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High Relative Accuracy With Collocation Matrices of q-Jacobi Polynomials

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ABSTRACT

Little q-Jacobi polynomials belong to the field of quantum calculus. This article obtains the bidiagonal decomposition of the collocation matrices of these polynomials, showing that, in many cases, it can be constructed to high relative accuracy (HRA). Then, it can be used to compute with HRA the inverses, eigenvalues, and singular values of these matrices. Numerical experiments are provided and illustrate the excellent results obtained when applying the presented methods.

1 | Introduction

Through the use of q-integers, q-factorials, q-binomial coefficients, and other q-analogues of classical calculus, quantum calculus (see [1, 2].), also called, q-calculus, has extended its applications to many fields. Although it dates back to Leonhard Euler and Carl Gustav Jacobi, it has only recently begun to find usefulness in quantum mechanics. Other fields of applications include quantum groups and algebras, combinatorics, approximation theory, probability and statistics, orthogonal polynomials, and matrices of q-integers. Our work deals with these last two fields, showing an important class of matrices of q-integers for which many algebraic computations, such as the computation of eigenvalues, singular values or inverses, can be carried out with high relative accuracy (HRA). The new methods presented in this work can also be applied to the solution of linear systems, which arise in some important problems such as interpolation or quadrature rules.

Jacobi polynomials form a classical family of orthogonal polynomials, containing important subfamilies of orthogonal polynomials such as Chebyshev and Legendre polynomials. A q-analogue is given by the little q-Jacobi polynomials, considered in this paper and which contain other important families as the little q-Legendre polynomials, also considered here. They have contributed to the rapidly growing field of basic hypergeometric series, or q-series (cf. [3].). Given a system of functions $(u_0, \dots, u_n)$, its *collocation matrix* at parameters $t_1 < \cdots < t_{n+1}$ is given by $(u_{j-1}(t_i))_{1\le i,j\le n+1}$. In this paper, we guarantee the accurate computation for the collocation matrices of some systems of functions. In particular, for collocation matrices of some little q-Jacobi polynomials. Previously, in [4]., this goal was achieved for collocation matrices of q-Bernstein polynomials, in [5]. for collocation matrices of h-Bernstein basis, in [6]. for collocation matrices of Jacobi polynomials, in [7]. for collocation matrices of Laguerre polynomials and in [8]. for collocation matrices of q-Laguerre polynomials. Let us observe that the previous families

Abbreviations: HRA, high relative accuracy; NE, Neville elimination; STP, strictly totally positive; TP, totally positive.

of polynomials do not include the families of polynomials studied in this paper.

A crucial help to assure the accurate computations has been to prove the total positivity of some matrices related to the collocation matrices. A matrix is called *totally positive* (*strictly totally positive*, respectively) if all its minors are nonnegative (positive, respectively), and we denote it by TP (STP, respectively). TP and STP matrices are also called totally nonnegative and totally positive, respectively. They have applications in many fields such as approximation theory, differential equations, statistics, finance, mechanics, biology, computer-aided geometric design, or combinatorics (see [9–11].). A remarkable property of nonsingular TP matrices is that they admit a bidiagonal decomposition. It is used as the initial parametrization for the algorithms of [12]. to perform the linear algebra operations mentioned previously with HRA. However, the accurate bidiagonal decomposition is a necessary requirement to obtain results to HRA with these algorithms.

The paper is organized as follows. Section 2 presents basic facts about the bidiagonal decomposition of nonsingular TP matrices and about high relative accuracy. Section 3 constructs the bidiagonal decompositions of collocation matrices of little q-Jacobi polynomials. The total positivity and strict total positivity of some related matrices are also proved. Computations with high relative accuracy are guaranteed in some cases, including collocation matrices of little q-Legendre polynomials. Section 4 illustrates the theoretical results of the paper with numerical experiments, showing the computation to HRA of inverses, eigenvalues, singular values, or solutions of some linear systems, in spite of the fact that these matrices are extremely ill-conditioned.

2 | Auxiliary Results

Let $D = (d_{ij})_{1\le i,j\le n}$ be a diagonal matrix, which can be denoted by $D = \mathrm{diag}(d_1, \dots, d_n)$, where $d_i := d_{ii}$ for $i = 1, \dots, n$. Using this notation, the $n \times n$ identity matrix is defined as $I_n = \mathrm{diag}(1, \dots, 1)$.

Neville elimination (NE) is an alternative procedure to Gaussian elimination that produces zeros in a column of a matrix by adding to each row an appropriate multiple of the previous one. Given a nonsingular matrix $A = (a_{ij})_{1\le i,j\le n}$, the NE procedure consists of $n-1$ steps and leads to the following sequence of matrices:

$$A =: A^{(1)} \to \widetilde{A}^{(1)} \to A^{(2)} \to \widetilde{A}^{(2)} \to \cdots \to A^{(n)} = \widetilde{A}^{(n)} = U \quad (1)$$

where U is an upper triangular matrix.

The matrix $\widetilde{A}^{(k)} = (\widetilde{a}_{ij}^{(k)})_{1\le i,j\le n}$ is obtained from the matrix $A^{(k)} = (a_{ij}^{(k)})_{1\le i,j\le n}$ by a row permutation that moves to the bottom the rows with a zero entry in column k below the main diagonal. For nonsingular TP matrices, it is always possible to perform NE without row exchanges (see [13].). If a row permutation is not necessary at the k-th step, we have that $\widetilde{A}^{(k)} = A^{(k)}$. The entries of $A^{(k+1)} = (a_{ij}^{(k+1)})_{1\le i,j\le n}$ can be obtained from $\widetilde{A}^{(k)} = (\widetilde{a}_{ij}^{(k)})_{1\le i,j\le n}$ using the formula:

$$a_{ij}^{(k+1)} = \begin{cases} \widetilde{a}_{ij}^{(k)} - \dfrac{\widetilde{a}_{ik}^{(k)}}{\widetilde{a}_{i-1,k}^{(k)}} \widetilde{a}_{i-1,j}^{(k)}, & \text{if } k \le j < i \le n \text{ and } \widetilde{a}_{i-1,k}^{(k)} \ne 0 \\ \widetilde{a}_{ij}^{(k)}, & \text{otherwise} \end{cases} \quad (2)$$

for $k = 1, \dots, n-1$. The (i,j) *pivot* of the NE of A is given by

$$p_{ij} = \widetilde{a}_{ij}^{(j)}, \quad 1 \le j \le i \le n$$

If $i = j$ we say that p_{ii} is a *diagonal pivot*. The (i,j) *multiplier* of the NE of A, with $1 \le j \le i \le n$, is defined as

$$m_{ij} = \begin{cases} \dfrac{\widetilde{a}_{ij}^{(j)}}{\widetilde{a}_{i-1,j}^{(j)}} = \dfrac{p_{ij}}{p_{i-1,j}}, & \text{if } \widetilde{a}_{i-1,j}^{(j)} \ne 0 \\ 0, & \text{if } \widetilde{a}_{i-1,j}^{(j)} = 0 \end{cases}$$

The multipliers satisfy that

$$m_{ij} = 0 \Rightarrow m_{hj} = 0 \quad \forall h > i$$

NE is a very useful method to study TP matrices. In fact, NE can be used to characterize nonsingular TP matrices. In [13]., the following characterization of nonsingular TP matrices was provided in terms of NE.

