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Error bounds for linear complementarity problems of B_π^R-matrices

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Abstract It is proved that any B_π^R-matrix has positive determinant. For $\pi > 0$, norm bounds for the inverses of B_π^R-matrices and error bounds for linear complementarity problems (LCPs) associated with B_π^R-matrices are provided. In this last case, the bounds are simpler than previous bounds and also have the advantage that they can be used without previously knowing whether we have a B_π^R-matrix. Some numerical examples show that these new bounds can be considerably sharper than previous ones.

Keywords Error bounds, Linear complementarity problems, Norm bounds for the inverse, B_π^R-matrices

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1 Introduction

This paper provides error bounds for linear complementarity problems (LCPs) associated with B_π^R-matrices as well as norms for the inverses of these matrices.

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The LCP (see Section 4) has many important applications, for instance, to problems in linear and quadratic programming, network equilibrium problems, or to the Nash equilibrium of a bimatrix game (see [1, 2, 4, 22]). A principal minor is the determinant of a submatrix involving the same rows and columns, and P-matrices are square matrices with all their principal minors positive. Let us recall a remarkable property of P-matrices: the solution to a LCP exists and is unique if and only if its associated matrix is a P-matrix [4].

Error bounds for LCPs associated to several subclasses of P-matrices are presented in [2, 5–8, 10, 11, 14, 17, 24]. In particular, error bounds for LCPs associated with B_π^R-matrices with $\pi > 0$ were presented in [9,13]. The class of B_π^R-matrices was introduced by Neumann et al. in [19], generalizing the class of B-matrices (see [8, 10, 18, 21]). If we do not know whether a given matrix is a B_π^R-matrix with a fixed $\pi > 0$, then we cannot apply the bounds of [9, 13]. In this paper we shall provide alternative bounds for any matrix with positive row sums that is a B_π^R-matrix with $\pi \geq 0$. Moreover, we shall characterize B_π^R-matrices with $\pi \geq 0$ and provide $\pi > 0$. In contrast to [9, 13], our new bound does not depend on an additional parameter ε, so that its application is simpler. In addition, we show in Section 4 with some test matrices used in [9, 13] that our new bound considerably improves those of [9, 13].

In Section 2 we first introduce B_π^R-matrices and clarify a result of [19], where it was claimed that any B_π^R-matrix is a P-matrix but the proof assumed that $\pi \geq 0$. We show in Example 1 that there exist B_π^R-matrices that are not P-matrices when π has a negative component. However, Theorem 2 proves that any B_π^R-matrix has positive determinant. We also present in Section 2 a characterization to determine whether a given matrix is a B_π^R-matrix with $\pi \geq 0$. This characterization also provides a positive vector π. Section 3 is devoted to bound the infinity norm of the inverse of B_π^R-matrices. Results of Section 3 are used in Section 4 to derive the new error bounds of LCPs associated to B_π^R-matrices with $\pi > 0$. Numerical examples are included at the end of Section 4.

Finally, let us recall some matrix definitions. We say that a matrix A is *nonnegative* (respectively, *positive*) if all its entries are nonnegative (respectively, positive) and we write $A \geq 0$ (respectively, $A > 0$). The same notation applies to vectors considering them as column matrices. A matrix $M = (m_{ij})_{1 \leq i,j \leq n}$ is a *strictly diagonally dominant matrix* if $|m_{ii}| > \sum_{j \neq i} |m_{ij}|$, for each $i = 1, \ldots, n$. A *Z-matrix* is a square real matrix with nonpositive off-diagonal entries. A nonsingular *M-matrix* is a Z-matrix with nonnegative inverse. Nonsingular M-matrices form an important subclass of P-matrices and some fields where these matrices arise are dynamic systems, economics or the discretization of partial differential equations.

2 Some basic results on B_π^R-matrices

Let us start by recalling the definition of a B_π^R-matrix given in [19].

Definition 1 Let $\pi = (\pi_1, \ldots, \pi_n)^T$ be a vector such that

$$0 < \sum_{j=1}^{n} \pi_j \leq 1. \tag{1}$$

Let $M = (m_{ij})_{1\leq i,j\leq n}$ be a real matrix with positive row sums and let $R = (R_1, \ldots, R_n)^T$ be the vector formed by the row sums of M. Then we say that M is a B_π^R-matrix if for all $i = 1, \ldots, n$,

$$\pi_j R_i > m_{ij}, \quad \forall j \neq i. \tag{2}$$

When $\pi_j = 1/n$ for all j, the previous definition coincides with that of a B-matrix (see [21]). The close relationship of P-matrices with the LCP was recalled in the Introduction. In fact, in Theorem 3.4 of [19] it was proved that a B_π^R-matrix is also a P-matrix whenever the vector π is nonnegative. However, the condition on the sign of π is omitted as a hypothesis in the statement of that theorem. Precisely, as was commented in page 251 of [20], the nonnegativity of the vector π is sufficient to ensure that a B_π^R-matrix is also a P-matrix. So, we state the result that was proved in fact in Theorem 3.4 of [19].

