Bachelor Thesis

Technische Universität Wien

STUDY OF SELF-EXCITED WHEEL HOP

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Summary

In the context of vehicle dynamics, this work focuses on a specific section: the study of self-excited wheel hop. self-excited wheel hop is a phenomenon in which a rotating wheel can be lifted off a flat surface without any external force. This study deals with the stability of a dynamic system as a function of different parameters: the effective coupling coefficient of the rolling radius η; the β slope, which indicates the inclination of the steering surface of the wheel axle; and the damping coefficient k of the spring damper. The main goal is to assess the range of the slope β where instability will occur, getting a plot of the stability boundary in the *tan(*β*)* versus η diagram.

In order to achieve this goal, different steps have been taken. First of all, the equations modelling the dynamic system have been obtained. Afterwards, based on these equations, the Routh Hurwitz method has been studied and used in order to get the results wanted, obtaining also an explicit equation that relates the three parameters and gives the results of the stability boundary. Finally, based on these results, the stability boundary has been plotted in the *tan(*β*)* versus η diagram, using also different values of the damping coefficient k to see how this stability changes.

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1 Introduction

1.1 General Introduction

Tyre mechanics is a very broad field influenced by many factors and where great progress has been made during the last decades. The operational properties of the road vehicle are the result of the dynamic interaction of the various components of the vehicle structure, which may include modern control elements.

The introduction of new and more refined experimental techniques, coupled with the advent of the electronic computer, has enabled the formulation and utilisation of more realistic mathematical models of the tyre in a wide range of operational conditions, leading to a better understanding of tyre behaviour and its role as a vehicle component.

One of the key elements in tyre dynamics is the phenomenon of self-excited wheel hopping. Self-excited wheel hopping is a phenomenon whereby a rotating wheel is able to jump on a flat surface without an external impulse, thus defying the laws of conventional physics. This phenomenon is the result of a complex interaction between a number of factors, including wheel geometry, surface friction properties, mass distribution and rotational dynamics. It is typically observed during periods of high acceleration, particularly when the vehicle is equipped with a substantial amount of power or torque. The occurrence of wheel hopping can result in the loss of traction, instability, and potential damage to the vehicle. This phenomenon is attributed to a complex interplay between various factors, including the interaction between the tires, suspension system, drivetrain, and road surface. Therefore, to mitigate self-excited wheel hop, engineers employ various techniques, including the optimisation of suspension geometry, the use of stiffer bushings, the adjustment of damping rates, and the implementation of traction control systems. Additionally, the selection of appropriate tyres and the maintenance of smooth road surfaces can also help to reduce the likelihood of encountering this phenomenon.

In order to optimize road vehicles and other applications, this phenomenon must be deeply studied. The main objective of this thesis is to perform a comprehensive analysis, addressing both theoretical and practical aspects. In order to achieve this goal, an approach that combines theoretical tools from classical mechanics and dynamical systems theory with computational techniques, as MATLAB, will be employed. Therefore, the existing literature on the subject will be investigated and a study of the behaviour of self-excited wheel hopping in different conditions and configurations will be carried out.

In summary, this work aims to provide a deeper and more detailed understanding of the physical foundations and potential applications of the self-excited wheel hopping phenomenon.

1.2 What causes wheel hopping?

As it has been mentioned before, wheel hopping is the result of a complex interaction between a number of factors, and it can occur in different situations. These include the following:

- Aggressive acceleration is a phenomenon whereby a driver accelerates in a manner that results in the torque applied to the driven wheels exceeding the traction available, which in turn can result in wheel hop.

- Irregularities on the surface: when a wheel encounters an irregularity on the road and the others remain on a smooth surface, this phenomenon can be caused.

- Inappropriate tyre pressure or inflation: this can affect the traction and generate oscillations.

- Inadequate suspension can also lead to wheel hopping, particularly in instances of aggressive or high-performance driving. Consequently, it is of paramount importance to ascertain the correct values of the spring and damper coefficients in these systems.

For all these reasons above, it is crucial to optimize all components of a car related to this phenomenon in order to reduce its frequency of occurrence.

1.3 Different areas in which self-excited wheel hop occurs

The phenomenon of self-excited wheel hop can be observed in a number of contexts, including racing, military vehicles, off-road vehicles, and agricultural machinery. In racing, the vehicle must perform at a high level, which presents a challenge due to the high power outputs. Therefore, it is crucial to optimise the vehicle's behaviour, maximising traction during launches and corner exits.

Similarly, off-road vehicles are also susceptible to this phenomenon, as they are driven on highly irregular terrain, which places a significant burden on the suspension system. This phenomenon also occurs in construction and agricultural machinery, such as excavators and tractors, as well as military vehicles, which require the maximum performance in a variety of environments.

1.4 How to correct wheel hopping?

Wheel hopping is a frustrating problem that can damage a vehicle. Therefore, it is important to mitigate this problem in order to reduce this damage and improve vehicle's performance. Some solutions to this problem are:

- Adjusting suspension system: it is crucial to have a correct balance between shock absorber characteristics and spring stiffness, apart from other parameters of the suspension system.
- Upgrade tyres and having a correct inflation of them in order to have better traction.
- Upgrade drivetrain
- Reduce engine torque
- Obtaining a correct wheel alignment

Implementing these solutions not only mitigates wheel hopping, but also improves vehicle durability and the driving experience.

2 Description of the mechanical model

The mechanical model consists of a system formed by a wheel anchored to a mass by its axis of rotation, this mass remaining under the action of two springs and a damper. This mass can move in a direction which is at a slope β with respect to the z-axis. The model is shown in *Figure 1*.