Theorem 1. (Theorem 5.4 of [13].). *Let A be a nonsingular matrix. Then, A is TP if and only if there are no row exchanges in the NE of A and U^T and the pivots of both NE are nonnegative.*

Nonsingular TP matrices can be expressed as a product of nonnegative bidiagonal matrices. The following theorem (see Theorem 4.2 and p. 120 of [14].) introduces this representation, which is called the *bidiagonal decomposition*.

Theorem 2. (cf. Theorem 4.2 of [14].). *Let $A = (a_{ij})_{1\le i,j\le n}$ be a nonsingular matrix. Then, A is TP if and only if it admits the following representation*:

$$A = F_{n-1}F_{n-2}\cdots F_1 D G_1 \cdots G_{n-2}G_{n-1} \quad (3)$$

where D is the diagonal matrix $\mathrm{diag}(p_{11}, \dots, p_{nn})$ *with positive diagonal entries and F_i, G_i are the nonnegative bidiagonal matrices given by*

$$F_i = \begin{pmatrix} 1 & & & & & & \\ 0 & 1 & & & & & \\ & \ddots & \ddots & & & & \\ & & 0 & 1 & & & \\ & & & m_{i+1,1} & 1 & & \\ & & & & \ddots & \ddots & \\ & & & & & m_{n,n-i} & 1 \end{pmatrix}, \quad G_i = \begin{pmatrix} 1 & 0 & & & & & \\ & 1 & \ddots & & & & \\ & & \ddots & 0 & & & \\ & & & 1 & \widetilde{m}_{i+1,1} & & \\ & & & & 1 & \ddots & \\ & & & & & \ddots & \widetilde{m}_{n,n-i} \\ & & & & & & 1 \end{pmatrix} \quad (4)$$

for all $i \in \{1, \dots, n-1\}$. If, in addition, the entries m_{ij} and $\widetilde{m}_{ij}$ satisfy

$$\begin{aligned} m_{ij} = 0 &\Rightarrow m_{hj} = 0 \quad \forall h > i \\ \widetilde{m}_{ij} = 0 &\Rightarrow \widetilde{m}_{hj} = 0 \quad \forall h > i \end{aligned} \quad (5)$$

then, the decomposition is unique.

In the bidiagonal decomposition given by (3) and (4), the entries m_{ij} and p_{ii} are the multipliers and diagonal pivots, respectively, corresponding to the NE of A (see Theorem 4.2 of [14]. and the comment below it) and the entries $\widetilde{m}_{ij}$ are the multipliers of the NE of A^T (see p. 116 of [14].). The following result shows that the bidiagonal decomposition also characterizes STP matrices.

Theorem 3. (cf. Theorem 4.3 of [14].). *A nonsingular $n \times n$ matrix A is STP if and only if it can be factorized in the form (3) with D a diagonal matrix with positive diagonal entries, F_i, G_i given by (4), and the entries m_{ij} and $\widetilde{m}_{ij}$ positive numbers. This factorization is unique.*

The bidiagonal decomposition can be used to represent more classes of matrices. The following remark shows which hypotheses of Theorem 2 are sufficient for the uniqueness of a factorization following (3).

Remark 1. If we consider the factorization given by (3–5) without any further requirement than the nonsingularity of D, by Proposition 2.2 of [15]., the uniqueness of (3) holds.

In [16]., the matrix notation $\mathcal{BD}(A)$ was introduced to represent the bidiagonal decomposition of a nonsingular TP matrix,

$$(\mathcal{BD}(A))_{ij} = \begin{cases} m_{ij}, & \text{if} \quad i > j \\ \widetilde{m}_{ji}, & \text{if} \quad i < j \\ p_{ii}, & \text{if} \quad i = j \end{cases} \tag{6}$$

Throughout this paper, $\mathcal{BD}(A)$ will denote the bidiagonal decomposition of a matrix that satisfies the hypotheses of Remark 1. The following remark gives the relationship between the bidiagonal decompositions of a matrix and of its transpose.

Remark 2. If A is a TP matrix, then A^T is also TP. Transposing formula (3) of Theorem 2 we obtain the unique bidiagonal decomposition of A^T:

$$A^T = G_{n-1}^T \cdots G_1^T D F_1^T \cdots F_{n-1}^T$$

where F_i and G_i, $i \in \{1, \dots, n-1\}$, are the bidiagonal lower and upper triangular nonnegative matrices given in (4). It can also be checked that

$$\mathcal{BD}(A^T) = \mathcal{BD}(A)^T \tag{7}$$

An algorithm can be performed with high relative accuracy (HRA) if it does not include subtractions (except of the initial data), that is, if it only includes products, divisions, sums of numbers of the same sign, subtractions of numbers of opposite sign and subtractions of the initial data (cf. [16, 17].). In particular, a subtraction-free algorithm provides results with HRA. In [16]., assuming that the parameters of $\mathcal{BD}(A)$ are known with HRA, Koev presented algorithms for computing to HRA the eigenvalues of the matrix A, the singular values of the matrix A, the inverse of the matrix A and the solution of linear systems of equations $Ax = b$ where b has a pattern of alternating signs.

