



An adiabatic approximation to the path integral for relativistic fermionic fields

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Abstract

A new approach to the path integral over fermionic fields, based on the extension of a reformulation of the adiabatic approximation to some quantum-mechanical systems, is presented. A novel non-analytic contribution to the effective fermionic action for a fermion field coupled to a non-Abelian vector field is identified. The possible interpretation of this contribution as a violation of the decoupling theorem in quantum field theory (QFT) is discussed. The generalization of the approach to the case of finite temperature and density suggests the possibility to apply it to the understanding of non-perturbative properties in QFT and their dependence on temperature and density.

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1. Introduction

The adiabatic approximation is one of the most important methods going beyond perturbation theory in quantum mechanics. In QFT, the necessity of non-perturbative methods is clear in many cases (low energy limit of asymptotically free theories, high energy limit of infrared safe theories). Unfortunately, the attempts to translate the adiabatic approximation to QFT have been very limited and the main results are the identification of Wess–Zumino terms and anomalies as geometric phases [1]. The complexity of QFT (a quantum-mechanical system with infinite degrees of freedom) has been an obstacle to the possible use of the adiabatic approximation as the starting point to an alternative to the perturbative expansion. In this Letter we attempt to give a first step in this direction by considering the path integral over a relativistic fermionic field system. In the next section we take as a starting point

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a very simple quantum-mechanical system, a spin coupled to a time-dependent magnetic field, and the adiabatic approximation is reformulated in an appropriate way to be generalized to other systems. In Section 3, we point out the difficulties found when one tries a generalization of this reformulation in QFT. In Section 4, a possible way to circumvent these problems is presented and our proposal is applied to the path integral over a fermion field in the fundamental representation of $SU(2)$ coupled to a vector field in the adjoint representation. The leading term in the adiabatic approximation is determined and its possible relation with non-perturbative properties of the theory are discussed at a qualitative level. In Section 5, a generalization of the results at finite density and temperature is obtained and finally in Section 6 a discussion of possible physical realizations of the adiabatic approximation in QFT is presented.

2. A quantum-mechanical example

In order to formulate the adiabatic approximation in QFT, let us first discuss it at the level of quantum mechanics in the most simple non-trivial example, namely, a half-integer spin (j) coupled to an external magnetic field ($\vec{B}(t)$) varying periodically in time ($\vec{B}(T) = \vec{B}(-T)$). Let us review the well-known solution of this problem [2]. One has a Hamiltonian

$$H = \sum_{\alpha, \beta} \vec{B}(t) \cdot \vec{J}_{\alpha\beta} a_{\alpha}^{\dagger} a_{\beta}, \quad (1)$$

where $\alpha, \beta = -j, -j+1, \dots, j-1, j$ and a, a^{\dagger} are operators satisfying the (anti)commutation relations

$$\{a_{\alpha}, a_{\beta}^{\dagger}\} = \delta_{\alpha\beta}. \quad (2)$$

We want to determine the probability amplitude that the system remains in the ground state ($|0\rangle$) during time evolution, i.e., the matrix element $\langle 0(T)|0(-T)\rangle$. In order to do this calculation it is convenient to use at each time, the direction of the magnetic field as the spin quantization axis to rewrite the Hamiltonian as

$$H(t) = \sum_{m=-j}^j mB(t) a_m^{\dagger}(t) a_m(t). \quad (3)$$

The ground state of the system, $|0(t)\rangle$ is the state satisfying the conditions

$$a_m(t)|0(t)\rangle = 0 \quad \text{for } m > 0, \quad a_m^{\dagger}(t)|0(t)\rangle = 0 \quad \text{for } m < 0. \quad (4)$$

In the adiabatic approximation one has two contributions to the matrix element $\langle 0(T)|0(-T)\rangle$, one due to the energy of the ground state

$$E_0(t) = \sum_{m<0} mB(t) = -\frac{(j+1/2)^2}{2} B(t), \quad (5)$$

and a contribution due to the time evolution of the phase of the ground state $|0(t)\rangle$

$$\int_{-T}^T dt \langle 0(t)|i\partial_t|0(t)\rangle = -\mathcal{I}m \int d\vec{S} \sum_{k \neq 0} \frac{\langle 0(t)|\nabla_{\vec{B}} H|k(t)\rangle \wedge \langle k(t)|\nabla_{\vec{B}} H|0(t)\rangle}{(E_k - E_0)^2}. \quad (6)$$

Stoke's theorem has been applied to rewrite the time integral as a surface integral in magnetic field space and the matrix elements of the gradient of the Hamiltonian can be directly read from the matrix representation of the angular momentum operator.

