Imperfect inspection and replacement of a system with a defective state: a cost and reliability analysis

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ABSTRACT: We consider a system with three possible states, good, defective and failed. Failures are detected as soon as they occur; the defective state, which is only revealed by inspection, does not prevent the system from fulfilling the function for which it was designed. We present a maintenance model consisting of periodic inspections to check the state of the system, in which inspections are subject to error. At a false positive inspection the system is unnecessarily replaced; at a false negative inspection a defect remains unrevealed with reliability implications for future operation. The model is illustrated with an example from the railways. In this context, we suppose that system lifetime is heterogeneous so that the time the system spends in the defective state is a random variable from a mixed distribution. We determine under what circumstances the cost of maintenance cannot be justified by its efficacy, and suggest that when there is the possibility that replacement is poorly executed (lifetime heterogeneity) the natural response to imperfect inspection of increasing the inspection frequency can be counter-productive.

Keywords: *maintenance, misclassification, reliability, condition based maintenance.*

1. INTRODUCTION

We study a system comprising of a single, non-repairable component and a socket that together perform an operational function (Ascher and Feingold, 1984). The system can be in one of three states: good, defective and failed. In the defective state the system has entered a wear-out phase but continues to perform its operational function; that is, the function for which it was designed. A failure of the system is immediately revealed because the system ceases to function. However, the defective state is revealed only by inspection. If a failure of the system has significant consequences, either in cost or safety terms, then it would normally be desirable to replace the system if the defective state is detected at inspection rather than to continue operation until failure. By system replacement we shall mean replacement of the component in the socket by an identical, "new" component. This component might in practice be re-cycled and re-conditioned; for our model the important point is that notionally successive components installed at replacement arise from a common population of components. In particular, we are interested in the case in which the population of components is heterogeneous; that is, the population of components is a mixture of two sub-populations, the first sub-population comprising of weak components and the second strong components. The mixing proportion p is then the probability that at replacement the component is weak. In this context, weak components have a short time to defect arrival; strong components have a long time to defect arrival. Maintenance polices in the context of such mixtures have been studied by Scarf et al (2009) and Scarf and Cavalcante (2010). In practice, components may be weak for a number of reasons: they may be poorly manufactured; they may be damaged in transit; they may be poorly installed; they may be imperfectly reconditioned. However, the model of a heterogeneous population remains the same whatever the underlying cause of the weakness, and in this way, we can model poorly executed component replacement. The cost-rate and reliability calculations that we develop in the paper are however general and do not depend on the form of the component lifetime distribution.

We consider a maintenance policy for the system in which periodic inspections are performed at instants $kT, k=1,...,M$. The cost of inspection is c_0. At inspection, if the system is found to be in the defective state it is replaced. If the system fails, a cost c_f is incurred. In this paper in particular, we suppose that inspections are subject to error; a false positive or a false negative may occur. A false positive occurs when the inspection says the system is defective when in fact it is good. A false negative occurs when the inspection says the system is good when in fact it is defective. We can see that inspection errors will have an effect upon the reliability of the system and upon cost. Under perfect inspection, reliability and cost only depend on the frequency of inspection, and more frequent inspection necessarily implies greater reliability. In our model, reliability and cost depend on the frequency and quality of inspection. In fact, when inspection is perfect, increasing the frequency of inspection is a natural procedure in order to reduce the probability of failure. However, when false positives and negatives are possible, increasing on the frequency may be very cost-inefficient and, when the component population is heterogenous, may in fact reduce system reliability (Scarf et al. 2009). In this way our article explores some maintenance planning situations where common sense indicates some direction when the best thing to do is exactly the contrary.

From a modelling point of view, positive inspections may generate two types of situation depending on whether an additional source of information about the real state of the system is available; that is, whether an initial positive inspection can be followed with a deeper investigation. In the first situation, a positive inspection (system state is indicated as defective) is investigated further at additional cost and the actual state of the system is revealed; if it is good then the system is not replaced; if the system is defective, it is replaced. In the second situation, when there is no opportunity to gain other information apart from that derived directly from the inspection, that is, the result of inspections is the only knowledge we have about the system, then the system will be replaced at a positive inspection. If this positive inspection is a false positive then the system is unnecessarily replaced. In this paper we will deal with this second situation and therefore assume that a positive inspection always leads to replacement of the system with cost c_p, with $c_p < c_f$.

In the first situation discussed above where a positive inspection is followed by a further investigation that reveals the true system state, arguably false positives do not occur. In this situation more precisely it is that a positive inspection implies additional cost, so that a better description of a model for this situation would be one in which false negatives can occur and positive inspections incur additional cost. If such additional cost was $c*$ per event, then the cost structure would be as in Table 1. The implication of this argument is that if false positives can occur then, necessarily, positive inspection leads to replacement with cost structure as in Table 2. The probabilities of false negatives and false positives that we use in our notation are shown in Table 3.

Table 1: Classification of system and inspection status with associated inspection costs: situation 1, positive inspection followed by further investigation to reveal true system state.

Inspection outcome \ System status	System good	System defective
Inspection negative	c_0	c_0
Inspection positive	$c_0 + c*$	$c_0 + c* + c_p$

Table 2: Classification of system and inspection status with associated inspection costs: situation 2, case of false positive inspections.

Inspection outcome \ System status	System good	System defective
Inspection negative	c_0	c_0
Inspection positive	$c_0 + c_p$	$c_0 + c_p$

Table 3: Classification of system and inspection status with associated probabilities: situation 2, case of false positive inspections.

System status / Inspection outcome	System good	System defective
Inspection negative	$1-\alpha$	β
Inspection positive	α	$1-\beta$

We further assume that maintenance procedure is completed with a preventive replacement at the *M*th inspection that occurs at time *MT,* with cost c_p, provided that at an earlier moment there has not been a replacement.

