A database of rigorous and high-precision periodic orbits of the Lorenz model☆

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A B S T R A C T

A benchmark database of very high-precision numerical and validated initial conditions of periodic orbits for the Lorenz model is presented. This database is a "computational challenge" and it provides the initial conditions of all periodic orbits of the Lorenz model up to multiplicity 10 and guarantees their existence via computer-assisted proofs methods. The orbits are computed using high-precision arithmetic and mixing several techniques resulting in 1000 digits of precision on the initial conditions of the periodic orbits, and intervals of size 10^{100} that prove the existence of each orbit.

Program summary

Program title: Lorenz-Database

Nature of problem: Database of all periodic orbits of the Lorenz model up to multiplicity 10 with 1000 precision digits.

Solution method: Advanced search methods for locating unstable periodic orbits combined with the Taylor series method for multiple precision integration of ODEs and interval methods for providing Computer-Assisted proofs of the periodic orbits.

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1. Introduction

In computational physics and dynamics new developments in numerical techniques appear continuously. As a consequence, there is a need to validate the correctness and effectiveness of these new methods. Therefore, it is advisable to have a top level numerical database that may serve as a common benchmark for all these new studies. A general belief is that it is not possible to perform a reliable numerical simulation on a chaotic system, but this is clearly a misunderstanding. In fact, using suitable techniques and sufficiently high precision, it is possible to perform a very precise simulation for deterministic dynamical systems. So, a top and real challenge is to state correct and useful numerical data that may be used by everybody. Thus, this paper focuses on answering and providing results on the following question: is it possible to provide useful data for very high-precision simulations of deterministic chaotic systems? The answer is yes, and the most suitable set of data corresponds to information about some invariants of the system; in our case the set of unstable periodic orbits. This set has several advantages: first, it is clear how to use these data as a test of accuracy—simply try to follow one or several periodic orbits. In addition to this, during the construction of the set of benchmarks, we have reconfirmed some previous results on the proposed model, the Lorenz model.

The Lorenz model [1] is the most classical and paradigmatic low-dimensional chaotic problem since it is one of the first models with the presence of chaotic behavior and chaotic attractors. This nonlinear model has been analyzed by a large number of researchers, but it is still an important dynamical system to be studied. Based on a more complicated model by Saltzman [2], Lorenz achieved his famous equations:

$$\dot{x} = \sigma(y - x), \qquad \dot{y} = -xz + rx - y, \qquad \dot{z} = xy - bz, \tag{1}$$

where σ (the Prandtl number), r (the relative Rayleigh number) and b are three dimensionless control parameters. It is well known that a good knowledge of the set of periodic orbits (POs) of the Lorenz model (unstable periodic orbits, UPOs, foliated to the attractor) provides some more general information about the system, and gives critical information in chaotic regions [3–10]. Therefore, having complete information of all UPOs of low–medium multiplicity is highly desirable. Some partial data have been already published in the literature, but we focus on completing the references, giving at the same time a useful benchmark for analytical and numerical techniques in both dynamical systems analysis of low-dimensional chaotic systems, and in high-precision numerical methods for ODEs.

The location of UPOs has been an important and a well studied problem by physicists [11–15] and mathematicians using a vast number of numerical algorithms. Obtaining accurate information of UPOs is thus a very interesting task. Another interesting point is related to the question of computability of chaotic systems. As commented above, deterministic chaotic systems can be accurately numerically integrated, given sufficiently high precision; yet this is scarcely done in the literature. Moreover, some very recent publications state as a "computational challenge" the task of obtaining numerical solutions of the Lorenz system in some "long" time intervals [16–18]. The reported methods are extremely expensive, e.g. high-order implicit methods or simple implementations of the Taylor series method. As such they require thousands of CPU-hours on massive parallel computers. Let us remark that this issue – high-precision numerical solutions of ODEs – nowadays is handled without any problem by several freely available softwares, such as TIDES[1] [19] that uses a highly optimized Taylor-series method [20]. As an example, using this software, a periodic orbit (with 500 digits of precision) of the Lorenz system was shown in [21]. Of course, locating the initial conditions of the UPOs, and proving their existence with high precision become a much more complex problem. In this paper we have used a fast and accurate algorithm for the correction of approximate periodic orbits [22] that allows us to locate UPOs for any dynamical system up to any arbitrary precision and, in particular, to compute UPOs with 1000 precision digits for low-dimensional problems such as the Lorenz model. To our knowledge this is the only available method (Taylor series method) capable of reaching arbitrary high precision (for instance 1000 digits) for ordinary differential equations (ODEs) in a reasonable computing time.