Theorem 1 *If A is a B_π^R-matrix with $\pi \geq 0$, then A is a P-matrix.*

With the following example we show that the condition $\pi \geq 0$ can not be omitted to assure that a B_π^R-matrix is a P-matrix.

Example 1 Let us consider the vector $\pi = (1.1, -2.9, 2.1)^T$. Then the matrix

$$A := \begin{pmatrix} 2 & -3 & 2 \\ -1 & 1 & 1 \\ 0.1 & -1 & 1 \end{pmatrix}$$

is a B_π^R-matrix. However, A is not a P-matrix since the principal minor using the first and second rows and columns is -1.

Let us also mention that, in order to derive bounds for LCPs associated to B_π^R-matrices, the condition $\pi > 0$ was used in [9, 13] as well as in the bounds that we shall present later. In contrast to the loss of the property of being a P-matrix seen in Example 1, we can see that $\det A > 0$ holds for any B_π^R-matrix A for any vector π.

Theorem 2 *Let $M = (m_{ij})_{1\leq i,j\leq n}$ be a real matrix with positive row sums. If M is a B_π^R-matrix, then* $\det M > 0$.

Proof By (1) there exists $k \in \{1, \ldots, n\}$ such that $\pi_k > 0$. Let us choose $\varepsilon > 0$ such that $\pi_k - \varepsilon > 0$ and $m_{ik} - (\pi_k - \varepsilon)R_i < 0$ for $i \neq k$. Then we can define a new parameter vector $\hat{\pi} = (\hat{\pi}_1, \ldots, \hat{\pi}_n)^T$ with

$$\hat{\pi}_i = \begin{cases} \pi_i, & i \neq k, \\ \pi_k - \varepsilon, & i = k, \end{cases}$$

and use it to decompose M as

$$M = B^+ + C, \quad B^+ := (m_{ij} - \hat{\pi}_j R_i)_{1\le i,j\le n}, \quad C := R\hat{\pi}^T. \tag{3}$$

Then B^+ is a Z-matrix with row sums $\bar{R} = (\bar{R}_1, \dots, \bar{R}_n)^T$. Observe that, by (1) and the definition of $\hat{\pi}$, $\sum_{j=1}^n \hat{\pi}_j < 1$. Hence, for $i = 1, \dots, n$, the row sum $\bar{R}_i$ is given by:

$$\bar{R}_i = \sum_{j=1}^n (m_{ij} - \hat{\pi}_j R_i) = R_i(1 - \sum_{j=1}^n \hat{\pi}_j) > 0. \tag{4}$$

Since B^+ is a Z-matrix with positive diagonal entries, the positivity of its row sums implies that it is also strictly diagonally dominant. Hence, B^+ is a nonsingular M-matrix and so $\det(B^+) > 0$. By the decomposition (3) and the relationship between R and $\bar{R}$ given by (4), we have that

$$\begin{aligned}\det M &= \det(B^+ + C) = \det(B^+ + R\hat{\pi}^T) = \det(B^+)(1 + \hat{\pi}^T(B^+)^{-1}R) \\ &= \det(B^+)(1 + \hat{\pi}^T(B^+)^{-1}(1 - \sum_{j=1}^n \hat{\pi}_j)^{-1}\bar{R}).\end{aligned}$$

Given $e = (1, \dots, 1)^T$, observe that $B^+e = \bar{R}$, and so

$$\det M = \det(B^+)(1 + \hat{\pi}^T(B^+)^{-1}(1 - \sum_{j=1}^n \hat{\pi}_j)^{-1}B^+e).$$

Therefore, we deduce that

$$\begin{aligned}\det M &= \det(B^+)(1 + \hat{\pi}^T(1 - \sum_{j=1}^n \hat{\pi}_j)^{-1}e) = \det(B^+)(1 + \sum_{j=1}^n \hat{\pi}_j(1 - \sum_{j=1}^n \hat{\pi}_j)^{-1}) \\ &= \det(B^+)(1 - \sum_{j=1}^n \hat{\pi}_j)^{-1},\end{aligned}$$

and, since $\det(B^+) > 0$, we conclude that $\det M > 0$. □

By Proposition 3.5 of [19], the class of matrices satisfying Definition 1 is closed under positive linear combinations. Then, by Theorem 2, it has positive determinant. Finally, Theorem 1 gives a sufficient condition to assure that the positive combination is a P-matrix. This information is gathered in the following corollary.

Corollary 1 *Let $A = (a_{ij})_{1\le i,j\le n}$ and $B = (b_{ij})_{1\le i,j\le n}$ be a B_π^R-matrix and a B_ψ^R-matrix, respectively. Let s and t be nonnegative numbers with $s + t > 0$. Then:*

i) $\det(sA + tB) > 0$.
ii) If $\pi, \psi > 0$, then $sA + tB$ is a P-matrix.

