An efficient numerical method for 2D elliptic singularly perturbed systems with different magnitude parameters in the diffusion and the convection terms

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Abstract

In this work we are interested in constructing a uniformly convergent method to solve a 2D elliptic singularly perturbed weakly system of convection-diffusion type. We assume that small positive parameters appear at both the diffusion and the convection terms of the partial differential equation. Moreover, we suppose that both the diffusion and the convection parameters can be distinct and also they can have a different order of magnitude. Then, the nature of the overlapping regular or parabolic boundary layers, which, in general, appear in the exact solution, is much more complicated. To solve the continuous problem, we use the classical upwind finite difference scheme, which is defined on piecewise uniform Shishkin meshes, which are given in a different way depending on the value and the ratio between the four singular perturbation parameters which appear in the continuous problem. So, the numerical algorithm is an almost first order uniformly convergent method. The numerical results obtained with our algorithm for a test problem are presented; these results corroborate in practice the good behavior and the uniform convergence of the algorithm, aligning with the theoretical results.

Keywords: 2D problems, elliptic coupled systems, singularly perturbed problems, diffusion and convection parameters, upwind scheme, Shishkin meshes
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1. Introduction

1.1. Model Problem

In this work, we consider a singularly perturbed 2D elliptic weakly-coupled system of convection-diffusion type, which is given by

$$\begin{cases} \vec{\boldsymbol{\mathcal{L}}}_{\vec{\boldsymbol{\varepsilon}},\vec{\boldsymbol{\mu}}}\vec{\mathbf{z}}(x,y) \equiv \vec{\boldsymbol{\varepsilon}}\Delta\vec{\mathbf{z}}(x,y) + \vec{\boldsymbol{\mu}}\vec{\mathbf{A}}(x,y)\cdot\nabla\vec{\mathbf{z}}(x,y) - \vec{\mathbf{B}}(x,y)\vec{\mathbf{z}}(x,y) = \vec{\mathbf{f}}(x,y),\ \forall(x,y)\in\Omega := (0,1)^2, \\ \vec{\mathbf{z}}(x,y) = \vec{\mathbf{g}}_1(x,y), \quad \forall(x,y)\in\Gamma_1, \quad \vec{\mathbf{z}}(x,y) = \vec{\mathbf{g}}_2(x,y), \quad \forall(x,y)\in\Gamma_2, \\ \vec{\mathbf{z}}(x,y) = \vec{\mathbf{g}}_3(x,y), \quad \forall(x,y)\in\Gamma_3, \quad \vec{\mathbf{z}}(x,y) = \vec{\mathbf{g}}_4(x,y), \quad \forall(x,y)\in\Gamma_4, \end{cases} \tag{1.1}$$

where the problem (1.1) is defined on the domain $\overline{\Omega} := \Omega \cup \partial\Omega$, where $\partial\Omega$ denote the four sides of Ω defined by

$$\partial\Omega := \begin{cases} \Gamma_1 = \big\{(0,y)\,|\,(0\le y\le 1)\big\}, \quad \Gamma_2 = \big\{(x,0)\,|\,(0\le x\le 1)\big\}, \\ \Gamma_3 = \big\{(1,y)\,|\,(0\le y\le 1)\big\}, \quad \Gamma_4 = \big\{(x,1)\,|\,(0\le x\le 1)\big\}, \end{cases}$$

and $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$.

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The convection and the reaction coefficients of the problem (1.1), along both x- and y-directions, are given by the matrices $\vec{\mathbf{A}}(x,y)=(\vec{\mathbf{A}}_1(x,y),\vec{\mathbf{A}}_2(x,y))$ and $\vec{\mathbf{B}}(x,y)$, respectively. We assume that it holds

$\vec{\mathcal{L}}_{\vec{\varepsilon},\vec{\mu}}=(\mathcal{L}^1_{\varepsilon_1,\mu_1},\mathcal{L}^2_{\varepsilon_2,\mu_2})^T, \vec{\varepsilon}=(\varepsilon_1,\varepsilon_2)^T, \vec{\mu}=(\mu_1,\mu_2)^T, \vec{\mathbf{z}}(x,y)=(z_1(x,y),z_2(x,y))^T, \vec{\mathbf{f}}(x,y)=(f_1(x,y),f_2(x,y))^T,$
$\vec{\mathbf{g}}_i(x,y)=(g_{i1}(x,y),g_{i2}(x,y))^T, i=1,2,3,4,\ \vec{\mathbf{A}}_1(x,y)=\begin{pmatrix} a_1^1(x,y) & 0\\ 0 & a_1^2(x,y)\end{pmatrix}, \vec{\mathbf{A}}_2(x,y)=\begin{pmatrix} a_2^1(x,y) & 0\\ 0 & a_2^2(x,y)\end{pmatrix},$
$\vec{\mathbf{B}}(x,y)=\begin{pmatrix} b_{11}(x,y) & b_{12}(x,y)\\ b_{21}(x,y) & b_{22}(x,y)\end{pmatrix}.$

The small perturbation parameters hold $0<\varepsilon_1,\varepsilon_2,\mu_1,\mu_2\ll 1$ and, without loss of generality, we assume that $\varepsilon_1\le\varepsilon_2$ and $\mu_1\le\mu_2$. Finally, we suppose that the following assumptions on the components of the convection and the reaction matrices hold

$$\begin{cases} a_1^i(x,y)\ge\vartheta_1>0,\ a_2^i(x,y)\ge\vartheta_2>0,\ \ i=1,2,\\ b_{ii}(x,y)\ge\beta>0,\ i=1,2,\\ b_{ii}(x,y)>|b_{ij}(x,y)|,\quad b_{ij}(x,y)\le 0,\ \ i,j=1,2,\ i\ne j,\end{cases} \tag{1.2}$$

for some positive constants ϑ_1, ϑ_2 and β. From previous values, we define the constants

$$\vartheta=\min(\vartheta_1,\vartheta_2),\ \ \Lambda=\min_{i,j}\left\{\frac{b_{ii}-b_{ij}}{2a_1^i},\frac{b_{ii}-b_{ij}}{2a_2^i}\right\},\ \text{for } i,j=1,2,\ i\ne j. \tag{1.3}$$

We also assume that the components of $\vec{\mathbf{A}}_1$, $\vec{\mathbf{A}}_2$, $\vec{\mathbf{f}}$ and $\vec{\mathbf{B}}$ are sufficiently smooth functions on Ω, $\vec{\mathbf{g}}_i\in C^{3,\gamma}(\Gamma_i)$, $i=1,2,3,4$, for some $\gamma\in(0,1]$, and they satisfy sufficient compatibility conditions in order that the continuous problem has a solution $\vec{\mathbf{z}}$; moreover, this solution satisfies $\vec{\mathbf{z}}\in C^{3,\gamma}(\bar{\Omega})$ (in [1] and [2], Theorem 3.2, appear the compatibility conditions that guarantee this regularity).

The efficient numerical solution of singularly perturbed systems, for both elliptic and parabolic cases, has a increasing interest in the last years. The reason is that this type of problems is a good mathematical model for many physical phenomena in different areas, as such diffusion process in electro-analytic chemistry, turbulent interactions of waves and currents, bio-fluids mechanics, combustion process, saturated flow in fractured porous media, reaction-diffusion enzyme model or tubular model in chemical reactor theory (see [3, 4, 5, 6, 7]). For instance, consider the following model for saturated flow in fractured porous media from [3]:

$$\begin{cases} (\gamma_{c_1}+n_1\gamma)\frac{\partial\rho_1}{\partial t}-\frac{k_1}{\nu}\Delta\rho_1+\frac{\gamma}{\nu}(\rho_1-\rho_2)=f_1(x,t),\\ (\gamma_{c_2}+n_2\gamma)\frac{\partial\rho_2}{\partial t}-\frac{k_2}{\nu}\Delta\rho_2+\frac{\gamma}{\nu}(\rho_2-\rho_1)=f_2(x,t),\end{cases}$$

where ρ_1,ρ_2 are the pressure of liquid in the pores of the first and second-order respectively, γ is the coefficient of compressibility of the liquid and ν is the viscosity of the liquid. Here γ_{c_1} and γ_{c_2} are positive constants, whereas k_1,k_2 are the porosity of the system of pores of first and second-order respectively, and n_1,n_2 are the values of the first and second-order porosity at standard pressure.

It is well known that for singularly perturbed problems, when the diffusion and the convection parameters, which appear in the differential equation of the continuous problem, are sufficiently small, boundary layers and/or internal layers of different types appear on the boundary of the domain or at some interior points. To solve efficiently those problems for any value of the singular perturbation parameters, uniformly convergent methods are needed, i.e., numerical methods which calculate a good approximation independently of the value of the parameters.

A type of problems which have been deep studied in the last time, is this one where small parameters affect to both the convection and the diffusion terms of the differential equation, see, for instance, [8, 9, 10, 11, 12, 13, 14], where uniformly convergent methods were developed for different scalar elliptic or parabolic problems. Nevertheless, when coupled singularly perturbed systems are considered, the analysis and the construction of efficient numerical methods is a more difficult task. This difficulty increases notably when the small parameters in the differential equation has a different order of magnitude; then, overlapping boundary layers appear in the exact solution of the continuous problem.

In the literature there exists many works where uniformly convergent methods were constructed to solve elliptic and parabolic singularly perturbed coupled systems, see, for instance, [15, 16, 17, 18, 19, 20, 21, 22, 23], where 1D and 2D convection-diffusion or reaction-diffusion systems were considered. Nevertheless, the case of coupled system having small parameters affecting to both the diffusion and the convection terms, is a special case, which is less considered in the literature. For instance, in [24], a parabolic one dimensional weakly coupled system of convection-diffusion type was studied. In [25], a 2D elliptic singularly perturbed weakly-coupled system of convection-reaction-diffusion type, was analyzed. In [26] a 1D elliptic singularly perturbed weakly-coupled system, for which the diffusion parameters at each equation are different and the convection parameters are the same at both equations, was considered. Finally, in [27], a 2D elliptic system where the diffusion parameters are different and the convection parameter is the same in both equations, was studied. In all those works, uniformly convergent methods were construct on special nonuniform meshes of Shishkin type. Nevertheless, up to our knowledge, there is not work where a 2D singularly perturbed system of convection-diffusion type, which have different small parameters at both the diffusion and the convection terms, was analyzed. So, our main objective here is to construct a numerical algorithm to solve efficiently problem (1.1); to do that, we will make a detailed analysis of how the mesh, used in the construction of the algorithm, must be defined and also we will prove the uniform convergence of the method, which is the main property required for the numerical methods in the context of singularly perturbed problems.

Note that, due that ε_1 and ε_2 are distinct and they can have a very different order of magnitude, and the same occurs for μ_1 and μ_2, many different ratio between those four parameters appear; therefore, the structure of overlapping boundary layers, on both the inflow and the outflow boundary of the domain, is considerably much more complicated than in previous works. For the case $\vartheta{\mu_1}^2 \leq \vartheta{\mu_2}^2 \leq \Lambda\varepsilon_1 \leq \Lambda\varepsilon_2$, the components z_1 and z_2 of the solution exhibit boundary layers at both the outflow and inflow of the domain with width $O(\sqrt{\varepsilon_2})$ and the component z_1 exhibits additional boundary layers at both the inflow and outflow with width $O(\sqrt{\varepsilon_1})$. In the case of $\Lambda\varepsilon_1 \leq \Lambda\varepsilon_2 < \vartheta{\mu_1}^2 \leq \vartheta{\mu_2}^2$, inflow and outflow boundary layers of width $O(\frac{\varepsilon_2}{\mu_2})$ and $O(\mu_2)$, respectively are expected for both components. The component z_1 exhibits additional inflow and outflow boundary layers of the widths $O(\frac{\varepsilon_1}{\mu_1}), O(\frac{\varepsilon_1}{\mu_2}), O(\frac{\varepsilon_2}{\mu_1})$ and $O(\mu_1)$, respectively. Similarly, we can define the widths of the outflow and inflow boundary layers for another two cases.

The paper is structured as follows. In Section 2, we analyze which is the asymptotic behavior of the exact solution of the continuous problem and we also prove appropriated bounds for its partial derivatives, which show the behavior of the exact solution with respect to the four singular perturbation parameters $\varepsilon_1, \varepsilon_2, \mu_1$ and μ_2 and the ratio between them. In Section 3, we define the numerical finite difference scheme; the most important aspect, to dispose of an efficient and uniformly convergent method, is the construction of adequate piecewise uniform Shishkin meshes; their definition is related with the value of the singular perturbation parameters and also the ratio between them. In Section 4, we prove the uniform convergence of the numerical method, with respect to the four singular perturbation parameters; we obtain that the numerical algorithm is an almost first-order uniformly convergent method. In Section 5, we show the numerical results obtained for a particular test problem; these results corroborate in practice the theoretical results proved. Finally, in Section 6, some conclusions are given.

Henceforth, we denote by $\|\cdot\|$ the continuous maximum norm; moreover, for a function $\vec{\mathbf{\Psi}} = (\Psi_1, \Psi_2)^T$, $|\vec{\mathbf{\Psi}}| = (|\Psi_1|, |\Psi_2|)^T$, and C denotes a generic positive constant which is independent of the diffusion parameters $\varepsilon_1, \varepsilon_2$, the convection parameters μ_1, μ_2 and also of the discretization parameter N.

2. Asymptotic behavior of the exact solution of the continuous problem

In this section, we give some results which show the asymptotic behavior of the exact solution of the problem (1.1); also, we prove appropriated estimates for its partial derivatives with respect to the diffusion and the convection parameters.

To shorten the long of the paper, we only consider the following cases:

$$\begin{array}{ll} \textbf{Case 1:} \text{ it holds } \vartheta{\mu_1}^2 \leq \vartheta{\mu_2}^2 \leq \Lambda\varepsilon_1 \leq \Lambda\varepsilon_2, & \textbf{Case 2:} \text{ it holds } \Lambda\varepsilon_1 \leq \Lambda\varepsilon_2 < \vartheta{\mu_1}^2 \leq \vartheta{\mu_2}^2, \\ \textbf{Case 3:} \text{ it holds } \Lambda\varepsilon_1 < \vartheta{\mu_1}^2 \leq \vartheta{\mu_2}^2 < \Lambda\varepsilon_2, & \textbf{Case 4:} \text{ it holds } \vartheta{\mu_1}^2 < \Lambda\varepsilon_1 \leq \Lambda\varepsilon_2 < \vartheta{\mu_2}^2. \end{array} \tag{2.1}$$

There exist other two cases, when $\Lambda\varepsilon_1 < \vartheta\mu_1 \le \Lambda\varepsilon_2 < \vartheta{\mu_2}^2$ or $\vartheta{\mu_1}^2 < \Lambda\varepsilon_1 \le \vartheta{\mu_2}^2 < \Lambda\varepsilon_2$ hold; these two cases are not considered in this work and they will be analyzed in a future work.

Lemma 2.1 (Minimum principle)**.** *Let* $\vec{\mathbf{\Phi}} := (\Phi_1, \Phi_2) \in C^{3,\gamma}(\overline{\Omega})$ *and we assume that (1.2) holds. If* $\vec{\mathbf{\Phi}}(x,y) \ge \vec{\mathbf{0}}$ *on* $\partial\Omega$ *and* $\vec{\mathcal{L}}_{\vec{\varepsilon},\vec{\mu}}\vec{\mathbf{\Phi}}(x,y) \le \vec{\mathbf{0}}$ *for all* $(x,y) \in \Omega$, *then it holds* $\vec{\mathbf{\Phi}}(x,y) \ge \vec{\mathbf{0}}$ *for all* $(x,y) \in \overline{\Omega}$.

Proof. Let us consider that a point $(x^*, y^*) \in \Omega$ is such that

$$\min\{\Phi_1(x^*,y^*), \Phi_2(x^*,y^*)\} = \min\left\{\min_{(x,y)\in\Omega} \Phi_1(x,y), \min_{(x,y)\in\Omega} \Phi_2(x,y)\right\} < 0.$$

Without loss of generality we assume that $\Phi_1(x^*,y^*) \le \Phi_2(x^*,y^*)$. Then, it holds $\mathcal{L}^1_{\varepsilon_1,\mu_1}\Phi_1$ holds $\dfrac{\partial\Phi_1}{\partial x} = \dfrac{\partial\Phi_1}{\partial y} = 0$ and $\dfrac{\partial^2\Phi_1}{\partial x^2} > 0$, $\dfrac{\partial^2\Phi_1}{\partial y^2} > 0$. Then, we have

$$\begin{aligned}\mathcal{L}^1_{\varepsilon_1,\mu_1}\Phi_1(x^*,y^*) =& \varepsilon_1\left(\frac{\partial^2\Phi_1}{\partial x^2}(x^*,y^*) + \frac{\partial^2\Phi_1}{\partial y^2}(x^*,y^*)\right) + \mu_1\left(a_1^1(x^*,y^*)\frac{\partial\Phi_1}{\partial x}(x^*,y^*) + a_2^1(x^*,y^*)\frac{\partial\Phi_1}{\partial y}(x^*,y^*)\right)\\ &-b_{11}(x^*,y^*)\Phi_1(x^*,y^*) - b_{12}(x^*,y^*)\Phi_2(x^*,y^*) > 0,\end{aligned}$$

which contradicts the hypothesis of the lemma. Similarly, we can prove the contradiction for the second component Φ_2. This completes the proof of the result. □

Lemma 2.2 (Stability result)**.** *Let* $\vec{\mathbf{\Phi}} \in C^{3,\gamma}(\bar{\Omega})$*; then, it holds*

$$|\vec{\mathbf{\Phi}}(x,y)| \le \frac{1}{\vartheta}\|\vec{\mathcal{L}}_{\vec{\varepsilon},\vec{\mu}}\vec{\mathbf{\Phi}}\| + \max\{\|\vec{\mathbf{\Phi}}\|_{\Gamma_1}, \|\vec{\mathbf{\Phi}}\|_{\Gamma_2}, \|\vec{\mathbf{\Phi}}\|_{\Gamma_3}, \|\vec{\mathbf{\Phi}}\|_{\Gamma_4}\},$$

where ϑ *is the constant defined in (1.3).*

Proof. We define the barrier function

$$\vec{\mathbf{\Psi}}^{\pm}(x,y) = \frac{1}{\vartheta}\|\vec{\mathcal{L}}_{\vec{\varepsilon},\vec{\mu}}\vec{\mathbf{\Phi}}\| + \max\{\|\vec{\mathbf{\Phi}}\|_{\Gamma_1}, \|\vec{\mathbf{\Phi}}\|_{\Gamma_2}, \|\vec{\mathbf{\Phi}}\|_{\Gamma_3}, \|\vec{\mathbf{\Phi}}\|_{\Gamma_4}\} \pm \vec{\mathbf{\Phi}}(x,y).$$

Then, using Lemma 2.1, easily the result follows. □

Theorem 2.3. *Let* $\vec{\mathbf{z}} := (z_1, z_2)$ *be the exact solution of the continuous problem (1.1). Then, its derivatives satisfy the following bounds on* $\overline{\Omega}$*:*

$$\left|\frac{\partial^{(l_1+l_2)}z_i(x,y)}{\partial x^{l_1}\partial y^{l_2}}\right| \le C(\varepsilon_i)^{(-l_1-l_2)/2}\left\{1 + \left(\frac{\mu_i}{\sqrt{\varepsilon_i}}\right)^{(l_1+l_2)}\right\}\max\{z_i,\ f_i\},\ 1 \le l_1+l_2 \le 2,\ i=1,2, \tag{2.2a}$$

$$\begin{aligned}\left|\frac{\partial^{(l_1+l_2)}z_1(x,y)}{\partial x^{l_1}\partial y^{l_2}}\right| &\le C(\varepsilon_1)^{(-l_1-l_2)/2}\left\{1 + \left(\frac{\mu_1}{\sqrt{\varepsilon_1}}\right)^{(l_1+l_2)}\right\}\max\{z_1,\ f_1\}\\ &+ C\varepsilon_1^{2-l_1-l_2}\max\left\{z_1,\ f_1,\ \frac{\partial f_1}{\partial x},\ \frac{\partial f_1}{\partial y}\right\},\ 3 \le l_1+l_2 \le 4,\end{aligned} \tag{2.2b}$$

$$\begin{aligned}\left|\frac{\partial^{(l_1+l_2)}z_2(x,y)}{\partial x^{l_1}\partial y^{l_2}}\right| &\le C(\varepsilon_2)^{(-l_1-l_2)/2}\left\{1 + \left(\frac{\mu_2}{\sqrt{\varepsilon_2}}\right)^{(l_1+l_2)}\right\}\max\{z_2,\ f_2\}\\ &+ C\varepsilon_1^{1-(l_1+l_2)/2}\varepsilon_2^{-1}\max\left\{z_2,\ f_2,\ \frac{\partial f_2}{\partial x},\ \frac{\partial f_2}{\partial y}\right\},\ 3 \le l_1+l_2 \le 4.\end{aligned} \tag{2.2c}$$

Proof. To establish the aforementioned inequalities, we use the technique of extended domains, as it is detailed in [28]. We introduce the stretching variables for the first component as $\mathfrak{R}_1 = x/\nu_1$, $\mathfrak{T}_1 = y/\nu_1$, and for the second component as $\mathfrak{R}_2 = x/\nu_2$, $\mathfrak{T}_2 = y/\nu_2$, where $\nu_1 = \varepsilon_1(\mu_1 + \sqrt{\varepsilon_1})^{-1}$, $\nu_2 = \varepsilon_2(\mu_2 + \sqrt{\varepsilon_2})^{-1}$. The transformed domains for both components are denoted by $\Omega_1^* = (0, 1/\nu_1)^2$ and $\Omega_2^* = (0, 1/\nu_2)^2$," respectively.

On both transformed domains, the stretched functions are expressed as $z_1^*(\mathfrak{R}_1,\mathfrak{T}_1)=z_1(x,y)$, $z_2^*(\mathfrak{R}_2,\mathfrak{T}_2)=z_2(x,y)$, ${a_1^1}^*(\mathfrak{R}_1,\mathfrak{T}_1)=a_1^1(x,y)$, ${a_1^2}^*(\mathfrak{R}_2,\mathfrak{T}_2)=a_1^2(x,y)$, ${a_2^1}^*(\mathfrak{R}_1,\mathfrak{T}_1)=a_2^1(x,y)$, ${a_2^2}^*(\mathfrak{R}_2,\mathfrak{T}_2)=a_2^2(x,y)$, $b_{11}^*(\mathfrak{R}_1,\mathfrak{T}_1)=b_{11}(x,y)$, $b_{12}^*(\mathfrak{R}_1,\mathfrak{T}_1)=b_{12}(x,y)$, $b_{21}^*(\mathfrak{R}_2,\mathfrak{T}_2)=b_{21}(x,y)$, $b_{22}^*(\mathfrak{R}_2,\mathfrak{T}_2)=b_{22}(x,y)$ and $f_1^*(\mathfrak{R}_1,\mathfrak{T}_1)=f_1(x,y)$, $f_2^*(\mathfrak{R}_2,\mathfrak{T}_2)=f_2(x,y)$. We use the above transformation to solve the governing problem (1.1) in the new variables, which gives us the solutions $z_1^*(\mathfrak{R}_1,\mathfrak{T}_1)$, and $z_2^*(\mathfrak{R}_2,\mathfrak{T}_2)$. Further, we consider the rectangule $\mathfrak{G}_{\varsigma_1^*,\varrho_1^*}=(\varsigma_1^*-\varrho_1^*,\varsigma_1^*+\varrho_1^*)^2\cap\Omega_1^*$, $\mathfrak{G}_{\varsigma_2^*,\varrho_2^*}=(\varsigma_2^*-\varrho_2^*,\varsigma_2^*+\varrho_2^*)^2\cap\Omega_2^*$, and $\overline{\mathfrak{G}}_{\varsigma_1^*,\varrho_1^*}$, $\overline{\mathfrak{G}}_{\varsigma_2^*,\varrho_2^*}$ are the closure of $\mathfrak{G}_{\varsigma_1^*,\varrho_1^*}$, $\mathfrak{G}_{\varsigma_2^*,\varrho_2^*}$, where $\varsigma_1^*\in\Omega_1^*$, $\varsigma_2^*\in\Omega_2^*$ and $\varrho_1^*>0$, $\varrho_2^*>0$. Then, for all $(\varsigma_1^*,\varrho_1^*)$ and $(\varsigma_2^*,\varrho_2^*)$ the given differential equation gives the following estimates for $1\le l_1+l_2\le 4$, (see [11], Lemma 2.2).

$$\left|\frac{\partial^{(l_1+l_2)}z_1}{\partial\mathfrak{R}_1^{l_1}\partial\mathfrak{T}_1^{l_2}}\right|_{\mathfrak{G}_{\varsigma_1^*,\varrho_1^*}}\le C,\qquad \left|\frac{\partial^{(l_1+l_2)}z_2}{\partial\mathfrak{R}_2^{l_1}\partial\mathfrak{T}_2^{l_2}}\right|_{\mathfrak{G}_{\varsigma_2^*,\varrho_2^*}}\le C.$$

These estimates hold for any point $(\mathfrak{R}_1,\mathfrak{T}_1)\in\Omega_1^*$ and $(\mathfrak{R}_2,\mathfrak{T}_2)\in\Omega_2^*$. Then, as it is usual, if we come back to the original variables $(x,y)\in\overline{\Omega}$, the required estimates in (2.2) follow. □

Previous estimates do not reflect the presence of boundary layers in the exact solution of the continuous problem. To obtain better estimates, we decompose the exact solution of (1.1), in its regular component $\vec{\mathbf{v}}$, boundary layer components $\vec{\mathbf{w}}$ and corner layer components $\vec{\mathbf{s}}$. Moreover, those functions can be decomposed into $\vec{\mathbf{w}}_l,\vec{\mathbf{w}}_r,\vec{\mathbf{w}}_b,\vec{\mathbf{w}}_t$ and $\vec{\mathbf{s}}_{lb},\vec{\mathbf{s}}_{br},\vec{\mathbf{s}}_{rt},\vec{\mathbf{s}}_{lt}$, respectively. These components are the solutions of the following problems:

$$\begin{cases}\vec{\mathcal{L}}_{\vec{\varepsilon},\vec{\mu}}\vec{\mathbf{v}}(x,y)=\vec{\mathbf{f}}, & \forall(x,y)\in\Omega,\\ \vec{\mathbf{v}}(x,y)=\vec{\xi}_1(y), \quad \forall(x,y)\in\Gamma_1, & \vec{\mathbf{v}}(x,y)=\vec{\xi}_2(x), \quad \forall(x,y)\in\Gamma_2,\\ \vec{\mathbf{v}}(x,y)=\vec{\xi}_3(y), \quad \forall(x,y)\in\Gamma_3, & \vec{\mathbf{v}}(x,y)=\vec{\xi}_4(x), \quad \forall(x,y)\in\Gamma_4,\end{cases}\tag{2.3}$$

where $\vec{\xi}_i$, $i=1,2,3,4$ are specially chosen functions (see the analysis below), for the regular component,

$$\begin{cases}\vec{\mathcal{L}}_{\vec{\varepsilon},\vec{\mu}}\vec{\mathbf{w}}_k(x,y)=0,\ k=l,r,b,t,\quad \forall(x,y)\in\Omega,\\ \vec{\mathbf{w}}_l(x,y)=(\vec{\mathbf{z}}-\vec{\mathbf{v}})(x,y),\ \forall(x,y)\in\Gamma_1,\ \vec{\mathbf{w}}_l(x,y)=0,\ \forall(x,y)\in\Gamma_2\cup\Gamma_3\cup\Gamma_4,\\ \vec{\mathbf{w}}_r(x,y)=(\vec{\mathbf{z}}-\vec{\mathbf{v}})(x,y),\ \forall(x,y)\in\Gamma_3,\ \vec{\mathbf{w}}_r(x,y)=0,\ \forall(x,y)\in\Gamma_1\cup\Gamma_2\cup\Gamma_4,\\ \vec{\mathbf{w}}_t(x,y)=(\vec{\mathbf{z}}-\vec{\mathbf{v}})(x,y),\ \forall(x,y)\in\Gamma_4,\ \vec{\mathbf{w}}_t(x,y)=0,\ \forall(x,y)\in\Gamma_1\cup\Gamma_2\cup\Gamma_3,\end{cases}\tag{2.4}$$

for the boundary layer components, and

$$\begin{cases}\vec{\mathcal{L}}_{\vec{\varepsilon},\vec{\mu}}\vec{\mathbf{s}}_k(x,y)=0,\ k=lb,br,rt,lt,\quad \forall(x,y)\in\Omega,\\ \vec{\mathbf{s}}_{lb}(x,y)=-\vec{\mathbf{w}}_l(x,y),\ \forall(x,y)\in\Gamma_1,\ \vec{\mathbf{s}}_{lb}(x,y)=-\vec{\mathbf{w}}_b(x,y),\ \forall(x,y)\in\Gamma_2,\\ \vec{\mathbf{s}}_{br}(x,y)=-\vec{\mathbf{w}}_b(x,y),\ \forall(x,y)\in\Gamma_2,\ \vec{\mathbf{s}}_{br}(x,y)=-\vec{\mathbf{w}}_r(x,y),\ \forall(x,y)\in\Gamma_3,\\ \vec{\mathbf{s}}_{rt}(x,y)=-\vec{\mathbf{w}}_r(x,y),\ \forall(x,y)\in\Gamma_3,\ \vec{\mathbf{s}}_{rt}(x,y)=-\vec{\mathbf{w}}_t(x,y),\ \forall(x,y)\in\Gamma_4,\\ \vec{\mathbf{s}}_{lt}(x,y)=-\vec{\mathbf{w}}_l(x,y),\ \forall(x,y)\in\Gamma_1,\ \vec{\mathbf{s}}_{lt}(x,y)=-\vec{\mathbf{w}}_t(x,y),\ \forall(x,y)\in\Gamma_4,\\ \vec{\mathbf{s}}_{lb}(x,y)=0,\ \forall(x,y)\in\Gamma_3\cup\Gamma_4,\ \vec{\mathbf{s}}_{br}(x,y)=0,\ \forall(x,y)\in\Gamma_1\cup\Gamma_4,\\ \vec{\mathbf{s}}_{rt}(x,y)=0,\ \forall(x,y)\in\Gamma_1\cup\Gamma_2,\ \vec{\mathbf{s}}_{lt}(x,y)=0,\ \forall(x,y)\in\Gamma_2\cup\Gamma_3,\end{cases}\tag{2.5}$$

for the corner layer components, respectively.

Firstly, we study the behavior of the smooth component $\vec{\mathbf{v}}=(v_1,v_2)^T$, distinguishing various cases.