Another important application of the Taylor method is that it can be made to use interval arithmetic, which allows us to obtain validated numerical methods for differential equations. This is a cornerstone of Computer-Assisted Proofs for proving the existence of periodic orbits. Therefore, using interval methods, we give rigor to the numerically obtained high-precision results. In other words, the results rigorously enclose the exact invariants in small sets. And therefore, we have not only some numerical results but, we will have a rigorous result that states the skeleton of UPOs of the system. This kind of information is an important complement to numerical studies as it provides rigor to some simulations [23,24].

As a concrete benchmark, the values of the coordinates of nine periodic orbits (one per multiplicity) along their complete period – at fixed output times – are provided with 1000 precision digits for comparison purposes for computational dynamics tests.

The work reported here gives a complete database of high-precision and validated numerical data. We hope that these data can act as a serious benchmark for new numerical and analytical techniques aimed at dissipative chaotic systems.

The paper is organized as follows. In Section 2, we present the low-precision location of unstable periodic orbits in chaotic systems. In order to improve these, we explain in Section 3 the computation of high-precision initial conditions of the periodic orbits applied to the chaotic Lorenz system. Moreover, in this section we show the results of some numerical tests using different ODE solvers to illustrate their behavior for the Lorenz model. Another important point that we deal with in Section 4 is the rigorous location of unstable periodic orbits in chaotic systems. In Section 5, we detail the contents of the developed database, which is available to the scientific community. Finally, we present the conclusions of this work, and in the Appendix we show an example of the files of the database.

[1] http://cody.unizar.es/software.html

http://sourceforge.net/projects/tidesodes/.

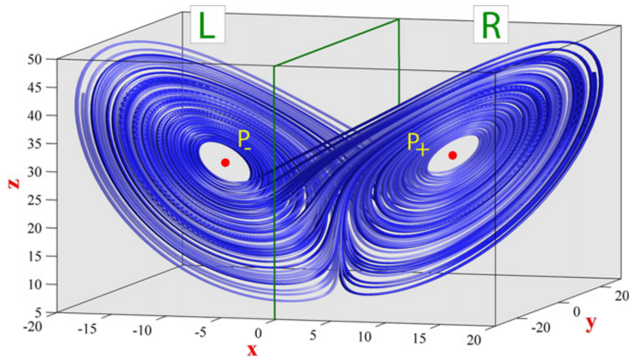

Fig. 1. The Lorenz attractor and the symbolic notation.

2. Low-precision location of unstable periodic orbits in chaotic systems

In this section, we describe how to locate low-precision unstable UPOs in the Lorenz model, for details see [25,26].

The Lorenz system (1) is well understood in terms of geometric models [27]. It has been shown to be chaotic in the topological sense for the non-classical [28] and classical [29] parameter values. The existence of the Lorenz attractor has been verified using Computer-Assisted Proofs techniques in [4].

Given the Lorenz model (1), let $\boldsymbol{x}(0) = \boldsymbol{y}$ be the initial conditions and

$$\boldsymbol{x} = \boldsymbol{x}(t; \boldsymbol{y}), \quad t \in \mathbb{R}, \ \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3, \tag{2}$$

the solution of the above autonomous differential system.

A periodic orbit, which is characterized by the vector $\boldsymbol{y}$ of initial conditions and its period T, verifies the periodicity condition

$$\boldsymbol{x}(T; \boldsymbol{y}) - \boldsymbol{y} = \boldsymbol{0}. \tag{3}$$

The chaotic attractor of the Lorenz model is illustrated in Fig. 1. Here the classical Saltzman parameter values, $b = 8/3, \sigma = 10$, $r = 28$ are used. The Lorenz system has three equilibria: one of them is the origin $P_0 = (0, 0, 0)$, and the other two are symmetric: P_+ and P_-, with coordinates $(\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1) \approx (\pm 8.485, \pm 8.485, 27)$ [25]. In order to classify the orbits densely filling the chaotic attractor, we use symbolic dynamics notation [27]. Every time a trajectory passes through the left side of the attractor the letter L is assigned to the trajectory. Likewise, if the trajectory passes through the right side, the letter R is assigned. It is known that any such infinite string of symbols uniquely characterizes each periodic orbit. Periodic orbits repeat indefinitely the finite sequence of symbols of its period, and can therefore be characterized by a finite number of symbols. For example, the LR periodic orbit does a loop on the left and another one on the right. If a trajectory does two consecutive loops on the left, one on the right, another one on the left and finally one on the right, it corresponds to the notation LLRLR. Note that for the Saltzman parameter values, two symbols are enough to describe the orbits as the first return map is unimodal [27].

Now we describe the numerical techniques to obtain a low precision location of the periodic orbits. Let us define a two-dimensional section Σ—which is a rectangular subset of the plane $z = r - 1$. We define the Poincaré map $P \colon \Sigma \to \Sigma$ as the planar map $P(\boldsymbol{x}) = \varphi(T_\Sigma(\boldsymbol{x}); \boldsymbol{x})$, where φ denotes the flow, and $T_\Sigma(\boldsymbol{x})$ is the return time, i.e., the time it takes for the trajectory starting at $\boldsymbol{x} \in \Sigma$ to intersect the section Σ. Note that a periodic orbit of the flow corresponds to a periodic orbit of the discrete map P. Our goal here is to find all periodic orbits of P up to multiplicity 10.

