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Probabilistic fatigue life analysis of wind turbine blade composite structures

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PROBABILISTIC FATIGUE LIFE ANALYSIS OF WIND TURBINE BLADE COMPOSITE STRUCTURES

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Doctoral Degree in
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Preface

This thesis was prepared in the Department of Mechanical Engineering of the Technical University of Zaragoza, in fulfillment of the requirements for acquiring a PhD degree. The PhD project was carried out from January 2015 to September 2022 at the Mechanical Engineering department of the University of Zaragoza. The project has been supervised by José Antonio Bea Cascarosa from the University of Zaragoza.

First and foremost it is with gratitude that I acknowledge the support and help of my supervisor at the University of Zaragoza, José Antonio Bea of the Department of Structural Mechanics, for the useful comments, remarks and engagement through the learning process. I would like to thank further the members of the committee for the time they took to go through the manuscript and for their valuable comments.

Most importantly, none of this could have happened without my family. Both my parents from which my father could see me starting the thesis but unfortunately not finalizing it. My family in-law for their endless encouragement through the years and great support.

Finally and most importantly I would like to thank my family. My two little angels Teresa and Marlon Jr. that came to this world during the period dedicated to this thesis. My wife Paula, as her support, encouragement and patience were undeniably important if not essential to finish this thesis. It would be an understatement to say that, combining working on the thesis with both full-time busy jobs and in the mean time building a family including the pregnancies, birth and raising of two of our kids, we have experienced some ups and downs in the past three years. This dissertation stands as a testament to your unconditional love and encouragement.

Probabilistic fatigue life analysis of wind turbine blade composite structures

Abstract

With the increasing length of wind turbine blades together with the use of carbon fiber composite which exhibit an even larger scatter on its properties than glass fiber composites (as test results clearly reveals), fatigue damage is becoming one of the most critical forms of damage potentially occurring in wind turbine blades. Most of the fatigue damage evaluation is still done using deterministic method where high design safety factors as demanded by certification organs are used to cover the uncertainties of inherent randomness in blade loading, material properties, and geometric blade parameters used for blade design.

Probabilistic fatigue life analysis of composite blades gives the ability to actually quantify these uncertainties and use them as part of the analysis input. As deterministic analysis only results in a single reserve factor which does not quantify when the design will fail, probabilistic analysis will give as a result the sensitivity factors of the structure to the different input variables (load, material properties, etc.) and the cumulative distribution function of the reliability showing the probability that a structure (or part) will fail during its entire analyzed life, which can be used for example by Operations and Maintenance (O&M) defining when in the component lifetime the probability of failure is high enough in order to perform more inspections or decide to change a component before catastrophic failure.

This thesis provides a review of the current state of the art of deterministic methods used to analyze the fatigue life of a wind turbine blade subject to fatigue loading, followed by a step-by-step explanation of probabilistic fatigue methods implementation using a finite element based structural model to increase the quality and reliability of the analysis input.

Since structural analysis is mainly performed these days using Finite Element Method (FEM), methods are presented to perform Probabilistic analysis integrated with FEM, using fast probabilistic analysis algorithms (compared to for example Monte Carlo Simulations) that can be used with general purpose finite element programs compute the probabilistic response and the reliability of wind turbine blade structures.

Still the very simplistic Miner's linear damage rule is used as main fatigue damage growth model, combined with rain-flow counting to simplify the fatigue timeseries loads binning them to Markov matrices, thereby losing the sequence of load amplitudes (and their

means) actually occurring. Fatigue damage growth is a rather non-linear process, therefore non-linear damage rules are studied to be practically used to model the complex damage growth process having further the ability to take in account the load sequence effect which actually influence the damage growth evolution process in structures.

A stochastic phenomenological damage evolution model using Markov chains is presented. The statistical prediction of the complete damage process is introduced into the Markov chain reliability formulation, leading to a coherent prediction of the reliability of composite laminates inside a wind turbine.

This study aims to give an overview on how to perform probabilistic fatigue life analysis on wind turbine blades with the main structural components made out of glass and/or carbon fiber reinforced composites. A short status of deterministic fatigue life assessment of wind turbine blades is presented, followed by an overview of probabilistic fatigue life analysis methods using a probabilistic damage model and methods to model the damage accumulation processes.

Análisis probabilístico de la vida fatiga de estructuras compuestas de palas de turbinas eólicas

Resumen

A raíz del aumento en la longitud de las palas de las turbinas eólicas y el empleo en su construcción de los materiales compuestos de fibra de carbono, los cuales muestran una mayor dispersión en sus propiedades que los compuestos de fibra de vidrio tal y como revelan los resultados de las pruebas, el daño ocasionado por la fatiga de los materiales se está convirtiendo en una de las causas más críticas para las palas de los aerogeneradores.

La mayor parte de la evaluación del daño ocasionado por la fatiga de los materiales se lleva a cabo a través de un modelo determinista en el que se emplean elevados factores de seguridad que son exigidos por los organismos internacionales de certificación para cubrir las incertidumbres en la aleatoriedad inherente de la carga aplicada a la pala, las propiedades de los materiales empleados así como los parámetros geométricos empleados en el diseño de las palas.

El análisis probabilista de la fatiga en la vida útil de las palas de los aerogeneradores ofrece la posibilidad de cuantificar estas incertidumbres y utilizarlas como parte del input del mismo.

A diferencia del análisis determinista en el que únicamente se ofrece como resultado un único factor de reserva que no cuantifica/determina cuándo fallará el diseño, el análisis probabilista nos permite obtener como resultado los factores de sensibilidad de la estructura en relación con las diferentes variables e entrada tales como las cargas o las propiedades de los materiales utilizados, así como la función de distribución acumulativa de la fiabilidad que muestra la probabilidad de que una estructura o parte de ella, falle durante toda su vida útil.

Este análisis nos permite emplearlo, por ejemplo, en Operaciones y Mantenimiento (O&M) definiendo en qué momento de la vida útil del componente existe la probabilidad de fallo lo suficientemente alta como para realizar más inspecciones o decidir introducir cambios en los componentes al objeto de evitar un fallo catastrófico.

La presente tesis proporciona una revisión sobre el estado actual de los métodos deterministas utilizados para analizar la vida útil de las palas de las turbinas eólicas sujetas a cargas de fatiga, junto a una explicación paso a paso de la implementación de métodos

de fatiga probabilista utilizando un modelo estructural basado en elementos finitos para aumentar la calidad y fiabilidad del input del análisis.

Dado que el análisis estructural se realiza en la actualidad principalmente mediante el método de elementos finitos, en la tesis se presentan métodos para realizar el análisis probabilista integrado con el método de elementos finitos, utilizando algoritmos de análisis probabilista rápidos en comparación con las simulaciones de Monte Carlo que pueden emplearse con elementos finitos de carácter general al objeto de calcular una respuesta probabilista así como la fiabilidad de las estructuras de las palas de los aerogeneradores.

Todavía se sigue empleando la regla simplista de Miner para evaluar el daño lineal como modelo principal de crecimiento del daño ocasionado por la fatiga de los materiales, combinado con el método rainflow de conteo de ciclos para simplificar las series temporales de cargas en fatiga, agrupándolas para obtener sus matrices de Markov, perdiendo de esta forma la información del orden de la secuencia de cargas. El aumento/crecimiento del daño producido por fatiga es un proceso no lineal, por lo que, se han estudiado las reglas de daño no lineal al objeto de poder ser utilizadas para adecuar el complejo proceso de aumento/crecimiento del daño que tiene en cuenta el efecto de la secuencia de carga influyendo en la evolución del daño causado en las estructuras.

A partir del presente trabajo se presenta un modelo de evolución del fenómeno del daño estocástico utilizando cadenas de Markov basado en el modelo clásico.

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Acronyms

AEP Annual Energy Production

AI Artificial Intelligence

B-model Bogdanoff Markov chain model

BC Boundary Condition

CA Constant Amplitude

CAPEX CAPital EXpense

CC Compression - Compression loading

CD Cumulative Damage

CDF Cumulative Distribution Function

CFRP Carbon Fiber-Reinforced Plastic

CLD Constant Life Diagram

CLT Classical Lamination Theory

DC Duty Cycle

DOF Degrees Of Freedom

CV Coefficient of Variation

ECDF Empirical Cumulative Distribution Function

FE Finite Element

FEA Finite Element Analysis

FEM Finite Element Method

FLH Full Load Hours

FRF Fatigue Reserve Factor

FRP Fiber-Reinforced Plastic

FORM First Order Reliability Method

FOSM First Order Second Moment

GFRP Glass Fiber-Reinforced Plastic

GSM Gram per Square Meter

IS Inner Skin

LB Load Block

LC Load Case

LCF Low Cycle Fatigue

LCOE Levelized Cost Of Energy

LDR Linear Damage Rule

LE Leading Edge

LRSM Linear Residual Strength Model

LSF Limit State Function

MAPE Mean Absolute Percentage Error

MC Markov Chain

MCS Monte Carlo Simulation

MM Method of Moments

MW Mega Watt

MSE Mean Square Error

NDR Non-linear Damage Rule

NRSM Non-linear Residual Strength Model

OPEX OPerating EXpense

OS Outer Skin

PCLD Probabilistic Constant Life Diagram

PDE Partial Differential Equation

PDF Probability Density Function

PET Poly Ethylene Terephthalate

PMF Probability Mass Function

PTM Probability Transition Matrix

PFEM Probabilistic Finite Element Method

PFEMpy Probabilistic Finite Element Method Python code

PSFEM Probabilistic Stochastic Finite Element Method

PSF Partial Safety Factor

PVC Poly Vinyl Chloride

PVW Principle Virtual Work

RFEM Random Finite Element Method

RMSE Root Mean Square Error

RV Random input Variable

RP Random output Parameter

Q4 Quadrilateral 4-noded

QLLL Quadrilateral, biLinear deflection, biLinear rotations and Linear transverse shear strain fields

SAN Styrene Acrylo Nitrile

SC Spar Cap

SFEM Stochastic Finite Element Method

S-N Stress against N cycles to failure curve

SSFEM Spectral Stochastic Finite Element Method

SORM Second Order Reliability Method

SW Shear Web

TC Tension - Compression loading

TT Tension - Tension loading

TE Trailing Edge

TEC Trailing Edge Cap

UD Uni Directional

USA United States of America

VA Variable Amplitude

WTB Wind Turbine Blade

Symbols

a Crack length

b Failure state

b_j Failure at state j

B strain-displacement matrix

Cov Covariance

D Damage

D Constitutive matrix

D_{cum} Cumulative (total) damage

E Expectation (mean)

F_a Load amplitude of cycle

F_m Load mean of cycle

F Force vector

K Global Stiffness Matrix

K_e Element Stiffness Matrix

m S-N curve slope parameter

N_f Number of cycles to failure

P_f Probability of failure

R Load or stress ratio

r Ratio of stay to jump probabilities

S_a Stress amplitude of the load cycle

S_m Mean stress of the fatigue cycle

S_{max} Maximum load of the fatigue cycle

S_{min} Minimum load of the fatigue cycle

S_r Range of the load cycle

UTS Ultimate Tensile Strength

UCS Ultimate Compressive Strength

u Displacement vector

Var Variance

k Weibull shape parameter

λ Weibull scale parameter

θ Weibull location parameter

Document

Chapter 1

Introduction

1.1 Wind turbine blade fatigue

Somewhat three to four decades ago, the blades of the early Kilo Watt (KW) range wind turbines were only a few meters long, straight and very stiff structures on which only basic static analysis were performed to validate the designs. The blades of today's modern multi Mega Watt (MW) wind turbines are now longer than 100 meters for offshore wind turbines. These modern blades are much longer, slender structures designed towards the lowest mass and material cost possible, resulting in very flexible and highly loaded and strained structures [Mølholt Jensen et al., 2013].

These modern wind turbine blades are further subjected to highly stochastic random loading from the wind as well as the cyclic loadings caused by for example the tower shadow effects [Burton et al., 2001] or wave and other ocean motions for offshore turbines [Suzuki et al., 2015],[Zhou et al., 2012], during their operation, which makes fatigue of the structure and its materials critical [Nijssen and Brøndsted, 2013]. Wind turbine blades must be able to survive during its lifetime several orders of magnitude more cycles and load variability than for example cars, aeroplanes, or helicopters, as illustratively shown in Fig. 1.1.

Designers have been for years aware of fatigue failure of blades [Katsaparakakis et al., 2021] and the impact that these fatigue failures can have on the operational cost of a wind turbine [Dao et al., 2019]. Therefore many investigations have been conducted already on fatigue failure and analysis, as by Kensche [2006], Sutherland et al. [1992], Nijssen [2006], van Delft and van Leeuwen [2002], but there is still a lot of room for improvement to achieve more accurate methods to predict when and where these fatigue failures are expected to occur, so both design as well as maintenance schedules can be improved to prevent failure.

Blades of a modern wind turbine are critical components, capturing and transmitting most of the load experienced by the wind turbine. Blades can be complex structural items

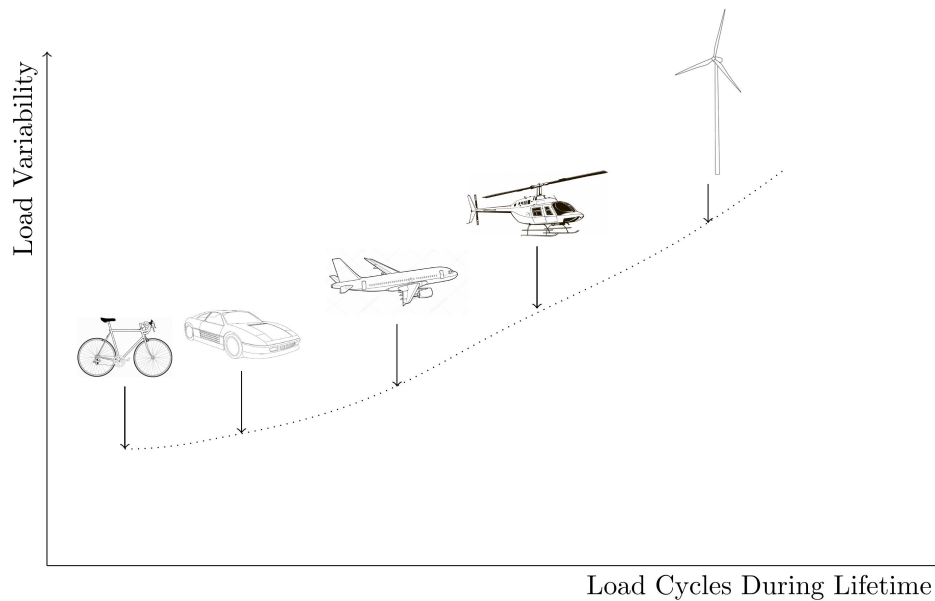


Figure 1.1: Load regimes for different structures

composed of advanced composite material parts. Accurate representation of these features in structural models is vital to simulation of turbine blade response to defined loading conditions. Current trends in wind turbines show that in order to capture more energy for each turbine, the length and swept area of turbine blades are increasing. Current and future blades are using structures that are as stiff and light as possible [Zangenberg and Brøndsted, 2015]. Large turbine blades being developed today are beyond the point of effective trial and error design as has been done during the pioneering years of the past. In addition, design for product reliability is always an important requirement in the overall blade design and manufacturing process. As turbine blades increase in size, modeling tools that enable accurate and reliable assessment of their manufactured condition become increasingly important.

1.2 Research aims

On today's wind turbine blades and other composite structures probabilistic fatigue analysis has still not been widely applied, although its input properties such as loads and material properties are highly stochastic of nature. The aim of this thesis was to gain an understanding of the probabilistic analysis methods and fatigue damage models for composite structures and research how to apply them to analyze the probabilistic fatigue life of wind turbine blades.

