Ghost condensation and subluminal propagation on low derivative backgrounds

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We show a new class of interaction terms with higher derivatives that can be added to every low derivative real scalar, such that the first order perturbations induced by the higher derivative terms on the low derivative background are ghost-free. This follows without imposing additional constraints. Furthermore, we show a related class of theories with an additional stabilizer variable and a constraint which are ghost-free without restricting to a perturbative expansion. In this case the field equation followed *by the stabilizer variable* may have interesting physical applications: namely, in contrast to some models with *first-order derivative interactions* with applications for dark energy and inflation, these *constrained second-order derivative self-interactions* do not necessarily affect the luminal propagation, hence, avoiding the common superluminality issues of the former.

I. INTRODUCTION

In spite of the phenomenological success of the standard model of particle physics (SM) and Einstein gravity in a wide range of energies, it is understood that they do not give the full picture. New effects are explored beyond the currently tested energy scales, for instance, by adding effective terms to the SM which must cause *negligible corrections* in at least some energy regimes.

In a similar reasoning, it would be desirable to explore corrections to low derivative theories by adding higher time derivative terms. However, in striking contrast, the latter cannot induce small deformations on the low derivative theory because they radically modify the physics through non-perturbative effects, as they bear a fundamental instability [1]. They enlarge the dimensionality of phase space including a ghost that catasthropically destabilizes the low derivative degrees of freedom (*dof*s) upon interaction [1–6]. Hence, higher derivative extensions are usually only considered as effective terms, for instance, by the method of "perturbative constraints" [1–4, 7–12].

This instability is sometimes called as "of Ostrogradsky". The "theorem of Ostrogradsky" states that non-degenerate higher derivative theories entail an unbounded energy from below, which is ultimately seen in the propagation of an additional ghosty *dof* [1, 2, 5, 22]. A comprehensive introduction to the Ostrogradsky's instability is given, for instance, in [2], or in [23], where further results on this instability with multiple higher derivative dynamical variables are shown.

Among some exceptions to this instability it is possible to consider Galileons and degenerate theories [6, 13–21].

The case of interest in this letter are degenerate theories with a single higher derivative dynamical variable. For this class, the issue of the instability is resolved at the expense of introducing an additional *ad-hoc* dynamical variable with low derivatives, besides the higher derivative variable, and by imposing constraints *by fiat* among the latter two variables such that the ghosty *dof* is eliminated.

In this letter, we first show a new class of higher derivative terms that can be added to a given low derivative theory, such that the first order perturbations induced by the higher derivative deformation on the low derivative background are degenerate and hence, they are ghost free. This follows *without* the need to introduce additional constraints.

On a second step, we consider related theories with an additional constraint and an auxiliary variable that are degenerate and ghost free without restricting to a perturbative expansion. In this case, some physical consequences of the higher derivative sector are explicitly contained in the equation that specifies the *stabilizer* auxiliary variable. In field theory, its equation of motion has the interesting feature that *it can be strictly luminal and still be sourced by derivative interactions.*

We proceed as follows: in section II we specify the class of theories to be analyzed in this letter. We first state the assumptions and notation in section II A, while in section II B we briefly review the signatures of the Ostrogradskian instability in a non degenerate higher derivative theory.

In section III A, we show in first place the class of higher derivative deformations that could have stable, arbitrarily small fluctuations on the low derivative background *without the need of constraints nor additional variables.* Then, in section III B, we specify a straightforward constraint on the previous class of theories, such that they are degenerate and ghost free without restricting to perturbative expansions. Finally, a first, trivial example is given in section III C, where we verify directly that there is no Ostrogradsky ghost both in the Lagrangian and Hamiltonian formalisms.

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In section IV, the absence of the signatures of the instability for the constrained theories is verified in the general case.

In section V, we show with a non trivial example the two physical consequences on the energy of the *stabilized* variable and on the motion of the *stabilizer* variable. We show that the field equation for the latter includes ghost-free *derivative interactions* without modifying the cone of influence nor the speed of propagation that is fixed by the low derivative part of the theory. Hence, avoiding some of the common issues for strictly *low derivative* self-interactions [24–30]: namely, the breakdown of the Cauchy problem, loss hyperbolicity [24], as well as causality issues [30], which could be, for instance, problematic in k-essence models [25].

We give the conclusions in section VI.

II. THE ASSUMPTIONS: A BASIS OF UNSTABLE HIGHER DERIVATIVE THEORIES

First, we introduce notation and delimit the class of theories to be analyzed in this note. In section II B, we briefly review the issue with the stability in non degenerate theories with higher derivatives.

A. The assumptions and notation

Let us first consider a theory with Lagrangian depending on up to first time derivatives of the dynamical variable ϕ, and possibly, spatial derivatives $\mathcal{L}^{(1)}(\partial\phi, \phi)$. For the moment, unless stated otherwise in the next sections, and to avoid distractions with non-essential field formalism, we will first analyze mechanics of the sole dynamical variable ϕ depending only on time.

We will assume that *only* this low derivative sector $\left(\mathcal{L}^{(1)}\right)$ is nondegenerate. That is, with $\dot{\phi}$ time derivative of ϕ,

$$\frac{\partial^2 \mathcal{L}^{(1)}}{\partial\dot{\phi}\partial\dot{\phi}} \neq 0\,, \tag{1}$$

such that the Euler-Lagrange equation for $\mathcal{L}^{(1)}$ depends linearly on $\ddot{\phi}$. Since two initial conditions must be given in order to solve the second order equation of motion, the dimensionality of phase space is two. Thus, we are implicitly assuming that with $\mathcal{L}^{(1)}$ we describe only one *degree of freedom* (*dof*).

Now, we will consider a deformation of $\mathcal{L}^{(1)}$ by adding higher derivative terms of the sole dynamical variable ϕ. Let us denote with $\mathcal{L}(\phi, \dot{\phi}, \ddot{\phi})$ such a theory. We will canonically normalize ϕ taking as reference the standard low derivative kinetic term in $\mathcal{L}^{(1)}$

$$\dot{\phi}^2 \tag{2}$$

in such a way that with $\mathcal{L}$, we introduce a new physical scale Λ at which the higher derivative sector is probed. We assume Λ (*e.g.* energy scale) to be high with respect to the characteristic scale of $\mathcal{L}^{(1)}$:

more precisely, *we assume that ignoring the terms scaled by* Λ *in the Lagrangian* $\mathcal{L}$, *we recover* $\mathcal{L}^{(1)}$ (namely, as $\Lambda \to \infty$).

Occasionally, to give a concrete analysis, we also consider an explicit form for the Lagrangian $\mathcal{L}$

$$\mathcal{L} = \mathcal{L}^{(1)} + \mathcal{L}^{(2)}\,, \tag{3}$$

where we explicitly add a higher derivative sector $\mathcal{L}^{(2)}$ to the low derivative sector $\mathcal{L}^{(1)}$. Further, assume that the higher derivatives in $\mathcal{L}$ cannot be eliminated by integration by parts.

Finally, let us introduce the following notation: for a Lagrangian F that possibly depends on up to second time derivatives of more than one dynamical variable $\psi_1, \psi_2, \ldots$, consider the operator

$$\Theta(F;\psi_1) = \left(\frac{d^2}{dt^2}\frac{\partial}{\partial\ddot{\psi}_1} - \frac{d}{dt}\frac{\partial}{\partial\dot{\psi}_1} + \frac{\partial}{\partial\psi_1}\right) F\,, \tag{4}$$

also known in the literature as *Euler derivative* of F with respect to the variable ψ_1, which can be non zero *off-shell*, and such that *on-shell* the Euler-Lagrange equation for ψ_1 is written as

$$\Theta(F;\psi_1) = 0\,. \tag{5}$$

1. *Non degeneracy*

In the case that $\mathcal{L}$ is *non degenerate* with respect to the higher derivative terms, the number of *dof*s is increased. A ghost is integrated-in when passing from $\mathcal{L}^{(1)}$ to $\mathcal{L}$.

The higher derivative theory $\mathcal{L}$ is *non degenerate in the higher derivative sector* if the term

$$\frac{\partial^2 \mathcal{L}}{\partial\ddot{\phi}\partial\ddot{\phi}} \tag{6}$$

does not vanish. With this non degenerate condition, the theory $\mathcal{L}$ is unstable and non unitary upon quantization [1–6, 14, 15, 17]. For definiteness, we will assume that for the theory $\mathcal{L}$, the term (6) is not *identically* zero. However, in order to satisfy a degeneracy condition, below we will construct a constrained version of $\mathcal{L}$, which we denote $\mathcal{L}'$, where the term (6) *vanishes only on-shell*, on the surface of constraints in phase space. A first example is given in section III C.

All in all, below we will consider three different Lagrangians $\mathcal{L}^{(1)}$, $\mathcal{L}$ and $\mathcal{L}'$, which are related as follows: $\mathcal{L}^{(1)}(\phi)$ is a given non degenerate low derivative theory for the sole dynamical variable ϕ. $\mathcal{L}$ is a deformation of $\mathcal{L}^{(1)}$ that includes higher derivative terms of ϕ and is *non degenerate* with respect to the higher derivatives. Finally, $\mathcal{L}'$ is a constrained version the theory $\mathcal{L}$ that is *degenerate only on-shell* in the higher derivative sector. Let us also note that in section III we will give an additional condition on the as yet very general structure of $\mathcal{L}$. This is summarized in table I.