3 | Little q-Jacobi Polynomials

Let us start by recalling some standard notation about q-calculus. The q-shifted factorial is defined as

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{k=1}^{n} (1 - aq^{k-1}); \quad n \in N, \quad a \in R \quad q \in (0,1) \tag{8}$$

In this section, we consider the little q-Jacobi polynomials $J_{n,q}(x; a, b)$ and collocation matrices of families of these polynomials. The explicit expression of the little q-Jacobi polynomial is given by

$$J_{n,q}(x;a,b) := \sum_{k=0}^{n} \frac{(q^{-n};q)_k (abq^{n+1};q)_k}{(aq;q)_k} \frac{(qx)^k}{(q;q)_k} \tag{9}$$

This definition comes from a particular case of basic hypergeometric series. For further details, see p. 181 of [3]. Moreover, in Section 7.3 of [3]., there is a discussion of the little q-Jacobi polynomials where their orthogonality is shown under the assumptions that $0 < q, aq < 1$. Our case of study will be given by $0 < q < 1$, $|a|, |b| \leq 1$.

Let $M := (J_{j-1,q}(t_{i-1}))_{1 \leq i,j \leq n+1}$ be the collocation matrix of the little q-Jacobi polynomials at the nodes $(0 >) t_0 > t_1 > \dots > t_n$ and let $R_q(a)$, be the following $(n+1) \times (n+1)$ diagonal matrix:

$$R_q(a) = \operatorname{diag}\left(\frac{q^{j-1}}{(aq;q)_{j-1} (q;q)_{j-1}} \right)_{1 \leq j \leq n+1} \tag{10}$$

Let us now consider the matrix $A_q(a,b) := (a_{ij})_{1 \leq i,j \leq n+1}$ whose entries are the coefficients appearing in (9), that is,

$$a_{i+1,j+1} = \begin{cases} \frac{(q^{-i};q)_j (abq^{i+1};q)_j}{(aq;q)_j} \frac{q^j}{(q;q)_j} & \text{if } i \geq j \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

Hence, the matrix $A_q(a,b)$ defines the change of basis between the monomial basis and the little q-Jacobi polynomials,

$$\begin{aligned} &\left(J_{0,q}(t;a,b), J_{1,q}(t;a,b), \dots, J_{n,q}(t;a,b) \right) \\ &\quad = (1, t, \dots, t^n) A_q(a,b)^T \end{aligned} \tag{12}$$

In the proof of the following proposition, it will be used that

$$\begin{aligned} &(1 - q^{-i})(1 - abq^{i+j}) - (1 - q^{-i+j})(1 - abq^i) \\ &\quad = (1 - q^j)\left(abq^i - \frac{1}{q^i} \right) \end{aligned} \tag{13}$$

Proposition 1. *Let $A_q(a,b) := (a_{ij})_{1 \leq i,j \leq n+1}$ be the triangular matrix defined by (11). Then, we have that $A_q(a,b)$ has the following bidiagonal decomposition,*

$$\mathcal{BD}(A_q(a,b))_{i+1,j+1} = \begin{cases} \frac{\prod_{r=0}^{j-1} \left(abq^i - \frac{1}{q^{i-r}} \right)}{\prod_{r=0}^{j-1} \left(abq^{i-1} - \frac{1}{q^{i-r-1}} \right)}, & \text{if } i > j \\ \frac{q^j}{(aq;q)_j} \prod_{r=0}^{j-1} \left(abq^j - \frac{1}{q^{j-r}} \right), & \text{if } i = j \\ 0, & \text{if } i < j \end{cases} \tag{14}$$

Proof. Let $A_q(a,b) := (a_{ij})_{1\le i,j\le n+1}$ be the matrix given by (11) and let $R_q(a)$ be the diagonal matrix defined by (10). Let us consider a factorization of $A_q(a,b) = B_q(a,b)R_q(a)$, where the entries of $B_q(a,b) := (b_{ij})_{1\le i,j\le n+1}$ are

$$b_{i+1,j+1} = \begin{cases} (q^{-i};q)_j(abq^{i+1};q)_j, & \text{if } i\ge j \\ 0, & \text{otherwise} \end{cases} \tag{15}$$

In order to deduce the bidiagonal decomposition of $A_q(a,b)$, we will start computing the bidiagonal decomposition of $B_q(a,b)$. Let $B^{(1)} := B_q(a,b)$ and $B^{(t)} = (b_{ij}^{(t)})_{1\le i,j\le n+1}$ be the matrix obtained after performing $t-1$ steps of the NE of $B_q(a,b)$ for $t=2,\dots,n+1$. Let us first show that

$$b_{i+1,j+1}^{(t)} = (q^{-i+t-1};q)_{j-t+1}(abq^{i+1};q)_{j-t+1}\prod_{r=0}^{t-2}(1-q^{j-r}) \prod_{r=0}^{t-2}\left(abq^i - \frac{1}{q^{i-r}}\right), \quad i\ge j\ge t-1 \tag{16}$$

by induction over t. Let us notice that $B_q(a,b)$ is a lower triangular matrix. Hence, when we apply NE to $B_q(a,b)$, only the entries below the main diagonal are changed. We will start by performing the first step of the NE of $B_q(a,b)$ and deduce the expression for the entries of $B_q^{(2)}(a,b)$. We have that

$$b_{i+1,j+1}^{(2)} = b_{i+1,j+1} - b_{i,j+1}\frac{b_{i+1,1}}{b_{i,1}} = b_{i+1,j+1} - b_{i,j+1}$$

Hence, using (13) and (15), we see that

$$\begin{aligned} b_{i+1,j+1}^{(2)} &= (q^{-i};q)_j(abq^{i+1};q)_j - (q^{-i+1};q)_j(abq^i;q)_j \\ &= (q^{-i+1};q)_{j-1}(abq^{i+1};q)_{j-1} \\ &\quad \left[(1-q^{-i})(1-abq^{i+j}) - (1-q^{-i+j})(1-abq^i)\right] \\ &= (q^{-i+1};q)_{j-1}(abq^{i+1};q)_{j-1}(1-q^j)\left(abq^i - \frac{1}{q^i}\right) \end{aligned}$$

and we have that $b_{i+1,j+1}^{(2)}$ verifies (16). Let us now assume that $b_{i+1,j+1}^{(t)}$ is given by (16) and let us prove that the formula is true for $b_{i+1,j+1}^{(t+1)}$. For that, we will perform another step of the NE of $B_q(a,b)$:

$$\begin{aligned} b_{i+1,j+1}^{(t+1)} = b_{i+1,j+1}^{(t)} - \frac{b_{i+1,t}^{(t)}}{b_{i,t}^{(t)}} b_{i,j+1}^{(t)} = b_{i+1,j+1}^{(t)} \\ - \frac{\prod_{r=0}^{t-2}\left(abq^i - \frac{1}{q^{i-r}}\right)}{\prod_{r=0}^{t-2}\left(abq^{i-1} - \frac{1}{q^{i-r-1}}\right)} b_{i,j+1}^{(t)} \end{aligned}$$