1.3 Research objectives

The main objective of this thesis is to explore, formulate and apply probabilistic fatigue life analysis methods to analyze the typical composite laminates of wind turbine blades. The novelty of this work lies on the implementation of probabilistic based finite element methods together with Markov chain based non-linear fatigue damage accumulation models, using custom made computational tools. Different goals have been established to present the formulation, advantages and limitations of the proposed methods.

1. **Investigate methods to perform probabilistic structural analysis to obtain probabilistic model responses:** Hereby using state of the art FE models of a full and detailed part of a wind turbine blade structure. This PFEM using perturbation method, must result in probabilistic model response properties, such as the statistical moments of probabilistic displacement, strain and stress states that can be applied for further probabilistic fatigue life analysis;
2. **Develop tools to perform the probabilistic structural analysis:** Develop tools with a custom programmed FE code to perform sensitivity and probabilistic finite element analysis. The tools can also be used in combination with commercial Finite Element Analysis (FEA) software.
3. **Investigate Markov chain based probabilistic fatigue damage accumulation model:** Damage models which can take into account and model correctly both the non-linear fatigue accumulation evolution, typical for composite materials as well as load sequence effects, so a model that is sensitive on the order of load application; Compare the probabilistic fatigue damage accumulation model against commonly applied and studied linear and non-linear fatigue damage accumulation models for composite laminates.
4. **Define methods and develop tools to apply the probabilistic fatigue damage accumulation models for probabilistic fatigue life analysis:** Program tools to analyze fatigue damage accumulation.
5. **Develop probabilistic fatigue models:** Define a probabilistic version of the deterministic Stress against N cycles to failure curve (S-N) combined for the different mean stress levels in a PCLD, for the analysis of the different stress levels caused by a full wind turbine blade fatigue load spectrum. Identify the key parameters that influence fatigue life, identified in the sensitivity analysis to determine the probabilistic fatigue life.

1.4 Scope

The intention of this thesis is to define and describe all the applied methods in such a way that they can be easily applied and adapted to probabilistic fatigue life analysis to full Wind Turbine Blade (WTB) structures, however due to time constraints some simplifications had to be applied:

- **Structural models:** for WTBs very large detailed models are used that nowadays typically have > 100k nodes and elements and therefore even more Degrees Of Freedom (DOFs), therefore models of small WTB parts are used instead to demonstrate and apply the methods used. Further a custom programmed FE code was applied, which already took quite some time and effort for a simple flat plate to program and validate, including for probabilistic Finite Element Method (FEM) and MCS analysis;
- **Loads:** WTB are subjected to very complex stochastic Variable Amplitude (VA) loads during its more than 20 years of design lifetime. Simplified loads, by representative load blocks at different representative CA cycle load levels were used as analysis loads;

1.5 Thesis outline

The thesis will be divided in 10 different chapters:

In Chapter 2 "State of the art" an overview of the state of the art on Wind turbines, WTB, fatigue analysis methods for WTB and composite materials is presented. Describing the most commonly used fatigue analysis methods and models for the verification of wind turbine blades. After discussing the main shortcomings for a few of the important analysis parts, some new improved methods are proposed and discussed.

In Chapter 3 "Probabilistic Analysis", an introduction and overview is given on probabilistic analysis in general and specific on wind turbine blades, the difference with deterministic analysis and the many benefits of probabilistic analysis with respect to deterministic analysis. An explanation of important analysis terms for probabilistic analysis are given in this Chapter.

In Chapter 4 "PFEM python code (PFEMpy)" a custom Python based FEM code is discussed that has been used for performing the PFEM analysis.

In Chapter 5 "Probabilistic Wind Turbine Blade Data" the analysis input data and models that will be used for the different probabilistic analysis demonstrated in this thesis will be discussed.

In Chapter 6 "Probabilistic FEM analysis on composite plate" a method for performing probabilistic analysis applying PFEM is discussed, which takes into consideration the analysis input uncertainties such as on material stiffness properties and loads. PFEM analysis methods are defined to compute the sensitivities followed by perturbation analysis using first order Taylor series expansions to compute the mean values and variances of the stress tensor measuring the random input variables contribution to the model random output parameters. MCS will be applied to validate the sensitivities and the first moments of the response random variables obtained from the PFEM analysis.

In Chapter 7 "Probabilistic Constant Life Diagram" the application of a probabilistic model PCLD of the deterministic Constant Life Diagram (CLD) using the perturbation method was proposed and demonstrated on a 'shifted' Goodman CLD model.

In Chapter 8 "Markov chain based non-linear fatigue damage accumulation model", a cumulative damage model based on Markov chains as developed by Bogdanoff and Kozin [1985], to perform the fatigue damage accumulation calculation for fatigue life analysis is proposed and discussed.

In Chapter 9 "Probabilistic fatigue life analysis on wind turbine blade", a probabilistic approach to blade fatigue life analysis on on WTB and composite structures in general is presented. The different methods discussed in the previous chapters to perform probabilistic analysis are joined together in order to be able to apply probabilistic fatigue life analysis on composite structural models with block spectrum loading.

Conclusions will be drawn in Chapter 10 "General Conclusions, Recommendations and Future Works" and recommendations on practical use for the proposed probabilistic analysis methods. Perspectives of future developments and applications of the procedure are shortly discussed. Furthermore, some aspects related to the range of applicability of the proposed procedure are discussed.

Chapter 2

State of the art

2.1 Wind turbines

Wind turbines convert the kinetic energy in the wind captured by the rotor blades attached to a hub that is mounted on a cylindrical shaft that bears the mechanical torque applied to the drive train (main shaft/bearing, gearbox), this mechanical energy is converted into electricity by means of the attached generator, followed by the transformer and converter, which are converting the electrical energy generated to the correct voltage and current level to pass it finally to the electricity grid.

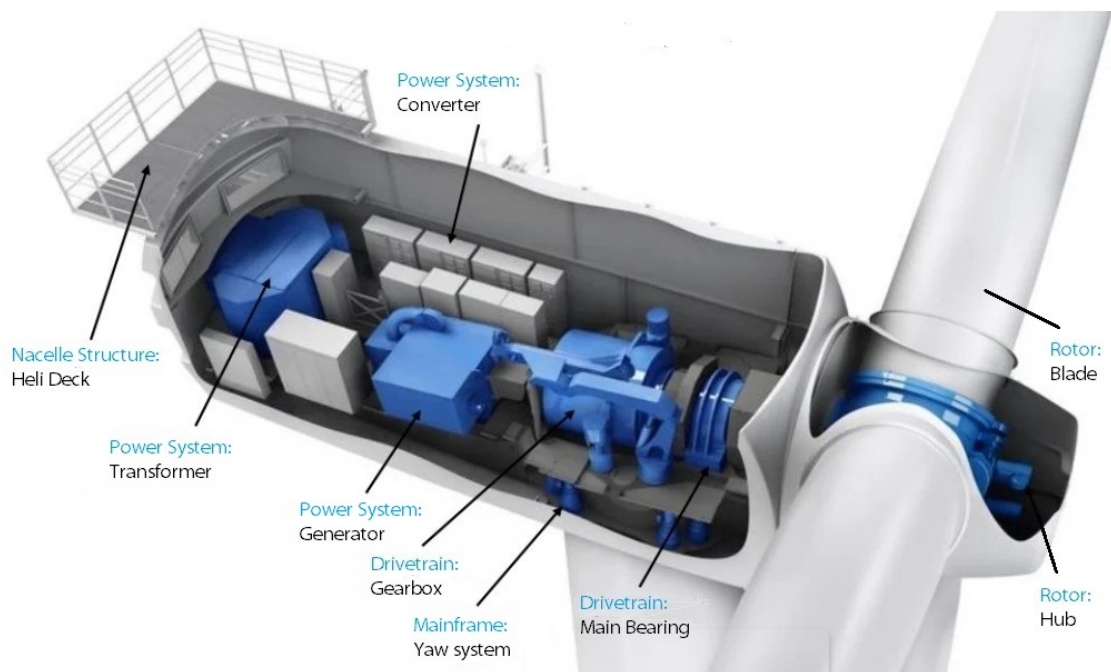


Figure 2.1: Typical wind turbine rotor and nacelle layout

Wind turbines are at the moment the most newly installed renewable energy source

on the planet as clearly visible at the annual new installed energy sources plot shown in Fig. 2.2.

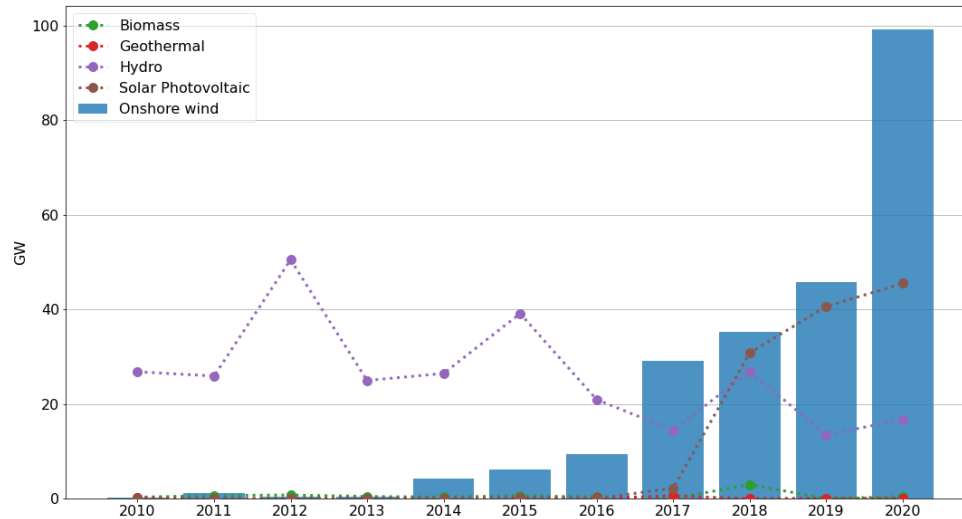


Figure 2.2: Annual new installed renewable power generation capacity 2010-2020 IRENA [2021]

The most common method to compare the cost of energy sources is using the Levelized Cost Of Energy (LCOE) parameter, where the total cost per kWh produced (unit cost) is calculated by discounting and levelizing the CApital EXpense (CAPEX) investment and OPERating EXpense (OPEX) operations and maintenance costs over the lifetime of the turbine, and then dividing them by Annual Energy Production (AEP), see Eq. 2.1;

$$\text{LCOE} = \frac{\text{levelized lifetime costs}}{\text{levelized lifetime energy produced}} = \frac{\text{CAPEX} + \text{OPEX}}{\text{AEP}} \quad (2.1)$$

Wind power LCOE data for the year 2021, see Fig. 2.3, show especially for onshore wind turbines the LCOE was as low as 35 €/MWh, making it one of the cheapest technology for electricity generation today. Offshore wind turbines are more expensive at 73 €/MWh, despite the higher annual mean Full Load Hours (FLH) of up to 4500 hours per year. The energy generation costs of offshore wind are higher because of the more costly installation, operating and financing (on average around 3500 €/kW), [IRENA, 2021].

Wind power is nowadays not only one of the most important renewable energy sources worldwide, but moreover it can cost wise compete very well against fossil fuel, nuclear and also other renewable based energy sources, as shown in Fig. 2.4 for the United States of America (USA), where LCOE for different sources to create electrical energy are compared [LAZARD, 2021].

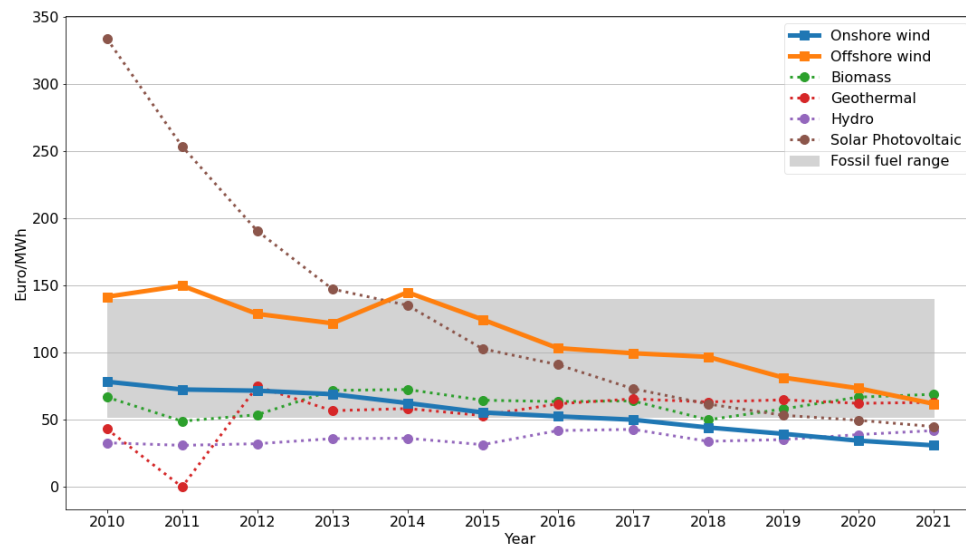


Figure 2.3: Global weighted average utility scale LCOE for Renewable Energy during 2010-2021 [IRENA, 2021]

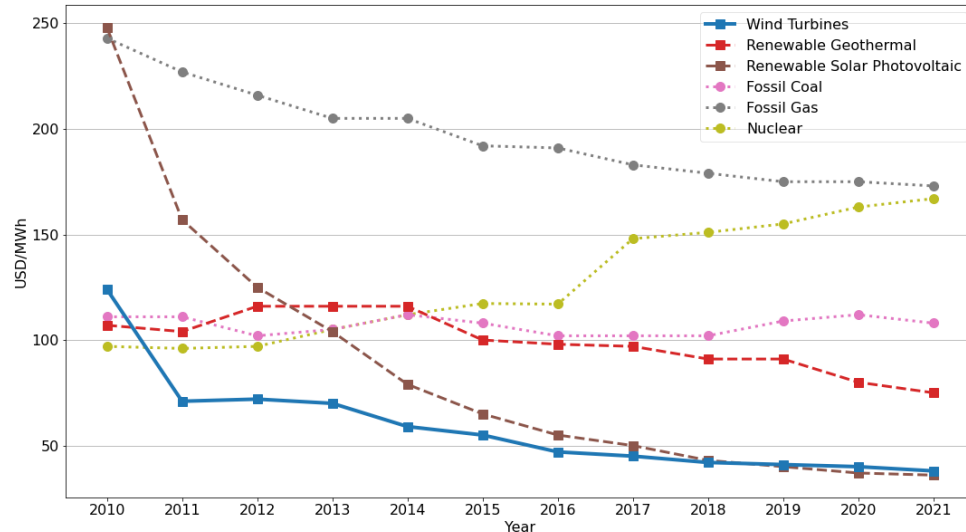


Figure 2.4: Historical mean unsubsidized LCOE values for USA 2010-2021 [LAZARD, 2021]

The LCOE values of Fig. 2.3 and Fig. 2.4 indicate significant historical cost declines for utility-scale wind energy generation for onshore as well as offshore wind turbines driven by, among other factors:

- Decreasing wind turbine cost;
- Improving wind turbine technologies;
- Increased competition within the wind turbine market.

Therefore to keep wind turbine and more specific the WTB composite structure cost competitive in order to compete both against other energy sources as well as inside the wind turbine market, blades have to be designed closer to its structural and material design strength limits, not only for static ultimate extreme load strength but especially also for fatigue load strength. Moreover without compromising on reliability, as the LCOE contains the OPEX cost to run and maintain the turbine, therefore premature failure of the blades before they have reached their design life must be avoided.