Components that can be in one of three states (good, failed and defective) are considered in articles on delay time modelling developed by Christer and co-workers (e.g. Baker and Wang, 1991; Christer and Wang 1995; Baker et al. , 1997). For a recent review of delay time modelling see Wang (2012). Berrade et al. (2012) analyze the quality of inspections in a two-phase inspection policy for a protection system, in which failure is typically revealed through inspection. Our paper here develops the ideas of Berrade et al. (2012) in the context of a three state failure model for a system with a single, non-repairable component. This consideration makes the model in Berrade et al. (2012) in some sense a special case, where the transition from the defective to the failed state is instantaneous. Of course, in our paper here, failures are immediately revealed whereas for the protection system they are unrevealed. Also, with our three-state approach we can see the effect of the condition observation (state observed at inspection) on the maintenance policy effectiveness, manifest for example as a cost saving or as an increase in reliability. The current paper is a development of work in Berrade et al. (2011) and Okumura et al. (1996).

Jia and Christer (2002) consider the periodic testing of a preparedness system with a defective state using the delay time concept, but this work considers a perfect testing procedure. A perfect testing procedure is also assumed in Cavalcante et al. (2011) who present a two-phase inspection policy for an heterogeneous population where components can be weak, representing early failures, or strong components affected only by wear-out. Okumura et al. (1996) consider imperfect testing with false positives and negatives but with perfect replacement. Others have investigated imperfect inspection, although not explicitly through the concepts of false positives and false negatives. For example, recently Wang et al. (2011) present a delay time model with imperfect maintenance at inspection and carry out numerical simulations to study the influence of imperfect maintenance on the long-run availability. An earlier paper that considers false negatives is Baker and Wang (1991). Our paper is also related to recent work on the quality of maintenance (Scarf and Cavalcante; 2012).

The purpose of our model in this paper is to consider the effect of imperfect inspection and replacement on the efficacy of the inspection. We envisage an application in which the user would investigate, for example, the values for the probabilities of a false positive, a false negative and poorly executed replacement for which it is cost-optimal to perform inspection. The operational reliability implications of imperfect inspection and replacement can also be determined with our model. The model can also consider in a rather simple way the cost-benefit of condition based maintenance. Thus it might be used to consider the investment decision for a monitoring system. The model itself is described in detail in the following section. Then in section 3 we develop the reliability calculations. A numerical illustration for an application to railway maintenance is described in section 4, and we finish with a discussion.

2. THE MODEL

In the development of the maintenance model, we assume that replacement renews the system. Replacement occurs on failure or at a positive inspection or at *MT*, whichever occurs first. Replacement will signify the start of a new (replacement) cycle, and for cost purposes we will use the long run average cost per unit time (the ratio of the average cost per replacement cycle to average replacement cycle length, Ross (1996)) as our decision criterion for optimising the inspection interval *T* and the number of inspections until preventive replacement *M.* We will use the term "cost-rate" for this long run average cost per unit time. Also, we will use the term "cycle" synonymously with the term "replacement cycle". At the start of a new cycle we set *t*=0. We will use the terms time and age (of the system) interchangeably therefore.

The failure model is as follows. Denote by *X* the time from replacement until the moment that the system enters the defective state, assuming no inspection or replacement, with *X* a continuous non-negative valued random variable with distribution function $F_X(x)$, and reliability (survival) function $\overline{F}_X(x) = 1 - F_X(x)$. Further, denote by *D* the delay time from defect arrival to subsequent failure with *D,* a continuous non-negative valued random variable with distribution function $F_D(x)$, and reliability (survival) function $\overline{F}_D(x) = 1 - F_D(x)$. Thus *X* and *D* are properties of the underlying component reliability behaviour, and the (underlying, assuming no inspection or replacement) time to failure $Y = X + D$. It is standard to assume that *X* and *D* are independent variables, so $F_Y(y) = \int_0^y F_D(y-x)\mathrm{d}F_X$. Relaxation of this assumption might make an interesting study.

The system is inspected at times $kT, k = 1,\ldots,M$ and preventively replaced at *MT*. *M* and *T* are decision variables. Costs are as follows: cost of inspection, c_0; cost of preventive replacement (either at *MT* or at positive inspection), c_p; cost of failure, c_f. Inspection error (misclassification) probabilities are as in Table 3. We also assume that the system is inspected at *MT,* instantaneously before the replacement. The inclusion of the relevant inspection cost in the model might be justified in practice if, for example, the inspection cost accrues as a set-up cost, that is, as the cost of the preparation to maintain (bringing manpower and parts on site). Relaxing the assumption of an inspection at *MT* leads to minor modifications of the formulae that follow, and these are outlined.

Other notation: τ is the length of a replacement cycle, a random variable that depends on *X, D, M* and *T* in a non-trivial way; if $M = 1$, $\tau = \min(X + D, T)$. *K* is the number of inspections in a cycle, a random variable.

The cost-rate is given by

$$C(T, M) = E(C(\tau)) / E(\tau) \tag{1}$$

where the expected cost of a cycle is given by

$$E(C(\tau)) = c_0 E(K) + c_p P(c_p) + c_f (1 - P(c_p)). \tag{2}$$

In the foregoing formula $P(c_p)$ represents the probability of incurring a cost c_p in a replacement cycle, that is, if the probability that the cycle ends in preventive replacement (at *MT* or at a positive inspection). Further, $P(c_f) = 1 - P(c_p)$ represents the probability of incurring a cost c_f in a cycle, that is, the probability that the cycle ends in failure.