and so, applying (13) and (15) we see that

$$\begin{aligned} b_{i+1,j+1}^{(t+1)} &= (q^{-i+t-1};q)_{j-t+1}(abq^{i+1};q)_{j-t+1}\prod_{r=0}^{t-2}(1-q^{j-r}) \\ &\quad \prod_{r=0}^{t-2}\left(abq^i - \frac{1}{q^{i-r}}\right) \\ &\quad - (q^{-i+t};q)_{j-t+1}(abq^i;q)_{j-t+1}\prod_{r=0}^{t-2}(1-q^{j-r}) \\ &\quad \prod_{r=0}^{t-2}\left(abq^i - \frac{1}{q^{i-r}}\right) \\ &= (q^{-i+t};q)_{j-t}(abq^{i+1};q)_{j-t}\prod_{r=0}^{t-2}(1-q^{j-r}) \\ &\quad \prod_{r=0}^{t-2}\left(abq^i - \frac{1}{q^{i-r}}\right) \\ &\quad \left[(1-q^{-i+t-1})(1-abq^{i+j-t+1}) - (1-q^{-i+j})(1-abq^i)\right] \\ &= (q^{-i+t};q)_{j-t}(abq^{i+1};q)_{j-t}\prod_{r=0}^{t-1}(1-q^{j-r}) \\ &\quad \prod_{r=0}^{t-1}\left(abq^i - \frac{1}{q^{i-r}}\right). \end{aligned}$$

Let us recall that the multipliers and diagonal pivots of the NE of $B_q(a,b)$ define its bidiagonal decomposition (cf. Theorem 2 and the discussion below it). Therefore, we have that $\mathcal{BD}(B_q(a,b))$ is given by

$$\mathcal{BD}(B_q(a,b))_{i+1,j+1} = \begin{cases} \dfrac{\prod_{r=0}^{j-1}\left(abq^i - \frac{1}{q^{i-r}}\right)}{\prod_{r=0}^{j-1}\left(abq^{i-1} - \frac{1}{q^{i-r-1}}\right)}, & \text{if } i>j \\ (q;q)_j\prod_{r=0}^{j-1}\left(abq^j - \frac{1}{q^{j-r}}\right), & \text{if } i=j \\ 0, & \text{if } i<j \end{cases}$$

Finally, let us deduce $\mathcal{BD}(A_q(a,b))$. Since we have that $A_q(a,b) = B_q(a,b)R_q(a)$, we can write $A_q(a,b)$ as the following product of matrices

$$A_q(a,b) = \overline{F}_n\overline{F}_{n-1}\cdots\overline{F}_1\overline{D}R_q(a)$$

where $\overline{F}_i$ and D are the lower bidiagonal matrices and the diagonal matrix that give the bidiagonal decomposition of $B_q(a,b)$ (see (3)). By the uniqueness of the bidiagonal decomposition, we can deduce that the bidiagonal decomposition of $A_q(a,b)$ is given by the same lower bidiagonal matrices $\overline{F}_i$ and the diagonal matrix $\overline{D}\cdot R_q(a)$. And so, $\mathcal{BD}(A_q(a,b))$ is given by (14). □

In Proposition 1, we have obtained the bidiagonal decomposition of the matrix $A_q(a,b)$ that defines the change of basis between the little q-Jacobi polynomials and the monomial basis. This proposition will be used to deduce the bidiagonal decomposition and total positivity of some collocation matrices of little q-Jacobi polynomials. Depending on the parameters defining $A_q(a,b)$, we can also compute the bidiagonal decomposition of these matrices to HRA, and hence, solve many linear algebra problems with these matrices to HRA. Let us first study the relationship of the matrix of change of basis $A_q(a,b)$ with total positivity.

Proposition 2. *Given* $|a|,|b|\le 1$, $0<q<1$ *and* $J_{n+1} = \operatorname{diag}((-1)^{i+1})_{1\le i\le n+1}$, *the matrix* $A_q(a,b)J_{n+1}$ *is a nonsingular TP matrix.*

Proof. By Proposition 1, we know that $\mathcal{BD}(A_q(a,b))$ is given by (14). Since J_{n+1} is a diagonal matrix, the multipliers of the bidiagonal decomposition of $A_q(a,b)J_{n+1}$ coincide with the ones of $\mathcal{BD}(A_q(a,b))$, and only the diagonal pivots are changed by the product. Hence, the bidiagonal decomposition of $A_q(a,b)J_{n+1}$ is given by:

$$\mathcal{BD}(A_q(a,b)J_{n+1})_{i+1,j+1} = \begin{cases} \dfrac{\prod_{r=0}^{j-1}\left(abq^i - \frac{1}{q^{i-r}}\right)}{\prod_{r=0}^{j-1}\left(abq^{i-1} - \frac{1}{q^{i-r-1}}\right)}, & \text{if } i > j \\ \frac{q^j}{(aq;q)_j}\prod_{r=0}^{j-1}\left(\frac{1}{q^{j-r}} - abq^j\right), & \text{if } i = j \\ 0, & \text{if } i < j \end{cases} \tag{17}$$

Now, all the diagonal pivots are positive and the multipliers are nonnegative. Hence, by Theorem 2 $A_q(a,b)J_{n+1}$ is a nonsingular TP matrix. □

Thanks to the property shown in Proposition 2 we can compute collocation matrices of the little q-Jacobi polynomials by the product of a nonsingular TP matrix and the collocation matrix of the monomial basis. The collocation matrix of the monomial basis is the well-known Vandermonde matrix, which is strictly totally positive when the nodes are positive and increasingly ordered. Using these two facts, we can characterize the strict total positivity of collocation matrices of the little q-Jacobi polynomials. We deduce a method to compute the bidiagonal decomposition and show that it can be computed to HRA. Then, we use the bidiagonal decomposition to achieve accurate computations with this class of matrices.

Theorem 4. *Let $M := (J_{j-1,q}(t_{i-1};a,b))_{1\le i,j\le n+1}$ for $(0>)t_0 > t_1 > \ldots > t_n$ with $|a|,|b| \le 1$ and $0<q<1$. Then*

i) M is an STP matrix.

ii) Given the nodes t_i $(0 \le i \le n)$, we can compute $\mathcal{BD}(M)$ with HRA and hence, the following computations can be performed with HRA: all the eigenvalues and singular values, the inverse of M, and the solution of the linear systems $Mx = b$ where $b = (b_0,\ldots,b_n)$ has alternating signs.

