Crisis curves in nonlinear business cycles

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A B S T R A C T

Intermittent behavior of economic dynamics is studied by a nonlinear model of business cycles when two of its control parameters, amplitude and frequency of a periodic exogenous driving force, are changed. This study points out how similar wanted situations may be reached by changing any of both parameters, and it gives a deeper understanding of what happens in actual economic data when some control parameters are changed at the same time.

Keywords:
Nonlinear business cycles
Intermittent behavior
Bifurcations
Routes to chaos

1. Introduction

Macroeconomic theory has been limited for decades to the study of models where the dynamics are focused on the analysis of the local equilibrium points [1]. This analysis is rigorous when considering linear models, but misleading results arise when dynamics are inherently nonlinear. Economic systems frequently show large amplitude and aperiodic fluctuations, evidencing the complex dynamics that control them. Different methods have been proposed with the aim of testing whether a series is random or it has an underlying general structure [2] as a first step to show that the system may have been generated within the framework of a chaotic deterministic system. However, the line between fluctuations caused by random-shocks and endogenous fluctuations defined by the nonlinear nature of the relationship between different economic variables is not clearly defined. So, the analysis of nonlinear models is a powerful tool for patterns recognition and for increasing the understanding of business cycles. Besides, economic systems are controlled by different intrinsic parameters. Some of them can be modified by economic agents' actions. Therefore, to know how the behavior of the model changes for different values of the parameters it is necessary to decide the right way of getting a profit as high as possible. Finally, finding out any dynamical or structural invariant that would serve as the key signature uniting often diverse nonlinear systems in one class, allow us to easily deduce some properties demonstrated for the former model.

The current economy is composed of a variety of separate activities but closely linked, directly or indirectly, with other sectors. Thus, international trade has grown faster than national one, and the proliferation and reduction in transport costs have increased the interdependence. In this paper we analyze a very simplified model of two interrelated economies by using different state-of-the-art numerical techniques to study nonlinear systems. The results of this analysis show the dependence of the qualitative behavior of the system with respect to its control parameters. This example shows how the ideas and techniques in nonlinear science and complexity help to study problems in theoretical economy.

The present paper is organized as follows. In Section 2, we introduce the nonlinear model used in this paper to analyze business cycles. Section 3 presents a uniparametric study of the behavior of the system and Section 4 shows a global vision

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by means of a careful analysis of the complete Lyapunov spectrum in the biparametric space. Finally, in Section 5 we present some conclusions.

2. The nonlinear economic model

Given a certain economy, let us denote its income by x; savings by $s = \sigma x$ and investments by I. Following Phillips' assumptions [3], we will suppose that income increases proportionally to the difference of investments and savings. Besides, a delay will be assumed in the adjustment of investments. Moreover, we introduce a nonlinear term in the expression of variation of investments to ensure that the system has a persistent cyclic movement of finite amplitude. We remark that we have multiple ways of choosing this nonlinear term. For example, Hicks [4] used linear constraints, Goodwin [5] simplified the restrictions through a sigmoid function and Puu [10] introduced a cubic term $(dx/dt')^3$. Here, we modify the original constant proportionality factor v by adding a quadratic term in x. So we reach the model

$$\frac{dI}{dt'} = \left(v + (1+\sigma - v)x^2\right)\frac{dx}{dt'} - I. \tag{1}$$

Considering identical unit lags for all adjustments, we have

$$\frac{dx}{dt'} = I - \sigma x. \tag{2}$$

Now, differentiating Eq. (2) and using expression (1), we have

$$\frac{d^2x}{dt'^2} = \frac{dI}{dt'} - \sigma\frac{dx}{dt'} = (v + (1+\sigma - v)x^2)\frac{dx}{dt'} - I - \sigma\frac{dx}{dt'}.$$

Finally, by substituting I from (2), we arrive to

$$\frac{d^2x}{dt'^2} = (v - 1 - \sigma)(1 - x^2)\frac{dx}{dt'} - \sigma x. \tag{3}$$

If now we consider two regions and we introduce interregional trade by a linear import–export multiplier, then we have to add the difference between imports and exports [10]. Taking into account that exports from *region 1* are imports of *region 2*, we have that this difference is $m_i x_i - m_j x_j$, where m_i is the constant propensity to import of region i.