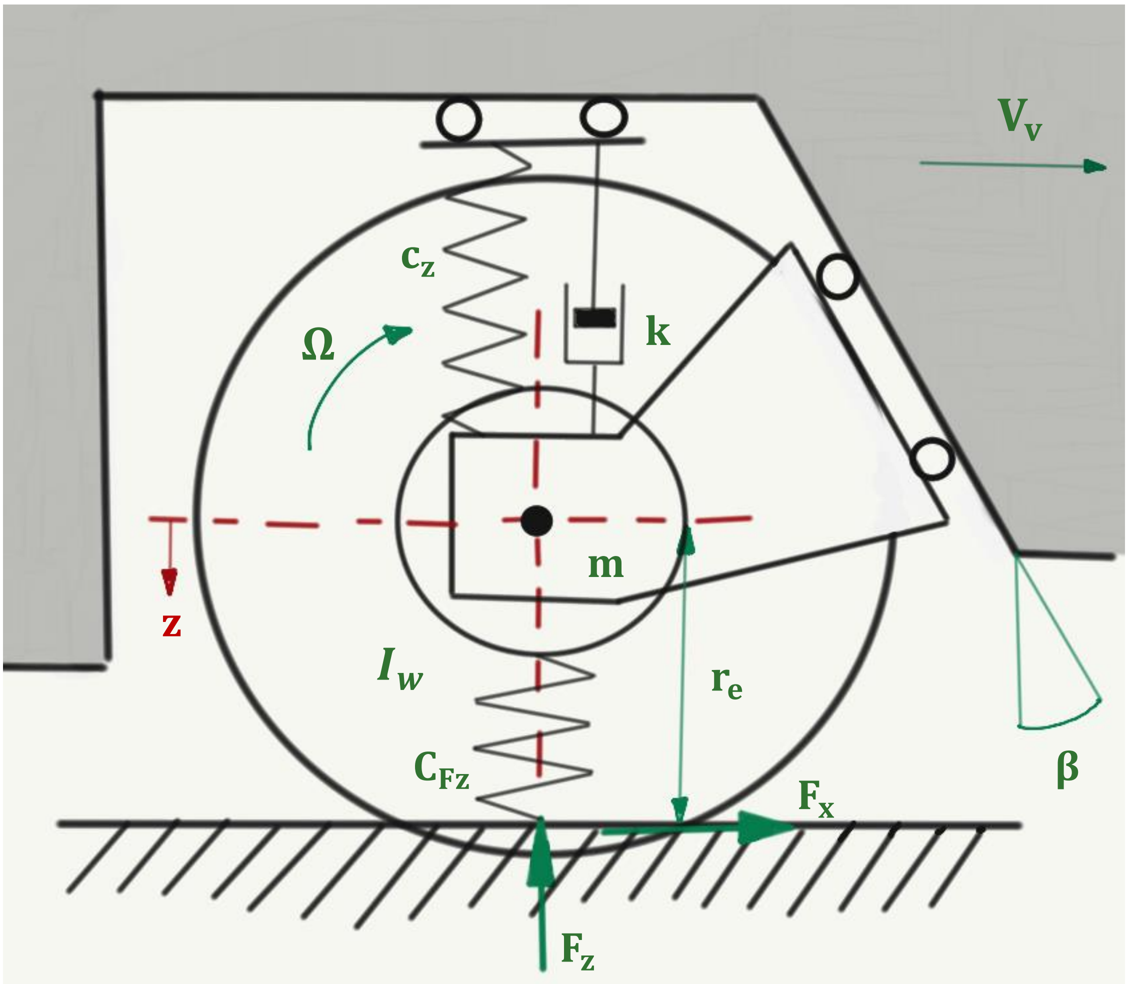

Figure 1. Model of the system

In this problem, the effective rolling radius coupling coefficient η is the decisive parameter. This coefficient quantifies how much the effective rolling radius changes in response to changes in vehicle speed or load.

It is assumed that $M_y = 0$ and we consider a flat road and a uniform tyre. The main goal is to study the stability of this model using the Hurwitz criterion, assessing the range of the slope β of the wheel axle guidance surface where instability will occur. For this, a graphic of the stability boundary must be plotted in the $\tan(\beta)$ versus η diagram, varying

η between the values 0 and 1, and using different values of the damping coefficient k in order to see how the stable and unstable areas change depending on this parameter.

The parameter values employed to solve the model are the following:

$$r_0 = r_{eo} = 0.3 \text{ m}$$

$$\sigma_k = \frac{C_{F_k}}{C_{F_x}} = 0.3 \text{ m}$$

$$C_{F_z} + c_z = C_{F_x} = 2 * 10^5 \text{ N/m}$$

$$m = 30 \text{ kg},\ I_w = 0.5 m r_0^2 \text{ kgm}^2$$

Furthermore, we assume a forward speed of the vehicle $V_v = 23$ m/s.

Given this model, the equations that model the system must be firstly obtained and then, the Hurwitz criterion must be applied in order to get the results wanted.

3 Resolution of the mechanical model

In this section, the resolution of the mechanical model given in the previous section will be presented. Starting with governing equations that model the system, the needed assumptions to simplify them will be introduced and, then, the stability of the model will be presented after the introduction of the Routh-Hurwitz criterion.

3.1 Equations modelling the system

Based on the model and the equations above, the equations of the fourth-order system must be derived.

The derivation starts with the following equation:

$$\frac{du}{dt} + \frac{1}{\sigma_k} Vu = V\kappa = -V_{sx} \Rightarrow \dot{u} = -\frac{1}{\sigma_k} Vu - (V - r_e \Omega) \tag{3.1}$$

This relaxation equation describes how the system returns to equilibrium after being disturbed, and it is the differential equation for the fore and aft deflection. Here, u represents the displacement or deflection in the longitudinal (fore-aft) direction; σ_k is the relaxation length, and relates the longitudinal tyre stiffness C_{F_x} and longitudinal slip stiffness C_{F_k}; V is the forward speed of the wheel; κ is the longitudinal slip ratio and, in this case, $\kappa \approx -V_{sx}/V$; and V_{sx} is the longitudinal slip velocity defined as $V_{sx} = V - r_e \Omega$.

Furthermore, $V = v + V_v$. Here, V_v is the forward speed of the vehicle and v is the forward speed of the wheel due to the movement of the mass m in the s direction.

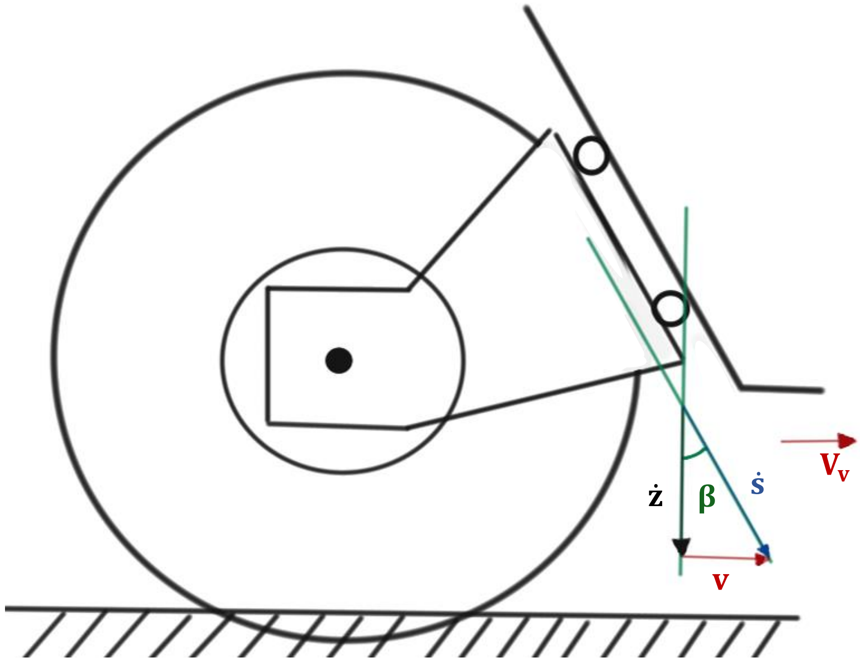

Figure 2. Forward speed of the wheel

$$\dot{u} = -\frac{1}{\sigma_k}(v + V_v)u - ((v + V_v) - r_e\Omega) \tag{3.2}$$

Here, $v = \dot{z}\tan(\beta)$:

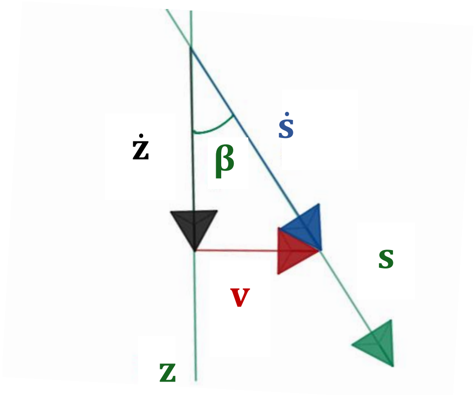

Figure 3. Relation between v and ż

And from the following equation:

$$\tilde{r}_e = \tilde{r}_f - \varepsilon\tilde{d}_t - \eta\tilde{\rho} = \tilde{r}_c + (1 - \varepsilon)\tilde{d}_t - \eta\tilde{\rho} \tag{3.3}$$

and the relationship $r_e = r_{eo} + \tilde{r}_e$, we get:

$$r_e = r_{eo} + \tilde{r}_e = r_{eo} + \tilde{r}_f - \varepsilon\tilde{d}_t - \eta\tilde{\rho} \tag{3.4}$$

The tilde $\sim$ means small variations, r_e is the effective rolling radius, r_f is the free (undeformed) radius; ε is the tread incoherence parameter; d_t is the tread thickness; r_C is the carcass radius; and ρ is the tyre deflection.