After some straightforward algebra one finds

$$\int_{-T}^T dt \langle 0(t) | i \partial_t | 0(t) \rangle = \frac{(j + 1/2)^2}{2} \Omega[\hat{B}], \tag{7}$$

where

$$\Omega[\hat{B}] = \int d\vec{S} \cdot \frac{\vec{B}}{B^3}, \tag{8}$$

is the solid angle that \vec{B} subtends. This is the standard derivation of the adiabatic approximation including Berry’s phase (7) [2].

Let us now see how these results can be rederived in an alternative formulation based on a Grassmann path integral representation of the evolution, which will be useful in the discussion of QFT in Section 4. One can represent the operators a, a^\dagger by Grassmann variables $\psi, \bar{\psi}$ and the probability amplitude that the system remains in the ground state by a Euclidean Grassmann path integral,

$$\lim_{T \rightarrow \infty} \langle 0(T) | 0(-T) \rangle = \frac{\int d\psi d\bar{\psi} e^{-S}}{\int d\psi d\bar{\psi} e^{-S}|_{\vec{B}=0}} = e^{-\Gamma[\vec{B}]}, \tag{9}$$

where the Euclidean action for the fermionic system is

$$S = \int d\tau \bar{\psi} (\partial_\tau + \vec{J} \cdot \vec{B}) \psi. \tag{10}$$

In order to simplify the action, one introduces new Grassmann variables c_m, \bar{c}_m through the expansion

$$\psi = \sum_{m=-j}^j c_m f_m, \quad \bar{\psi} = \sum_{m=-j}^j f_m^\dagger \bar{c}_m, \tag{11}$$

where $f_m(t)$ are the eigenspinors defined by

$$\vec{J} \cdot \vec{B} f_m = m B f_m. \tag{12}$$

The action as a function of the new variables is given by,

$$S = \int d\tau \left\{ \sum_m [\bar{c}_m (\partial_\tau + m B) c_m] + \sum_{m,m'} \bar{c}_m i \mathcal{A}_{m,m'} c_{m'} \right\}, \tag{13}$$

where $i \mathcal{A}_{m,m'} = f_m^\dagger \partial_\tau f_{m'}$. In the adiabatic approximation, one neglects off-diagonal terms ($m \neq m'$) and the path integral becomes a product of independent integrals

$$e^{-\Gamma^{(\text{ad})}[\vec{B}]} = \prod_{m=-j}^j \int dc_m d\bar{c}_m \exp \left\{ - \int_{-\beta/2}^{\beta/2} d\tau \bar{c}_m (\partial_\tau + i \mathcal{A}_m + m B) c_m \right\}, \tag{14}$$

with $\mathcal{A}_m = \mathcal{A}_{m,m}$ and the limits on the Euclidean time ($\tau = it$) incorporate finite temperature ($1/\beta$) effects in the imaginary time formalism [3].

The integral on each pair of variables (c_m, \bar{c}_m), is a standard quantum-mechanical determinant [4]

$$\det[\partial_\tau + i \mathcal{A}_m(\tau) + m B(\tau)] = \mathcal{N} \cosh \left[\frac{\beta}{2} (m \tilde{B} + i \tilde{\mathcal{A}}_m) \right], \tag{15}$$

where \mathcal{N} is an infinite constant that will cancel in the ratio of Grassmann integrals in (9) and we have introduced the notation

$$\tilde{f} = \frac{1}{\beta} \int_{-\beta/2}^{\beta/2} d\tau f(\tau). \tag{16}$$

In the zero temperature limit the quantum-mechanical determinants (15) become exponentials and the effective fermionic action in the adiabatic approximation takes the simple form

$$-\Gamma^{(\text{ad})}[\vec{B}] = -\frac{i}{2} \sum_m \left(|m| \int dt B(t) + \frac{m}{|m|} \int dt \mathcal{A}_m(t) \right), \tag{17}$$

where the first term can be recognized as the dynamical phase $-i \int dt E_0(t)$ with E_0 the energy of the ground state of the quantum-mechanical system (5). The second term in (17) reproduces Berry’s phase as one can show by using once more Stokes theorem and the definition of f_m in (12), i.e.,

$$\int dt i \mathcal{A}_m(t) = \int dt f_m^\dagger \partial_t f_m = -im \Omega[\hat{B}], \tag{18}$$

with $\Omega[\hat{B}]$ the solid angle that the magnetic field subtends in its evolution.