For the parameter values we are considering, all periodic orbits of the Lorenz equations are unstable. Therefore, we cannot rely upon any simple contraction principle for the direct flow. Instead, we use a variant of Newton's method which brings contraction into play. More precisely, we will consider the global Poincaré map $F \colon \Sigma^m \to \Sigma^m$ defined by

$$F_k(\boldsymbol{z}) = \boldsymbol{x}_{(k+1 \bmod m)} - P(\boldsymbol{x}_k), \quad k = 1, \ldots, m \tag{4}$$

where $\boldsymbol{z} = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m) \in \Sigma^m \, (\boldsymbol{x}_k \in \Sigma)$. Note that a zero of F corresponds to a multiplicity m (or period-m) orbit of P. Applying Newton's method on F makes the simple zeros of F super-attracting, and thus numerically stable.

In order to make our numerical computations rigorous, we use set-valued methods (sometimes known as interval analysis, see [30,31]). In this framework, the interval Newton method becomes

$$N([\boldsymbol{z}]) = \check{z} - [DF([\boldsymbol{z}])]^{-1} F(\check{z}), \tag{5}$$

where $[\boldsymbol{z}] = ([\boldsymbol{x}_1], \ldots, [\boldsymbol{x}_m])$ is an interval vector, and $\check{z}$ is the midpoint of $[\boldsymbol{z}]$. If $N([\boldsymbol{z}]) \subset [\boldsymbol{z}]$, then F has a unique zero in $[\boldsymbol{z}]$, and therefore P has a unique periodic orbit of multiplicity m, with each iterate $\boldsymbol{x}_k$ inside the rectangle $[\boldsymbol{x}_k] \subset \Sigma$. That is, in the conditions of the interval Newton operator:

1. If $N([\boldsymbol{z}]) \subset [\boldsymbol{z}]$, then $\exists! y \in [\boldsymbol{z}]$ such that $F(y) = 0$.
2. If $N([\boldsymbol{z}]) \cap [\boldsymbol{z}] = \emptyset$, then $F(z) \neq 0$ in $[\boldsymbol{z}]$.
3. If $\exists y \in [\boldsymbol{z}]$ such that $F(y) = 0$, then $y \in N([\boldsymbol{z}])$.

Note that we do not prove the existence of periodic orbits using the basic definition given by Eq. (3) (this is in general not possible); instead we resort to well-established fixed-point theorems that involve only open conditions. From these conditions we see that, if we have a good approximation of a zero of a function F, then if that approximation is enclosed inside some small interval vector (with width 10^{-100}, for example), and the property of inclusion is satisfied (condition 1), then we can claim that the interval vector contains exactly one zero of the function, and automatically the result gives us a mathematical proof of the existence and uniqueness of a periodic orbit.

In order to find good candidate enclosures $[\boldsymbol{z}]$ containing true periodic orbits, we use the fact that – for the Lorenz system – the periodic orbits are uniquely characterized by their symbolic dynamics. In effect, this means that we know exactly how many low-period orbits to expect, and roughly where to find them. Using a very long trajectory, we can search amongst its iterates for a best-approximate match for any particular periodic orbit. Applying Newton's method to this approximation, followed by a small inflation into a set produces the desired candidate enclosure $[\boldsymbol{z}]$. For details, see [26].

3. High-precision location of unstable periodic orbits in chaotic systems

This section reviews briefly the numerical algorithm that permits to compute periodic orbits with very high-precision.

In order to compute the roots of Eq. (3), equivalently, to find the initial conditions of a periodic orbit with high-precision, we use an iterative corrector of UPOs based on some modifications of the Newton method and the key use of an ODE solver able to solve differential systems with arbitrary precision. The Newton method begins with a set of approximated initial conditions $(\boldsymbol{y}_0, T_0)$, obtained in the previous section, being $(\boldsymbol{y}_i, T_i)$ at step i of the iterative process. Our aim is improve them, in such a way that

$$\|\boldsymbol{x}(T_i + \Delta T_i; \boldsymbol{y}_i + \Delta \boldsymbol{y}_i) - (\boldsymbol{y}_i + \Delta \boldsymbol{y}_i)\| < \|\boldsymbol{x}(T_i; \boldsymbol{y}_i) - \boldsymbol{y}_i\|.$$

For this purpose, we calculate the approximate corrections $(\Delta \boldsymbol{x}_i, \Delta T_i)$, which are obtained by expanding

$$\boldsymbol{x}(T_i + \Delta T_i; \boldsymbol{y}_i + \Delta \boldsymbol{y}_i) - (\boldsymbol{y}_i + \Delta \boldsymbol{y}_i) = 0,$$

Table 1
CPU time (seconds) for the computation of some UPOs depending on the precision digits.