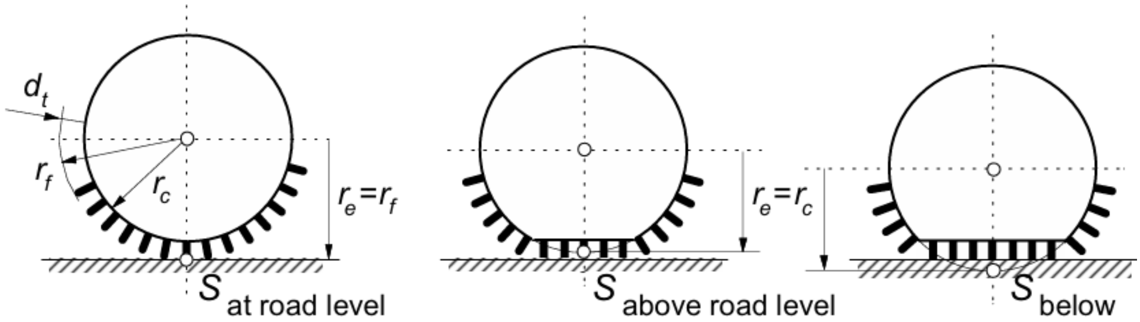

Figure 4. Different radius of the tyre

Furthermore, the following relation for the small variations of the tyre deflection ρ is known:

$$\tilde{\rho} = z + \tilde{r}_C + \tilde{d}_t - w \tag{3.5}$$

In this equation, w express the road height changes. We can neglect $\tilde{r}_C$, $\tilde{d}_t$ due to the assumption of a uniform tyre; and w, as we assume a flat horizontal road. Therefore, we obtain the following relation:

$$\tilde{\rho} = z \tag{3.6}$$

So, including this in Eq. (3.4) and neglecting the terms $\tilde{r}_f$ and $\varepsilon\tilde{d}_t$, we get:

$$r_e = r_{eo} + \tilde{r}_e = r_{eo} - \eta z \tag{3.7}$$

Furthermore, for this problem we use the relation $r_{eo} = r_f$, so Eq. (3.7) looks like this:

$$r_e = r_f - \eta z \tag{3.8}$$

As a next step we include this relation in Eq. (3.2) and continue deriving the equation:

$$\dot{u} = -\frac{1}{\sigma_k}(\dot{z}\tan(\beta) + V_v)u - ((\dot{z}\tan(\beta) + V_v) - (r_f - \eta z)\Omega) \tag{3.9}$$

Expanding this equation, we obtain the following non-linear equation:

$$\dot{u} = -\frac{1}{\sigma_k}u\dot{z}\tan(\beta) - \frac{1}{\sigma_k}V_v u - \dot{z}\tan(\beta) - V_v + \Omega r_f - \eta z\Omega \tag{3.10}$$

where $\Omega = \Omega_0 + \tilde{\Omega}$:

$$\dot{u} = -\frac{1}{\sigma_k} u\dot{z}\tan(\beta) - \frac{1}{\sigma_k} V_v u - \dot{z}\tan(\beta) - V_v + \Omega_0 r_f + \tilde{\Omega} r_f - \eta z \Omega_0 - \eta z \tilde{\Omega}$$

(3.11)

and $\Omega_0 = \frac{V_v}{r_f}$:

$$\dot{u} = -\frac{1}{\sigma_k} u\dot{z}\tan(\beta) - \frac{1}{\sigma_k} V_v u - \dot{z}\tan(\beta) + \tilde{\Omega} r_f - \eta z \frac{V_v}{r_f} - \eta z \tilde{\Omega} \quad (3.12)$$

Now, in order to linearize the equation in the simplest possible way, we dismiss the term $\frac{1}{\sigma_k} u\dot{z}\tan(\beta)$ as u and $\dot{z}\tan(\beta)$ are zero in the reference configuration; and we also dismiss the term $\eta z \tilde{\Omega}$. So, the equation is now linear and looks like this:

$$\dot{u} = -\frac{1}{\sigma_k} V_v u - \dot{z}\tan(\beta) + \tilde{\Omega} r_f - \eta z \frac{V_v}{r_f} \quad (3.13)$$

To get the second equation, we start from the balance of angular momentum:

$$I_w \dot{\Omega} = -rF_x - M_y \quad (3.14)$$

where I_w is the wheel polar moment of inertia; r the loaded radius; F_x the forces in the x direction; and M_y the moment in the y-axis.

From here, we consider a constant vehicle speed and we assume $M_y = 0$. Therefore, the only force acting in the x-axis is the one due to the variation of the tangential slip force, so:

$$F_x = \frac{C_{F_k} u}{\sigma_k} \quad (3.15)$$

Introducing this relation in Eq. (3.14):

$$I_w \dot{\Omega} = -r \frac{C_{F_k} u}{\sigma_k} \quad (3.16)$$

Furthermore, it is known that $\sigma_k = \frac{C_{F_k}}{C_{F_x}}$ and that the derivative of Ω is $\dot{\tilde{\Omega}}$, so the equation finally looks like this:

$$I_w \dot{\tilde{\Omega}} = -r C_{F_x} u \quad (3.17)$$

Now, to get the last equation, we have to derive the equation of motion for $z(t)$. In order to do this, we are getting the forces in the s-axis:

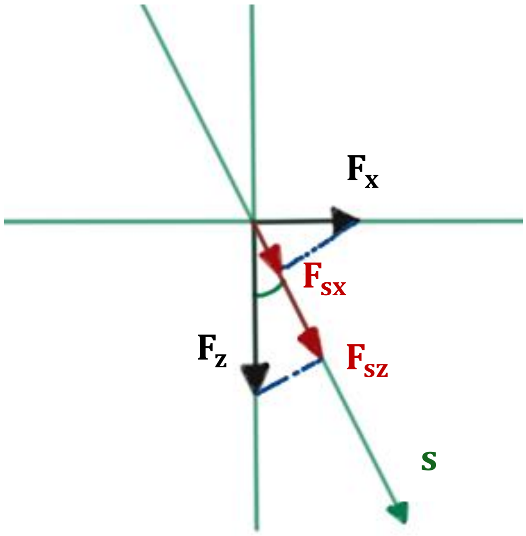

Figure 5. Forces on the different axes

$$m\ddot{s} = F_{sz} + F_{sx} \tag{3.18}$$

$$F_{sz} = F_z \cos(\beta) \tag{3.19}$$

$$F_{sx} = F_x \cos\left(\frac{\pi}{2} - \beta\right) = F_x \sin(\beta) \tag{3.20}$$

$$m\ddot{s} = F_z \cos(\beta) + F_x \sin(\beta) \tag{3.21}$$

In order to get the acceleration in the z-axis, the diagram of the *Figure 6* is needed:

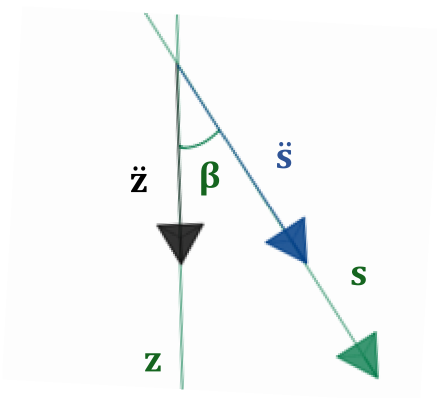

Figure 6. Acceleration in the z-axis

So, the acceleration in the z-axis is:

$$\ddot{s} = \frac{1}{\cos(\beta)} \ddot{z} \tag{3.22}$$

As it is shown in the *Figure 7*, a damper and two springs act on the mass, so the equation for the vertical force in the z-axis looks like this:

$$F_z = -c_z z - C_{F_z} z - k\dot{z} \qquad (3.23)$$

where it is assumed for simplicity that $c_z + C_{F_z} = C_{F_x}$:

$$F_z = -C_{F_x} z - k\dot{z} \qquad (3.24)$$

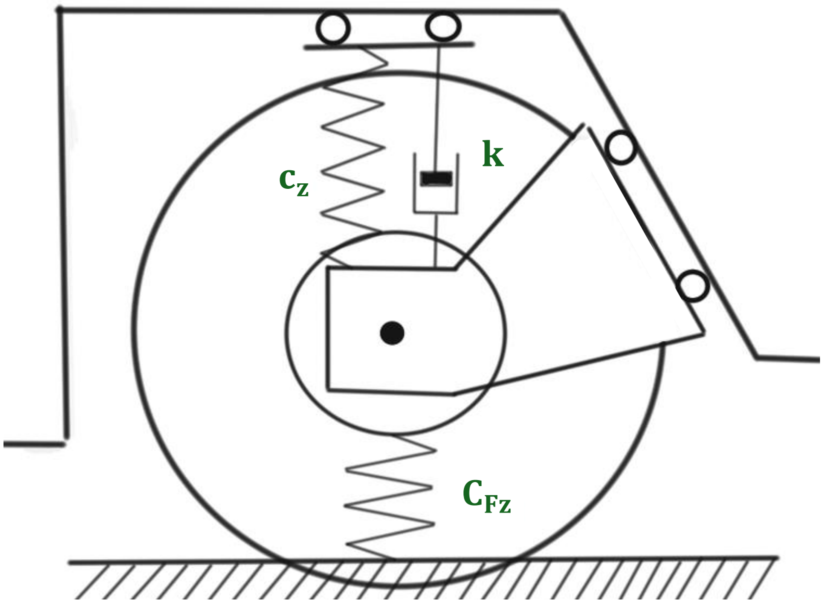

Figure 7. Forces in the z-axis

Now, we replace $\ddot{s}$, F_z and F_x in Eq.(3.21):

$$m\frac{1}{\cos(\beta)}\ddot{z} = \left(-C_{F_x} z - k\dot{z}\right)\cos(\beta) + C_{F_x} u \sin(\beta) \qquad (3.25)$$

and reorganize the terms:

$$m\ddot{z} = \left(-C_{F_x} z - k\dot{z}\right)\cos^2(\beta) + C_{F_x} u \sin(\beta)\cos(\beta) \qquad (3.26)$$

Therefore, the linearized system is as follows:

$$\left\{\begin{array}{l} \dot{u} = -\dfrac{1}{\sigma_k} V_v u - \dot{z}\tan(\beta) + \tilde{\Omega} r_f - \eta z \dfrac{V_v}{r_f} \\ \\ I_w \dot{\tilde{\Omega}} = -r C_{F_x} u \\ \\ m\ddot{z} = \left(-C_{F_x} z - k\dot{z}\right)\cos^2(\beta) + C_{F_x} u \sin(\beta)\cos(\beta) \end{array}\right.$$

Here, we have obtained the linearized system by cancelling the quadratic terms, which is a correct and simpler way, but the formal way to obtain the linearized system would be to consider the trivial configuration and derive the Jacobian at this state. In order to do this, we firstly identify the steady state, where $\dot{\tilde{\Omega}}, \dot{u}, \dot{z}$ and $\ddot{z}$ are 0. The trivial configuration is made up of the Equations (3.12), (3.17) and (3.26).

Therefore, for Eq.(3.17):

$$0 = -rC_{F_x}u \;\; => \; u = 0$$

For Eq.(3.26):

$$0 = \left(-C_{F_x}z\right)\cos^2(\beta) + C_{F_x}u\sin(\beta)\cos(\beta) \;\; => \; z = 0$$

For Eq.(3.12):

$$0 = -\frac{1}{\sigma_k}V_v u + \tilde{\Omega}r_f - \eta z\frac{V_v}{r_f} - \eta z\tilde{\Omega} \;\; => \;\; \tilde{\Omega} = 0$$

Now, once we have obtained the steady state, we calculate the partial derivatives of each equation with respect to each state variable and we evaluate them in the equilibrium point $\left[\tilde{\Omega} \quad u \quad z \quad \dot{z}\right] = [0 \;\; 0 \;\; 0 \;\; 0]$:

$$\frac{d\dot{u}}{d\tilde{\Omega}}\Big|_{\mathrm{e}} = r_f, \;\; \frac{d\dot{u}}{du}\Big|_{\mathrm{e}} = -\frac{1}{\sigma_k}V_v, \;\; \frac{d\dot{u}}{dz}\Big|_{\mathrm{e}} = -\eta\frac{V_v}{r_f}, \;\; \frac{d\dot{u}}{d\dot{z}}\Big|_{\mathrm{e}} = -\tan(\beta)$$

$$\frac{d\dot{\tilde{\Omega}}}{d\tilde{\Omega}}\Big|_{\mathrm{e}} = 0, \;\; \frac{d\dot{\tilde{\Omega}}}{du}\Big|_{\mathrm{e}} = -\frac{rC_{F_x}}{I_w}, \;\; \frac{d\dot{\tilde{\Omega}}}{dz}\Big|_{\mathrm{e}} = 0, \;\; \frac{d\dot{\tilde{\Omega}}}{d\dot{z}}\Big|_{\mathrm{e}} = 0$$

$$\frac{d\dot{z}}{d\tilde{\Omega}}\Big|_{\mathrm{e}} = 0, \;\; \frac{d\dot{z}}{du}\Big|_{\mathrm{e}} = 0, \;\; \frac{d\dot{z}}{dz}\Big|_{\mathrm{e}} = 0, \;\; \frac{d\dot{z}}{d\dot{z}}\Big|_{\mathrm{e}} = 1$$

$$\frac{d\ddot{z}}{d\tilde{\Omega}}\Big|_{\mathrm{e}} = 0, \;\; \frac{d\ddot{z}}{du}\Big|_{\mathrm{e}} = \frac{C_{F_x}\sin(\beta)\cos(\beta)}{m}, \;\; \frac{d\ddot{z}}{dz}\Big|_{\mathrm{e}} = \frac{-C_{F_x}\cos^2(\beta)}{m},$$

$$\frac{d\ddot{z}}{d\dot{z}}\Big|_{\mathrm{e}} = \frac{-k\cos^2(\beta)}{m}$$

Therefore, the Jacobian in the equilibrium point is:

$$J = \begin{bmatrix} \left.\frac{\partial \dot{\tilde{\Omega}}}{\partial \tilde{\Omega}}\right|_e & \left.\frac{\partial \dot{\tilde{\Omega}}}{\partial u}\right|_e & \left.\frac{\partial \dot{\tilde{\Omega}}}{\partial z}\right|_e & \left.\frac{\partial \dot{\tilde{\Omega}}}{\partial \dot{z}}\right|_e \\ \left.\frac{\partial \dot{u}}{\partial \tilde{\Omega}}\right|_e & \left.\frac{\partial \dot{u}}{\partial u}\right|_e & \left.\frac{\partial \dot{u}}{\partial z}\right|_e & \left.\frac{\partial \dot{u}}{\partial \dot{z}}\right|_e \\ \left.\frac{\partial \dot{z}}{\partial \tilde{\Omega}}\right|_e & \left.\frac{\partial \dot{z}}{\partial u}\right|_e & \left.\frac{\partial \dot{z}}{\partial z}\right|_e & \left.\frac{\partial \dot{z}}{\partial \dot{z}}\right|_e \\ \left.\frac{\partial \ddot{z}}{\partial \tilde{\Omega}}\right|_e & \left.\frac{\partial \ddot{z}}{\partial u}\right|_e & \left.\frac{\partial \ddot{z}}{\partial z}\right|_e & \left.\frac{\partial \ddot{z}}{\partial \dot{z}}\right|_e \end{bmatrix}$$