Furthermore, let us highlight the essence of the problem with the theory $\mathcal{L}$: because $\mathcal{L}$ is *non degenerate*, it cannot induce small corrections to the low derivative dynamics of $\mathcal{L}^{(1)}$ regardless whether there is any small parameter Λ^{-1} suppressing the higher derivative sector in $\mathcal{L}$ [1]. This can be seen in that the Euler Lagrange equation derived from $\mathcal{L}$

$$\Theta(\mathcal{L};\phi) = 0 \tag{7}$$

is a differential equation of fourth order, such that four initial conditions are required to specify the dynamics, as opposed to the Euler-Lagrange equation obtained with a theory $\mathcal{L}^{(1)}$

$$\Theta(\mathcal{L}^{(1)};\phi) = 0\,, \tag{8}$$

which only requires two initial conditions to describe the physical system. As noted long ago, the Ostrogradsky's instability is a non-perturbative effect [1–5] and perturbative expansions for $\mathcal{L}$ with expansion parameter Λ can at best hide the ghost [1–4, 7–12].

A thorough review shows that upon interaction, the low derivative *dof* that would be propagated with $\mathcal{L}^{(1)}$ dynamics is catastrophically destabilized in a theory $\mathcal{L}$ [1–6, 14, 15, 17]. Below, we provide an overview of the essential signature of the instability.

B. Brief review of the Ostrogradskian, linear instability

The instability of non degenerate higher derivative theories is most evident in the unboundedness from below of the energy. Hence, let us consider the conserved quantity derived from the time homogeneity of a second-order time derivative action with Lagrangian $\mathcal{L}(\phi,\dot\phi,\ddot\phi)$, which we associate with the energy for a standard low derivative theory:

$$E[\mathcal{L}] = \ddot\phi\frac{\partial\mathcal{L}}{\partial\ddot\phi} + \dot\phi\frac{\partial\mathcal{L}}{\partial\dot\phi} - \mathcal{L} - \dot\phi\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\ddot\phi}\,. \tag{9}$$

Because we have assumed no constraints in $\mathcal{L}$, the configuration space and energy are determined by the four coordinates ϕ, $\dot\phi$, $\ddot\phi$ and $\dddot\phi$, as we can see from the four initial conditions that are necessary to specify a particular solution to the fourth order equation of motion for ϕ (7).

Now, all terms on the right hand side of (9) with the exception of the rightmost, depend only on $\phi,\dot\phi$ and $\ddot\phi$ in a possibly non linear way. However, expanding the total time derivative in the rightmost term, it is easy to see that the energy depends *strictly in a linear* way on $\dddot\phi$ as

$$-\frac{\partial^2\mathcal{L}}{\partial\ddot\phi\partial\ddot\phi}\dot\phi\,\dddot\phi\,. \tag{10}$$

Hence, $E[\mathcal{L}]$ is never bounded from below with respect to the coordinate $\dddot\phi$ in configuration space if the theory is non-degenerate, because the critical term (6) does not vanish on-shell.

Alternatively, let us note that the linear dependance of the energy on the coordinate $\dddot\phi$ implies that on passing to the Hamiltonian formalism, there must exist a canonical coordinate that appears linearly in the Hamiltonian, if the theory is non degenerate. Since the latter is the most common exposition of the instability, let us briefly consider the Hamiltonian and show the common signature of the instability between both formalisms: the four canonical coordinates are [2, 22]

$$\begin{array}{llll} x_1 = \phi & p_1 & = & \frac{\partial\mathcal{L}}{\partial\dot\phi} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\ddot\phi} \\ x_2 = \dot\phi & p_2 & = & \frac{\partial\mathcal{L}}{\partial\ddot\phi}\,. \end{array} \tag{11}$$

Notice that the assumption of non-degeneracy

$$\frac{\partial^2\mathcal{L}}{\partial\ddot\phi\partial\ddot\phi} = \frac{\partial p_2}{\partial\ddot\phi} \neq 0 \tag{12}$$

implies that the conjugate momentum $p_2 = p_2(\ddot\phi,\dot\phi,\phi)$ depends on the acceleration $\ddot\phi$. Thus, it is possible that $\ddot\phi$ can be *uniquely* expressed in terms of the canonical coordinates p_2, x_1, x_2, which we will *assume* below.

All in all, for three of the coordinates in configuration space, ϕ, $\dot\phi$, $\ddot\phi$, we define the three canonical coordinates x_1, x_2, p_2.

For the remaining coordinate in configuration space $\dddot\phi$, we must define p_1, which as expected, is linear in $\dddot\phi$,

$$p_1 = G(x_1,x_2,p_2) - \frac{\partial^2\mathcal{L}}{\partial\ddot\phi\partial\ddot\phi}\,\dddot\phi\,, \tag{13}$$

where G is a possibly non linear function of x_1, x_2, p_2, whose form is not relevant in the discussion below. Notably, the coordinate $\dddot\phi$ can only be inverted in terms of p_1 if the theory is non degenerate (12). Namely, if the rightmost term in the equation (13) does not vanish. Only then, p_1 indeed depends on the coordinate $\dddot\phi$, and hence, in such a case, p_1 is an independent coordinate in phase space besides x_1, x_2, p_2.

Since neither the Lagrangian nor the other canonical coordinates x_1, x_2, p_2 depend on $\dddot\phi$, the linear dependance of p_1 on $\dddot\phi$ will *remain linear* upon the Legendre transform that gives the Hamiltonian[1],

$$H = p_1x_2 + p_2\ddot\phi(x_1,x_2,p_2) - \mathcal{L}(x_1,x_2,\ddot\phi(x_1,x_2,p_2))\,. \tag{14}$$

Namely, since we have assumed with the non degeneracy that p_1 and x_2 are independent canonical coordinates, the term p_1x_2 on the right hand side of H is *strictly*

[1] It is easy to verify that this choice of canonical coordinates and Hamiltonian generates correct (lagrangean) time evolution; hence, (14) is the right functional form for the energy [2, 22].

linear in p_1, while $p_2\ddot{\phi} - \mathcal{L}$ can be a bounded function of x_1, x_2, p_2.

As has been widely discussed in the literature [1–6, 14, 15, 23], unless a constraint expresses x_2 in terms of p_1, the term $p_1 x_2$ in the Hamiltonian is the most basic signal of the Ostrogradsky's instability: H is *strictly linear* in the conjugate momentum p_1 and it renders the Hamiltonian unbounded from below.

Finally, writing the problematic term $p_1 x_2$ in the Hamiltonian in terms of the coordinates in configuration space,

$$p_1 x_2 \equiv G(\phi, \dot{\phi}, \ddot{\phi})\dot{\phi} - \frac{\partial^2 \mathcal{L}}{\partial\ddot{\phi}\partial\ddot{\phi}}\dot{\phi}\,\dddot{\phi}\,, \tag{15}$$

we identify only the rightmost term as the critical term in the energy function $E[\mathcal{L}]$ (10). The remaining part in $p_1 x_2$, $G(\phi, \dot{\phi}, \ddot{\phi})\dot{\phi}$, is not critical since it may still be bounded depending on the specifics of theory. In other words, we narrowed the signature of this type of instability to the same term (10) in the Lagrangian and Hamiltonian formalisms. All the issues arise when the theory is such that the term (10), or more precisely the term (6) does not vanish on-shell.

In what follows, for simplicity, we will mainly discuss this type of instability only in the Lagrangian formalism.

III. GHOST CONDENSATION ON LOW DERIVATIVE BACKGROUNDS

The key assumption that induces the Ostrogradskian instability in the dynamics is *nondegeneracy* [2, 3].

Section III A is dedicated to motivate a class of *non degenerate* higher derivative theories $\mathcal{L}$ which *nevertheless* lead to *stable small fluctuations* on low derivative backgrounds. This is relevant because, even in a higher derivative theory, we would like to have a regime where the low derivative theory is most relevant, and the first order pertubations should not grow without control.

From this analysis, nonlinearities turn out to be a necessary feature of these type of higher derivative theories, where the elimination of ghosts follows in a similar mechanism as in "ghost condensation" for low derivative theories [31].

In section III B we define a fully *degenerate* version of the *non degenerate* theories $\mathcal{L}$, which we call $\mathcal{L}'$, that is ghost-free without restricting to a linearization. We do this following the strategy proposed in [14] of introducing constraints and an auxiliary variable with low derivatives.

We give a first trivial example in section III C, where we directly verify the absence of the signatures of the instability. A non trivial example is shown in section V.

A. Stabilizers of small perturbations: a restricted class of effective, higher derivative theories

Given a low derivative theory $\mathcal{L}^{(1)}$, we show below a special kind of deformations with higher derivative terms, $\mathcal{L}$, with the property that the *first order* perturbations about solutions to the dynamics of $\mathcal{L}^{(1)}$ are degenerate and can be stable. Although the full theory $\mathcal{L}$ is for the moment non degenerate and unstable, the degeneracy and stability of the first order perturbations -exclusively- is a basic requirement because we would like to have a regime where the low derivative theory $\mathcal{L}^{(1)}$ is most relevant, and the first order pertubations should not grow without control. Below, we will refer to a given solution of the dynamics derived from $\mathcal{L}^{(1)}$ as *the background*.