Orbit	Precision digits			
	50	100	500	1000
LR	0.04	0.22	22.29	251.49
LLR	0.08	0.31	33.32	372.27
LLLR	0.10	0.40	43.05	478.18
LLLLR	0.13	0.47	50.38	579.47
LLLLLR	0.14	0.54	59.10	666.32
LLLLLLR	0.17	0.63	66.95	760.35
LLLLLLLR	0.18	0.69	75.20	845.65
LLLLLLLLR	0.20	0.77	85.17	929.85
LLLLLLLLLR	0.21	0.83	90.16	1034.80

in a multi-variable Taylor series up to the first order

$$\boldsymbol{x}(T_i;\boldsymbol{y}_i)-\boldsymbol{y}_i+\left(\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{y}}-I\right)\Delta\boldsymbol{y}_i+\frac{\partial \boldsymbol{x}}{\partial t}\Delta T_i=\boldsymbol{0}, \tag{6}$$

where I is the identity matrix of order 3. The 3×3 matrix $\partial\boldsymbol{x}/\partial\boldsymbol{y}$ is the fundamental matrix, i.e. the solution of the variational equations. This matrix evaluated at $(\boldsymbol{y}_i, T_i)$ is an approximation M_i of the monodromy matrix M. And, $\partial\boldsymbol{x}/\partial t$ represents the derivative of the solution with respect to the time, i.e., $\dot{\boldsymbol{x}}=\boldsymbol{f}(\boldsymbol{x})$. This vector, evaluated at the corrected initial conditions $(\boldsymbol{y}_i, T_i)$, corresponds to $\boldsymbol{f}(\boldsymbol{y}_{T_i})$ where $\boldsymbol{y}_{T_i}=\boldsymbol{x}(T_i,\boldsymbol{y}_i)$.

In order to compute new values, the correction algorithm imposes an orthogonal displacement

$$(\boldsymbol{f}(\boldsymbol{y}_i))^T\Delta\boldsymbol{y}_i=0. \tag{7}$$

In this way, the next $(n+1)\times(n+1)$ linear system is obtained

$$\begin{pmatrix} M_i-I & \mathbf{f}(\boldsymbol{y}_{T_i}) \\ (\mathbf{f}(\boldsymbol{y}_i))^T & 0\end{pmatrix}\begin{pmatrix}\Delta\boldsymbol{y}_i\\ \Delta T_i\end{pmatrix}=\begin{pmatrix}\boldsymbol{y}_i-\boldsymbol{y}_{T_i}\\ 0\end{pmatrix}. \tag{8}$$

The linear system (8) is solved using singular value decomposition (SVD) techniques which provide a stable numerical method [22], and this gives us the corrected initial conditions.

To be able to compute the correction we use the software TIDES [19], that computes simultaneously the solution and the partial derivatives of the solution of (3), in double or multiple precision (using the multiple precision libraries `gmp` and `mpfr` [32]). This software is a key technique for computing the database as this is one of the few available softwares capable to solve ODEs in arbitrary precision. In [33], due to the lack of arbitrary precision numerical ODE solvers at that time, a much more cumbersome approach (based on the Lindstedt–Poincaré technique) is used to obtain high-precision periodic orbits.

The performance of the correction method can be seen in Table 1. Each row, which corresponds to a periodic orbit of multiplicity m ($m=2,\dots,10$), shows the CPU time in seconds vs. the number of digits of the computational relative error (precision digits). All the numerical tests have been carried out using a personal computer PC Intel quad-core i7, CPU 860, 2.80 GHz under a 2.6.32-29-generic SMP x86 64 Linux system.

The behavior of the method in the determination of the periodic orbits of the Lorenz model is quite similar for all of them, as we obtain our goal of 1000 digits of precision in just 10 iterations. Therefore, we illustrate the process in Fig. 2 just for the LR and LLRLR periodic orbits. As expected, our algorithm is quadratically convergent since it is mainly based on the Newton method.