Case 1: If $\vartheta{\mu_1}^2\le\vartheta{\mu_2}^2\le\Lambda\varepsilon_1\le\Lambda\varepsilon_2$, we decompose the smooth component $\vec{\mathbf{v}}$ as

$$\vec{\mathbf{v}}=\vec{\mathbf{r}}_0+\sqrt{\varepsilon_2}\vec{\mathbf{r}}_1+(\sqrt{\varepsilon_2})^2\vec{\mathbf{r}}_2+(\sqrt{\varepsilon_2})^3\vec{\mathbf{r}}_3,\tag{2.6}$$

where $\vec{\mathbf{r}}_i=(r_{i1},r_{i2})^T$, $i=0,1,2,3$, and their respective equations on $\overline{\Omega}$ are given by

$$-\vec{\mathbf{B}}\vec{\mathbf{r}}_0=\vec{\mathbf{f}},\quad \vec{\mathbf{B}}\vec{\mathbf{r}}_1=\frac{\vec{\varepsilon}}{\sqrt{\varepsilon_2}}\Delta\vec{\mathbf{r}}_0+\frac{\vec{\mu}}{\sqrt{\varepsilon_2}}\vec{\mathbf{A}}\nabla\vec{\mathbf{r}}_0,\quad \vec{\mathbf{B}}\vec{\mathbf{r}}_2=\frac{\vec{\varepsilon}}{\sqrt{\varepsilon_2}}\Delta\vec{\mathbf{r}}_1+\frac{\vec{\mu}}{\sqrt{\varepsilon_2}}\vec{\mathbf{A}}\nabla\vec{\mathbf{r}}_1,\tag{2.7a}$$

$$\vec{\varepsilon}\Delta\vec{\mathbf{r}}_3+\vec{\mu}\vec{\mathbf{A}}\nabla\vec{\mathbf{r}}_3-\vec{\mathbf{B}}\vec{\mathbf{r}}_3=-\frac{\vec{\varepsilon}}{\sqrt{\varepsilon_2}}\Delta\vec{\mathbf{r}}_2-\frac{\vec{\mu}}{\sqrt{\varepsilon_2}}\vec{\mathbf{A}}\nabla\vec{\mathbf{r}}_2,\quad \vec{\mathbf{r}}_3(x,y)=\vec{\mathbf{0}},\ \forall(x,y)\in\Gamma.\tag{2.7b}$$

Note that $\vec{\mathbf{r}}_0, \vec{\mathbf{r}}_1$ and $\vec{\mathbf{r}}_2$ satisfy the zeroth order differential equations (2.7a) and therefore there are no issue of compatibility. The term $\vec{\mathbf{r}}_3$ is the solution of the problem (2.7b) on the domain Ω. Since $\vec{\mathbf{r}}_2 \in C^{3,\gamma}(\overline{\Omega})$, we get $\Delta\vec{\mathbf{r}}_2 \in C^{1,\gamma}(\overline{\Omega})$. Applying Lemma 2.2 and Lemma 2.3 to the problem (2.7b), hence $\vec{\mathbf{v}} \in C^{3,\gamma}(\bar{\Omega})$ and also it holds

$$\left\|\frac{\partial^{l_1+l_2}v_1}{\partial x^{l_1}\partial y^{l_2}}\right\| \leq C, \quad 0 \leq l_1+l_2 \leq 2, \; \left\|\frac{\partial^{l_1+l_2}v_1}{\partial x^{l_1}\partial y^{l_2}}\right\| \leq C\varepsilon_1^{-1/2}, \; l_1+l_2 = 3, \tag{2.8a}$$

$$\left\|\frac{\partial^{l_1+l_2}v_2}{\partial x^{l_1}\partial y^{l_2}}\right\| \leq C, \quad 0 \leq l_1+l_2 \leq 3. \tag{2.8b}$$

Case 2: If $\vartheta{\mu_2}^2 \geq \vartheta{\mu_1}^2 > \Lambda\varepsilon_2 \geq \Lambda\varepsilon_1$, we can decompose the smooth component $\vec{\mathbf{v}}$ as

$$\vec{\mathbf{r}} = \vec{\mathbf{r}}_0 + \varepsilon_2\vec{\mathbf{r}}_1 + {\varepsilon_2}^2\vec{\mathbf{r}}_2 + {\varepsilon_2}^3\vec{\mathbf{r}}_3, \tag{2.9}$$

where $\vec{\mathbf{r}}_i = (r_{i1}, r_{i2})^T$, $i = 0,1,2,3$, and their respective equations on $\overline{\Omega}$ are

$$\vec{\mathcal{L}}_{\vec{\mu}}\vec{\mathbf{r}}_0 = \vec{\mathbf{f}},\; \vec{\mathbf{r}}_0(x,y) = \vec{\mathbf{z}}(x,y),\; \forall(x,y) \in \Gamma_3 \cup \Gamma_4, \tag{2.10a}$$

$$\vec{\mathcal{L}}_{\vec{\mu}}\vec{\mathbf{r}}_1 = -\frac{\vec{\varepsilon}}{\varepsilon_2}\Delta\vec{\mathbf{r}}_0,\; \vec{\mathbf{r}}_1(x,y) = \vec{\mathbf{z}}(x,y),\; \forall(x,y) \in \Gamma_3 \cup \Gamma_4, \tag{2.10b}$$

$$\vec{\mathcal{L}}_{\vec{\mu}}\vec{\mathbf{r}}_2 = -\frac{\vec{\varepsilon}}{\varepsilon_2}\Delta\vec{\mathbf{r}}_1,\; \vec{\mathbf{r}}_2(x,y) = \vec{\mathbf{0}},\; \forall(x,y) \in \Gamma_3 \cup \Gamma_4, \tag{2.10c}$$

$$\vec{\mathcal{L}}_{\vec{\varepsilon},\vec{\mu}}\vec{\mathbf{r}}_3 = -\frac{\vec{\varepsilon}}{\varepsilon_2}\Delta\vec{\mathbf{r}}_2, \quad \vec{\mathbf{r}}_3(x,y) = \vec{\mathbf{0}},\; \forall(x,y) \in \Gamma, \tag{2.10d}$$

where

$$\vec{\mathcal{L}}_{\vec{\mu}}\vec{\psi} \equiv \vec{\boldsymbol{\mu}}\vec{\mathbf{A}}\nabla\vec{\psi} - \vec{\mathbf{B}}\vec{\psi}. \tag{2.11}$$

To obtain the bounds for the derivatives of such components, we need new decompositions. The first one is

$$\vec{\mathbf{r}}_0 = \vec{\mathbf{p}}_0 + \mu_2\vec{\mathbf{p}}_1 + {\mu_2}^2\vec{\mathbf{p}}_2 + {\mu_2}^3\vec{\mathbf{p}}_3,$$

where, $\vec{\mathbf{p}}_i = (p_{i1}, p_{i2})^T$, $i = 0,1,2,3$, and their associated equations on $\overline{\Omega}$ are given by

$$-\vec{\mathbf{B}}\vec{\mathbf{p}}_0 = \vec{\mathbf{f}}, \quad -\vec{\mathbf{B}}\vec{\mathbf{p}}_1 = -\vec{\mathbf{A}}\nabla\vec{\mathbf{p}}_0, \quad -\vec{\mathbf{B}}\vec{\mathbf{p}}_2 = -\vec{\mathbf{A}}\nabla\vec{\mathbf{p}}_1, \tag{2.12a}$$

$$\vec{\boldsymbol{\mu}}\vec{\mathbf{A}}\nabla\vec{\mathbf{p}}_3 - \vec{\mathbf{B}}\vec{\mathbf{p}}_3 = -\vec{\mathbf{A}}\nabla\vec{\mathbf{p}}_2, \quad \vec{\mathbf{p}}_3(x,y) = \vec{\mathbf{0}},\; \forall(x,y) \in \Gamma_3 \cup \Gamma_4. \tag{2.12b}$$

From (2.12a), it follows

$$\left\|\frac{\partial^{l_1+l_2}\vec{\mathbf{p}}_k}{\partial x^{l_1}\partial y^{l_2}}\right\| \leq C,\; k = 0,1,2,\; 0 \leq l_1+l_2 \leq 3.$$

To obtain the bounds for the derivatives of $\vec{\mathbf{p}}_3$, it is to be observed that using (1.2), the following Lemma can be proved using a standard argument.

Lemma 2.4. *Let $\vec{\mathcal{L}}_{\vec{\mu}}$ be the differential operator given in (2.11). If $\vec{\mathbf{\Phi}}(x,y) \geq \vec{\mathbf{0}}$ on Γ_i, $i = 3,4$, $\vec{\mathcal{L}}_{\vec{\mu}}\vec{\mathbf{\Phi}}(x,y) \leq \vec{\mathbf{0}}$ for all $(x,y) \in \Omega$, then it holds $\vec{\mathbf{\Phi}}(x,y) \geq \vec{\mathbf{0}}$ for all $(x,y) \in \overline{\Omega}$.*

Then, from (2.12b) and with the help of Lemma 2.4, it can be proved that $\|\vec{\mathbf{p}}_3\| \leq C$. Also, from (2.12b) we have

$$\left\|\frac{\partial^{l_1+l_2}\vec{\mathbf{p}}_3}{\partial x^{l_1}\partial y^{l_2}}\right\| \leq C\vec{\boldsymbol{\mu}}^{-l_1-l_2},\; 1 \leq l_1+l_2 \leq 3.$$

Hence, taking the decomposition

$$\vec{\mathbf{r}}_0(x,y) = \vec{\mathbf{p}}_0(x,y) + \mu_2\vec{\mathbf{p}}_1(x,y) + {\mu_2}^2\vec{\mathbf{p}}_2(x,y),\; \forall(x,y) \in \Gamma_3 \cup \Gamma_4, \tag{2.13}$$

we can deduce

$$\left\| \frac{\partial^{l_1+l_2}\vec{\mathbf{r}}_0}{\partial x^{l_1}\partial y^{l_2}} \right\| \leq C,\ 0 \leq l_1 + l_2 \leq 3.$$

The decomposition for the component $\vec{\mathbf{r}}_1$ continues as follows. Let be

$$\vec{\mathbf{r}}_1 = \vec{\boldsymbol{\varrho}}_0 + \mu_2 \vec{\boldsymbol{\varrho}}_1 + {\mu_2}^2 \vec{\boldsymbol{\varrho}}_2,$$

where $\vec{\boldsymbol{\varrho}}_k = (\varrho_{k1}, \varrho_{k2})^T$, $k = 0, 1, 2$. Their respective equations on $\overline{\Omega}$ are given by

$$-\vec{\mathbf{B}}\vec{\boldsymbol{\varrho}}_0 = -\frac{\vec{\boldsymbol{\varepsilon}}}{\varepsilon_2}\Delta \vec{\mathbf{r}}_0, \quad -\vec{\mathbf{B}}\vec{\boldsymbol{\varrho}}_1 = -\vec{\mathbf{A}}\nabla \vec{\boldsymbol{\varrho}}_0, \tag{2.14a}$$

$$\vec{\boldsymbol{\mu}}\vec{\mathbf{A}}\nabla \vec{\boldsymbol{\varrho}}_2 - \vec{\mathbf{B}}\vec{\boldsymbol{\varrho}}_2 = -\vec{\mathbf{A}}\nabla \vec{\boldsymbol{\varrho}}_1,\ \vec{\boldsymbol{\varrho}}_2(x,y) = \vec{\mathbf{0}},\ \forall (x,y) \in \Gamma_3 \cup \Gamma_4. \tag{2.14b}$$

From (2.14a), we can obtain

$$\left\| \frac{\partial^{l_1+l_2}\varrho_{0i}}{\partial x^{l_1}\partial y^{l_2}} \right\| \leq C(1 + {\mu_i}^{1-l_1-l_2}),\ \left\| \frac{\partial^{l_1+l_2}\varrho_{1i}}{\partial x^{l_1}\partial y^{l_2}} \right\| \leq C{\mu_i}^{-l_1-l_2},\ 0 \leq l_1 + l_2 \leq 3,\ i = 1, 2.$$

From (2.14b) and Lemma 2.4, we have $\|\vec{\boldsymbol{\varrho}}_2\| \leq C$ and, applying in (2.14b) the bounds

$$\left\| \frac{\partial^{l_1+l_2}\varrho_{2i}}{\partial x^{l_1}\partial y^{l_2}} \right\| \leq C{\mu_i}^{-1-l_1-l_2},\ 0 \leq l_1 + l_2 \leq 3,\ i = 1, 2.$$

Now, writing

$$\vec{\mathbf{r}}_1(x,y) = \vec{\boldsymbol{\varrho}}_0(x,y) + \mu_2 \vec{\boldsymbol{\varrho}}_1(x,y),\ \forall (x,y) \in \Gamma_3 \cup \Gamma_4, \tag{2.15}$$

and using the above estimations, it can be proved that it holds

$$\left\| \frac{\partial^{l_1+l_2}r_{1i}}{\partial x^{l_1}\partial y^{l_2}} \right\| \leq C(1 + {\mu_i}^{1-l_1-l_2}),\ 0 \leq l_1 + l_2 \leq 3,\ i = 1, 2.$$

From (2.10c), we deduce that

$$\left\| \frac{\partial^{l_1+l_2}r_{2i}}{\partial x^{l_1}\partial y^{l_2}} \right\| \leq C{\mu_i}^{-1-l_1-l_2}),\ 0 \leq l_1 + l_2 \leq 3,\ i = 1, 2.$$

Taking into account (2.10d), with help of the bounds given in Theorem 2.3, it holds

$$\left\| \frac{\partial^{l_1+l_2}r_{32}}{\partial x^{l_1}\partial y^{l_2}} \right\| \leq C(\varepsilon_2^{-l_1-l_2}{\mu_2}^{-(3-l_1-l_2)}),\ 0 \leq l_1 + l_2 \leq 3.$$

Also, using the equation for r_{31}, the first component of $\vec{\mathbf{r}}_3$, in (2.10d), to deduce estimates on v_{31} and its derivatives, we split it in the form

$$r_{31} = \varphi_0 + \varepsilon_1\varphi_1 + {\varepsilon_1}^2\varphi_2 + {\varepsilon_1}^3\varphi_3,$$

where the equations for φ_i, $i = 0, 1, 2, 3$ are given by

$$\mu_1(a_1^1, a_2^1)\nabla\varphi_0 - b_{11}\varphi_0 = -\frac{\varepsilon_1}{\varepsilon_2}\Delta r_{21} - b_{12}r_{32},\ \varphi_0(x,y) = 0,\ \forall (x,y) \in \Gamma_3 \cup \Gamma_4, \tag{2.16a}$$

$$\mu_1(a_1^1, a_2^1)\nabla\varphi_1 - b_{11}\varphi_1 = -\Delta\varphi_0,\ \varphi_1(x,y) = 0,\ \forall (x,y) \in \Gamma_3 \cup \Gamma_4, \tag{2.16b}$$

$$\mu_1(a_1^1, a_2^1)\nabla\varphi_2 - b_{11}\varphi_2 = -\Delta\varphi_1,\ \varphi_2(x,y) = 0,\ \forall (x,y) \in \Gamma_3 \cup \Gamma_4, \tag{2.16c}$$

$$\varepsilon_1\Delta\varphi_3 + \mu(a_1^1, a_2^1)\nabla\varphi_3 - b_{11}\varphi_3 = -\Delta\varphi_2,\ \varphi_3(x,y) = 0,\ \forall (x,y) \in \Gamma. \tag{2.16d}$$

From (2.16a)-(2.16c), it can be proved that

$$\left\|\frac{\partial^{l_1+l_2}\varphi_0}{\partial x^{l_1}\partial y^{l_2}}\right\| \le C{\mu_1}^{-(3-l_1-l_2)},\ 0\le l_1+l_2\le 1,\ \left\|\frac{\partial^{l_1+l_2}\varphi_0}{\partial x^{l_1}\partial y^{l_2}}\right\| \le C(\varepsilon_2^{1-l_1-l_2}{\mu_1}^{-(5-l_1-l_2)}),\ 2\le l_1+l_2\le 3,$$
$$\left\|\frac{\partial^{l_1+l_2}\varphi_1}{\partial x^{l_1}\partial y^{l_2}}\right\| \le C\varepsilon_2^{-1}{\mu_1}^{-(3-l_1-l_2)},\ 0\le l_1+l_2\le 1,\ \left\|\frac{\partial^{l_1+l_2}\varphi_1}{\partial x^{l_1}\partial y^{l_2}}\right\| \le C(\varepsilon_2^{-l_1-l_2}{\mu_1}^{-(5-l_1-l_2)}),\ 2\le l_1+l_2\le 3,$$
$$\left\|\frac{\partial^{l_1+l_2}\varphi_2}{\partial x^{l_1}\partial y^{l_2}}\right\| \le C\varepsilon_2^{-2}{\mu_1}^{-(3-l_1-l_2)},\ 0\le l_1+l_2\le 1,$$
$$\left\|\frac{\partial^{l_1+l_2}\varphi_2}{\partial x^{l_1}\partial y^{l_2}}\right\| \le C\varepsilon_2^{-1-l_1-l_2}{\mu_1}^{-3}+C\varepsilon_1^{1-l_1-l_2}\varepsilon_2^{-1}{\mu_1}^{-3},\ 2\le l_1+l_2\le 3.$$

Moreover, from (2.16d) and using the bounds in [11, 29], the following estimates can be obtained

$$\left\|\frac{\partial^{l_1+l_2}\varphi_3}{\partial x^{l_1}\partial y^{l_2}}\right\| \le C\varepsilon_1^{-l_1-l_2}\varepsilon_2^{-3}{\mu_1}^{-3+l_1+l_2}+C\varepsilon_1^{1-l_1-l_2}\varepsilon_2^{-1}{\mu_1}^{-3+l_1+l_2},\ 0\le l_1+l_2\le 3.$$

Now, writing

$$r_{31}(x,y)=\varphi_0+\varepsilon_1\varphi_1+\varepsilon_1^2\varphi_2,\ \forall(x,y)\in\Gamma_1\cup\Gamma_2, \tag{2.17}$$

we deduce that it holds

$$\|r_{31}\|\le C{\mu_1}^{-3},\ \left\|\frac{\partial^{l_1+l_2}v_{31}}{\partial x^{l_1}\partial y^{l_2}}\right\| \le C{\mu_1}^{-4},\ l_1+l_2=1,\ \left\|\frac{\partial^{l_1+l_2}r_{31}}{\partial x^{l_1}\partial y^{l_2}}\right\| \le C\varepsilon_2^{-1}{\mu_1}^{-3},\ l_1+l_2=2,$$
$$\left\|\frac{\partial^{l_1+l_2}r_{31}}{\partial x^{l_1}\partial y^{l_2}}\right\| \le C\varepsilon_1^{-1}+C\varepsilon_2^{-2}{\mu_1}^{-3},\ l_1+l_2=3.$$

Applying the bounds on r_{31} and its derivatives to the equation of r_{32}, the second component of $\vec{\mathbf{r}}_3$, in (2.10d), we have

$$\left\|\frac{\partial^{l_1+l_2}r_{32}}{\partial x^{l_1}\partial y^{l_2}}\right\| \le C\varepsilon_2^{-3}+C\varepsilon_2^{-1}{\mu_1}^{-4}+C\varepsilon_2^{-2}{\mu_1}^{-2},\ l_1+l_2=3.$$

Finally, writing

$$\vec{\mathbf{v}}=\vec{\mathbf{r}}_0+\varepsilon_2\vec{\mathbf{r}}_1+\varepsilon_2^2\vec{\mathbf{r}}_2+\varepsilon_2^3\vec{\mathbf{r}}_3,\ \forall(x,y)\in\Gamma_1\cup\Gamma_2, \tag{2.18a}$$
$$\vec{\mathbf{v}}=\vec{\mathbf{r}}_0+\varepsilon_2\vec{\mathbf{r}}_1,\ \forall(x,y)\in\Gamma_3\cup\Gamma_4, \tag{2.18b}$$

and using all the above bounds, the following estimates for the smooth component are obtained

$$\left\|\frac{\partial^{l_1+l_2}\vec{\mathbf{v}}}{\partial x^{l_1}\partial y^{l_2}}\right\| \le C,\ 0\le l_1+l_2\le 2,\ \left\|\frac{\partial^{l_1+l_2}v_1}{\partial x^{l_1}\partial y^{l_2}}\right\| \le C\varepsilon_1^{-1},\ l_1+l_2=3, \tag{2.19a}$$
$$\left\|\frac{\partial^{l_1+l_2}v_2}{\partial x^{l_1}\partial y^{l_2}}\right\| \le C,\ l_1+l_2=3. \tag{2.19b}$$

Case 3: If $\Lambda\varepsilon_1<\vartheta{\mu_1}^2\le\vartheta{\mu_2}^2<\Lambda\varepsilon_2$, we now decompose the components of $\vec{\mathbf{v}}$ as

$$v_1=r_{01}+\varepsilon_1r_{11}+\varepsilon_1^2r_{21}+\varepsilon_1^3r_{31}, \tag{2.20}$$
$$v_2=r_{02}+\sqrt{\varepsilon_2}r_{12}+(\sqrt{\varepsilon_2})^2r_{22}+(\sqrt{\varepsilon_2})^3r_{32}, \tag{2.21}$$

where $\vec{\mathbf{r}}_i=(r_{i1},r_{i2})^T$, $i=0,1,2,3$, and their corresponding defining equations on $\overline{\Omega}$ are

$$\mathcal{L}^1_{\mu_1}r_{01}=f_1,\ r_{01}(x,y)=z_1(x,y),\ \forall(x,y)\in\Gamma_3\cup\Gamma_4, \tag{2.22a}$$
$$\mathcal{L}^1_{\mu_1}r_{11}=-\Delta r_{01},\ r_{11}(x,y)=z_1(x,y),\ \forall(x,y)\in\Gamma_3\cup\Gamma_4, \tag{2.22b}$$
$$\mathcal{L}^1_{\mu_1}r_{21}=-\Delta r_{11},\ r_{21}(x,y)=0,\ \forall(x,y)\in\Gamma_3\cup\Gamma_4, \tag{2.22c}$$
$$\mathcal{L}^1_{\varepsilon_1,\mu_1}r_{31}=-\Delta r_{21},\quad r_{31}(x,y)=0,\ \forall(x,y)\in\Gamma, \tag{2.22d}$$

and

$$-b_{21}r_{01} - b_{22}r_{02} = f_2, \quad b_{21}r_{11} + b_{22}r_{12} = \Delta r_{02} + \frac{\mu_2}{\sqrt{\varepsilon_2}}(a_1^2, a_2^2)\nabla r_{02}, \tag{2.23a}$$

$$b_{21}r_{21} + b_{22}r_{22} = \Delta r_{12} + \frac{\mu_2}{\sqrt{\varepsilon_2}}(a_1^2, a_2^2)\nabla r_{12}, \tag{2.23b}$$

$$\mathcal{L}^2_{\varepsilon_2,\mu_2} r_{32} = -\Delta r_{22} - \frac{\mu_2}{\sqrt{\varepsilon_2}}(a_1^2, a_2^2)\nabla r_{22}, \quad r_{32}(x,y) = 0,\ \forall (x,y) \in \Gamma. \tag{2.23c}$$

We use a similar methodology that in the previous cases; then, applying Lemmas 2.2 and 2.3 to the problem (2.22d) and (2.23c), it holds

$$\left\| \frac{\partial^{l_1+l_2}\vec{\mathbf{v}}}{\partial x^{l_1}\partial y^{l_2}} \right\| \le C,\ 0 \le l_1 + l_2 \le 2,\ \left\| \frac{\partial^{l_1+l_2}v_1}{\partial x^{l_1}\partial y^{l_2}} \right\| \le C\varepsilon_1^{-1},\ l_1 + l_2 = 3, \tag{2.24a}$$

$$\left\| \frac{\partial^{l_1+l_2}v_2}{\partial x^{l_1}\partial y^{l_2}} \right\| \le C\varepsilon_2^{-1/2},\ l_1 + l_2 = 3. \tag{2.24b}$$

Case 4: If $\vartheta{\mu_1}^2 < \Lambda\varepsilon_1 \le \Lambda\varepsilon_2 < \vartheta{\mu_2}^2$, we now decompose the components of $\vec{\mathbf{v}}$ as

$$v_1 = r_{01} + \sqrt{\varepsilon_1}r_{11} + (\sqrt{\varepsilon_1})^2 r_{21} + (\sqrt{\varepsilon_1})^3 r_{31}, \tag{2.25}$$

$$v_2 = r_{02} + \varepsilon_2 r_{12} + \varepsilon_2^2 r_{22} + \varepsilon_2^3 r_{32}, \tag{2.26}$$

where $\vec{\mathbf{r}}_i = (r_{i1}, r_{i2})^T$, $i = 0,1,2,3$, and their corresponding defining equations on $\overline{\Omega}$ are given by

$$-b_{11}r_{01} - b_{12}r_{02} = f_1, \quad b_{11}r_{11} + b_{12}r_{12} = \Delta r_{01} + \frac{\mu_1}{\sqrt{\varepsilon_1}}(a_1^1, a_2^1)\nabla r_{01}, \tag{2.27a}$$

$$b_{11}r_{21} + b_{12}r_{22} = \Delta r_{11} + \frac{\mu_1}{\sqrt{\varepsilon_1}}(a_1^1, a_2^1)\nabla r_{11}, \tag{2.27b}$$

$$\mathcal{L}^1_{\varepsilon_1,\mu_1} r_{31} = -\Delta r_{21} - \frac{\mu_1}{\sqrt{\varepsilon_1}}(a_1^1, a_2^1)\nabla r_{21}, \quad r_{31}(x,y) = 0,\ \forall (x,y) \in \Gamma, \tag{2.27c}$$

and

$$\mathcal{L}^1_{\mu_2} r_{02} = f_2,\ r_{02}(x,y) = z_2(x,y),\ \forall (x,y) \in \Gamma_3 \cup \Gamma_4, \tag{2.28a}$$

$$\mathcal{L}^1_{\mu_2} r_{12} = -\Delta r_{02},\ r_{12}(x,y) = z_2(x,y),\ \forall (x,y) \in \Gamma_3 \cup \Gamma_4, \tag{2.28b}$$

$$\mathcal{L}^1_{\mu_2} r_{22} = -\Delta r_{12},\ r_{22}(x,y) = 0,\ \forall (x,y) \in \Gamma_3 \cup \Gamma_4, \tag{2.28c}$$

$$\mathcal{L}^2_{\varepsilon_2,\mu_2} r_{32} = -\Delta r_{22}, \quad r_{32}(x,y) = 0,\ \forall (x,y) \in \Gamma. \tag{2.28d}$$

Then, we can follows the same technique that we have used for the analysis of the previous cases, and using the results of Lemmas 2.2 and 2.3, applied now to the problem (2.27c) and (2.28d), we can obtain

$$\left\| \frac{\partial^{l_1+l_2}\vec{\mathbf{v}}}{\partial x^{l_1}\partial y^{l_2}} \right\| \le C,\ 0 \le l_1 + l_2 \le 2,\ \left\| \frac{\partial^{l_1+l_2}v_1}{\partial x^{l_1}\partial y^{l_2}} \right\| \le C\varepsilon_1^{-1/2},\ l_1 + l_2 = 3, \tag{2.29a}$$

$$\left\| \frac{\partial^{l_1+l_2}v_2}{\partial x^{l_1}\partial y^{l_2}} \right\| \le C\varepsilon_2^{-1},\ l_1 + l_2 = 3. \tag{2.29b}$$

Then, from previous bounds, the required result for the regular component follows.

In second place, to study the asymptotic behavior of the layer functions, we define the functions $\mathcal{B}_i^l(x)$, $\mathcal{B}_i^r(x)$

and $\mathcal{B}_i^l(y)$, $\mathcal{B}_i^r(y)$, $i=1,2$, which, on the domain Ω, are given by

$$\mathcal{B}_1^l(x)=\begin{cases} e^{-\theta_1 x}, \ \vartheta{\mu_1}^2\le\vartheta{\mu_2}^2\le\Lambda\varepsilon_1\le\Lambda\varepsilon_2,\\ e^{-\lambda_1 x}, \ \vartheta{\mu_1}^2\ge\vartheta{\mu_2}^2>\Lambda\varepsilon_2\ge\Lambda\varepsilon_1,\\ e^{-\lambda_1 x}, \ \Lambda\varepsilon_1<\vartheta{\mu_1}^2\le\vartheta{\mu_2}^2<\Lambda\varepsilon_2,\\ e^{-\theta_1 x}, \ \vartheta{\mu_1}^2<\Lambda\varepsilon_1\le\Lambda\varepsilon_2<\vartheta{\mu_2}^2,\end{cases} \qquad \mathcal{B}_1^r(x)=\begin{cases} e^{-\theta_1 (1-x)}, \ \vartheta{\mu_1}^2\le\vartheta{\mu_2}^2\le\Lambda\varepsilon_1\le\Lambda\varepsilon_2,\\ e^{-\kappa_1 (1-x)}, \ \vartheta{\mu_1}^2\ge\vartheta{\mu_2}^2>\Lambda\varepsilon_2\ge\Lambda\varepsilon_1,\\ e^{-\kappa_1 (1-x)}, \ \Lambda\varepsilon_1<\vartheta{\mu_1}^2\le\vartheta{\mu_2}^2<\Lambda\varepsilon_2,\\ e^{-\theta_1 (1-x)}, \ \vartheta{\mu_1}^2<\Lambda\varepsilon_1\le\Lambda\varepsilon_2<\vartheta{\mu_2}^2,\end{cases} \tag{2.30a}$$

$$\mathcal{B}_2^l(x)=\begin{cases} e^{-\theta_2 x}, \ \vartheta{\mu_1}^2\le\vartheta{\mu_2}^2\le\Lambda\varepsilon_1\le\Lambda\varepsilon_2,\\ e^{-\lambda_2 x}, \ \vartheta{\mu_1}^2\ge\vartheta{\mu_2}^2>\Lambda\varepsilon_2\ge\Lambda\varepsilon_1,\\ e^{-\theta_2 x}, \ \Lambda\varepsilon_1<\vartheta{\mu_1}^2\le\vartheta{\mu_2}^2<\Lambda\varepsilon_2,\\ e^{-\lambda_1 x}, \ \vartheta{\mu_1}^2<\Lambda\varepsilon_1\le\Lambda\varepsilon_2<\vartheta{\mu_2}^2,\end{cases} \qquad \mathcal{B}_2^r(x)=\begin{cases} e^{-\theta_2 (1-x)}, \ \vartheta{\mu_1}^2\le\vartheta{\mu_2}^2\le\Lambda\varepsilon_1\le\Lambda\varepsilon_2,\\ e^{-\kappa_2 (1-x)}, \ \vartheta{\mu_1}^2\ge\vartheta{\mu_2}^2>\Lambda\varepsilon_2\ge\Lambda\varepsilon_1,\\ e^{-\theta_2 (1-x)}, \ \Lambda\varepsilon_1<\vartheta{\mu_1}^2\le\vartheta{\mu_2}^2<\Lambda\varepsilon_2,\\ e^{-\kappa_2 (1-x)}, \ \vartheta{\mu_1}^2<\Lambda\varepsilon_1\le\Lambda\varepsilon_2<\vartheta{\mu_2}^2,\end{cases} \tag{2.30b}$$

where $\theta_i=\sqrt{\dfrac{\vartheta\Lambda}{\varepsilon_i}}$, $\lambda_i=\dfrac{\vartheta\mu_i}{\varepsilon_i}$, $\kappa_i=\dfrac{\Lambda}{2\mu_i}$, for $i=1,2$. Similarly, we can describe the functions corresponding to the y-direction $\mathcal{B}_i^b(y)$, $\mathcal{B}_i^t(y)$, $i=1,2$.