The adiabatic approximation to this simple quantum-mechanical system in the Grassmann path integral representation, will reappear as an ingredient in some approximation to a QFT with fermionic fields as we will see later. Also this reformulation of the adiabatic approximation is interesting because it allows to go beyond the zero temperature limit by using the quantum-mechanical determinants in (15).

3. Formal direct approach and its difficulties

The purpose of this section is to introduce a direct extension to QFT of the reformulation of the previous section. Before doing that we will show the problems of a direct implementation of the adiabatic approximation.

The most natural way to formulate the adiabatic approximation and the related Born–Oppenheimer approach in QFT is based on the use of the Schrödinger representation, where the wave function of quantum mechanics is replaced by a functional in the space of field configurations. After this replacement, all the standard results of the adiabatic quantum-mechanical expansion can be applied directly in QFT [5]. This formulation gives a new perspective of the anomaly in chiral gauge theories which appears as a geometric phase in the space of gauge field configurations related to the gauge non-invariance of the phase of the fermionic Fock states [6]. Unfortunately, a quantum-mechanical system with infinite degrees of freedom is too complicated to go beyond the study of a few topological properties and the adiabatic approximation remains as a reformulation of the theory at a formal level.

An alternative way to implement the adiabatic formulation is based on a direct use of the reformulation of the quantum-mechanical example of the previous section in a QFT system. For definiteness, let us consider the action in Euclidean space for a fermionic field Ψ coupled to a vector field A_μ ,

$$\mathcal{S} = \int d^D x \{ \bar{\Psi} \gamma_\mu (\partial_\mu + ieA_\mu) \Psi \}. \tag{19}$$

Using the decomposition

$$\Psi = \begin{pmatrix} \Psi_R \\ \Psi_L \end{pmatrix}, \quad \bar{\Psi} = (\Psi_L^\dagger \quad \Psi_R^\dagger), \tag{20}$$

for the fermionic field in a representation where all gamma matrices are off-diagonal, one has

$$S = \int d^4x \{ \Psi_L^\dagger (\partial_4 + ieA_4) \Psi_L + \Psi_R^\dagger (\partial_4 + ieA_4) \Psi_R + \Psi_L^\dagger \vec{\sigma} \cdot (-i\vec{\nabla} + e\vec{A}) \Psi_L - \Psi_R^\dagger \vec{\sigma} \cdot (-i\vec{\nabla} + e\vec{A}) \Psi_R \}. \tag{21}$$

Following the steps of the previous section, we introduce the eigenfunctions $\Phi_n(\vec{x})$

$$[\vec{\sigma} \cdot (-i\vec{\nabla} + e\vec{A})] \Phi_n(\vec{x}) = \epsilon_n \Phi_n(\vec{x}). \tag{22}$$

These eigenfunctions and the eigenvalues ϵ_n are, in fact, functionals of the vector field at a given time and a more precise notation for them is $\Phi_n[\vec{A}(\tau)](\vec{x})$ and $\epsilon_n[\vec{A}(\tau)]$.

Next step is to use the decomposition of the fermionic fields in terms of the eigenfunctions in (22),

$$\Psi_L(\vec{x}, \tau) = \sum_n c_{L_n}(\tau) \Phi_n[\vec{A}(\tau)](\vec{x}), \quad \Psi_R(\vec{x}, \tau) = \sum_n c_{R_n}(\tau) \Phi_n[\vec{A}(\tau)](\vec{x}), \tag{23}$$

$$\Psi_L^\dagger(\vec{x}, \tau) = \sum_n c_{L_n}^\dagger(\tau) \Phi_n^\dagger[\vec{A}(\tau)](\vec{x}), \quad \Psi_R^\dagger(\vec{x}, \tau) = \sum_n c_{R_n}^\dagger(\tau) \Phi_n^\dagger[\vec{A}(\tau)](\vec{x}). \tag{24}$$