Having a database of periodic orbits of the Lorenz system has two important applications. The first one is to serve as benchmark of high-precision numerical ODE solvers. In the literature there are quite a few high-precision numerical integrations of chaotic dynamical systems that can be used to that purpose. Therefore,

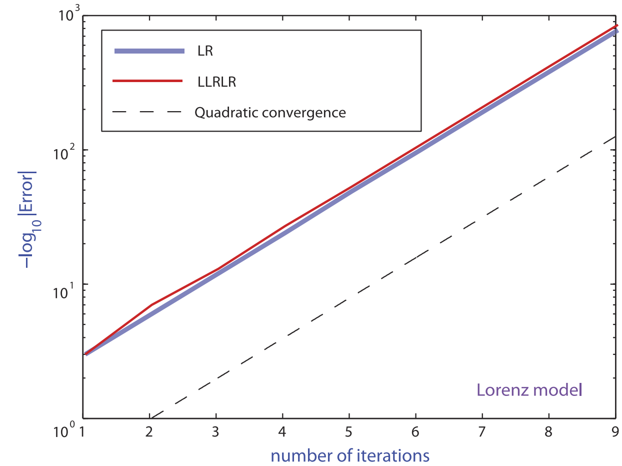

Fig. 2. Computational relative error vs. number of iterations in the computation of high-precision initial conditions of periodic orbits.

it is quite useful for that community to dispose of such an information, as to have correct data of initial conditions of periodic orbits permits to compare easily different numerical methods. For instance, in Fig. 3 we show some comparisons on the numerical integration of the LR periodic orbit of the Lorenz model using the well established codes `dop853` (a Runge–Kutta code) and `odex` (an extrapolation code) developed by Hairer and Wanner [34], and the Taylor series method implemented on the TIDES code. We observe that the RK code `dop853` becomes the fastest option for low-precision requests. Nevertheless, in quadruple precision the `odex` code is by far more efficient than the RK code because it is a variable order code, as the Taylor series method. Finally, for very high-precision requests the Taylor series method is the only reliable method amongst the standard methods, and is capable to solve ODE systems up to thousands of precision digits in a reasonable CPU time. In our benchmark test, Fig. 3, it has been of great help to have as reference orbit the precise initial conditions and period of several periodic orbits.

We remark that these tests are also related to the computability of a deterministic chaotic system using a given precision (the round-off unit of the computations). The Lyapunov exponent λ of a periodic orbit is defined as $\log(m_1)/T$, where m_1 is the magnitude of its leading characteristic multiplier and T is its period. As an example, for the orbit LR we have $\lambda\simeq 0.99465$. So, with this value we may estimate the number of laps that we may follow the periodic orbit with some precision. This total time $T_{\rm total}$, the Lyapunov time that reflects the limits of the predictability of the system at a given precision, is obtained from $\exp(\lambda T_{\rm total})\simeq 1/u$, with u the round-off unit of the computations. If we take as example the LR orbit with 1000 digits ($u\approx 3.8055\times 10^{-1000}$ in `mpfr`) we obtain

$$T_{\rm total}=-\frac{\log u}{\lambda}\approx 2313.74$$

that is, we can follow the periodic orbit $\lfloor T_{\rm total}/T\rfloor = 1484$ laps, approximately. Note that this is the limit of the computability at the precision u of the orbit LR of the Lorenz system, its *Lyapunov time*. The computability of the Lorenz system itself is also obtained in the same way, as it is already well known in the dynamical systems literature.

Another application of having the database is to provide a computer-assisted verified topological template of the Lorenz attractor, whose existence was established in [4,5]. Here, we just comment that, having a rigorous set of UPOs embedded in the attractor, we can guarantee the values of the linking matrix obtained considering the knots formed by the UPOs of the chaotic

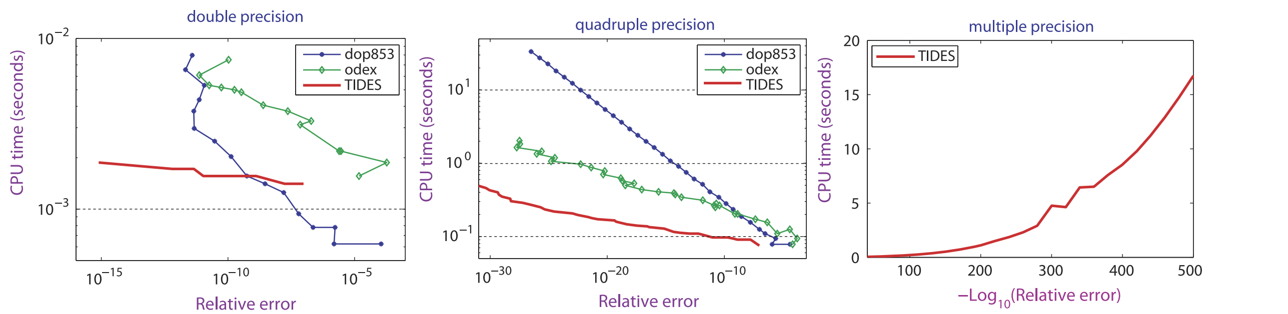

Fig. 3. Precision vs. CPU time diagram (benchmark test) in double, quadruple and multiple precision for the numerical integration of the LR periodic orbit of the Lorenz model using a Runge–Kutta code (`dop853`), an extrapolation code (`odex`) and a Taylor series method (TIDES code).