Theorem 2.5. *Let $\vec{\boldsymbol{w}}_k$, $k=l,r,b,t$, where $\vec{\boldsymbol{w}}_k=(w_{k_1},w_{k_2})^T$ satisfy the problem (2.4) for the the four cases defined in(2.1). Then, the following bounds hold for the singular components.*
For the **Case 1***, it holds*

$$\begin{aligned}
&|w_{l_1}(x,y)|\le C\mathcal{B}_2^l(x),\ |w_{l_2}(x,y)|\le C\mathcal{B}_2^l(x),\ \ |w_{b_1}(x,y)|\le C\mathcal{B}_2^b(y),\ |w_{b_2}(x,y)|\le C\mathcal{B}_2^b(y),\\
&\left|\frac{\partial^{i+j}w_{l_1}}{\partial x^i\partial y^j}\right|\le C(\varepsilon_1^{-i/2}\mathcal{B}_1^l(x)+\varepsilon_2^{-i/2}\mathcal{B}_2^l(x)),\ \left|\frac{\partial^{i+j}w_{b_1}}{\partial x^i\partial y^j}\right|\le C(\varepsilon_1^{-j/2}\mathcal{B}_1^b(y)+\varepsilon_2^{-j/2}\mathcal{B}_2^b(y)),\ i,j=1,2,3,\\
&\left|\frac{\partial^{i+j}w_{l_2}}{\partial x^i\partial y^j}\right|\le C\varepsilon_2^{-i/2}\mathcal{B}_2^l(x),\ i,j=1,2,\ \left|\frac{\partial^{i+j}w_{l_2}}{\partial x^i\partial y^j}\right|\le C\varepsilon_2^{-1}(\varepsilon_1^{-1/2}\mathcal{B}_1^l(x)+\varepsilon_2^{-1/2}\mathcal{B}_2^l(x)),\ i,j=3,\\
&\left|\frac{\partial^{i+j}w_{b_2}}{\partial x^i\partial y^j}\right|\le C\varepsilon_2^{-j/2}\mathcal{B}_2^b(y),\ i,j=1,2,\ \left|\frac{\partial^{i+j}w_{b_2}}{\partial x^i\partial y^j}\right|\le C\varepsilon_2^{-1}(\varepsilon_1^{-1/2}\mathcal{B}_1^b(y)+\varepsilon_2^{-1/2}\mathcal{B}_2^b(y)),\ i,j=3,\\
&|w_{r_1}(x,y)|\le C\mathcal{B}_2^r(x),\ |w_{r_2}(x,y)|\le C\mathcal{B}_2^r(x),\ \ |w_{t_1}(x,y)|\le C\mathcal{B}_2^t(y),\ |w_{t_2}(x,y)|\le C\mathcal{B}_2^l(y),\\
&\left|\frac{\partial^{i+j}w_{r_1}}{\partial x^i\partial y^j}\right|\le C(\varepsilon_1^{-i/2}\mathcal{B}_1^r(x)+\varepsilon_2^{-i/2}\mathcal{B}_2^r(x)),\ \left|\frac{\partial^{i+j}w_{t_1}}{\partial x^i\partial y^j}\right|\le C(\varepsilon_1^{-j/2}\mathcal{B}_1^t(y)+\varepsilon_2^{-j/2}\mathcal{B}_2^t(y)),\ i,j=1,2,3,\\
&\left|\frac{\partial^{i+j}w_{r_2}}{\partial x^i\partial y^j}\right|\le C\varepsilon_2^{-i/2}\mathcal{B}_2^r(x),\ i,j=1,2,\ \left|\frac{\partial^{i+j}w_{r_2}}{\partial x^i\partial y^j}\right|\le C\varepsilon_2^{-1}(\varepsilon_1^{-1/2}\mathcal{B}_1^r(x)+\varepsilon_2^{-1/2}\mathcal{B}_2^r(x)),\ i,j=3,\\
&\left|\frac{\partial^{i+j}w_{t_2}}{\partial x^i\partial y^j}\right|\le C\varepsilon_2^{-j/2}\mathcal{B}_2^t(y),\ i,j=1,2,\ \left|\frac{\partial^{i+j}w_{t_2}}{\partial x^i\partial y^j}\right|\le C\varepsilon_2^{-1}(\varepsilon_1^{-1/2}\mathcal{B}_1^t(y)+\varepsilon_2^{-1/2}\mathcal{B}_2^t(y)),\ i,j=3.
\end{aligned}$$

For the **Case 2***, we have*

$$\begin{aligned}
&|w_{l_1}(x,y)|\le C\mathcal{B}_2^l(x),\ |w_{l_2}(x,y)|\le C\mathcal{B}_2^l(x),\ \ |w_{b_1}(x,y)|\le C\mathcal{B}_2^b(y),\ |w_{b_2}(x,y)|\le C\mathcal{B}_2^b(y),\\
&\left|\frac{\partial^{i+j}w_{l_1}}{\partial x^i\partial y^j}\right|\le C({\mu_1}^i\varepsilon_1^{-i}\mathcal{B}_1^l(x)+{\mu_2}^i\varepsilon_2^{-i}\mathcal{B}_2^l(x)),\ \left|\frac{\partial^{i+j}w_{b_1}}{\partial x^i\partial y^j}\right|\le C({\mu_1}^j\varepsilon_1^{-j}\mathcal{B}_1^b(y)+{\mu_2}^j\varepsilon_2^{-j}\mathcal{B}_2^b(y)),\ i,j=1,2,3,\\
&\left|\frac{\partial^{i+j}w_{l_2}}{\partial x^i\partial y^j}\right|\le C{\mu_2}^i\varepsilon_2^{-i}\mathcal{B}_2^l(x),\ i,j=1,2,\ \left|\frac{\partial^{i+j}w_{l_2}}{\partial x^i\partial y^j}\right|\le C\left(\frac{{\mu_1}^3}{\varepsilon_1\varepsilon_2^2}\mathcal{B}_1^l(x)+\frac{{\mu_2}^3}{\varepsilon_2^3}\mathcal{B}_2^l(x)\right),\ i,j=3,\\
&\left|\frac{\partial^{i+j}w_{b_2}}{\partial x^i\partial y^j}\right|\le C{\mu_2}^j\varepsilon_2^{-j}\mathcal{B}_2^b(y),\ i,j=1,2,\ \left|\frac{\partial^{i+j}w_{b_2}}{\partial x^i\partial y^j}\right|\le C\left(\frac{{\mu_1}^3}{\varepsilon_1\varepsilon_2^2}\mathcal{B}_1^b(y)+\frac{{\mu_2}^3}{\varepsilon_2^3}\mathcal{B}_2^b(y)\right),\ i,j=3,\\
&|w_{r_1}(x,y)|\le C,\ |w_{r_2}(x,y)|\le C,\ \ |w_{t_1}(x,y)|\le C,\ |w_{t_2}(x,y)|\le C,
\end{aligned}$$

$$\left|\frac{\partial^{i+j} w_{r_1}}{\partial x^i \partial y^j}\right| \le C{\mu_1}^{-i},\ i,j=1,2,\ \left|\frac{\partial^{i+j} w_{r_1}}{\partial x^i \partial y^j}\right| \le C{\mu_1}^{-3} + C\varepsilon_1^{-1},\ i,j=3,\ \left|\frac{\partial^{i+j} w_{r_2}}{\partial x^i \partial y^j}\right| \le C{\mu_2}^{-i},\ i,j=1,2,3,$$
$$\left|\frac{\partial^{i+j} w_{t_1}}{\partial x^i \partial y^j}\right| \le C{\mu_1}^{-j},\ j=1,2,\ \left|\frac{\partial^{i+j} w_{t_1}}{\partial x^i \partial y^j}\right| \le C{\mu_1}^{-3} + C\varepsilon_1^{-1},\ i,j=3,\ \left|\frac{\partial^{i+j} w_{t_2}}{\partial x^i \partial y^j}\right| \le C{\mu_2}^{-j},\ i,j=1,2,3.$$

For the **Case 3**, *it holds*

$$|w_{l_1}(x,y)| \le C\mathcal{B}_1^l(x),\ |w_{l_2}(x,y)| \le C\mathcal{B}_2^l(x),\ |w_{b_1}(x,y)| \le C\mathcal{B}_1^b(y),\ |w_{b_2}(x,y)| \le C\mathcal{B}_2^b(y),$$
$$\left|\frac{\partial^{i+j} w_{l_1}}{\partial x^i \partial y^j}\right| \le C({\mu_1}^i\varepsilon_1^{-i}\mathcal{B}_1^l(x) + \varepsilon_2^{-i/2}\mathcal{B}_2^l(x)),\ \left|\frac{\partial^{i+j} w_{b_1}}{\partial x^i \partial y^j}\right| \le C({\mu_1}^j\varepsilon_1^{-j}\mathcal{B}_1^b(y) + \varepsilon_2^{-j/2}\mathcal{B}_2^b(y)),\ i,j=1,2,3,$$
$$\left|\frac{\partial^{i+j} w_{l_2}}{\partial x^i \partial y^j}\right| \le C\varepsilon_2^{-i/2}\mathcal{B}_2^l(x),\ i,j=1,2,\ \left|\frac{\partial^{i+j} w_{l_2}}{\partial x^i \partial y^j}\right| \le C\varepsilon_2^{-1}(\mu_1\varepsilon_1^{-1}\mathcal{B}_1^l(x) + \varepsilon_2^{-1/2}\mathcal{B}_2^l(x)),\ i,j=3,$$
$$\left|\frac{\partial^{i+j} w_{b_2}}{\partial x^i \partial y^j}\right| \le C\varepsilon_2^{-j/2}\mathcal{B}_2^b(y),\ i,j=1,2,\ \left|\frac{\partial^{i+j} w_{b_2}}{\partial x^i \partial y^j}\right| \le C\varepsilon_2^{-1}(\mu_1\varepsilon_1^{-1}\mathcal{B}_1^b(y) + \varepsilon_2^{-1/2}\mathcal{B}_2^b(y)),\ i,j=3,$$
$$|w_{r_1}(x,y)| \le C\mathcal{B}_1^r(x),\ |w_{r_2}(x,y)| \le C\mathcal{B}_2^r(x),\ |w_{t_1}(x,y)| \le C\mathcal{B}_1^t(y),\ |w_{t_2}(x,y)| \le C\mathcal{B}_2^l(y),$$
$$\left|\frac{\partial^{i+j} w_{r_1}}{\partial x^i \partial y^j}\right| \le C({\mu_2}^{-i} + \varepsilon_2^{-i/2}\mathcal{B}_2^r(x)),\ \left|\frac{\partial^{i+j} w_{t_1}}{\partial x^i \partial y^j}\right| \le C({\mu_2}^{-j} + \varepsilon_2^{-j/2}\mathcal{B}_2^t(y)),\ i,j=1,2,3,$$
$$\left|\frac{\partial^{i+j} w_{r_2}}{\partial x^i \partial y^j}\right| \le C\varepsilon_2^{-i/2}\mathcal{B}_2^r(x),\ i,j=1,2,\ \left|\frac{\partial^{i+j} w_{r_2}}{\partial x^i \partial y^j}\right| \le C\varepsilon_2^{-1}({\mu_2}^{-3} + \varepsilon_2^{-1/2}\mathcal{B}_2^r(x)),\ i,j=3,$$
$$\left|\frac{\partial^{i+j} w_{t_2}}{\partial x^i \partial y^j}\right| \le C\varepsilon_2^{-j/2}\mathcal{B}_2^t(y),\ i,j=1,2,\ \left|\frac{\partial^{i+j} w_{t_2}}{\partial x^i \partial y^j}\right| \le C\varepsilon_2^{-1}({\mu_2}^{-3} + \varepsilon_2^{-1/2}\mathcal{B}_2^t(y)),\ i,j=3.$$

Finally, for the case **Case 4**, *we have*

$$|w_{l_1}(x,y)| \le C\mathcal{B}_1^l(x),\ |w_{l_2}(x,y)| \le C\mathcal{B}_2^l(x),\ |w_{b_1}(x,y)| \le C\mathcal{B}_1^b(y),\ |w_{b_2}(x,y)| \le C\mathcal{B}_2^b(y),$$
$$\left|\frac{\partial^{i+j} w_{l_1}}{\partial x^i \partial y^j}\right| \le C\varepsilon_1^{-i/2}\mathcal{B}_1^l(x),\ \left|\frac{\partial^{i+j} w_{b_1}}{\partial x^i \partial y^j}\right| \le C\varepsilon_1^{-j/2}\mathcal{B}_1^b(y),\ i,j=1,2,$$
$$\left|\frac{\partial^{i+j} w_{l_2}}{\partial x^i \partial y^j}\right| \le C(\varepsilon_1^{-i/2}\mathcal{B}_1^l(x) + {\mu_2}^i\varepsilon_2^{-i}\mathcal{B}_2^l(x)),\ i,j=1,2,3,\ \left|\frac{\partial^{i+j} w_{l_1}}{\partial x^i \partial y^j}\right| \le C\varepsilon_1^{-1}(\varepsilon_1^{-1/2}\mathcal{B}_1^l(x) + \mu_2\varepsilon_2^{-1}\mathcal{B}_2^l(x)),\ i,j=3,$$
$$\left|\frac{\partial^{i+j} w_{b_2}}{\partial x^i \partial y^j}\right| \le C(\varepsilon_1^{-j/2}\mathcal{B}_1^b(y) + {\mu_2}^j\varepsilon_2^{-j}\mathcal{B}_2^b(y)),\ i,j=1,2,3,\ \left|\frac{\partial^{i+j} w_{b_1}}{\partial x^i \partial y^j}\right| \le C\varepsilon_1^{-1}(\varepsilon_1^{-1/2}\mathcal{B}_1^b(y) + \mu_2\varepsilon_2^{-1}\mathcal{B}_2^b(y)),\ i,j=3,$$
$$|w_{r_1}(x,y)| \le C\mathcal{B}_1^r(x),\ |w_{r_2}(x,y)| \le C\mathcal{B}_2^r(x),\ |w_{t_1}(x,y)| \le C\mathcal{B}_1^t(y),\ |w_{t_2}(x,y)| \le C\mathcal{B}_2^l(y),$$
$$\left|\frac{\partial^{i+j} w_{r_1}}{\partial x^i \partial y^j}\right| \le C\varepsilon_1^{-i/2}\mathcal{B}_1^r(x),\ \left|\frac{\partial^{i+j} w_{t_1}}{\partial x^i \partial y^j}\right| \le C\varepsilon_1^{-j/2}\mathcal{B}_1^t(y),\ i,j=1,2,$$
$$\left|\frac{\partial^{i+j} w_{r_2}}{\partial x^i \partial y^j}\right| \le C({\mu_1}^{-i/2} + \varepsilon_2^{-i}\mathcal{B}_2^r(x)),\ i,j=1,2,\ \left|\frac{\partial^{i+j} w_{r_1}}{\partial x^i \partial y^j}\right| \le C\varepsilon_2^{-1}({\mu_1}^{-3/2} + \varepsilon_2^{-1}\mathcal{B}_2^r(x)),\ i,j=3,$$
$$\left|\frac{\partial^{i+j} w_{t_2}}{\partial x^i \partial y^j}\right| \le C({\mu_1}^{-j/2} + \varepsilon_2^{-j}\mathcal{B}_2^t(y)),\ i,j=1,2,\ \left|\frac{\partial^{i+j} w_{t_1}}{\partial x^i \partial y^j}\right| \le C\varepsilon_2^{-1}({\mu_1}^{-3/2} + \varepsilon_2^{-1}\mathcal{B}_2^t(y)),\ i,j=3.$$

Proof. To prove the estimates for the derivatives of the layer components, we again use the idea of extended domains (see [11, 29, 30] for more details). Here, we discuss only the bounds for the left layer component for the four cases defined in (2.1). To do that,we consider the change of variable for both components given by $\varPsi_1 = x/\nu_1$, $\varPsi_2 = x/\nu_2$, where $\nu_1 = \varepsilon_1(\mu_1 + \sqrt{\varepsilon_1})^{-1}$, $\nu_2 = \varepsilon_2(\mu_2 + \sqrt{\varepsilon_2})^{-1}$. Then, the resulting problems after this transform of variables, are given by

$$\begin{cases} \left(\varepsilon_1\nu_1^{-2}\dfrac{\partial^2 w_{l_1}^*}{\partial \Psi_1^2}+\varepsilon_1\dfrac{\partial^2 w_{l_1}^*}{\partial y^2}\right)+\mu_1\nu_1^{-1}a_1^{1^*}(\Psi_1,y)\dfrac{\partial w_{l_1}^*}{\partial \Psi_1}+\mu_1 a_2^{1^*}(\Psi_1,y)\dfrac{\partial w_{l_1}^*}{\partial y}- \quad b_{11}^*(\Psi_1,y)w_{l_1}^*-b_{12}^*(\Psi_1,y)w_{l_2}^* \\ \qquad\qquad = f_1^*(\Psi_1,y), \quad \forall(\Psi_1,y)\in\Omega_{\nu_1}^{*,1}, \\ w_{l_1}^*(\Psi_1,y)=(z_1^*-v_1^*)(\Psi_1,y), \quad \forall(\Psi_1,y)\in\Gamma_{1,\nu_1}^{*,1}, \\ w_{l_1}^*(\Psi_1,y)=0, \quad \forall(\Psi_1,y)\in\Gamma_{2,\nu_1}^{*,1}\cup\Gamma_{3,\nu_1}^{*,1}\cup\Gamma_{4,\nu_1}^{*,1}, \end{cases} \tag{2.31a}$$

$$\begin{cases} \left(\varepsilon_2\nu_2^{-2}\dfrac{\partial^2 w_{l_2}^*}{\partial \Psi_2^2}+\varepsilon_2\dfrac{\partial^2 w_{l_2}^*}{\partial y^2}\right)+\mu_2\nu_2^{-1}a_1^{2^*}(\Psi_2,y)\dfrac{\partial w_{l_2}^*}{\partial \Psi_2}+\mu_2 a_2^{2^*}(\Psi_2,y)\dfrac{\partial w_{l_2}^*}{\partial y}- \quad b_{21}^*(\Psi_2,y)w_{l_1}^*-b_{22}^*(\Psi_2,y)w_{l_2}^* \\ \qquad\qquad = f_2^*(\Psi_2,y), \quad \forall(\Psi_2,y)\in\Omega_{\nu_2}^{*,1}, \\ w_{l_2}^*(\Psi_2,y)=(z_2^*-v_2^*)(\Psi_2,y), \quad \forall(\Psi_2,y)\in\Gamma_{1,\nu_2}^{*,1}, \\ w_{l_2}^*(\Psi_2,y)=0, \quad \forall(\Psi_2,y)\in\Gamma_{2,\nu_2}^{*,1}\cup\Gamma_{3,\nu_2}^{*,1}\cup\Gamma_{4,\nu_2}^{*,1}, \end{cases} \tag{2.31b}$$

where, $\Omega_{\nu_1}^{*,1}=(0,1/\nu_1)\times(0,1)$, $\Omega_{\nu_2}^{*,1}=(0,1/\nu_2)\times(0,1)$ and $\Gamma_{i,\nu_1}^{*,1}$, $\Gamma_{i,\nu_2}^{*,1}$, $i=1,2,3,4$ are the corresponding boundaries of the extended domain $\Omega_{\nu_1}^{*,1}$ and $\Omega_{\nu_2}^{*,1}$, respectively. Then, using the similar methodology given in [28, 31], we can get the bounds for the left layer component $\vec{\mathbf{w}}_l{}^*$.

Analogously, we can consider the relative boundary layer components $\vec{\mathbf{w}}_k$, $k=r,b,t$, for the four cases defined in (2.1). □

Theorem 2.6. *Let $\vec{\mathbf{s}}_{k_1k_2}$, $k_1,k_2=l,b,r,t$, where $\vec{\mathbf{s}}_{k_1k_2}=(s_{(k_1k_2)_1},s_{(k_1k_2)_2})^T$ satisfy problem (2.5) for the four cases defined in (2.1). Then, the following bounds are satisfied by the corner components.*
For the **Case 1**, *it holds*

$$\begin{aligned}
&|\vec{\mathbf{s}}_{k_1k_2}(x,y)|\le C\mathcal{B}_2^{k_1}(x)\mathcal{B}_2^{k_2}(y),\ k_1=l,r,\ k_2=b,t,\\
&\left|\frac{\partial^{i+j}s_{(k_1k_2)_1}}{\partial x^i\partial y^j}\right|\le C(\varepsilon_1^{(-i-j)/2}\mathcal{B}_1^{k_1}(x)\mathcal{B}_1^{k_2}(y)+\varepsilon_2^{(-i-j)/2}\mathcal{B}_2^{k_1}(x)\mathcal{B}_2^{k_2}(y)),\ 1\le i+j\le 3,\ k_1=l,r,\ k_2=b,t,\\
&\left|\frac{\partial^{i+j}s_{(k_1k_2)_2}}{\partial x^i\partial y^j}\right|\le C\varepsilon_2^{(-i-j)/2}\mathcal{B}_2^{k_1}(x)\mathcal{B}_2^{k_2}(y),\ 1\le i+j\le 2,\ k_1=l,r,\ k_2=b,t,\\
&\left|\frac{\partial^{i+j}s_{(k_1k_2)_2}}{\partial x^i\partial y^j}\right|\le C\varepsilon_2^{-2}(\varepsilon_1^{-1}\mathcal{B}_1^{k_1}(x)\mathcal{B}_1^{k_2}(y)+\varepsilon_2^{-1}\mathcal{B}_2^{k_1}(x)\mathcal{B}_2^{k_2}(y)),\ i+j=3,\ k_1=l,r,\ k_2=b,t.
\end{aligned}$$

For the **Case 2**, *we have*

$$\begin{aligned}
&|\vec{\mathbf{s}}_{k_1k_2}(x,y)|\le C\mathcal{B}_2^{k_1}(x)\mathcal{B}_2^{k_2}(y),\ k_1=l,r,\ k_2=b,t,\\
&\left|\frac{\partial^{i+j}s_{(lb)_1}}{\partial x^i\partial y^j}\right|\le C({\mu_1}^{i+j}\varepsilon_1^{-i-j}\mathcal{B}_1^{l}(x)\mathcal{B}_1^{b}(y)+{\mu_2}^{i+j}\varepsilon_2^{-i-j}\mathcal{B}_2^{l}(x)\mathcal{B}_2^{b}(y)),\ 1\le i+j\le 3,\\
&\left|\frac{\partial^{i+j}s_{(lb)_2}}{\partial x^i\partial y^j}\right|\le C{\mu_2}^{i+j}\varepsilon_2^{-i-j}\mathcal{B}_2^{l}(x)\mathcal{B}_2^{b}(y),\ 1\le i+j\le 2,\\
&\left|\frac{\partial^{i+j}s_{(lb)_2}}{\partial x^i\partial y^j}\right|\le C{\mu_2}^{6}\varepsilon_2^{-4}(\varepsilon_1^{-2}\mathcal{B}_1^{l}(x)\mathcal{B}_1^{b}(y)+\varepsilon_2^{-2}\mathcal{B}_2^{l}(x)\mathcal{B}_2^{b}(y)),\ i+j=3,\\
&\left|\frac{\partial^{i+j}s_{(br)_1}}{\partial x^i\partial y^j}\right|\le C({\mu_1}^{-i+j}\varepsilon_1^{-j}\mathcal{B}_1^{b}(y)+{\mu_2}^{-i+j}\varepsilon_2^{-j}\mathcal{B}_2^{b}(y)),\ 1\le i+j\le 3,\\
&\left|\frac{\partial^{i+j}s_{(br)_2}}{\partial x^i\partial y^j}\right|\le C{\mu_2}^{-i+j}\varepsilon_2^{-j}\mathcal{B}_2^{b}(y),\ 1\le i+j\le 3,\ \left|\frac{\partial^{i+j}s_{(rt)_1}}{\partial x^i\partial y^j}\right|\le C{\mu_1}^{-i-j},\ 1\le i+j\le 2,
\end{aligned}$$

$$\left|\frac{\partial^{i+j}s_{(rt)_1}}{\partial x^i\partial y^j}\right| \le C({\mu_1}^{-6}+\varepsilon_1^{-2}),\ i+j=3,\ \left|\frac{\partial^{i+j}s_{rt_2}}{\partial x^i\partial y^j}\right| \le C{\mu_2}^{-i-j},\ 1\le i+j\le 3,$$
$$\left|\frac{\partial^{i+j}s_{(lt)_1}}{\partial x^i\partial y^j}\right| \le C(({\mu_1}^{i-j}\varepsilon_1^{-i}\mathcal{B}_1^l(x)+{\mu_2}^{i-j}\varepsilon_2^{-i}\mathcal{B}_2^l(x))),\ 1\le i+j\le 3,$$
$$\left|\frac{\partial^{i+j}s_{(lt)_2}}{\partial x^i\partial y^j}\right| \le C{\mu_2}^{i-j}\varepsilon_2^{-i}\mathcal{B}_2^l(x),\ 1\le i+j\le 3.$$

For the **Case 3**, *it holds*

$$|\vec{s}_{(lb)_1}(x,y)| \le C\mathcal{B}_1^l(x)\mathcal{B}_1^b(y),\ |\vec{s}_{(lb)_2}(x,y)| \le C\mathcal{B}_2^l(x)\mathcal{B}_2^b(y),\ |\vec{s}_{(br)_1}(x,y)| \le C\mathcal{B}_1^r(x)\mathcal{B}_1^b(y),$$
$$|\vec{s}_{(br)_2}(x,y)| \le C\mathcal{B}_2^r(x)\mathcal{B}_2^b(y),\ |\vec{s}_{(rt)_1}(x,y)| \le C\mathcal{B}_1^r(x)\mathcal{B}_1^t(y),\ |\vec{s}_{(rt)_2}(x,y)| \le C\mathcal{B}_2^r(x)\mathcal{B}_2^t(y),$$
$$|\vec{s}_{(lt)_1}(x,y)| \le C\mathcal{B}_1^l(x)\mathcal{B}_1^t(y),\ |\vec{s}_{(lt)_2}(x,y)| \le C\mathcal{B}_2^l(x)\mathcal{B}_2^t(y),$$
$$\left|\frac{\partial^{i+j}s_{(lb)_1}}{\partial x^i\partial y^j}\right| \le C({\mu_1}^{i+j}\varepsilon_1^{-i-j}\mathcal{B}_1^l(x)\mathcal{B}_1^b(y)+\varepsilon_2^{(-i-j)/2}\mathcal{B}_2^l(x)\mathcal{B}_2^b(y)),\ 1\le i+j\le 3,$$
$$\left|\frac{\partial^{i+j}s_{(lb)_2}}{\partial x^i\partial y^j}\right| \le C\varepsilon_2^{(-i-j)/2}\mathcal{B}_2^l(x)\mathcal{B}_2^b(y),\ 1\le i+j\le 2,$$
$$\left|\frac{\partial^{i+j}s_{(lb)_2}}{\partial x^i\partial y^j}\right| \le C\varepsilon_2^{-2}({\mu_1}^2\varepsilon_1^{-2}\mathcal{B}_1^l(x)\mathcal{B}_1^b(y)+\varepsilon_2^{-1}\mathcal{B}_2^l(x)\mathcal{B}_2^b(y),\ i+j=3,$$
$$\left|\frac{\partial^{i+j}s_{(br)_1}}{\partial x^i\partial y^j}\right| \le C({\mu_1}^{-i+j}\varepsilon_1^{-j}\mathcal{B}_1^b(y)+\varepsilon_2^{(-1-j)/2}\mathcal{B}_2^r(x)\mathcal{B}_2^b(y),\ 1\le i+j\le 3,$$
$$\left|\frac{\partial^{i+j}s_{(br)_2}}{\partial x^i\partial y^j}\right| \le C\varepsilon_2^{(-i-j)/2}\mathcal{B}_2^r(x)\mathcal{B}_2^b(y),\ 0\le i+j\le 2,$$
$$\left|\frac{\partial^{i+j}s_{(br)_2}}{\partial x^i\partial y^j}\right| \le C\varepsilon_2^{-2}({\mu_2}^{-2}\varepsilon_1^{-1}\mathcal{B}_1^b(y)+\varepsilon_2^{-1}\mathcal{B}_2^r(x)\mathcal{B}_2^b(y),\ i+j=3,$$
$$\left|\frac{\partial^{i+j}s_{(rt)_1}}{\partial x^i\partial y^j}\right| \le C({\mu_2}^{-i-j}+\varepsilon_2^{(-i-j)/2}\mathcal{B}_2^r(x)\mathcal{B}_2^t(y)),\ 1\le i+j\le 3,$$
$$\left|\frac{\partial^{i+j}s_{(rt)_2}}{\partial x^i\partial y^j}\right| \le C\varepsilon_2^{(-i-j)/2}\mathcal{B}_2^r(x)\mathcal{B}_2^t(y),\ 1\le i+j\le 2,$$
$$\left|\frac{\partial^{i+j}s_{(rt)_2}}{\partial x^i\partial y^j}\right| \le C\varepsilon_2^{-2}({\mu_2}^{-6}+\varepsilon_2^{-1}\mathcal{B}_2^r(x)\mathcal{B}_2^t(y)),\ i+j=3,$$
$$\left|\frac{\partial^{i+j}s_{(lt)_1}}{\partial x^i\partial y^j}\right| \le C({\mu_1}^{i-j}\varepsilon_1^{-i}\mathcal{B}_1^l(x)+\varepsilon_2^{(-i-j)/2}\mathcal{B}_2^l(x)\mathcal{B}_2^t(y)),\ 1\le i+j\le 3,$$
$$\left|\frac{\partial^{i+j}s_{(lt)_2}}{\partial x^i\partial y^j}\right| \le C\varepsilon_2^{(-i-j)/2}\mathcal{B}_2^l(x)\mathcal{B}_2^t(y),\ 1\le i+j\le 2,$$
$$\left|\frac{\partial^{i+j}s_{(lt)_2}}{\partial x^i\partial y^j}\right| \le C\varepsilon_2^{-2}({\mu_2}^{-2}\varepsilon_1^{-1}\mathcal{B}_1^l(x)+\varepsilon_2^{-1}\mathcal{B}_2^l(x)\mathcal{B}_2^t(y)),\ i+j=3.$$

Finally, for the **Case 4**, *we have*

$$|\vec{s}_{(lb)_1}(x,y)| \le C\mathcal{B}_1^l(x)\mathcal{B}_1^b(y),\ |\vec{s}_{(lb)_2}(x,y)| \le C\mathcal{B}_2^l(x)\mathcal{B}_2^b(y),\ |\vec{s}_{(br)_1}(x,y)| \le C\mathcal{B}_1^r(x)\mathcal{B}_1^b(y),$$
$$|\vec{s}_{(br)_2}(x,y)| \le C\mathcal{B}_2^r(x)\mathcal{B}_2^b(y),\ |\vec{s}_{(rt)_1}(x,y)| \le C\mathcal{B}_1^r(x)\mathcal{B}_1^t(y),\ |\vec{s}_{(rt)_2}(x,y)| \le C\mathcal{B}_2^r(x)\mathcal{B}_2^t(y),$$
$$|\vec{s}_{(lt)_1}(x,y)| \le C\mathcal{B}_1^l(x)\mathcal{B}_1^t(y),\ |\vec{s}_{(lt)_2}(x,y)| \le C\mathcal{B}_2^l(x)\mathcal{B}_2^t(y),$$
$$\left|\frac{\partial^{i+j}s_{(lb)_1}}{\partial x^i\partial y^j}\right| \le C\varepsilon_1^{(-i-j)/2}\mathcal{B}_1^l(x)\mathcal{B}_1^b(y),\ 1\le i+j\le 2,$$

$$\left|\frac{\partial^{i+j}s_{(lb)_2}}{\partial x^i\partial y^j}\right| \le C({\mu_1}^{i+j}\varepsilon_1^{-i-j}\mathcal{B}_1^b(y)+\varepsilon_2^{(-i-j)/2}\mathcal{B}_2^l(x)\mathcal{B}_2^b(y)),\ 1\le i+j\le 3,$$
$$\left|\frac{\partial^{i+j}s_{(lb)_1}}{\partial x^i\partial y^j}\right| \le C\varepsilon_1^{-2}(\varepsilon_1^{-2}\mathcal{B}_1^l(x)\mathcal{B}_1^b(y)+{\mu_2}^2\varepsilon_2^{-1}\mathcal{B}_2^l(x)\mathcal{B}_2^b(y),\ i+j=3,$$
$$\left|\frac{\partial^{i+j}s_{(br)_1}}{\partial x^i\partial y^j}\right| \le C\varepsilon_1^{(-i-j)/2}\mathcal{B}_1^r(x)\mathcal{B}_1^b(y),\ 1\le i+j\le 2,$$
$$\left|\frac{\partial^{i+j}s_{(br)_2}}{\partial x^i\partial y^j}\right| \le C({\mu_1}^{-i+j}\varepsilon_1^{-j}\mathcal{B}_1^b(y)+\varepsilon_2^{(-1-j)/2}\mathcal{B}_2^r(x)\mathcal{B}_2^b(y),\ 0\le i+j\le 3,$$
$$\left|\frac{\partial^{i+j}s_{(br)_1}}{\partial x^i\partial y^j}\right| \le C\varepsilon_1^{-2}(\varepsilon_1^{-1}\mathcal{B}_1^r(x)\mathcal{B}_1^b(y)+{\mu_2}^{-2}\varepsilon_2^{-1}\mathcal{B}_2^r(x)\mathcal{B}_2^b(y)),\ i+j=3,$$
$$\left|\frac{\partial^{i+j}s_{(rt)_1}}{\partial x^i\partial y^j}\right| \le C\varepsilon_1^{(-i-j)/2}\mathcal{B}_1^r(x)\mathcal{B}_1^t(y),\ 1\le i+j\le 2,$$
$$\left|\frac{\partial^{i+j}s_{(rt)_2}}{\partial x^i\partial y^j}\right| \le C({\mu_2}^{-i-j}+\varepsilon_2^{(-i-j)/2}\mathcal{B}_2^r(x)\mathcal{B}_2^t(y)),\ 1\le i+j\le 2,$$
$$\left|\frac{\partial^{i+j}s_{(rt)_1}}{\partial x^i\partial y^j}\right| \le C\varepsilon_1^{-2}({\mu_2}^{-6}+\varepsilon_2^{-1}\mathcal{B}_2^r(x)\mathcal{B}_2^t(y)),\ i+j=3,$$
$$\left|\frac{\partial^{i+j}s_{(lt)_1}}{\partial x^i\partial y^j}\right| \le C\varepsilon_1^{(-i-j)/2}\mathcal{B}_1^l(x)\mathcal{B}_1^t(y),\ 1\le i+j\le 2,$$
$$\left|\frac{\partial^{i+j}s_{(lt)_2}}{\partial x^i\partial y^j}\right| \le C({\mu_1}^{i-j}\varepsilon_1^{-i}\mathcal{B}_1^l(x)+\varepsilon_2^{(-i-j)/2}\mathcal{B}_2^l(x)\mathcal{B}_2^t(y)),\ 1\le i+j\le 3,$$
$$\left|\frac{\partial^{i+j}s_{(lt)_1}}{\partial x^i\partial y^j}\right| \le C\varepsilon_1^{-2}({\mu_2}^{-2}\varepsilon_1^{-1}\mathcal{B}_1^l(x)+\varepsilon_2^{-1}\mathcal{B}_2^l(x)\mathcal{B}_2^t(y)),\ i+j=3.$$