Using the orthogonality of the eigenfunctions Φ_n , the action takes a compact form in terms of the (Grassmann) coefficients c_L, c_R

$$S = \int d\tau \left\{ \sum_n [c_{L_n}^\dagger(\tau) \partial_\tau c_{L_n}(\tau) + c_{L_n}^\dagger(\tau) \epsilon_n c_{L_n}(\tau) + i c_{L_n}^\dagger(\tau) \mathcal{A}_n c_{L_n}(\tau)] \right. \\ \left. + \sum_n [c_{R_n}^\dagger(\tau) \partial_\tau c_{R_n}(\tau) - c_{R_n}^\dagger(\tau) \epsilon_n c_{R_n}(\tau) + i c_{R_n}^\dagger(\tau) \mathcal{A}_n c_{R_n}(\tau)] \right. \\ \left. + \sum_{n \neq m} i [c_{L_n}^\dagger(\tau) \mathcal{A}_{nm} c_{L_m}(\tau) + c_{R_n}^\dagger(\tau) \mathcal{A}_{nm} c_{R_m}(\tau)] \right\}, \tag{25}$$

where we have introduced the connection \mathcal{A}_n ,

$$\mathcal{A}_n[A(\tau)] = \int d\vec{x} \Phi_n^\dagger[\vec{A}(\tau)](\vec{x}) (-i\partial_\tau + eA_4) \Phi_n[\vec{A}(\tau)](\vec{x}), \tag{26}$$

and \mathcal{A}_{nm} for $n \neq m$,

$$\mathcal{A}_{nm}[A(\tau)] = \int d\vec{x} \Phi_n^\dagger[\vec{A}(\tau)](\vec{x}) (-i\partial_\tau + eA_4) \Phi_m[\vec{A}(\tau)](\vec{x}). \tag{27}$$

In the adiabatic approximation one neglects the off-diagonal terms ($n \neq m$) and then one has infinite copies of quantum-mechanical systems each of them similar to the one discussed in Section 2. However, for a general vector field configuration the spectrum (ϵ_n) will be continuous and the difference of energy levels can be arbitrarily small rendering the adiabatic expansion out of control. Besides that, the eigenvalues ϵ_n and eigenfunctionals Φ_n are not known, except for very special choices of the vector field, and the formulation remains once more at a formal level.

4. A new approach and its application to an $SU(2)$ gauge theory

The only way we have found to use the reformulation of the adiabatic approximation to get a useful expansion in QFT, is based on the introduction of variables independently at each point. In order to do that, one has to select an operator at each point and use its eigenfunctions in the expansion of some of the fields at this point. We can then identify two ingredients in the formulation of the new approach. The first one is a separation of the fields in two sets, one of them corresponding to the spin degrees of freedom of the quantum-mechanical example. The second

one is the choice of the operator at each point whose eigenfunctions are used to introduce new variables for the spin-like fields.

Several requirements constraint the ambiguities in this two ingredients. The action should be quadratic in the spin-like fields either directly or after the introduction of appropriate auxiliary fields. The expression for some of the terms in the action as a function of the new variables should be as simple as possible. The contribution from the remaining terms in the action (including the space derivatives) as well as the corrections to the “adiabatic” approximation (off-diagonal contributions in the new variables) should be small. The search of a good set of fields and local operators defining the new variables has to be done, however, case by case. The usefulness of the approach will be established if one finds examples where all these requirements are satisfied yielding to a dominant contribution with interesting results.

In order to illustrate our approach let us consider the Lagrangean of a fermionic system coupled to a vector field

$$\mathcal{L} = \bar{\Psi} i \gamma^\mu \partial_\mu \Psi - g \bar{\Psi} \gamma^\mu A_\mu^a T^a \Psi - m \bar{\Psi} \Psi, \quad (28)$$

where T^a are the generators of $SU(2)$ acting on the fermionic fields in the fundamental representation. It is convenient to use the Dirac representation for the γ matrices with