To achieve these optimal designed light weight and cost effective blades, new technologies must be utilized for current and future blade designs and moreover the application of improved methodology are required that model and analyze more realistic the very complex Wind Turbine Blade lightweight composite structures to prevent fatigue failure during its lifetime.

2.2 Wind Turbine Blades

2.2.1 Construction And Materials

Due to the strict mass and cost to high strength requirements for wind turbine blade designs, the main structural parts are build up out of laminates, structural cores and joints from Fiber-Reinforced Plastic (FRP) and other polymer materials.

A typical blade structure is build up, as illustrated in Fig. 2.5, out of:

- **Main load carrying parts:**

- **SC:** placed in or connected to each shell side, ideally around the highest profile thickness, these carry the larger part of the main blade loads in the flapwise direction towards the wind. Mainly UD high strength/stiffness fibers are used from glass, carbon or a hybrid between glass and carbon fiber laminates;
- **Shear Web (SW):** One or two (sometimes a 3rd extra) shear webs placed perpendicular on and between the SCs, which carry the shear loads transferred between the SCs and give resistance to sectional compression between the SCs of the two shell halves; Mostly a sandwich from $\pm 45^\circ$ biax laminates placed in the shear load direction with in between sandwich core are used that provide extra buckling resistance to the SW;
- **Trailing Edge Cap (TEC):** placed in each shell side, these carry and give resistance to the edgewise loads by adding stiffness in edgewise direction to the blade. They further strengthen the Trailing Edge (TE) to prevent buckling instability.

- **Aerodynamic shell:** Carries the torsional and shear loads caused by the aerodynamic and gravitational loads.

- **Outer Skin (OS):** Has the gel coat and/or paint applied on top. Mainly build up out of a few layers of $\pm 45^\circ$ biax or $0^\circ \pm 45^\circ$ triax laminates placed in the shear load direction;
- **Sandwich cores:** Are lightweight cores that stabilize the aerodynamic profile when the blade is loaded and provide buckling resistance to the shells. Balsa or PVC or higher density PET are typical used for higher loaded structural blade panels, lower density PET or SAN at slightly lower loaded areas.
- **Inner Skin (IS):** Mainly build up out of a few layers of $\pm 45^\circ$ biax or $0^\circ \pm 45^\circ$ triax laminates placed in the shear load direction, often the same layup as the OS to for with the cores sandwich panels;

- **Blade Root:**

- **Root connection:** to connect the blade to the pitch bearing in the hub, by laminated or bonded studs or a mechanical T-bolt connection.
- **Root barrel laminate:** Starting with a very thick laminate to accommodate the root connection, tapering down towards a thin laminate where the SC, TEC and skin laminates start.
- **Root transition laminate:** In order to transfer smoothly the loads between the round root barrel taper towards the SC, TEC and skin laminates, Typical a mix of 0° and $\pm 45^\circ$ plies are used here, often combined in a TRIAX .
- **Adhesive Bondlines:** to join the separate manufactured structural blade parts, in order to form a single blade structure.

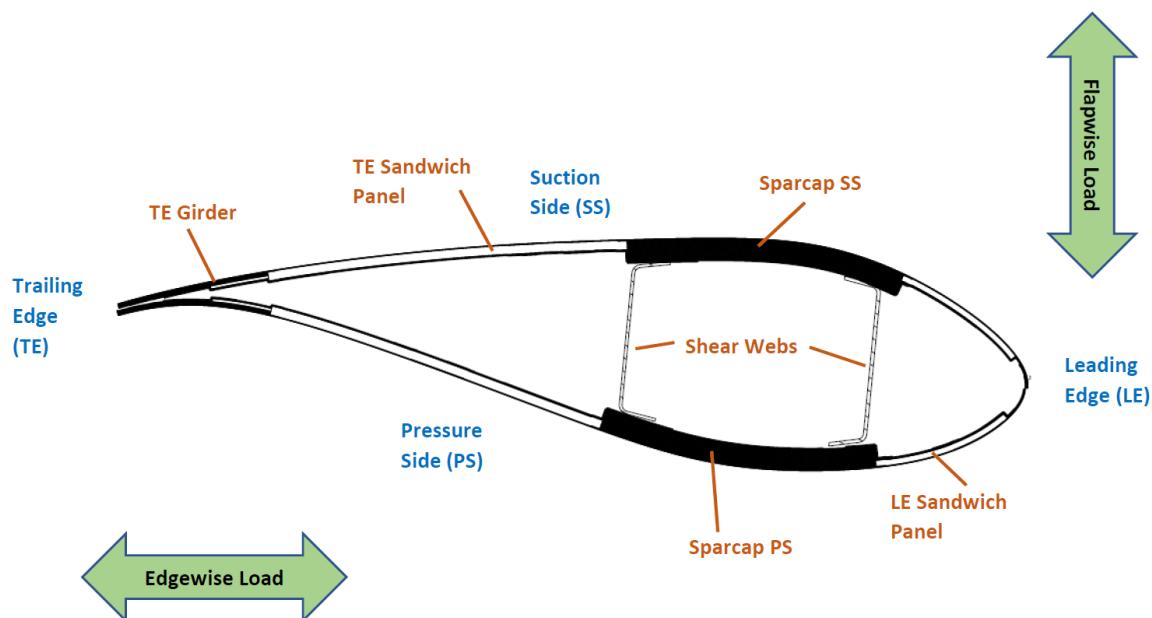


Figure 2.5: Blade structure cross-section overview

FRP laminates used for blades have a advantage that they have a large flexibility in strength specific tailoring, such as variation of the fiber direction, type and content and other parameters, such as matrix, matrix fillers, fiber sizing, stacking sequence, etc. Although this comes at a price compared with for instance metals of a more complex characterization of all the material properties, notably with respect to fatigue properties. A FRP structure represents a convolution of manufacturing processes, micro- and macro geometry, structural loading and component properties and interactions. No two FRP structures are the same, and evaluation of a design is subject to many uncertainties, like fiber orientation, fibre-matrix adhesion, presence of residual stress and strain. The fatigue properties of glass and carbon fiber reinforced FRP structures are in general superior to those of many other materials like metals, this is reflected in their relative 'flat' S-N curve, especially for

carbon fiber, shown in Fig. 2.6. This relative 'flat' S-N curve for composite laminates can be typified as relative robust against fatigue failure, since the sensitivity to fatigue loads drops significantly as the load range drops, but on the other hand this also means that a small increase in load results in substantial fatigue life shortening, as shown in Fig. 2.6.

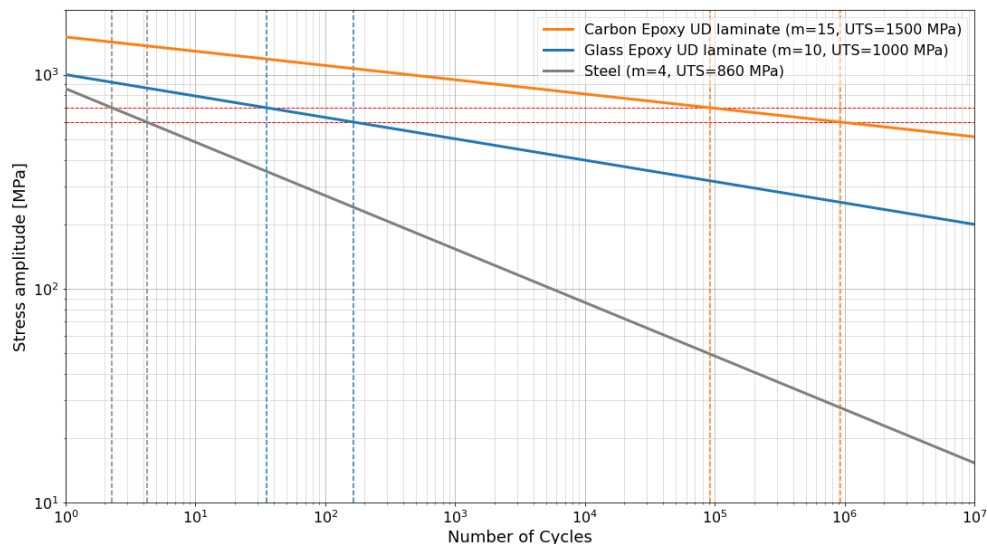


Figure 2.6: Steel, glass/epoxy and carbon/epoxy laminate SN-curves [IEC61400-5, 2020]

2.2.2 WTB loading

The main load directions are in flapwise direction from the main wind loads and edgewise direction from rotational and gravity loading.

Blades twist and bend due to the forces from the wind, thereby altering their aerodynamic properties. The load cases used for blade design and verification analysis are usually derived from time domain simulations of the entire wind turbine, with the relevant loads output as time histories.

2.3 WTB Fatigue life assessment

Fatigue is a localized and progressive process of continuous structural damage accumulation due to the repetitive application of external loading, which may be well below the structural strength capacity.

Fatigue failure is generally considered a critical failure mode for a Wind Turbine Blade (WTB) [Nijssen and Brøndsted, 2013], as it must endure a large number of fatigue cycles

that are highly irregular during its entire design lifetime [Sutherland et al., 1992]. In the early days of fatigue analysis on metal structures like for example trains and bridges, the dimensions of a machine or structure would simply be increased up to a level that the stress would stay below a defined fatigue endurance limit under constant amplitude for the metal used. For modern lightweight structures such as wind turbine blades the amount and cost of the materials used must be kept to a minimum to keep a commercially viable product, resulting in higher design stress and strain levels applied on the structural laminates of a WTB.

Designers have been for years aware of fatigue failure of WTB and the impact that these fatigue failures can have on a wind turbine during its design lifetime. Therefore many investigations have been conducted already on fatigue failure [Nijssen, 2006],[Mandell et al., 1992],[Kensche, 2006] but mostly deterministic analysis, but there is still room for improvement to achieve more accurate methods to predict when and where these fatigue failures are expected to occur, so both design as well as maintenance schedules can be improved to prevent catastrophic failure.

2.3.1 Deterministic fatigue life assessment

Deterministic or empirical approaches, using data to and applying relative simple design rules are still the prevalent methods used for fatigue analysis and design of blades.

Some suitable loads and material properties are assumed and an analysis is then performed to a more or less detailed description of the structure. The main approach is still, as shown in Fig. 2.7, the traditional stress based S-N curve or combined for different load/stress ratios in a CLD model method, where the relationship between the stress amplitude S_a and cycles to failure N_f , is determined by appropriate fatigue experiments that describe these S-N curves. The Palmgren-Miner Linear Damage Rule (LDR) also called Miner's rule [Miner, 1945] extends this approach to calculate the fatigue damage accumulation for VA loading spectrum's. The total fatigue damage D equals to the total sum of damage resulting for each individual stress cycle in a load spectrum. The LDR formula, was initially derived for the simple case of constant amplitude cycling and afterwards extended to the case of more complex fatigue loading. Therefore some of the more fundamental shortcomings of this method of fatigue analysis are the shortcomings of this linear damage model used, as it has been shown that the LDR does not predict failure under variable amplitude loading spectrum very well, as it does not take for example load sequence effects into account [Nijssen, 2006].

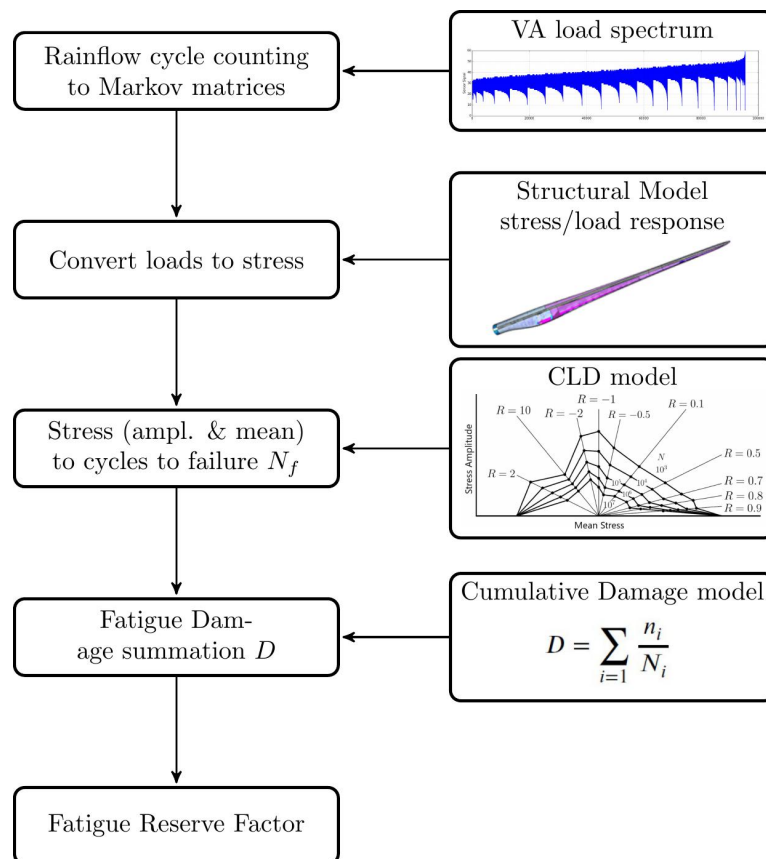


Figure 2.7: Deterministic fatigue life analysis flow

Constant Life Diagram

To incorporate the combined effect of mean stress and material anisotropy and to be able to use a family of S-N curves, a CLD is used that depict the focus of all stress states that produce a given fatigue life [Sendekyj, 2001],[Anastasios et al., 2010]. The CLD reflect the combined effect of S_m (on the horizontal CLD axis) and the S_a (on the vertical CLD axis) on the fatigue life of the material analysed, so a designer can determine quickly and accurately the effects on fatigue lifetime of changes in the stress or strain in a component under design. In metallic fatigue it was frequently assumed that compression stresses were of no significance because they acted only to close fatigue cracks, unlike tensile forces. Diagrams of this kind are available in different form, with the most familiar one being the 'Goodman' diagram, as shown in Fig. 2.8, it is the linear relationship that is associated with the name of Goodman and is often used because of its simplicity.

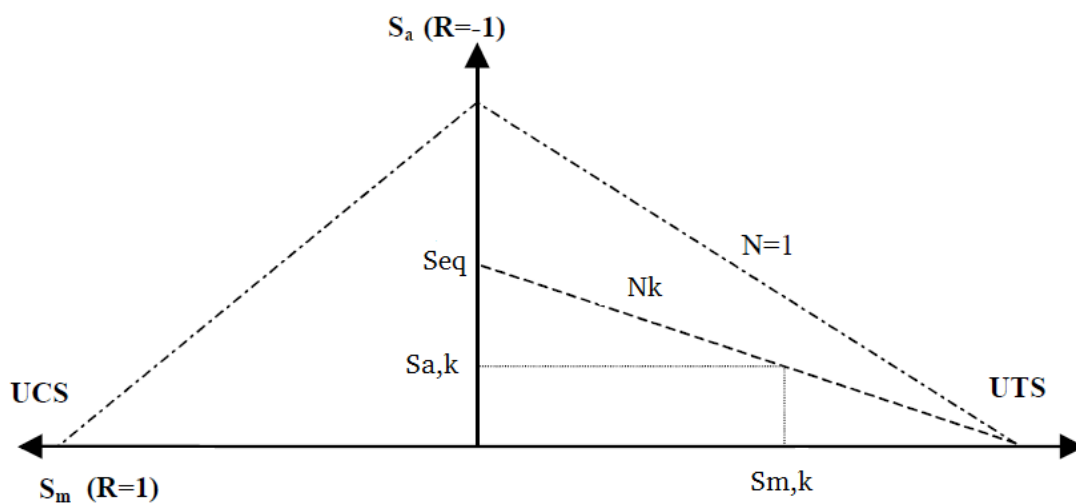


Figure 2.8: Linear Goodman Constant Life Diagram

However, to make it suitable for composite materials the tensile strength part needs to be extended with a separate compressive strengths part that this type of material exhibits. A CLD constructed from test data for different stress ratios R , is for example the piecewise linear model shown in Fig. 2.9, is basically the interpolations of the S-N curves for all the different loading conditions (combination of tension and compression loadings), and therefore offer a predictive tool for the estimation of the fatigue life of the material under load combinations relationship between S_a and S_m for which no experimental data exist.