Proof. Here, only we discuss the bounds on the derivatives of the corner layer component $\vec{\mathbf{s}}_{lb}$; for this component, we consider the new variables $\Psi_1 = x/\nu_1$, $\varphi_1 = y/\nu_1$, and $\Psi_2 = x/\nu_2$, $\varphi_2 = y/\nu_2$, where $\nu_1 = \varepsilon_1(\mu_1+\sqrt{\varepsilon_1})^{-1}$, $\nu_2 = \varepsilon_2(\mu_2+\sqrt{\varepsilon_2})^{-1}$. After changing the variables, the resulting continuous problems are given by

$$\begin{cases} \varepsilon_1\nu_1^{-2}\left(\dfrac{\partial^2 s^*_{lb_1}}{\partial\Psi_1^2}+\dfrac{\partial^2 s^*_{lb_1}}{\partial\varphi_1^2}\right)+\mu_1\nu_1^{-1}\left(a_1^{1^*}(\Psi_1,\varphi_1)\dfrac{\partial s^*_{lb_1}}{\partial\Psi_1}+a_2^{1^*}(\Psi_1,\varphi_1)\dfrac{\partial s^*_{lb_1}}{\partial\varphi_1}\right)-b^*_{11}(\Psi_1,\varphi_1)s^*_{lb_1}-b^*_{12}(\Psi_1,\varphi_1)s^*_{lb_2} \\ \qquad\qquad = f_1^*(\Psi_1,\varphi_1),\ \forall(\Psi_1,\varphi_1)\in\Omega^{*,2}_{\nu_1}, \\ s^*_{lb_1}(\Psi_1,\varphi_1) = -w^*_{l_1}(\Psi_1,\varphi_1),\quad \forall(\Psi_1,\varphi_1)\in\Gamma^{*,2}_{1,\nu_1}, \\ s^*_{lb_1}(\Psi_1,\varphi_1) = -w^*_{b_1}(\Psi_1,\varphi_1),\quad \forall(\Psi_1,\varphi_1)\in\Gamma^{*,2}_{2,\nu_1},\quad s^*_{lb_1}(\Psi_1,\varphi_1)=0,\quad \forall(\Psi_1,\varphi_1)\in\Gamma^{*,2}_{3,\nu_1}\cup\Gamma^{*,2}_{4,\nu_1}, \end{cases}$$

$$\begin{cases} \varepsilon_2\nu_2^{-2}\left(\dfrac{\partial^2 s^*_{lb_2}}{\partial\Psi_2^2}+\dfrac{\partial^2 p^*_{lb_2}}{\partial\varphi_2^2}\right)+\mu_2\nu_2^{-1}\left(a_1^{2^*}(\Psi_2,\varphi_2)\dfrac{\partial s^*_{lb_2}}{\partial\Psi_2}+a_2^{2^*}(\Psi_2,\varphi_2)\dfrac{\partial s^*_{lb_2}}{\partial\varphi_2}\right)-b^*_{21}(\Psi_2,\varphi_2)s^*_{lb_1}-b^*_{22}(\Psi_2,\varphi_2)s^*_{lb_2} \\ \qquad\qquad = f_2^*(\Psi_2,\varphi_2),\ \forall(\Psi_2,\varphi_2)\in\Omega^{*,2}_{\nu_2}, \\ s^*_{lb_2}(\Psi_2,\varphi_2) = -w^*_{l_2}(\Psi_2,\varphi_2),\quad \forall(\Psi_2,\varphi_2)\in\Gamma^{*,2}_{1,\nu_2}, \\ s^*_{lb_2}(\Psi_2,\varphi_2) = -w^*_{b_2}(\Psi_2,\varphi_2),\quad \forall(\Psi_2,\varphi_2)\in\Gamma^{*,2}_{2,\nu_2},\quad s^*_{lb_2}(\Psi_2,\varphi_2)=0,\quad \forall(\Psi_2,\varphi_2)\in\Gamma^{*,2}_{3,\nu_2}\cup\Gamma^{*,2}_{4,\nu_2}, \end{cases}$$

where, $\Omega^{*,2}_{\nu_1} = (0,1/\nu_1)^2$, $\Omega^{*,2}_{\nu_2} = (0,1/\nu_2)^2$ and $\Gamma^{*,2}_{i,\nu_1}$, $\Gamma^{*,2}_{i,\nu_2}$ $i = 1,...,4$ are the relative boundaries of the extended domains $\Omega^{*,2}_{\nu_1}$, and $\Omega^{*,2}_{\nu_2}$, respectively. Then, using a similar methodology that in [11, 28, 29, 31], we can get the bounds for the corner layer component $\vec{\mathbf{s}^*}_{lb}$.

Analogously, we can consider the relative corner layer components $\vec{\mathbf{s}}_k$, $k = br, rt, lt$, for which we can get analogous bounds for the same four cases defined in (2.1). ☐

3. Discretization of the Problem

In this section, we construct the numerical method that we use to solve numerically the problem (1.1), which is defined on a special piecewise uniform Shishkin mesh. To define the mesh we distinguish the four cases defined in (2.1), which depend on the ratio between the diffusion and the convection parameters.

Then, on the corresponding mesh, the finite difference method (FDM) used to solve the problem (1.1) will be defined on the domain $\bar{\Omega}^{N,N} = \{(x_i, y_j) : 0 \leq i, j \leq N\}$. For simplicity in the presentation, we take the same number of mesh points for the x and y variables.

3.1. The Shishkin mesh

The first step to construct the numerical method is the definition of the mesh, which is given as a tensorial product of appropriated one dimensional Shishkin meshes. We define the one dimensional mesh associated to the x-variable, $x_i,\ i = 0, 1, \ldots, N$, and similarly, we can define the mesh points associated to y-variable, $y_j,\ j = 0, 1, \ldots, N$.

Case 1: If $\vartheta {\mu_1}^2 \leq \vartheta {\mu_2}^2 \leq \Lambda\varepsilon_1 \leq \Lambda\varepsilon_2$, we construct the appropriate fitted non-uniform Shishkin mesh and subdivide the unit interval into five subintervals each as

$$[0,1] = [0,\tau_1] \cup [\tau_1,\tau_2] \cup [\tau_2, 1-\tau_2] \cup [1-\tau_2, 1-\tau_1] \cup [1-\tau_1, 1],$$

where the transition points τ_1 and τ_2 are defined by

$$\tau_1 = \min\left\{\frac{\tau_2}{2}, 2\sqrt{\frac{\varepsilon_1}{\Lambda\vartheta}}\ln N\right\},\ \tau_2 = \min\left\{\frac{1}{4}, 2\sqrt{\frac{\varepsilon_2}{\Lambda\vartheta}}\ln N\right\}. \tag{3.1}$$

On each one of the subintervals $[0,\tau_1], [\tau_1,\tau_2], [1-\tau_2, 1-\tau_1]$ and $[1-\tau_1, 1]$, there are $N/8+1$ uniformly spaced grid points, while the rest subinterval $[\tau_2, 1-\tau_2]$ there are $N/2+1$ uniformly spaced grid points. Then, the grid points along the x-axis are given by

$$x_i = \begin{cases} \frac{8}{N}\tau_1 i, & \text{if} \quad 0 \leq i \leq N/8, \\ \tau_1 + \frac{8}{N}(\tau_2-\tau_1)(i-\frac{N}{8}), & \text{if} \quad N/8+1 \leq i \leq N/4, \\ \tau_2 + \frac{2}{N}(1-2\tau_2)(i-\frac{N}{4}), & \text{if} \quad N/4+1 \leq i \leq 6N/8, \\ 1-\tau_2 + \frac{8}{N}(\tau_2-\tau_1)(i-\frac{6N}{8}), & \text{if} \quad 6N/8+1 \leq i \leq 7N/8, \\ 1-\tau_1 + \frac{8}{N}\tau_1(i-\frac{7N}{8}), & \text{if} \quad 7N/8+1 \leq i \leq N. \end{cases}$$

Case 2: If $\vartheta {\mu_2}^2 \geq \vartheta {\mu_1}^2 > \Lambda\varepsilon_2 \geq \Lambda\varepsilon_1$, then, the appropriate fitted non-uniform Shishkin mesh is constructed and split the unit interval into seven subintervals each as

$$[0,1] = [0,\tau_1] \cup [\tau_1,\tau_2] \cup [\tau_2,\tau_3] \cup [\tau_3,\tau_4] \cup [\tau_4, 1-\sigma_1] \cup [1-\sigma_1, 1-\sigma_2] \cup [1-\sigma_2, 1],$$

where now the transition points $\tau_i,\ i = 1,2,3,4$ and $\sigma_1,\ \sigma_2$ are defined as

$$\tau_1 = \min\left\{\frac{\tau_2}{2}, 2\frac{\varepsilon_1}{\mu_1\vartheta}\ln N\right\},\ \tau_2 = \min\left\{\frac{2\tau_3}{3}, 2\frac{\varepsilon_1}{\mu_2\vartheta}\ln N\right\}, \tag{3.2a}$$

$$\tau_3 = \min\left\{\frac{3\tau_4}{4}, 2\frac{\varepsilon_2}{\mu_1\vartheta}\ln N\right\},\ \tau_4 = \min\left\{\frac{1}{4}, 2\frac{\varepsilon_2}{\mu_2\vartheta}\ln N\right\}, \tag{3.2b}$$

$$\sigma_1 = \min\left\{\frac{1}{4}, 2\frac{\mu_2}{\Lambda}\ln N\right\},\ \sigma_2 = \min\left\{\frac{\sigma_1}{2}, 2\frac{\mu_1}{\Lambda}\ln N\right\}. \tag{3.2c}$$

On the subinterval $[1-\sigma_1, 1-\sigma_2]$, $[1-\sigma_2, 1]$ there are $N/8+1$ uniformly spaced grid points, while the rest subinterval $[0,\tau_1]$, $[\tau_1,\tau_2]$, $[\tau_2,\tau_3]$, $[\tau_3,\tau_4]$ there are $N/16+1$ uniformly spaced grid points and in the subinterval $[\tau_4, 1-\sigma_1]$ there are $N/2+1$ uniformly spaced grid points.

Then, the mesh points along the x-axis are given by

$$x_i = \begin{cases} \frac{16}{N}\tau_1 i, & \text{if} \quad 0 \le i \le N/16, \\ \tau_1 + \frac{16}{N}(\tau_2 - \tau_1)(i - \frac{N}{16}), & \text{if} \quad N/16+1 \le i \le N/8, \\ \tau_2 + \frac{16}{N}(\tau_3 - \tau_2)(i - \frac{N}{8}), & \text{if} \quad N/8+1 \le i \le 3N/16, \\ \tau_3 + \frac{16}{N}(\tau_4 - \tau_3)(i - \frac{3N}{16}), & \text{if} \quad 3N/16+1 \le i \le N/4, \\ \tau_4 + \frac{2}{N}(1 - \sigma_1 - \tau_4)(i - \frac{N}{4}), & \text{if} \quad N/4+1 \le i \le 3N/4, \\ 1 - \sigma_1 + \frac{8}{N}(\sigma_1 - \sigma_2)(i - \frac{3N}{4}), & \text{if} \quad 3N/4+1 \le i \le 7N/8, \\ 1 - \sigma_1 + \frac{8}{N}\sigma_2(i - \frac{7N}{8}), & \text{if} \quad 7N/8+1 \le i \le N. \end{cases}$$

Case 3: If $\Lambda\varepsilon_1 < \vartheta{\mu_1}^2 \le \vartheta{\mu_2}^2 < \Lambda\varepsilon_2$, then, the piecewise uniform Shishkin mesh is developed and split the unit interval $[0, 1]$ into seven subintervals each as

$$[0,1] = [0,\tau_1] \cup [\tau_1,\tau_2] \cup [\tau_2,\tau_3] \cup [\tau_3, 1-\tau_3] \cup [1-\tau_3, 1-\sigma_1] \cup [1-\sigma_1, 1-\sigma_2] \cup [1-\sigma_2, 1],$$

where the transition points τ_i, $i = 1, 2, 3$ and σ_1, σ_2 now are defined as

$$\tau_1 = \min\left\{\frac{\tau_2}{2}, \frac{2\varepsilon_1}{\mu_1\vartheta}\ln N\right\},\ \tau_2 = \min\left\{\frac{2\tau_3}{3}, \frac{2\varepsilon_1}{\mu_2\vartheta}\ln N\right\},\ \tau_3 = \min\left\{\frac{1}{4}, 2\sqrt{\frac{\varepsilon_2}{\Lambda\vartheta}}\ln N\right\}, \tag{3.3a}$$

$$\sigma_1 = \min\left\{\frac{2\tau_3}{3}, \frac{2\mu_2}{\Lambda}\ln N\right\},\ \sigma_2 = \min\left\{\frac{\sigma_1}{2}, \frac{2\mu_1}{\Lambda}\ln N\right\}. \tag{3.3b}$$

On each subintervals $[0,\tau_1], [\tau_1,\tau_2], [\tau_2,\tau_3], [1-\tau_3, 1-\sigma_1], [1-\sigma_1, 1-\sigma_2], [1-\sigma_2, 1]$, there are $N/12+1$ uniformly spaced grid points, while the rest subinterval $[\tau_3, 1-\tau_3]$ there are $N/2+1$ uniformly spaced grid points.
Then, the grid points along the x-axis are given by

$$x_i = \begin{cases} \frac{12}{N}\tau_1 i, & \text{if} \quad 0 \le i \le N/12, \\ \tau_1 + \frac{12}{N}(\tau_2 - \tau_1)(i - \frac{N}{12}), & \text{if} \quad N/12+1 \le i \le N/6, \\ \tau_2 + \frac{12}{N}(\tau_3 - \tau_2)(i - \frac{N}{6}), & \text{if} \quad N/6+1 \le i \le N/4, \\ \tau_3 + \frac{2}{N}(1 - 2\tau_3)(i - \frac{N}{4}), & \text{if} \quad N/4+1 \le i \le 3N/4, \\ 1 - \tau_3 + \frac{12}{N}(\tau_3 - \sigma_1)(i - \frac{3N}{4}), & \text{if} \quad 3N/4+1 \le i \le 5N/6, \\ 1 - \sigma_1 + \frac{12}{N}(\sigma_1 - \sigma_2)(i - \frac{5N}{6}), & \text{if} \quad 5N/6+1 \le i \le 11N/12, \\ 1 - \sigma_2 + \frac{12}{N}\sigma_2(i - \frac{11N}{12}), & \text{if} \quad 11N/12+1 \le i \le N. \end{cases}$$

Case 4: If $\vartheta{\mu_1}^2 < \Lambda\varepsilon_1 \le \Lambda\varepsilon_2 < \vartheta{\mu_2}^2$, then, the piecewise uniform Shishkin mesh is developed and split the unit interval $[0, 1]$ into six subintervals each as

$$[0,1] = [0,\tau_1] \cup [\tau_1,\tau_2] \cup [\tau_2,\tau_3] \cup [\tau_3, 1-\tau_3] \cup [1-\tau_3, 1-\sigma_1] \cup [1-\sigma_1, 1],$$

where the transition points τ_i, $i = 1, 2, 3$ and σ_1 now are defined as

$$\tau_1 = \min\left\{\frac{\tau_2}{2}, \frac{2\varepsilon_1}{\mu_2\vartheta}\ln N\right\},\ \tau_2 = \min\left\{\frac{2\tau_3}{3}, \frac{2\varepsilon_2}{\mu_2\vartheta}\ln N\right\},\ \tau_3 = \min\left\{\frac{1}{4}, 2\sqrt{\frac{\varepsilon_1}{\Lambda\vartheta}}\ln N\right\}, \tag{3.4a}$$

$$\sigma_1 = \min\left\{\frac{\tau_3}{2}, \frac{2\mu_2}{\Lambda}\ln N\right\}. \tag{3.4b}$$

On each subintervals $[0,\tau_1], [\tau_1,\tau_2], [\tau_2,\tau_3]$, there are $N/12+1$ uniformly spaced grid points, while the rest subinterval $[\tau_3, 1-\tau_3]$ there are $N/2+1$ and $[1-\tau_3, 1-\sigma_1], [1-\sigma_1, 1]$ there are $N/8+1$ uniformly spaced grid points.

Then, the grid points along the x-axis are given by

$$x_i = \begin{cases} \frac{12}{N}\tau_1 i, & \text{if} \quad 0 \le i \le N/12, \\ \tau_1 + \frac{12}{N}(\tau_2 - \tau_1)(i - \frac{N}{12}), & \text{if} \quad N/12+1 \le i \le N/6, \\ \tau_2 + \frac{12}{N}(\tau_3 - \tau_2)(i - \frac{N}{6}), & \text{if} \quad N/6+1 \le i \le N/4, \\ \tau_3 + \frac{2}{N}(1 - 2\tau_3)(i - \frac{N}{4}), & \text{if} \quad N/4+1 \le i \le 3N/4, \\ 1 - \tau_3 + \frac{8}{N}(\tau_3 - \sigma_1)(i - \frac{3N}{4}), & \text{if} \quad 3N/4+1 \le i \le 7N/8, \\ 1 - \sigma_1 + \frac{8}{N}\sigma_1(i - \frac{7N}{8}), & \text{if} \quad 7N/8+1 \le i \le N. \end{cases}$$

For each one of the cases, the step sizes are defined as

$$h_i = x_i - x_{i-1},\ i = 1,2,...,N,\ \bar{h}_i = h_i + h_{i+1},\ i = 1,2,...,N-1.$$

The boundaries of the domain $\bar{\Omega}^{N,N}$ are denoted as

$$\Gamma_1^{N,N} = \Big\{(0,y_j)\,\Big|\,0 \le j \le N\Big\}, \Gamma_2^{N,N} = \Big\{(x_i,0)\,\Big|\,0 \le i \le N\Big\},$$
$$\Gamma_3^{N,N} = \Big\{(1,y_j)\,\Big|\,0 \le j \le N\Big\}, \Gamma_4^{N,N} = \Big\{(x_i,1)\,\Big|\,0 \le i \le N\Big\},$$

and $\Gamma^{N,N} = \Gamma_1^{N,N} \cup \Gamma_2^{N,N} \cup \Gamma_3^{N,N} \cup \Gamma_4^{N,N}$.

3.2. Finite difference method (FDM)

On an arbitrary mesh, $\bar{\Omega}^{N,N}$, in order to discretize the problem (1.1), we define the standard upwind finite difference scheme, which is given by

$$\begin{cases} \vec{\mathcal{L}}_{\vec{\varepsilon},\vec{\mu}}^{N,N}\vec{\mathbf{Z}}(x_i,y_j) = \vec{\mathbf{f}}(x_i,y_j),\ \forall (x_i,y_j) \in \Omega^{N,N}, \\ \vec{\mathbf{Z}}(x_i,y_j) = \vec{\mathbf{g}}_1(y_j),\ (x_i,y_j) \in \Gamma_1^{N,N}, \quad \vec{\mathbf{Z}}(x_i,y_j) = \vec{\mathbf{g}}_2(x_i),\ (x_i,y_j) \in \Gamma_2^{N,N}, \\ \vec{\mathbf{Z}}(x_i,y_j) = \vec{\mathbf{g}}_3(y_j),\ (x_i,y_j) \in \Gamma_3^{N,N}, \quad \vec{\mathbf{Z}}(x_i,y_j) = \vec{\mathbf{g}}_4(x_i),\ (x_i,y_j) \in \Gamma_4^{N,N}, \end{cases} \tag{3.5}$$

where

$$\vec{\mathcal{L}}_{\vec{\varepsilon},\vec{\mu}}^{N,N}\vec{\mathbf{Z}}(x_i,y_j) = \vec{\varepsilon}(\delta_{xx}^2 + \delta_{yy}^2)\vec{\mathbf{Z}}(x_i,y_j) + \vec{\mu}(\vec{\mathbf{A}}_1(x_i,y_j)D_x^+ + \vec{\mathbf{A}}_2(x_i,y_j)D_y^+)\vec{\mathbf{Z}}(x_i,y_j) - \vec{\mathbf{B}}(x_i,y_j)\vec{\mathbf{Z}}(x_i,y_j).$$

As it is usual, the discrete differential operators D_x^+, D_y^+, δ_{xx}^2, and δ_{yy}^2 are given by

$$D_x^+\vec{\mathbf{Z}}(x_i,y_j) = \frac{\vec{\mathbf{Z}}(x_{i+1},y_j) - \vec{\mathbf{Z}}(x_i,y_j)}{h_{i+1}},\ D_x^-\vec{\mathbf{Z}}(x_i,y_j) = \frac{\vec{\mathbf{Z}}(x_i,y_j) - \vec{\mathbf{Z}}(x_{i-1},y_j)}{h_i},$$
$$D_y^+\vec{\mathbf{Z}}(x_i,y_j) = \frac{\vec{\mathbf{Z}}(x_i,y_{j+1}) - \vec{\mathbf{Z}}(x_i,y_j)}{k_{j+1}},\ D_y^-\vec{\mathbf{Z}}(x_i,y_j) = \frac{\vec{\mathbf{Z}}(x_i,y_j) - \vec{\mathbf{Z}}(x_i,y_{j-1})}{k_j},$$
$$\delta_{xx}^2\vec{\mathbf{Z}}(x_i,y_j) = \frac{2}{\bar{h}_i}(D_x^+\vec{\mathbf{Z}}(x_i,y_j) - D_x^-\vec{\mathbf{Z}}(x_i,y_j)),\ \delta_{yy}^2\vec{\mathbf{Z}}(x_i,y_j) = \frac{2}{\bar{k}_j}(D_y^+\vec{\mathbf{Z}}(x_i,y_j) - D_y^-\vec{\mathbf{Z}}(x_i,y_j)),$$

for $i,j = 1,2,\ldots,N-1$. Similarly to the continuous problem, we can prove a discrete minimum principle for the discrete operator $\vec{\mathcal{L}}_{\vec{\varepsilon},\vec{\mu}}^{N,N}$.

Lemma 3.1 (Discrete minimum principle)**.** *Let $\vec{\mathcal{L}}_{\vec{\varepsilon},\vec{\mu}}^{N,N}$ be the discrete operator given in (3.5), If $\vec{\mathbf{\Phi}}(x_i,y_j) \ge \vec{\mathbf{0}}$ on $\Gamma^{N,N}$ and $\vec{\mathcal{L}}_{\vec{\varepsilon},\vec{\mu}}^{N,N}\vec{\mathbf{\Phi}}(x_i,y_j) \le \vec{\mathbf{0}}$, $\forall (x_i,y_j) \in \Omega^{N,N}$, then $\vec{\mathbf{\Phi}}(x_i,y_j) \ge \vec{\mathbf{0}}$, $\forall (x_i,y_j) \in \overline{\Omega}^{N,N}$.*

Proof. The proof is standard in the context of singularly perturbed problems, and it follows the ideas of [16, 20, 21, 22], where parabolic singularly perturbed systems of convection-diffusion type were considered. □

Lemma 3.2 (Discrete stability result)**.** *Let $\vec{\boldsymbol{Z}}(x_i, y_j)$ be the solution of (3.5). Then it holds*

$$\|\vec{\boldsymbol{Z}}(x_i,y_j)\|_{\overline{\Omega}^{N,N}} \leq \frac{1}{\vartheta}\|\vec{\boldsymbol{\mathcal{L}}}^{N,N}_{\vec{\boldsymbol{\varepsilon}},\vec{\boldsymbol{\mu}}}\vec{\boldsymbol{Z}}\|^{N,N}_{\Omega} + \max\left\{\|\vec{\boldsymbol{Z}}\|_{\Gamma_1^{N,N}}, \|\vec{\boldsymbol{Z}}\|_{\Gamma_2^{N,N}},\ \|\vec{\boldsymbol{Z}}\|_{\Gamma_3^{N,N}}, \|\vec{\boldsymbol{Z}}\|_{\Gamma_4^{N,N}}\right\},$$

where $\|.\|^{N,N}_{\Omega}$ denotes the discrete pointwise maximum norm on $\overline{\Omega}^{N,N}$ and ϑ is defined in (1.3).

Proof. It can be proved easily using Lemma 3.1. □

4. Uniform convergence of the numerical method

In this section, we analyze the error of our proposed numerical method and we prove that it is a uniformly convergent method. To approximate the nodal errors, we decompose the discrete solution into regular (smooth), layer and corner components, similarly to the approach used for the continuous solutions. Then, we have

$$\vec{\mathbf{Z}}(x_i,y_j) = \vec{\mathbf{V}}(x_i,y_j) + \vec{\mathbf{W}}(x_i,y_j) + \vec{\mathbf{S}}(x_i,y_j).$$

Moreover, the layer and the corner layer components are decomposed in the form

$$\begin{aligned}\vec{\mathbf{W}}(x_i,y_j) &= \vec{\mathbf{W}}_l(x_i,y_j) + \vec{\mathbf{W}}_r(x_i,y_j) + \vec{\mathbf{W}}_b(x_i,y_j) + \vec{\mathbf{W}}_t(x_i,y_j),\\ \vec{\mathbf{S}}(x_i,y_j) &= \vec{\mathbf{S}}_{lb}(x_i,y_j) + \vec{\mathbf{S}}_{br}(x_i,y_j) + \vec{\mathbf{S}}_{rt}(x_i,y_j) + \vec{\mathbf{S}}_{lt}(x_i,y_j).\end{aligned}$$

The regular component $\vec{\mathbf{V}}(x_i, y_j)$ is the solution of the discrete problem

$$\begin{cases}\vec{\boldsymbol{\mathcal{L}}}^{N,N}_{\vec{\boldsymbol{\varepsilon}},\vec{\boldsymbol{\mu}}}\vec{\mathbf{V}}(x_i,y_j) = \vec{\mathbf{f}}(x_i,y_j), \quad \forall (x_i,y_j)\in\Omega^{N,N},\\ \vec{\mathbf{V}}(x_i,y_j) = \vec{\mathbf{v}}(x_i,y_j), \quad \forall (x_i,y_j)\in\Gamma^{N,N},\end{cases} \tag{4.1}$$

and the layer and corner components $\vec{\mathbf{W}}(x_i, y_j)$ and $\vec{\mathbf{S}}(x_i, y_j)$ are the solutions of the following discrete problems:

$$\begin{cases}\vec{\boldsymbol{\mathcal{L}}}^{N,N}_{\vec{\boldsymbol{\varepsilon}},\vec{\boldsymbol{\mu}}}\vec{\mathbf{W}}(x_i,y_j) = \vec{\mathbf{0}}, \quad \forall (x_i,y_j)\in\Omega^{N,N},\\ \vec{\mathbf{W}}(x_i,y_j) = \vec{\mathbf{w}}(x_i,y_j), \quad \forall (x_i,y_j)\in\Gamma^{N,N},\end{cases} \tag{4.2}$$

for the layer component, and

$$\begin{cases}\vec{\boldsymbol{\mathcal{L}}}^{N,N}_{\vec{\boldsymbol{\varepsilon}},\vec{\boldsymbol{\mu}}}\vec{\mathbf{S}}(x_i,y_j) = 0, \quad \forall (x_i,y_j)\in\Omega^{N,N},\\ \vec{\mathbf{S}}(x_i,y_j) = \vec{\mathbf{s}}(x_i,y_j), \quad \forall (x_i,y_j)\in\Gamma^{N,N},\end{cases} \tag{4.3}$$

for the corner component, respectively.

Then, we study the contribution to the error of each one of these components. First, we consider the error associated to the regular component.

Lemma 4.1. *Let $\vec{\boldsymbol{v}}(x, y)$ be the solution of (2.3) and $\vec{\boldsymbol{V}}(x_i, y_j)$ the numerical solution of (4.1). Then, for the four cases defined in (2.1), it holds*

$$|\,\vec{\boldsymbol{V}}(x_i,y_j) - \vec{\boldsymbol{v}}(x_i,y_j)| \leq CN^{-1}.$$

Proof. Using standard Taylor expansions, it is easy to see that the truncation error associated to the regular component (4.1) satisfies

$$|\vec{\boldsymbol{\mathcal{L}}}^{N,N}_{\vec{\boldsymbol{\varepsilon}},\vec{\boldsymbol{\mu}}}(\vec{\mathbf{V}}(x_i,y_j) - \vec{\mathbf{v}}(x_i,y_j))| \leq C\left[(h_i+h_{i+1})\left(\vec{\boldsymbol{\varepsilon}}\left\|\frac{\partial^3\vec{\mathbf{v}}}{\partial x^3}\right\| + \vec{\boldsymbol{\mu}}\left\|\frac{\partial^2\vec{\mathbf{v}}}{\partial x^2}\right\|\right) + (k_j+k_{j+1})\left(\vec{\boldsymbol{\varepsilon}}\left\|\frac{\partial^3\vec{\mathbf{v}}}{\partial y^3}\right\| + \vec{\boldsymbol{\mu}}\left\|\frac{\partial^2\vec{\mathbf{v}}}{\partial y^2}\right\|\right)\right]. \tag{4.4}$$

Therefore, from (2.8), (2.18) and (2.24), it is straightforward to prove that holds

$$|\vec{\mathcal{L}}^{N,N}_{\vec{\varepsilon},\vec{\mu}}(\vec{\mathbf{V}}(x_i,y_j)-\vec{\mathbf{v}}(x_i,y_j))| \leq \begin{cases} \begin{pmatrix} CN^{-1}(\sqrt{\varepsilon_1}+\mu_1) \\ CN^{-1}(\varepsilon_2+\mu_1) \end{pmatrix}, & \text{if } \vartheta{\mu_1}^2 \leq \vartheta{\mu_2}^2 \leq \Lambda\varepsilon_1 \leq \Lambda\varepsilon_2, \\ \begin{pmatrix} CN^{-1}(1+\mu_1) \\ CN^{-1}(\varepsilon_2+\mu_2) \end{pmatrix}, & \text{if } \vartheta{\mu_2}^2 \geq \vartheta{\mu_1}^2 > \Lambda\varepsilon_2 \geq \Lambda\varepsilon_1, \\ \begin{pmatrix} CN^{-1}(1+\mu_1) \\ CN^{-1}(\sqrt{\varepsilon_2}+\mu_2) \end{pmatrix}, & \text{if } \Lambda\varepsilon_1 < \vartheta{\mu_1}^2 \leq \vartheta{\mu_2}^2 < \Lambda\varepsilon_2, \\ \begin{pmatrix} CN^{-1}(\sqrt{\varepsilon_1}+\mu_1) \\ CN^{-1}(1+\mu_2) \end{pmatrix}, & \text{if } \vartheta{\mu_1}^2 < \Lambda\varepsilon_1 \leq \Lambda\varepsilon_2 < \vartheta{\mu_2}^2. \end{cases}$$

Further, using the Lemma 3.1, for all cases, we easily can obtain

$$|\vec{\mathbf{V}}(x_i,y_j)-\vec{\mathbf{v}}(x_i,y_j)| \leq CN^{-1}, \tag{4.5}$$

which is the required result. □

Next step is to use the mesh functions defined on $\bar{\Omega}^{N,N}$, in order to prove appropriate bounds for the error associated to the singular component $\vec{\mathbf{W}}$.