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad (29)$$

and the decomposition in bispinors of the Dirac field

$$\Psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}. \quad (30)$$

We neglect for a moment, the terms proportional to the space components of the vector field (\vec{A}^a) and to space derivatives of the fermionic field. The remaining terms take a simple form if we use the eigenvectors f_\pm of the operator $\hat{A}_0^a T^a$

$$(\hat{A}_0^a T^a) f_\pm = \pm \frac{1}{2} f_\pm, \quad (31)$$

where we have used the parametrization $A_0^a = A_0 \hat{A}_0^a$ with $\sum_a \hat{A}_0^a \hat{A}_0^a = 1$. With these eigenvectors, one can introduce the new fermionic variables $\varphi_{n,i}, \chi_{n,i}$

$$\varphi = \sum_{n=\pm} \sum_{i=1,2} \varphi_{n,i} f_{n,i}, \quad (32)$$

$$\chi = \sum_n \sum_{i=1,2} \chi_{n,i} f_{n,i}, \quad (33)$$

where the bispinors $f_{n,i}$ are given by

$$f_{n,1} = \begin{pmatrix} f_n \\ 0 \end{pmatrix}, \quad f_{n,2} = \begin{pmatrix} 0 \\ f_n \end{pmatrix}. \quad (34)$$

Note that the new fermionic variables have been introduced independently at each point in space. We then have Grassmann variables $\varphi_{n,i}, \chi_{n,i}$ at each space–time point.

In the new representation for the fermionic variables, one has

$$\Psi^\dagger \partial_\tau \Psi = \sum_{n,i} [\varphi_{n,i}^\dagger \partial_\tau \varphi_{n,i} + \chi_{n,i}^\dagger \partial_\tau \chi_{n,i}] - \sum_{n,n',i} [\varphi_{n,i}^\dagger \varphi_{n',i} + \chi_{n,i}^\dagger \chi_{n',i}] i \mathcal{A}_{n,n'} \quad (35)$$

with $i \mathcal{A}_{n,n'} = f_n^\dagger \partial_\tau f_{n'}$ and $\tau = it$ is the Euclidean time.

For the interaction term one has

$$-g\Psi^\dagger A_0^a T^a \Psi = -\sum_n g_n A_0 \sum_i [\varphi_{n,i}^\dagger \varphi_{n,i} + \chi_{n,i}^\dagger \chi_{n,i}], \tag{36}$$

with $g_\pm = \pm \frac{g}{2}$.

The Lagrangian density before including space derivatives and the space components of the vector field as a function of the new variables is

$$\begin{aligned} \mathcal{L}(A_0^a) = & \sum_{n,i} [\varphi_{n,i}^\dagger (\partial_\tau - g_n A_0 - m + i\mathcal{A}_n) \varphi_{n,i} + \chi_{n,i}^\dagger (\partial_\tau - g_n A_0 + m + i\mathcal{A}_n) \chi_{n,i}] \\ & + \sum_{n \neq n', i} [\varphi_{n,i}^\dagger \varphi_{n',i} + \chi_{n,i}^\dagger \chi_{n',i}] i\mathcal{A}_{n,n'}. \end{aligned} \tag{37}$$

We then have at each point, a generalization of the quantum-mechanical example in Section 2 with A_0^a playing the role of the magnetic field \vec{B} (the direction in the internal $SU(2)$ space is the analog to the orientation of the magnetic field) and four instead of one spin variable (corresponding to the four components of the Dirac spinor). Then, the energies associated to the new variables are

$$E(\varphi_{n,i}) = g_n A_0 + m, \quad E(\chi_{n,i}) = g_n A_0 - m. \tag{38}$$

The corrections to the adiabatic approximation due to the off-diagonal terms ($n \neq n'$) in (37) involve levels separated by a gap gA_0 . Then, the adiabatic approximation will be justified when the product of the coupling and the time component of the vector field is large compared to the time derivative of its orientation ($\dot{\hat{A}}_0^a$).

With respect to the consistency of treating the space derivatives and space components of the vector field as a correction, we have a sum of two contributions

$$\begin{aligned} \Psi^\dagger \vec{\alpha} \cdot (-i\vec{\nabla} + g\vec{A}^a T^a) \Psi = & \sum_{n,i,i'} [\varphi_{n,i}^\dagger (-i\vec{\nabla}) \chi_{n,i'} + \chi_{n,i}^\dagger (-i\vec{\nabla}) \varphi_{n,i'}] (f_{n,i}^\dagger \vec{\sigma} f_{n,i'}) \\ & + \sum_{n,n',i,i'} [\varphi_{n,i}^\dagger \chi_{n',i'} + \chi_{n,i}^\dagger \varphi_{n',i'}] (f_{n,i}^\dagger \vec{\sigma} \cdot (-i\vec{\nabla} + g\vec{A}^a T^a) f_{n',i'}). \end{aligned} \tag{39}$$