So, we obtain the coupled system:

$$\frac{d^2x_1}{dt'^2} - (v_1 - 1 - \sigma_1)(1 - x_1^2)\frac{dx_1}{dt'} + (\sigma_1 + m_1)x_1 - m_2x_2 = 0, \tag{4}$$

$$\frac{d^2x_2}{dt'^2} - (v_2 - 1 - \sigma_2)(1 - x_2^2)\frac{dx_2}{dt'} + (\sigma_2 + m_2)x_2 - m_1x_1 = 0. \tag{5}$$

Note that this system is similar to the one given in pp. 81 of [10].

If we suppose that the region 1 is very small with respect to the region 2, then m_1x_1 is negligible in Eq. (5), that is to say, region 1 is a small open economy influenced by the rest of the world but the first has practically no influence on the second [10]. So, x_2 converges to a limit cycle that can be approximated by a simple periodic motion, $x_2 \approx a^*\cos(\omega^* t')$. Therefore, we just have to study the behavior of region 1 by means of Eq. (4) with $a^*\cos(\omega^* t')$ instead of x_2.

Now, just for convenience, we make a change in time scale, taking a new time $t = \sqrt{\sigma_1 + m_1}\,t'$ and obtaining

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = a\cos(\omega t), \tag{6}$$

where we have removed the subscript 1 and $\dot{x}$ represents the derivative with respect to t. The three control parameters are $\mu = (v_1 - 1 - \sigma_1)/\sqrt{\sigma_1 + m_1}$, an endogenous damping factor; $a = m_2a^*/(\sigma_1 + m_1)$, the driven amplitude of the periodic exogenous driving force; and $\omega = \omega^*/\sqrt{\sigma_1 + m_1}$, its driven frequency. Eq. (6) corresponds with the forced (or driven) van der Pol oscillator [9–12]. This equation can also describe market fluctuations driven by, for example, climate variabilities, pig cycle or building cycle [5]. In many other fields of science, the van der Pol equation arises as a typical model of self-excited oscillations and a lot of work has already been done to investigate its properties [13–18], however bifurcations of this system have been less studied.

3. Uniparametric dependency

In [6–8] the authors perform a numerical analysis of Eq. (6), and two explosions of chaotic sets [19] are studied: merging crisis and type-I intermittency boundary crisis. As in all these works, throughout this paper the value of the damping factor μ is fixed to be $\mu = 1$. In this section, first we show several bifurcation diagrams to study a 1D-parametric evolution of the system pointing out such boundary crisis. The links between chaotic saddles, unstable periodic orbits and chaotic attractors are studied within a small window changing a or ω and fixing the values of the other two parameters.