4.1. Error estimates for the boundary layer components

Lemma 4.2. *Let $\vec{\boldsymbol{w}}_k(x_i,y_j)$ be the true solution of* (2.4) *and $\vec{\boldsymbol{W}}_k(x_i,y_j)$ be the numerical solution of* (4.2)*; then, for* **Case 1** *defined in (2.1), the error associated to the boundary layer functions satisfies*

$$|\vec{\boldsymbol{W}}_k(x_i,y_j)-\vec{\boldsymbol{w}}_k(x_i,y_j)| \leq CN^{-1}\ln N,\ k=l,b,r,t.$$

Proof. If $\tau_1=1/8$, $\tau_2=1/4$, the proof can be obtained by applying standard methods (on uniform meshes) by taking into consideration that it holds $\varepsilon_1^{-1/2} \leq C\ln N$ and $\varepsilon_2^{-1/2} \leq C\ln N$. Hence, by using (4.2) and Theorem 2.5, we have

$$\begin{aligned} \|\vec{\mathcal{L}}^{N,N}_{\vec{\varepsilon},\vec{\mu}}(\vec{\mathbf{W}}_l-\vec{\mathbf{w}}_l)\| &\leq C\left[(h_i+h_{i+1})\left(\vec{\boldsymbol{\varepsilon}}\left\|\frac{\partial^3\vec{\mathbf{w}}_l}{\partial x^3}\right\|+\vec{\boldsymbol{\mu}}\left\|\frac{\partial^2\vec{\mathbf{w}}_l}{\partial x^2}\right\|\right)+(k_j+k_{j+1})\left(\vec{\boldsymbol{\varepsilon}}\left\|\frac{\partial^3\vec{\mathbf{w}}_l}{\partial y^3}\right\|+\vec{\boldsymbol{\mu}}\left\|\frac{\partial^2\vec{\mathbf{w}}_l}{\partial y^2}\right\|\right)\right] \\ &\leq CN^{-1}({\varepsilon_1}^{-1/2}+\varepsilon_1{\varepsilon_2}^{-3/2}) \leq CN^{-1}{\varepsilon_2}^{-1/2} \leq CN^{-1}\ln N. \end{aligned}$$

Further, if $\tau_1=\sqrt{\dfrac{\varepsilon_1}{\Lambda\vartheta}}\ln N$, $\tau_2=1/4$, and consider the intervals $(x_i,y_j)\in(\tau_2,1-\tau_2)\times(0,1)$, $(\tau_1,\tau_2)\times(0,1)$ or $(1-\tau_2,1-\tau_1)\times(0,1)$, then we have

$$|\vec{\mathcal{L}}^{N,N}_{\vec{\varepsilon},\vec{\mu}}(\vec{\mathbf{W}}_l-\vec{\mathbf{w}}_l)(x_i,y_j)| \leq CN^{-1}\begin{pmatrix} \varepsilon_2^{-1/2}\mathcal{B}_2^l(x_{i-1}) \\ \varepsilon_2^{-1/2}\mathcal{B}_2^l(x_{i-1})) \end{pmatrix} \leq CN^{-1}\ln N.$$

When $(x_i,y_j)\in(0,\tau_1]\times(0,1)$ or $[1-\tau_1,1]\times(0,1)$, it follows

$$|\vec{\mathcal{L}}^{N,N}_{\vec{\varepsilon},\vec{\mu}}(\vec{\mathbf{W}}_l-\vec{\mathbf{w}}_l)(x_i,y_j)| \leq CN^{-1}\ln N\sqrt{\varepsilon_1}\begin{pmatrix} \varepsilon_1^{-1/2}+\varepsilon_2^{-1/2} \\ \varepsilon_2^{-1/2}+\varepsilon_2^{-1/2} \end{pmatrix} \leq CN^{-1}\ln N.$$

If $\tau_2=\sqrt{\dfrac{\varepsilon_2}{\Lambda\vartheta}}\ln N$, $\tau_1=\dfrac{\tau_2}{2}$ and $\dfrac{\sqrt{\varepsilon_2}}{2} \leq \sqrt{\varepsilon_1} < \sqrt{\varepsilon_2}$, hence, it holds $\tau_2 \leq C\sqrt{\varepsilon_1}\ln N$.

To prove the error estimate on the region $(x_i,y_j)\in[\tau_2,1-\tau_2]\times(0,1)$, we consider the following barrier functions:

$$\mathcal{G}_1^{l,N}(x_i)=\prod_{\iota=1}^{i}\left(1+\sqrt{\left(\frac{\Lambda\vartheta}{\varepsilon_1}\right)}h_\iota\right)^{-1},\quad \mathcal{G}_2^{l,N}(x_i)=\prod_{\iota=1}^{i}\left(1+\sqrt{\left(\frac{\Lambda\vartheta}{\varepsilon_2}\right)}h_\iota\right)^{-1}, \tag{4.6}$$

with $\mathcal{G}_1^{l,N}(x_0) = \mathcal{G}_2^{l,N}(x_0) = 1$. After applying Theorem 2.5, we can conclude that it holds

$$|(\vec{\mathbf{W}}_l - \vec{\mathbf{w}}_l)(x_i, y_j)| \le |\vec{\mathbf{W}}_l(x_i, y_j)| + |\vec{\mathbf{w}}_l(x_i, y_j)| \le C\mathcal{G}_2^{l,N}(\tau_2) + C\mathcal{G}_2^{l,N}(\tau_2) \le CN^{-1}.$$

To get appropriate bounds for the error in the region $(x_i, y_j) \in (\tau_1, \tau_2) \times (0, 1)$ or in the region $(1 - \tau_2, 1 - \tau_1) \times (0, 1)$ using that $(h_i + h_{i+1}) \le N^{-1} \ln N \sqrt{\varepsilon_1}$, it follows

$$|\vec{\mathcal{L}}_{\vec{\varepsilon},\vec{\mu}}^{N,N}(\vec{\mathbf{W}}_l - \vec{\mathbf{w}}_l)(x_i, y_j)| \le CN^{-1} \ln N \sqrt{\varepsilon_1} \begin{pmatrix} \varepsilon_1^{-1/2} + \varepsilon_2^{-1/2} \\ \varepsilon_2^{-1/2} + \varepsilon_2^{-1/2} \end{pmatrix} \le CN^{-1} \ln N.$$

In the cases where (x_i, y_j) belongs to either $(0, \tau_1) \times (0, 1)$ or $(1 - \tau_1, 1) \times (0, 1)$, we have $h_i + h_{i+1} \le CN^{-1} \ln N \sqrt{\varepsilon_1}$. Similarly, by following the same approach as mentioned earlier, we obtain the required bounds.

Finally, assuming that $\tau_1 = \sqrt{\dfrac{\varepsilon_1}{\Lambda\vartheta}} \ln N$ and $\tau_2 = \sqrt{\dfrac{\varepsilon_2}{\Lambda\vartheta}} \ln N$, in cases where $(x_i, y_j) \in [\tau_2, 1 - \tau_2] \times (0, 1)$ or $(0, \tau_1] \times (0, 1)$, or $(1 - \tau_1, 1) \times (0, 1)$, we can obtain the required bounds by using a method similar to that used in the corresponding intervals of the previous cases. When (x_i, y_j) is within either $(\tau_1, \tau_2) \times (0, 1)$ or $(1 - \tau_2, 1 - \tau_1) \times (0, 1)$, we have $h_i + h_{i+1} \le CN^{-1}\tau_2 \le CN^{-1} \ln N \sqrt{\varepsilon_2}$. Therefore, we can deduce that it holds

$$|\vec{\mathcal{L}}_{\vec{\varepsilon},\vec{\mu}}^{N,N}(\vec{\mathbf{W}}_l - \vec{\mathbf{w}}_l)(x_i, y_j)| \le CN^{-1} \ln N.$$

So, by using Lemma 3.1, we obtain

$$|\vec{\mathbf{W}}_l(x_i, y_j) - \vec{\mathbf{w}}_l(x_i, y_j)| \le CN^{-1}, \tag{4.7}$$

which is the required result.

Analogously, we can derive similar bounds for the errors related to the other layer components $\vec{\mathbf{w}}_r$, $\vec{\mathbf{w}}_b$, and $\vec{\mathbf{w}}_t$ for **Case 1** defined in (2.1). □

Next, we consider the **Case 2** defined in (2.1). Let be the usual functions, in the context of singularly perturbed problems, given by

$$\mathcal{G}_1^{l,N}(x_i) = \prod_{\iota=1}^{i} \left(1 + \frac{\vartheta\mu_1 h_\iota}{2\varepsilon_1}\right)^{-1}, \qquad \mathcal{G}_2^{l,N}(x_i) = \prod_{\iota=1}^{i} \left(1 + \frac{\vartheta\mu_2 h_\iota}{2\varepsilon_2}\right)^{-1},$$

with $\mathcal{G}_1^{l,N}(x_0) = \mathcal{G}_2^{l,N}(x_0) = 1$.

Lemma 4.3. *For* **Case 2** *defined in (2.1), the layer components W_{l_1} and W_{l_2} satisfy the following bounds on $\bar{\Omega}^{N,N}$*

$$|W_{l_1}(x_i, y_j)| \le C\mathcal{G}_1^{l,N}(x_i), \quad |W_{l_2}(x_i, y_j)| \le C\mathcal{G}_2^{l,N}(x_i).$$

Proof. The result follows directly from [17]. □

Similar bounds can be proved for W_{k_1}, W_{k_2} with $k = b, r, t$.

Lemma 4.4. *Let $\vec{\boldsymbol{w}}_k(x_i, y_j)$ be the solution of* (2.4) *and $\vec{\boldsymbol{W}}_k(x_i, y_j)$ be the numerical solution of* (4.2). *Then, for* **Case 2** *defined in (2.1), it holds*

$$|\,\vec{\boldsymbol{W}}_k(x_i, y_j) - \vec{\boldsymbol{w}}_k(x_i, y_j)| \le CN^{-1} \ln N, \; k = l, b, r, t.$$

Proof. Here, we provide only the details corresponding to the layer function $\vec{\mathbf{w}}_l$, and similarly we can proceed for the other ones. Based on the results in Theorem 2.5, the truncation error for the singular component $\vec{\mathbf{w}}_l$ satisfies the following estimates:

$$\|\vec{\mathcal{L}}_{\vec{\varepsilon},\vec{\mu}}^{N,N}(\vec{\mathbf{W}}_l - \vec{\mathbf{w}}_l)\| \le C\left[(h_i + h_{i+1})\left(\vec{\boldsymbol{\varepsilon}}\left\|\frac{\partial^3 \vec{\mathbf{w}}_l}{\partial x^3}\right\| + \vec{\boldsymbol{\mu}}\left\|\frac{\partial^2 \vec{\mathbf{w}}_l}{\partial x^2}\right\|\right) + (k_j + k_{j+1})\left(\vec{\boldsymbol{\varepsilon}}\left\|\frac{\partial^3 \vec{\mathbf{w}}_l}{\partial y^3}\right\| + \vec{\boldsymbol{\mu}}\left\|\frac{\partial^2 \vec{\mathbf{w}}_l}{\partial y^2}\right\|\right)\right]. \tag{4.8}$$

If $\tau_1 = 1/16$, $\tau_2 = 1/8$, $\tau_3 = 3/16$, $\tau_4 = 1/4$, $\sigma_1 = 1/4$ and $\sigma_2 = 1/8$, the proof can be obtained easily by applying standard methods (on uniform meshes) by taking into consideration that it holds $\mu_1\varepsilon_1^{-1} \leq C\ln N$, $\mu_2\varepsilon_2^{-1} \leq C\ln N$, and $\mu_2{}^{-1} \leq \mu_1{}^{-1} \leq C\ln N$. Then, using Theorem 2.5 in (4.8), we have

$$|\vec{\mathcal{L}}^{N,N}_{\vec{\varepsilon},\vec{\mu}}(\vec{W}_l - \vec{w}_l)| \leq CN^{-1}\begin{pmatrix} \mu_1^3\varepsilon_1^{-2}\mathcal{B}_1^l(x_{i-1}) + \mu_2^3\varepsilon_2^{-2}\mathcal{B}_2^l(x_{i-1}) \\ \mu_1^3\varepsilon_1^{-1}\varepsilon_2^{-1}\mathcal{B}_1^l(x_{i-1}) + \mu_2^3\varepsilon_2^{-2}\mathcal{B}_2^l(x_{i-1}) \end{pmatrix}. \tag{4.9}$$

Now, let us consider the mesh functions along x- and y-directions $\vec{\psi}(x_i)$, $\vec{\psi}(y_j)$ defined on $\bar{\Omega}^{N,N}$ by

$$\psi_1(x_i) = CN^{-1}\bigg(\exp\left(\frac{2h_i\mu_1}{\varepsilon_1}\right)\varepsilon_1^{-1}\mu_1\mathfrak{R}_i + \exp\left(\frac{2h_i\mu_1}{\varepsilon_2}\right)\varepsilon_2^{-1}\mu_2\mathfrak{P}_i\bigg), \tag{4.10a}$$

$$\psi_2(x_i) = CN^{-1}\bigg(\exp\left(\frac{2h_i\mu_2}{\varepsilon_2}\right)\varepsilon_1^{-1}\mu_1\mathfrak{P}_i\bigg), \tag{4.10b}$$

where

$$\mathfrak{R}_i = \frac{\upsilon^{N-i}-1}{\upsilon^N - 1}, \text{ with } \upsilon = 1 + \frac{\mu_1 h_i}{\varepsilon_1},\ \mathfrak{P}_i = \frac{\lambda^{N-i}-1}{\lambda^N-1}, \text{ with } \lambda = 1 + \frac{\mu_2 h_i}{\varepsilon_2}.$$

Similarly, we can consider the mesh function $\vec{\psi}(y_j)$ along y-direction also.

Then, it is straightforward to prove that $0 \leq \mathfrak{R}_i, \mathfrak{P}_i \leq 1$, and also that it holds

$$(\varepsilon_1\delta^2_{xx} + \mu_1 D^+_x)\mathfrak{R}_i = 0,\ (\varepsilon_1\delta^2_{xx} + \mu_1 D^+_x)\mathfrak{P}_i = 0,$$

$$D^+_x\mathfrak{R}_i \leq -\frac{\mu_1}{\varepsilon_1}\exp\left(\frac{\mu_1 x_{i+1}}{\varepsilon_1}\right),\ D^+_x\mathfrak{P}_i \leq -\frac{\mu_2}{\varepsilon_2}\exp\left(\frac{\mu_2 x_{i+1}}{\varepsilon_2}\right).$$

Therefore, it is clear that we have

$$\vec{\mathcal{L}}^{N,N}_{\vec{\varepsilon},\vec{\mu}}\vec{\psi}(x_i) \leq \begin{pmatrix} -CN^{-1}\varepsilon_1^{-2}\mu_1^3\mathcal{B}_1^l(x_{i-1}) - CN^{-1}\varepsilon_2^{-2}\mu_2^3\mathcal{B}_2^l(x_{i-1}) \\ -CN^{-1}\mu_2^3\varepsilon_1^{-1}\varepsilon_2^{-1}\mathcal{B}_2^l(x_{i-1}) \end{pmatrix}, \tag{4.11a}$$

and similarly we can prov

$$\vec{\mathcal{L}}^{N,N}_{\vec{\varepsilon},\vec{\mu}}\vec{\psi}(y_j) \leq \begin{pmatrix} -CN^{-1}\varepsilon_1^{-2}\mu_1^3\mathcal{B}_1^l(y_{j-1}) - CN^{-1}\varepsilon_2^{-2}\mu_2^3\mathcal{B}_2^l(y_{j-1}) \\ -CN^{-1}\mu_2^3\varepsilon_1^{-1}\varepsilon_2^{-1}\mathcal{B}_2^l(y_{j-1}) \end{pmatrix}. \tag{4.12}$$

Now we define the barrier function $\vec{\boldsymbol{\Psi}}^{\pm}(x_i, y_j) = \vec{\psi}(x_i) + \vec{\psi}(y_j) \pm (\vec{\mathbf{W}}_l - \vec{\mathbf{w}}_l)(x_i, y_j)$. Clearly, it holds $\vec{\boldsymbol{\Psi}}(x_i, y_j) \geq \vec{\mathbf{0}}$, $(x_i, y_j) \in \Gamma^{N,N}$ and $\vec{\mathcal{L}}^{N,N}_{\vec{\varepsilon},\vec{\mu}}\vec{\boldsymbol{\Psi}}(x_i, y_j) \leq \vec{\mathbf{0}}$, $(x_i, y_j) \in \Omega^{N,N}$ (from (4.9) and (4.11)). Hence, from Lemma 3.1, it follows

$$|(\vec{\mathbf{W}}_l - \vec{\mathbf{w}}_l)(x_i, y_j)| \leq CN^{-1}\begin{pmatrix} \mu_1\varepsilon_1^{-1} + \mu_2\varepsilon_2^{-1} \\ \mu_2\varepsilon_1^{-1} \end{pmatrix} \leq CN^{-1}\ln N.$$

Next, from (4.2) and Lemma 4.3, when $\tau_4 = \dfrac{2\varepsilon_2}{\mu_2\vartheta}\ln N$, for the grid points (x_i, y_j), $N/4 \leq i \leq N$, $0 \leq j \leq N$, we have

$$\begin{aligned} |(W_{l_1} - w_{l_1})(x_i, y_j)| &\leq |W_{l_1}(x_i, y_j)| + |w_{l_1}(x_i, y_j)| \leq C\mathcal{G}_2^{l,N}(x_i) + C\mathcal{G}_2^{l,N}(x_i) \\ &\leq C\mathcal{G}_2^{l,N}(\tau_4) + C\mathcal{G}_2^{l,N}(\tau_4) \leq CN^{-1}. \end{aligned} \tag{4.13}$$

Analogously, we can deduce that it holds

$$|(W_{l_2} - w_{l_2})(x_i, y_j)| \leq CN^{-1}. \tag{4.14}$$

When $\tau_4 = \dfrac{2\varepsilon_2}{\mu_2\vartheta}\ln N$, there are two different cases: $2\varepsilon_1 \geq \varepsilon_2$ and $2\varepsilon_1 < \varepsilon_2$, respectively. In the case $\dfrac{\varepsilon_2}{2} \leq \varepsilon_1 \leq \varepsilon_2$, for $3N/16 \leq i \leq N/4$, $0 \leq j \leq N$, it follows that $\tau_4 \leq \dfrac{2\varepsilon_2}{\mu_2\vartheta}\ln N$. Hence, we have

$$|\vec{\mathcal{L}}^{N,N}_{\vec{\varepsilon},\vec{\mu}}(\vec{\mathbf{W}}_l - \vec{\mathbf{w}}_l)(x_i, y_j)| \leq CN^{-1}\ln N \begin{pmatrix} \varepsilon_1^{-1}\mu_1^2\mathcal{B}_1^l(x_{i-1}) + \varepsilon_2^{-1}\mu_2^2\mathcal{B}_2^l(x_{i-1}) \\ \varepsilon_2^{-1}\mu_2^2\mathcal{B}_2^l(x_{i-1}) \end{pmatrix}.$$

In the case that $\varepsilon_2 > 2\varepsilon_1$ for $3N/16 \leq i \leq N/4$, $0 \leq j \leq N$, we can obtain

$$|\vec{\mathcal{L}}^{N,N}_{\vec{\varepsilon},\vec{\mu}}(\vec{\mathbf{W}}_l - \vec{\mathbf{w}}_l)(x_i, y_j)| \leq \begin{pmatrix} C\varepsilon_1^{-1}\mu_1^2\mathcal{B}_1^l(x_{i-1}) + CN^{-1}\ln N\varepsilon_2^{-1}\mu_2^2\mathcal{B}_2^l(x_{i-1}) \\ C\varepsilon_2^{-1}\mu_2^2\mathcal{B}_1^l(x_{i-1}) + CN^{-1}\ln N\varepsilon_2^{-1}\mu_2^2\mathcal{B}_2^l(x_{i-1}) \end{pmatrix}.$$

For $N/8 \leq i \leq 3N/16$, $0 \leq j \leq N$, $\tau_3 \leq \dfrac{2\varepsilon_2}{\mu_1\vartheta}\ln N$, and therefore the truncation error satisfies

$$|\vec{\mathcal{L}}^{N,N}_{\vec{\varepsilon},\vec{\mu}}(\vec{\mathbf{W}}_l - \vec{\mathbf{w}}_l)(x_i, y_j)| \leq CN^{-1}\ln N \begin{pmatrix} C\varepsilon_1^{-1}\mu_1^2\mathcal{B}_1^l(x_{i-1}) + \varepsilon_2^{-1}\mu_2^2\mathcal{B}_2^l(x_{i-1}) \\ \varepsilon_2^{-1}\mu_2^2\mathcal{B}_2^l(x_{i-1}) \end{pmatrix}.$$

The same truncation error condition holds for $N/16 \leq i \leq N/8$, $0 \leq j \leq N$, $\tau_2 \leq \dfrac{2\varepsilon_1}{\mu_2\vartheta}\ln N$, as well as for $0 \leq i \leq N/16$, $0 \leq j \leq N$, $\tau_1 \leq \dfrac{2\varepsilon_1}{\mu_1\vartheta}\ln N$.

For $0 \leq i \leq N/16$, $0 \leq j \leq N$, we consider the mesh functions $\vec{\boldsymbol{\psi}}(x_i)$, $\vec{\boldsymbol{\psi}}(y_j)$ as follows:

$$\psi_1(x_i) = CN^{-1}\ln N\left(\exp\left(\frac{2\vartheta h_i\mu_1}{\varepsilon_1}\right)\mathcal{G}_1^{l,N}(x_i) + \exp\left(\frac{2\vartheta h_i\mu_2}{\varepsilon_2}\right)\mathcal{G}_2^{l,N}(x_i)\right), \tag{4.15a}$$

$$\psi_2(x_i) = CN^{-1}\ln N\left(\exp\left(\frac{2\vartheta h_i\mu_2}{\varepsilon_2}\right)\mathcal{G}_2^{l,N}(x_i)\right), \tag{4.15b}$$

and similarly we can define the mesh function $\vec{\boldsymbol{\psi}}(y_j)$ along $y-$direction. Now, for $N/16 \leq i \leq N/8$, $0 \leq j \leq N$, we consider the functions

$$\psi_1(x_i) = CN^{-1}\ln N\left(\mathcal{G}_1^{l,N}(x_i) + \exp\left(\frac{2\vartheta h_i\mu_2}{\varepsilon_2}\right)\mathcal{G}_2^{l,N}(x_i)\right), \tag{4.15c}$$

$$\psi_2(x_i) = CN^{-1}\ln N\left(\exp\left(\frac{2\vartheta h_i\mu_2}{\varepsilon_2}\right)\mathcal{G}_2^{l,N}(x_i)\right) + CN^{-1}((\tau_2 - x_i)\varepsilon_1^{-1} + 1), \tag{4.15d}$$

and analogously we can define the mesh function $\vec{\boldsymbol{\psi}}(y_j)$ along $y-$direction.

For $N/8 \leq i \leq 3N/16$, $0 \leq j \leq N$ and $3N/16 \leq i \leq N/2$, we can choose the similar appropriate mesh functions, as we have for $0 \leq i \leq N/16$, $0 \leq j \leq N$ and $N/16 \leq i \leq N/8$, $0 \leq j \leq N$.

Now, constructing the barrier function

$$\vec{\boldsymbol{\Psi}}^{\pm}(x_i, y_j) = \vec{\boldsymbol{\psi}}(x_i) + \vec{\boldsymbol{\psi}}(y_j) \pm (\vec{\mathbf{W}}_l - \vec{\mathbf{w}}_l)(x_i, y_j), \quad \text{for } 0 \leq i \leq N/4,\ 0 \leq j \leq N,$$

by using Lemma 3.1, we can prove that $\vec{\boldsymbol{\Psi}}^{\pm}(x_i, y_j) \geq \vec{\mathbf{0}}$ for all $0 \leq i \leq N/4$, $0 \leq j \leq N$. Hence, it follows

$$|(\vec{\mathbf{W}}_l - \vec{\mathbf{w}}_l)(x_i, y_j)| \leq CN^{-1}\ln N, \ 0 \leq i \leq N/4,\ 0 \leq j \leq N. \tag{4.16}$$

From equations (4.13), (4.14) and (4.16), for the case $\tau_4 = \dfrac{\varepsilon_2}{\mu_2\vartheta}\ln N$, it has been established that it holds

$$|(\vec{\mathbf{W}}_l - \vec{\mathbf{w}}_l)(x_i, y_j)| \leq CN^{-1}\ln N.$$

Next, the case $\tau_2 = 1/8$, $\tau_3 = 3/16$, $\tau_4 = 1/4$, $\sigma_1 = 1/4$, $\sigma_2 = 1/8$, $\tau_1 = \dfrac{2\varepsilon_1}{\mu_1\vartheta}\ln N$ is considered; then, $\mu_2\varepsilon_2^{-1} \le C\ln N$ holds.

For $(x_i, y_j) \in (0, \tau_1] \times (0, 1)$ it holds $h_i \le C\varepsilon_1\mu_1^{-1}N^{-1}\ln N$, and therefore from the truncation error estimate given in (4.8), we can obtain

$$|\vec{\mathcal{L}}^{N,N}_{\vec{\varepsilon},\vec{\mu}}(\vec{\mathbf{W}}_l - \vec{\mathbf{w}}_l)(x_i, y_j)| \le CN^{-1}\ln N \begin{pmatrix} \varepsilon_1^{-1}\mu_1^2\mathcal{B}_1^l(x_{i-1}) + \varepsilon_2^{-1}\mu_2^2\mathcal{B}_2^l(x_{i-1}) \\ \varepsilon_2^{-1}\mu_2^2\mathcal{B}_2^l(x_{i-1}) \end{pmatrix}.$$

For $(x_i, y_j) \in [\tau_1, \tau_2] \times (0, 1)$, from (4.8) and Lemma 2.5, we have

$$|\vec{\mathcal{L}}^{N,N}_{\vec{\varepsilon},\vec{\mu}}(\vec{\mathbf{W}}_l - \vec{\mathbf{w}}_l)(x_i, y_j)| \le \begin{pmatrix} C\varepsilon_1^{-1}\mu_1^2\mathcal{B}_1^l(x_{i-1}) + CN^{-1}\ln N\varepsilon_2^{-1}\mu_2^2\mathcal{B}_2^l(x_{i-1}) \\ C\varepsilon_2^{-1}\mu_2^2\mathcal{B}_2^l(x_{i-1}) + CN^{-1}\ln N\varepsilon_2^{-1}\mu_2^2\mathcal{B}_2^l(x_{i-1}) \end{pmatrix}.$$

Similarly, for $(x_i, y_j) \in [\tau_2, 1) \times (0, 1)$, from (4.8) and Lemma 2.5, we can deduce

$$|\vec{\mathcal{L}}^{N,N}_{\vec{\varepsilon},\vec{\mu}}(\vec{\mathbf{W}}_l - \vec{\mathbf{w}}_l)(x_i, y_j)| \le \begin{pmatrix} C\varepsilon_1^{-1}\mu_1^2\mathcal{B}_1^l(x_{i-1}) + CN^{-1}\ln N\varepsilon_2^{-1}\mu_2^2\mathcal{B}_2^l(x_{i-1}) \\ C\varepsilon_2^{-1}\mu_2^2\mathcal{B}_2^l(x_{i-1}) + CN^{-1}\ln N\varepsilon_2^{-1}\mu_2^2\mathcal{B}_2^l(x_{i-1}) \end{pmatrix}.$$

Now we consider the suitable barrier function for the edge layer component

$$\vec{\boldsymbol{\Psi}}^{\pm}(x_i, y_j) = \vec{\boldsymbol{\psi}}(x_i) + \vec{\boldsymbol{\psi}}(y_j) \pm (\vec{\mathbf{W}}_l - \vec{\mathbf{w}}_l)(x_i, y_j) \text{ for } \ 0 \le i \le N/16,\ 0 \le j \le N,$$

where

$$\begin{aligned} \psi_1(x_i) &= CN^{-1}\ln N\bigg(\exp\left(\frac{2\vartheta h_i\mu_1}{\varepsilon_1}\right)\mathcal{G}_1^{l,N}(x_i) + \exp\left(\frac{2\vartheta h_i\mu_2}{\varepsilon_2}\right)\mathcal{G}_2^{l,N}(x_i)\bigg), \\ \psi_2(x_i) &= CN^{-1}\ln N\bigg(\exp\left(\frac{2\vartheta h_i\mu_2}{\varepsilon_2}\right)\mathcal{G}_2^{l,N}(x_i)\bigg), \end{aligned}$$

for $N/16 \le i \le N/8$, $0 \le j \le N$, and as

$$\begin{aligned} \psi_1(x_i) &= C\mathcal{G}_1^{l,N}(x_i) + CN^{-1}\ln N\exp\left(\frac{2\vartheta h_i\mu_2}{\varepsilon_2}\right)\mathcal{G}_2^{l,N}(x_i), \\ \psi_2(x_i) &= CN^{-1}\ln N\exp\left(\frac{2\vartheta h_i\mu_2}{\varepsilon_2}\right)\mathcal{G}_2^{l,N}(x_i) + CN^{-1}((\tau_2 - x_i)\varepsilon_2^{-1} + 1), \end{aligned}$$

and finally for $N/8 \le i \le N$, $0 \le j \le N$, as

$$\begin{aligned} \psi_1(x_i) &= C\mathcal{G}_1^{l,N}(x_i) + CN^{-1}\ln N\exp\left(\frac{2\vartheta h_i\mu_2}{\varepsilon_2}\right)\mathcal{G}_2^{l,N}(x_i), \\ \psi_2(x_i) &= C\mathcal{G}_1^{l,N}(x_i) + CN^{-1}\ln N\exp\left(\frac{2\vartheta h_i\mu_2}{\varepsilon_2}\right)\mathcal{G}_2^{l,N}(x_i), \end{aligned}$$

and, similarly, we can define the mesh function $\vec{\boldsymbol{\psi}}(y_j)$ along $y-$direction. Therefore, summarizing, from all previous estimates for each one of the cases, it holds

$$|(\vec{\mathbf{W}}_l - \vec{\mathbf{w}}_l)(x_i, y_j)| \le CN^{-1}\ln N.$$

which is the required result. □

In the same way we can prove similar bounds for the errors related to the other layer components $\vec{\mathbf{w}}_r$, $\vec{\mathbf{w}}_b$, and $\vec{\mathbf{w}}_t$ for **Case 2** defined in (2.1).