$$J = \begin{bmatrix} 0 & -\frac{C_{F_x} r_f}{I_w} & 0 & 0 \\ r_f & -\frac{1}{\sigma_k} V_v & -\eta \frac{V_v}{r_f} & \tan(\beta) \\ 0 & 0 & 0 & 1 \\ 0 & \frac{C_{F_x} u \sin(\beta)\cos(\beta)}{m} & -\frac{C_{F_x}\cos^2(\beta)}{m} & -\frac{k\cos^2(\beta)}{m} \end{bmatrix}$$

So, the linear equation system can be expressed as $X' = JX$, with X the state vector $X = [\tilde{\Omega} \quad u \quad z \quad \dot{z}]$.

As it can be seen, the result obtained is the same that we had obtained by cancelling the quadratic terms, but this form is more complex and correct.

And now, the stability of the system can be studied as a function of the three parameters mentioned above.

3.2 Routh-Hurwitz stability criterion

3.2.1 Explanation

In order to study the stability of the fourth-order system, the Routh-Hurwitz criterion is going to be employed. But, before solving the stability of the model, this criterion will be explained.

The Routh–Hurwitz stability principle is a mathematical examination that serves as a necessary and sufficient condition for the stability of a linear dynamic system or control setup. The Routh examination constitutes a technique first introduced by British mathematician Edward John Routh in 1876 to ascertain if all roots of the characteristic polynomial of a linear system possess negative real components. In 1895, German mathematician Adolf Hurwitz proposed independently to organise the coefficients of the polynomial into a square array, termed the Hurwitz array, demonstrating that the polynomial retains stability only when the determinants of its core sub-arrays are positive. Both methodologies are interchangeable; the Routh assessment provides a more effective approach to computing the Hurwitz determinants compared to direct calculation. A polynomial that fulfils the Routh–Hurwitz standard is designated a Hurwitz polynomial.

So, in order to study the stability using this criterion, the following steps are necessary:

1. Obtaining the characteristic polynomial of the system.
2. If the characteristic polynomial is not of very high order, it is enough to check that all the coefficients of the polynomial are positive and the determinants of the submatrices $H_{n-1}, H_{n-3}\ldots$ are positive too, with n the order of the polynomial. If this happens, all the roots of the characteristic polynomial will have negative real components and the system will be stable. Given a characteristic polynomial $p(s) = a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n$, the Hurwitz matrix is constructed as follows:

   $$H = \begin{vmatrix} a_1 & a_0 & 0 & 0 & \ldots & \ldots \\ a_3 & a_2 & a_1 & a_0 & \ldots & \ldots \\ a_5 & a_4 & a_3 & a_2 & \ldots & \ldots \\ a_7 & a_6 & a_5 & a_4 & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & a_{n-2} \\ 0 & 0 & \ldots & \ldots & 0 & a_n \end{vmatrix}$$

   So, the submatrix H_i would be the square matrix formed by the first i rows and columns of this matrix H.

Afterwards, it will be necessary to know the exact boundary where the system changes from stable to unstable.

3.2.2 Stability of the model

Once the criterion has been explained, we can study now the stability of the system.

First of all, in order to obtain the characteristic polynomial, we express the system with the form $X' = AX$, with X the state vector and A the coefficient matrix, which is the same as the Jacobian J calculated above.

Now, the characteristic equation can be obtained by calculating the determinant of $(A - sI) = 0$. After doing this in MATLAB, the following characteristic polynomial is obtained:

$$a_0 s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4 = 0 \tag{3.27}$$

Once the characteristic polynomial has been obtained, the next step is to check that all the coefficients a_i are positive and the determinant of the Hurwitz sub-matrix H_3 is also positive. The Hurwitz matrix for our characteristic polynomial is:

$$H_4 = \begin{bmatrix} a_1 & a_0 & 0 & 0 \\ a_3 & a_2 & a_1 & a_0 \\ a_5 & a_4 & a_3 & a_2 \\ a_7 & a_6 & a_5 & a_4 \end{bmatrix}$$

Here, a_7, a_6 and a_5 are 0.

So, the sub-matrix H_3 is the following:

$$H_3 = \begin{bmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ 0 & a_4 & a_3 \end{bmatrix}$$

So, we will check the coefficients taking into account the possible range of values of our parameters $\beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), k > 0$ and $\eta \in [0,1]$:

$$a_0 = 1 > 0 \tag{3.28}$$

$$a_1 = W * k * \cos^2(\beta) + K > 0 \tag{3.29}$$

$$a_2 = L * k * \cos^2(\beta) + P > 0 \tag{3.30}$$

$$a_3 = Q\big(\cos^2(\beta) + \eta * \cos(\beta) * \sin(\beta)\big) + S * k * \cos^2(\beta) > 0 \tag{3.31}$$

$$a_4 = T * \cos^2(\beta) > 0 \tag{3.32}$$

Here, the values for the variables are:

$$W = \frac{1}{30}$$

$$K = \frac{230}{30}$$

$$L = \frac{23}{9}$$

$$P = 20000$$

$$Q = \frac{4600000}{9}$$

$$S = \frac{6108397932088889}{13743895347200}$$

$$T = \frac{7635497415111111125}{8589934592}$$

As it can be seen above, a_0, a_1 and a_2 are always positive for the range of values of β, k and η.

Checking the coefficient a_4, we can see that it can only be 0 when $\beta = \pm\frac{\pi}{2}$, which is a value of β that it is not in the range mentioned.

On the other hand, the coefficient a_3 becomes negative for some values of η and k when β gets close to $-\frac{\pi}{2}$. Therefore, for these values, the system will be unstable. However, this coefficient only gives us part of all the unstable values, so this will not be enough to establish the stability boundary we are looking for and, as we will see later, the determinant of the sub-matrix H_3 will be the decisive factor determining stability.

After checking all the coefficients, the determinant of the sub-matrix H_3 must be checked:

$$|H_3| = a_1 a_2 a_3 - a_1^2 a_4 - a_3^2 a_0 > 0 \tag{3.33}$$

Therefore, for our data, we have:

$$|H_3| = A - A * \eta^2 + B * \eta + C = -A * \eta^2 + B * \eta + D > 0 \tag{3.34}$$

The values of the parameters are:

$$A = M\cos^2(\beta)\sin^2(\beta) \tag{3.35}$$

$$B = \sin(\beta)\cos(\beta)\,(2G - 2M\cos^2(\beta) - Ik\cos^2(\beta) + Zk^2\cos^4(\beta)) \tag{3.36}$$

$$C = k\cos^2(\beta)(kN\cos^2(\beta) - kR\cos^4(\beta) + k^2Y\cos^4(\beta) + E - F\cos^2(\beta)) \tag{3.37}$$

$$D = A + C \tag{3.38}$$

$$E = 6.8148 * 10^8$$

$$F = 4.6776 * 10^8$$

$$G = 3.9185 * 10^{11}$$

$$I = 1.3440 * 10^7$$

$$Z = 4.3539 * 10^4$$

$$M = 2.6123 * 10^{11}$$

$$N = 1.8584 * 10^5$$

$$R = 5.5226 * 10^4$$

$$Y = 37.8601$$

So, in order to find the critical values, we need to solve the following equation for η:

$$|H_3| = -A * \eta^2 + B * \eta + D = 0 \tag{3.39}$$

As we can see, this is a second degree equation and we can solve it as it follows:

$$\eta = \frac{-B^2 \pm \sqrt{B^2 + 4AD}}{-2A} \tag{3.40}$$

Therefore, this is the explicit equation that governs the stability of the system as a function of the different parameters. This equation gives us two possible solutions for η, so only the solutions between the values [0,1] are the ones we are looking for, as the values that are not in this range have no sense for η.