Proof. Let $A_q(a,b) = (a_{ij})_{1\le i,j\le n+1}$ be the lower triangular matrix defined by (11). Then we have that $A_q(a,b)^T$ is the matrix of change of basis between the basis of the little q-Jacobi polynomials and the monomial basis:

$$\big(J_{0,q}(t;a,b), J_{1,q}(t;a,b), \ldots, J_{n,q}(t;a,b)\big) = (1,t,\ldots,t^n)A_q(a,b)^T$$

Given $V := (t_{i-1}^{j-1})_{1\le i,j\le n+1}$, the collocation matrix M can be written as

$$M = VJ_{n+1}J_{n+1}A_q(a,b)^T \tag{18}$$

where $J_{n+1} = \mathrm{diag}((-1)^{i+1})_{1\le i\le n+1}$. Since $0 < -t_0 < \ldots < -t_n$, $VJ_{n+1} = ((-t_{i-1})^{j-1})_{1\le i,j\le n+1}$ is a Vandermonde matrix with strictly increasing positive nodes. Hence, VJ_{n+1} is STP (see p. 12 of [10].). Since we have that the collocation matrix is the product of a nonsingular TP matrix $J_{n+1}A_q(a,b)^T$ (by Proposition 2) and an STP matrix VJ_{n+1}, we deduce from Theorem 3.1 of [9]. that M is also an STP matrix and i) holds.

As for the high relative accuracy, we first need to study whether $\mathcal{BD}(J_{n+1}A_q(a,b)^T)$ can be obtained to HRA. By (17), we have that

$$\mathcal{BD}(A_q(a,b)J_{n+1})_{i+1,j+1} = \frac{\prod_{r=0}^{j-1}\left(abq^i - \frac{1}{q^{i-r}}\right)}{\prod_{r=0}^{j-1}\left(abq^{i-1} - \frac{1}{q^{i-r-1}}\right)}$$
whenever $i > j$.

This expression can be simplified if $j \ge 1$ to:

$$\mathcal{BD}(A_q(a,b)J_{n+1})_{i+1,j+1} = q^{-j}\frac{(1-abq^{2i})(1-abq^{2i-1})}{(1-abq^{2i-j})(1-abq^{2i-j-1})}$$

For the diagonal pivots, we can use the following formula:

$$\mathcal{BD}(A_q(a,b)J_{n+1})_{j+1,j+1} = \frac{q^j}{(aq;q)_j}\prod_{r=0}^{j-1}\left(\frac{1-abq^{2j-r}}{q^{j-r}}\right)$$

Let us notice that, for both the diagonal and off-diagonal entries of $\mathcal{BD}(A_q(a,b)J_{n+1})$, we need to assure that $(1-abq^k)$ can be computed to HRA for a positive $k \in \mathbb{Z}$. When $|a|,|b| \ne 1$, we can use the formula

$$1 + cd = \frac{1}{2}[(1+c)(1+d) + (1-c)(1-d)]$$

to decompose $(1-abq^k)$ in products and sums of the numbers $1\pm a$, $1\pm b$ and $1\pm q^k$. Additions of numbers of the same sign and products are carried out to high relative accuracy, as well as subtractions of initial data like $1-|a|$, $1-|b|$. For $1-q^k$ we can use the fact that $1-q^k = (\sum_{j=0}^{k-1} q^j)(1-q)$. Hence, all the computations involved can be carried out to high relative accuracy. By (7), obtaining $\mathcal{BD}(J_{n+1}A_q(a,b)^T)$ is straightforward from $\mathcal{BD}(A_q(a,b)J_{n+1})$.

Moreover, since VJ_{n+1} is a Vandermonde matrix, $\mathcal{BD}(VJ_{n+1})$ is known to HRA and can be computed to HRA (see Section 3 of [18].). Finally, we can obtain the bidiagonal decomposition of the matrix M thanks to the routine introduced in Algorithm 5.1 of [16]. (which is implemented with the name `TNProduct` in the library `TNTool` available in [12].). This algorithm takes as input $\mathcal{BD}(C)$, $\mathcal{BD}(D)$ and it provides $\mathcal{BD}(CD)$. If $\mathcal{BD}(C)$ and $\mathcal{BD}(D)$ are provided with HRA, then the result $\mathcal{BD}(CD)$ is also computed to HRA. Hence, $\mathcal{BD}(M)$ can be computed to HRA by $\texttt{TNProduct}\big(\mathcal{BD}(VJ_{n+1}), \mathcal{BD}(J_{n+1}A_q(a,b)^T)\big)$. Finally, the construction of $\mathcal{BD}(M)$ with HRA assures that the linear algebra problems mentioned in the statement of this theorem can be performed to HRA with the algorithms devised in [16, 18, 19]. □

Theorem 4 assures that the collocation matrices of little q-Jacobi polynomials on nodes $(0>)t_0 > t_1 > \ldots > t_n$ are STP and that their bidiagonal decomposition can be computed to HRA. However, the computation of the bidiagonal decomposition of the matrix of change of basis $(J_{n+1}A_q(a,b))^T$ might require a possible large computational effort depending on the arbitrary parameters a and b. But, for many interesting values of the parameters, the computations involved are greatly simplified. Let us illustrate some of the most interesting cases:

- The case that $a = q^k$ for some $k \in \mathbb{N}$ and $b = 1$. See for example [20, 21]., where this case appears.

- The particular case in which $a = b = 1$, which corresponds to the *little q-Legendre polynomials*.

From now on, we consider this last case, that is, the little q-Legendre polynomials and show how the results presented in this section are adapted for this family of orthogonal polynomials. The little q-Legendre polynomials, $P_{n,q}(x) := J_{n,q}(x;1,1)$, are hence given by

$$P_{n,q}(x) := \sum_{k=0}^{n} \frac{(q^{-n};q)_k (q^{n+1};q)_k}{(q;q)_k} \frac{(qx)^k}{(q;q)_k} \tag{19}$$

We consider the matrix of change of basis $C_q = (c_{ij})_{1\le i,j\le n+1}$ between the monomial basis and the little q-Legendre polynomials,

$$(P_{0,q}(t), P_{1,q}(t), \ldots, P_{n,q}(t)) = (1, t, \ldots, t^n) C_q^T \tag{20}$$

This matrix is equal to the matrix $A_q(1,1)$ defined by (11) Hence, the bidiagonal decomposition of C_q is given by formula (14) of Proposition 1 for $a = b = 1$.