We now present a characterization that allows us to determine whether a given matrix is a B_π^R-matrix with $\pi \geq 0$ and so, in the affirmative case, that it is in particular a P-matrix. Moreover, the characterization gives a suitable positive vector π satisfying (1) and so we can apply the bounds that will be presented later. This characterization will allow us to obtain bounds for B_π^R-matrices whenever the vector π is unknown. A characterization of a B_π^R-matrix for any π was obtained in Observation 3.2 of [19], but we are going to adapt it by imposing the additional condition $\pi \geq 0$.

Proposition 1 *Let A be a square matrix with positive row sums and let $R = (R_1, \ldots, R_n)^T$ be the vector formed from the row sums of A. Then there exists a nonnegative vector π satisfying* (1) *such as A is a B_π^R-matrix if and only if*

$$\sum_{j=1}^{n} \max_{i \neq j} \left(\frac{a_{ij}}{R_i}, 0 \right) < 1. \tag{5}$$

Proof Let us first suppose that A is a B_π^R-matrix for a given nonnegative vector π satisfying (1). By (1) there exists $k \in \{1, \ldots, n\}$ such that $\pi_k > 0$. Then we have that $\max_{i \neq k} \left(\frac{a_{ik}}{R_i}, 0 \right) < \pi_k$ and, since $\max_{i \neq j} \left(\frac{a_{ij}}{R_i}, 0 \right) \leq \pi_j$ for all $j \neq k$, we also have that

$$\sum_{j=1}^{n} \max_{i \neq j} \left(\frac{a_{ij}}{R_i}, 0 \right) < \sum_{j=1}^{n} \pi_j \leq 1. \tag{6}$$

Conversely, let us now suppose that (5) holds. If we define

$$k := 1 - \sum_{j=1}^{n} \max_{i \neq j} \left(\frac{a_{ij}}{R_i}, 0 \right), \tag{7}$$

then we have that the vector $\pi = (\pi_1, \ldots, \pi_n)$ with

$$\pi_j := \max_{i \neq j} \left(\frac{a_{ij}}{R_i}, 0 \right) + \frac{k}{n} \quad \text{for } j = 1, \ldots, n \tag{8}$$

is positive and satisfies (1). Hence, A is a B_π^R-matrix. □

Remark 1 Let us observe that the choice of π in (8) agrees with the natural parameter vector $\pi = (\frac{1}{n}, \ldots, \frac{1}{n})^T$ of an $n \times n$ B-matrix in some extremal examples of B-matrices (see [19]). A first example of these B-matrices is provided by any positive diagonal matrix. In this case, (7) gives $k = 1$ and so (8) gives $\pi_j = \frac{1}{n}$ for all $j = 1, \ldots, n$. The other extremal example of a B-matrix is provided by a matrix of the form

$$A = \begin{pmatrix} 1+\varepsilon & 1 & \ldots & 1 \\ 1 & 1+\varepsilon & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \ldots & 1 & 1+\varepsilon \end{pmatrix},$$

where $\varepsilon > 0$. In this case, $\frac{a_{ij}}{R_i} = \frac{1}{n+\varepsilon}$ for any $i \neq j$, and so (7) gives $k = 1 - \frac{n}{n+\varepsilon} = \frac{\varepsilon}{n+\varepsilon}$ and (8) gives $\pi_j = \frac{1}{n+\varepsilon} + \frac{\varepsilon}{n(n+\varepsilon)} = \frac{1}{n}$ for all $j = 1, \ldots, n$.

Remark 2 Observe that in the proof of Proposition 1 we prove that, if the matrix A satisfies (5), then the vector π given by (8) is positive.

3 Norm bounds for the inverses of B_π^R-matrices

Given a B_π^R-matrix $M = (m_{ij})_{1\leq i,j\leq n}$, in [13] a decomposition of M depending on a parameter ε was obtained and applied to derive error bounds of LCPs when the involved matrix is a B_π^R-matrix with $\pi_j > 0$ for all j. In the following result, we provide another decomposition of a B_π^R-matrix with $\pi_j > 0$ for all j, which will not depend on any parameter and which will be very useful in this paper.

Proposition 2 *Let $M = (m_{ij})_{1\leq i,j\leq n}$ be a B_π^R-matrix with $\pi_j > 0$ for all j and for each $i = 1, \ldots, n$ let $\gamma_i := \max_{j\neq i}\{0, \frac{m_{ij}}{\pi_j}\}$. Then we can write $M = B^+ + C$, where $B^+ := (m_{ij} - \pi_j\gamma_i)_{1\leq i,j\leq n}$ is a strictly diagonally dominant Z-matrix with positive diagonal entries and C is the rank one matrix given by $C := (\gamma_1, \ldots, \gamma_n)^T(\pi_1, \ldots, \pi_n)$.*

Proof We only have to prove that the Z-matrix B^+ has positive row sums. As usual, let us denote by $R = (R_1, \ldots, R_n)$ the vector of row sums of M, which are positive because M is a B_π^R-matrix. For each $i = 1, \ldots, n$, from (1) we deduce that the sum of the ith row of B^+ is $R_i - \gamma_i(\sum_{j=1}^n \pi_j) \geq R_i - \gamma_i$. Then, by definition of γ_i, we conclude that it is bounded below by either R_i (and so, it is positive) or by $R_i - \frac{m_{ij}}{\pi_j}$ for some $j \in \{1, \ldots, n\}$ (which is also positive by (2)). □

The following result gives an upper bound for $\|M^{-1}\|_\infty$.