For $i = 1, \ldots, M - 1$, the conditional mean length of a cycle if $(i-1)T < X < iT$ is given by

$$E(\tau | (i-1)T < X < iT) =$$
$$q^{i-1} \int_0^{iT-x} (x + d) \mathrm{d}F_D + q^{i-1} \sum_{j=i}^{M-1} \beta^{j-i+1} \int_{jT-x}^{(j+1)T-x} (x + d) \mathrm{d}F_D +$$
$$q^{i-1} T \sum_{j=i}^{M-1} j \beta^{j-i} (1 - \beta) \overline{F}_D (jT - x) + q^{i-1} MT \beta^{M-i} \overline{F}_D (MT - x) + \alpha T \sum_{j=1}^{i-1} j q^{j-1},$$

where $q = 1 - \alpha$ and *X=x*. The first term relates to the conditional mean length of a cycle when the failure occurs before the first inspection that follows the arrival of the defective state. The second term relates to a failure that occurs after the first inspection that follows the arrival of a defect and when inspections have failed to detect it. The third term corresponds to the case when the defective state is detected at inspection and necessarily prior to a failure occurring. The fourth term is the conditional mean length of a cycle that ends with the preventive maintenance at *MT*. The fifth term relates to a cycle that ends with a false positive before or at $(i-1)T$. This final term can be written as

$$T \frac{1 + (i-1) q^i - i q^{i-1}}{1 - q}.$$

If $(M-1)T < X < MT$ the conditional mean length of a cycle is given by

$$E(\tau | (M-1)T < X < MT) =$$
$$q^{M-1} \left\{ \int_0^{MT-x} (x + d) \mathrm{d}F_D + MT \overline{F}_D (MT - x) \right\} + T \frac{1 + (M-1) q^M - M q^{M-1}}{1 - q}.$$

The first term is the conditional expectation when a failure occurs before MT, the second when the cycle ends with the preventive maintenance at MT and the third corresponds to the case when the cycle ends with a false positive before or at $(M-1)T$; a false positive at MT has no implications, however, since the system is renewed at MT.

In addition we have that

$$E(\tau|X > MT) = q^{M-1}MT + T\frac{1+(M-1)q^{M} - Mq^{M-1}}{1-q}.$$

The mean length of a cycle turns out to be

$$\begin{aligned}
E(\tau) &= \sum_{i=1}^{M} q^{i-1}\int_{(i-1)T}^{iT}\int_{0}^{iT-x}(x+d)\mathrm{d}F_D\mathrm{d}F_X \\
&+ \sum_{i=1}^{M-1} q^{i-1}\int_{(i-1)T}^{iT}\left\{\sum_{j=i}^{M-1}\beta^{j-i+1}\int_{jT-x}^{(j+1)T-x}(x+d)\mathrm{d}F_D\right\}\mathrm{d}F_X \\
&+ \sum_{i=1}^{M-1} q^{i-1}T(1-\beta)\int_{(i-1)T}^{iT}\left\{\sum_{j=i}^{M-1} j\beta^{j-i}\overline{F}_D(jT-x)\right\}\mathrm{d}F_X \\
&+ \sum_{i=1}^{M} q^{i-1}MT\beta^{M-i}\int_{(i-1)T}^{iT}\overline{F}_D(MT-x)\mathrm{d}F_X + q^{M-1}MT\overline{F}_X(MT) + \alpha T\sum_{i=1}^{M-1} iq^{i-1}\overline{F}_X(iT).
\end{aligned}$$

The first term in the foregoing expression represents the mean length of a cycle ending in a failure before the first inspection that takes place after the defective state has occurred. The second term is the mean length of a cycle that ends with a failure that occurs after the first inspection that follows the defective state and before detecting it. The third term corresponds to the mean length of a cycle that ends when the defective state is detected at an inspection before a failure occurs. The fourth term is the case when the cycle ends with the preventive maintenance at MT when the defective state has occurred but is not detected at any of the following inspections and no failure occurs. The fifth term is the corresponding mean value when the defective state has not arrived by MT and thus the component undergoes preventive replacement. In the first five terms, no false positive occurs. The sixth term represents the mean value of a cycle that ends with a false positive.

The number of inspections in a cycle, K, takes values in $\{0,1,...,M\}$ with the following probabilities.

$$P(K=0) = \int_0^T\int_0^{T-x}\mathrm{d}F_D\mathrm{d}F_X .$$

For $i = 1,...,M-1$ we have

$$\begin{aligned}
P(K=i) &= \sum_{j=1}^{i} q^{j-1}\int_{(j-1)T}^{jT}\left\{\beta^{i-j}(1-\beta)\overline{F}_D(iT-x) + \int_{iT-x}^{(i+1)T-x}\beta^{i-j+1}\mathrm{d}F_D\right\}\mathrm{d}F_X \\
&+ q^{i}\int_{iT}^{(i+1)T}\mathrm{d}F_X\int_0^{(i+1)T-x}\mathrm{d}F_D + \alpha q^{i-1}\overline{F}_X(iT).
\end{aligned}$$

The first term reflects the case when a defect arrives before the ith inspection and is detected at the ith inspection before a failure takes place. In the second term the defect is not detected at the ith inspection and failure happens before the i+1th inspection. The third term corresponds to the case when both the defect arrival and the failure occur between the ith and the i+1th inspections. The fourth term relates to a false positive occurring at the ith inspection.

Furthermore,

$$P(K=M)=\sum_{j=1}^{M} q^{j-1}\int_{(j-1)T}^{jT}\beta^{M-j}\overline{F}_D(MT-x)\mathrm{d}F_X+q^{M-1}\overline{F}_X(MT).$$

Then it follows

$$\begin{aligned}E(K)=&\sum_{i=1}^{M-1} q^{i-1}\int_{(i-1)T}^{iT}\left\{\sum_{j=i}^{M-1} j\beta^{j-i}(1-\beta)\overline{F}_D(jT-x)+\sum_{j=i}^{M-1} j\beta^{j-i+1}\int_{jT-x}^{(j+1)T-x}\mathrm{d}F_D\right\}\mathrm{d}F_X\\&+M\sum_{i=1}^{M} q^{i-1}\int_{(i-1)T}^{iT}\beta^{M-i}\overline{F}_D(MT-x)\mathrm{d}F_X+\sum_{i=1}^{M-1} iq^{i}\int_{iT}^{(i+1)T}\mathrm{d}F_X\int_0^{(i+1)T-x}\mathrm{d}F_D\\&+\sum_{i=1}^{M-1} iq^{i-1}\alpha\overline{F}_X(iT)+Mq^{M-1}\overline{F}_X(MT).\end{aligned}$$