attractor. The topological structure of the Lorenz attractor [6–8] is described in terms of a paper-sheet model, called a template made of "normal" and twisted, like a Möbius band, stripes. The topological model can be quantified by a set of linking numbers—the local torsions. The torsions are, locally, the crossings number of the stripes in the template, i.e., the number of twists of the layering graph between any two unstable periodic orbits in the chaotic attractor. The local torsions determine the linking matrices and hence the template of the attractor. In practice, the template may be derived using a Poincaré return map of trajectories in the chaotic attractor, and by studying the unstable periodic orbits of the attractor [35]. This information is briefly summarized in Fig. 4. On the top we show the classical topological "mask" [6–8,36]. On the bottom we show the unimodal FRM (First Return Map) for the attractor of the Lorenz model (for the classical Saltzman values) defined on successive local maxima, $z(i)$. This is a unimodal map; it has only one relative extremum. As a consequence, the topological template has just two branches and therefore, we need two symbols to describe all orbits. Moreover, the Linking Matrix (LM), the Insertion Matrix and the topological template of the equivariant fundamental domain (one wing) [37] are given to complete the information. This information is checked with our rigorous database, obtaining the same results.

This is just one example of the use of high-precision validated data (apart from the use also as a benchmark test of validated numerical ODE integrators).

4. Rigorous location of unstable periodic orbits in chaotic systems

The rigorous computations needed to validate the high-precision periodic orbits were carried out along the strategy outlined in Section 2. All computations were performed using the `CAPD` library, see [38], which uses `gmp` and `mpfr` libraries for its multiple precision. Given a specific high-precision approximation of the initial conditions at the m different iterates of the Poincaré map of a periodic orbit $(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m)$, we begin by inflating the trajectory into an interval vector $[\boldsymbol{z}] = ([\boldsymbol{x}_i], \ldots, [\boldsymbol{x}_m])$, each component having width 10^{-100}. This will be our candidate enclosure for applying the interval Newton method for the global Poincaré map, as described earlier. We note that to obtain a rigorous enclosure we need a validated solution of the ODE system (see the review [39] or the book [40] for mode details about suitable numerical methods based on interval arithmetic and rigorous computing).

The `CAPD` library can compute rigorous enclosures of both the Poincaré map P and its partial derivatives DP over a given initial set $[\boldsymbol{z}] \subset \Sigma^m$. Looking at the structure of the global Poincaré map (4), this is all information we need to form the Newton image of the enclosure set $[\boldsymbol{z}]$ according to (5). For each enclosure, we verify that $N([\boldsymbol{z}]) \subset [\boldsymbol{z}]$, and thus validate the existence (and uniqueness)

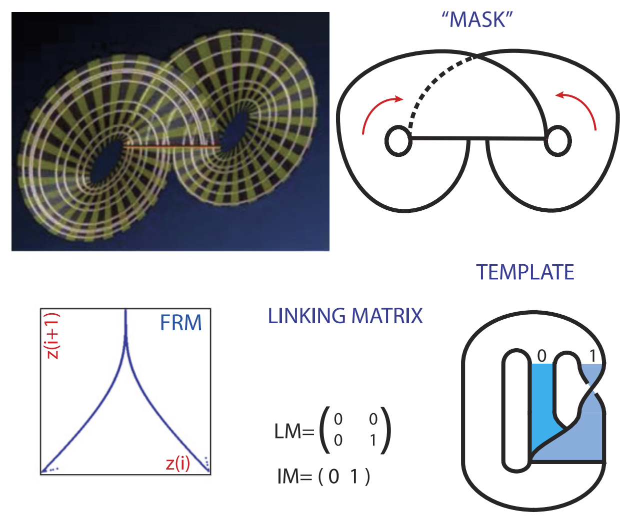

Fig. 4. Top: Orbits on the Lorenz template [36] and its topological mask. Bottom: FRM, linking matrix and the topological classical template of the equivariant fundamental domain (one wing) of the chaotic attractor for the Lorenz model.

of a multiplicity-m orbit within distance 10^{-100} of its given high-precision approximation.

All in all, we validated all 116 periodic orbits with a tolerance of 10^{-100}. For this, we performed the computations with 400 bits of precision, and used a Taylor integration scheme of order 90. The CPU time varied between 35 s and 25 min per orbit for a single thread running on a AMD Opteron 6274 @ 2.2 GHz. Theoretically, it should be possible to reduce the computation time for a multiplicity-m orbit by a factor m by parallelizing the component-bound computations. We did not pursue this option; instead we parallelized over the orbits, and launched all 116 computations concurrently on 64 threads. The total computation time was less than 50 min.

As a final result, we conclude that all 116 files with high-precision initial conditions include also a validated periodic orbit; in this case the 100 first initial digits are correct. In this sense, each file gives rise to a theorem. Below we show one example using the data file `lor_2_LR.txt` shown in the Appendix.