Consider a perturbative expansion in $\mathcal{L}$ with the distinctive feature that the leading, 0-th order approximation is the standard, low derivative dynamics. Namely, denoting a solution for the low derivative sector $\mathcal{L}^{(1)}(\phi)$ as ϕ_0

$$\Theta(\mathcal{L}^{(1)};\phi)\Big|_{\phi_0} = 0\,, \tag{16}$$

where we have used notation (4), we decompose ϕ in terms of a fluctuation (π) about this 0-th order solution (ϕ_0) as

$$\phi = \phi_0 + \epsilon\pi\,. \tag{17}$$

ϵ is a *small* dimensionless parameter scaling as the inverse of a positive power of Λ (the physical scale at which the higher derivative sector is probed).

At quadratic order in ϵ in the Lagrangian ($\mathcal{O}(\epsilon^2)$), the expansion reads

$$\mathcal{L}(\phi) = \mathcal{L}(\phi_0) + \mathcal{L}_\pi\,. \tag{18}$$

The background part $\mathcal{L}(\phi_0)$ in expression (18) is satisfied on-shell by ϕ_0, because at 0-th order the Λ^{-1} terms in $\mathcal{L}$ are subleading, namely,

$$\mathcal{L}(\phi_0) \approx \mathcal{L}^{(1)}(\phi_0)\,. \tag{19}$$

This follows by the assumption stated above equation (3) and by definition (16) of ϕ_0.

The linearized dynamics $\mathcal{L}_\pi$ is the only potentially ghosty contribution,

$$\mathcal{L}_\pi = \epsilon^2 \frac{1}{2}\, \frac{\partial^2 \mathcal{L}}{\partial\ddot{\phi}\partial\ddot{\phi}}\bigg|_{\phi_0} \ddot{\pi}^2 + \tilde{\mathcal{L}}_\pi + \mathcal{O}(\epsilon^3)\,, \tag{20}$$

where the term $\tilde{\mathcal{L}}_\pi$ is, up to a total time derivative, a polynomial of up to second order in $\epsilon\pi$ and $\epsilon\dot{\pi}$.

Now, as explained in the previous section, we do not expect such an expansion to be meaningful for the *non degenerate* theory $\mathcal{L}$ at $\mathcal{O}(\epsilon^3)$. However, *for every healthy theory* even a $\mathcal{O}(\epsilon^2)$ expansion must be meaningful in the sense that small fluctuations

$$\epsilon\|\pi\| \ll \|\phi_0\| \tag{21}$$

about the low derivative solution ϕ_0 must not become large arbitrarily fast. Hence, let us read out from the signatures of the instability in (20) the structure that $\mathcal{L}$ must have: at order $\mathcal{O}(\epsilon^2)$ the only term that signals the Ostrogradsky's instability is the first on the right hand side of equation (20), namely, the term proportional to $\ddot{\pi}^2$. It leads to a linearized equation of 4-th order for π.

Clearly, the constrained, stable dynamics $\mathcal{L}'$ should not have such terms in the perturbative expansion about the background ϕ_0. Thus, it is clear that the critical term (6), which controls the instability of the expansion (20) at $\mathcal{O}(\epsilon^2)$, needs to vanish when it is evaluated on the 0-th order solution. By definition of ϕ_0 (16), a natural option for the term (6) is to consider only theories $\mathcal{L}$ that satisfy

$$\frac{\partial^2 \mathcal{L}}{\partial\ddot{\phi}\partial\ddot{\phi}} = c(\phi)\Theta(\mathcal{L}^{(1)};\phi)\,, \tag{22}$$

where $c(\phi)$ is some non singular function which is non zero off-shell and may depend on up to second derivatives of ϕ. As a consequence of the choice (22), the perturbative expansion (20) reduces to

$$\mathcal{L}_\pi = \tilde{\mathcal{L}}_\pi + \mathcal{O}(\epsilon^3)\,, \tag{23}$$

which contrasts to the case of higher derivative Lagrangians that do not satisfy (22), because now there are no $\propto \ddot{\pi}^2$ contributions in $\mathcal{L}_\pi$. Indeed, with the restriction (22) on the theories $\mathcal{L}$, the fluctuations (π) at $\mathcal{O}(\epsilon^2)$ solve a *second order* equation. In other words, the small fluctuations π about the ϕ_0 low derivative background are ghost-free at $\mathcal{O}(\epsilon^2)$ and accordingly, in the energy function $E[\mathcal{L}_\pi]$ computed as in (9), the signature of the instability (10) vanishes by the same choice (22)

$$-\dot{\pi}\dddot{\pi}\frac{\partial^2 \mathcal{L}_\pi}{\partial\ddot{\pi}\partial\ddot{\pi}} = -\epsilon^2 c\;\Theta(\mathcal{L}^{(1)};\phi)\Big|_{\phi_0}\dot{\pi}\dddot{\pi} = 0\,. \tag{24}$$

To sum up, given a low derivative theory $\mathcal{L}^{(1)}$, below we will restrict to a class of higher derivative theories $\mathcal{L}$ that can be regarded as a deformation of $\mathcal{L}^{(1)}$ and that satisfy (22) because we want the *arbitrarily small fluctuations about the solutions to the low derivative theory to be meaningful.*

Such theories are the physically sound choice because, at least, the first order corrections about the low derivative theory are free of Ostrogradsky ghosts without additional constraints.

B. Fully degenerate theories $\mathcal{L}'$: The physically motivated constraint

Although the theories $\mathcal{L}$ satisfying (22) are naturally healthy *without additional constraints at linear order*, we can explore a class of *related healthy theories* without restricting to perturbative expansions, which we denote as $\mathcal{L}'$.

By extension of the previous physical motivation to consider theories of the form (22), we impose on $\mathcal{L}'$

$$\frac{\partial^2 \mathcal{L}'}{\partial\ddot{\phi}\partial\ddot{\phi}} = c(\phi)\Theta(\mathcal{L}^{(1)};\phi)\,. \tag{25}$$

Now, to eliminate ghosts in $\mathcal{L}'$ is simple because *the degeneracy of the first order perturbations* has also restricted with the equation (25) the term of the kinetic matrix that is relevant for *the degeneracy of the full theory $\mathcal{L}'$*.

We can observe the generalities of the physical consequences for *fully degenerate* theories that satisfy (25) without proposing the specifics of a physical mechanism that causes the degeneracy itself. Namely, with an *ad-hoc* constraint, using a Lagrange multiplier $a(t)$

$$\mathcal{L}'(\phi,a) = \mathcal{L}(\phi,\partial\phi,\partial^2\phi) + a(t)\Theta(\mathcal{L}^{(1)};\phi)\,, \tag{26}$$

where $a\Theta(\mathcal{L}^{(1)};\phi)$ is strictly linear in $\ddot{\phi}$.

As we show in section V, in the theories $\mathcal{L}'$ defined by (25-26) important physical implications of the higher derivative sector are embedded in the *stabilizer Lagrange multiplier* $a(t)$. Furthermore, ϕ is fixed by the constraint $\Theta(\mathcal{L}^{(1)};\phi)$, but there are still *signatures* of its interaction with $a(t)$ left in the energy. A clear explicit example is shown in Figure 1.

We check in section IV that this structure is enough to eliminate the signatures of the ghost.

This setup can be generalized to the case of field theory. For instance, if $\phi(x)$ is a real scalar field, the generalization of condition (25) for a relativistic theory is straightforward:

$$\frac{\partial^2 \mathcal{L}'}{\partial\phi_{,\mu\nu}\partial\phi_{,\rho\sigma}} = c^{\mu\nu\rho\sigma}(\phi)\Theta(\mathcal{L}^{(1)};\phi)\,, \tag{27}$$

where $\phi_{,\mu\nu} \equiv \partial_\mu\partial_\nu\phi$. The assumptions on $c(\phi)$ given in section II are extended to $c^{\mu\nu\rho\sigma}(\phi)$. An example for a real scalar field is given in section V 3.

In table I we summarize the related theories $\mathcal{L}^{(1)}$, $\mathcal{L}$, $\mathcal{L}'$.

C. A trivial example

Let us first give a trivial example to put the notation in practice. *A non trivial example is given in section* V.

Consider a free point particle as the low derivative sector

$$\mathcal{L}^{(1)} = \frac{1}{2}\dot{\phi}^2\,. \tag{28}$$

A possible deformation of this low derivative theory according to equations (25), (26) is

$$\mathcal{L}'(\phi,a) = \mathcal{L}(\phi) - a\ddot{\phi}\,. \tag{29}$$

$$\mathcal{L}(\phi) = \frac{1}{12\,\Lambda^5}\ddot{\phi}^4 + \frac{1}{2}\dot{\phi}^2 \tag{30}$$

where Λ is a constant and ϕ^2, a^2, Λ^{-1} have units of time.

TABLE I. Summary of the Lagrangians considered in this letter and the *relevant entry of the Kinetic matrix concerned with the degeneracy of*† the low derivative sector in $\mathcal{L}^{(1)}$, and the higher derivative sectors for $\mathcal{L}$, $\mathcal{L}'$. Notice that $\mathcal{L}$ is degenerate and free of Ostrogradsky ghost at first order *without* additional *ad-hoc* constraints. The term $\Theta(\mathcal{L}^{(1)};\phi)$ is defined in equation (4).