Theorem 3 *Let $M = (m_{ij})_{1\leq i,j\leq n}$ be a B_π^R-matrix with $\pi_j > 0$ for all j and let R_j, γ_j be given as in Definition 1 and Proposition 2, respectively. Then*

$$\|M^{-1}\|_\infty \leq \frac{\max_{1\leq i\leq n}\left\{\frac{1}{\pi_i} - 1\right\}}{\min_{1\leq i\leq n}\left\{R_i - \gamma_i \sum_{j=1}^n \pi_j\right\}}. \tag{9}$$

Proof By Proposition 2 and Theorem (2.3) of Chapter 6 of [1], B^+ is a nonsingular M-matrix. So, we can write $(B^+)^{-1} =: (\bar{b}_{ij})_{1\leq i,j\leq n}$ with $\bar{b}_{ij} \geq 0$ for all i, j. Then we can express $M = B^+(I + (B^+)^{-1}C)$ and so

$$\|M^{-1}\|_\infty \leq \|(I + (B^+)^{-1}C)^{-1}\|_\infty \|(B^+)^{-1}\|_\infty. \tag{10}$$

Let us now provide an upper bound for $\|(B^+)^{-1}\|_\infty$. By Proposition 2, B^+ is a strictly diagonally dominant matrix with positive diagonal entries and so it has positive row sums:

$$R_i - \gamma_i \sum_{j=1}^n \pi_j > 0, \quad i = 1, \ldots, n.$$

By Theorem 1 of [23], we deduce that

$$\|(B^+)^{-1}\|_\infty \leq \frac{1}{\min_{1\leq i\leq n}\left\{R_i - \gamma_i \sum_{j=1}^n \pi_j\right\}}. \tag{11}$$

Now we bound the other factor of (10). Observe that

$$I + (B^+)^{-1}C = \begin{pmatrix} 1+a_1\pi_1 & a_1\pi_2 & \dots & a_1\pi_n \\ a_2\pi_1 & 1+a_2\pi_2 & \dots & a_2\pi_n \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ a_n\pi_1 & a_n\pi_2 & \dots & 1+a_n\pi_n \end{pmatrix}, \tag{12}$$

where $a_i := \sum_{j=1}^n \bar{b}_{ij}\gamma_j \geq 0$ for $i = 1,\dots,n$. Then (12) can be written as

$$I + (B^+)^{-1}C = I + AP \tag{13}$$

where $A := (a_1,\dots,a_n)^T e^T (\geq 0), P := \text{diag}(\pi_1,\pi_2,\dots,\pi_n)$ and $e := (1,\dots,1)^T$. By our hypothesis on π, P is nonsingular and so $I + AP = P^{-1}(I + PA)P$. Denoting by $\bar{C} := PA$, we have

$$(I + AP)^{-1} = P^{-1}(I + \bar{C})^{-1}P. \tag{14}$$

Observe that $\bar{C} = \bar{a}e^T$, where $\bar{a}_i := \pi_i a_i \geq 0$, for each $i = 2,\dots,n$ and $\bar{a} := (\bar{a}_1,\dots,\bar{a}_n)^T$. So, since $e^T\bar{a} = \sum_{i=1}^n \bar{a}_i \geq 0$, we can derive from the Sherman-Morrison formula (see formula (2.1.5) of page 65 of [15])

$$(I + \bar{C})^{-1} = (I + \bar{a}e^T)^{-1} = I - \frac{\bar{a}e^T}{1 + e^T\bar{a}}. \tag{15}$$

Hence, by (14), we get that

$$(I + AP)^{-1} = \begin{pmatrix} 1 - \frac{\bar{a}_1}{1+\sum_{i=1}^n \bar{a}_i} & \frac{\pi_2}{\pi_1}\left(\frac{-\bar{a}_1}{1+\sum_{i=1}^n \bar{a}_i}\right) & \dots & \frac{\pi_n}{\pi_1}\left(\frac{-\bar{a}_1}{1+\sum_{i=1}^n \bar{a}_i}\right) \\ \frac{\pi_1}{\pi_2}\left(\frac{-\bar{a}_2}{1+\sum_{i=1}^n \bar{a}_i}\right) & 1 - \frac{\bar{a}_2}{1+\sum_{i=1}^n \bar{a}_i} & \dots & \frac{\pi_n}{\pi_2}\left(\frac{-\bar{a}_2}{1+\sum_{i=1}^n \bar{a}_i}\right) \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\pi_1}{\pi_n}\left(\frac{-\bar{a}_n}{1+\sum_{i=1}^n \bar{a}_i}\right) & \frac{\pi_2}{\pi_n}\left(\frac{-\bar{a}_n}{1+\sum_{i=1}^n \bar{a}_i}\right) & \dots & 1 - \frac{\bar{a}_n}{1+\sum_{i=1}^n \bar{a}_i} \end{pmatrix}. \tag{16}$$