Theorem 1. *For the Lorenz system (1) with the Saltzman parameter values ($b = 8/3, \sigma = 10, r = 28$) there exists a unique periodic orbit with symbolic notation LR (multiplicity $m = 2$) whose initial conditions are*

$$x_0 = \check{x}_0 \pm 10^{-100},$$
$$y_0 = \check{y}_0 \pm 10^{-100},$$
$$z_0 = 27,$$

Table 2
Number of periodic orbits *nm* depending on their multiplicity *m* and total number of computed UPOs.

m	2	3	4	5	6	7	8	9	10	TOTAL
nm	1	1	2	3	5	9	16	28	51	116

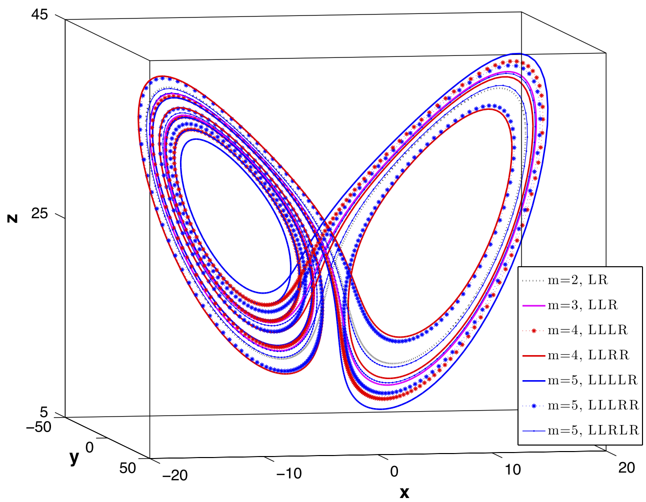

Fig. 5. Periodic orbits of the Lorenz model of multiplicities between 2 and 5.

with

$$\check{x}_0 = -2.147367631918116125647657994834426364\,539183126377307270606358273648286122\,240899658325767107886028868,$$

$$\check{y}_0 = 2.078048211461249400317478579765812352\,432481250078273367551283626639574888\,006207392602813065110324916.$$

Note that the above result is a theorem; it is rigorously proved via Computer-Assisted techniques. The 1000 digits in the files are very high-precision results checked numerically with a carefully done numerical study (they have been checked with more precision digits and we have some "guarantee" on the correctness of the presented digits, but all the digits have not been proved theoretically). In [16,18] some long numerical integrations with high-precision (thousands of digits) of one particular set of initial conditions for the Lorenz system are also presented.

Although the techniques used in this paper are not new, the systematic high-precision computations can serve as a benchmark. Indeed, the 100 leading digits of the coordinates of each reported periodic orbit can serve as a must-pass-test for any new ODE-solver equipped with high-precision arithmetic as they are all proved to be correct. The remaining 900 digits – which we believe are accurate, but not proven – can be used for extended comparisons. Finally, the timings reported here can also serve as a benchmark for efficiency.

5. The Lorenz database

The goal of this work is to develop a database that consists of two kinds of files. The complete database is provided as a complementary folder of this paper. In the first set of files, we provide the initial conditions of one periodic orbit per file with 1000 precision digits and the values of these coordinates validated with 100 digits that prove the existence of the periodic orbit. Recall that in all cases the z-coordinate has a fixed value, $z = 27$ (the same as the equilibria $P_{\pm}$).

In total there are 116 files with initial conditions (all the UPOs of the Lorenz attractor of multiplicity $m \leq 10$) as we show in Table 2, which specifies the number of UPOs, nm, depending on their multiplicity, m, and the total number of computed UPOs. These files are denoted by `lor_m_symb.txt` where m is the multiplicity and `symb` the symbolic sequence of the orbit. The format of these files is shown in the Appendix. First of all, the individual number of the orbit (`num`) is specified; then the multiplicity (`orbit-mult`), the number of the orbit among all the orbits with the same multiplicity (`num-same-mult`) and the symbolic sequence of the orbit (`symb`). After that we find, with 1000 precision digits, the period T and the initial conditions x and y of the orbit ($z = 27$ in all cases). Finally, the rigorous intervals for the same variables (x, y, z) in high-precision are obtained just by taking the first 100 digits (these are the rigorously proved digits).

Besides, there are nine files with complete data of one periodic orbit each, one per each multiplicity ($2 \leq m \leq 10$), which are denoted as `symb_orbit.txt`. In these files, we give with 1100 digits the values of the coordinates of the orbit at fixed output times with time increment $h = 0.01$. The format of the files is to give in each line one complete point, that is, the values of t_i, $x(t_i)$, $y(t_i)$, $z(t_i)$. Most of the shown digits are most likely correct (the computations have been done with an error tolerance of 10^{-1090} and each data has been carefully checked for the first 1000 digits), and again the first 100 digits are rigorously proved digits.