If we modify the policy slightly so that there is no inspection at MT, then the above expression changes. Let K^* denote the number of inspections in a cycle when there is no inspection at MT. Then K^* takes values on $\{0,1,...,M-1\}$ and

$$\begin{aligned}E(K^*)=&\sum_{i=1}^{M-1} q^{i-1}\int_{(i-1)T}^{iT}\left\{\sum_{j=i}^{M-1} j\beta^{j-i}(1-\beta)\overline{F}_D(jT-x)+\sum_{j=i}^{M-1} j\beta^{j-i+1}\int_{jT-x}^{(j+1)T-x}\mathrm{d}F_D\right\}\mathrm{d}F_X\\&+(M-1)\sum_{i=1}^{M-1} q^{i-1}\int_{(i-1)T}^{iT}\beta^{M-i}\overline{F}_D(MT-x)\mathrm{d}F_X+\sum_{i=1}^{M-2} iq^{i}\int_{iT}^{(i+1)T}\mathrm{d}F_X\int_0^{(i+1)T-x}\mathrm{d}F_D\\&+\sum_{i=1}^{M-2} iq^{i-1}\alpha\overline{F}_X(iT)+(M-1)q^{M-2}\overline{F}_X((M-1)T).\end{aligned}$$

The cost incurred in a cycle will be c_p in any of the following cases.

1. The defective state does not occur, no false positive occurs and the system undergoes the preventive renewal at MT.

2. No false positive occurs before the defective state occurs. The defective state is detected in a subsequent inspection before MT and before a failure occurs.

3. No false positive occurs before the defective state occurs. The defective state is not detected but no failure occurs before MT. Then, the preventive replacement is carried out at MT.

4. A false positive occurs.

Therefore, the probability that a cycle ends with cost of c_p is given by

$$P(c_p) = q^{M-1}\overline{F}_X(MT)$$
$$+\sum_{i=1}^{M-1} q^{i-1}\int_{(i-1)T}^{iT}\left\{\sum_{j=i}^{M-1}\beta^{j-i}(1-\beta)\overline{F}_D(jT-x)\right\}\mathrm{d}F_X$$
$$+\sum_{i=1}^{M} q^{i-1}\int_{(i-1)T}^{iT}\beta^{M-i}\overline{F}_D(MT-x)\mathrm{d}F_X + \sum_{i=1}^{M-1} q^{i-1}\alpha\overline{F}_X(iT).$$

The first, second, third and fourth terms in the expression above are the corresponding probabilities for the cases numbered from 1 to 4.

The cost-rate is then determined by substituting $P(c_p)$, $P(c_f)$, and $E(K)$ in equation (2) to obtain $E(C(\tau))$, and then substituting this and $E(\tau)$ in equation (1).

In case of $M=\infty$, the pure inspection policy given in Okumura et al.(1996) is obtained.

When M=1, the above expressions simplify considerably. Then

$$E(\tau|0<X<T) = \int_0^{T-x}(x+d)\mathrm{d}F_D + T\overline{F}_D(T-x),$$

$$E(\tau|X>T) = T.$$

Therefore

$$E(\tau) = \int_0^T\left\{\int_0^{T-x}(x+d)\mathrm{d}F_D + T\overline{F}_D(T-x)\right\}\mathrm{d}F_X + T\overline{F}_X(T).$$

In the general case we consider that there is an inspection at MT so we also assume that there is an inspection at T when M=1. Hence K takes values in {0,1}, recalling that K is the number of inspections in a cycle, with the following probabilities

$$P(K=0) = \int_0^T\left\{\int_0^{T-x}\mathrm{d}F_D\right\}\mathrm{d}F_X,$$

$$P(K=1) = \int_0^T R_D(T-x)\mathrm{d}F_X + \overline{F}_X(T),$$

(K=0 if the system fails, K=1 otherwise) so that

$$E(K) = \int_0^T \overline{F}_D(T-x)\mathrm{d}F_X + \overline{F}_X(T).$$

Regarding the costs we have $P(c_f) = P(K=0)$ and $P(c_p) = P(K=1)$. Hence the cost-rate turns out to be

$$\frac{E(C(\tau))}{E(\tau)} = \frac{(c_0+c_p)\{\int_0^T \overline{F}_D(T-x)\mathrm{d}F_X + \overline{F}_X(T)\} + c_f\{\int_0^T\int_0^{T-x}\mathrm{d}F_D\mathrm{d}F_X\}}{\int_0^T\left\{\int_0^{T-x}(x+d)\mathrm{d}F_D + T\overline{F}_D(T-x)\right\}\mathrm{d}F_X + T\overline{F}_X(T)}.$$

3. RELIABILITY

We now determine the reliability of the system in the sense of Lewis (1987) and described in detail in Scarf et al. (2005). Christer (1987) analyzes the reliability of a single component under three possible states subject to periodic testing, using the delay time concept and assuming perfect inspection. The current work extends this study to the case of imperfect inspections. For a renewal process, the (operational) reliability is defined as the survival function (reliability function) of the failure inter-arrival times. For our model, renewals occur on failure, at preventive replacements (at *MT*), at replacements following a defect revealed at inspection, and at replacements due to false positives, and we derive the reliability (survival function of the failure inter-arrival-times) given such renewals.

The reliability of the system when there is neither inspection nor maintenance is:

$$R(t) = \int_0^t \overline{F}_D(t-x)\mathrm{d}F_X + \overline{F}_X(t)\,.$$

We denote by $R_{MT}(T)$ the reliability under the inspection and maintenance procedure. Following Christer (1987) we introduce $r_{MT}^{(s)}(t)$ as the reliability at time t where $(s-1)T \le t < sT$, for s a positive integer. Now if $0 \le t < T$

$$r_{MT}^{(1)}(t) = \int_0^t \overline{F}_D(t-x)\mathrm{d}F_X + \overline{F}_X(t)$$

That is, for $0 \le t < T$, $r_{MT}^{(1)}(t) = R(t)$. If $T \le t < 2T$

$$r_{MT}^{(2)}(t) = \left\{\int_0^T (1-\beta)\overline{F}_D(T-x)\mathrm{d}F_X\right\} r_{MT}^{(1)}(t-T) + \beta\int_0^T \overline{F}_D(t-x)\mathrm{d}F_X + \left\{\alpha\overline{F}_X(T)\right\} r_{MT}^{(1)}(t-T)$$
$$+(1-\alpha)\int_T^t \overline{F}_D(t-x)\mathrm{d}F_X + (1-\alpha)\overline{F}_X(t).$$

The first term relates to the defective state detected at T and before a failure occurs. In the second term the defective state is not detected at T but no failure occurs before t. In the third term the system is replaced due to a false alarm at T. In the fourth term no false alarm occurs at T, the defect arises after T without failure before t. In the fifth term no false alarm occurs at T and no defect arises before t.