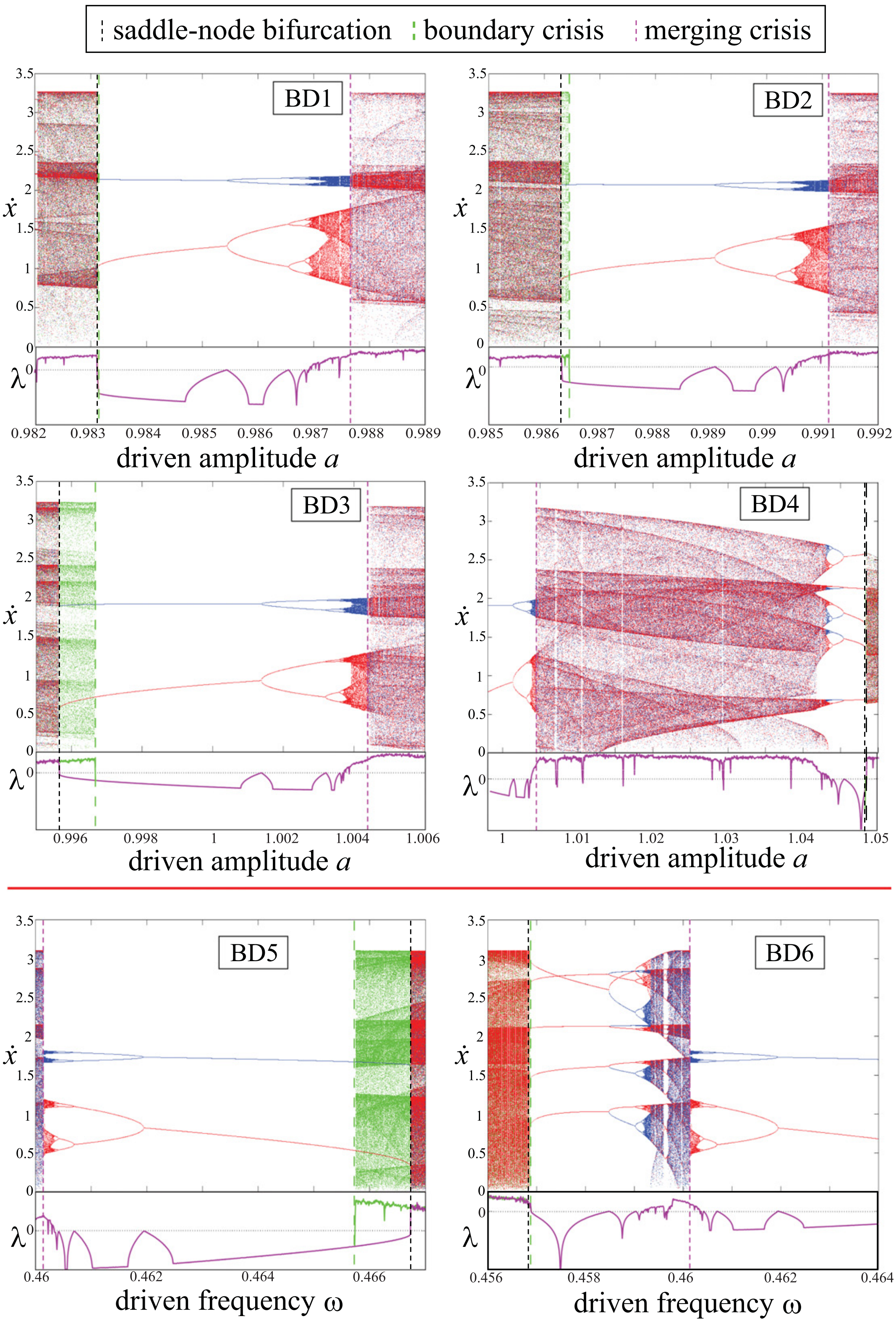

Fig. 1. Bifurcation diagrams of $\dot{x}$ as a function of the driven amplitude a for different values of driven frequency ω (BD1: $\omega = 0.45$; BD2: $\omega = 0.455$; BD3 and BD4: $\omega = 0.467$); or a function of ω for $a = 0.9954$ (BD5 and BD6). Different colors represent different attractors. Under each bifurcation diagram, the maximum Lyapunov exponent different from zero is shown. Boundary crisis, merging crisis and saddle-node bifurcations are also marked.