Lemma 4.5. *Let $\vec{\boldsymbol{w}}_k(x_i, y_j)$ be the true solution of* (2.4) *and $\vec{\boldsymbol{W}}_k(x_i, y_j)$ be the numerical solution of* (4.2)*. Then, for* **Case 3** *defined in (2.1), we have*

$$|\,\vec{\boldsymbol{W}}_k(x_i, y_j) - \vec{\boldsymbol{w}}_k(x_i, y_j)| \le CN^{-1}\ln N,\;\; k = l, b, r, t.$$

Proof. If $\tau_1 = 1/12$, $\tau_2 = 1/6$ and $\tau_3 = 1/4$, the proof follows by using standard methods for uniform meshes, because it holds $\mu_2\varepsilon_1^{-1} \le C\ln N$, $\varepsilon_2^{-1/2} \le C\ln N$ and $\mu_1^{-1} \le C\ln N$. Hence, by using (4.2) and Theorem 2.5, we have

$$\begin{aligned}\|\vec{\mathcal{L}}^{N,N}_{\vec{\varepsilon},\vec{\mu}}(\vec{\mathbf{W}}_l - \vec{\mathbf{w}}_l)\| &\le C\left[(h_i + h_{i+1})\left(\vec{\varepsilon}\left\|\frac{\partial^3\vec{\mathbf{s}}_l}{\partial x^3}\right\| + \vec{\boldsymbol{\mu}}\left\|\frac{\partial^2\vec{\mathbf{s}}_l}{\partial x^2}\right\|\right) + (k_j + k_{j+1})\left(\vec{\varepsilon}\left\|\frac{\partial^3\vec{\mathbf{s}}_l}{\partial y^3}\right\| + \vec{\boldsymbol{\mu}}\left\|\frac{\partial^2\vec{\mathbf{s}}_l}{\partial y^2}\right\|\right)\right]\\ &\le CN^{-1}\ln N.\end{aligned}$$

Further, if $\tau_1 = \dfrac{\varepsilon_1}{\mu_1\varepsilon}\ln N$, $\tau_2 = 1/6$, $\tau_3 = 1/4$, $\sigma_1 = 1/6$, $\sigma_2 = 1/12$, and we consider the intervals $(x_i, y_j) \in (\tau_1, \tau_2) \times (0, 1), (\tau_2, \tau_3) \times (0, 1), (\tau_3, 1 - \tau_3) \times (0, 1)$, $(1 - \tau_3, 1 - \sigma_1) \times (0, 1)$ or $(1 - \sigma_1, 1 - \sigma_2) \times (0, 1)$, then the truncation error satisfies

$$|\vec{\mathcal{L}}^{N,N}_{\vec{\varepsilon},\vec{\mu}}(\vec{\mathbf{W}}_l - \vec{\mathbf{w}}_l)(x_i, y_j)| \le CN^{-1}\begin{pmatrix}\varepsilon_2^{-1/2}\mathcal{B}^l_2(x_{i-1})\\ \varepsilon_2^{-1/2}\mathcal{B}^l_2(x_{i-1})\end{pmatrix} \le CN^{-1}\ln N.$$

On the other hand, when $(x_i, y_j) \in (0, \tau_1] \times (0, 1)$ or $[1 - \sigma_2, 1] \times (0, 1)$, it follows

$$|\vec{\mathcal{L}}^{N,N}_{\vec{\varepsilon},\vec{\mu}}(\vec{\mathbf{W}}_l - \vec{\mathbf{w}}_l)(x_i, y_j)| \le CN^{-1}\ln N\varepsilon_1\mu_1^{-1}\begin{pmatrix}\mu_1^3\varepsilon_1^{-2} + \varepsilon_2^{-1/2}\\ \mu_1\varepsilon_1^{-1} + \varepsilon_2^{-1/2}\end{pmatrix} \le CN^{-1}\ln N.$$

In the case that $\tau_2 = \dfrac{2\varepsilon_1}{\mu_2\vartheta}$, $\tau_1 = 1/12$, $\tau_3 = 1/4$, $\sigma_1 = 1/6$ or $\sigma_2 = 1/12$, and we consider the intervals $(x_i, y_j) \in (\tau_2, \tau_3) \times (0, 1)$, $(\tau_3, 1 - \tau_3) \times (0, 1)$, or $(1 - \tau_3, 1 - \sigma_1) \times (0, 1)$, we have

$$|\vec{\mathcal{L}}^{N,N}_{\vec{\varepsilon},\vec{\mu}}(\vec{\mathbf{W}}_l - \vec{\mathbf{w}}_l)(x_i, y_j)| \le CN^{-1}\begin{pmatrix}\varepsilon_2^{-1/2}\mathcal{B}^l_2(x_{i-1})\\ \varepsilon_2^{-1/2}\mathcal{B}^l_2(x_{i-1})\end{pmatrix} \le CN^{-1}\ln N.$$

When $(x_i, y_j) \in (0, \tau_1) \times (0, 1)$, $(\tau_1, \tau_2] \times (0, 1)$, $(1 - \sigma_1, 1 - \sigma_2) \times (0, 1)$ or $(1 - \sigma_2, 1) \times (0, 1)$, it follows

$$|\vec{\mathcal{L}}^{N,N}_{\vec{\varepsilon},\vec{\mu}}(\vec{\mathbf{W}}_l - \vec{\mathbf{w}}_l)(x_i, y_j)| \le CN^{-1}\ln N\varepsilon_1\mu_2^{-1}\begin{pmatrix}\mu_1^3\varepsilon_1^{-2} + \varepsilon_2^{-1/2}\\ \mu_1\varepsilon_1^{-1} + \varepsilon_2^{-1/2}\end{pmatrix} \le CN^{-1}\ln N.$$

In the case that $\tau_3 = \sqrt{\dfrac{\varepsilon_2}{\Lambda\vartheta}}\ln N$, $\tau_2 = \dfrac{2\tau_3}{3}$ and $\tau_1 = \dfrac{\tau_2}{2}$, $\dfrac{\sqrt{\varepsilon_2}}{2} \le \sqrt{\varepsilon_1} < \sqrt{\varepsilon_2}$, it follows that $\tau_3 \le C\sqrt{\varepsilon_1}\ln N$.

To prove the error estimate on the region $(x_i, y_j) \in [\tau_3, 1 - \tau_3] \times (0, 1)$, we consider the following barrier functions:

$$\mathcal{B}^{l,N}_1(x_i) = \prod_{\iota=1}^{i}\left(1 + \left(\frac{\vartheta\mu}{2\varepsilon_1}\right)h_\iota\right)^{-1},\quad \mathcal{B}^{l,N}_2(x_i) = \prod_{\iota=1}^{i}\left(1 + \sqrt{\left(\frac{\Lambda\vartheta}{\varepsilon_2}\right)}h_\iota\right)^{-1}, \tag{4.17}$$

with $\mathcal{B}^{l,N}_1(x_0) = \mathcal{B}^{l,N}_2(x_0) = 1$. After applying Theorem 2.5, we can conclude that it holds

$$|(\vec{\mathbf{W}}_l - \vec{\mathbf{w}}_l)(x_i, y_j)| \le |\vec{\mathbf{W}}_l(x_i, y_j)| + |\vec{\mathbf{w}}_l(x_i, y_j)| \le C\mathcal{B}^{l,N}_2(\tau_3) + C\mathcal{B}^{l,N}_2(\tau_3) \le CN^{-1}.$$

To get appropriate bounds for the error in the region $(x_i, y_j) \in (\tau_1, \tau_2) \times (0,1)$, $(\tau_2, \tau_3) \times (0,1)$, $(1-\tau_3, 1-\sigma_1) \times (0,1)$ or $(1-\sigma_1, 1-\sigma_2) \times (0,1)$ and $(h_i + h_{i+1}) \le N^{-1} \ln N \sqrt{\varepsilon_2}$, we take into account that now the truncation error satisfies

$$|\vec{\mathcal{L}}^{N,N}_{\vec{\varepsilon},\vec{\mu}}(\vec{\mathbf{W}}_l - \vec{\mathbf{w}}_l)(x_i, y_j)| \le CN^{-1} \ln N \sqrt{\varepsilon_2} \begin{pmatrix} \mu_1^3 \varepsilon_1^{-2} + \varepsilon_2^{-1/2} \\ \mu_1 \varepsilon_1^{-1} + \varepsilon_2^{-1/2} \end{pmatrix},$$

and therefore, using the suitable barrier functions, we can deduce that

$$|\vec{\mathcal{L}}^{N,N}_{\vec{\varepsilon},\vec{\mu}}(\vec{\mathbf{W}}_l - \vec{\mathbf{w}}_l)(x_i, y_j)| \le CN^{-1} \ln N.$$

In the cases where (x_i, y_j) belongs to either $(0, \tau_1) \times (0,1)$ or $(1-\sigma_2, 1) \times (0,1)$, we have $h_i + h_{i+1} \le CN^{-1} \ln N \sqrt{\varepsilon_2}$, following the same approach as mentioned earlier, we obtain the required bounds.

Finally, assuming that $\tau_1 = \dfrac{\varepsilon_1}{\mu_1 \alpha} \ln N$, $\tau_2 = \dfrac{2\varepsilon_1}{\mu_2 \vartheta} \ln N$ and $\tau_3 = \sqrt{\dfrac{\varepsilon_2}{\Lambda \vartheta}} \ln N$, in cases where $(x_i, y_j) \in [\tau_3, 1-\tau_3] \times (0,1)$ or $(0, \tau_1] \times (0,1)$, or $(1-\sigma_2, 1) \times (0,1)$, we can obtain the required bounds by using a method similar to that used in the corresponding intervals of the previous cases. Also, when (x_i, y_j) is within either $(\tau_1, \tau_2) \times (0,1)$, $(\tau_2, \tau_3) \times (0,1)$, $(1-\tau_3, 1-\sigma_1) \times (0,1)$ or $(1-\sigma_1, 1-\sigma_2) \times (0,1)$, we have $h_i + h_{i+1} \le CN^{-1} \ln N \mu_1^{-1} \varepsilon_1$. Therefore, from all previous estimates, we obtain

$$|\vec{\mathcal{L}}^{N,N}_{\vec{\varepsilon},\vec{\mu}}(\vec{\mathbf{W}}_l - \vec{\mathbf{w}}_l)(x_i, y_j)| \le CN^{-1} \ln N.$$

So, by using Lemma 3.1, we obtain

$$|\vec{\mathbf{W}}_l(x_i, y_j) - \vec{\mathbf{w}}_l(x_i, y_j)| \le CN^{-1}, \tag{4.18}$$

which is the required result. □

Finally, we analyze when **Case 4** defined in (2.1) holds.

Lemma 4.6. *Let $\vec{\boldsymbol{w}}_k(x_i, y_j)$ be the true solution of* (2.4) *and $\vec{\boldsymbol{W}}_k(x_i, y_j)$ be the numerical solution of* (4.2)*; then, for* **Case 4** *defined in (2.1), the error associated to the boundary layer functions satisfies*

$$|\vec{\boldsymbol{W}}_k(x_i, y_j) - \vec{\boldsymbol{w}}_k(x_i, y_j)| \le CN^{-1} \ln N, \quad k = l, b, r, t.$$

Proof. Using the similar argument shown in Lemmas 4.2, 4.4 and 4.5, we can prove the error estimates for the case $\vartheta {\mu_1}^2 < \Lambda \varepsilon_1 \le \Lambda \varepsilon_2 < \vartheta {\mu_2}^2$. □

The last step concerns with the analysis of the error associated to the corner components. As before, we only consider the corner component $\vec{\mathbf{s}}_{lb}$ and similarly we can proceed for the other ones. Again, we must consider the four cases defined in (2.1).

4.2. Error estimates for the corner layer components

Lemma 4.7. *Let $\vec{\boldsymbol{s}}_k(x_i, y_j)$ be the true solution of* (2.5) *and $\vec{\boldsymbol{S}}_k(x_i, y_j)$ the numerical solution of (4.3), for $k = lb, br, rt, lt$. Then, for* **Case 1** *defined in (2.1), we have*

$$|\vec{\boldsymbol{S}}_k(x_i, y_j) - \vec{\boldsymbol{s}}_k(x_i, y_j)| \le CN^{-1} \ln N, k = lb, br, rt, lt.$$

Proof. If $\tau_1 = 1/8$, $\tau_2 = 1/4$, the proof can be obtained by applying standard methods (on uniform meshes) by taking into consideration that $\varepsilon_1^{-1/2} \le C \ln N$ and $\varepsilon_2^{-1/2} \le C \ln N$. Hence, by using (4.3) and Theorem 2.6, we have

$$\begin{aligned} \|\vec{\mathcal{L}}^{N,N}_{\vec{\varepsilon},\vec{\mu}}(\vec{\mathbf{S}}_{lb} - \vec{\mathbf{s}}_{lb})\| &\le C\left[(h_i + h_{i+1})\left(\vec{\boldsymbol{\varepsilon}}\left\|\frac{\partial^3 \vec{\mathbf{s}}_{lb}}{\partial x^3}\right\| + \vec{\boldsymbol{\mu}}\left\|\frac{\partial^2 \vec{\mathbf{s}}_{lb}}{\partial x^2}\right\|\right) + (k_j + k_{j+1})\left(\vec{\boldsymbol{\varepsilon}}\left\|\frac{\partial^3 \vec{\mathbf{s}}_{lb}}{\partial y^3}\right\| + \vec{\boldsymbol{\mu}}\left\|\frac{\partial^2 \vec{\mathbf{s}}_{lb}}{\partial y^2}\right\|\right)\right] \\ &\le CN^{-1} \ln N. \end{aligned}$$

Further, if $\tau_1 = \sqrt{\frac{\varepsilon_1}{\Lambda\vartheta}}\ln N$, $\tau_2 = 1/4$, and we consider the intervals $(x_i, y_j) \in (\tau_2, 1-\tau_2)\times(0,1)$, $(\tau_1,\tau_2)\times(0,1)$, $(0,\tau_1)\times(\tau_2,1)$, $(1-\tau_1,1)\times(\tau_2,1)$ or $(1-\tau_2,1-\tau_1)\times(0,1)$, then the truncation error satisfies

$$|\vec{\mathcal{L}}^{N,N}_{\vec{\varepsilon},\vec{\mu}}(\vec{\mathbf{S}}_{lb} - \vec{\mathbf{s}}_{lb})(x_i, y_j)| \leq CN^{-1}\begin{pmatrix}\varepsilon_2^{-1/2}(\mathcal{B}_2^l(x_{i-1})\mathcal{B}_2^b(y_{j-1}))\\ \varepsilon_2^{-1/2}(\mathcal{B}_2^l(x_{i-1})\mathcal{B}_2^b(y_{j-1}))\end{pmatrix} \leq CN^{-1}\ln N.$$

On the other hand, when $(x_i, y_j) \in (0,\tau_1]\times(0,\tau_2]$ or $[1-\tau_1,1)\times(0,\tau_2]$, it holds

$$|\vec{\mathcal{L}}^{N,N}_{\vec{\varepsilon},\vec{\mu}}(\vec{\mathbf{S}}_{lb} - \vec{\mathbf{s}}_{lb})(x_i, y_j)| \leq CN^{-1}\ln N\sqrt{\varepsilon_1}\begin{pmatrix}\varepsilon_1^{-1/2}+\varepsilon_2^{-1/2}\\ \varepsilon_2^{-1/2}+\varepsilon_2^{-1/2}\end{pmatrix} \leq CN^{-1}\ln N.$$

In the case that $\tau_2 = \sqrt{\frac{\varepsilon_2}{\Lambda\vartheta}}\ln N$ and $\tau_1 = \frac{\tau_2}{2}$, $\frac{\sqrt{\varepsilon_2}}{2} \leq \sqrt{\varepsilon_1} < \sqrt{\varepsilon_2}$, it follows $\tau_2 \leq C\sqrt{\varepsilon_1}\ln N$.
In order to establish the error estimate for the region $(x_i, y_j) \in [\tau_2, 1-\tau_2]\times(0,1)$, $(0,\tau_1]\times[\tau_2,1)$, $[1-\tau_2,1)\times[\tau_2,1)$, we examine the appropriate barrier function described in Lemma 4.2 and defined in equation (4.6). As a result, we can deduce that

$$|(\vec{\mathbf{S}}_{lb} - \vec{\mathbf{s}}_{lb})(x_i, y_j)| \leq |\vec{\mathbf{S}}_{lb}(x_i, y_j)| + |\vec{\mathbf{s}}_{lb}(x_i, y_j)| \leq C\min\{\mathcal{G}_1^{l,N}(\tau_2), \mathcal{G}_2^{b,N}(\tau_2)\} \leq CN^{-1}.$$

In order to obtain suitable bounds for the error in the region $(x_i, y_j) \in (\tau_1,\tau_2)\times(0,\tau_2)$ or $(1-\tau_2,1-\tau_1)\times(0,\tau_2)$, along with the conditions $(h_i + h_{i+1}) \leq N^{-1}\ln N\sqrt{\varepsilon_1}$ and $(k_j + k_{j+1}) \leq N^{-1}\ln N\sqrt{\varepsilon_1}$, the following holds

$$|\vec{\mathcal{L}}^{N,N}_{\vec{\varepsilon},\vec{\mu}}(\vec{\mathbf{S}}_{lb} - \vec{\mathbf{s}}_{lb})(x_i, y_j)| \leq CN^{-1}\ln N\sqrt{\varepsilon_1}\begin{pmatrix}\varepsilon_1^{-1/2}+\varepsilon_2^{-1/2}\\ \varepsilon_2^{-1/2}+\varepsilon_2^{-1/2}\end{pmatrix} \leq CN^{-1}\ln N.$$

In the cases where (x_i, y_j) belongs to either $(0,\tau_1)\times(0,\tau_2)$ or $(1-\tau_1,1)\times(0,\tau_2)$, we have $h_i + h_{i+1} \leq CN^{-1}\ln N\sqrt{\varepsilon_1}$, $k_j + k_{j+1} \leq CN^{-1}\ln N\sqrt{\varepsilon_1}$. By using a similar methodology as described previously, we can prove the necessary bounds.
Finally, assuming that $\tau_1 = \sqrt{\frac{\varepsilon_1}{\Lambda\vartheta}}\ln N$ and $\tau_2 = \sqrt{\frac{\varepsilon_2}{\Lambda\vartheta}}\ln N$, and in cases where $(x_i, y_j) \in [\tau_2, 1-\tau_2]\times(0,1)$ or $(0,\tau_1]\times(0,\tau_2)$, or $(1-\tau_1,1)\times(0,\tau_2)$, we can obtain the required bounds by using a method similar to that used in the corresponding intervals of the previous cases. When (x_i, y_j) is within either $(\tau_1,\tau_2)\times(0,1)$ or $(1-\tau_2,1-\tau_1)\times(0,1)$ or $(0,\tau_1)\times(\tau_2,1)$ or $(1-\tau_1,1)\times(\tau_2,1)$, we have $h_i+h_{i+1} \leq CN^{-1}\tau_2 \leq CN^{-1}\ln N\sqrt{\varepsilon_2}$. Therefore, from all previous estimates, we obtain

$$|\vec{\mathcal{L}}^{N,N}_{\vec{\varepsilon},\vec{\mu}}(\vec{\mathbf{S}}_{lb} - \vec{\mathbf{s}}_{lb})(x_i, y_j)| \leq CN^{-1}\ln N.$$

So, by using Lemma 3.1, we obtain

$$|\vec{\mathbf{S}}_{lb}(x_i, y_j) - \vec{\mathbf{s}}_{lb}(x_i, y_j)| \leq CN^{-1}, \tag{4.19}$$

which is the required result. □

Analogously, we can derive similar bounds for the errors related to the other corner layer components $\vec{\mathbf{s}}_{br}$, $\vec{\mathbf{s}}_{rt}$, and $\vec{\mathbf{s}}_{lt}$ for **Case 1** defined in (2.1).

Lemma 4.8. *Let $\vec{\mathbf{s}}_k(x_i, y_j)$ be the true solution of* (2.5) *and $\vec{\boldsymbol{S}}_k(x_i, y_j)$ the numerical solution of (4.3), for $k = lb, br, rt, lt$. Then, for* **Case 2** *defined in (2.1), we have*

$$|\vec{\boldsymbol{S}}_k(x_i, y_j) - \vec{\boldsymbol{s}}_k(x_i, y_j)| \leq CN^{-1}\ln N, \quad k = lb, br, rt, lt.$$

Proof. Here, we present the specific details related to the corner layer component $\vec{\mathbf{s}}_{lb}$ exclusively. The same procedure can be followed for the remaining corner layer components. According to the findings mentioned in Theorem 2.6, the truncation error for the singular component $\vec{\mathbf{s}}_{lb}$ can be estimated as follows:

$$\|\vec{\mathcal{L}}^{N,N}_{\vec{\varepsilon},\vec{\mu}}(\vec{\mathbf{S}}_{lb}-\vec{\mathbf{s}}_{lb})\| \leq C\Bigg[(h_i+h_{i+1})\left(\vec{\varepsilon}\left\|\frac{\partial^3\vec{\mathbf{s}}_{lb}}{\partial x^3}\right\|+\vec{\mu}\left\|\frac{\partial^2\vec{\mathbf{s}}_{lb}}{\partial x^2}\right\|\right)+(k_j+k_{j+1})\left(\vec{\varepsilon}\left\|\frac{\partial^3\vec{\mathbf{s}}_{lb}}{\partial y^3}\right\|+\vec{\mu}\left\|\frac{\partial^2\vec{\mathbf{s}}_{lb}}{\partial y^2}\right\|\right)\Bigg]. \tag{4.20}$$

If $\tau_1 = 1/15$, $\tau_2 = 1/8$, $\tau_3 = 3/16$, $\tau_4 = 1/4$, $\sigma_2 = 1/8$, and $\sigma_1 = 1/4$, the proof can be obtained easily by applying standard methods (on uniform meshes) by taking into consideration that $\mu_2\varepsilon_1^{-1} \leq C\ln N$, $\mu_2\varepsilon_2^{-1} \leq C\ln N$, and $\mu_1{}^{-1} \leq C\ln N$. Then, using Theorem 2.6 in (4.20), the truncation error satisfies

$$|\vec{\mathcal{L}}^{N,N}_{\vec{\varepsilon},\vec{\mu}}(\vec{\boldsymbol{S}}_{lb}-\vec{\boldsymbol{s}}_{lb})| \leq CN^{-1}\begin{pmatrix}\mu_1{}^3\varepsilon_1^{-2}(\mathcal{B}_1^l(x_{i-1})\mathcal{B}_1^b(y_{j-1}))+\mu_2^3\varepsilon_2^{-2}(\mathcal{B}_2^l(x_{i-1})\mathcal{B}_2^b(y_{j-1}))\\ \mu_2^3\varepsilon_1^{-1}\varepsilon_2^{-1}(\mathcal{B}_1^l(x_{i-1})\mathcal{B}_1^b(y_{j-1}))+\mu_2^3\varepsilon_2^{-2}(\mathcal{B}_2^l(x_{i-1})\mathcal{B}_2^b(y_{j-1}))\end{pmatrix}. \tag{4.21}$$

Now, let us consider the appropriate barrier functions, as indicated in Lemma 4.4 and defined in equations (4.10), within the domain $\bar{\Omega}^{N,N}$. Therefore, by using Lemma 3.1, we can deduce that it holds

$$|(\vec{\mathbf{S}}_{lb}-\vec{\mathbf{s}}_{lb})(x_i,y_j)| \leq CN^{-1}\begin{pmatrix}\mu_1\varepsilon_1^{-1}+\mu_2\varepsilon_2^{-1}\\ \mu_2\varepsilon_1^{-1}\end{pmatrix} \leq CN^{-1}\ln N.$$

Next, from (4.3), when $\tau_4 = \dfrac{\varepsilon_2}{\mu_2\vartheta}\ln N$, for the grid points $\{(x_i,y_j), |\,(0<i,j<N)/(0<i,j<N/4)\}$, we have

$$\begin{aligned}|(S_{lb_1}-s_{lb_1})(x_i,y_j)| &\leq |S_{lb_1}(x_i,y_j)|+|s_{lb_1}(x_i,y_j)| \leq C\min(\mathcal{G}_1^{l,N}(x_i),\mathcal{G}_2^{b,N}(y_j))\\ &\leq C\min(\mathcal{G}_1^{l,N}(\tau_2),\mathcal{G}_2^{b,N}(\tau_2)) \leq CN^{-1}.\end{aligned} \tag{4.22}$$

Analogously, we can deduce

$$|(S_{lb_2}-s_{lb_2})(x_i,y_j)| \leq CN^{-1}. \tag{4.23}$$

When $\tau_4 = \dfrac{\varepsilon_2}{\mu_2\vartheta}\ln N$, there are two different cases: $2\varepsilon_1 \geq \varepsilon_2$ and $2\varepsilon_1 < \varepsilon_2$, respectively. In the case $\dfrac{\varepsilon_2}{2} \leq \varepsilon_1 \leq \varepsilon_2$, for $3N/16 \leq i \leq N/4$, $0 \leq j \leq N/4$, $\tau_4 \leq \dfrac{2\varepsilon_1}{\mu_1\vartheta}\ln N$, it holds

$$|\vec{\mathcal{L}}^{N,N}_{\vec{\varepsilon},\vec{\mu}}(\vec{\mathbf{S}}_{lb}-\vec{\mathbf{s}}_{lb})(x_i,y_j)| \leq CN^{-1}\ln N\begin{pmatrix}\varepsilon_1^{-1}\mu_1^2(\mathcal{B}_1^l(x_{i-1})\mathcal{B}_1^b(y_{j-1}))+\varepsilon_2^{-1}\mu_2^2(\mathcal{B}_2^l(x_{i-1})\mathcal{B}_2^b(y_{j-1}))\\ \varepsilon_2^{-1}\mu_2^2(\mathcal{B}_2^l(x_{i-1})\mathcal{B}_2^b(y_{j-1}))\end{pmatrix}.$$

In the second case, $\varepsilon_2 > 2\varepsilon_1$, for $3N/16 \leq i \leq N/4$, $0 \leq j \leq N/4$, we have

$$|\vec{\mathcal{L}}^{N,N}_{\vec{\varepsilon},\vec{\mu}}(\vec{\mathbf{S}}_{lb}-\vec{\mathbf{s}}_{lb})(x_i,y_j)| \leq \begin{pmatrix}C\varepsilon_1^{-1}\mu_1^2(\mathcal{B}_1^l(x_{i-1})\mathcal{B}_1^b(y_{j-1}))+CN^{-1}\ln N\varepsilon_2^{-1}\mu_2^2(\mathcal{B}_2^l(x_{i-1})\mathcal{B}_2^b(y_{j-1}))\\ C\varepsilon_2^{-1}\mu^2(\mathcal{B}_1^l(x_{i-1})\mathcal{B}_1^b(y_{j-1}))+CN^{-1}\ln N\varepsilon_2^{-1}\mu_2^2(\mathcal{B}_2^l(x_{i-1})\mathcal{B}_2^b(y_{j-1}))\end{pmatrix}.$$

For $N/8 \leq i \leq 3N/16$, $0 \leq j \leq N/4$, $\tau_3 \leq \dfrac{\varepsilon_2}{\mu_1\vartheta}\ln N$, it follows that the local error satisfies

$$|\vec{\mathcal{L}}^{N,N}_{\vec{\varepsilon},\vec{\mu}}(\vec{\mathbf{S}}_{lb}-\vec{\mathbf{s}}_{lb})(x_i,y_j)| \leq CN^{-1}\ln N\begin{pmatrix}C\varepsilon_1^{-1}\mu_1^2(\mathcal{B}_1^l(x_{i-1})\mathcal{B}_1^b(y_{j-1}))+\varepsilon_2^{-1}\mu_2^2(\mathcal{B}_2^l(x_{i-1})\mathcal{B}_2^b(y_{j-1}))\\ \varepsilon_2^{-1}\mu_2^2(\mathcal{B}_2^l(x_{i-1})\mathcal{B}_2^b(y_{j-1}))\end{pmatrix}.$$

Similarly, we can define the local error for the region $0 \leq i \leq N/8$, $0 \leq j \leq N/4$. For $0 \leq i \leq N/4$, $0 \leq j \leq N/4$, we consider the suitable barrier functions $\vec{\boldsymbol{\psi}}(x_i)$, $\vec{\boldsymbol{\psi}}(y_j)$ as mentioned in Lemma 4.4 of the equations (4.15). Hence, it follows

$$|(\vec{\mathbf{S}}_{lb}-\vec{\mathbf{s}}_{lb})(x_i,y_j)| \leq CN^{-1}\ln N,\ 0 \leq i \leq N/4,\ 0 \leq j \leq N/4. \tag{4.24}$$

From equations (4.22), (4.23) and (4.24), for the case $\tau_4 = \dfrac{\varepsilon_2}{\mu_2\vartheta}\ln N$, it has been proven that

$$|(\vec{\mathbf{S}}_{lb} - \vec{\mathbf{s}}_{lb})(x_i, y_j)| \leq CN^{-1}\ln N.$$

Next, the case $\tau_2 = 1/8$, $\tau_3 = 3/16$, $\tau_4 = 1/4$, $\sigma_2 = 1/8$, and $\sigma_1 = 1/4$, $\tau_1 = \dfrac{\varepsilon_1}{\mu_1\vartheta}\ln N$ is considered; then, $\mu_2\varepsilon_1^{-1} \leq C\ln N$ holds. For $(x_i, y_j) \in (0, \tau_1] \times (0, \tau_4]$, $h_i, k_j \leq C\varepsilon_1\mu_1^{-1}N^{-1}\ln N$. Therefore, by using the truncation error estimate (4.20), we obtain

$$|\vec{\mathcal{L}}^{N,N}_{\vec{\varepsilon},\vec{\mu}}(\vec{\mathbf{S}}_{lb} - \vec{\mathbf{s}}_{lb})(x_i, y_j)| \leq CN^{-1}\ln N \begin{pmatrix} \varepsilon_1^{-1}\mu_1^2(\mathcal{B}_1^l(x_{i-1})\mathcal{B}_1^b(y_{j-1})) + \varepsilon_2^{-1}\mu_2^2(\mathcal{B}_2^l(x_{i-1})\mathcal{B}_2^b(y_{j-1})) \\ \varepsilon_2^{-1}\mu_2^2(\mathcal{B}_2^l(x_{i-1})\mathcal{B}_2^b(y_{j-1})) \end{pmatrix}.$$

For $(x_i, y_j) \in [\tau_1, \tau_2] \times (0, \tau_4]$, from (4.20) and Lemma 2.6, we have

$$|\vec{\mathcal{L}}^{N,N}_{\vec{\varepsilon},\vec{\mu}}(\vec{\mathbf{S}}_{lb} - \vec{\mathbf{s}}_{lb})(x_i, y_j)| \leq \begin{pmatrix} C\varepsilon_1^{-1}\mu_1^2(\mathcal{B}_1^l(x_{i-1})\mathcal{B}_1^b(y_{j-1})) + CN^{-1}\ln N\varepsilon_2^{-1}\mu_2^2(\mathcal{B}_2^l(x_{i-1})\mathcal{B}_2^b(y_{j-1})) \\ C\varepsilon_2^{-1}\mu_2^2(\mathcal{B}_2^l(x_{i-1})\mathcal{B}_2^b(y_{j-1})) + CN^{-1}\ln N\varepsilon_2^{-1}\mu_2^2(\mathcal{B}_2^l(x_{i-1})\mathcal{B}_2^b(y_{j-1})) \end{pmatrix}.$$

For $(x_i, y_j) \in [\tau_2, \tau_3]) \times (0, \tau_2]$, from (4.20) and Lemma 2.6, we can deduce

$$|\vec{\mathcal{L}}^{N,N}_{\vec{\varepsilon},\vec{\mu}}(\vec{\mathbf{S}}_{lb} - \vec{\mathbf{s}}_{lb})(x_i, y_j)| \leq \begin{pmatrix} C\varepsilon_1^{-1}\mu_1^2(\mathcal{B}_1^l(x_{i-1})\mathcal{B}_1^b(y_{j-1})) + CN^{-1}\ln N\varepsilon_2^{-1}\mu_2^2(\mathcal{B}_2^l(x_{i-1})\mathcal{B}_2^b(y_{j-1})) \\ C\varepsilon_2^{-1}\mu_2^2(\mathcal{B}_2^l(x_{i-1})\mathcal{B}_2^b(y_{j-1})) + CN^{-1}\ln N\varepsilon_2^{-1}\mu_2^2(\mathcal{B}_2^l(x_{i-1})\mathcal{B}_2^b(y_{j-1})) \end{pmatrix}.$$

Similarly, we can define the error for the region $(\tau_3, 1) \times (0, 1)$. Taking the suitable barrier function for the corner layer component for $0 \leq i \leq N$, $0 \leq j \leq N$, it has been deduced for each of the cases that it holds

$$|(\vec{\mathbf{S}}_{lb} - \vec{\mathbf{s}}_{lb})(x_i, y_j)| \leq CN^{-1}\ln N,$$

which is the required result. □

In the same way we can prove similar bounds for the errors related to the other corner layer components $\vec{\mathbf{s}}_{br}$, $\vec{\mathbf{s}}_{rt}$, and $\vec{\mathbf{s}}_{lt}$ for **Case 2** defined in (2.1).