The first term is a non-diagonal contribution between energy levels separated by $2m$. Since the eigenvectors f_n depend only on the direction in the internal space of A_0 , then the corrections due to these terms will be proportional to $\vec{\nabla} \hat{A}_0^a / 2m$. The second term has non-diagonal contributions between energy levels separated by $2m$, $2m + gA_0$ or $|2m - gA_0|$ and there are two types of terms, ones proportional to $\vec{\nabla} \hat{A}_0^a$ and the others proportional to $g\vec{A}^a$. From these simple arguments one can see what are the conditions on the vector field and the fermion mass in order to treat (39) as a small correction to (37). It should be noted that considering spacial derivatives as corrections does not mean that we are making use of the usual derivative expansion method [7].

Then, in the approximation where we keep only the terms diagonal in (37) and using the result (17) and (18) we have a new approximation to the effective fermionic action,

$$\begin{aligned} \Gamma_{\text{ad}} = & \int d^3\vec{x} \sum_n \text{sgn}(g_n A_0 + m) \left[\int d\tau (g_n A_0 + m) + \frac{g_n}{g} i\Omega[\hat{A}_0^a] \right] \\ & + \int d^3\vec{x} \sum_n \text{sgn}(g_n A_0 - m) \left[\int d\tau (g_n A_0 - m) + \frac{g_n}{g} i\Omega[\hat{A}_0^a] \right], \end{aligned} \tag{40}$$

where $\Omega[\hat{A}_0^a]$ is the solid angle that \hat{A}_0^a subtends in internal space in its time evolution.

We can distinguish two different regions. In the weak coupling region ($gA_0 \ll 2m$) there is a cancellation of contributions from the two sets of new variables $\varphi_{n,i}$ and $\chi_{n,i}$ and Γ_{ad} reduces to a constant. On the other hand, in

the strong coupling region ($gA_0 \gg 2m$) the effective fermionic action in the adiabatic approximation is

$$\Gamma_{\text{ad}}^{(s)} = 2g \int d\tau d^3\vec{x} A_0 + 2i\Omega[\hat{A}_0^a]. \quad (41)$$

The presence of a contribution in the effective fermionic action which disappears in the weak coupling region, suggest a possible relation between the presence of such contribution and the non-perturbative properties of the theory.

One should note that the limit $m \rightarrow \infty$ for fixed gA_0 corresponds to the weak coupling region, where the new non-perturbative contribution disappears as expected from the decoupling theorem [8]. On the other hand, for an arbitrarily large mass, one can be in the strong coupling limit if one has a sufficiently large coupling g and/or vector field A_0 and then the non-perturbative trace of the fermionic system remains.

5. Finite density and temperature effects

The generalization of the adiabatic approximation to the $SU(2)$ gauge theory in the case of finite density is trivial. All one has to do is to include the chemical potential (μ) through a term $\mu\bar{\Psi}\gamma^0\Psi$ in the Lagrangian which modifies the energies associated to the new variables

$$E(\varphi_{n,i}) = g_n A_0 + m + \mu, \quad E(\chi_{n,i}) = g_n A_0 - m + \mu. \quad (42)$$

The energy differences are not modified and all the estimates of the corrections due to space derivatives and the space components of the vector field are not changed.

The effective fermionic action in the adiabatic approximation is now

$$\begin{aligned} \Gamma_{\text{ad}}(\mu) = & \int d\vec{x} \sum_n \text{sgn}(g_n A_0 + m + \mu) \left[\int dt (g_n A_0 + m + \mu) + \frac{g_n}{g} i\Omega[\hat{A}_0^a] \right] \\ & + \int d\vec{x} \sum_n \text{sgn}(g_n A_0 - m + \mu) \left[\int dt (g_n A_0 - m + \mu) + \frac{g_n}{g} i\Omega[\hat{A}_0^a] \right]. \end{aligned} \quad (43)$$

We consider $\mu > 0$ for definiteness. In this case, we can consider three different regions. The first one corresponds to $gA_0 < 2|m - \mu|$, where once more there is a cancellation of contributions and one has a trivial adiabatic effective action as in the weak coupling region of the case without chemical potential. There is also an analog of the strong coupling region where $gA_0 > 2|m + \mu|$ and the adiabatic effective action is (41). Finally, there is an intermediate region, $2|m - \mu| < gA_0 < 2|m + \mu|$, where there is a cancellation of the contributions of half of the new fermionic variables and the result for the effective action is

$$\Gamma_{\text{ad}}^{(i)} = g \int d\tau d^3\vec{x} A_0 + i\Omega[\hat{A}_0^a] + \text{const.} \quad (44)$$