With the bidiagonal decomposition of the matrix of change of basis, it is possible to perform accurate computations with collocation matrices of little q-Legendre polynomials. The following remark adapts Theorem 4 to the case of the little q-Legendre polynomials. In iii) of Remark 3, we can see that the computation of $\mathcal{BD}(M)$ to HRA is easier for the little q-Legendre polynomials than for the general case.

Remark 3. (Little q-Legendre polynomials). Let $M := (P_{j-1,q}(t_{i-1}))_{1\le i,j\le n+1}$ for $(0>)t_0 > t_1 > \ldots > t_n$ with $0 < q < 1$. Then

i. M is an STP matrix.

ii. Given the nodes t_i $(0 \le i \le n)$, we can compute $\mathcal{BD}(M)$ with HRA, and hence, the following computations can be performed with HRA: all the eigenvalues and singular values, the inverse of M, and the solution of the linear systems $Mx = b$ where $b = (b_0, \ldots, b_n)$ has alternating signs.

iii. Let $C_q = A_q(1,1)$ be the matrix of change of basis between the monomial basis and the little q-Legendre polynomials. Then, we can use the following formulas to compute $\mathcal{BD}(C_q J_{n+1})$ to HRA:
   - If $j \ge 1$,

$$\mathcal{BD}(C_q J_{n+1})_{i+1,j+1} = q^{-j} \frac{(1-q^{2i})(1-q^{2i-1})}{(1-q^{2i-j})(1-q^{2i-j-1})}$$

   - If $i = j$,

$$\mathcal{BD}(C_q J_{n+1})_{j+1,j+1} = \frac{q^j}{(q;q)_j} \prod_{r=0}^{j-1} \left(\frac{1-q^{2j-r}}{q^{j-r}} \right)$$

In this case, we only need to use the formulas $(1-q^{2k}) = (1+q^k)(1-q^k)$ and $1-q^k = (\sum_{j=0}^{k-1} q^j)(1-q)$ to avoid subtractions when computing $\mathcal{BD}(C_q J_{n+1})$ (with the exception of $1-q$, which is a subtraction of initial data). Taking into account this, the previous formulas can be expressed in the following way:

   - Diagonal pivots:

$$\mathcal{BD}(C_q J_{n+1})_{j+1,j+1} = q^{-\frac{j(j-1)}{2}} \frac{\prod_{k=j}^{2j-1} \sum_{r=0}^{k} q^r}{\prod_{k=1}^{j-1} \sum_{r=0}^{k} q^r}, \quad j = 0, 1, \ldots, n-1.$$

   - Multipliers:

$$\mathcal{BD}(C_q J_{n+1})_{i+1,j+1} = q^{-j} \frac{\left(\sum_{k=0}^{2i-1} q^k\right)\left(\sum_{k=0}^{2i-2} q^k\right)}{\left(\sum_{k=0}^{2i-j-1} q^k\right)\left(\sum_{k=0}^{2i-j-2} q^k\right)}, \quad n \ge i > j > 0$$

4 | Numerical Experiments

In order to illustrate the good accuracy of the methods presented in this work, the collocation matrices, A_n, of the basis of the space of polynomials of degree at most n, $(P_{0,0.5}(x), \ldots, P_{1,0.5}(x), \ldots, P_{n,0.5}(x))$, at the sequence of nodes $(-i)_{i=1}^{n}$ for $n = 4, 6, 8, \ldots, 24$ have been considered. Figure 1

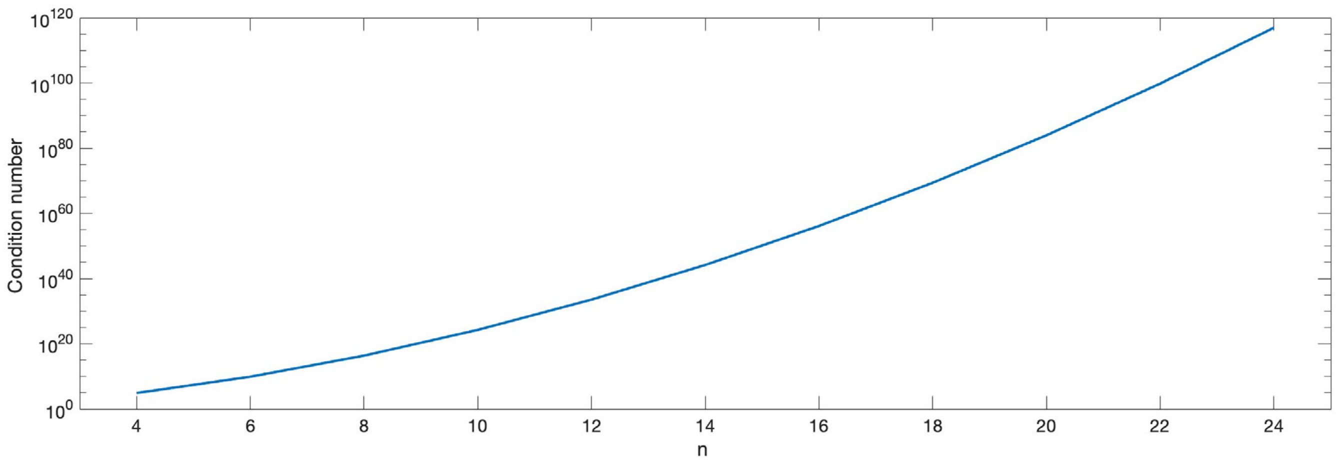

FIGURE 1 | Condition numbers $k_\infty(A_n) = \|A_n\|_\infty \cdot \|A_n^{-1}\|_\infty$.