Now in order to get this stability in a $\eta - \tan(\beta)$ diagram, we just have to solve this equation for different values of k and β, and we will obtain the values of η where the system goes from stable to unstable.

4 Discussion of the obtained results

After calculating the explicit equation to get the exact boundary, we solve now this equation for different values of k and β.

The range of the values of the three parameters are:

- $k \in [0,6000]$
- $\beta \in (-\frac{\pi}{2}, \frac{\pi}{2})$
- $\eta \in [0,1]$

First of all, we are going to see how the eigenvalues change when η increases from 0 to 1 for fixed values of k and β. In this case, $k = 800$ and $\beta = -0.85$.

So, for these values we get the following diagram:

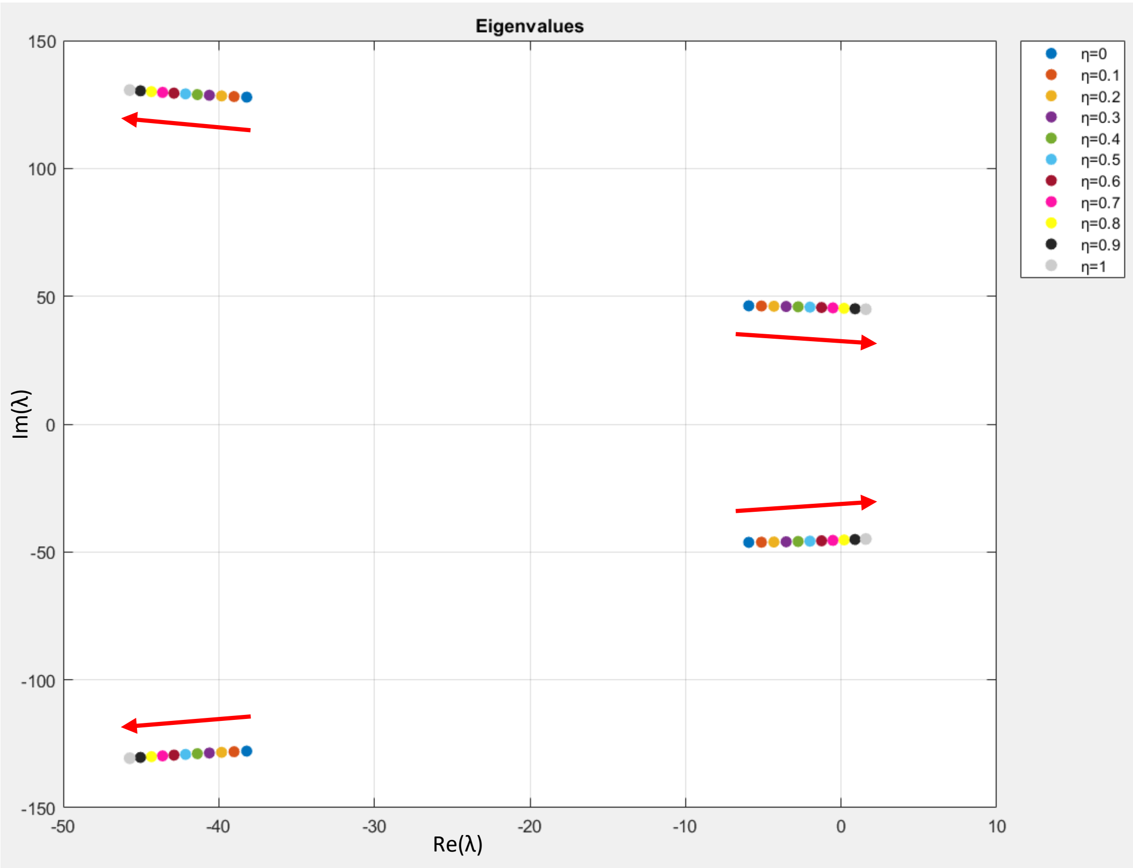

Figure 8. Eigenvalues for k=800 and β=-0.85

As we can see in *Figure 8*, the larger η is, the closer the real parts of the eigenvalues of the right side are to 0, following the direction of the red arrows shown. Therefore, the larger η is, the more likely the system is to be unstable, as it can be seen in some of the eigenvalues of the diagram, as they get to have a positive real part, thus being unstable.

If we take higher values of k, the system will be more stable. This can be seen in *Figure 9*, where we use $k = 1500$ and $\beta = -0.85$. Here, the eigenvalues are further to the left, and we have a stable system as all of the eigenvalues have a negative real part.

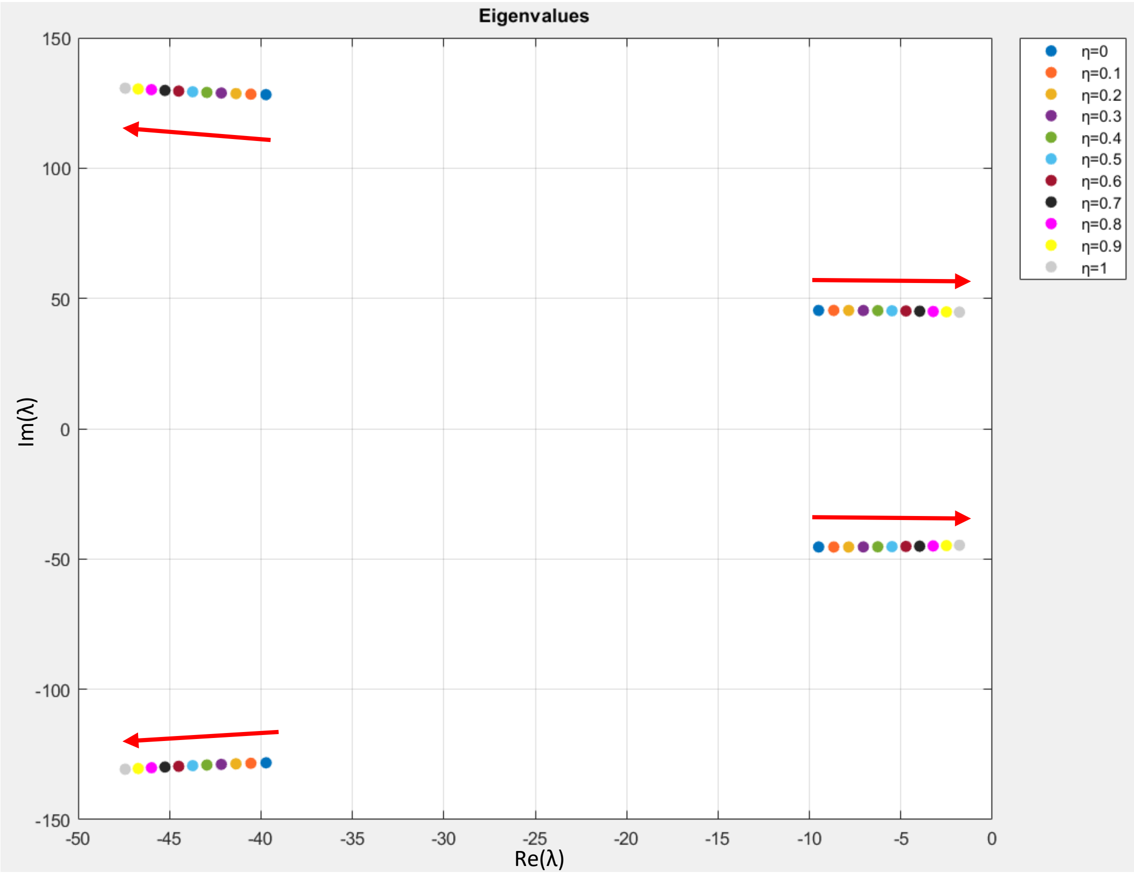

Figure 9. Eigenvalues for k=1500 and β=-0.85

Then, for the range of values of the three parameters, the stability boundary will be shown in a $\eta - \tan(\beta)$ diagram.