Lemma 4.9. *Let $\vec{\mathbf{s}}_k(x_i, y_j)$ be the true solution of* (2.5) *and $\vec{\boldsymbol{S}}_k(x_i, y_j)$ the numerical solution of (4.3), for $k = lb, br, rt, lt$. Then, for* **Case 3** *defined in (2.1), we have*

$$|\vec{\boldsymbol{S}}_k(x_i, y_j) - \vec{\boldsymbol{s}}_k(x_i, y_j)| \leq CN^{-1}\ln N, \quad k = lb, br, rt, lt.$$

Proof. Here, we present the specific details related to the corner layer component $\vec{\mathbf{s}}_{lb}$ exclusively. The same procedure can be followed for the remaining corner layer components. According to the findings mentioned in Theorem 2.6, the truncation error for the singular component $\vec{\mathbf{s}}_{lb}$ can be estimated as follows:

$$\|\vec{\mathcal{L}}^{N,N}_{\vec{\varepsilon},\vec{\mu}}(\vec{\mathbf{S}}_{lb} - \vec{\mathbf{s}}_{lb})\| \leq C\left[(h_i + h_{i+1})\left(\vec{\boldsymbol{\varepsilon}}\left\|\frac{\partial^3\vec{\mathbf{s}}_{lb}}{\partial x^3}\right\| + \vec{\boldsymbol{\mu}}\left\|\frac{\partial^2\vec{\mathbf{s}}_{lb}}{\partial x^2}\right\|\right) + (k_j + k_{j+1})\left(\vec{\boldsymbol{\varepsilon}}\left\|\frac{\partial^3\vec{\mathbf{s}}_{lb}}{\partial y^3}\right\| + \vec{\boldsymbol{\mu}}\left\|\frac{\partial^2\vec{\mathbf{s}}_{lb}}{\partial y^2}\right\|\right)\right]. \tag{4.25}$$

If $\tau_1 = 1/12$, $\tau_2 = 1/6$, $\tau_3 = 1/4$, $\sigma_1 = 1/6$ and $\sigma_2 = 1/12$, the proof can be obtained easily by applying standard methods (on uniform meshes) by taking into consideration that $\mu_1\varepsilon_1^{-1} \leq C\ln N$, $\varepsilon_2^{-1/2} \leq C\ln N$, and ${\mu_1}^{-1} \leq C\ln N$. Then, using Theorem 2.6 in (4.25), we have

$$|\vec{\mathcal{L}}^{N,N}_{\vec{\varepsilon},\vec{\mu}}(\vec{\boldsymbol{S}}_{lb} - \vec{\boldsymbol{s}}_{lb})| \leq CN^{-1} \begin{pmatrix} {\mu_1}^3\varepsilon_1^{-2}(\mathcal{B}_1^l(x_{i-1})\mathcal{B}_1^b(y_{j-1})) + \varepsilon_2^{-1/2}(\mathcal{B}_2^l(x_{i-1})\mathcal{B}_2^b(y_{j-1})) \\ \varepsilon_2^{-1/2}(\mathcal{B}_2^l(x_{i-1})\mathcal{B}_2^b(y_{j-1})) \end{pmatrix}. \tag{4.26}$$

Now, let us consider the appropriate barrier functions, as indicated in Lemma 4.4 and defined in equations (4.10), within the domain $\bar{\Omega}^{N,N}$. Therefore, by using Lemma 3.1, we can deduce that it holds

$$|(\vec{\mathbf{S}}_{lb} - \vec{\mathbf{s}}_{lb})(x_i, y_j)| \leq CN^{-1} \begin{pmatrix} \mu_1 \varepsilon_1^{-1} + \varepsilon_2^{-1/2} \\ \varepsilon_2^{-1/2} \end{pmatrix} \leq CN^{-1} \ln N.$$

Next, from (4.3), when $\tau_3 = \sqrt{\dfrac{\varepsilon_2}{\Lambda\vartheta}} \ln N$, for the grid points $\{(x_i, y_j), | (0 < i, j < N)/(0 < i, j < N/4)\}$, we have

$$\begin{aligned} |(S_{lb_1} - s_{lb_1})(x_i, y_j)| &\leq |S_{lb_1}(x_i, y_j)| + |s_{lb_1}(x_i, y_j)| \leq C \min(\mathcal{G}_1^{l,N}(x_i), \mathcal{G}_2^{b,N}(y_j)) \\ &\leq C \min(\mathcal{G}_1^{l,N}(\tau_3), \mathcal{G}_2^{b,N}(\tau_3)) \leq CN^{-1}. \end{aligned} \tag{4.27}$$

Analogously, we can deduce

$$|(S_{lb_2} - s_{lb_2})(x_i, y_j)| \leq CN^{-1}. \tag{4.28}$$

In the case $\tau_3 = \sqrt{\dfrac{\varepsilon_2}{\Lambda\vartheta}} \ln N$, for $N/6 \leq i \leq N/4$, $0 \leq j \leq N/4$, $\tau_3 \leq \sqrt{\dfrac{\varepsilon_2}{\Lambda\vartheta}} \ln N$. Hence, it holds

$$|\vec{\mathcal{L}}_{\vec{\varepsilon},\vec{\mu}}^{N,N} (\vec{\mathbf{S}}_{lb} - \vec{\mathbf{s}}_{lb})(x_i, y_j)| \leq CN^{-1} \ln N \begin{pmatrix} \varepsilon_1^{-1} \mu_1^2 (\mathcal{B}_1^l(x_{i-1}) \mathcal{B}_1^b(y_{j-1})) + \varepsilon_2^{-1/2} (\mathcal{B}_2^l(x_{i-1}) \mathcal{B}_2^b(y_{j-1})) \\ \varepsilon_2^{-1/2} (\mathcal{B}_2^l(x_{i-1}) \mathcal{B}_2^b(y_{j-1})) \end{pmatrix}.$$

For $N/12 \leq i \leq N/6$, $0 \leq j \leq N/4$, $\tau_2 \leq \dfrac{\varepsilon_1}{\mu_2 \vartheta} \ln N$, it follows that the truncation error satisfies

$$|\vec{\mathcal{L}}_{\vec{\varepsilon},\vec{\mu}}^{N,N} (\vec{\mathbf{S}}_{lb} - \vec{\mathbf{s}}_{lb})(x_i, y_j)| \leq CN^{-1} \ln N \begin{pmatrix} C\varepsilon_1^{-1} \mu_1^2 (\mathcal{B}_1^l(x_{i-1}) \mathcal{B}_1^b(y_{j-1})) + \varepsilon_2^{-1/2} (\mathcal{B}_2^l(x_{i-1}) \mathcal{B}_2^b(y_{j-1})) \\ \varepsilon_2^{-1/2} (\mathcal{B}_2^l(x_{i-1}) \mathcal{B}_2^b(y_{j-1})) \end{pmatrix}.$$

Similarly, we can define the local error for the region $0 \leq i \leq N/12$, $0 \leq j \leq N/4$. For $0 \leq i \leq N/4$, $0 \leq j \leq N/4$, we consider the suitable barrier functions $\vec{\boldsymbol{\psi}}(x_i)$, $\vec{\boldsymbol{\psi}}(y_j)$ as mentioned in Lemma 4.4 of the equations (4.15). Hence, it follows

$$|(\vec{\mathbf{S}}_{lb} - \vec{\mathbf{s}}_{lb})(x_i, y_j)| \leq CN^{-1} \ln N,\ 0 \leq i \leq N/4,\ 0 \leq j \leq N/4. \tag{4.29}$$

From equations (4.27), (4.28) and (4.29), for the case $\tau_3 = \sqrt{\dfrac{\varepsilon_2}{\Lambda\vartheta}} \ln N$, it has been proven that

$$|(\vec{\mathbf{S}}_{lb} - \vec{\mathbf{s}}_{lb})(x_i, y_j)| \leq CN^{-1} \ln N.$$

Next, the case $\tau_2 = 1/6$, $\tau_3 = 1/4$, $\sigma_1 = 1/6$, $\sigma_2 = 1/12$, and $\sigma_1 = 1/4$, $\tau_1 = \dfrac{\varepsilon_1}{\mu_1 \vartheta} \ln N$ is considered; then, $\mu_2 \varepsilon_1^{-1} \leq C \ln N$, $\varepsilon_2^{-1/2} \leq C \ln N$ holds. For $(x_i, y_j) \in (0, \tau_1] \times (0, \tau_3]$, $h_i, k_j \leq C\varepsilon_1 \mu_1^{-1} N^{-1} \ln N$. Therefore, by using the truncation error estimate (4.25), we obtain

$$|\vec{\mathcal{L}}_{\vec{\varepsilon},\vec{\mu}}^{N,N} (\vec{\mathbf{S}}_{lb} - \vec{\mathbf{s}}_{lb})(x_i, y_j)| \leq CN^{-1} \ln N \begin{pmatrix} \varepsilon_1^{-1} \mu_1^2 (\mathcal{B}_1^l(x_{i-1}) \mathcal{B}_1^b(y_{j-1})) + \varepsilon_2^{-1/2} (\mathcal{B}_2^l(x_{i-1}) \mathcal{B}_2^b(y_{j-1})) \\ \varepsilon_2^{-1/2} (\mathcal{B}_2^l(x_{i-1}) \mathcal{B}_2^b(y_{j-1})) \end{pmatrix}.$$

For $(x_i, y_j) \in [\tau_1, \tau_2] \times (0, \tau_3]$, from (4.25) and Theorem 2.6, we have

$$|\vec{\mathcal{L}}_{\vec{\varepsilon},\vec{\mu}}^{N,N} (\vec{\mathbf{S}}_{lb} - \vec{\mathbf{s}}_{lb})(x_i, y_j)| \leq \begin{pmatrix} C\varepsilon_1^{-1} \mu_1^2 (\mathcal{B}_1^l(x_{i-1}) \mathcal{B}_1^b(y_{j-1})) + CN^{-1} \ln N \varepsilon_2^{-1/2} (\mathcal{B}_2^l(x_{i-1}) \mathcal{B}_2^b(y_{j-1})) \\ C\varepsilon_2^{-1} \mu_2^2 (\mathcal{B}_2^l(x_{i-1}) \mathcal{B}_2^b(y_{j-1})) + CN^{-1} \ln N \varepsilon_2^{-1/2} (\mathcal{B}_2^l(x_{i-1}) \mathcal{B}_2^b(y_{j-1})) \end{pmatrix}.$$

For $(x_i, y_j) \in [\tau_2, \tau_3]) \times (0, \tau_3]$, from (4.25) and Theorem 2.6, we can deduce

$$|\vec{\mathcal{L}}^{N,N}_{\vec{\varepsilon},\vec{\mu}}(\vec{\mathbf{S}}_{lb} - \vec{\mathbf{s}}_{lb})(x_i, y_j)| \leq \begin{pmatrix} C\varepsilon_1^{-1}\mu_1^2(\mathcal{B}_1^l(x_{i-1})\mathcal{B}_1^b(y_{j-1})) + CN^{-1}\ln N\varepsilon_2^{-1/2}(\mathcal{B}_2^l(x_{i-1})\mathcal{B}_2^b(y_{j-1})) \\ C\varepsilon_2^{-1}\mu_2^2(\mathcal{B}_2^l(x_{i-1})\mathcal{B}_2^b(y_{j-1})) + CN^{-1}\ln N\varepsilon_2^{-1/2}(\mathcal{B}_2^l(x_{i-1})\mathcal{B}_2^b(y_{j-1})) \end{pmatrix}.$$

Similarly, we can define the error for the region $(\tau_3, 1) \times (0, 1)$. Taking the suitable barrier function for the corner layer component for $0 \leq i \leq N$, $0 \leq j \leq N$, it has been deduced for each of the cases that

$$|(\vec{\mathbf{S}}_{lb} - \vec{\mathbf{s}}_{lb})(x_i, y_j)| \leq CN^{-1}\ln N.$$

□

Finally, we analyze the fourth case, when **Case 4** defined in (2.1) holds.

Lemma 4.10. *Let $\vec{\boldsymbol{s}}_k(x_i, y_j)$ be the true solution of (2.5) and $\vec{\boldsymbol{S}}_k(x_i, y_j)$ be the numerical solution of (4.3); then, for* **Case 4** *defined in (2.1), the error associated to the corner layer functions satisfies*

$$|\vec{\boldsymbol{S}}_k(x_i, y_j) - \vec{\boldsymbol{s}}_k(x_i, y_j)| \leq CN^{-1}\ln N, \ k = lb, br, rt, lt.$$

Proof. Using the similar argument shown in Lemmas 4.7, 4.8 and 4.9, we can prove the error estimates for the case $\vartheta{\mu_1}^2 < \Lambda\varepsilon_1 \leq \Lambda\varepsilon_2 < \vartheta{\mu_2}^2$. □

Combining all previous results, we obtain the main result of the work.

Theorem 4.11. *Let $\vec{\boldsymbol{z}}(x_i, y_j)$ be the true solution of the continuous problem (1.1) and $\vec{\boldsymbol{Z}}(x_i, y_j)$ the numerical solution of (3.5) defined on the corresponding Shishkin mesh. Then, the error satisfies*

$$|\vec{\boldsymbol{Z}}(x_i, y_j) - \vec{\boldsymbol{z}}(x_i, y_j)| \leq CN^{-1}\ln N, \tag{4.30}$$

and therefore, the numerical scheme is an almost uniformly convergent method.

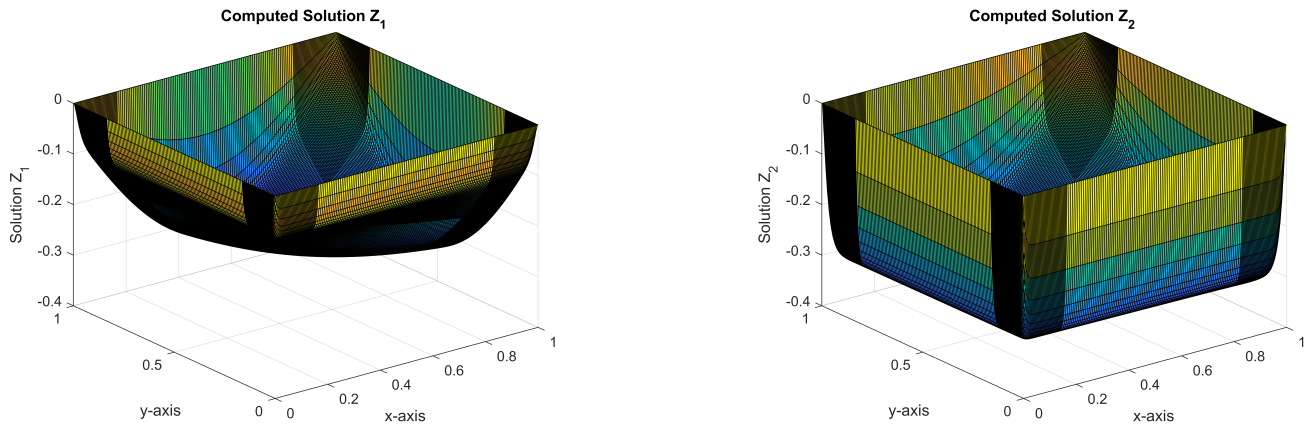
5. Numerical Experiments

In Section 3, we describe the implementation of the standard 5-point finite difference method using an S-mesh. To demonstrate the performance of the numerical method developed in the preceding sections, we focus on one example of the form given in (1.1). These problems are solved using MATLAB R2023a on a system with 32 GB of RAM and an i5 processor operating at 1.8 GHz. We observed that the computational time to obtain results is relatively high due to the complexity of the problem. However, to mitigate this, a sparse matrix representation is employed within the algorithm to reduce both computational time and memory usage. To solve the test problems, we have ordered the numerical solution in the following form:

$$Z = \begin{cases} Z_1(x_0, y_0), Z_1(x_1, y_0), \ldots, Z_1(x_N, y_0), \\ Z_1(x_0, y_1), Z_1(x_1, y_1), \ldots, Z_1(x_N, y_1), \\ Z_1(x_0, y_2), Z_1(x_1, y_2), \ldots, Z_1(x_N, y_2), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ Z_1(x_0, y_N), Z_1(x_1, y_N), \ldots, Z_1(x_N, y_N), \\ Z_2(x_0, y_0), Z_2(x_1, y_0), \ldots, Z_2(x_N, y_0), \\ Z_2(x_0, y_1), Z_2(x_1, y_1), \ldots, Z_2(x_N, y_1), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ Z_2(x_0, y_N), Z_2(x_1, y_N), \ldots, Z_2(x_N, y_N), \end{cases}$$

where $Z_1(x_0, y_i), Z_1(x_i, y_0), Z_1(x_N, y_i)$ and $Z_1(x_i, y_N)$, $i = 0, ..., N$, are calculated by using the boundary conditions. Then, the resulting linear system can be written as

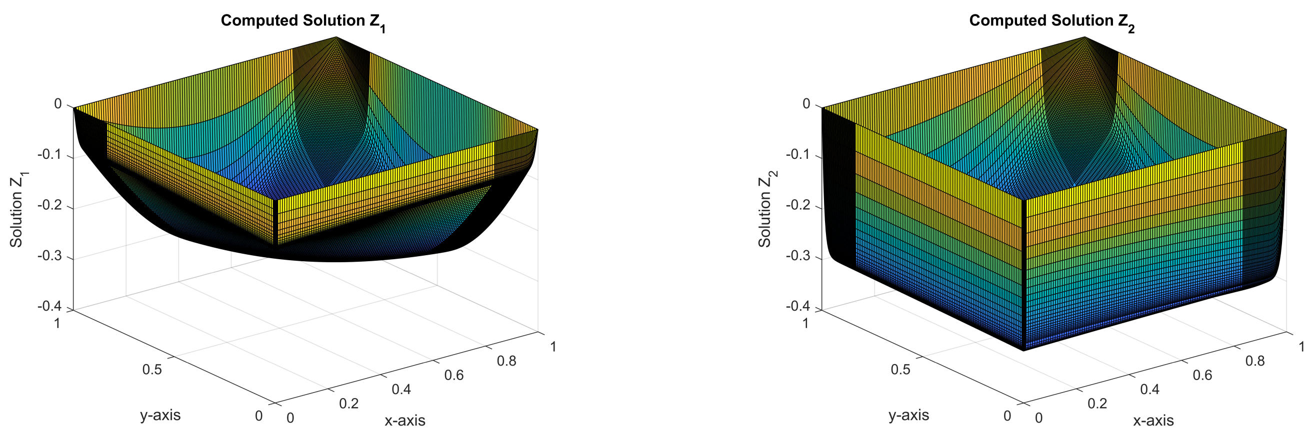
$$[A]_{(2(N+1)^2, 2(N+1)^2)}[Z]_{(2(N+1)^2, 1)} = [F]_{(2(N+1)^2, 1)},$$



(a) Surface graph of the numerical solution z_1;

(b) Surface graph of the numerical solution z_2;

Figure 1: When $\varepsilon_1 = 2^{-14}, \varepsilon_2 = 2^{-12}$, $\mu_1^2 = 2^{-18}$, $\mu_2^2 = 2^{-16}, N = 128$ for Example 5.1;



(a) Surface graph of the numerical solution z_1;

(b) Surface graph of the numerical solution z_2;

Figure 2: When $\varepsilon_1 = 2^{-18}, \varepsilon_2 = 2^{-16}$, $\mu_1^2 = 2^{-14}$, $\mu_2^2 = 2^{-12}, N = 128$ for Example 5.1;

and we solve this system by using MATLAB taking into account that the matrix A is sparse.

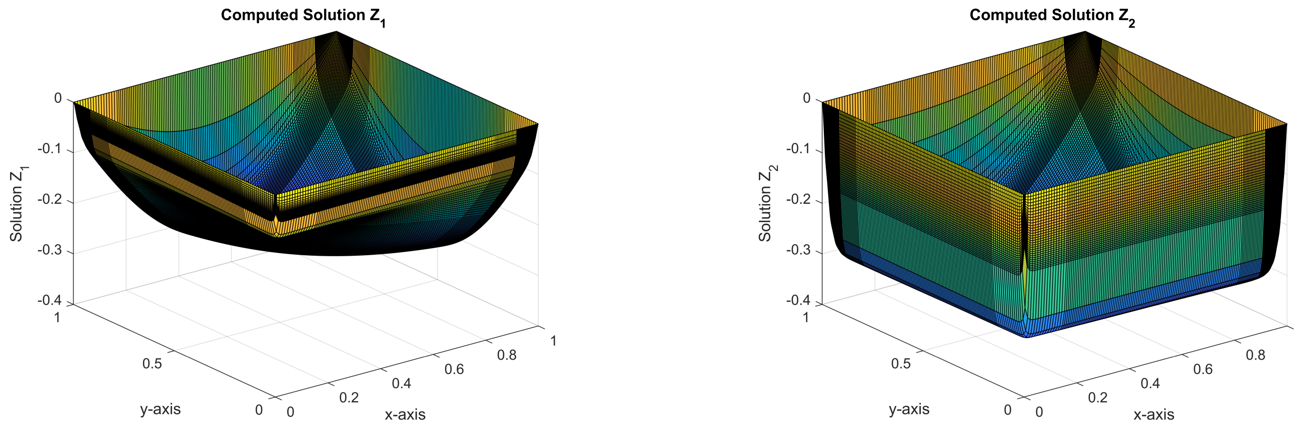
In this section we show the numerical results obtained by using our algorithm to solve a test problem, which is given in the following form:

Example 5.1.

$$\vec{\boldsymbol{\varepsilon}}\left(\frac{\partial^2 \vec{\boldsymbol{z}}}{\partial x^2} + \frac{\partial^2 \vec{\boldsymbol{z}}}{\partial y^2}\right) + \vec{\boldsymbol{\mu}}\left(\vec{\boldsymbol{A}}_1(x,y)\frac{\partial \vec{\boldsymbol{z}}}{\partial x} + \vec{\boldsymbol{A}}_2(x,y)\frac{\partial \vec{\boldsymbol{z}}}{\partial y}\right) - \vec{\boldsymbol{B}}(x,y)\vec{\boldsymbol{z}} = \vec{\boldsymbol{f}}(x,y), \quad \forall (x,y) \in \Omega,$$

where the boundary conditions as well as convection, reaction coefficients and source terms are given by

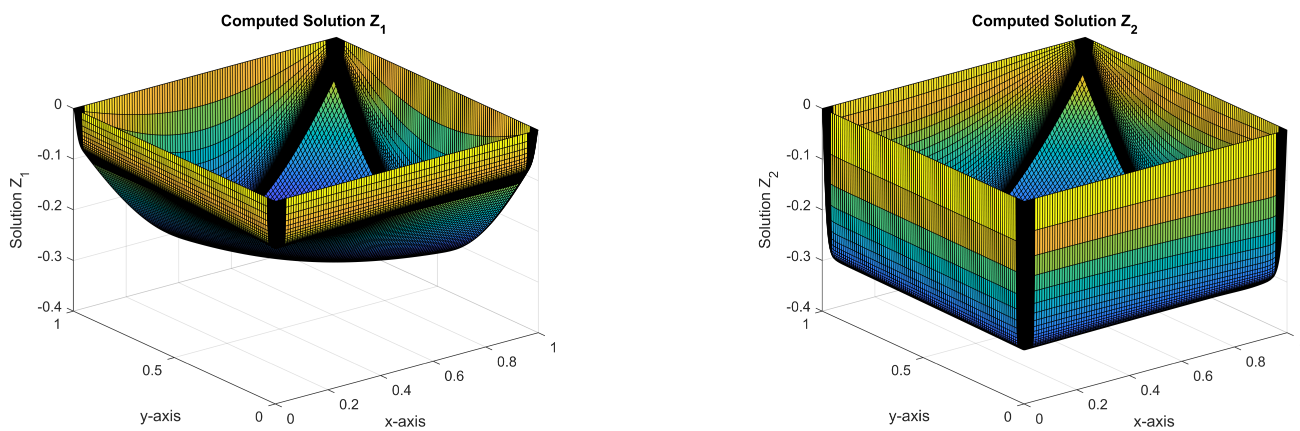
$$\begin{aligned}
&\vec{\boldsymbol{z}}(x,0) = \vec{\boldsymbol{z}}(x,1) = \vec{\boldsymbol{z}}(0,y) = \vec{\boldsymbol{z}}(1,y) = 0,\\
&\vec{\boldsymbol{A}}_1(x,y) = \begin{pmatrix} 1 + x^2y^2 & 0 \\ 0 & 2 - xy \end{pmatrix}, \; \vec{\boldsymbol{A}}_2(x,y) = \begin{pmatrix} 2 + \sin(x+y) & 0 \\ 0 & 2 - \cos(x+y) \end{pmatrix},\\
&\vec{\boldsymbol{B}}(x,y) = \begin{pmatrix} 3 + x + y & -1 - x^2y^2 \\ -1 - \exp(xy) & 3 + \exp(xy) \end{pmatrix}, \quad \vec{\boldsymbol{f}}(x,y) = \Big(\sin(\pi xy), \cos(\pi xy/2)\Big)^T.
\end{aligned}$$



(a) Surface graph of the numerical solution z_1;

(b) Surface graph of the numerical solution z_2;

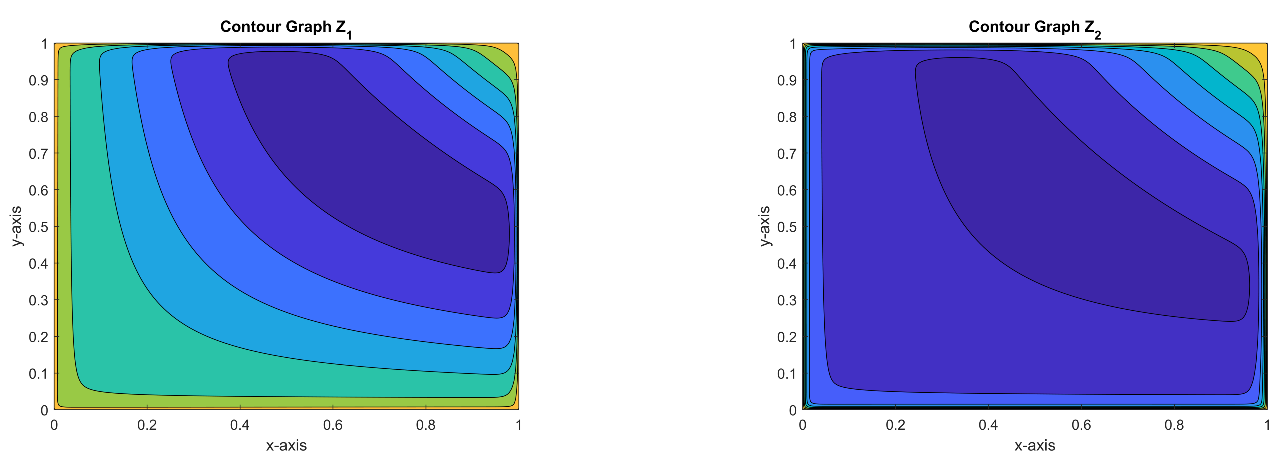
Figure 3: When $\varepsilon_1 = 2^{-18}, \varepsilon_2 = 2^{-12}$, $\mu_1^2 = 2^{-16}$, $\mu_2^2 = 2^{-14}$, $N = 128$ for Example 5.1;



(a) Surface graph of the numerical solution z_1;

(b) Surface graph of the numerical solution z_2;

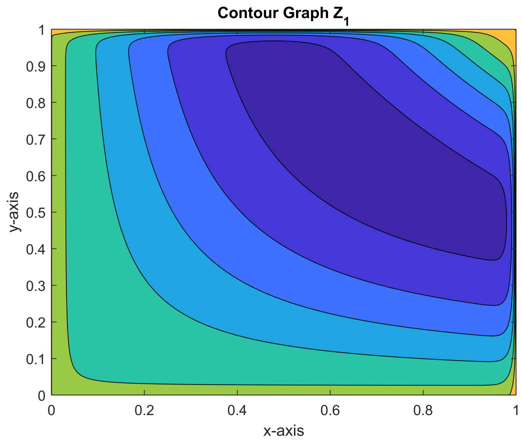
Figure 4: When $\varepsilon_1 = 2^{-16}, \varepsilon_2 = 2^{-14}$, $\mu_1^2 = 2^{-18}$, $\mu_2^2 = 2^{-12}$, $N = 128$ for Example 5.1;



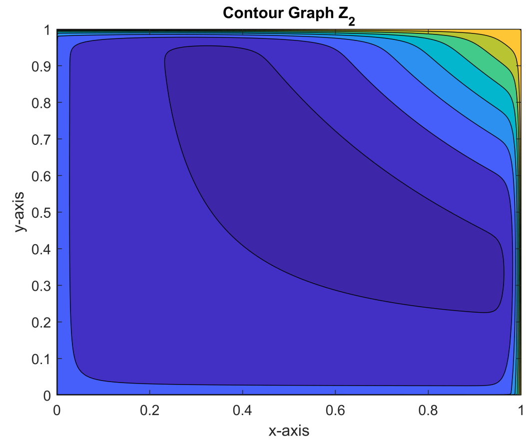
(a) Contour plot of the numerical solution z_1;

(b) Contour plot of the numerical solution z_2;

Figure 5: When $\varepsilon_1 = 2^{-14}, \varepsilon_2 = 2^{-12}$, $\mu_1^2 = 2^{-18}$, $\mu_2^2 = 2^{-16}$, $N = 128$ for Example 5.1;

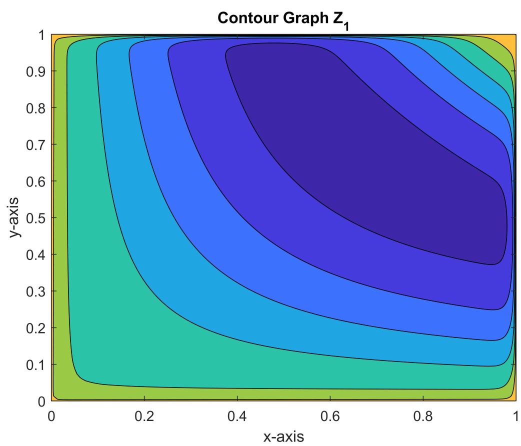


(a) Contour plot of the numerical solution z_1;

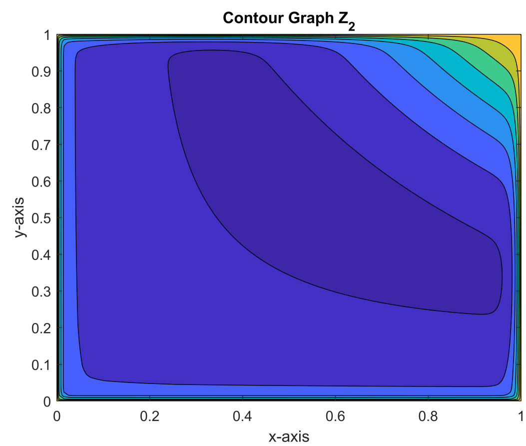


(b) Contour plot of the numerical solution z_2;

Figure 6: When $\varepsilon_1 = 2^{-18}, \varepsilon_2 = 2^{-16}$, $\mu_1^2 = 2^{-14}$, $\mu_2^2 = 2^{-12}, N = 128$ for Example 5.1;