Theory	**Depends on**	**Kinetic Matrix† off-shell**	**K. Matrix† on-shell**	**Degenerate**
$\mathcal{L}^{(1)}$	$\phi,\, \dot{\phi}$	$\frac{\partial^2\mathcal{L}^{(1)}}{\partial\dot{\phi}\partial\dot{\phi}} \neq 0$	$\frac{\partial^2\mathcal{L}^{(1)}}{\partial\dot{\phi}\partial\dot{\phi}} \neq 0$	No
$\mathcal{L}$	$\phi,\, \dot{\phi},\, \ddot{\phi}$	$\frac{\partial^2\mathcal{L}}{\partial\ddot{\phi}\partial\ddot{\phi}} = c\Theta(\mathcal{L}^{(1)};\phi) \neq 0$	$\frac{\partial^2\mathcal{L}_\pi}{\partial\ddot{\pi}\partial\ddot{\pi}} = \epsilon^2 c\Theta(\mathcal{L}^{(1)};\phi)\vert_{\phi_0} = 0$	Yes, at $\mathcal{O}(\epsilon^2)$
$\mathcal{L}'$	$\phi,\, \dot{\phi},\, \ddot{\phi}, a$	$\frac{\partial^2\mathcal{L}'}{\partial\ddot{\phi}\partial\ddot{\phi}} = c\Theta(\mathcal{L}^{(1)};\phi) \neq 0$	$\frac{\partial^2\mathcal{L}'}{\partial\ddot{\phi}\partial\ddot{\phi}} = c\Theta(\mathcal{L}^{(1)};\phi) \equiv 0$	Yes

Let us explain the parts in (29). The rightmost term corresponds to $a\Theta(\mathcal{L}^{(1)};\phi)$ in (26), where

$$\Theta(\mathcal{L}^{(1)};\phi) = -\frac{d}{dt}\frac{\partial\mathcal{L}^{(1)}}{\partial\dot{\phi}} + \frac{\partial\mathcal{L}^{(1)}}{\partial\phi} = -\ddot{\phi}\,. \tag{31}$$

The remaining terms (30) are a higher derivative deformation of the free point particle (28) suppressed by the new scale Λ. It is essential to note that $\mathcal{L}'(\phi, a)$ satisfies (22), (25)

$$\frac{\partial^2\mathcal{L}'}{\partial\ddot{\phi}\partial\ddot{\phi}} = \frac{\partial^2\mathcal{L}}{\partial\ddot{\phi}\partial\ddot{\phi}} = c(\phi)\Theta(\mathcal{L}^{(1)};\phi) = -\frac{\ddot{\phi}}{\Lambda^5}\left(-\ddot{\phi}\right)\,, \tag{32}$$

where $c = -\ddot{\phi}/\Lambda^5$.

First, notice that the theory $\mathcal{L}$ (30), *without* additional constraint, is already free of ghosts *at linear order* because it satisfies (32): namely,

$$\mathcal{L}(\phi_0) = \left.\frac{1}{12\,\Lambda^5}\ddot{\phi}_0^4 + \frac{1}{2}\dot{\phi}_0^2\right|_{\Lambda\to\infty} \approx \frac{1}{2}\dot{\phi}_0^2 \tag{33}$$

$$\mathcal{L}_\pi = \frac{1}{2\,\Lambda^5}\ddot{\phi}_0^2\ddot{\pi}^2 + \frac{1}{2}\dot{\pi}^2 = \frac{1}{2}\dot{\pi}^2\,, \tag{34}$$

where the second Lagrangian for the first order perturbation π follows by the 0-th order solution to the first line, $\ddot{\phi}_0 = 0$ (by the assumptions above equation (3), and (17 - 19)). The absence of ghosts *holds without imposing constraints*.

On the other hand, let us check the ghost elimination in the related *constrained* theory $\mathcal{L}'$.

Because of (32) the theory $\mathcal{L}'(\phi, a)$ is degenerate in the higher derivative sector once the Lagrange multiplier $a(t)$ imposes the low derivative dynamics $\ddot{\phi} = 0$ as a constraint. As a consequence, there are no ghosty contributions to the energy: indeed, the energy function computed with (9) is

$$E[\mathcal{L}'] = -\frac{1}{\Lambda^5}\dot{\phi}\ddot{\phi}^2\dddot{\phi} + \frac{1}{4\Lambda^5}\ddot{\phi}^4 + \frac{1}{2}\dot{\phi}^2 + \dot{a}\dot{\phi}\,. \tag{35}$$

However, because the energy is valued on solutions to the equations of motion

$$\ddot{\phi} = 0\;,\; \ddot{a} = 0\,, \tag{36}$$

we note the elimination of the critical term proportional to $\dddot{\phi}$ (10), such that the energy reads

$$E[\mathcal{L}'] = \frac{1}{2}\dot{\phi}^2 + \dot{a}\dot{\phi}\,. \tag{37}$$

A less obvious fact is that the linear term in the Lagrange multiplier $\dot{a}\dot{\phi}$ is harmless: using the equations of motion (36) we see that

$$K = \dot{a}\dot{\phi} \tag{38}$$

is independently conserved, thus, K is an initially fixed *finite* constant. Hence, the energy budget for the nonlinear field ϕ can be explicitly written without linear instabilities up to an unmeasurable constant shift $-K$:

$$E[\mathcal{L}'] - K = \frac{1}{2}\dot{\phi}^2\,, \tag{39}$$

which is effectively the energy of a free particle. In other words, the free point particle remains effectively free despite these higher derivative self-interactions. Thus, although the example is trivial, it is clear that there is no ghost that can turn ϕ dynamics unstable. This is an oversimplified case of this scheme with a more interesting non trivial example given in section V.

Let us also point out that the elimination of the linear term in $\dot{a}(t)$ is not a simple coincidence of this model and the effectively free dynamics (36). Indeed, the elimination of linear terms in $a(t)$, $\dot{a}$ in the energy comes from more general principles that will be put to use in section IV to verify the stability of the setup (25), (26) in full generality.

Furthermore, let us consider the Hamiltonian analysis for this example:

The theory $\mathcal{L}'$ (29) corresponds to $\mathcal{L}$ (30) subjected to the primary constraint $\chi_1 = \ddot{\phi} \approx 0$. Hence, to compute the total Hamiltonian with primary constraints H_T, let us first consider the Hamiltonian on the primary constrained surface, denoted as H_0

$$H_0 = p_1\dot{x}_1 + p_2\dot{x}_2 - \mathcal{L}(p_1, x_1, p_2, x_2)\,, \tag{40}$$

where $\mathcal{L}$ is given by (30) and the four canonical variables were defined in equation (11). Explicitly, the momenta conjugate to $x_1 = \phi$ and $x_2 = \dot{\phi}$ are respectively,

$$p_1 = x_2 - \frac{1}{\Lambda^5}\left(\dot{x}_2\right)^2\ddot{x}_2 \tag{41}$$

$$p_2 = \frac{\dot{x}_2^3}{3\,\Lambda^5}\,. \tag{42}$$

The momentum p_2 depends on $\dot{x}_2$ and can be inverted. The principal root gives

$$\dot{x}_2 = \left(3\Lambda^5 p_2\right)^{1/3}\,, \tag{43}$$

and (40) reads

$$H_0 = p_1\, x_2 + \frac{3}{4}\left(3\Lambda^5 p_2^4\right)^{1/3} - \frac{x_2^2}{2}\,. \tag{44}$$

To write the total Hamiltonian let us notice that the definition of momentum p_1 is itself an independent primary constraint. Namely, provided $\chi_1 = \dot{x}_2 = \dddot{\phi} \approx 0$, rewriting equation (41), we find

$$\chi_2 = p_1 - x_2 + \frac{1}{\Lambda^5}\,\left(\chi_1\right)^2\, \ddot{x}_2 \approx p_1 - x_2 \approx 0\,. \tag{45}$$

All in all, χ_2 constraining p_1 and x_2, and χ_1 constraining p_2 are two primary constraints and the total Hamiltonian reads

$$H_T = H_0 + a_1\chi_1 + a_2\chi_2\,. \tag{46}$$

Demanding the conservation of both constraints fixes both Lagrange multipliers a_1, a_2. For the latter it is more appropriate to choose an *equivalent* and nonsingular primary constraint surface defined with $\chi_1 = p_2 \approx 0$. Since $p_2 \approx 0$ implies $\dot{x}_2 = \left(3\Lambda^5 p_2\right)^{1/3} \approx 0$ and vice versa, they are indeed equivalent. The advantage of the former is that its time evolution ($\dot{\chi}_1 \approx 0$) does not lead to division over constraints, as opposed to the constraint with a cube root (See for instance [32]). Then,

$$\begin{aligned} &a_1 + \left(3\Lambda^5 p_2\right)^{1/3} \approx 0 \\ &a_2 - p_1 + x_2 \approx 0\,. \end{aligned} \tag{47}$$

Hence, there are no more constraints, and χ_1, χ_2 are of second class.

Because there are 4 canonical variables x_1, p_1, x_2, p_2 and 2 second class constraints, there is $(4-2)/2 = 1$ degree of freedom and no Ostrogradsky ghost.

Using all constraints in H_T we recover the total energy (39) that was previously found in the Lagrangian formalism

$$H_T = \frac{1}{2}x_2^2\,. \tag{48}$$

As discussed in section II B, the Ostrogradsky ghost elimination is due to the constraint χ_2 that relates the would be linear momenta p_1 to x_2 on the constraint surface.