TABLE 1 | Relative errors when computing the eigenvalues and singular values of the matrix A_{20}.

i	λ_i	HRA	`eig`	σ_i	HRA	`svd`
1	5.9958e+76	2.1441e-16	0	6.4712e+76	1.9866e-16	0
2	2.2829e+68	2.0978e-16	6.2096e-14	2.8576e+68	1.0055e-15	1.8435e-15
3	3.8704e+60	1.8438e-16	1.3200e-12	5.5807e+60	1.6624e-15	2.1577e-03
4	2.205e+53	1.9290e-16	8.4368e+00	3.6358e+53	8.1894e-16	1.0807e+07
5	3.7779e+46	9.3953e-16	5.8365e+06	7.0696e+46	5.7379e-16	4.6823e+06
6	1.8391e+40	5.2588e-16	2.0504e+06	3.875e+40	3.7438e-16	7.4777e+11
7	2.4637e+34	2.6206e-15	2.1625e+06	5.7967e+34	1.5911e-16	1.1290e+12
8	8.9197e+28	1.9723e-16	2.0936e+11	2.3239e+29	7.5701e-16	2.9613e+15
9	8.6514e+23	1.5514e-16	4.7819e+11	2.4746e+24	2.1696e-16	2.0574e+20
10	2.2441e+19	7.3008e-16	2.8296e+15	6.9861e+19	1.1726e-16	4.7120e+20
11	1.5642e+15	6.3931e-16	1.3668e+18	5.2520e+15	0	6.1653e+21
12	2.9647e+11	1.4411e-15	9.1088e+18	1.0623e+12	2.2982e-16	7.6855e+23
13	1.5620e+08	7.6319e-16	1.5605e+19	5.8761e+08	0	6.4439e+25
14	2.3793e+05	6.1160e-16	7.3006e+19	9.0953e+05	7.6797e-16	7.4390e+24
15	1.1267e+03	2.0180e-16	3.5467e+21	4.0182e+03	1.3581e-15	3.4259e+25
16	1.8959e+01	1.1244e-15	9.5027e+19	5.7318e+01	8.6776e-16	3.9006e+26
17	1.4073e+00	9.4667e-16	2.8943e-01	1.1497e+00	1.3520e-15	1.2683e+28
18	1.6376e-01	3.3899e-16	1.2020e+23	1.1733e-02	1.3307e-15	1.5633e+26
19	7.2215e-03	1.0810e-15	4.9870e+35	7.4056e-05	5.4901e-16	1.2856e+28
20	1.1009e-04	3.6933e-16	1.8840e+52	1.4223e-07	3.7221e-16	4.2747e+05

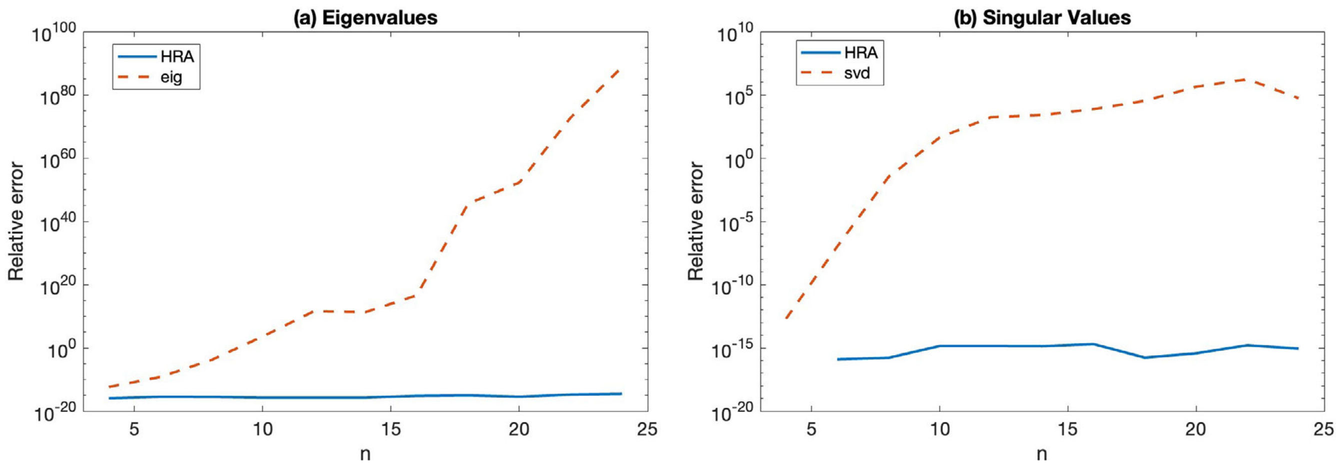

FIGURE 2 | Relative errors for the smallest eigenvalue and singular value of A_n for $n = 4, 6, \dots, 24$.

shows the condition number of these matrices. As we can observe, the matrices are very ill-conditioned. So, we cannot expect that the usual numerical algebra algorithms provide accurate results for them.

First, the eigenvalues of the considered matrices have been computed with Mathematica with a 200-digit precision. Then, the eigenvalues have also been computed with Matlab in two ways. The first one by using its `eig` function. The second one by computing the bidiagonal decomposition of the matrices to HRA and then using the function `TNEigenValues` of the software library TNTool available for download in [12]. In an analogous way, the singular values of the matrices A_n have been computed by using Mathematica and Matlab (`svd` and `TNSingularValues` in [12].). Then, the relative errors for the eigenvalues and the singular values computed with Matlab have been calculated by considering the results obtained with Mathematica as exact. Table 1 shows the relative errors for the eigenvalues and singular values of A_{20}. It can be observed that the lower the eigenvalues or singular values are, the greater the relative error is for the standard MATLAB algorithms `eig` and `svd`. Taking into account this, Figure 2 shows the relative error for the smallest eigenvalues and singular values of A_n for $n = 4, 6, \dots, 24$.

TABLE 2 | Relative errors for A_n^{-1}.

	TNInverseExpand		inv	
n	Mean	Maximal	Mean	Maximal
4	5.3837e-17	2.1000e-16	1.0968e-16	3.8572e-16
6	1.1032e-16	2.7715e-16	7.5809e-16	3.1524e-15
8	9.4107e-17	3.2382e-16	1.5233e-12	2.5186e-12
10	1.4371e-16	7.0733e-16	6.6379e-10	1.0598e-09
12	1.3025e-16	4.4672e-16	1.6719e-08	2.8871e-08
14	1.9913e-16	6.2054e-16	2.0866e-06	3.0435e-06
16	1.7360e-16	6.1761e-16	6.7176e-06	2.2795e-05
18	2.2661e-16	8.1351e-16	5.5412e-04	1.4863e-03
20	2.3674e-16	8.5452e-16	1.5748e-02	7.9772e-02
22	2.3597e-16	9.0500e-16	7.9096e-01	1.2834e+00
24	2.5444e-16	9.5972e-16	9.5620e-01	1.0163e+00

TABLE 3 | Componentwise relative errors for $A_{20}x = b$.