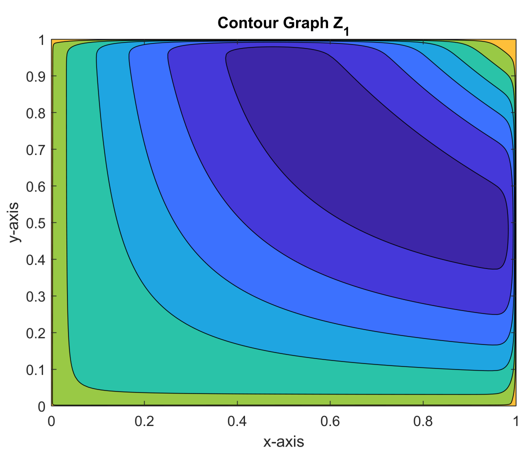


(a) Contour plot of the numerical solution z_1;

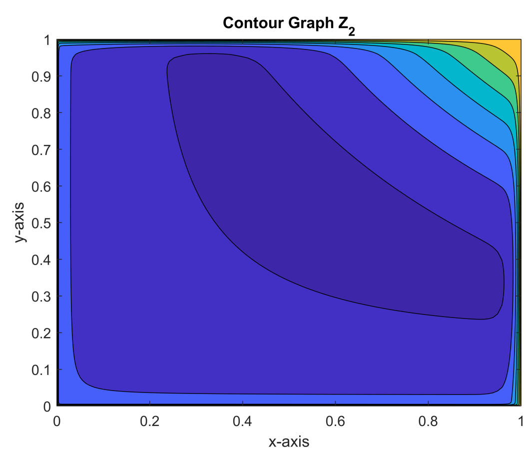


(b) Contour plot of the numerical solution z_2;

Figure 7: When $\varepsilon_1 = 2^{-18}, \varepsilon_2 = 2^{-12}$, $\mu_1^2 = 2^{-16}$, $\mu_2^2 = 2^{-14}, N = 128$ for Example 5.1;

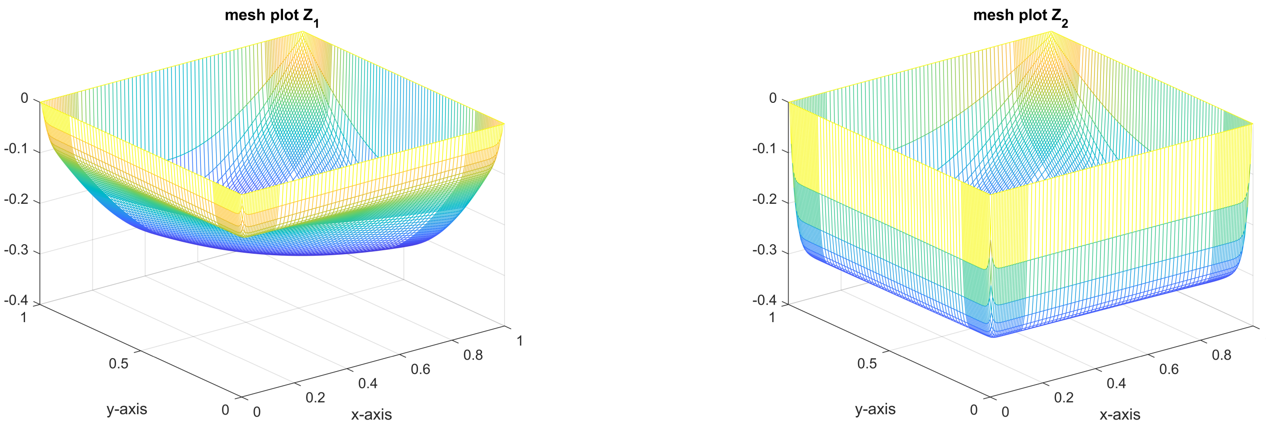


(a) Contour plot of the numerical solution z_1;



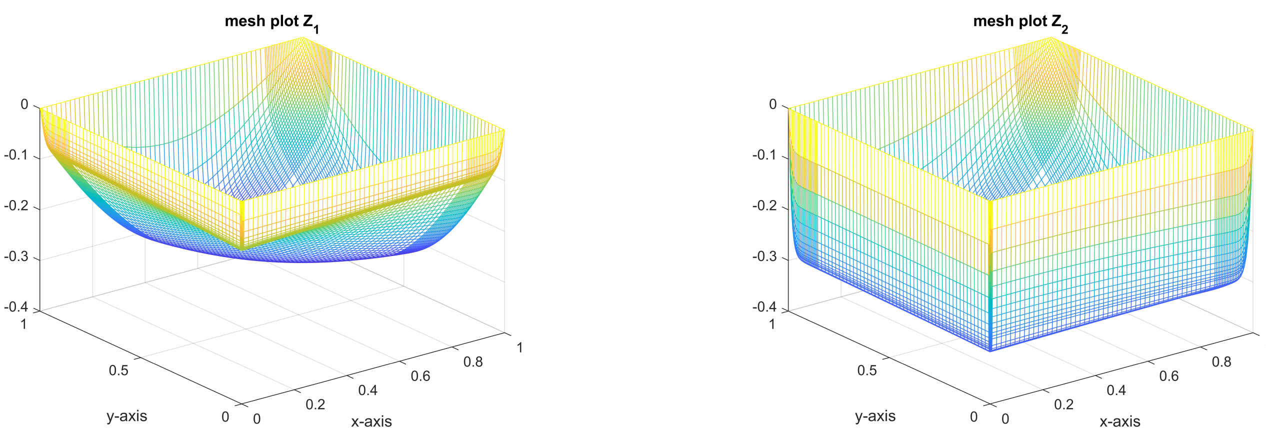
(b) Contour plot of the numerical solution z_2;

Figure 8: When $\varepsilon_1 = 2^{-16}, \varepsilon_2 = 2^{-14}$, $\mu_1^2 = 2^{-18}$, $\mu_2^2 = 2^{-12}, N = 128$ for Example 5.1;



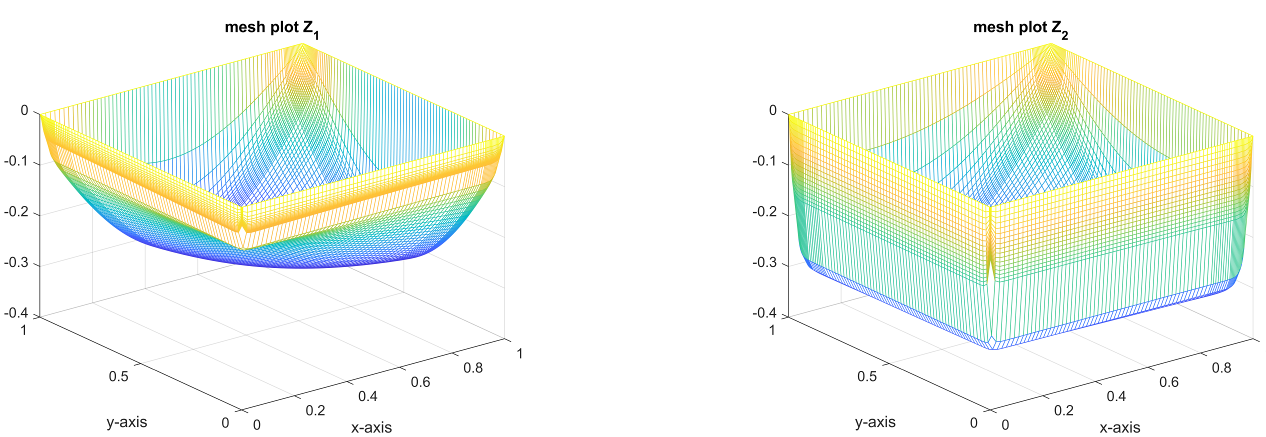
(a) Mesh plot of the numerical solution z_1; (b) Contour plot of the numerical solution z_2;

Figure 9: When $\varepsilon_1 = 2^{-14}, \varepsilon_2 = 2^{-12}$, $\mu_1^2 = 2^{-18}$, $\mu_2^2 = 2^{-16}, N = 128$ for Example 5.1;



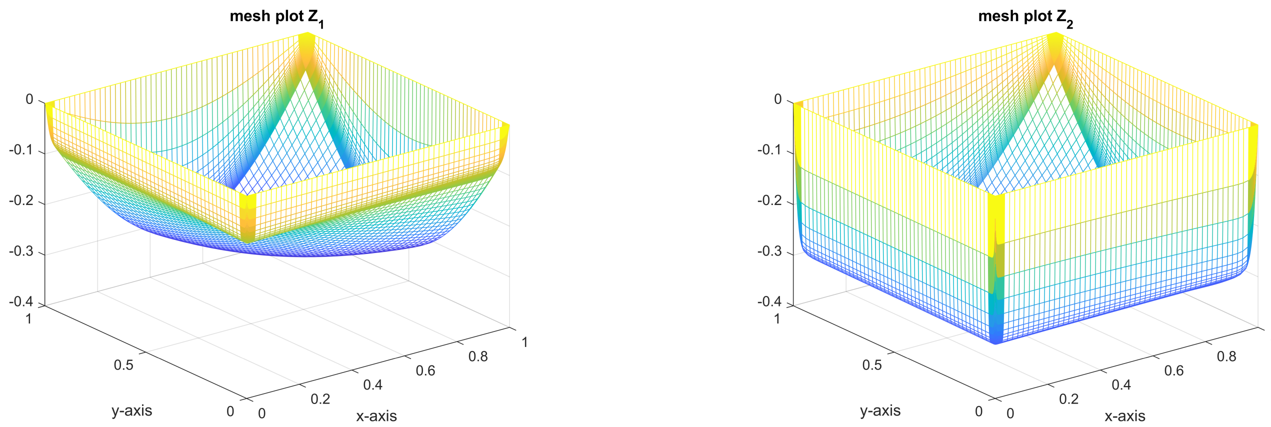
(a) Mesh plot of the numerical solution z_1; (b) Mesh plot of the numerical solution z_2;

Figure 10: When $\varepsilon_1 = 2^{-18}, \varepsilon_2 = 2^{-16}$, $\mu_1^2 = 2^{-14}$, $\mu_2^2 = 2^{-12}, N = 128$ for Example 5.1;



(a) Mesh plot of the numerical solution z_1; (b) Mesh plot of the numerical solution z_2;

Figure 11: When $\varepsilon_1 = 2^{-18}, \varepsilon_2 = 2^{-12}$, $\mu_1^2 = 2^{-16}$, $\mu_2^2 = 2^{-14}, N = 128$ for Example 5.1;



(a) Mesh plot of the numerical solution z_1;

(b) Mesh plot of the numerical solution z_2;

Figure 12: When $\varepsilon_1 = 2^{-16}, \varepsilon_2 = 2^{-14}$, $\mu_1^2 = 2^{-18}$, $\mu_2^2 = 2^{-12}$, $N = 128$ for Example 5.1;

Figures 1, 2, 3 and 4 display the two components of the numerical solution for the Example 5.1, taking different values of the diffusion and the convection parameters ε_1, ε_2, μ_1 and μ_2, respectively, and a value of the discretization parameter N. Figure 1 corresponds to case 1, Figure 2 corresponds to case 2, Figure 3 corresponds to case 3 and Figure 4 corresponds to case 4. From them, we clearly observe the boundary layers on the inflow and outflow boundary of the corresponding numerical solution.

To see clearer the aspect of the different boundary layers, Figures 5, 6, 7 and 8 display the contour plot of the solution, for the same values of the diffusion and convection parameters. To observe the mesh plotting of the square domain for the problem (5.1), Figures 9, 10, 11 and 12 display the mesh plot of the solution, for the same values of the diffusion and convection parameters.

As the exact solution of this problem is unknown, to approximate the maximum point-wise errors we use, in a usual way, the double mesh technique (see [32]). Then, we calculate

$$E^{N,N}_{\vec{\varepsilon},\vec{\mu}} = \max_{(x_i,y_j)\in\overline{\Omega}^{N,N}} |\widehat{Z}^{2N,2N}(x_{2i}, y_{2j}) - Z^{N,N}(x_i, y_j)|,$$

where $\widehat{Z}^{2N,2N}$ is the numerical solution obtained on a mesh with $2N$ subintervals taking the mesh points of the coarse mesh and also their midpoints on each spatial direction. Then, the parameter uniform maximum point-wise errors are calculated applying the formula

$$E^{N,N} = \max_{\vec{\varepsilon},\vec{\mu}} E^{N,N}_{\vec{\varepsilon},\vec{\mu}}.$$

From the previous values, the uniform numerical orders of convergence are given by

$$Q^{N,N} = \log_2\left(E^{N,N}/E^{2N,2N}\right).$$

Tables 1 and 2 show the maximum errors for some values of η and the discretization parameter N, and also the uniform-errors and numerical uniform orders of convergence for the corresponding values of the singular perturbation parameters, in **Case 1**; from them, we clearly observe the first order of uniform convergence. Note that this order is better that the order proved theoretically; this fact is common in singularly perturbed problems; the reason is that we use the double mesh principle to approximate the maximum errors because we do not have an exact solution known and therefore we only can obtain approximated maximum errors.

Tables 3 and 4 show the maximum errors for the same values of η and N that in Table 1, and also the uniform-errors and numerical uniform orders of convergence for the corresponding values of the singular perturbation parameters, in **Case 2**; from them, we again see the almost first order of uniform convergence, according with the theoretical result given in Theorem 4.11.

Table 1: For Example 5.1, maximum point-wise errors $E^{N,N}$ and orders of convergence $Q^{N,N}$ calculated for z_1 when $\vartheta{\mu_1}^2 \leq \vartheta{\mu_2}^2 \leq \Lambda\varepsilon_1 \leq \Lambda\varepsilon_2$

$\varepsilon_1 = 2^{-4}\eta$ $\varepsilon_2 = 2^{-2}\eta$ ${\mu_1}^2 = 2^{-8}\eta$ ${\mu_2}^2 = 2^{-6}\eta$				
η/N	48	96	192	384
2^{0}	7.479e-04	3.572e-04	1.743e-04	8.609e-05
2^{-2}	1.486e-03	6.822e-04	3.248e-04	1.582e-04
2^{-4}	3.316e-03	1.455e-03	6.671e-04	3.173e-04
2^{-6}	7.614e-03	3.306e-03	1.450e-03	6.644e-04
2^{-8}	1.385e-02	7.611e-03	3.304e-03	1.449e-03
2^{-10}	1.384e-02	8.912e-03	4.465e-03	2.307e-03
2^{-12}	1.383e-02	8.911e-03	4.465e-03	2.307e-03
2^{-14}	1.382e-02	8.909e-03	4.465e-03	2.307e-03
2^{-16}	1.382e-02	8.908e-03	4.464e-03	2.307e-03
2^{-18}	1.381e-02	8.908e-03	4.464e-03	2.307e-03
2^{-20}	1.381e-02	8.908e-03	4.464e-03	2.307e-03
2^{-22}	1.381e-02	8.908e-03	4.464e-03	2.307e-03
2^{-24}	1.381e-02	8.908e-03	4.464e-03	2.307e-03
2^{-26}	1.381e-02	8.908e-03	4.464e-03	2.307e-03
2^{-28}	1.381e-02	8.908e-03	4.464e-03	2.307e-03
2^{-30}	1.381e-02	8.908e-03	4.464e-03	2.307e-03
$E^{N,N}$	1.385e-02	8.912e-03	4.465e-03	2.307e-03
$Q^{N,N}$	0.6361	0.9971	0.9526	-

Tables 5 and 6 show the maximum errors for the same values of η and N that in Table 1, and also the uniform-errors and numerical uniform orders of convergence for the corresponding values of the singular perturbation parameters, in **Case 3**; from them, we observe the almost first order of uniform convergence, which again is in agreement with Theorem 4.11.

Finally, Tables 7 and 8 show the maximum errors for the same values of η and N that in Table 1, and also the uniform-errors and numerical uniform orders of convergence for the corresponding values of the singular perturbation parameters, in **Case 4**; from them, we again see the almost first order of uniform convergence, in agreement with Theorem 4.11. Then, we can conclude that, in all cases analyzed in this work, the numerical algorithm constructed is efficient and uniformly convergent with respect to all parameters.

6. Conclusions

In this work we have considered the numerical approximation of the exact solution of a two-parameter weakly-coupled elliptic system of singularly perturbed 2D convection-reaction-diffusion problems, having small positive parameters at both the diffusion and the convection terms in the differential equation associated to the boundary value problem. The case of the more general systems, for which different diffusion and convection parameters are in the two equations of the coupled system, is analyzed. In this case, different types of overlapping boundary layers appear on the outflow and the inflow boundary, depending on the value and the ratio between the diffusion and the convection parameters. To solve numerically this problem, a finite difference scheme is constructed on an adequate special piecewise uniform mesh of Shishkin type. The resulting numerical scheme is an almost first-order uniformly convergent method with respect to all singular perturbation parameters. Some numerical results for a test problem are showed; from these results, clearly the uniform convergence of the algorithm is corroborated in practice, according with the theoretical results.

Table 2: For Example 5.1, maximum point-wise errors $E^{N,N}$ and orders of convergence $Q^{N,N}$ calculated for z_2 when$\vartheta{\mu_1}^2 \le \vartheta{\mu_2}^2 \le \Lambda\varepsilon_1 \le \Lambda\varepsilon_2$

$\varepsilon_1 = 2^{-4}\eta$ $\varepsilon_2 = 2^{-2}\eta$ ${\mu_1}^2 = 2^{-8}\eta$ ${\mu_2}^2 = 2^{-6}\eta$				
η/N	48	96	192	384
2^{0}	5.935e-04	2.854e-04	1.399e-04	6.923e-05
2^{-2}	9.889e-04	4.624e-04	2.231e-04	1.095e-04
2^{-4}	1.981e-03	8.734e-04	4.066e-04	1.958e-04
2^{-6}	4.479e-03	1.862e-03	8.211e-04	3.852e-04
2^{-8}	1.038e-02	4.355e-03	1.802e-03	8.135e-04
2^{-10}	1.027e-02	5.126e-03	2.520e-03	1.272e-03
2^{-12}	1.023e-02	5.096e-03	2.500e-03	1.274e-03
2^{-14}	1.020e-02	5.081e-03	2.490e-03	1.275e-03
2^{-16}	1.019e-02	5.074e-03	2.485e-03	1.275e-03
2^{-18}	1.018e-02	5.070e-03	2.482e-03	1.275e-03
2^{-20}	1.018e-02	5.068e-03	2.481e-03	1.275e-03
2^{-22}	1.018e-02	5.067e-03	2.480e-03	1.275e-03
2^{-24}	1.018e-02	5.067e-03	2.480e-03	1.275e-03
2^{-26}	1.018e-02	5.066e-03	2.480e-03	1.275e-03
2^{-28}	1.018e-02	5.066e-03	2.480e-03	1.275e-03
2^{-30}	1.018e-02	5.066e-03	2.480e-03	1.275e-03
$E^{N,N}$	1.038e-02	5.126e-03	2.520e-03	1.275e-03
$Q^{N,N}$	1.0179	1.0244	0.9829	-

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7. Declarations

7.1. Availability of supporting data

Not applicable.

7.2. Competing Interests

The authors have no competing interests to declare that are relevant to the content of this article.

7.3. Funding

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7.4. Author's contribution

Both authors are responsible of all sections and mathematical details in the manuscript; they wrote and reviewed the all manuscript. Ram Shiromani prepared the Figures and Tables included in the manuscript.

7.5. Ethical Approval

Not applicable.

Table 3: For Example 5.1, maximum point-wise errors $E^{N,N}$ and orders of convergence $Q^{N,N}$ calculated for z_1 when $\vartheta{\mu_2}^2 \geq \vartheta{\mu_1}^2 > \Lambda\varepsilon_1 \geq \Lambda\varepsilon_2$

$\varepsilon_1 = 2^{-8}\eta$ $\varepsilon_2 = 2^{-6}\eta$ ${\mu_1}^2 = 2^{-4}\eta$ ${\mu_2}^2 = 2^{-2}\eta$				
η/N	48	96	192	384
2^{0}	1.196e-02	7.978e-03	5.556e-03	3.468e-03
2^{-2}	1.074e-02	7.245e-03	5.061e-03	3.163e-03
2^{-4}	1.181e-02	6.867e-03	4.329e-03	2.703e-03
2^{-6}	1.934e-02	1.168e-02	6.793e-03	3.651e-03
2^{-8}	2.394e-02	1.926e-02	1.165e-02	6.777e-03
2^{-10}	2.397e-02	2.059e-02	1.484e-02	9.571e-03
2^{-12}	2.395e-02	2.059e-02	1.484e-02	9.573e-03
2^{-14}	2.397e-02	2.058e-02	1.484e-02	9.574e-03
2^{-16}	2.396e-02	2.059e-02	1.484e-02	9.574e-03
2^{-18}	2.395e-02	2.059e-02	1.484e-02	9.575e-03
2^{-20}	2.395e-02	2.059e-02	1.484e-02	9.575e-03
2^{-22}	2.395e-02	2.059e-02	1.484e-02	9.575e-03
2^{-24}	2.395e-02	2.058e-02	1.484e-02	9.575e-03
2^{-26}	2.395e-02	2.058e-02	1.484e-02	9.575e-03
2^{-28}	2.395e-02	2.058e-02	1.484e-02	9.575e-03
2^{-30}	2.395e-02	2.058e-02	1.484e-02	9.575e-03
$E^{N,N}$	2.397e-02	2.059e-02	1.484e-02	9.575e-03
$Q^{N,N}$	0.2193	0.4725	0.6321	-

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Table 4: For Example 5.1, maximum point-wise errors $E^{N,N}$ and orders of convergence $Q^{N,N}$ calculated for z_2 when $\vartheta {\mu_2}^2 \geq \vartheta {\mu_1}^2 > \Lambda\varepsilon_1 \geq \Lambda\varepsilon_2$

$\varepsilon_1 = 2^{-8}\eta$ $\varepsilon_2 = 2^{-6}\eta$ ${\mu_1}^2 = 2^{-4}\eta$ ${\mu_2}^2 = 2^{-2}\eta$				
η/N	48	96	192	384
2^{0}	1.672e-02	1.058e-02	6.332e-03	3.667e-03
2^{-2}	1.595e-02	1.025e-02	6.241e-03	3.638e-03
2^{-4}	1.509e-02	9.759e-03	5.953e-03	3.476e-03
2^{-6}	1.612e-02	9.957e-03	5.875e-03	3.431e-03
2^{-8}	2.314e-02	1.504e-02	8.686e-03	4.728e-03
2^{-10}	2.307e-02	1.660e-02	1.095e-02	6.787e-03
2^{-12}	2.305e-02	1.657e-02	1.094e-02	6.782e-03
2^{-14}	2.303e-02	1.656e-02	1.093e-02	6.779e-03
2^{-16}	2.301e-02	1.655e-02	1.093e-02	6.777e-03
2^{-18}	2.300e-02	1.654e-02	1.093e-02	6.777e-03
2^{-20}	2.300e-02	1.654e-02	1.093e-02	6.776e-03
2^{-22}	2.299e-02	1.654e-02	1.092e-02	6.776e-03
2^{-24}	2.299e-02	1.654e-02	1.092e-02	6.776e-03
2^{-26}	2.299e-02	1.654e-02	1.092e-02	6.776e-03
2^{-28}	2.299e-02	1.654e-02	1.092e-02	6.776e-03
2^{-30}	2.299e-02	1.654e-02	1.092e-02	6.776e-03
$E^{N,N}$	2.314e-02	1.660e-02	1.095e-02	6.787e-03
$Q^{N,N}$	0.4792	0.6003	0.6901	-

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Table 5: For Example 5.1, maximum point-wise errors $E^{N,N}$ and orders of convergence $Q^{N,N}$ calculated for z_1 when $\Lambda\varepsilon_1 < \vartheta{\mu_1}^2 \leq \vartheta{\mu_2}^2 < \Lambda\varepsilon_2$

$\varepsilon_1 = 2^{-8}\eta$ $\varepsilon_2 = 2^{-2}\eta$ ${\mu_1}^2 = 2^{-6}\eta$ ${\mu_2}^2 = 2^{-4}\eta$				
η/N	48	96	192	384
2^{0}	4.540e-03	3.006e-03	1.979e-03	1.129e-03
2^{-2}	7.877e-03	4.222e-03	2.110e-03	1.048e-03
2^{-4}	1.440e-02	7.757e-03	4.172e-03	2.088e-03
2^{-6}	1.825e-02	1.436e-02	7.738e-03	4.165e-03
2^{-8}	2.036e-02	1.787e-02	1.432e-02	7.739e-03
2^{-10}	2.035e-02	1.788e-02	1.433e-02	8.740e-03
2^{-12}	2.033e-02	1.789e-02	1.433e-02	8.742e-03
2^{-14}	2.032e-02	1.788e-02	1.433e-02	8.742e-03
2^{-16}	2.032e-02	1.788e-02	1.433e-02	8.742e-03
2^{-18}	2.032e-02	1.788e-02	1.433e-02	8.742e-03
2^{-20}	2.032e-02	1.788e-02	1.433e-02	8.742e-03
2^{-22}	2.032e-02	1.788e-02	1.433e-02	8.742e-03
2^{-24}	2.032e-02	1.788e-02	1.433e-02	8.742e-03
2^{-26}	2.032e-02	1.788e-02	1.433e-02	8.742e-03
2^{-28}	2.032e-02	1.788e-02	1.433e-02	8.742e-03
2^{-30}	2.032e-02	1.788e-02	1.433e-02	8.742e-03
$E^{N,N}$	2.036e-02	1.789e-02	1.433e-02	8.742e-03
$Q^{N,N}$	0.1866	0.3201	0.7130	-

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Table 6: For Example 5.1, maximum point-wise errors $E^{N,N}$ and orders of convergence $Q^{N,N}$ calculated for z_2 when $\Lambda\varepsilon_1 < \vartheta{\mu_1}^2 \leq \vartheta{\mu_2}^2 < \Lambda\varepsilon_2$

$\varepsilon_1 = 2^{-8}\eta$ $\varepsilon_2 = 2^{-2}\eta$ ${\mu_1}^2 = 2^{-6}\eta$ ${\mu_2}^2 = 2^{-4}\eta$				
η/N	48	96	192	384
2^{0}	1.376e-03	6.660e-04	3.268e-04	1.629e-04
2^{-2}	2.793e-03	1.346e-03	6.380e-04	3.023e-04
2^{-4}	4.557e-03	2.543e-03	1.320e-03	6.566e-04
2^{-6}	1.271e-02	4.121e-03	2.403e-03	1.283e-03
2^{-8}	2.662e-02	1.373e-02	4.408e-03	2.249e-03
2^{-10}	2.662e-02	1.639e-02	7.518e-03	2.854e-03
2^{-12}	2.662e-02	1.639e-02	7.517e-03	2.859e-03
2^{-14}	2.663e-02	1.639e-02	7.517e-03	2.862e-03
2^{-16}	2.663e-02	1.639e-02	7.517e-03	2.864e-03
2^{-18}	2.663e-02	1.639e-02	7.517e-03	2.864e-03
2^{-20}	2.663e-02	1.639e-02	7.517e-03	2.865e-03
2^{-22}	2.663e-02	1.639e-02	7.517e-03	2.865e-03
2^{-24}	2.663e-02	1.639e-02	7.517e-03	2.865e-03
2^{-26}	2.663e-02	1.639e-02	7.517e-03	2.865e-03
2^{-28}	2.663e-02	1.639e-02	7.517e-03	2.865e-03
2^{-30}	2.663e-02	1.639e-02	7.517e-03	2.865e-03
$E^{N,N}$	2.663e-02	1.639e-02	7.517e-03	2.865e-03
$Q^{N,N}$	0.7002	1.1246	1.3916	-

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Table 7: For Example 5.1, maximum point-wise errors $E^{N,N}$ and orders of convergence $Q^{N,N}$ calculated for z_1 when $\vartheta{\mu_1}^2 < \Lambda\varepsilon_1 \leq \Lambda\varepsilon_2 < \vartheta{\mu_2}^2$

$\varepsilon_1 = 2^{-6}\eta$ $\varepsilon_2 = 2^{-4}\eta$ ${\mu_1}^2 = 2^{-8}\eta$ ${\mu_2}^2 = 2^{-2}\eta$				
η/N	48	96	192	384
2^{0}	1.993e-03	1.006e-03	5.164e-04	2.618e-04
2^{-2}	4.031e-03	1.934e-03	9.357e-04	4.765e-04
2^{-4}	7.965e-03	4.014e-03	1.922e-03	9.303e-04
2^{-6}	7.960e-03	4.699e-03	2.575e-03	1.417e-03
2^{-8}	7.957e-03	4.699e-03	2.574e-03	1.417e-03
2^{-10}	7.956e-03	4.698e-03	2.574e-03	1.417e-03
2^{-12}	7.955e-03	4.697e-03	2.574e-03	1.417e-03
2^{-14}	7.955e-03	4.697e-03	2.573e-03	1.417e-03
2^{-16}	7.954e-03	4.697e-03	2.573e-03	1.417e-03
2^{-18}	7.954e-03	4.696e-03	2.573e-03	1.417e-03
2^{-20}	7.954e-03	4.696e-03	2.573e-03	1.417e-03
2^{-22}	7.954e-03	4.696e-03	2.573e-03	1.417e-03
2^{-24}	7.954e-03	4.696e-03	2.573e-03	1.417e-03
2^{-26}	7.954e-03	4.696e-03	2.573e-03	1.417e-03
2^{-28}	7.954e-03	4.696e-03	2.573e-03	1.417e-03
2^{-30}	7.954e-03	4.696e-03	2.573e-03	1.417e-03
$E^{N,N}$	7.965e-03	4.699e-03	2.575e-03	1.417e-03
$Q^{N,N}$	0.7613	0.8678	0.8617	-

Table 8: For Example 5.1, maximum point-wise errors $E^{N,N}$ and orders of convergence $Q^{N,N}$ calculated for z_2 when $\vartheta{\mu_1}^2 < \Lambda\varepsilon_1 \leq \Lambda\varepsilon_2 < \vartheta{\mu_2}^2$

$\varepsilon_1 = 2^{-6}\eta$ $\varepsilon_2 = 2^{-4}\eta$ ${\mu_1}^2 = 2^{-8}\eta$ ${\mu_2}^2 = 2^{-2}\eta$				
η/N	48	96	192	384
2^{0}	7.499e-03	4.065e-03	2.122e-03	1.085e-03
2^{-2}	1.246e-02	7.228e-03	3.927e-03	2.052e-03
2^{-4}	1.539e-02	1.080e-02	6.885e-03	3.790e-03
2^{-6}	1.526e-02	1.073e-02	6.852e-03	4.151e-03
2^{-8}	1.524e-02	1.072e-02	6.845e-03	4.146e-03
2^{-10}	1.525e-02	1.072e-02	6.849e-03	4.148e-03
2^{-12}	1.526e-02	1.073e-02	6.852e-03	4.150e-03
2^{-14}	1.526e-02	1.073e-02	6.854e-03	4.152e-03
2^{-16}	1.526e-02	1.073e-02	6.856e-03	4.152e-03
2^{-18}	1.527e-02	1.074e-02	6.856e-03	4.153e-03
2^{-20}	1.527e-02	1.074e-02	6.857e-03	4.153e-03
2^{-22}	1.527e-02	1.074e-02	6.857e-03	4.153e-03
2^{-24}	1.527e-02	1.074e-02	6.857e-03	4.153e-03
2^{-26}	1.527e-02	1.074e-02	6.857e-03	4.153e-03
2^{-28}	1.527e-02	1.074e-02	6.857e-03	4.153e-03
2^{-30}	1.527e-02	1.074e-02	6.857e-03	4.153e-03
$E^{N,N}$	1.527e-02	1.074e-02	6.857e-03	4.153e-03
$Q^{N,N}$	0.5077	0.6473	0.7234	-