Then, since $\bar{a}_i \geq 0$ for all $i = 1,\dots,n$, we conclude that $\|(I + AP)^{-1}\|_\infty$ is given by

$$\|(I + AP)^{-1}\|_\infty = 1 - \frac{\bar{a}_i}{1 + \sum_{j=1}^n \bar{a}_j} + \sum_{j\neq i} \frac{\pi_j}{\pi_i}\,\frac{\bar{a}_i}{1 + \sum_{j=1}^n \bar{a}_j} \tag{17}$$

for some $i = 1, \dots, n$. Since $\sum_{j=1}^n \pi_j \le 1$ and $\bar{a}_i \ge 0$ for all i, formula (17) can be bounded above by

$$\frac{\bar{a}_i}{1 + \sum_{j=1}^n \bar{a}_j} \left(\sum_{j \ne i} \frac{\pi_j}{\pi_i} - 1 \right) + 1 \le \frac{1 - \pi_i}{\pi_i} - 1 + 1 = \frac{1 - \pi_i}{\pi_i}$$

and so,

$$\|(I + AP)^{-1}\|_\infty \le \max_i \left\{ \frac{1}{\pi_i} - 1 \right\}. \tag{18}$$

Now the result follows from (10), (11), (13) and (18). □

Proposition 1, Remark 2 and Theorem 3 allow us to deduce the following corollary.

Corollary 2 *Let M be a square matrix with positive row sums $R = (R_1, \dots, R_n)^T$ satisfying* (5)*, let $\pi = (\pi_1, \dots, \pi_n)$ be the positive vector given by* (8) *and let γ_j be given as in Proposition 2 for $j = 1, \dots, n$. Then M is a B_π^R-matrix and formula* (9) *holds.*

In the proof of Theorem 3 we have bounded the second factor of (10) by using Varah's bound for strictly diagonally dominant matrices of Theorem 1 of [23]. If we use a sharper bound, then we obtain sharper bounds for the norm of the inverse of a B_π^R-matrix. In order to illustrate this fact, we are going to use the bound introduced in [16] for Nekrasov matrices, which in particular improves Varah's bound for SDD matrices (as proven in Theorem 2.4 of [16]):

$$\left\|A^{-1}\right\|_\infty \le \max_{i \in N} \frac{z_i(A)}{|a_{ii}| - h_i(A)}, \tag{19}$$

where $z_i(A)$ and $h_i(A)$ are defined recursively for $i = 1, \dots, n$ by

$$z_1(A) := 1, \;\; z_i(A) := \sum_{j=1}^{i-1} |a_{ij}| \frac{z_j(A)}{|a_{jj}|} + 1, \quad i = 2, \dots, n.$$

$$h_1(A) := \sum_{j \ne 1} |a_{1j}|, \;\; h_i(A) := \sum_{j=1}^{i-1} |a_{ij}| \frac{h_j(A)}{|a_{jj}|} + \sum_{j=i+1}^{n} |a_{ij}|, \quad i = 2, \dots, n.$$

In particular, if we apply bound (19) to the second factor of (10) we deduce the following result:

Theorem 4 *Let $M = (m_{ij})_{1 \le i,j \le n}$ be a B_π^R-matrix with $\pi_j > 0$ for all j and let R_j, γ_j be given as in Definition 1 and Proposition 2, respectively. Then*

$$\|M^{-1}\|_\infty \le \max_{1 \le i \le n} \left\{ \frac{1}{\pi_i} - 1 \right\} \max_{1 \le i \le n} \frac{z_i(B^+)}{m_{ii} - \gamma_i \pi_i - h_i(B^+)}, \tag{20}$$

where B^+ is given in Proposition 2, $h_i(B^+) = \sum_{j=1}^{i-1} \frac{\gamma_i \pi_j - m_{ij}}{m_{jj} - \gamma_j \pi_j} h_j(B^+) + \sum_{j=i+1}^{n} (\gamma_i \pi_j - m_{ij})$ and $z_i(B^+) = \sum_{j=1}^{i-1} \frac{\gamma_i \pi_j - m_{ij}}{m_{jj} - \gamma_j \pi_j} z_j(B^+) + 1$.

The next result follows from Proposition 1, Remark 2 and Theorem 4.