Once more, we can discuss the decoupling of the fermion degrees of freedom. There are once more cases with arbitrarily large mass and density where for sufficiently large gA_0 a fermionic signal remains. On the other hand, for fixed mass and gA_0 the fermion decouples in the limit $\mu \rightarrow \infty$. This shows that the non-perturbative properties of the non-Abelian gauge theory related to the presence of a non-trivial adiabatic effective action disappear in the infinite density limit.

It is also very easy to generalize the adiabatic approximation to the case of finite temperature. As we have seen in Section 2, the modification of the fermionic integral for each quantum-mechanical system is very simple and

the effective fermionic action in the adiabatic approximation at finite temperature (and also including a chemical potential) is given by

$$\begin{aligned}
 -\Gamma_{\text{ad}}(\mu, \beta) = & \int d\vec{x} \sum_n \ln \cosh \left[\int_{-\beta/2}^{\beta/2} d\tau (g_n A_0 + m + \mu) + i\Omega [\hat{A}_0^a] \right] \\
 & + \int d\vec{x} \sum_n \ln \cosh \left[\int_{-\beta/2}^{\beta/2} d\tau (g_n A_0 - m + \mu) + i\Omega [\hat{A}_0^a] \right], \tag{45}
 \end{aligned}$$

where β is the inverse of the temperature. If one takes the high temperature limit, $\beta \rightarrow 0$ for fixed (μ, m, gA_0) , then the adiabatic effective action is proportional to β^2 and then the new non-perturbative contribution disappears.

6. Discussion

The result for the effective fermionic action in the adiabatic approximation (40) is not gauge-invariant neither Lorentz-invariant. This should not be a surprise. A similar situation happens in the simplest quantum-mechanical example of the spin coupled to a magnetic field [9], where each term in the adiabatic expansion is not separately rotational invariant, the variation of each term being canceled by that of subsequent terms in the expansion. This makes difficult to understand how one can find a physical situation where the adiabatic approximation to the effective fermionic action (40) can give a consistent first-order approximation to a relativistic QFT with fermionic fields.

One possibility is to consider a gauge field theory, identifying the vector field A_μ^a with the gauge field of the $SU(2)$ gauge theory. In that case one should add a term in the action depending on the dynamical gauge field. The Lorentz and gauge invariance of the theory is inconsistent with the adiabatic approximation; in fact, this could have been anticipated because by applying an arbitrary gauge transformation on a vector field whose space components are sufficiently small and whose time component orientation in internal space varies sufficiently slowly to justify the approximation one gets a gauge field configuration where the approximation is not justified. The only way to look for a consistent realization of the adiabatic approximation in a gauge field theory is by considering the vector field as the gauge field satisfying an appropriate non-covariant gauge fixing condition. It is not possible with our present understanding of gauge field theories at the non-perturbative level to check whether there is a situation were a gauge fixing condition can be found such that the relevant gauge field configurations in that gauge satisfy all the conditions to justify the adiabatic approximation to the effective fermionic action considered in this work. All one can do is to explore the consequences of the assumption that this is the case. In all cases, where the effective action in the adiabatic approximation is non-trivial, one has a term proportional to $\int d\tau d^3\vec{x} A_0$ and then, in order to have a finite action, the gauge field A_0 should be concentrated in a finite region in space–time. This fact, together with the disappearance of the new contribution in the high temperature or high density limits suggests a relation of the presence of a non-trivial adiabatic approximation and the confinement in the non-Abelian gauge theory. Note that if this relation holds, then the confinement at low energies would be due to the presence of heavy quarks which do not decouple as one would naively expect due to non-perturbative effects.

Other possibilities for a physical realization of the adiabatic approximation to the effective fermionic action could correspond to a situation where the vector field is not a dynamical field but a background field which parametrizes some of the non-perturbative properties of the vacuum of QFT or an auxiliary field introduced to linearize a fermion self-coupling. In those cases the Lorentz and gauge non-invariance of the result inherent to the adiabatic approximation should be related with the details of the introduction of the background or auxiliary field.

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