The main parameters that define a CLD are the stress amplitude S_a , the mean stress S_m , cycles to failure N_f , and the load/stress ratio's R . A typical CLD annotation is presented in Figure 2.10. As shown, the positive ($S_m - S_a$)-half-plane is divided into three sectors, the central one comprising combined tensile and compressive loading. The Tension - Tension

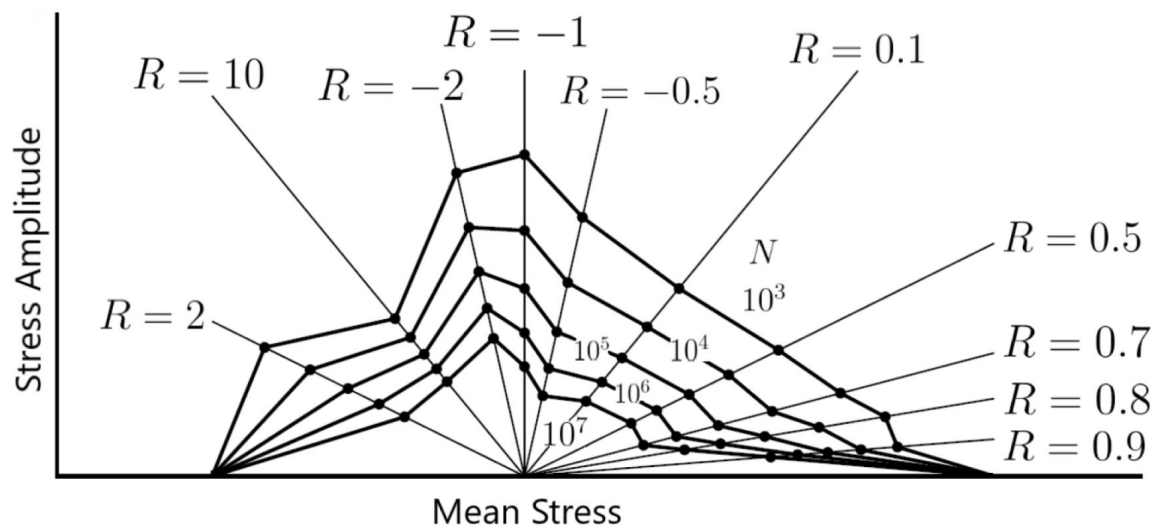


Figure 2.9: Piece-wise linear Constant Life Diagram

loading (TT) sector is bounded by the radial lines, representing the S-N curves at $R = 1$ and $R = 0$, the former corresponding to static fatigue and the latter to tensile cycling with $S_{min} = 0$. S-N curves belonging to this sector have positive R-values less than unity. Similar comments regarding the other sectors can be derived from the annotations shown in Figure 2.10. Every radial line with $0 < R < 1$, in the TT sector, has a corresponding symmetric line with respect to the S_a axis, which lies in the Compression - Compression loading (CC) sector and whose R-value is the inverse of the tensile one, $R = 0.1$ and $R = 10$.

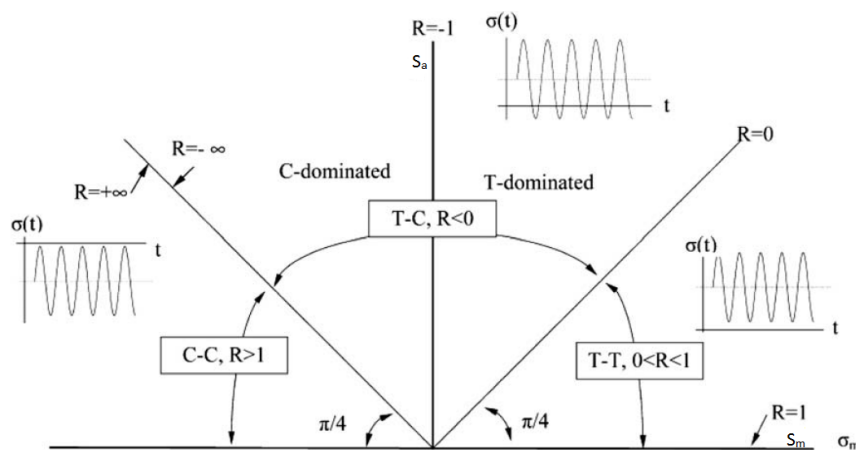


Figure 2.10: CLD definition [Anastasios et al., 2010]

2.3.2 Variability and uncertainty in deterministic analysis

In a deterministic fatigue design verification of a WTB submitted to fatigue loading according to the current design and certification standards, see for example IEC61400-1 [2019], IEC61400-5 [2020], the cumulative damage level D during its fatigue life is calculated, which has to stay below 1 in order not to fail therefore and have a Fatigue Reserve Factor (FRF) higher than 1 to have sufficient safety margin. But what does it actually mean to have a FRF of for example 2.5? How much better is this in for example lifetime than a FRF of 2, further how does it relate to the risk and reliability of your design, and at what cost?

The fact that WTB structures usually operate in hostile and very random service environments, makes it difficult to predict the structural performance. There is a lot of variability and uncertainties associated with WTB material properties, loading, analysis and geometric variables [Toft, 2010], [Stensgaard Toft et al., 2013], [Bacharoudis, 2018].

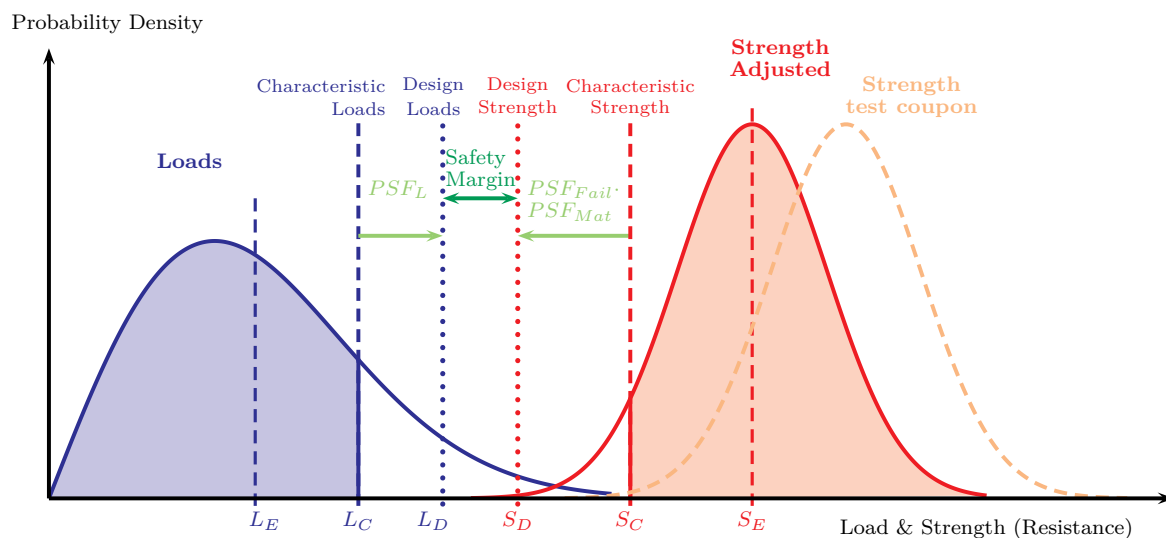


Figure 2.11: Partial safety factors for design

To account for a part of this variability in a deterministic fatigue design verification according to the current design guidelines standards, the following steps need to be taken, as illustrated in Fig. 2.11:

1. Material strength design value:

- (a) Adjust (correct) the material test coupon strength data for scale effects, creep, degradation and defects;
- (b) Apply the characteristic material strength, from the lower bound characteristic

material strength S_C and (stiffness) properties values, for a 95% survival probability with a 95% confidence level;

- (c) Apply Partial Safety Factors (PSFs) for consequence of failure PSF_{Fail} and material strength uncertainty (manufacturing effects) PSF_{Mat} to get the design strength S_D ;

2. Loads design value:

- (a) Apply the upper bound characteristic loads values L_C ;
- (b) Apply the loads with PSF for loads uncertainties PSF_L to get the design load L_D ;

In other words, referring to Fig. 2.11, safe design requires that in a deterministic fatigue approach $L_D < S_D$ with a specified Safety Margin. The allowable strength design method uses PSFs to compute the allowable fatigue strength in members from the adjusted coupon fatigue strength, and successful design ensures that the strength caused by the loads do not exceed the allowable strength.

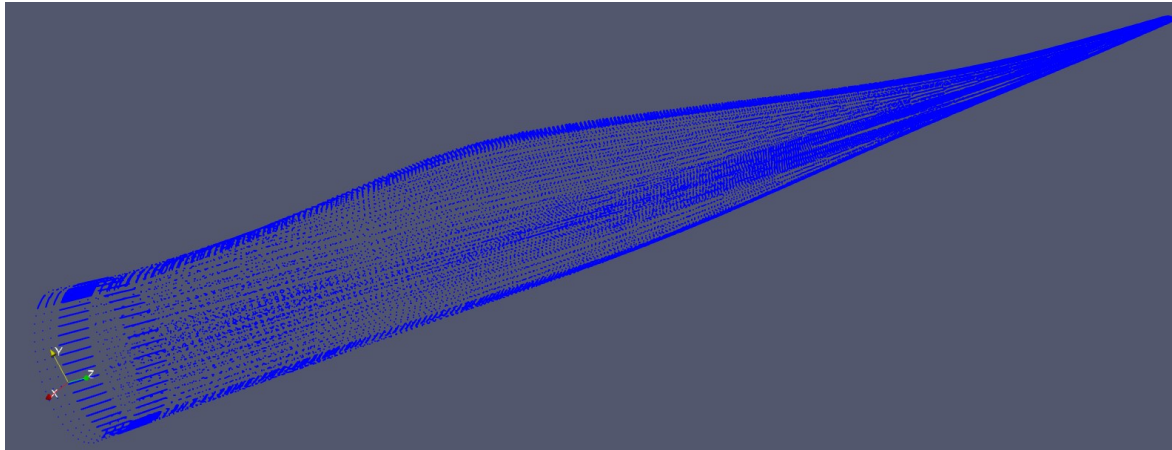
A large disadvantage is that in this way the actual variability and uncertainties are ignored, as the PSFs cannot quantify or predict the exact influences and sources of randomness in a structure [Sriramula and Chryssanthopoulos, 2009], materials and loads.

2.4 Finite element method and analysis

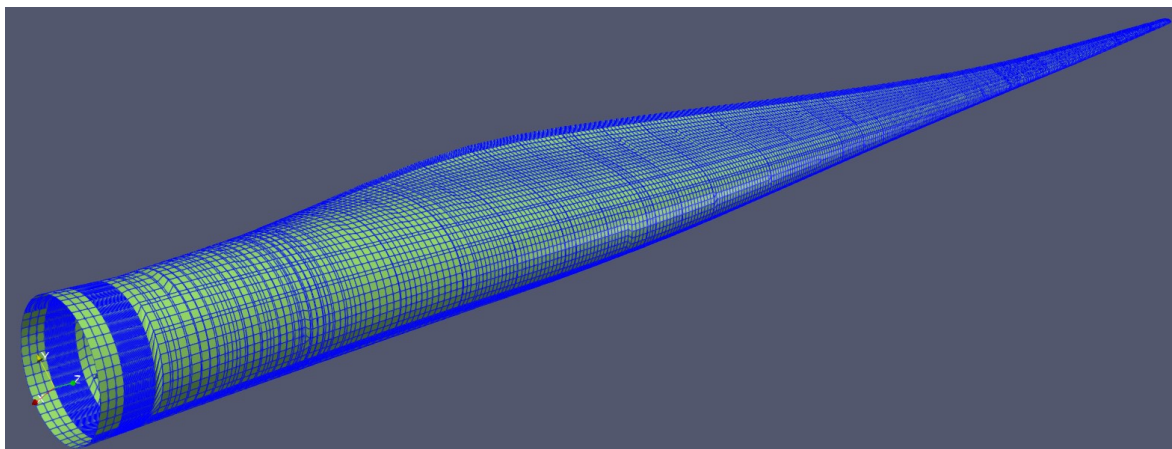
The FEM and FEA are widely used numerical method for solving problems in science and engineering. The adaptive virtue of this method offers simple ways to solve complex problems in structural analysis, heat transfer, fluid mechanics and electromagnetic fields among other applications. The advantages of FEM are well known: it can be applied to complex geometries with mixed material and boundary conditions. It is also suitable for time dependent problems and nonlinear geometric as well as material behavior. However, FEM is deterministic by nature and is therefore limited to describe the general characteristics of a system. In particular, it cannot directly study a system reliably where there exists some degree of uncertainty.

In practice a designer that uses deterministic FEM is typically restricted to the E mean values of loads and material properties applied to a model with an idealized geometry, therefore reducing the physical significance of the model. For substantial variations and randomness these E mean values of the properties of a physical system are only a rough representation of the system.

There is a large amount of literature available on general FEM and FEA, such as Bathe [2010], Zienkiewicz et al. [2013] and Hughes [2000].



(a) WTB full FE model nodes



(b) WTB full FE model element mesh

Figure 2.12: WTB FE model

A typical FE model of a full +70m long Wind Turbine Blade (WTB) for a multi-Mega Watt (MW) wind turbine is shown in Fig. 2.12. The geometric model typically starts with a round root section, followed by thick 30% – 50% airfoils for the round root transition towards the blade maximum chord section, followed by thinner 15% – 30% airfoils for the out-board part towards the blade tip. Typically element mesh sizes of around 100×100mm (or smaller) are used, with most commonly quad shell elements (with some triangle elements for tapering sections) are applied. But also 8 node solid or hybrid solid-shell elements are more and more applied, both to model details such as adhesive bondlines, but also to model sections or even a full WTB, as computer power becomes less of an issue these days.

Chapter 3

Probabilistic Analysis

3.1 Introduction

If Mathematics is the logic of certainty, then probability is the logic of uncertainty. Probability is extremely useful in a wide variety of fields, since it provides tools for understanding and explaining variation, separating signal from noise, and modeling complex phenomena. Probability is important in for example:

- **Life:** Life is very uncertain, and probability is the logic of uncertainty. Although it is impractical to carry out a probability calculation for every decision made in life, thinking about and applying probability on life situations can help us avert some common fallacies, shed light on coincidences, and make better predictions.
- **Physics:** Although Einstein famously said "God does not play dice with the universe", current understanding of quantum physics heavily involves probability at the most fundamental level of nature.
- **Computer science:** Probability plays an essential role in studying randomized algorithms, machine learning, and Artificial Intelligence (AI).
- **Statistics:** Probability is the foundation and language for statistics, enabling many powerful methods for using data to learn about the world.
- **Structural analysis:** Use probability to perform probabilistic structural analysis to obtain probabilistic model responses.

The objective of probability or probabilistic analysis of a structure is to determine the probability of an event occurring during the operational life of a structure, or for certification purposed when using reliability of an event not occurring during its design life.