Corollary 3 *Let M be a square matrix with positive row sums $R = (R_1, \dots, R_n)^T$ satisfying (5), let $\pi = (\pi_1, \dots, \pi_n)$ be the positive vector given by (8) and let γ_j be given as in Proposition 2 for $j = 1, \dots, n$. Then M is a B_π^R-matrix and formula (20) holds.*

We now present some numerical examples in order to illustrate our new results. Our test matrices were introduced in previous articles that studied error bounds for LCPs of B_π^R-matrices. The matrix $A_1(m)$ corresponds to Example 1 from [13]. $M_1(k)$, $M_2(h)$ and $M_3(m)$ are examples from [9]:

$$A_1(m) = \begin{pmatrix} 10m & -10m & 1 \\ -10m+1 & 10m & 0 \\ 2 & 3 & 3 \end{pmatrix}, \qquad M_1(k) = \begin{pmatrix} 4k & k & 0 & -k \\ k & 6k & 0 & 0 \\ 0 & k & 4k & -k \\ k & 0 & -k & 7k \end{pmatrix},$$

$$M_2(h) = \begin{pmatrix} 3h & h & -h \\ 3h & 10h & 3h \\ -h & h & 3h \end{pmatrix}, \qquad M_3(m) = \begin{pmatrix} 3m & m & 0 & 0 \\ 0.5m & 4m & 0 & -0.5m \\ 0.5m & m & 3m & -0.5m \\ 0.5m & m & -0.5m & 3m \end{pmatrix}.$$

We have computed bounds for the infinity norm of the inverse using theorems 3 and 4 and corollaries 2 and 3. The previous theorems need a given vector π, so we are going to use the parameter vectors given in the original articles. In Table 1 we gather these parameters and we present our results in Table 2.

Matrix	π	source
$A_1(8)$	$(19/50, 19/50, 6/25)$	[13]
$M_1(21/25)$	$(7/24, 7/24, 1/4, 1/6)$	[9]
$M_2(8/9)$	$(3/8, 3/8, 1/4)$	[9]
$M_3(1/2)$	$(9/24, 7/24, 1/6, 1/6)$	[9]

Table 1 Examples of B_π^R-matrices with their parameter vector π.

Matrix	$A_1(8)$	$M_1(21/25)$	$M_2(8/9)$	$M_3(1/2)$
$\|\|A^{-1}\|\|_\infty$	2.0000	0.40668	0.8839	1.0333
Theorem 3	30.083	10.4167	10.1250	17.500
Theorem 4	27.226	10.4167	10.1250	17.500
Corollary 2	7	4.4025	4.1720	7.0200
Corollary 3	7	4.4025	4.1720	7.0200

Table 2 Bounds to $||A^{-1}||_\infty$.

We can see that Theorem 4 only improves Theorem 3 for the matrix A_1 and that Corollary 2 (and Corollary 3) considerably improve theorems 3 and 4.

4 Error bounds for LCPs involving B_π^R-matrices.

Let us recall that the linear complementarity problem (LCP) looks for a vector $x \in \mathbf{R}^n$ such that

$$x \geq 0, \quad Mx + q \geq 0, \quad x^T(Mx + q) = 0, \tag{21}$$

where M is the $n \times n$ associated real matrix and $q \in \mathbf{R}^n$. Some important applications of this problem have been mentioned in the Introduction.

By Theorem 2.3 of [2], if M is a P-matrix, then the solution x^* of the LCP (21) satisfies

$$\|x - x^*\|_\infty \leq \max_{d\in[0,1]^n} \|M_D^{-1}\|_\infty \|r(x)\|_\infty, \tag{22}$$

where

$$M_D := I - D + DM, \tag{23}$$

I is the $n\times n$ identity matrix, D is the diagonal matrix $\mathrm{diag}(d_i)$ with $0 \leq d_i \leq 1$, for all $i = 1, \ldots, n$ and $r(x) := \min(x, Mx+q)$, where the min operator denotes the componentwise minimum of two vectors.

In [13], another decomposition of a B_π^R-matrix involving a parameter ε was obtained and applied to derive bounds for the error of the LCP when the associated matrix is a B_π^R-matrix with $\pi_j > 0$ for all j. It was also used in [9]. Let us now recall it in order to compare it with our new decomposition.