D. Nonlinearities in $\mathcal{L}$ and $\mathcal{L}'$

Finally, let us show that nonlinearities are built-in for the setup following equations (22), or (25): integrating for $\mathcal{L}$ from equation (25), assuming that $c(\phi)$ is a polynomial function of $\ddot{\phi}$, and because $\Theta(\mathcal{L}^{(1)};\phi)$ is linear in $\ddot{\phi}$, there is a highest order for $\ddot{\phi}$ in $\mathcal{L}$ that scales at least as

$$\ddot{\phi}^p \tag{49}$$

with $p \geq 3$.

IV. THE STABILITY OF $\mathcal{L}'$

The theories $\mathcal{L}'(\phi, a)$ that satisfy (25), (26) do not suffer from the linear instability discussed in section II B, because they have the built-in term $a\Theta(\mathcal{L}^{(1)};\phi)$ which imposes degeneracy *only* on-shell

$$\left.\frac{\partial^2\mathcal{L}'}{\partial\ddot{\phi}\partial\ddot{\phi}}\right|_0 = c(\phi)\;\Theta(\mathcal{L}^{(1)};\phi)\Big|_0 = 0\,, \tag{50}$$

where we have used (25), (26) and denoted with $|_0$ evaluation on solutions to the equations of motion for $a(t)$ and ϕ. Below, the important difference between on-shell and off-shell expressions will be understood from the context, even without the explicit symbol $|_0$ for the former.

That there is no linear instability can be seen directly in the energy function

$$\begin{aligned} E[\mathcal{L}'] &= \left(\ddot{\phi}\frac{\partial}{\partial\ddot{\phi}} + \dot{\phi}\frac{\partial}{\partial\dot{\phi}} - 1 - \dot{\phi}\frac{d}{dt}\frac{\partial}{\partial\ddot{\phi}}\right)\left(\mathcal{L} + a\Theta(\mathcal{L}^{(1)};\phi)\right) \\ E[\mathcal{L}'] &= E[\mathcal{L}] + E\left[a\Theta(\mathcal{L}^{(1)};\phi)\right]\,. \end{aligned} \tag{51}$$

The would-be linear terms with respect to the coordinate $\dddot{\phi}$ arise only from the term $E[\mathcal{L}]$ in the last line, which contains the term of the form (10) that vanishes on-shell using (50). This verifies the absence of the linear instability in the coordinate $\dddot{\phi}$ for the theories $\mathcal{L}'(\phi, a)$.

Let us stress that the vanishing of equation (50) *only on-shell* is a sufficient condition because the energy function, which is where the linear instability would become relevant, is valued on solutions to the equations of motion.

On the other hand, let us discuss the rightmost contribution to the energy in equation (51): at first glance, the energy of the term $a\Theta(\mathcal{L}^{(1)};\phi)$ seems to include a new kind of linearities in the energy due to the Lagrange multiplier $a(t)$, which appears linearly. Below, we show that this is only a naive appearence from the constrained setup and that no new instabilities are introduced:

First, let us note that linear terms in a, $\dot{a}$ are not truly an issue because Lagrange multipliers can always be solved as a functional of ϕ. In this case, because the constraint contains time derivatives of ϕ (non holonomic) this can be done only *on-shell*. For instance, below, we compute an explicit functional relation between ϕ and $a(t)$.

Second, that $a(t)$ can be solved in terms of ϕ clarifies the issue with the number of degrees of freedom: namely, the Euler-Lagrange equations for $a(t)$ and ϕ derived from $\mathcal{L}'$ lead to a system of two coupled differential equations of second order for ϕ and $a(t)$ respectively

$$\Theta(\mathcal{L}';a) = \frac{\partial\mathcal{L}'}{\partial a} = \Theta(\mathcal{L}^{(1)};\phi) = 0 \tag{52}$$

$$\Theta(\mathcal{L}';\phi) = \Theta\left(a\Theta(\mathcal{L}^{(1)};\phi);\phi\right) + \Theta(\mathcal{L};\phi) = 0\,. \tag{53}$$

In particular, the Euler-Lagrange equation for ϕ (53) always contains a linear term in $\ddot{a}$. This comes from the fact that $a\Theta(\mathcal{L}^{(1)};\phi)$ is linear in $\dddot{\phi}$ by assumption (1). Besides, $\dddot{\phi}$ and higher derivatives of ϕ in equation (53) can be written in terms of ϕ, $\dot{\phi}$ using (52). Thus, naively, without an analysis of constraints (See the Hamiltonian analysis in section III C), it would seem that four initial conditions are necessary to specify the dynamics.

These two issues are addressed below, where we explicitly compute the only direct contribution from $a(t)$ to the total energy of the system. Namely, we compute the term $E\left[a\Theta(\mathcal{L}^{(1)};\phi)\right]$ in equation (51):

With this computation below, we show that the apparently linear terms in a, $\dot{a}$ in $E\left[a\Theta(\mathcal{L}^{(1)};\phi)\right]$ can be expressed as a *functional of* ϕ, $\dot{\phi}$ *only*, while the dependance on the initial conditions chosen to solve for $a(t)$ can be explicitly isolated into a constant K. Such a constant is a *physically irrelevant shift* to the energy. Hence, the apparently free choice of two initial conditions to solve equation (53) for $a(t)$ are in reality *spurious to determine the dynamics*, as it should be for a Lagrange multiplier.

Indeed, using equation (9) we can write the energy functional E on the function $a\Theta(\mathcal{L}^{(1)};\phi)$ as

$$E\left[a\Theta(\mathcal{L}^{(1)};\phi)\right] = \tag{54}$$
$$\left(\ddot{\phi}\frac{\partial}{\partial\ddot{\phi}}+\dot{\phi}\frac{\partial}{\partial\dot{\phi}}+\dot{a}\frac{\partial}{\partial\dot{a}}-1-\dot{\phi}\frac{d}{dt}\frac{\partial}{\partial\ddot{\phi}}\right)\left(a\Theta(\mathcal{L}^{(1)};\phi)\right)$$

Let us note that the third term on the right hand side vanishes, while the last term contains $\dot{a}$. In order to use the equation of motion (53) which contains $\ddot{a}$, let us take a time derivative on both sides of equation (54). Namely,

$$\frac{d}{dt}E\left[a\Theta(\mathcal{L}^{(1)};\phi)\right] = a\left(\dddot{\phi}\frac{\partial}{\partial\ddot{\phi}}+\ddot{\phi}\frac{\partial}{\partial\dot{\phi}}-\frac{d}{dt}\right)\Theta(\mathcal{L}^{(1)};\phi)$$
$$-\dot{a}\Theta(\mathcal{L}^{(1)};\phi)+\dot{\phi}\left(\frac{d}{dt}\frac{\partial}{\partial\dot{\phi}}-\frac{d^2}{dt^2}\frac{\partial}{\partial\ddot{\phi}}\right)a\Theta(\mathcal{L}^{(1)};\phi) \tag{55}$$

The last equation can be written as

$$\frac{d}{dt}E\left[a\Theta(\mathcal{L}^{(1)};\phi)\right] = \tag{56}$$
$$-\dot{\phi}\left(\frac{\partial}{\partial\phi}-\frac{d}{dt}\frac{\partial}{\partial\dot{\phi}}+\frac{d^2}{dt^2}\frac{\partial}{\partial\ddot{\phi}}\right)a\Theta(\mathcal{L}^{(1)};\phi)\,,$$

where we have used that

$$\frac{d}{dt}\Theta(\mathcal{L}^{(1)};\phi) = \left(\dddot{\phi}\frac{\partial}{\partial\ddot{\phi}}+\ddot{\phi}\frac{\partial}{\partial\dot{\phi}}+\dot{\phi}\frac{\partial}{\partial\phi}\right)\Theta(\mathcal{L}^{(1)};\phi) \tag{57}$$

because $\Theta(\mathcal{L}^{(1)};\phi)$ depends only on ϕ, $\dot{\phi}$ and $\ddot{\phi}$ by the assumption given in the equation (1). Furthermore, on-shell and after computing all derivatives, we have also used equation (52) to eliminate the term $-\dot{a}\Theta(\mathcal{L}^{(1)};\phi)$ in equation (55).

Now, using the Euler derivative notation (4), we can rewrite the right hand side of the equation (56) as

$$\frac{d}{dt}E\left[a\Theta(\mathcal{L}^{(1)};\phi)\right] = -\dot{\phi}\Theta\left(a\Theta(\mathcal{L}^{(1)};\phi);\phi\right)\,. \tag{58}$$

Finally, using the Euler-Lagrange equation (53) to get rid of all $\ddot{a}$ dependance, equation (58) takes the form

$$\frac{d}{dt}E\left[a\Theta(\mathcal{L}^{(1)};\phi)\right] = \dot{\phi}\,\Theta(\mathcal{L};\phi)\,. \tag{59}$$

Equation (59) leads to the important conclusion that the time derivative of $E\left[a\Theta(\mathcal{L}^{(1)};\phi)\right]$ does not depend on any way on $a(t)$ itself, nor on $\dot{a}(t)$, because on the right of (59) $\mathcal{L}$ depends only on ϕ.