The fluctuations on the variability of the structural model properties, and uncertainties regarding the analytical models used, contribute to a rather large probability that, when

using only deterministic design approaches, the structure does not perform as it was intended during design. In response to this problem, methods have been developed to deal with the statistical nature of stochastic loads and material properties, and more recently, probabilistic analysis frameworks for comparing and combining these statistical effects have emerged.

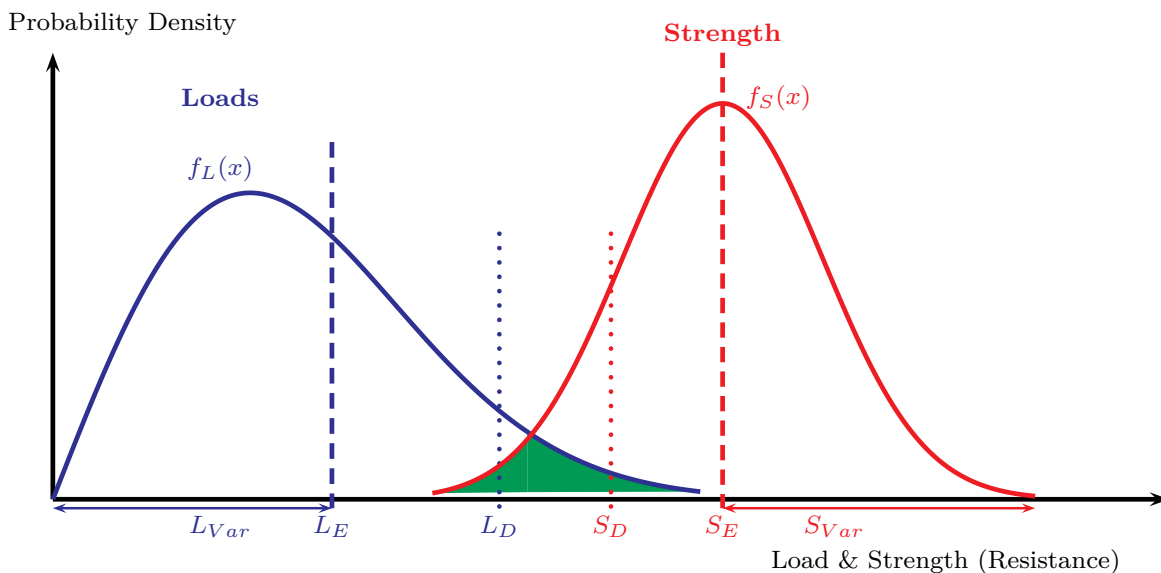


Figure 3.1: Failure probability evaluation

Consider, as shown in Fig. 3.1, two variables: load L , on a structure and the strength (resistance) S of the structure, which are random in nature characterized by their corresponding probability density functions $f_L(x)$ and $f_S(x)$, respectively. In a deterministic approach, the design safety is assured by requiring that the allowable design strength S_D is greater than the maximum design load L_D with a specified margin of safety. The **area of overlap** between the two curves, provides a qualitative measure of the probability of failure P_f , which depends on three factors:

1. **Relative positions of the two curves:** As the distance between the two curves increases, the probability of failure decreases. The positions of the curves may be represented by the Expectation (mean) L_E and S_E of the two variables;
2. **Dispersion of the two curves:** If the two curves are narrow, then the area of overlap and the probability of failure are small. The dispersion is characterized by the Variance (L_{Var} and S_{Var}) of the two variables;
3. **Shapes of the two curves:** represented by the Probability Density Functions (PDFs) $f_L(x)$ (Weibull) and $f_S(x)$ (Normal).

For a safe design, the design variables must be chosen so that the area of overlap between the two curves is as small as possible.

Probabilistic analysis is a good tool to obtain estimates of the failure probability and to quantify the relative influence of variability and uncertainty in all of the input parameters so that the life drivers can be identified. The reliability analysis of structures can be differentiated as deterministic or probabilistic (stochastic) analysis. Where deterministic analysis using a deterministic model is one in which every set of variable states is uniquely determined by parameters in the model and by sets of previous states of these variables; therefore, a deterministic model always performs the same way for a given set of initial conditions. A probabilistic model on the other hand is usually called a "statistical model", where randomness is present and variable states are not described by unique values, but rather by probability distributions.

On the base of probabilistic methodology the following analysis will be realized in this thesis:

- **Probabilistic analysis:** the probability of failure is defined in comparison with the simulated quantities. The statistical second moments mean E , variance Var and the covariance Cov of the variable quantities are calculated analytically or numerically, giving more accurate fatigue life results than deterministic values.
- **Sensitivity analysis:** Sensitivity analysis reveals the physics underlying the system response, so the sensitivity of the input RV analysis parameters, such as the material or loads applied to a model, to the output quantities Random output Parameter (RP) of the system response (displacement, strain, stress). It is used for gradient-based optimization, uncertainty quantification and reliability analysis. In this thesis the focus is on computational sensitivity analysis of mechanical systems modeled using Partial Differential Equations (PDEs) and solved using FEM. Further gradient-based sensitivities are derived to validate the latter.

Probabilistic analysis is starting to be used more frequently for wind turbine and blade design, but so far mainly for the analysis of turbine loads. The more complex theory behind it and algorithms to implement has caused that many attempts failed to implement probabilistic analysis at a larger scale for structural fatigue life analysis. Further to be able to use probabilistic analysis for (type) certification of wind turbine parts, there are challenges to standardize the methodologies and how to cover unknown uncertainties such as human error. In the past computer power was often lacking for large probabilistic simulations, but with today's state of the art computer power and probabilistic analysis methods available, this should not be an issue any more.

3.2 Probability Analysis Terms

Before going into more detail of the probabilistic analysis that are going to be performed, some general terms that are going to be used will be discussed.

3.2.1 Stochastic process

A stochastic process is a mathematical tool used for the modeling of time-dependent random phenomena, where the term "stochastic" means random and "process" refers to the time-evolving status of a given system. Stochastic processes have applications to multiple fields and can be useful anytime one recognizes the role of randomness and unpredictability of events that can occur at random times in, e.g., engineering, physical, biological, or financial system.

Basically, a stochastic process is a time-dependent family X_t of random variables, where t is the time index. The uncertainties on stochastic Random input Variable are modeled by a random variables X_t and random vectors α .

3.2.2 Statistical second moments

The concrete outcome of random modeling lies in the computation of expectations or expected values, which often turn out to be more useful than the probability values themselves. An average or expected lifetime, for example, can be easier to interpret than a (small) probability of default. The expected value E or mean represents the "center" of a random variable, an expected value of an experiment, or the average outcome of an experiment repeated many times,

$$E[X] = X^0 \quad (3.1)$$

The Variance Var of a random variable is a numerical measure of the spread, or dispersion of the PDF of X .

$$\text{Var}[X] = E [(X-E[X])^2] \quad (3.2)$$

The Covariance Cov is a measure of the joint variability of two random variables, for the Covariance Cov of two random variables X and Y follows,

$$\text{Cov}[X, Y] = E [(X-E[X])(Y-E[Y])] \quad (3.3)$$

If X and Y are independent random variables, then their covariance is zero. Further a special case of the covariance is in which the two variables are identical (in which one variable always takes the same value as the other),

$$\text{Cov}[X, X] = \text{Var}[X] \quad (3.4)$$

3.2.3 Failure probability

The failure probability P_f relates to the event occurring within a specified reference period, such as a structures design fatigue life. Where life safety, or where comparisons with failure rates for other types of events or hazards, are being considered the failure probability may be evaluated on an annual basis. Where the failure probability is being evaluated for economic concerns, e.g. risk-based cost-benefit assessments, 'lifetime' re-liabilities may be evaluated for some defined period, like the typical 20 years of design life for a wind turbine. For some types of events, typically temporary conditions, a reference period may be implied by the (short) duration of the event. It is particularly important to ascertain and understand the significance of the reference period when comparing evaluated probabilities with targets. The essence of the reliability problem is the probability integral,

$$P_f = \int_{\alpha|g(\alpha) \leq 0} f_X(\alpha) d(\alpha) \quad (3.5)$$

Where P_f is the probability of failure of an event, $f_X(\alpha)$ is the PDF of the vector of random variables α that represent uncertain quantities that influences the state of the structure by the Limit State Function (LSF) $g(\alpha) \leq 0$ denotes a subset of the outcome space where failure occurs.

Failure probability may also be defined in terms of non-exceedence, or non-failure or safety, therefore called reliability which is then defined as,

$$\text{Reliability} = 1.0 - P_f \quad (3.6)$$

For structural reliability, limit states are generally defined for ultimate failure, but other limit states may also be defined, including serviceability or operability criteria.

The PDF of a random variable α is the function that assigns probabilities to the possible outcomes of α , we get,

$$\text{Pr}(X = \alpha) \quad (3.7)$$

Where X_k are the possible values of RV α .

A continuous failure rate depends on the existence of a failure distribution, $F(\alpha)$, which is a CDF of a random variable α that is the function that accumulates the probabilities from a specified value. The CDF is intimately related to the PDF via our identity for the probability of disjoint events, so the CDF is given by a sum over values of the PDF by.

$$F_X(\alpha) = \sum_{k=1}^n Pr(X \leq \alpha) \quad (3.8)$$

It is important to note that the theoretically calculated failure probability encompasses only those risks and uncertainties that can be controlled or assessed by engineering design and analysis. Additional risks due to gross and human errors are not explicitly designed for, and are therefore not included.

3.2.4 Markov Chains

Markov chains [Norris, 1997] were first introduced in 1906 by Andrey Andreyevich Markov (14 June 1856 – 20 July 1922), who was a Russian mathematician best known for his work on stochastic processes. Markov chains are a widely used way to model random variables evolving over time [Privault, 2018]. They are being applied throughout engineering and social sciences, such as in machine learning, statistical physics, game theory, finance and biology. We can picture a Markov chain intuitively by imagining a system with states and someone randomly wandering around from state to state, where the stationary distribution of the chain helps us understand how the chain will behave in the long run.

A Markov chain is a type of Markov process, discrete in time and damage states, that satisfies the Markov property, i.e., the memory less property of a stochastic process. This means that the future states of the process (conditional on both past and present states) depends only upon the present state, not on the sequence of events that preceded it.

For a Markov chain, given the present, the past and the future are conditionally independent, as Markov chains have a form of one-step dependence. Markov chains "live" in both space and time: the set of possible values of the X_n is called the state space, and the index n represents the evolution of the process over time. The state space of a Markov chain can be either discrete or continuous, and time can also be either discrete or continuous. We will focus in this thesis entirely on discrete-state, discrete-time Markov chains, with a finite state space. Specifically, we will assume that the X_n take values in a finite set.

A discrete-time Markov chain is a sequence of random variables $X_1, X_2, X_3, \dots$ with the Markov property, namely that the probability of moving to the next state depends only on the present state and not on the previous states,

$$P(X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \dots, X_0 = i_0) = P(X_{n+1} = j \mid X_n = i) \quad (3.9)$$

The quantity $P(X_{n+1} = j \mid X_n = i)$ is called the transition probability from state to state. In this thesis, when referring to a Markov chain we will implicitly assume that it is time-homogeneous, which means that the transition probability is the same for all times.

The condition from Eq. 3.9 is called the Markov property, and it says that given the entire past history, so $X_0, X_1, X_2, X_3, \dots, X_n$, only the most recent term, X_n , matters for predicting X_{n+1} . If we think of time as the present, times before as the past, and times after as the future, the Markov property says that given the present, the past and future are conditionally independent. The Markov property greatly simplifies computations of conditional probability, as instead of having to condition on the entire past, we only need to condition on the most recent value.

To describe the dynamics of a Markov chain, we need to know the probabilities of moving from any state to any other state, that is, the probabilities $P(X_{n+1} = j \mid X_n = i)$ on the right-hand side of the Markov property. The probabilities associated with various state changes are called transition probabilities. The process is characterized by a state space, a transition matrix describing the probabilities of particular transitions, and an initial state (or initial distribution) across the state space.

This information can be encoded in a matrix, called the Probability Transition Matrix (PTM), whose entry is the probability of going from state i to state j in one step of the chain. Note that the PTM is a non-negative matrix in which each row sums to 1.

The Markov chain type that will be applied in this thesis is a $b \times b$ unit-jump Markov chain, which has $b - 1$ state-dependent transition states of the form, so for the PTM matrix follows,

$$\mathbf{P} = \begin{bmatrix} p_0 & q_0 & 0 & 0 & 0 \\ 0 & p_1 & q_1 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & p_{b-1} & q_{b-1} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (3.10)$$

Where conditional probability p represents the probability of remaining in the damage level and $q = 1 - p$ the probability of jumping to the next adjacent damage level. The PTM only has the diagonal terms consisting of p 's and q 's, with the ratio of transition probabilities r is $r = p/q$.

3.3 Probabilistic analysis methods

Once all the random quantities to be taken into account have been characterized, an appropriate computational or analytical stochastic analysis method that corresponds to the level of randomness of the input parameters needs to be used. In order to evaluate the random fatigue life, some of the most commonly used probabilistic methods for estimating failure probability are:

- **First Order Reliability Method (FORM):** The FORM Concept as developed by Hasofer and Lind [1974] is based on a description of the reliability problem in standard Gaussian space. Hence transformations from correlated non-Gaussian variables to uncorrelated Gaussian variables with zero mean and unit variance are required. The FORM algorithms compute relative fast but as penalty has a lower accuracy. FORM typically is applied when the tail of the response PDF is of interest.
- **Second Order Reliability Method (SORM):** The second-order methods, Der Kiureghian [2000] improve the accuracy of first-order probability estimates by including curvature information at the beta-point, and approximating the failure surface by a quadratic surface. The difference between the first- and second-order estimates of the probability gives an indication of the curvature of the failure surface. SORM is also typically applied when the tail of the response PDF is of interest.
- **Linear perturbation (using Taylor series):** Uses a Taylor series expansion of the quantities involved in the equilibrium equation of the system around their mean values. More details are explained in subsection 3.3.1 "Perturbation method using Taylor series". Linear perturbation is applied when the second moments so the expected (mean) and variance of the response are of interest.
- **Monte Carlo Simulation:** Repeated random sampling is computed to obtain numerical results. Monte Carlo methods are widely used in engineering for quantitative probabilistic analysis in process design Thomopoulos [2013]. MCS is typically applied when the entire PDF is of interest. More details are explained in subsection 3.3.2 "Monte Carlo Simulation".
- **Spectral stochastic method:** Using Karhunen-Loève or polynomial chaos expansions Ghanem and Spanos [2003] to model the RV parameters and the RP are approximated by the polynomial chaos expansion. Is an alternative stochastic approach but is mainly concerned with representing the random material properties of a structure. The representation of the random parameters in this form seeks to reduce the computational power used in other methodologies such as MCS. To increase the efficiency of spectral stochastic method the solution space is mapped with Fourier-type

series; This method is typically applied when the second moments so the expected (mean) and variance of the response are of interest.

The Linear perturbation using Taylor series expansions will be the method explored in this thesis together with the MCS as method to compare and validate the results, both will therefore be discussed in more detail below.

3.3.1 Perturbation method using Taylor series

In perturbation theory, the solution is expressed as a power series known as a perturbation series in some "small" parameter (here called α) that quantifies the deviation from the exactly solvable problem [Holmes, 2010]. An important feature of the technique is a middle step that breaks the problem into solvable and perturbative parts. An approximate perturbation solution is obtained by truncating the series, usually by keeping only the first or first two terms. The solution to the solvable known problem A_0 and the first-order A_1 and second-order A_2 perturbation terms then becomes,

$$A = A_0 + \alpha^1 A_1 + \alpha^2 A_2 \quad (3.11)$$

The perturbation method applied in this thesis uses Taylor series expansions as power series of a function which is an infinite sum of terms that are expressed in terms of the function's derivatives at a single point. Taylor series are named after Brook Taylor, who introduced them in 1715. Taylor series expansions are approximations of an analytical function by mean of power series, of the quantities involved in the equilibrium equation of the system around their mean values. Then the coefficients in the expansions of the left- and right-hand sides are identified and evaluated by perturbation analysis.