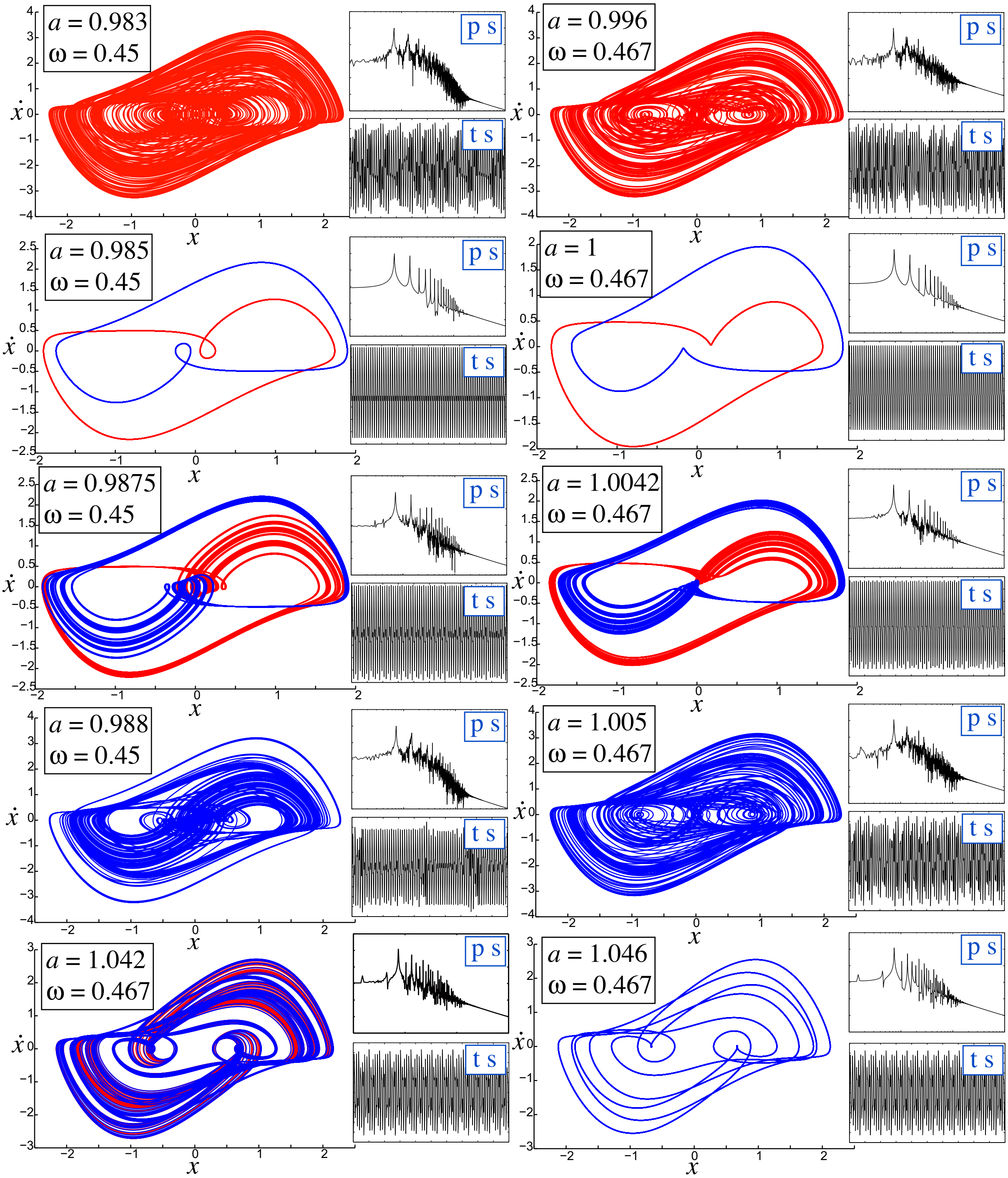

Fig. 2. Several attractors for different values of the driven amplitude and the driven frequency. Power spectra (ps) and time series (ts) of each attractor are shown in the small plots. When the attractor is not symmetric, two attractors (one in blue and another in red) are plotted (each attractor is the mirror image of the other one). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

To be precise, BD1 on Fig. 1 corresponds with the values used by [6–8]. This plot shows a bifurcation diagram with $\omega = 0.45$ and varying the driven amplitude a. For the starting value of a there is chaotic behavior. When a is increased, the system arrives at a saddle-node bifurcation, where two nonsymmetric periodic orbits appear and coexist with, at least, a symmetric chaotic attractor (each different attractor is plotted with a different color). Subsequently, the chaotic attractor disappears at a boundary crisis and only periodic orbits remain. Later, a periodic doubling chain ends into chaotic motion, with two nonsymmetric chaotic attractors, which join together in a merging crisis generating a symmetric chaotic attractor bigger than the simple union of two former nonsymmetric attractors. The first four attractors shown on the left in Fig. 2 correspond with values on BD1 which illustrate that parametric evolution.