In Fig. 5, we can see some of the orbits, of multiplicities between 2 and 5, that have been computed and that are included in the database. The LR and LLRR orbits are symmetric, while the LLR, LLLR, LLLLR, LLLRR and LLRLR orbits are non-symmetric.

6. Conclusions

The goal of this paper is to present a high-precision and validated database of periodic orbits useful to scientific community. This consists of hundreds of approximated initial conditions (with 1000 digits of precision) of all the periodic orbits of the Lorenz attractor with multiplicities between 2 and 10. To obtain this database, we have combined two different methods: a corrector of periodic orbits algorithm in arbitrary precision, which allows us to obtain the initial conditions of UPOs of any dynamical system with the required precision, and Computer-Assisted techniques to prove the existence of these orbits within a tolerance of 10^{-100}. This database is a "computational challenge" and it can be used as a benchmark for checking new numerical and theoretical techniques in computational physics and dynamics.

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Appendix. File `lor_2_LR.txt`

```
================================================
   Lorenz Database: High precision and
                    rigorous data file
   File:            lor_2_LR.txt
================================================
* Numerical data with 1000 precision digits  ***
*   (error < 10^(-1000))                      ***
* The first 100 digits are rigorously proved ***
*   via CAP techniques. They give a Theorem   ***
*   of existence of the Periodic Orbit.       ***
```

```
***********************************************
******************  FORMAT  ******************
***********************************************
**************** Data of the orbit  ***********
* num  |  orbit-mult  |  num-same-mult  |  symb
  1          2                1           LR
*********** Initial conditions (1000 digits) ***
 T =  1.5586522107161747275678702092126960705284
      805489972439358895215783190198756258880854
      355851082660142374227874628676588925856759
      114998565388913608713285011019327706322439
      313214649374465282053245397773437382070265
      038689003930946637574777949263393159437541
      623011610884405671641950154306933323049096
      162291405158398552430505472975477593892734
      206854673172738936026993914663645933054022
      189761784055859520883303114513251097516232
      856917878431617597363720748783672257211325
      662086758469406354756156487387177650044370
      454248933780710742659716738020422549262291
      333210964856082100412502113122061849916697
      973523392695931523265206965940137017550539
      699545689811043543162019034070112606824561
      903563007526645593784918880438439263450120
      625109594685133700759631857668509485055243
      223996057006402060530026179463699015245595
      033093872400297506789639255487515858437209
      627081541412899445444256582441991078147467
      765689395686384271173335081992134537066065
      694742760880531107895985179385297145888797
      830428111474345661639697827320258658080745
      546645389940133286984072568999426750324798
      267652782047614476776946084749296290153599
      326908602
 x = -2.1473676319181161256476579948344263645391
      831263773072706063582736482861222408996583
      257671078860288685519677370558563429547486
      605895676290598523687812781098722230972187
      262033850465367144898223380814452401981070
      814342093623677105339157133941615290986783
      571794270184327168026759931161103325194824
      164227655414132873997507004053059711216356
      574150375356858873365020048243430228835021
      836513730479627166658184017565158130895088
      274385428150842625062154742132607627769976
      978976419805208327908486164289266143982923
      606380647647044871033936630315467336826742
      843458831192176096824455498673117655818315
      383124497033769336819193724989102515222691
      900292841071936008530072491470578952381790
      051769917148434292675094735410259879438780
      966758152308238848109127435975809636611429
      188564166440038221628456041830667200064968
      436090251373045854367659518877238474359199
      744355074321354572981759029457060162234081
      997571617253506903724889072866646150027113
      706583673983028023296187777548562182909040
      415036997238022952647202830517650784159493
      114408580942249235770049714948136302482824
      667603017160138684083311963774670685336801
      763179293
 y =  2.0780482114612494003174785797658123524324
      812500782733675512836266395748880062073926
      028130651103249169852108870521308699282559
      469673015670838903747526759636485599294729
      432463904901202551341628059576690377511330
      638977113875846234568597404146956064096117
      239797849362718721864237093464022138068085
      394681323529698800475076250038669795767969
      360955427915532896483014016015919944636236
      662965349866267624234955157359579460952244
      654359517991870154559448825863685819061560
      961123926834254736653707830913133093954565
      833197795039610240479117866798032368314397
      223028690056106631450511033879996633563025
      708806934614955550764097498856242201673401
      712794022785983063271985029968280404584960
      027801129335843498299100475726125323052835
      943174177460023453331904583876396932802531
      016431720506859782544800224853690666966327
      941170192404700687891069617885714789404814
      934098731781680300758236420856811954640014
      979712396472715422991678177345324057959566
      642779652752526027994483211072185288125184
      730180543839514812217592209661253548414412
      422934425190796386507711054645070627748184
      823866114202952135376014293316533904495746
      013910176
 z =  27
===============================================
```

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