If $2T \le t < 3T$

$$r_{MT}^{(3)}(t) = \{\int_0^T \overline{F}_D(T-x)(1-\beta)\mathrm{d}F_X\} r_{MT}^{(2)}(t-T) + \{\int_0^T \overline{F}_D(2T-x)\beta(1-\beta)\mathrm{d}F_X\} r_{MT}^{(1)}(t-2T)$$
$$+\int_0^T \overline{F}_D(t-x)\beta^2\mathrm{d}F_X + \{(1-\alpha)\int_T^{2T} \overline{F}_D(2T-x)(1-\beta)\mathrm{d}F_X\} r_{MT}^{(1)}(t-2T)$$
$$+(1-\alpha)\int_T^{2T} \overline{F}_D(t-x)\beta\ \mathrm{d}F_X + (1-\alpha)^2\int_{2T}^t \overline{F}_D(t-x)\mathrm{d}F_X + (1-\alpha)^2\,\overline{F}_X(t)$$
$$+\{\alpha\overline{F}_X(T)\} r_{MT}^{(2)}(t-T) + \{(1-\alpha)\alpha\overline{F}_X(2T)\} r_{MT}^{(1)}(t-2T)$$

The general expression if $(s-1)T \le t < sT$ and $s = 1, 2, \ldots, M$ is

$$r_{MT}^{(S)}(t)=\sum_{j=1}^{s-1}(1-\alpha)^{j-1}\{\int_{(j-1)T}^{jT}(1\text{-}\beta)\{\sum_{\mathrm{i}=1}^{\mathrm{s-j}}\beta^{\mathrm{i-1}}\overline{F}_D((i+j-1)T-x)r_{MT}^{(s-i-j+1)}(t-(i+j-1)T)\}\mathrm{d}F_X\}$$
$$+\sum_{j=1}^{s-1}(1-\alpha)^{j-1}\beta^{s-j}\int_{(j-1)T}^{jT}\overline{F}_D(t-x)\mathrm{d}F_X+$$
$$+\sum_{j=1}^{s-1}((1-\alpha)^{j-1}\alpha\overline{F}_X(jT)r_{MT}^{(s-j)}(t-jT))+$$
$$\{(1-\alpha)^{s-1}\int_{(s-1)T}^{t}\overline{F}_D(t-x)\mathrm{d}F_X\}+$$
$$\{(1-\alpha)^{s-1}\overline{F}_X(t)\}$$

The meaning of the terms in the foregoing expression is as follows. In the first line: the defective state arises in $(0,(s-1)T)$ with no false positive beforehand, the defective state is detected and there is no failure before *t*. In the second line: the defective state arises in $(0,(s-1)T)$ with no false positive beforehand, the defective state is not detected but there is no subsequent failure before t. In the third line: the system is replaced due to a false alarm. In the fourth line: no false alarm occurs before $(s-1)T$, the defective state arises in $((s-1)T,t)$ and there is no failure before *t*. In the fifth line: no false alarm occurs before $(s-1)T$, and no defective state before *t*.

The (operational) reliability of the system under inspection and maintenance is then

$$R_{MT}(t)=r_{MT}^{(s)}(t),\qquad (s-1)T\le t<sT,\quad s=1,....,M.$$

In case that $\alpha=\beta=0$ and $M\to\infty$, the reliability of the system is that in Christer (1987).

We now complete the expression of the reliability for all *t*. If $MT\le t<(M+1)T$

$$R_{MT}(t)=r_{MT}^{(M+1)}(t)=r_{MT}^{(M)}(MT)r_{MT}^{(1)}(t-MT).$$

In general if $(kM+j-1)T\le t<(kM+j)T$ and $j=1,2,...,M$, $k=0,1,2,...$, then

$$R_{MT}(t)=r_{MT}^{(kM+j)}(t)=\left\{r_{MT}^{(M)}(MT)\right\}^k r_{MT}^{(j)}(t-kMT).$$

Note, when false positives do not imply system renewal, they have no consequence on the reliability of the system. Thus $\alpha=0$ leads to the corresponding reliability.

4. NUMERICAL EXAMPLE

To illustrate the model, we consider some data on defects in the contact surfaces of traction motor power switches. On a railway, 30 trains have been operated for a number of years. Broadly speaking, each train is made up of 4 motor cars and 8 trailed cars; in each motor car, each axle is driven by an electric traction motor and there is one power switch (the motor contactor) per motor, making 16 motor contactors per train. The consequence of motor contactor failure (contacts fused closed) is power shutdown to the motor car. Routine checks of contactors have been carried out every 25 days of service. At such an inspection, if the contactor looks overheated then contactor is preventively maintained. Such preventive maintenance may involve the simple replacement of the contact surfaces (these are consumable parts). If a consumable part cannot be easily replaced, then the contactor is replaced, overhauled and goes into the stock of spare contactors. For our purposes here, we shall assume that at a positive inspection (maintainer regards the contactor as overheated) the contactor system is renewed. Figure 1 shows the times to defect arrival for the collection of contactors used on these trains over approximately a two year period. Interpreting these data somewhat liberally, in the sense that an observed defect may correspond to a false positive and the recorded time corresponds to the time of inspection subsequent to a defect arrival, we have data on 392 "defect occurrences". There were also 34 survivals to the time at which the contactors are routinely changed-out for overhaul regardless of their condition; this change-out is scheduled to occur every 1095 days (3 years), and as maintenance does not always follow the schedule some contactors remain in use for longer periods.