As it has been mentioned in the previous section, we have to check the values of the parameters where the coefficient a_3 becomes negative and, when doing so, we see that the stability boundary given by this coefficient is contained in the boundary obtained from the study of the determinant of H_3. Therefore, we will keep this last boundary, as it is more stringent.

After solving the explicit equation obtained from H_3 for the different values of k and β we get the solution of *Figure 10.*

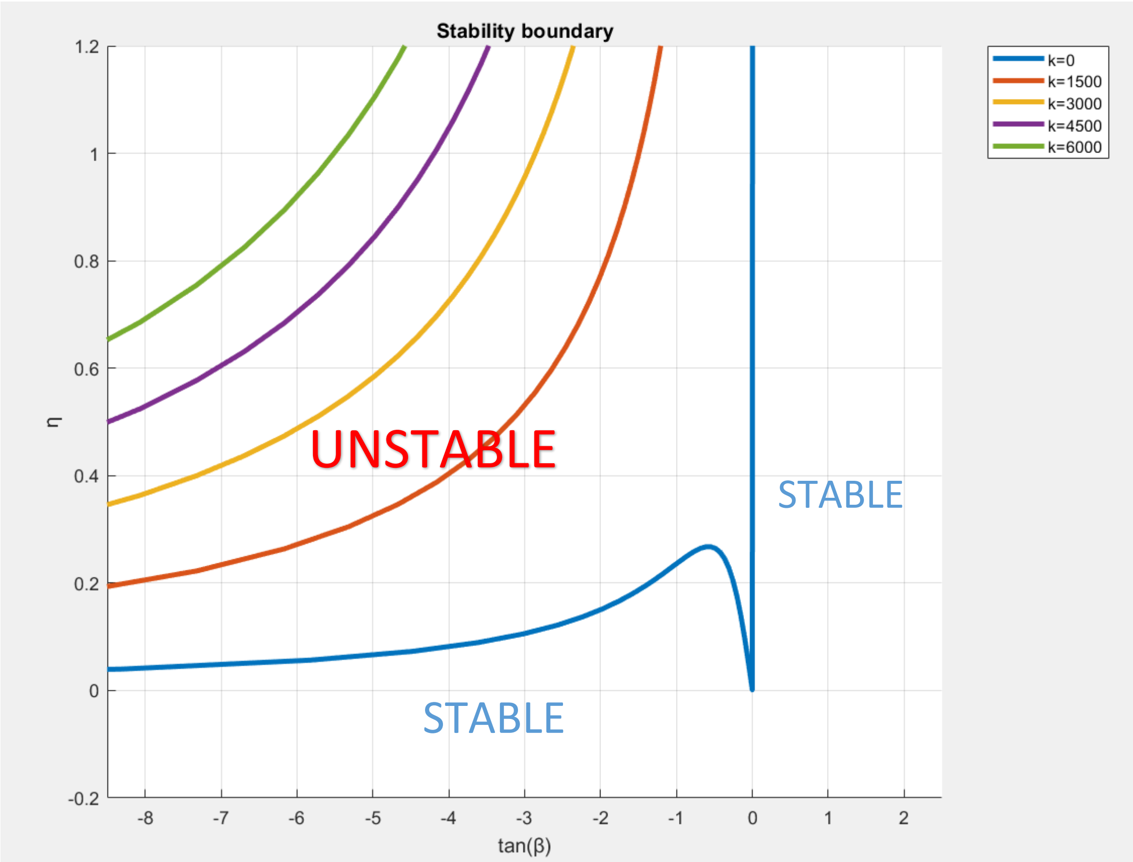

Figure 10. Stability boundary

The stable zones in the diagram are the ones below and on the right of the curve for different values of k. Therefore, as it can be seen in *Figure 10,* the higher k is, the higher β and η have to be for the system to be unstable.

The value of the damping coefficient k plays a crucial role in the stability of the system. A proper adjustment of this parameter can help to improve the self-excited wheel hop problem, as it affects to the stability, suspension and energy absorption.

So, it is crucial to achieve a balanced value of the damping coefficient, giving a higher or lower value depending on the vehicle and driving characteristics required.

Apart from the damping coefficient, it is also important having an appropriate value of the coupling coefficient η. As it can be seen in *Figure 8* and in *Figure 9*, the higher the coupling coefficient is for a fixed value of k and β, the more likely our system is to be unstable, as it have been discussed previously.

So, in order to have an efficient system, it is crucial to achieve a balance between these parameters so that driving precision and ride comfort are not adversely affected. They can be modified depending on the desired performance of the vehicle.

5 Conclusion

Self-excited wheel hop in vehicles is a significant challenge in automotive engineering, affecting both passenger comfort and vehicle structural integrity. This phenomenon consists of a loss of grip on the wheel of a vehicle, thus losing traction, and several factors influence the occurrence of it, including the suspension system, tyre dynamics and road surface conditions. In order to minimise these oscillations, it has been learned that the coupling between shock absorber characteristics and spring stiffness is critical.

Therefore, the damping coefficient k plays a fundamental role in the stability of the studied system. In this model, the higher the damping coefficient for a fixed value of η and β, the more likely our system is to be stable. The correct setting of this coefficient can have a significant effect on the vehicle, affecting stability, suspension and energy absorption.

On the other hand, the coupling coefficient η is also an important factor in this problem. The higher the coupling coefficient for a fixed value of k and β, the more likely the studied system is to be unstable.

These parameters can be modified according to the desired performance of the vehicle. But, in order to get an efficient system, it is essential to maintain a balance between them to ensure that driving accuracy and comfort are not negatively impacted.

In short, understanding this phenomenon and developing effective mitigation strategies is essential for the advancement of automotive technology. Improvements in this field not only contribute to a smoother and safer driving experience, but also extend vehicle life and reduce maintenance costs.

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7 List of Figures

8 Appendix

In order to solve the problem and plot the stability boundary, MATLAB has been used to make calculations easier and more efficient. Therefore, two different codes have been used in order to get the results obtained.