Evolution described for BD1 is not unique, in fact, BD2 and BD3 on Fig. 1 correspond with the same behavior, the only main difference is an increment of the region of coexistence of at least one chaotic attractor with two stable periodic orbits. The first four attractors shown on the right in Fig. 2 correspond with values on BD3 and have similar characteristics than those on the left (on BD1). BD4, however, begins in the middle of BD3 and continues to higher values of driven amplitude. Here boundary crisis and saddle-node bifurcation happen on the right, toward the left (decreasing the amplitude), a symmetric stable periodic orbit crosses a supercritical pitchfork bifurcation, where this orbit loses its stability and two nonsymmetric stable orbits appear; later a period doubling chain ends in chaotic motion. The bottom two attractors on Fig. 2 correspond with this last case. Next to each attractor, its power spectrum and time series light on the characteristics of each one. Both situations can be reproduced by changing the parameter ω, situation that can be seen in figures BD5 and BD6. The first one corresponds to the situation of plots BD1, BD2 and BD3; while the second one generates the same trend as BD4 (but in opposite direction).

Considering bifurcation diagrams BD1, BD2, BD3 and BD5, it seems logical that they are related in some way. Nevertheless, with these diagrams we can hardly go beyond. But, under each diagram, we have plotted the maximum Lyapunov exponent different from zero and this will be the key to find out the connection between the different diagrams with the same evolution. So, from these standard pictures of 1D-parametric bifurcation diagrams it is not possible to give a complete analysis of the system. Therefore, it seems quite important to study the influence of more parameters on the model.

In the next section we show how these bifurcations appear for different values of the control parameters, describing curves (even surfaces using more parameters). Therefore, it will give a more global idea of the behavior of the model.

4. Biparametric analysis

To perform a systematic search of different kinds of behavior of Eq. (6), we use as main technique the full spectrum of Lyapunov exponents [20,21], computed by using the algorithm given in [22]. In Fig. 3 we present a biparametric plot, varying driven amplitude a and frequency ω of the exogenous driving force. Although most of the parametric plane has a periodic or quasi-periodic motion, our analysis focuses on a rectangular window with a greater richness of bifurcations.

We plot in blue periodic motion region ($\lambda_1 = 0, \lambda_2 < 0$), in yellow quasi-periodic motion region ($\lambda_1 = \lambda_2 = 0$) and in red chaotic region ($\lambda_1 > 0$). We remark that there are regions of multistability, but here we just mark the behavior given by a fixed set of initial conditions ($x(0) = 0.2108, \dot{x}(0) = 0.0187$). Yellow curves represent period doubling or pitchfork (some of them marked with a circled P) bifurcations. These bifurcation curves are the skeleton of stable shrimp-shaped domains (SSD).

