 < \dots x_N = 1$ be a partition of $I = [0,1]$. Let $\Lambda := \{\lambda_i\}_{i=0}^{\infty}$ be a sequence of distinct real numbers greater than $\frac{-1}{p}$, and such that $\sum_{i=0}^{\infty} \frac{\lambda_i + \frac{1}{p}}{(\lambda_i + \frac{1}{p})^2 + 1} = \infty$. Then, the system $\{(x^{\lambda_i})_n^{(q,\alpha)} \;:\; i,n \in \mathbb{N}\}$ is fundamental in $\mathcal{L}^p(I)$.

7 | CONCLUSION

In the present paper, we have introduced a new approximation method using q-Bernstein polynomial as the base function in the structure of fractal interpolants. For a given function $f \in C(I)$, the convergence of the sequence of the quantum fractal functions towards f does not need any further condition on the scaling functions so that these approximants can be smooth or nonsmooth depending on the norm of the scaling functions. The shape of the proposed fractal approximants depends on the free variable $q \in (0,1)$ apart from the scaling functions. Hence, for the given continuous function f, the proposed quantum fractal approximants provide a large number of approximants than that would be obtained by the existing fractal approximants. It is observed that the multivalued quantum fractal operator $\mathcal{F}^{(q,\alpha)} \;:\; C(I) \rightrightarrows C(I)$ is not linear. The (q,α)-Kantorovich-Bernstein fractal functions in $\mathcal{L}^p$ spaces are developed and their approximation properties (quantum analogue of Müntz theorems) are studied.

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□