Code A

The code used to get the eigenvalues for fixed values of k and β of the *Figure 8* and *Figure 9* is:

```
clear ; close all; clc

C_Fx=200000 %N/m
r_f=0.3 %m
V_v=23 %m/s
sigma_k=0.3
m=30
I_w=0.5*m*r_f^2
k=1500
beta=-0.85
syms eta

for eta=0:+0.1:1
    A = [0, -C_Fx*r_f/I_w, 0, 0; r_f, -V_v/sigma_k, -V_v*eta/r_f, -tan(beta);
    0, 0, 0, 1; 0, C_Fx*cos(beta)*sin(beta)/m, -C_Fx*(cos(beta))^2/m, -
    k*(cos(beta))^2/m];
    char_eq = charpoly(A);
    eigenv=eig(A)
    plot(eigenv,'o','MarkerFaceColor','auto');
    hold on;
end
grid on;
```

Code B

The code used to get the $\eta - \tan(\beta)$ diagram of the *Figure 10* is the following:

```
clear ; close all; clc

C_Fx=200000 %N/m
r_f=0.3 %m
V_v=23 %m/s
sigma_k=0.3 %m
m=30 %kg
I_w=0.5*m*r_f^2  %kgm^2
s=1
i=1
j=1
```

```
l=1
g=1
syms eta beta k

A = [0, -C_Fx*r_f/I_w, 0, 0; r_f, -V_v/sigma_k, -V_v*eta/r_f, -tan(beta); 0,
0, 0, 1; 0, C_Fx*cos(beta)*sin(beta)/m, -C_Fx*(cos(beta))^2/m, -
k*(cos(beta))^2/m];
%A=coefficient matrix
char_eq = charpoly(A);
%char_eq=characteristic polynomial

        for beta=-1.551:+0.05:1.551
            fprintf('\n for:\n')
            k=0
            eta
            beta
           coeffVector = [1, (k*cos(beta)^2)/30 + 230/3,
           (20000*cos(beta)^2)/3 + (23*k*cos(beta)^2)/9 +
           (20000*cos(beta)*sin(beta)*tan(beta))/3 +
           18325193796266667/1374389534720, (4600000*cos(beta)^2)/9 +
           (6108397932088889*k*cos(beta)^2)/13743895347200 +
           (4600000*eta*cos(beta)*sin(beta))/9,
           (763549741511111125*cos(beta)^2)/8589934592]
            %coeffVector=characteristic polynomial
           M_3=[coeffVector(2),coeffVector(1),0;coeffVector(4),coeffVector(
           3),coeffVector(2);0,coeffVector(5),coeffVector(4)]
            H_3=det(M_3)
           %H_3=Hurwitz determinant
            eqn2=H_3==0
            sol2=solve(eqn2,eta)

            SOL2=transpose(sol2)*[0;1] %to get the solution we are
           interested in (as the other one is out of the range of the
           parameters)
            fbeta(s)=beta
            ftan(s)=tan(beta)
            feta(s)=SOL2
            fk(s)=k
            s=s+1
        end

        for beta=-1.56:+0.025:1.56
            fprintf('\n for:\n')
            k=1500
            eta
            beta
            coeffVector = [1, (k*cos(beta)^2)/30 + 230/3,
           (20000*cos(beta)^2)/3 + (23*k*cos(beta)^2)/9 +
           (20000*cos(beta)*sin(beta)*tan(beta))/3 +
           18325193796266667/1374389534720, (4600000*cos(beta)^2)/9 +
           (6108397932088889*k*cos(beta)^2)/13743895347200 +
           (4600000*eta*cos(beta)*sin(beta))/9,
           (763549741511111125*cos(beta)^2)/8589934592]

           M_3=[coeffVector(2),coeffVector(1),0;coeffVector(4),coeffVector(
           3),coeffVector(2);0,coeffVector(5),coeffVector(4)]
            H_3=det(M_3)
            eqn2=H_3==0
            sol2=solve(eqn2,eta)
```

```
    SOL2=transpose(sol2)*[0;1]
    fibeta(i)=beta
    fieta(i)=SOL2
    fitan(i)=tan(beta)
    fik(i)=k
    i=i+1
end

for beta=-1.56:+0.0125:1.56
    fprintf('\n for:\n')
    k=3000
    eta
    beta
    coeffVector = [1, (k*cos(beta)^2)/30 + 230/3,
    (20000*cos(beta)^2)/3 + (23*k*cos(beta)^2)/9 +
    (20000*cos(beta)*sin(beta)*tan(beta))/3 +
    18325193796266667/1374389534720, (4600000*cos(beta)^2)/9 +
    (6108397932088889*k*cos(beta)^2)/13743895347200 +
    (4600000*eta*cos(beta)*sin(beta))/9,
    (763549741511111125*cos(beta)^2)/8589934592]

    M_3=[coeffVector(2),coeffVector(1),0;coeffVector(4),coeffVector(
    3),coeffVector(2);0,coeffVector(5),coeffVector(4)]
    H_3=det(M_3)
    eqn2=H_3==0
    sol2=solve(eqn2,eta)
    SOL2=transpose(sol2)*[0;1]
    fjbeta(j)=beta
    fjeta(j)=SOL2
    fjtan(j)=tan(beta)
    fjk(j)=k
    j=j+1
end

for beta=-1.56:+0.0125:1.56
    fprintf('\n for:\n')
    k=4500
    eta
    beta
    coeffVector = [1, (k*cos(beta)^2)/30 + 230/3,
    (20000*cos(beta)^2)/3 + (23*k*cos(beta)^2)/9 +
    (20000*cos(beta)*sin(beta)*tan(beta))/3 +
    18325193796266667/1374389534720, (4600000*cos(beta)^2)/9 +
    (6108397932088889*k*cos(beta)^2)/13743895347200 +
    (4600000*eta*cos(beta)*sin(beta))/9,
    (763549741511111125*cos(beta)^2)/8589934592]

    M_3=[coeffVector(2),coeffVector(1),0;coeffVector(4),coeffVector(
    3),coeffVector(2);0,coeffVector(5),coeffVector(4)]
    H_3=det(M_3)
    eqn2=H_3==0
    sol2=solve(eqn2,eta)
    SOL2=transpose(sol2)*[0;1]
    flbeta(l)=beta
    fleta(l)=SOL2
    fltan(l)=tan(beta)
    flk(l)=k
    l=l+1
end
```

```
        for beta=-1.56:+0.0125:1.56
            fprintf('\n for:\n')
            k=6000
            eta
            beta
            coeffVector = [1, (k*cos(beta)^2)/30 + 230/3,
            (20000*cos(beta)^2)/3 + (23*k*cos(beta)^2)/9 +
            (20000*cos(beta)*sin(beta)*tan(beta))/3 +
            18325193796266667/1374389534720, (4600000*cos(beta)^2)/9 +
            (6108397932088889*k*cos(beta)^2)/13743895347200 +
            (4600000*eta*cos(beta)*sin(beta))/9,
            (763549741511111125*cos(beta)^2)/8589934592]

            M_3=[coeffVector(2),coeffVector(1),0;coeffVector(4),coeffVector(
            3),coeffVector(2);0,coeffVector(5),coeffVector(4)]
            H_3=det(M_3)
            eqn2=H_3==0
            sol2=solve(eqn2,eta)
            SOL2=transpose(sol2)*[0;1]
            fgbeta(g)=beta
            fgeta(g)=SOL2
            fgtan(g)=tan(beta)
            fgk(g)=k
            g=g+1
        end

plot3(ftan, feta,fk, 'LineWidth',3);
hold on;
plot3(fitan, fieta,fik, 'LineWidth',3);
hold on;
plot3(fjtan, fjeta,fjk, 'LineWidth',3);
hold on;
plot3(fltan, fleta,flk, 'LineWidth',3);
hold on;
plot3(fgtan, fgeta,fgk, 'LineWidth',3);
hold on

xlim([-8.5, 2.5])
ylim([-0.2, 1.2])
zlim([0, 7000])
xlabel('tan(Beta)');
ylabel('Eta');
zlabel('k');
title('Stability boundary');
grid on;
```