The Taylor series of a function $f(\alpha)$ that is infinitely differentiable at number α results based on Eq. 3.11 in the power series,

$$f(\alpha) = f(\alpha_0) + \frac{f'(\alpha_0)}{1!}(\alpha - \alpha_0) + \frac{f''(\alpha_0)}{2!}(\alpha - \alpha_0)^2 + \frac{f'''(\alpha_0)}{3!}(\alpha - \alpha_0)^3 + \dots, \quad (3.12)$$

Where $n!$ denotes the order of the series expansion (factorial of n). Eq. 3.12 can be written more compact as,

$$f(\alpha) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\alpha_0)}{n!}(\alpha - \alpha_0)^n, \quad (3.13)$$

where $f^{(n)}(\alpha)$ denotes the n -th derivative of f evaluated at the point α . For up to a second order Taylor series expansion we then get,

$$f(\alpha) = f(\alpha_0) + f^1(\alpha_0)(\alpha - \alpha_0) + \frac{1}{2}f^2(\alpha_0)(\alpha - \alpha_0)^2 \quad (3.14)$$

Taylor series are a particularly useful tool because if the function is smooth enough to let us expand about the point $\alpha = \alpha_0$, then any one of the resulting Taylor polynomials can be used as an asymptotic expansion. Moreover, Taylor series enables us to analyze the error very easily.

As uncertainties in the input parameters RV are modeled by a random vector α , from which the response becomes a random variable X ,

$$X = \mathcal{M}(\alpha) \quad (3.15)$$

Where $\mathcal{M}$ is the the model function.

Taylor series expansion of model

The Taylor series expansion up to second order for this model response quantity $\mathcal{M}(\alpha)$ around a random value α_i^0 is,

$$\begin{aligned} \mathcal{M}(\alpha) \approx & \mathcal{M}(\alpha^0) + \sum_{i=1}^n \frac{\partial \mathcal{M}}{\partial \alpha_i} \Big|_{\alpha_i=\alpha_i^0} (\alpha_i - \alpha_i^0) \\ & + \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 \mathcal{M}}{\partial \alpha_i \partial \alpha_j} \Big|_{\alpha_i=\alpha_i^0} (\alpha_i - \alpha_i^0)(\alpha_j - \alpha_j^0) + \mathcal{O}(\|\alpha - \alpha^0\|^2) \end{aligned} \quad (3.16)$$

In Eq. 3.16, α_i denotes the i -th coordinate of vector α and $\mathcal{O}(\cdot)$ means "negligible with respect to $(\cdot)$ ".

With Taylor series expansions, in theory one should be able to obtain as accurate a result as desired by simply adding together enough terms, but this has a penalty on the computational cost to solve the Taylor series expansion, as higher order derivatives need to be solved. For the perturbation based method using Taylor series applied in this thesis only first order truncated expansions will be used, so for continuation only approximations of first-order expansion will be demonstrated. Due to the first-order accuracy used to compute the statistical second moment i.e. the mean E and Variance Var , this method is therefore called First Order Second Moment (FOSM).

The first order truncated Taylor series expansion from Eq. 3.16 becomes,

$$\mathcal{M}(\alpha) \approx \mathcal{M}(\alpha^0) + \sum_{i=1}^n \mathcal{M}_i^I (\alpha_i - \alpha_i^0), \quad \mathcal{M}_i^I = \frac{\partial \mathcal{M}}{\partial \alpha_i} \Big|_{\alpha_i=\alpha_i^0} \quad (3.17)$$

The E value of the first-order expansion for the variable distribution of the model response quantity $\mathcal{M}(\alpha)$ can then be written as:

$$E[X] = E[\mathcal{M}(\alpha)] \approx \mathcal{M}(\alpha^0) + \sum_{i=1}^n \mathcal{M}_i^I E[\alpha_i - \alpha_i^0] \quad (3.18)$$

By definition, $E[\alpha_i - \alpha_i^0] = 0$, so Eq. 3.18 reduces to the approximation of the mean value,

$$E[\mathcal{M}(\alpha)] \approx \mathcal{M}(\alpha^0) \quad (3.19)$$

The Covariance matrices of the model function vector can then be written as,

$$\begin{aligned} \text{Cov}[\mathcal{M}(\alpha_i), \mathcal{M}(\alpha_j)] &\approx \sum_{i,j=1}^n \mathcal{M}_i^I (\mathcal{M}_i^I)^T \text{Cov}[\alpha_i, \alpha_j] \\ \mathcal{M}_i^I &= \left. \frac{\partial \mathcal{M}}{\partial \alpha_i} \right|_{\alpha_i=\alpha_i^0}, \quad \mathcal{M}_j^I = \left. \frac{\partial \mathcal{M}}{\partial \alpha_j} \right|_{\alpha_j=\alpha_j^0} \end{aligned} \quad (3.20)$$

For independent random variables the variance **Var** matrices of the displacement vector which indicates the spread of the distribution (the square of the standard deviation), the variance **Var** matrix is the diagonal of the covariance **Cov** matrix, therefore the variances vector for a first order Taylor series may be written as,

$$\begin{aligned} \text{Var}[\mathcal{M}(\alpha_i), \mathcal{M}(\alpha_j)] &\approx \sum_{i,j=1}^n \text{diag}(\mathcal{M}_i^I) (\mathcal{M}_i^I)^T \text{Cov}[\alpha_i, \alpha_j] \\ \mathcal{M}_i^I &= \left. \frac{\partial \mathcal{M}}{\partial \alpha_i} \right|_{\alpha_i=\alpha_i^0}, \quad \mathcal{M}_j^I = \left. \frac{\partial \mathcal{M}}{\partial \alpha_j} \right|_{\alpha_j=\alpha_j^0} \end{aligned} \quad (3.21)$$

In case of independent input variables, using Eq. 3.21 with Eq. 3.4, further can be simplified into,

$$\text{Var}[\mathcal{M}(\alpha_i)] \approx \sum_{i=1}^n (\mathcal{M}_i^I)^2 \text{Var}[\alpha_i], \quad \mathcal{M}_i^I = \left. \frac{\partial \mathcal{M}}{\partial \alpha_i} \right|_{\alpha_i=\alpha_i^0} \quad (3.22)$$

3.3.2 Monte Carlo Simulation

Monte Carlo Simulation MCS is a powerful tool for evaluating the probability of failure, for sensitivities analysis of example load effects combinations and for calculating statistical measures of the response random variables of a system [Thomopoulos \[2013\]](#).

In structural engineering, if at least one of the input variables is random, a stochastic system is obtained, in which all the responses will be also random. For this type of systems, the calculation of the exact probabilistic solution by analytical methods can only be achieved in most simple situations, therefore MCS plays a significant role in the evaluation

of data available from experiments, and it is a powerful tool for evaluating the probability of failure and for calculating statistical measures of the response random variables of a system. The method is computational costly, however the fast development of high speed computers occurring continuously nowadays makes the application of the method easier in order to assess the random response of stochastic systems.

Monte Carlo simulation will be used in this thesis in order to estimate the mean value E and the variance Var of the response quantity $X = \mathcal{M}(\alpha)$. Assume that a sample set of n input vectors has been generated, then the expected value E so the average outcome of an analysis/simulation repeated n times of Eq. 3.1 becomes,

$$E[X] = \frac{1}{n} \sum_{k=1}^n X^0 \quad (3.23)$$

$$E[\mathcal{M}(\alpha)] = \frac{1}{n} \sum_{k=1}^n \mathcal{M}(\alpha^0) \quad (3.24)$$

The variance Var of a random variable is a numerical measure of the spread, or dispersion of the PDF of X , so the variance Var outcome of an analysis/simulation repeated n times of Eq. 3.2 becomes,

$$Var[X] = \frac{1}{n-1} \sum_{k=1}^n (X - E[X])^2 \quad (3.25)$$

$$Var[\mathcal{M}(\alpha)] = \frac{1}{n-1} \sum_{k=1}^n (\mathcal{M}(\alpha) - \mathcal{M}(\alpha^0))^2 \quad (3.26)$$

The statistics computed on sample sets are random quantities in nature. In probabilistic engineering mechanics, it is common practice to run a single Monte Carlo simulation (possibly using a large size sample set), i.e. to provide a single realization of the estimator. A rigorous use of Monte Carlo simulation should provide confidence intervals on the results though.

The MCS is the most simple and direct approach and with the increase in the number of samples, the accuracy of the MCS improved gradually. However these improvements in the accuracy comes always at the cost of increased computational power and time, especially with systems that have great variability and involve complex models that include several random variables. Therefore MCS cannot be effectively applied to large-scale engineering problems, such as for example complex FE models of a WTB.

Even with this disadvantage, MCS is widely accepted, especially in the research community to validate other methods as will be done in this thesis to compare the MCS and validate against the PFEM Perturbation method, so to analyze the first moments (mean E ,

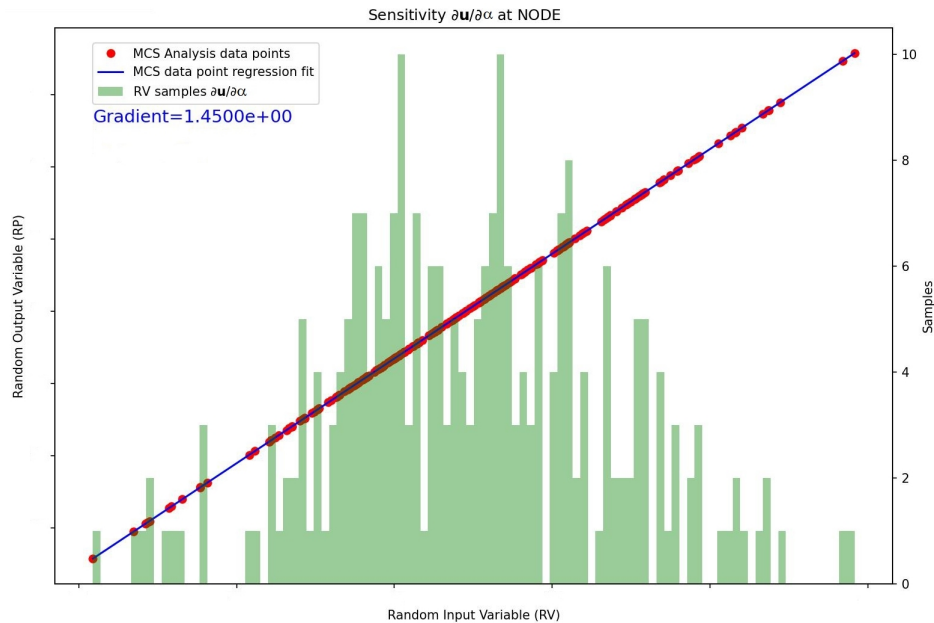


Figure 3.2: Sensitivity from gradient of regression line on data points

Variance Var) and the sensitivities. The sensitivities from the MCS analysis are defined by the gradient of the regression line that fits the RV with respect to the RP data points, as shown in Fig. 3.2. A crude MCS method is applied based on random number generation to obtain samples from a required distribution, where a large number of random variables are generated as samples and for each of the outcomes.

3.4 Probabilistic finite element method (PFEM)

The classic FEM methods have been combined with probabilistic and stochastic methodologies to create a new type of analysis to study stochastic systems with random variations and/or uncertainties in parameters [Kleiber and Hien, 1992],[Dasgupta, 1992]. To represent the stochastic nature of a system, random fields are introduced to the classic FEM to capture and create different stochastic scenarios. The influence of the random fluctuations is evaluated by calculating the statistical information of the response variables and evaluating the probability of an outcome of the system. Several variants of probabilistic and stochastic based FEM have been developed combining FEM with one of the probabilistic methods discussed in section 3.3 "Probabilistic analysis methods", with the most common used being the MCS, PFEM using perturbation method with Taylor series expansions and the Spectral Stochastic Finite Element Method (SSFEM) [Ghanem and Spanos, 2003]. Each of these methods adopts a different approach to represent, solve and study the randomness of a system. The main advantage of SSFEM is that the complete probability distributions of the RPs can be provided. But unfortunately, with the increase in the

number of terms, the computational burden of SSFEM increases sharply. This is why the application of SSFEM has been so far limited to the stochastic system with a small number of degrees of freedom. Furthermore, the SSFEM application range strongly depends on the adopted approach to calculate the coefficients of expansion.

The PFEM is an extension of the FEM that incorporates random parameters for one or more of the main classic FEM components such as the geometry, external forces and material properties. PFEM combines FEM with statistics and reliability methods and can be applied to linear and non-linear structural mechanics problems and fracture mechanics problems. The most common PFEM methods are Neumann expansions based method [Yamazaki et al., 1988] and the perturbation based method [Bea and Doblare, 2002] and [Pérez et al., 2006]. Where the first method is similar to MCS as it needs a sufficient large sample size to create accurate results and therefore computational costly. The perturbation based method [Kleiber and Hien, 1992] on the other hand doesn't need a sample set generation, therefore computes much faster and because its relative simplicity making it the preferred method PFEM has been applied by many researchers including Hisada and Nakagiri [1985], Handa and Kärholm [1975] for structural mechanics and Sakata et al. [2008], Sakata and Torigoe [2015] and Zhou et al. [2016] specific for composite materials.

3.4.1 PFEM Perturbation using Taylor series

The PFEM using perturbation method with Taylor series expansions, as discussed in subsection 3.3.1 "Perturbation method using Taylor series", will be applied in this thesis to introduce stochastic randomness [Bea and Doblare, 2002],[Bea et al., 1999], in order to solve and determine:

1. **Sensitivities:** of the input RV analysis parameters of the structural model (material and loads) with respect to the output quantities RP of the system response (displacement, strain, stress). This will be discussed in this section and demonstrated in Chapter 6 "Probabilistic FEM analysis on composite plate".
2. **Statistic second moment:** E mean and Var variance of the response variables (displacement, strain, stress). This will be discussed in this section and demonstrated in Chapter 6 "Probabilistic FEM analysis on composite plate".;

In the context of finite element analysis for quasi-static linear problems, the equilibrium equation obtained after discretizing the geometry in a standard FEA can be written as,

$$\mathbf{K}\mathbf{q} = \mathbf{F} \quad (3.27)$$

The solution of the system from Eq. 3.27 yields the unknown element nodal displacement $\mathbf{q}$.

The global stiffness matrix $\mathbf{K}$, depends upon random variables α_i , so the model of the random static system and its derivatives with respect to α_i , produced by the finite element method can be written as,

$$\mathbf{K}(\alpha_i)\mathbf{q}(\alpha_i) = \mathbf{F}(\alpha_i) \quad (3.28)$$

Then the partial derivatives of the terms in Eq. 3.27 with respect to the RV α_i becomes,

$$\begin{aligned} \frac{\partial \mathbf{K}}{\partial \alpha_i} \mathbf{q} + \mathbf{K} \frac{\partial \mathbf{q}}{\partial \alpha_i} &= \frac{\partial \mathbf{F}}{\partial \alpha_i} \\ \mathbf{K} \frac{\partial \mathbf{q}}{\partial \alpha_i} &= \frac{\partial \mathbf{F}}{\partial \alpha_i} - \frac{\partial \mathbf{K}}{\partial \alpha_i} \mathbf{q} \end{aligned} \quad (3.29)$$

Where $\alpha = (\alpha_1, \alpha_2, \alpha_i, \alpha_n)$ is a random vector, in which α_i is the random variable and n is the number of the random variables. $\mathbf{K}(\alpha)$ is the stochastic stiffness matrix and $\mathbf{q}(\alpha)$ and $\mathbf{F}(\alpha)$ are respectively the displacement and load vectors random fields. Using the PFEM for Eq. 3.29 involves inevitably the partial derivations of random quantities with respect to random variables.