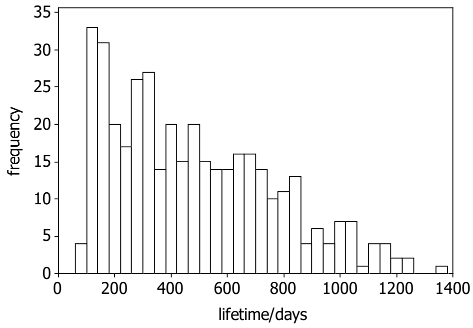


Figure 1. Histogram of observed motor contact "defect" arrival times (days): 392 "defects" and 34 survivals without defect to 1095 days.

The two parameter Weibull distribution ($F(x)=1-\exp\{-(x/\eta)^{\delta}\}$) was fitted to these data using the method of maximum likelihood, accounting for the censored observations. This gave estimates: $\hat{\eta}=614.2$ and $\hat{\delta}=1.57$, with a log-likelihood value of -2832. To consider quality of the replacement process and component lifetime heterogeneity in particular, the distribution of time to defect arrival was modelled also as a simple two-component mixture: $F_X(x)=pF_1(x)+(1-p)F_2(x)$, where $F_1(x)$ is the distribution function of time to defect arrival for a weak sub-population, $F_2(x)$ is the distribution function of time to defect arrival for a strong sub-population, and p is the mixing parameter. (By weak and strong here we mean merely that $\mu_1 << \mu_2$ where μ_1 and μ_2 are the corresponding sub-population means.) Results with Weibull components are: $\hat{p}=0.13$, $\hat{\eta}_1=150.4$ (7.08) and $\hat{\delta}_1=5.51$ (0.81), $\hat{\eta}_2=601.5$ (19.3) and $\hat{\delta}_2=2.08$ (0.11), with a log-likelihood value of -2716. (Standard errors are shown in parenthesis). On the basis of the log-likelihood values, the mixture is a much better fit than the single Weibull. In the numerical study that follows, we use parameter values for the mixture which are very similar to the above. For the delay time distribution (time in the defective state) we have very little information; we therefore arbitrarily choose an exponential distribution with mean 40 days for the base case.

We use cost parameter values as follows: we suppose the unit of cost is the cost of preventive replacement, that is $c_p = 1$; we then set the inspection cost $c_0 = 0.1$ and the cost of failure $c_f = 5$. We consider a range of values of the imperfect inspection parameters and set $\alpha = \beta = 0.2$ in the base case. Results for the cost-rate analysis are shown in Table 4 and Figures 2 and 3. The operational reliability is considered in Figures 4-6.

The inspection of Table 4 reveals some interesting points. Broadly speaking, when the quality of inspections is reduced, less inspection is cost-optimal (as $\alpha \uparrow$, $M^* \downarrow$ and $T^* \uparrow$). Also, there appears to be an interaction between maintenance quality (p) and the probability of a false positive (α): when inspection quality is good, as maintenance quality decreases ($p \uparrow$), much more inspection to mitigate against the higher probability of a defective system is required; however when inspection quality is poor, as maintenance quality decreases ($p \uparrow$) less frequent inspection is cost-optimal. Our interpretation is that this is due to the high probability of replacing a strong component by a weak one after a false positive. Similar effects are observed with respect to the number of inspections and the cost of failure. As $c_f \uparrow$ we might expect the number of inspections to also increase in order to avoid a bigger consequence of failure and this is the case when inspections are perfect. However, when inspections are not perfect, as $c_f \uparrow$ earlier replacement ($T^* \downarrow$) rather than more inspection is the cost-optimal response. Thus, if there is a possibility of mistakes at inspection, still using this response (increasing the number of inspections) could be costly in practice. The problem is that the presence of weak components, as well as the possibilities of false positives or negatives may be overlooked by maintainers. So, decisions that do not take account of these aspects may be harmful for the system, increasing the cost of maintenance and also reducing its reliability.

Figure 2 simply illustrates how the cost-rate varies with T and M in the base case; it provides a check that the results are sensible. Figure 3 illustrates the effect of the maintenance quality parameters upon the cost-rate and the optimum policy. Here we have plotted the cost-rate for the optimum policy at a number of combinations of α, β, and p. The cases with p=0 are identical in Figure 3 because the optimum policy in each case is M^*=1, so that there can be no cost implication of a positive inspection; hence the value of α does not influence the cost-rate. On the other hand, if we consider different maintenance parameter values that result in a large number of inspections in the optimum policy then we see sensitivity to the values of α and β. Thus, we can again see the interesting interaction between parameters that was discussed above in relation to Table 4.

Figures 4 and 5 show the reliability. Figure 4 itself illustrates an interesting point that the cost-optimal policy is also near to reliability-optimal. We claim that this is an effect of component heterogeneity. For homogeneous component replacements the reliability will typically increase with the frequency of preventive maintenance and cost will also increase, and therefore cost and reliability must be traded off (Cavalcante et al., 2010). However, when component reliability is heterogeneous, because of the possibility to introduce a weak component at a preventive replacement, very frequent preventive maintenance may reduce reliability. This effect was first noted by Scarf et al. (2009). In fact, component heterogeneity and imperfect inspection interact in a complex way. For example: when components are homogeneous (p=0), for some fixed policy (say, M=2, T=181) the reliability increases as the probability of a false positive inspection increases (as α ↑) as we would expect (Figure 5a); however, when components are heterogeneous (p=0.13), we can see that reliability is insensitive to α unless the policy changes (Figure 5b).