Given a B_π^R-matrix $M = (m_{ij})_{1\leq i,j\leq n}$, by (1) there exists $j \in \{1, \ldots, n\}$ such that $\pi_j > 0$. By (2) there exists an $\varepsilon > 0$ such that

$$\pi_j - \varepsilon > 0 \text{ and } m_{ij} - (\pi_j - \varepsilon)R_i < 0, \ \forall\, i \neq j. \tag{24}$$

Then we can write

$$M = B^+(\varepsilon) + C(\varepsilon), \tag{25}$$

where

$$B^+(\varepsilon) = \begin{pmatrix} m_{11} - \pi_1 R_1 & \ldots & m_{1j} - (\pi_j - \varepsilon)R_1 & \ldots & m_{1n} - \pi_n R_1 \\ \vdots & & \vdots & & \vdots \\ \vdots & & \vdots & & \vdots \\ m_{n1} - \pi_1 R_n & \ldots & m_{nj} - (\pi_j - \varepsilon)R_n & \ldots & m_{nn} - \pi_n R_n \end{pmatrix} \tag{26}$$

and

$$C(\varepsilon) = \begin{pmatrix} \pi_1 R_1 & \ldots & \pi_{j-1} R_1 & (\pi_j - \varepsilon)R_1 & \pi_{j+1} R_1 & \ldots \pi_n R_1 \\ \vdots & & \vdots & \vdots & \vdots & \vdots \\ \vdots & & \vdots & \vdots & \vdots & \vdots \\ \pi_1 R_n & \ldots & \pi_{j-1} R_n & (\pi_j - \varepsilon)R_n & \pi_{j+1} R_n & \ldots \pi_n R_n \end{pmatrix}. \tag{27}$$

In order to bound the error of the corresponding LCP, we have to provide an upper bound for $\|M_D^{-1}\|_\infty$, where M_D is given by (23) and M is a B_π^R-matrix for a vector $\pi = (\pi_1, \ldots, \pi_n)$ with $\pi_i > 0$ for all $i = 1, \ldots, n$. Let $B^+(\varepsilon)$ and $C(\varepsilon)$ be the matrices given by (26) and (27) and let

$$C_D := DC(\varepsilon), \quad B_D^+ := I - D + DB^+(\varepsilon). \tag{28}$$

By Proposition 2 of [13], $B_D^+(\varepsilon)$ is a strictly diagonally dominant Z-matrix with positive diagonal entries and so it has positive row sums. For each $i = 1, \ldots, n$, let us denote by $\beta_i > 0$ the sum of the entries of the ith row of $B_D^+(\varepsilon)$ and let $\beta(\varepsilon) := \min_i\{\beta_i\}$. The following result shows the mentioned upper bound for $\max_{d\in[0,1]^n} \|M_D^{-1}\|_\infty$ given in Theorem 1 of [13]. It uses the parameter ε.

Theorem 5 *Let M be a B_π^R-matrix for a vector $\pi = (\pi_1, \ldots, \pi_n)$ with $\pi_i > 0$ for all $i = 1, \ldots, n$ and let M_D, C_D, and B_D^+ be given by (23), (28) and $B^+(\varepsilon) =: (b_{ij})_{1\le i,j\le n}$. Then*

$$\max_{d\in[0,1]^n} \|M_D^{-1}\|_\infty \le \frac{\max_i \left\{\frac{1}{\pi_i} - 1\right\}}{\min\{\beta(\varepsilon), 1\}}, \tag{29}$$

where $\beta(\varepsilon) := \min_i\{\beta_i\}$ and $\beta_i := b_{ii} - \sum_{j\ne i} |b_{ij}|,\ i = 1, \ldots, n$.

In this section, we present a new bound for the error of the LCP associated to a B_π^R-matrix for a vector $\pi = (\pi_1, \ldots, \pi_n)$ with $\pi_i > 0$ for all $i = 1, \ldots, n$ by using the decomposition of Proposition 2. In contrast to the previous bound, it will not depend on a parameter.

Given M, a B_π^R-matrix for a vector $\pi = (\pi_1, \ldots, \pi_n)$ with $\pi_i > 0$ for all $i = 1, \ldots, n$, we can define again $M_D = (\bar{m}_{ij})_{1\le i,j\le n}$ by (23) for any diagonal matrix $D = \operatorname{diag}(d_i)$ with $0 \le d_i \le 1$ for all $i = 1, \ldots, n$. If B^+ and C are the matrices given by the decomposition of M given in Proposition 2, then we can define the corresponding matrices B_D^+, C_D by

$$C_D := DC, \quad B_D^+ := I - D + DB^+, \quad B^+ = (b_{ij})_{1\le i,j\le n}. \tag{30}$$

The following result gives an upper bound for $\|M_D^{-1}\|_\infty$.

Theorem 6 *Suppose that $M = (m_{ij})_{1\le i,j\le n}$ is a B_π^R-matrix for a vector π with $\pi_i > 0$ for all $i = 1, \ldots, n$ and let $M_D = (\bar{m}_{ij})_{1\le i,j\le n}, C_D$ and B_D^+ be the matrices given by (23) and (30). Then B_D^+ is a strictly diagonally dominant Z-matrix with positive diagonal entries and*

$$\max_{d\in[0,1]^n} \|M_D^{-1}\|_\infty \le \frac{\max_{1\le i\le n} \left\{\frac{1}{\pi_i} - 1\right\}}{\min_{1\le i\le n} \left\{1, R_i - \gamma_i \sum_{j=1}^n \pi_j\right\}}, \tag{31}$$

where, for each $i = 1, \ldots, n$, R_i and γ_i are given by Definition 1 and Proposition 2, respectively.