In other words,

$$E\left[a\Theta(\mathcal{L}^{(1)};\phi)\right] = \int^t dt'\dot{\phi}\Theta(\mathcal{L};\phi)+K \tag{60}$$

where $\int^t dt'$ denotes an antiderivative, which is a *local function* of t and we have written an explicit integration constant K. Let us stress that the first term on the right hand side of equation (60) does not depend on $a(t)$, hence, the whole dependance of $E\left[a\Theta(\mathcal{L}^{(1)};\phi)\right]$ on the initial conditions for $a(t)$ is at most confined to the constant K. Such a finite constant K, physically irrelevant shift to the energy, can be computed in some non trivial cases such as the example (65) in section V.

Let us summarize what we have done: we have rewritten all the linear terms of $a(t)$, $\dot{a}$ that naively appear when computing the left hand side of (60) as a non linear functional that depends *only* on the initial conditions used to fix the solution $\phi(t)$ and independent of any specific solution of $a(t)$ up to the constant K. Namely, with the right hand side of (60), the total energy is written independent of any specific solution $a(t)$ up to a constant unmeasurable shift to the energy K,

$$E[\mathcal{L}'] = E[\mathcal{L}]+\textstyle\int^t dt'\left(\dot{\phi}\Theta(\mathcal{L};\phi)\right)+K\,, \tag{61}$$

where $\mathcal{L}$ only depends on ϕ. In other words, in (60) we have verified that the *only* term containing the Lagrange multiplier $a(t)$ in the total Lagrangian $\mathcal{L}'$ does not introduce any *strictly linear*, unbounded contributions to the energy, because *the antiderivative of the function* $\dot{\phi}\Theta(\mathcal{L};\phi)$ is a general function of ϕ, $\dot{\phi}$ which *can be* bounded from below depending on the specifics of the model.

Let us also emphasize that we have computed (60) only using the Euler-Lagrange equations (52), (53). Hence, the expression (60) is by construction *independent of* the total energy $E[\mathcal{L}']$ and the other contribution to the energy $E[\mathcal{L}]$.

All in all, although in this section we have not explicitly counted the number of degrees of freedom, *we have verified indirectly that there can be no degree of freedom with the properties that one would expect from a "ghost"* because:

1. We can assure that there is a conserved quantity $E[\mathcal{L}']-K$, which is by all standards the measurable energy for the propagation of $\phi(t)$, up to couplings

to gravity, which *is not strictly* unbounded due to the elimination of linear terms of the form (10) on-shell.

Furthermore, the energy budget $E[\mathcal{L}'] - K$ is specified with the same *only two* pieces of initial data that are required to fix the low derivative mode.

2. At most, if there were an additional degree of freedom besides ϕ (a would-be ghost), its energy budget would be confined to the constant energy shift K in (61) that "does not talk" to the measurable energy budget $E[\mathcal{L}'] - K$ for ϕ, and hence, such a would-be ghost could not make ϕ dynamics unstable.

   Besides, the shift K can be *finite*, as we explicitly show for the non trivial example in section V.

Thus, from this section we can conclude that:

At least effectively, $\mathcal{L}'$ is a ghost-free deformation of the low derivative dynamics $\mathcal{L}^{(1)}$ of ϕ, which, however, includes non trivial contributions to the energy for the propagation of ϕ.

By the arguments given above on the *elimination of the strict linearity* in the energy function and on the fact that the measurable energy is determined by *the same two pieces of initial data* used to fix the low derivative ϕ, the cumbersome counting of number of degrees of freedom in the general case, which requires a painstaking analysis of constraints due to the built-in nonlinearities imposed by definition (25), is not strictly necessary to recognize the healthy dynamics of $\phi(t)$. See however, the counting of degrees of freedom with a Hamiltonian analysis for the example in section III C.

Although the degree of freedom in these models is ϕ and its energy is not affected by the initial conditions that one could choose for $a(t)$ (using the equations of motion for $a(t)$), *there is a new physical content* in the equation satisfied by $a(t)$:

namely, the motion of $a(t)$ can be interpreted as the necessary external action on the ϕ degree of freedom, as for every Lagrange multiplier, in order to keep the constraint that guarantees the low derivative and stable propagation of ϕ. In other words, the dynamics of the stabilizer Lagrange multiplier $a(t)$ is known indirectly by its stabilizer effect on the motion of the ϕ degree of freedom, despite the higher derivative terms of ϕ in the Lagrangian $\mathcal{L}'$. An explicit example and more about the new physical content in the equation that specifies *the stabilizer variable* $a(t)$ is shown in section V.

Let us stress that the dynamics of $\phi(t)$ can still be unstable but due to unrelated origins, for instance, due to an unbounded potential. In short, the construction given by equations (25), (26) only addresses the issue of the *strict linear* instability of the form (10).

A. The stability for the simplest example, revisited

Let us revisit the signatures of the stability for the trivial example in section III C to put to use the general expression for the energy (61).

Consider again the Lagrangian (29). With the two equations for ϕ and the Lagrange multiplier $a(t)$ (36), the first term in the energy (61) reads

$$E[\mathcal{L}]|_0 = \left(-\frac{1}{\Lambda^5}\dot{\phi}\ddot{\phi}^2\dddot{\phi} + \frac{1}{4\Lambda^5}\ddot{\phi}^4 + \frac{1}{2}\dot{\phi}^2 \right)\Bigg|_0 = \frac{1}{2}\dot{\phi}^2 \quad (62)$$

On the other hand, for the second term in (61), we have on-shell for the higher derivative Lagrangian $\mathcal{L}$

$$\Theta(\mathcal{L};\phi)|_0 = \left(\frac{\ddot{\phi}}{\Lambda^5}\left(2\,\dddot{\phi}^2 + \ddot{\phi}\phi^{(4)} \right) - \ddot{\phi} \right)\Bigg|_0 = 0\,. \quad (63)$$

Hence, the total energy given by expression (61) is

$$E[\mathcal{L}'] - K = \frac{1}{2}\dot{\phi}^2\,, \quad (64)$$

which coincides with the result obtained in (39) by a different, simpler argument. However, the advantage of the expression (61) is that it applies in full generality to all interacting theories of the type $\mathcal{L}'$. Indeed, we will use equation (61) in a non trivial example in section V.

V. DYNAMICS AND THE SPEED OF THE STABILIZER VARIABLE $a(x)$

Below we discuss the new physics that arise in this setup. As expected in $\mathcal{L}'$ the Lagrange multiplier can aquire physical meaning. Here a relevant property is related to the speed of the stabilizer variable $a(x)$ in field theories. Namely, the stabilizer Lagrange multiplier $a(x)$ can be sourced by *higher* derivative self-interactions of ϕ and yet $a(x)$ can be strictly luminal. As we will emphasize below, this is radically different from low derivative self-interacting scalars. We also stress on the Λ-suppressed modifications to the energy of the *stabilized* variable ϕ. We show these properties with a non trivial example.

First, to discuss the stability we start with the case in mechanics of a single particle. Then, we analyze the analogous scalar field theory case.

Let us consider $\mathcal{L}'$ written as

$$\mathcal{L}'(\phi, a) = \mathcal{L} + a\Theta(\mathcal{L}^{(1)};\phi)\,, \quad (65)$$

$$\mathcal{L} = \mathcal{L}^{(2)} + \mathcal{L}^{(1)} \quad (66)$$

where the low derivative sector is the harmonic oscillator

$$\mathcal{L}^{(1)} = \frac{1}{2}\dot{\phi}^2 - \frac{m^2}{2}\phi^2\,, \quad (67)$$

and we choose the following higher derivative sector

$$\mathcal{L}^{(2)} = \frac{1}{2\Lambda^5}\ddot{\phi}^2\left(-\frac{1}{6}\ddot{\phi}^2 + m^4\phi^2 \right)\,, \quad (68)$$