Table 4. Optimum inspection policy for a range of values of the model parameters. Unit cost is c_p. The base case is highlighted (shaded grey) and as are departures from the base case.

case	*mixture parameters*					*cost parameters*		*inspection parameters*		*mean delay time*	*cost-optimum policy*		
	δ_1	η_1	δ_2	η_2	p	c_f	c_0	β	α	$1/\lambda$	M^*	T^* (*days*)	*cost-rate (per 100 days)*
1	5.5	150	2.5	600	0.0	5	0.1	0.0	0.0	40	1	327	0.537
2	5.5	150	2.5	600	0.13	5	0.1	0.2	0.0	40	2	178	0.673
base	5.5	150	2.5	600	0.13	5	0.1	0.2	0.2	40	2	181	0.696
4	5.5	150	2.5	600	0.13	5	0.1	0.2	0.4	40	1	383	0.702
5	5.5	150	2.5	600	0.13	5	0.1	0.0	0.2	40	2	178	0.681
6	5.5	150	2.5	600	0.13	5	0.1	0.4	0.2	40	1	383	0.702
7	5.5	150	2.5	600	0.13	5	0.1	0.0	0.0	40	2	176	0.658
8	5.5	150	2.5	600	0.25	5	0.1	0.0	0.0	40	9	57	0.749
9	5.5	150	2.5	600	0.0	5	0.1	0.2	0.2	40	1	327	0.537
10	5.5	150	2.5	600	0.25	5	0.1	0.2	0.2	40	3	161	0.820
11	5.5	150	2.5	600	0.13	2	0.1	0.2	0.2	40	1	641	0.368
12	5.5	150	2.5	600	0.13	10	0.1	0.2	0.2	40	1	137	1.034
13	5.5	150	2.5	600	0.13	20	0.1	0.2	0.2	40	1	115	1.217
14	5.5	150	2.5	600	0.13	2	0.1	0.0	0.0	40	1	641	0.368
15	5.5	150	2.5	600	0.13	10	0.1	0.0	0.0	40	9	42	0.919
16	5.5	150	2.5	600	0.13	5	0.025	0.2	0.2	40	2	175	0.659
17	5.5	150	2.5	600	0.13	5	0.4	0.2	0.2	40	1	422	0.759
18	5.5	150	2.5	600	0.13	2.5	0.025	0.2	0.2	40	1	514	0.427
19	3.5	150	2.5	600	0.13	5	0.1	0.2	0.2	40	1	382	0.703
20	2	150	2.5	600	0.13	5	0.1	0.2	0.2	40	1	377	0.702
21	5.5	150	2.5	600	0.13	5	0.1	0.2	0.2	100	2	196	0.584
22	5.5	150	2.5	600	0.13	5	0.1	0.2	0.2	10	1	373	0.765

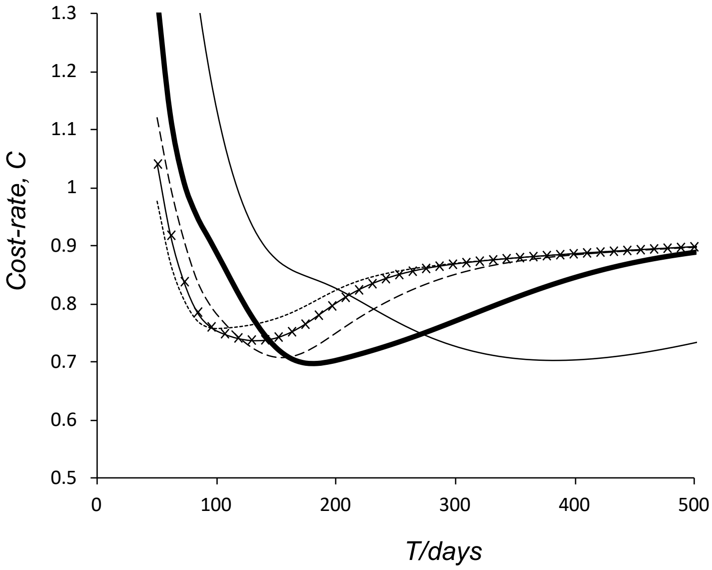


Figure 2. Cost-rate (per 100 days: $C = 100 \times E(C(\tau))/E(\tau)$), as a function of T for M=1 (——); M=2 (▬▬), M=3 (— — —); M=4 (—x—); M=5 (----). Parameter values: p = 0.13, β_1 =5.5, η_1 =150, β_2 =2.5 , η_2 =600, λ =0.025; β = 0.2 , α=0.2, c_f=5 , c_p =1, c_0 =0.1.

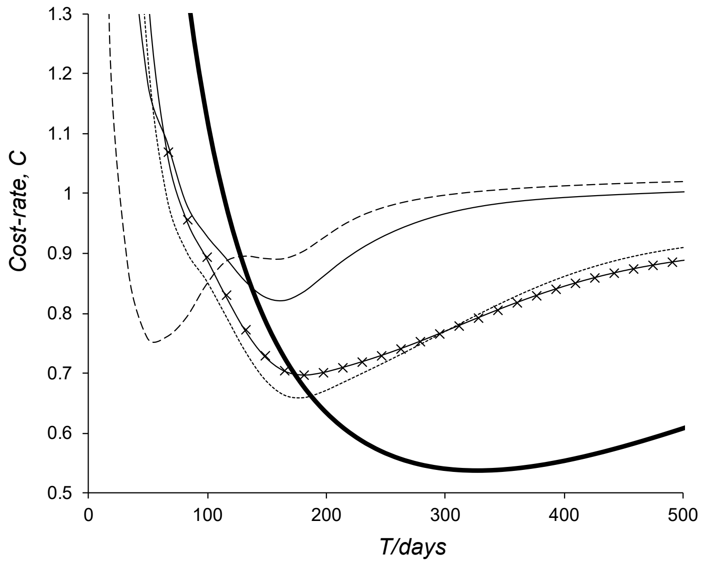


Figure 3. Cost-rate (per 100 days: $C = 100 \times E(C(\tau))/E(\tau)$), as a function of T with M at its optimum value, for (β = 0 , α=0, p=0: M^*=1; ▬▬), (β = 0 , α=0, p=0.13: M^*=2; ----), (β = 0 , α=0, p=0.25: M^*=9; — — —), (β = 0.2 , α=0.2, p=0 as case β = 0 , α=0, p=0—see table 4), (β = 0.2 , α=0.2, p=0.13: M^*=2; —x—), (β = 0.2 , α=0.2, p=0.25: M^*=3; ——). Other parameter values: δ_1 =5.5, η_1 =150, δ_2 =2.5 , η_2 =600, λ =0.025, c_f=5 , c_p =1, c_0 =0.1.