Proof It is easy to check that M_D is a $B_\pi^{\bar{R}}$-matrix where $\bar{R} = (\bar{R}_1, \ldots, \bar{R}_n)^T$ and that $\bar{R}_i = (1 - d_i) + d_i R_i$ for each $i = 1, \ldots, n$. We can observe that the decomposition (10) of M_D is given by the matrices B_D^+ and C_D of (30). Then

$$\max_{d\in[0,1]^n} \|M_D^{-1}\|_\infty \le \max_{d\in[0,1]^n} \|(I + (B_D^+)^{-1} C_D)^{-1}\|_\infty \max_{d\in[0,1]^n} \|(B_D^+)^{-1}\|_\infty. \tag{32}$$

Following the argumentation given in the proof of Theorem 3, we can give the same bound for the first factor of (32). So, we have that

$$\max_{d\in[0,1]^n} \|(I + (B_D^+)^{-1} C_D)^{-1}\|_\infty \le \max_{1\le i\le n} \left\{ \frac{1}{\pi_i} - 1 \right\}. \tag{33}$$

The matrix B_D^+ is a strictly diagonally dominant Z-matrix with positive diagonal entries, and so, taking into account (30), we can write

$$\alpha_i^D = (1-d_i) + d_i b_{ii} - \sum_{j\neq i}^{n} d_i |b_{ij}| = 1 - d_i + d_i \sum_{j=1}^{n} (m_{ij} - \gamma_i \pi_j) > 0.$$

By Theorem 1 of [23], we deduce that

$$\|(B_D^+)^{-1}\|_\infty \le \frac{1}{\min_{1\le i\le n} \alpha_i^D} = \frac{1}{\min_{1\le i\le n}\{1 - d_i + d_i \sum_{j=1}^{n} (m_{ij} - \gamma_i \pi_j)\}}. \tag{34}$$

Let us consider an index $k \in N$ such that $\alpha_k^D = \min_i\{\alpha_i^D\}$. Then

$$\alpha_k^D = 1 - d_k + d_k \sum_{j=1}^{n} (m_{kj} - \gamma_k \pi_j) = 1 - d_k + d_k (R_k - \sum_{j=1}^{n} \gamma_k \pi_j).$$

If $R_k - \sum_{j=1}^{n} \gamma_k \pi_j \ge 1$, then $\alpha_k^D \ge 1$ for any $d_k \in [0,1]$, and so, $\|(B_D^+)^{-1}\|_\infty \le 1$. Otherwise, we have that $\alpha_k^D \le R_k - \sum_{j=1}^{n} \gamma_k \pi_j$ for any $d_k \in [0,1]$. Taking into account these cases, we can bound (34) as follows

$$\|(B_D^+)^{-1}\|_\infty \le \frac{1}{\min_{1\le i\le n} \left\{1, R_i - \gamma_i \sum_{j=1}^{n} \pi_j \right\}}. \tag{35}$$

So we conclude that (31) holds since it is the product of the bound (35) for $\|(B_D^+)^{-1}\|_\infty$ and the bound (33) for $\|(I + (B_D^+)^{-1} C_D)^{-1}\|_\infty$. □

We can deduce the next result from Proposition 1, Remark 2 and Theorem 6.

Corollary 4 *Let M be a square matrix with positive row sums $R = (R_1, \ldots, R_n)^T$ satisfying* (5)*, let $\pi = (\pi_1, \ldots, \pi_n)$ be the positive vector given by* (8) *and let γ_j be given as in Proposition 2 for $j = 1, \ldots, n$. Then M is a B_π^R-matrix and formula* (31) *holds.*

Finally, we are going to present some numerical examples to compare our new results with previous ones. The test matrices are those used in the previous section. In this case, we have computed bounds for the error of the LCP using Theorem 6 (that used the given vector π in Table 1) and Corollary 4. We show the results obtained following this approach in the third and fourth rows of Table 3. We compare the results with those obtained using the bounds introduced in [13] and [9], which are included in the first two rows of Table 3. We borrowed the data from the original articles whenever possible, and we

Matrix	$A_1(8)$	$M_1(21/25)$	$M_2(8/9)$	$M_3(1/2)$
LCP [13]	26.389	10	6	10
LCP [9]	20.192	9.9125	6.6667	9
Theorem 6	30.083	10.4167	10.1250	17.500
Corollary 4	7	5.5882	4.1720	7.0200

Table 3 Bounds for the LCP.

computed the corresponding bound when it was not available. These bounds also use the parameter vector π given by Table 1.

Table 3 shows that the bounds obtained with Theorem 6 using a given vector π are not necessarily sharper. However, we can see that the new bounds given by Corollary 4 are sharper in all cases. Moreover, another advantage of this approach is that it can be applied to any matrix with positive row sums to first identify if it is a B_π^R-matrix. If so, it computes a compatible vector π and then we can apply our new bounds without further modifications.

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