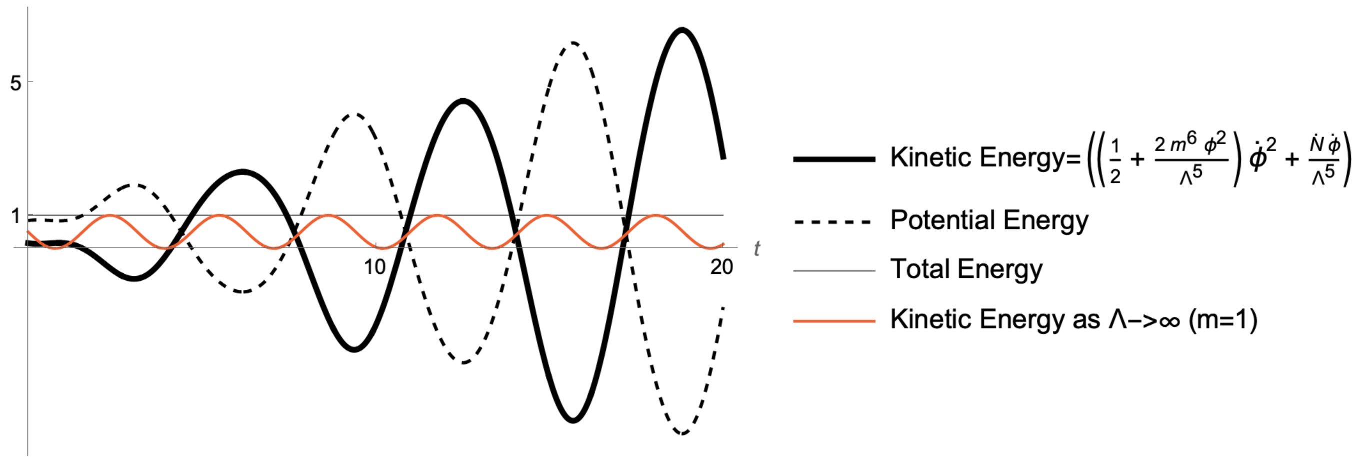

FIG. 1. Kinetic and potential energies (77), (78) for the motion of ϕ. Notice that even though ϕ *effectively* moves as an harmonic oscillator (73), the kinetic and potential energies (first two graphs, for finite Λ) are markedly different from a standard *harmonic oscillation*, as it is clear from the Λ^{-5} corrections. *It is not an isolated harmonic oscillation* of ϕ, but rather ϕ is constrained by the *interaction* with the Lagrange multiplier $a(t)$. The deviations from the usual kinetic and potential energies of an harmonic oscillation in equations (77), (78) grow as t^1 (in this example). The correction are suppresed by Λ, hence, they stem from the higher derivative sector $\mathcal{L}^{(2)}$. This difference from the usual harmonic motion highlights a forcing of the Lagrange multiplier in the system: namely, ϕ *is not an isolated degree of freedom.* (Graphs for $c_1 = c_2 = 1$, $m = 1/2$ with $\Lambda^5 = (2^{-9}\,11/3)$ such that $E_T = 1$). (Graph of case $\Lambda \to \infty$, which reduces to usual *isolated* harmonic motion, with $m = 1$ such that $E_T = 1$). Notice that only the two initial conditions c_1 and c_2 are required to specify the energy (76).

which satisfies the equations (22), (25) and (26)

$$\frac{\partial^2\mathcal{L}'}{\partial\ddot\phi\partial\ddot\phi} = \frac{\partial^2\mathcal{L}}{\partial\ddot\phi\partial\ddot\phi} = \frac{\left(\ddot\phi - m^2\phi\right)}{\Lambda^5}\Theta(\mathcal{L}^{(1)};\phi)\,, \qquad (69)$$

where,

$$\Theta(\mathcal{L}^{(1)};\phi) = -(\ddot\phi + m^2\phi)\,, \qquad (70)$$

and ϕ^2, a^2, Λ^{-1}, m^{-1} have units of time.

1. Stability of the Unconstrained theory $\mathcal{L}$ at linear order

The theory $\mathcal{L}$ *without* constraint (66) is free of ghosts at linear order: namely, with a perturbative expansion of the form (17 - 19), where the background 0th order solution (ϕ_0) is the harmonic oscillator $\mathcal{L}(\phi_0) \approx \mathcal{L}^{(1)}(\phi_0)$, the first line in $\mathcal{L}_\pi$ vanishes

$$\begin{aligned}\mathcal{L}_\pi = &\frac{1}{2}\ddot\pi^2\frac{\left(\ddot\phi_0 - m^2\phi_0\right)\left(\ddot\phi_0 + m^2\phi_0\right)}{\Lambda^5}\\ &+\frac{1}{2}\dot\pi^2\left(1+4\frac{m^6}{\Lambda^5}\phi_0^2\right)\\ &-\frac{1}{2}m^2\pi^2\left(1-\frac{5m^6\phi_0^2+4m^4\dot\phi_0^2}{\Lambda^5}\right)\end{aligned} \qquad (71)$$

and hence, the theory is free of ghosts for the first order perturbation π *without imposing additional constraints.* Let us stress that this follows by the critical definition of the higher derivative sector (22), designed in this case to be degenerate *only on the harmonic oscillator* background, namely by equation (69), and in first place, by the physical motivation on performing a perturbative expansion about solutions to the low derivative theory (17 - 19).

2. Stability of the Constrained theory $\mathcal{L}'$

In the case of $\mathcal{L}'$ (65), using the Euler-Lagrange equation for $a(t)$, the expression (70). Hence equation (69) also vanishes on-shell and the theory is degenerate. As a consequence, there are no ghosty contributions in the energy $E[\mathcal{L}']$ (9). Indeed, with the stabilizer Lagrange multiplier $a(t)$ specified on-shell as satisfying

$$\begin{aligned}&\Theta(\mathcal{L}';\phi) = 0\\ &\ddot a + m^2 a = \frac{m^5}{\Lambda^5}\left(3m^3\phi^3 - 4m\phi\dot\phi^2\right)\,.\end{aligned} \qquad (72)$$

and with (70) vanishing, let us consider explicit solutions

$$\phi(t) = c_1\cos(mt) + c_2\sin(mt) \qquad (73)$$

$$a(t) = c_3\,\cos(mt) + c_4\,\sin(mt) + \frac{1}{\Lambda^5}N(t)\,, \qquad (74)$$

where

$$\begin{aligned}\frac{1}{\Lambda^5}N(t) = &-\frac{1}{m}\cos(mt)\int^t dt'\,\sin(mt')\,\Theta(\mathcal{L}^{(2)};\phi)\\ &+\frac{1}{m}\sin(mt)\int^t dt'\,\cos(mt')\,\Theta(\mathcal{L}^{(2)};\phi)\,,\end{aligned} \qquad (75)$$

and $\Theta(\mathcal{L}^{(2)};\phi)$ is given by the right hand side of (72), such that $N(t)$ depends *only* on the initial conditions for ϕ (c_1

and c_2). Notice that $N(t)$ carries the higher derivative effects from $\mathcal{L}^{(2)}$.

With these solutions the total energy can be written *up to a physically irrelevant energy shift* as a kinetic part which vanishes when $\dot{\phi} = 0$ and $\dot{N}(t) = 0$, and the remaining potential

$$E_T = E[\mathcal{L}'] - m^2\,(c_1\,c_3 + c_2\,c_4) = E_{Kin} + E_{Pot} \tag{76}$$

$$E_{Kin} = \left(\frac{1}{2} + 2\frac{m^6}{\Lambda^5}\phi^2\right)\dot{\phi}^2 + \frac{1}{\Lambda^5}\dot{\phi}\dot{N} \tag{77}$$

$$E_{Pot} = \left(\frac{1}{2}m^2 + \frac{m^8}{4\Lambda^5}\phi^2\right)\phi^2 + \frac{m^2}{\Lambda^5}\phi N(t) \tag{78}$$

All in all, there are no linear terms $\dddot{\phi}$ in the total energy E_T that would signal a ghost (See section II B). Furthermore, the initial conditions that could be chosen for $a(t)$ (c_3 and c_4) are confined to *appear only* in the constant shift to the energy on the left hand side in (76)

$$K = m^2\,(c_1\,c_3 + c_2\,c_4)\ , \tag{79}$$

which by (76) does not affect in any way the dynamics of ϕ and hence, cannot destabilize it.

Indeed, this is an example of the general case proved in section IV, where we showed that for all theories $\mathcal{L}'$ the total energy E_T is fully fixed by the *only two* initial conditions chosen for ϕ, that E_T does not depend on any initial conditions that could be chosen for $a(t)$ and that it can always be written in the form (61), which for this example reads as

$$\begin{aligned} E_T = E[\mathcal{L}']\ -K = & \qquad\qquad (80)\\ & E[\mathcal{L}^{(1)}]+E[\mathcal{L}^{(2)}] + \frac{m^6}{\Lambda^5}\int^t dt'\dot{\phi}\left(3m^2\phi^3 - 4\phi\dot{\phi}^2\right)\ , \end{aligned}$$

where one can explicitly verify that the terms $\sim N$, $\sim \dot{N}$ in (76) coincide with the antiderivative in equation (80).

Furthermore, because the two initial conditions for $a(t)$ only affect with an unphysical shift the total energy of the ϕ degree of freedom, or in other words, because they *lack physical significance in the dynamics of* ϕ, for given initial conditions of c_1, c_2, we can choose an unphysical for ϕ energy shift corresponding to $c_3 = c_1$ and $c_4 = c_2$, such that we can better interpret the solution (74) for the auxiliary variable as

$$a(t) = \phi(t) + \frac{1}{\Lambda^5}N(t)\,. \tag{81}$$

In other words, in regards to the dynamics of ϕ, the *stabilizer variable* $a(t)$ can be viewed as a Λ-suppressed correction $N(t)$ superposed to the low derivative mode ϕ. This is reminiscent of the Ansatz $\phi = \phi_0 + \epsilon\pi$ that was taken for the perturbative expansion (17 - 19). Here we identify the solution to all orders

$$\epsilon\pi = \Lambda^{-5}N\,. \tag{82}$$

All in all, $a(t)$ carries the higher derivative effects as a Λ-suppressed correction to the low derivative mode ϕ. In general the equation for $a(t)$ (72) has a physical meaning and may have interesting applications, as we discuss below. Furthermore, although ϕ moves as an harmonic oscillator, there is a signature of the higher derivative sector and its interaction with $a(t)$ in the non trivial exchange of potential and kinetic energy, as is shown explicitly in Figure 1, and as it is clear from the Λ corrections in the energy of the system (77), (78).

3. *Field theory: the speed of the stabilizer variable* $a(x)$

In the field theory case the stabilizer Lagrange multiplier $a(x)$ is specified on-shell by a wave equation. The latter inherits important features from the *low* derivative mode ϕ. Specifically, the effective metric coincides for the wave equations for ϕ and $a(x)$.