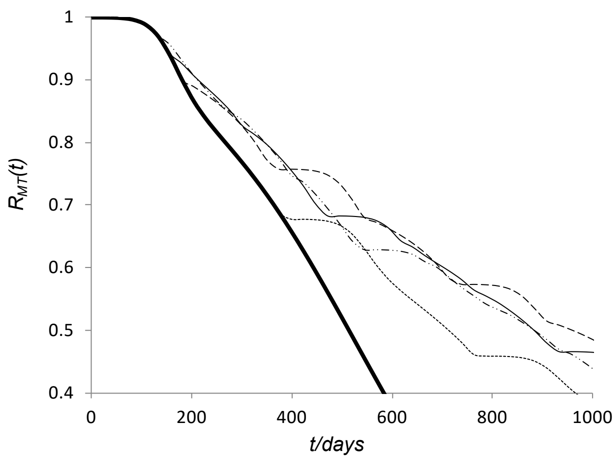
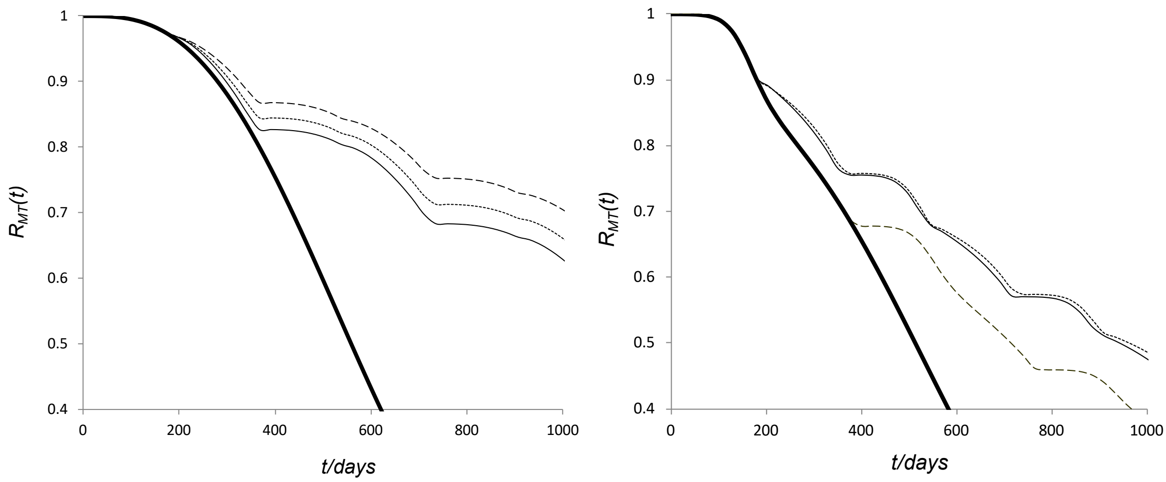


Figure 4. Reliability function, $R_{MT}(t)$ for different M and T. (----- M=1,T=383); (— — — M*=2,T*=181) (cost optimal policy); (——— M=3,T=154); (— ·· — M=4,T=133). Baseline reliability, $R(t)$, (▬▬▬). Parameter values as in the base case.



(a) (b)

Figure 5. Reliability function $R_{MT}(t)$ for various α with M and T at cost-optimal values. (a) p=0: α=0.05 (———); α=0.20 (-----); α=0.40 (— — —); M=2,T=181 in each case. (b) p=0.13: α=0.05 (———) M*=2,T*=178; α=0.20 (-----) M*=2,T*=181; α=0.40 (— — —); M*=1,T*=383. Baseline reliability, $R(t)$, (▬▬▬). Other parameter values as in the base case.

5 DISCUSSION

In this paper we consider a system that is subject to imperfect inspection and replacement. Our purpose is to explore the efficacy of inspection and replacement in circumstances in which they are subject to error. The system may be in one of three states: good, defective or failed, and the system is operational while in the defective state. The purpose of inspection is to prevent failure by allowing the replacement of the system while in the defective phase. However, if inspection is poorly executed then inspection may not be economic. Similarly, for a system that is subject to condition monitoring, if the monitored variable is not a good surrogate for the system state then false positives and false negatives will occur with high probability, and therefore monitoring may be uneconomic. In this way, the model can be used to consider the efficacy of a simple monitoring procedure. Furthermore, in our analysis replacement may not be perfect in that we allow the possibility of the introduction of a weak component at replacement. We then study the effect of such imperfect replacement and poor quality inspections upon the cost and reliability of the system.

We claim that looking at these two aspects of poor quality maintenance has practical implications for maintenance management. Broadly, we observe that if replacement is perfect (there is no possibility of introducing a weak component), the effect of false positives inspection errors is to only increase the cost of maintenance; reliability may improve because false positives imply early replacements. However when there is imperfect replacement, false positives inspection errors and the consequent unnecessary maintenance may not increase reliability in spite of increasing the cost of maintenance. Finally, if there is also the possibility of false negatives inspection errors and poorly executed replacement, then the natural response to these inspection errors (increasing the number of inspections) could be very costly in practice. Of course, the natural response is not necessarily the cost-optimal response. The problem is that the possibility of introducing a weak component at replacement as well as the possibilities of false positives or negatives are overlooked in practice. So, decisions that do not take account of these aspects may be harmful for the system. In fact, a common sense decision may increase the cost of maintenance and also damage the system more than contribute to avoiding failures. Thus, the emphasis in this paper is that the effort and resources used to manage maintenance in order to reduce the consequences of failure when poorly executed could be even more dangerous to the system than the failure mechanism itself, and that the complex interactions between poorly executed replacement and inspection errors require further investigation in general and careful understanding in particular practical cases.

ACKNOWLEDGEMENTS

The work of M.D. Berrade has been supported by the Spanish Ministry of Science and Innovation under project MTM2010-15311. The work of Cristiano A.V. Cavalcante has been supported by CNPq (Brazilian Research Council). Thanks are also due to Barry Tsui. The authors acknowledge the helpful assistance of two anonymous referees.

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