In order to apply stochastic randomness to evaluate the statistical second moments mean E and Variance Var in this thesis, a perturbation stochastic method is used applying first order Taylor series expansion of the stochastic global stiffness matrix $\mathbf{K}(\alpha)$, displacement $\mathbf{q}(\alpha)$ and force $\mathbf{F}(\alpha)$ around the mean of each random variable α_i , which are expressed as,

$$\mathbf{K}(\alpha) \approx \mathbf{K}^0 + \sum_{i=1}^n \mathbf{K}_i^I (\alpha_i - \alpha_i^0), \quad \mathbf{K}_i^I = \left. \frac{\partial \mathbf{K}}{\partial \alpha_i} \right|_{\alpha_i = \alpha_i^0} \quad (3.30)$$

$$\mathbf{q}(\alpha) \approx \mathbf{q}^0 + \sum_{i=1}^n \mathbf{q}_i^I (\alpha_i - \alpha_i^0), \quad \mathbf{q}_i^I = \left. \frac{\partial \mathbf{q}}{\partial \alpha_i} \right|_{\alpha_i = \alpha_i^0} \quad (3.31)$$

$$\mathbf{F}(\alpha) \approx \mathbf{F}^0 + \sum_{i=1}^n \mathbf{F}_i^I (\alpha_i - \alpha_i^0), \quad \mathbf{F}_i^I = \left. \frac{\partial \mathbf{F}}{\partial \alpha_i} \right|_{\alpha_i = \alpha_i^0} \quad (3.32)$$

By Substituting Eq. 3.30, Eq. 3.31 and Eq. 3.32 into Eq. 3.29 and equating equal terms gives,

$$\mathbf{K}^0 \cdot \mathbf{q}^0 = \mathbf{F}^0 \quad (3.33)$$

$$\mathbf{q}^0 = (\mathbf{K}^0)^{-1} \cdot \mathbf{F}^0 \quad (3.34)$$

$$\mathbf{q}_i^I = (\mathbf{K}^0)^{-1} \cdot \left(\frac{\partial \mathbf{F}}{\partial \alpha_i} - \mathbf{K}_i^I \cdot \mathbf{q}^0 \right) \quad (3.35)$$

Where $\mathbf{q}^0$, $\mathbf{K}^0$ and $\mathbf{F}^0$ are the mean values of the random variables for the displacement, stiffness matrix and the load respectively.

From the resulting nodal displacements $\mathbf{q}$ from Eq. 3.27, the element displacement field $\mathbf{u}_e$ becomes,

$$\mathbf{u}'_e = \mathbf{N}\mathbf{q} \quad (3.36)$$

Where $\mathbf{N}$ are the element shape functions that will be later on defined in detail in section 4.1.2 "Shape functions".

Applying Eq. 3.36 on Eq. 3.31, the compatible local displacement field for the element $\mathbf{u}'_e$ can be expanded as,

$$\mathbf{u}'_e(\alpha) \approx \mathbf{u}'_e{}^0 + \sum_{i=1}^n \mathbf{u}'_{ei}{}^I (\alpha_i - \alpha_i^0), \quad \mathbf{u}'_{ei}{}^I = \left. \frac{\partial \mathbf{u}'_e}{\partial \alpha_i} \right|_{\alpha_i = \alpha_i^0} \quad (3.37)$$

The second statistic moments E and Var of the local element displacement field vector $\mathbf{u}'_e(\alpha)$ can be written as,

$$E[\mathbf{u}'_e(\alpha)] \approx \mathbf{u}'_e{}^0 \approx \mathbf{N}\mathbf{q}^0 \quad (3.38)$$

For the case of independent input parameters, based on Eq. 3.22, the first order estimate of the variance matrix of the displacements vector $\mathbf{u}_e(\alpha)$ can be written as,

$$\text{Var}[\mathbf{u}'_e(\alpha)] \approx \sum_{i=1}^n (\mathbf{u}'_{ei}{}^I)^2 \text{Var}[\alpha_i] \quad (3.39)$$

The strains can be computed for the element e from the nodal displacement field $\mathbf{q}$ of Eq. 3.31. To obtain the strains the discrete version of the strain and displacement relation is used,

$$\boldsymbol{\varepsilon}' = \mathbf{B}\mathbf{q}' \quad (3.40)$$

Where $\mathbf{B}$ is the strain-displacement matrix as defined in detail in subsection 4.2.2 "B-matrix strain displacement". It is important to mention that matrix $\mathbf{B}$ only depends on the shape functions and geometry data of the element and therefore is not directly dependent on material and load randomness. The Cauchy small strains tensor of the element e is, therefore, computed from Eq. 3.31 getting,

The local strain vector $\boldsymbol{\varepsilon}'_e$ can be computed for the element e from the nodal displacements $\mathbf{q}$ using the FE analysis output Eq. 4.70 and Eq. 3.40. The first order Taylor series around the mean value for the strain vector $\boldsymbol{\varepsilon}'_e$ with respect to the RV α_i then becomes,

$$\boldsymbol{\varepsilon}'_e(\alpha) \approx \boldsymbol{\varepsilon}'^0_e + \sum_{i=1}^n \boldsymbol{\varepsilon}'^I_{ei}(\alpha_i - \alpha_i^0), \quad \boldsymbol{\varepsilon}'^I_{ei} = \left. \frac{\partial \boldsymbol{\varepsilon}'_e}{\partial \alpha_i} \right|_{\alpha_i = \alpha_i^0} \quad (3.41)$$

Being,

$$\boldsymbol{\varepsilon}'^0_e = \mathbf{B}\mathbf{q}^0; \quad \boldsymbol{\varepsilon}'^I_{ei} = \mathbf{B}\mathbf{q}^I \quad (3.42)$$

For the element strains second moments the E and Var of the element strain vector $\boldsymbol{\varepsilon}_e(\alpha)$ can then be written as.

$$\mathbb{E}[\boldsymbol{\varepsilon}'_e(\alpha)] \approx \boldsymbol{\varepsilon}'^0_e \quad (3.43)$$

The first order estimate of the covariance matrix becomes:

$$\text{Cov}[\boldsymbol{\varepsilon}'_e(\alpha_i), \boldsymbol{\varepsilon}'_e(\alpha_j)] \approx \sum_{i,j=1}^n \left(\frac{\partial \boldsymbol{\varepsilon}'_e}{\partial \alpha_i} \right) \left(\frac{\partial \boldsymbol{\varepsilon}'_e}{\partial \alpha_j} \right)^T \{ \mathbb{E}[\alpha_i, \alpha_j] - \alpha_i^0, \alpha_j^0 \} \quad (3.44)$$

The first order estimate of the variance matrix becomes:

$$\text{Var}[\boldsymbol{\varepsilon}'_e(\alpha_i), \boldsymbol{\varepsilon}'_e(\alpha_j)] \approx \sum_{i,j=1}^n \text{diag} \left(\frac{\partial \boldsymbol{\varepsilon}'_e}{\partial \alpha_i} \right) \left(\frac{\partial \boldsymbol{\varepsilon}'_e}{\partial \alpha_j} \right)^T \{ \mathbb{E}[\alpha_i, \alpha_j] - \alpha_i^0, \alpha_j^0 \} \quad (3.45)$$

For the case of independent input parameters, based on Eq. 3.22, the first order estimate of the variance matrix the element strain vector $\boldsymbol{\varepsilon}_e(\alpha)$ becomes,

$$\text{Var}[\boldsymbol{\varepsilon}'_e(\alpha)] \approx \sum_{i=1}^n (\boldsymbol{\varepsilon}'^I_{ei})^2 \text{Var}[\alpha_i] \quad (3.46)$$

As input for the fatigue life analysis using a probabilistic CLD damage model the element stress vector is required from the PFEM analysis, which we get from the stress in terms of their second moments mean E and variance Var. The stress tensor is obtained with,

$$\boldsymbol{\sigma}'_e \approx \mathbf{D}'_e \boldsymbol{\varepsilon}'_e \quad (3.47)$$

Where $\mathbf{D}'_e$ is the local element material constitutive coefficient matrix from Eq. 4.52, that will be discussed in detail in subsection 4.2.3 "D-matrix constitutive laminate stiffness matrix", which can be expanded with a Taylor expansion around the mean as,

$$\mathbf{D}'_e(\alpha) \approx \mathbf{D}'^0_e + \sum_{i=1}^n \mathbf{D}'^I_{ei}(\alpha_i - \alpha_i^0), \quad \mathbf{D}'^I_{ei} = \left. \frac{\partial \mathbf{D}'_e}{\partial \alpha_i} \right|_{\alpha_i = \alpha_i^0} \quad (3.48)$$

The first order Taylor series around the mean value for the stress tensor $\boldsymbol{\sigma}'_e$ with respect to the RV α_i then becomes,

$$\sigma'_e(\alpha) \approx \sigma_e'^0 + \sum_{i=1}^n \sigma_{ei}'^I (\alpha_i - \alpha_i^0), \quad \sigma_{ei}'^I = \left. \frac{\partial \sigma'_e}{\partial \alpha_i} \right|_{\alpha_i = \alpha_i^0} \quad (3.49)$$

Using Eq. 3.49 we can get by inserting in Eq. 3.47, the Taylor expansions of Eq. 3.41 and Eq. 3.48, we finally get for the first order Taylor series expansion for the local element stress tensor σ'_e ,

$$\sigma'_e(\alpha) \approx \left(\mathbf{D}_e'^0 + \sum_{i=1}^n \mathbf{D}_{ei}'^I (\alpha_i - \alpha_i^0) \right) \times \left(\boldsymbol{\varepsilon}_e'^0 + \sum_{i=1}^n \boldsymbol{\varepsilon}_{ei}'^I (\alpha_i - \alpha_i^0) \right) \quad (3.50)$$

From Eq. 3.50 and Eq. 3.47 we can get for the local element E mean stress vector $\sigma_e'^0$, from the mean strain vector as,

$$\sigma_e'^0 = \mathbf{D}_e'^0 \boldsymbol{\varepsilon}_e'^0 \quad (3.51)$$

The local element E mean stress vector $\sigma_e'^0$, can also directly be obtained from the nodal displacements using Eq. 3.51 and Eq. 3.40,

$$\sigma_e'^0 = \mathbf{D}'^0 \mathbf{B} \mathbf{q}^0 \quad (3.52)$$

And the local element stress vector sensitivity $\sigma_{ei}'^I$ can be obtained from the strain by,

$$\sigma_{ei}'^I = \mathbf{D}_e'^0 \boldsymbol{\varepsilon}_{ei}'^I + \mathbf{D}_{ei}'^I \boldsymbol{\varepsilon}_e'^0 \quad (3.53)$$

The local element stress vector sensitivity $\sigma_{ei}'^I$ can also directly be obtained from the nodal displacements using Eq. 3.53 and Eq. 3.42,

$$\sigma_{ei}'^I = \mathbf{D}'^0 \mathbf{B} \mathbf{q}^I + \mathbf{D}_{ei}'^I \mathbf{B} \mathbf{q}^0 \quad (3.54)$$

For the local element stress vector σ'_e the estimate of the mean E becomes.

$$E[\sigma'_e(\alpha)] \approx \sigma_e'^0 \quad (3.55)$$

The first order estimate of the covariance Cov matrix for the local element stress vector σ'_e becomes:

$$\text{Cov}[\sigma'_e(\alpha_i), \sigma'_e(\alpha_j)] \approx \sum_{i,j=1}^n \left(\frac{\partial \sigma'_e}{\partial \alpha_i} \right) \left(\frac{\partial \sigma'_e}{\partial \alpha_j} \right)^T \{E[\alpha_i, \alpha_j] - \alpha_i^0, \alpha_j^0\} \quad (3.56)$$

The first order estimate of the variance Var matrix for the local element stress vector σ'_e becomes:

$$\text{Var}[\sigma'_e(\alpha_i), \sigma'_e(\alpha_j)] \approx \sum_{i,j=1}^n \text{diag} \left(\frac{\partial \sigma'_e}{\partial \alpha_i} \right) \left(\frac{\partial \sigma'_e}{\partial \alpha_j} \right)^T \{E[\alpha_i, \alpha_j] - \alpha_i^0, \alpha_j^0\} \quad (3.57)$$

Again for the case of independent input parameters, the first order estimate of the variance matrix of the element stress vector $\sigma_e(\alpha_i)$, Eq. 3.57 can be simplified to,

$$\text{Var}[\sigma'_e(\alpha_i)] \approx \sum_{i=1}^n \left(\frac{\partial \sigma'_e}{\partial \alpha_i} \right)^2 \text{Var}[\alpha_i] \quad (3.58)$$

3.5 Analysis and design uncertainties

With probabilistic fatigue life assessment the aim is to take into account the uncertainties that appear in the modeling of physical systems chosen for the in this case fatigue life analysis and study the impact of those uncertainties onto the system responses Ditlevsen and Madsen [2005].

The following types of uncertainties can be taken into account for the probabilistic fatigue life analysis of structure and therefore be modeled as random variables with appropriate probability distribution:

- **Uncertainties in the mechanical model** which can be roughly divided in two groups:
 - **Geometric uncertainties:** can arise from manufacturing tolerances and operational wear of the blades. The variations of the blade shape and airfoil geometry lead to a significant scatter of the aerodynamic loads (lift and drag coefficients) which affects the blade fatigue loads and therefore affect the reliability not only of the WTB but also the rest of the wind turbine Ernst et al. [2014];
 - **Scatter in material properties:** such as the material stiffness Mustafa et al. [2015] strength Alazwari and Rao [2019] or **fatigue life** of a given material can be characterized as a random variable depending on for example the stress level. It is known that traditional structural materials like metals comprise of a large scatter in strength data, but experiments on glass and carbon fiber based FRP show an even larger dispersion on strength values of these composite materials Pfaller et al. [2012], as shown in Fig. 3.3.
- **Uncertainties in the analysis models:** such as the **fatigue damage accumulation model** applied Toft and Sørensen [2009], **aerodynamic model** Venkatesh et al. [2014], etc. can be modeled as random variables with appropriate probability density function.
- **Uncertainties in manufacturing:** The variability associated with **production and process**, which can introduce material and structural imperfection up to different degrees, can be considered as random variables Li et al. [2002].

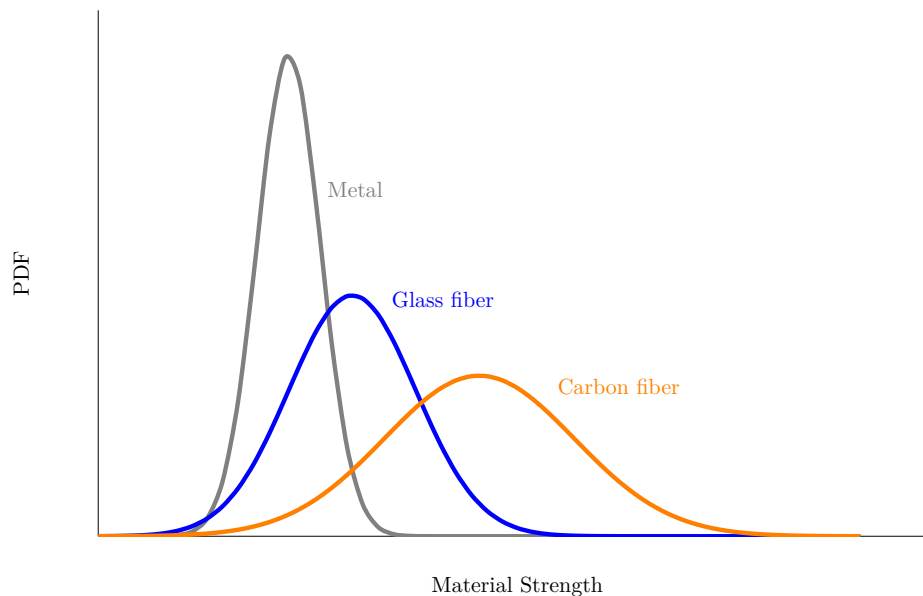


Figure 3.3: Steel, glass/epoxy and carbon/epoxy laminate strength probability density functions

- **Loading and boundary conditions:** environmental conditions as **external loads and/or internal loads** can be considered as random variables Robertson et al. [2019], Pourazarm et al. [2016].