In other words, the higher derivative terms in this setup *never* modify the speed of propagation of the low derivative theory. Hence, the higher derivative effects can be effectively seen *on-shell* as a wave equation that despite low derivative self-interactions can preserve sub/luminal speed for the stabilizer variable $a(x)$.

This is radically different from standard self interactions that are low in derivatives both on- and off-shell, which can carry issues like the modification of the speed of propagation and of the cone of influence.

Indeed, let us consider $\mathcal{L}'$ written as (65) where the low derivative sector is the most general self-interacting real scalar field $\phi(t,\vec{x})$ with Lorentz invariant Lagrangian $\mathcal{L}^{(1)}$ that depends only on powers of $\partial_\mu\phi\partial^\mu\phi$ and ϕ. The equation of motion for ϕ is in general

$$\begin{aligned} \Theta(\mathcal{L}';a) &= \Theta(\mathcal{L}^{(1)};\phi) \qquad\qquad (83)\\ &= -G^{\mu\nu}\partial_\mu\partial_\nu\phi - 2\mathcal{X}\frac{\partial^2\mathcal{L}^{(1)}}{\partial\phi\partial\mathcal{X}} + \frac{\partial\mathcal{L}^{(1)}}{\partial\phi} = 0\,, \end{aligned}$$

where $G^{\mu\nu}$ depends on ϕ and its first derivatives, $2\mathcal{X} = \partial_\mu\phi\partial^\mu\phi$ and $g^{\mu\nu}$ is flat space-time metric.

$$G^{\mu\nu} = \frac{\partial\mathcal{L}^{(1)}}{\partial\mathcal{X}}g^{\mu\nu} + \frac{\partial^2\mathcal{L}^{(1)}}{\partial\mathcal{X}^2}\partial^\mu\phi\partial^\nu\phi\,. \tag{84}$$

$G^{\mu\nu}$ defines the characteristic curves of the field equation and the propagating character of solutions to (83). Namely, whether the equation is hyperbolic, parabolic or elliptic. In the case it is hyperbolic, there are indeed propagating solutions. In other words, the characteristic curves are real and they serve to identify the wavefront and its velocity [26]. In short, $G^{\mu\nu}$ fixes the speed of sound for the wave equation (83) and the acoustic cone of influence [25–27]. Hence, it usually receives the name of effective, or emergent metric.

On the other hand, the equation that specifies the stabilizer Lagrange multiplier $a(x)$ *on-shell* is derived from $\Theta(\mathcal{L}';\phi) = 0$. Denoting with ϕ the solutions to the low derivative sector (83), the equation for $a(x)$ takes the form,

$$\Theta\left(a\,\Theta(\mathcal{L}^{(1)};\phi);\phi\right) = -\Theta(\mathcal{L}^{(2)};\phi)\,. \tag{85}$$

The left hand side of (85) is

$$\left(\partial_\mu\partial_\nu \frac{\partial}{\partial\left(\partial_\mu\partial_\nu\phi\right)} - \partial_\mu \frac{\partial}{\partial\left(\partial_\mu\phi\right)} + \frac{\partial}{\partial\phi}\right) a\,\Theta(\mathcal{L}^{(1)};\phi)\,. \tag{86}$$

Because $\Theta(\mathcal{L}^{(1)};\phi)$ (83) depends linearly on $\partial_\mu\partial_\nu\phi$, (86) takes the form of a second order differential operator acting on $a(t,\vec{x})$

$$\left(G^{\mu\nu}\partial_\mu\partial_\nu + v^\mu\partial_\mu + M^2\right)a\,, \tag{87}$$

where we encounter the same effective metric $G^{\mu\nu}(\phi)$ as for the propagation of the low derivative sector ϕ (83). Thus we can identify the same characteristic curves for the wave equation that specifies the stabilizer variable as for the respective low derivative theory (83). In other words, the higher derivative effects in this setup do not modify the speed of propagation, neither the (acoustic) cone of influence of the low derivative theory, because necessarily, the principal part of the differential operator $(G^{\mu\nu}\partial_\mu\partial_\nu)$ is kept invariant by the built-in constraint (83), given by the setup (25), (26).

All in all, if the effective metric $G^{\mu\nu}$ implied by the low derivative sector $\mathcal{L}^{(1)}$ of the corresponding higher derivative theory $\mathcal{L}'$ satisfies the hyperbolicity, stability and subluminality conditions that were recognized long ago by Aharonov, Komar and Susskind [24] (Appendix A),

$$\tfrac{\partial\mathcal{L}^{(1)}}{\partial\mathcal{X}} > 0\,,\;\; \tfrac{\partial^2\mathcal{L}^{(1)}}{\partial\mathcal{X}^2} \geq 0\,,\;\; \tfrac{\partial\mathcal{L}^{(1)}}{\partial\mathcal{X}} + 2\mathcal{X}\tfrac{\partial^2\mathcal{L}^{(1)}}{\partial\mathcal{X}^2} > 0\,,$$

then, the scalar stabilizer variable $a(x)$ inherits these properties. The higher derivative sector $\mathcal{L}^{(2)}$ in the constrained setup (25), (26) is limited to force $a(x)$ as in the right hand side of (85), but not to define the propagating character of solutions.

On the other hand, considering for definiteness the field theory in four dimensions, the mass of $a(x)$ is,

$$M^2(\phi) = -\Theta\left(\Theta(\mathcal{L}^{(1)};\phi);\phi\right)\,, \tag{88}$$

and the damping term is,

$$v^\mu(\phi) = \left(2\partial_\nu G^{\mu\nu} + \frac{\partial\Theta(\mathcal{L}^{(1)};\phi)}{\partial\left(\partial_\mu\phi\right)}\right)\,, \tag{89}$$

such that v^μ vanishes if the low derivative sector $\mathcal{L}^{(1)}$ contains no derivative self-interactions.

Let us consider the analogous example to (67) and (68) in field theory. Taking the massive real scalar field ϕ as the low derivative sector $\mathcal{L}^{(1)}$ and

$$\mathcal{L}^{(2)} = \frac{1}{2\Lambda^8}(\Box\phi)^2\left(-\frac{(\Box\phi)^2}{6} + m^4\phi^2\right) \tag{90}$$

which satisfies (27), where ϕ^{-1}, a^{-1}, Λ^{-1} and m^{-1} have units of length, and with $\mathcal{L}'$ in the standard form (65), the equation that specifies the stabilizer variable $a(x)$ is,

$$\Box a + m^2 a = \frac{m^5}{\Lambda^8}(3m^3\phi^3 - 4m\,\phi\,\partial_\mu\phi\partial^\mu\phi)\,, \tag{91}$$

where ϕ are solutions to the Klein-Gordon equation. Let us stress that as expected both equations for ϕ and $a(x)$ have the $\Box$ operator. In other words, the metric is flat for the $a(x)$ field equation, and the speed of light is not endangered by the derivative self-interactions of ϕ, whose only effect is to force $a(x)$ as on the right hand side of (91).

This contrasts to the effects in typical (unconstrained) low derivative self-interactions that can be obtained for a real scalar with Lagrangian $\mathcal{L}^{(1)}$, whose non-perturbative effects can have disastrous consequences such as superluminality [24–30].

All in all, in general, in this setup, it is possible to include ghost-free derivative self-interactions without modifying the cone of influence, nor the speed of propagation of the low derivative theory. In particular, in the case that $\mathcal{L}^{(1)}$ contains no derivative self-interactions, the right hand side of equation (85) still contains derivative interactions of the low derivative mode ϕ forcing the stabilizer variable $a(x)$, which are induced by the high derivative sector $\mathcal{L}^{(2)}$, and however, do not enclose contributions to the metric that could potentially spoil causality, or generate other undesirable effects (See related discussions in [24–30]). In such a case, $a(x)$ is *strictly luminal*.

VI. CONCLUSIONS

We showed a new class of higher derivative terms that can be added to a given low derivative theory, such that the first order perturbations induced by the higher derivative deformation are ghost free on the low derivative background. This follows *without* the need to introduce additional constraints.

We stressed on the phenomenological motivation to consider such a perturbative expansion on top of solutions to the standard low derivative theory: namely, it agrees with the expectation of an energy regime where the background -low derivative sector- is more relevant for the dynamics than the ghost-free fluctuations caused by the higher derivative terms. In brief, to account for the phenomenological success of low derivative theories at least at low energies.

On a second step we explored related theories that are degenerate and ghost free without restricting to a perturbative expansion, but at the expense of introducing a constraint and a stabilizer variable $a(t)$. In this case, some physical consequences of the higher derivative sector are, as expected, explicitly contained in this *stabilizer* Lagrange multiplier $a(t)$, and also on the energy for the motion of the *stabilized variable* ϕ.

We showed examples where *the modifications to solutions and to the energy* due to the higher derivative effects are suppressed by the new-physics energy scale. We explicitly verified in all examples that no signatures of the Ostrogradskian instability arise with this general setup.

In field theory we specialized to the equation for the *stabilizer variable* $a(x)$. We stressed that with this setup it is possible to include ghost-free *derivative self-interactions* and still preserve a luminal propagation for $a(x)$.

This is radically different from standard low-derivative self interactions, which commonly carry issues such as dangerous modifications to the speed of propagation and have found applications in dark energy and inflation.

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