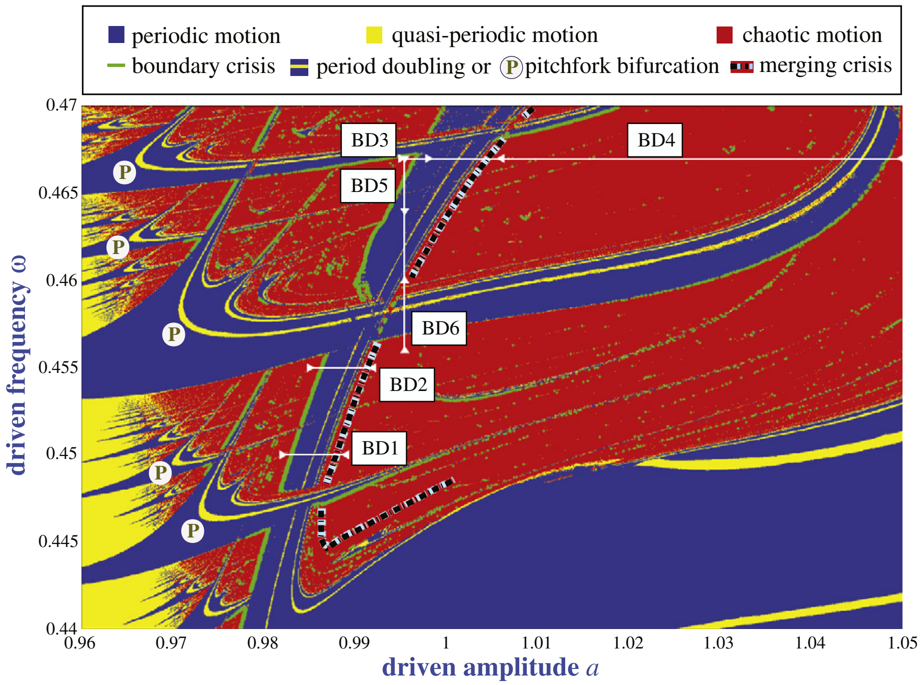

Fig. 3. Different kinds of behavior: periodic motion (blue), quasi-periodic motion (yellow), chaotic motion (red) and several bifurcation curves (see legend). Segments in white mark bifurcation diagrams shown in Fig. 1. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

These structures [23] appear in a great deal of dynamical systems in physics, chemistry [24] or biology [25], for example. Here we can see that, in this economic model, SSD appear in a typical self-similar fractal chain. Therefore, the large number of recent studies on SSD may shed light on the behavior of this system.

Back to the explosions of chaotic sets, analyzing the complete Lyapunov spectrum when boundary crisis or merging crisis take place, we have succeeded in locating the curves determined by such explosions (green for boundary crisis and light blue-black for merging crisis). In fact, if we worked in the three-parametric space, then we would find surfaces. The six intervals used to perform the plots of Fig. 1 are marked on Fig. 3 with white segments. Now, we can see that, with Fig. 3, the selection of intervals with the same qualitative behavior is very easy. For instance, the same situation is reproduced along the diagonal structure with $a \approx 0.98$ (where BD1, BD2, BD3 and BD5 are taken): on the left edge (smaller values of amplitude) a boundary crisis takes place where the chaotic attractor turns into a chaotic saddle; shortly before the boundary crisis takes place, two stable periodic orbits (and two unstable) appear in a saddle-node bifurcation and towards the right (increasing the values of amplitude) a period doubling chain leads those orbits to chaotic motion; later a merging crisis changes abruptly the size of the existing chaotic attractor. We observe that a similar route can be covered by decreasing the value of driven frequency ω. This was the situation of bifurcation diagrams BD1, BD2, BD3 (changing a) and BD5 (changing ω). However, plots BD4 and BD6 begin in the same strip and crosses over a different one, and in this strip the evolution is different, as it was commented in Section 3.

This study states that, to reach a certain behavior in a certain small open economy, we may play with several parameters and small changes of some of them may give rise to completely different situations. Thus, by estimating the actual state of a certain small open economy, its economy agents can decide whether it is advisable to modify the situation to get higher incomes or a more stable economy. Thus, given the meaning of the parameter a in this model, development of plans to increase (or decrease) its exports will generate an increased (or decreased) value of a and therefore a way to control its economy. For a small economy, to change the value of ω is a more delicate task because it only influences the values of σ_1 and m_1, and these values affect the other parameters in different ways. But a careful study of the situation, together with the parametric analysis performed in this section, can help to achieve the desired results.

5. Conclusions

In this paper we show how the same qualitative behavior on a model of business cycles can be obtained when we change driven amplitude a and frequency ω of a periodic exogenous driving force controlled, among others, by the propensity to import. This study points out the importance of understanding the dynamics of an economic model on its complete parametric space, when similar situations can be reached changing the values of the different control parameters of the economic model. And thus, the economy agents of the small economy can decide whether it is advisable to modify the situation to achieve the desired results (higher incomes, a more stable economy, ...). Also, this paper shows how the ideas and techniques in nonlinear science and complexity help to study problems in theoretical economy.

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