i	x_i	$\frac{\lvert x_i-\hat{x}_i\rvert}{\lvert x_i\rvert}$ for TNSolve	$\frac{\lvert x_i-\hat{x}_i\rvert}{\lvert x_i\rvert}$ for \
1	4.9861e+09	0	2.3229e-03
2	−6.4394e+09	1.4810e-16	2.7692e-03
3	2.1612e+09	4.4128e-16	3.4142e-03
4	−2.7719e+08	4.3007e-16	4.2220e-03
5	1.4259e+07	2.6127e-16	5.1735e-03
6	−2.9506e+05	3.9455e-16	6.2450e-03
7	2.4513e+03	3.7103e-16	7.4078e-03
8	−8.1735e+00	2.1733e-16	8.6317e-03
9	1.0952e-02	3.1679e-16	9.8883e-03
10	−5.9046e-06	2.869e-16	1.1152e-02
11	1.2811e-09	4.8425e-16	1.2402e-02
12	−1.1165e-13	4.5221e-16	1.3619e-02
13	3.8887e-18	1.9811e-16	1.4789e-02
14	−5.3654e-23	0	1.5903e-02
15	2.8906e-28	1.5513e-16	1.6951e-02
16	−5.9446e-34	4.3163e-16	1.7930e-02
17	4.5026e-40	0	1.8835e-02
18	−1.1829e-46	1.6439e-16	1.9666e-02
19	9.6215e-54	0	2.0422e-02
20	−1.8238e-61	1.8941e-16	2.1107e-02

The inverses A_n^{-1}, $n = 4, 6, \ldots, 24$, have also been computed with Mathematica by using a 200-digit precision. Then, the inverses have been calculated with Matlab in two ways. The first one by using the usual Matlab function `inv`. The second one by using $\mathcal{BD}(A_n)$ to HRA together with the algorithm devised in [19]. and implemented in the function `TNInverseExpand` of TNTool [12].. The componentwise relative errors for the approximations of the inverse obtained with Matlab have been computed considering Mathematica results as exact. Then, the mean and maximal componentwise relative errors have been calculated for each n. Table 2 shows these errors.

Finally, we have considered a linear system of equations $A_{20}x = b$, where the vector b has an alternating sign pattern. In particular, the absolute values of the entries of b have been randomly generated as integers in the interval $[1, 1000]$. The exact solution x of the system has been obtained with Mathematica. Then, approximations $\hat{x}$ to the exact solution have been obtained with the usual Matlab command \ and also with the function `TNSolve` of TNTool, which takes as input the bidiagonal decomposition $\mathcal{BD}(A_n)$ to HRA. Table 3 shows the corresponding componentwise relative errors. In all cases, we observe that our methods outperform the usual Matlab methods.

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Data Availability Statement

The data that support the findings of this study are available from the corresponding author upon reasonable request.

References

1. M. Ismail, "Classical and Quantum Orthogonal Polynomials in One Variable," in *Encyclopedia of Mathematics and Its Applications*, vol. 98 (Cambridge: Cambridge University Press, 2005).

2. V. Kac and P. Cheung, *Quantum calculus* (New York: Springer, 2002).

3. G. Gasper and M. Rahman, *Basic Hypergeometric Series*, 2nd ed. (Cambridge: Cambridge University Press, 1990), 2004.

4. J. Delgado and J. M. Peña, "Accurate Computations With Collocation Matrices of q-Bernstein Polynomials," *SIAM Journal on Matrix Analysis and Applications* 36 (2015): 880–893.

5. A. Marco, J. J. Martínez, and R. Viaña, "Accurate Bidiagonal Decomposition of Totally Positive h-Bernstein-Vandermonde Matrices and Applications," *Linear Algebra and Its Applications* 579 (2019): 320–335.

6. E. Mainar, J. M. Peña, and B. Rubio, "Accurate Computations With Collocations and Wronskian Matrices of Jacoby Polynomials," *Journal of Scientific Computing* 87 (2021): paper 77.

7. J. Delgado, H. Orera, and J. M. Peña, "Accurate Computations With Laguerre Matrices," *Numerical Linear Algebra With Applications* 26 (2019): e2217.

8. J. Delgado, H. Orera, and J. M. Peña, "High Relative Accuracy With Matrices of q-Integers," *Numerical Linear Algebra With Applications* 28 (2021): e2383.

9. T. Ando, "Totally Positive Matrices," *Linear Algebra and Its Applications* 90 (1987): 165–219.

10. S. M. Fallat and C. R. Johnson, "Totally Nonnegative Matrices," in *Princeton Series in Applied Mathematics 35* (Princeton, NJ: Princeton University Press, 2011).

11. A. Pinkus, *Totally Positive Matrices. Tracts in Mathematics*, vol. 181 (Cambridge, UK: Cambridge University Press, 2010).

12. P. Koev, https://math.mit.edu/~plamen/software/TNTool.html. accessed May 4th, 2024.

13. M. Gasca and J. M. Peña, "Total Positivity and Neville Elimination," *Linear Algebra and Its Applications* 165 (1992): 25–44.

14. M. Gasca and J. M. Peña, "On Factorizations of Totally Positive Matrices," in *Total Positivity and Its Applications*, eds. M. Gasca and C. A. Micchelli (Dordrecht, The Netherlands: Kluver Academic Publishers, 1996), 109–130.

15. A. Barreras and J. M. Peña, "Accurate Computations of Matrices With Bidiagonal Decomposition Using Methods for Totally Positive Matrices," *Numerical Linear Algebra With Applications* 20 (2013): 413–424.

16. P. Koev, "Accurate Computations With Totally Nonnegative Matrices," *SIAM Journal on Matrix Analysis and Applications* 29 (2007): 731–751.

17. J. Demmel and P. Koev, "The Accurate and Efficient Solution of a Totally Positive Generalized Vandermonde Linear System," *SIAM Journal on Matrix Analysis and Applications* 27 (2005): 142–152.

18. P. Koev, "Accurate Eigenvalues and SVDs of Totally Nonnegative Matrices," *SIAM Journal on Matrix Analysis and Applications* 27 (2005): 1–23.

19. A. Marco and J. J. Martínez, "Accurate Computation of the Moore-Penrose Inverse of Strictly Totally Positive Matrices," *Journal of Computational and Applied Mathematics* 350 (2019): 299–308.

20. J. E. Andersen and C. Berg, "Quantum Hilbert Matrices and Orthogonal Polynomials," *Journal of Computational and Applied Mathematics* 233 (2009): 723–729.

21. M. Ismail, "One Parameter Generalizations of the Fibonacci and Lucas Numbers," *Fibonacci Quarterly* 46/47 (2008/2009): 167–179.