Other sources of uncertainty can be human error and possible unknown failure mechanisms.

In addition to the uncertainties associated with the individual load and strength parameters (basic variables) which are mentioned above, it is well known that both the methods of global analysis and the equations used for assessing the strength of individual components are not exact.

In global structural analysis of a blade, the true properties of the materials and components often deviate from the idealizations on which the methods are based. Without exception, all practical structural systems exhibit behavior that (to a certain extent) is non-linear and dynamic and therefore have properties that are time-dependent, strain-rate dependent and spatially non-uniform. Furthermore, most practical structures are statically indeterminate and contain high levels of residual forces (and hence stresses) resulting from the particular fabrication and installation sequence adopted. These differences between real and predicted behaviour can be termed global analysis model uncertainty. In general, this is extremely difficult to quantify. Estimates of the magnitude of this type of model uncertainty can be obtained by comparisons using more refined analysis tools and sensitivity studies, or by full-scale physical testing.

The variability in load and strength parameters (including model uncertainty) arising from physical variability and inadequacies in modeling are allowed for in deterministic

design and assessment procedures by an appropriate choice of safety factors and by an appropriate degree of bias in the design equations. In probabilistic methods the variability in the basic design variables, including model uncertainty, is taken into account directly in the probabilistic modeling of the quantities.

3.6 WTB Probabilistic Fatigue Life Assessment

As discussed before, deterministic fatigue life analysis of WTBs is mostly performed using the lower bound material properties combined with applying PSF, to calculate the fatigue life. These PSF are dependent on the uncertainties in loads and materials, the uncertainties in the analysis methods, and the importance of structural components with respect to the consequences of their failure [Musial and Butterfield, 1997]. However these PSFs cannot quantify or predict the influence and sources of randomness in a structure [Sriramula and Chryssanthopoulos, 2009], so in this way ignore these variability and uncertainties. The interesting thing is that PSFs used in WTB design standards are actually based on probabilistic reliability analysis [Sørensen and Toft, 2014] performed using standards such as ISO2394 [2015], but based on rather very old data of blade designs, materials and loads.

Further for deterministic fatigue life analysis the cumulative damage level D is calculated, which has to stay below one ($D < 1$) in order not to fail which then results in a FRF > 1 . But what does it actually means to have a FRF of for example 2.5? How much longer fatigue life has a FRF of 2 compared to 2.5, moreover how does this value relate to the risk and reliability of your design, and at what costs? It is clear that fatigue life is a stochastic quantity that can not be specified by only a single number, as is the case for the traditional deterministic methods that result in a single design FRF.

Probabilistic fatigue life verification is a much better method to obtain estimates of the failure probability and to quantify the relative influence of variability and uncertainty in all of the input parameters so to quantify the likelihood of a fatigue failure to occur Lekou [2013] during its lifetime and consequently the critical fatigue life drivers can be identified.

The main outputs from the probabilistic analysis include the probability of failure data against fatigue life, as shown in Fig. 3.4, as well as probabilistic sensitivity factors. These sensitivity factors quantify the relative variability or uncertainty of each input variable. This allows you to concentrate your resources on the ones that are the most important and have the most influence on the reliability of the blade or other structure.

The probabilistic approach starts with setting the target P_f according to for example standard ISO2394 [2015], where both safety considerations as well as financial optimization are important. The fatigue P_f is not constant, it is zero at turbine installation and then rises during its operational (fatigue) life. This means that rather than set one fixed

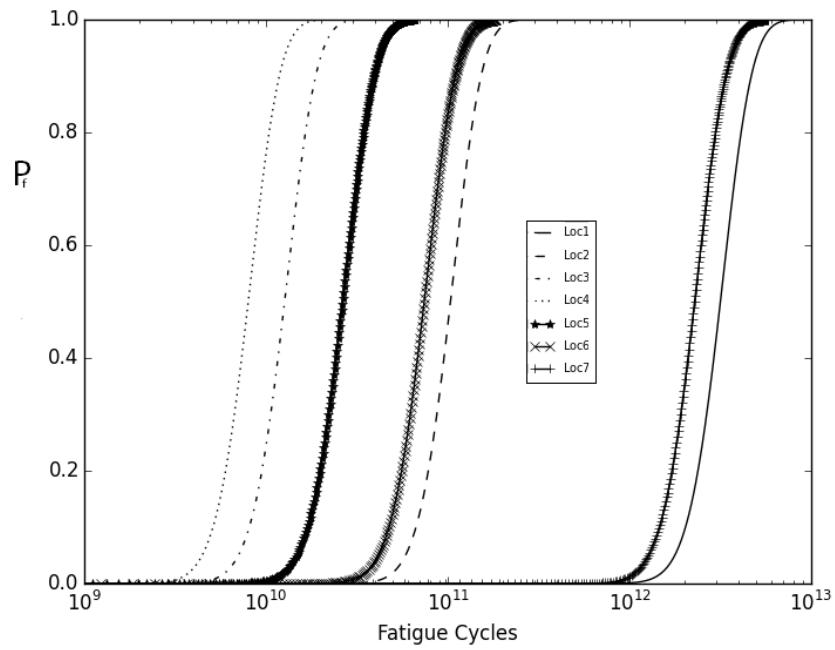


Figure 3.4: Probability of failure during fatigue life at different locations of a structure

number, one should define the desired behavior of the P_f over time, based on some optimization that involves initial investment, wear and tear during operational life and cost of inspections and repairs.

Chapter 4

PFEM python code (PFEMpy)

There are various commercial off the shelf Finite Element (FE) software available with some of the more popular ones being ANSYS [ANSYS Inc., 2022] and ABAQUS [Dassault Systèmes Simulia Corp, 2022]. The companies that develop and sell such software employ a one-size-fits-all approach where they sell the same set of services to all customers with minor levels of customization. While these products will be based on general best practices, they often do not fit very well for research and investigation needs.

As an important analysis to be performed was sensitivity analysis, many different derivatives needed to be calculated such as for the stiffness matrices which is very easily done utilizing symbolic computational techniques, therefore some of the needs and requirements on the to be applied Finite Element Method (FEM) were:

- Full element definitions and (local/global) stiffness matrices available and possible to manipulate. Commercial FE software have a theoretical description of their elements (behavior) available but not the full element description open source available;
- Able to apply symbolic computational techniques inside the FEM code;
- Easy integration of and performing of Monte Carlo Simulation (MCS) with the FEM code.

To meet all above requirements a custom FE software/code [PFEMpy: Rens, 2022] was developed by the thesis author, named Probabilistic Finite Element Method Python code (PFEMpy), which was written in the Python programming language [Van Rossum and Drake, 2009]. For symbolic mathematical computational techniques and symbolic manipulation, Python's SymPy module Meurer et al. [2017] was utilized.

The main FEM code consists of the following modules:

- **Pre-process:** to pre-process, so read and parse the analysis input data;
- **Model meshing:** to mesh the composite plate model;

- **BC:** to apply the Boundary Condition (BC) and loads to the model;
- **Element definition:** Contains the 4 node shell element description according to section 4.1 "Shell element definition".
- **Matrix module:** contains the matrix functions to construct:
 - **Composite laminate stiffness matrix:** as will be discussed in subsection 4.2.3 "D-matrix constitutive laminate stiffness matrix";
 - **Rotation matrices:** contains different rotation matrices for example between global-local stress and strains;
 - **Global stiffness matrix:** constructs the global stiffness matrices for the different elements;
- **FEM Solvers:** contains FEM solvers for basic linear analysis. Different solvers from SciPy [Virtanen et al., 2020] were implemented, such as `linalg.solve` to solve a linear equation set, `scipy.sparse.linalg.spsolve` to solve a sparse linear system and `scipy.sparse` to solve sparse matrices;
- **Post-process:** to post-process the analysis results to tables and plots;

Additional Probabilistic Finite Element Method (PFEM) code modules containing:

- **PFEM Solvers:** contains PFEM solvers for sensitivity analysis;
- **Perturbation analysis:** contains modules to create the Taylor expansion series and perform the perturbation analysis, as will be discussed in section 4.4 "Probabilistic Finite Element Method";
- **MCS:** contains modules to perform the MCS analysis according to subsection 3.3.2 "Monte Carlo Simulation":
 - **Random variables:** Creates RV and plot their distribution;
 - **MCS sampling:** loops for the amount of samples the simulations;
 - **Gradient based sensitivity analysis:** to analyze the sensitivities defining the gradients of the RV vs the RP analysis results;
 - **Post:** to post-process the analysis results to tables and plots
- **B-model:** contains modules to perform the probabilistic fatigue damage accumulation analysis using Bogdanoff Markov chain model (B-model).

A Reissner-Mindlin shell element formulation was used in combination with symbolic computational techniques, that were applied to analyze the many different derivatives for the PFEM analysis.

4.1 Shell element definition

As shell element for a layered composite layup, a 4-noded shell Quadrilateral, biLinear deflection, biLinear rotations and Linear transverse shear strain fields (QLLL) element [Oñate, 2013] with membrane, bending and shear was implemented in the PFEMpy. The formulation of this element is based on Reissner-Mindlin multi-field variational principle and the shape functions are formulated directly for the 4-node element. This element was initially developed by Dvorkin and Bathe [1984] and is considered robust for practical applications. Reissner-Mindlin plate theory is a good compromise between Classical Lamination Theory (CLT) and higher order theories for practical analysis of composite laminated plates and shells. The bending moments and shear forces are constant along each natural direction.

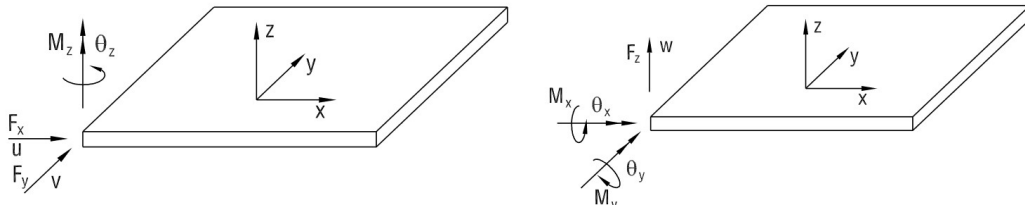


Figure 4.1: Shell element subject to plane membrane and bending action

Symbolic computational techniques [Meurer et al., 2017] were used that offers especially for sensitivity analysis significant computational savings over the traditional numerical integration approaches and thereby makes these elements attractive from both a computation and accuracy point of view.

Shear locking was eliminated for the 4-noded QLLL element element based on an assumed transverse shear approach, see Eq. 4.53.

4.1.1 Reissner-Mindlin principle

When a body is subjected to deformations, the process is commonly mathematical described in any point of the body (in the volume V of the body) by a displacement vector $\mathbf{u}$, strain tensor $\boldsymbol{\varepsilon}$ and stress tensor $\boldsymbol{\sigma}$. The Principle Virtual Work (PVW) is written in terms of the generalized strains, the resultant stresses and the external distributed loads t as,

$$\underbrace{\int \int \int_V (\delta \boldsymbol{\varepsilon})^T \boldsymbol{\sigma} dV}_{\text{virtual internal work}} = \int \int_A (\delta \mathbf{u})^T \mathbf{t} dA \quad (4.1)$$

where A is the mid-plane surface of the shell element over which the surface traction's $\mathbf{t}$ acts,

$$\delta \mathbf{t} = [f_x, f_y, f_z, m_x, m_y]^T \quad (4.2)$$

$\mathbf{u}$ represents the field of surface displacements,

$$\delta \mathbf{u} = [\delta u_0, \delta v_0, \delta w_0, \delta \theta_x, \delta \theta_y]^T \quad (4.3)$$

In vector $\mathbf{t}$, f_x and f_y are in-plane distributed loads acting in the direction of the x and y axes, respectively.

The virtual internal work, or left hand side integral of Eq. 4.1, is written in terms of the work of the resultant stresses on the virtual generalized strains over the plate middle surface as follows.

$$\delta \boldsymbol{\varepsilon} = \mathbf{S} \delta \hat{\boldsymbol{\varepsilon}} \quad (4.4)$$

Where $\mathbf{S}$ is the elastic compliance matrix.

Further a "hat" ($\hat{\cdot}$) symbol on the constitutive parameters denotes integrated (also called resultant or generalized) values over the section thickness for plates and shells.

Substituting Eq. 4.4 into the left hand side integral of Eq. 4.1 gives,

$$\int \int \int_V (\delta \boldsymbol{\varepsilon})^T \boldsymbol{\sigma} dV = \int \int \int_V (\delta \hat{\boldsymbol{\varepsilon}})^T \mathbf{S} \boldsymbol{\sigma} dV = \int \int_A (\delta \hat{\boldsymbol{\varepsilon}})^T \hat{\boldsymbol{\sigma}} dA \quad (4.5)$$

The PVW can now be written with only terms of integrals over the plate mid-plane A as,

$$\int \int_A (\delta \hat{\boldsymbol{\varepsilon}})^T \hat{\boldsymbol{\sigma}} dA = \int \int_A (\delta \mathbf{u})^T \mathbf{t} dA \quad (4.6)$$

The effect of the initial resultant stresses in the PVW can be taken into account giving,

$$\int \int_A (\delta \hat{\boldsymbol{\varepsilon}})^T \hat{\mathbf{D}} \hat{\boldsymbol{\varepsilon}} dA + \int \int_A (\delta \hat{\boldsymbol{\varepsilon}})^T \hat{\boldsymbol{\sigma}}^0 dA = \int \int_A (\delta \mathbf{u})^T \mathbf{t} dA \quad (4.7)$$

In Eq. 4.7 the displacement derivatives in the integrals are of first order, as usual for Reissner-Mindlin plate theory. Therefore, a C^0 continuous finite element interpolation can be used for all the displacement variables.

The discretized equilibrium equations relating the nodal forces and the nodal displacements are obtained by substituting the discretization equations into the PVW in the standard manner. The equivalent nodal force matrix for an element is given by,

$$\mathbf{f}_i^e = \int \int_A^e N_i \mathbf{t} dA - \int \int_A^e (\mathbf{B}_i)^T \hat{\boldsymbol{\sigma}}^0 dA \quad (4.8)$$

Where N_i is the matrix of the displacement interpolation functions for node i , or also called shape functions matrix, see also Eq. 4.12. Further $\mathbf{B}_i$ is the linear differential operator for node i or also called strain-displacement matrix that transforms the displacements $\mathbf{u}$ to strains. The $\mathbf{D}$ is the constitutive matrix, derived for composite laminates in Eq. 4.33.

The local element stiffness matrix $\mathbf{K}_{ij}^{le}$ in the global coordinate system, which is the output of the shell element function in the element definition module of the PFEMpy code is given by,

$$\mathbf{K}_{eij}^{'} = \int \int_A^e (\mathbf{B}_i')^T \hat{\mathbf{D}}' \mathbf{B}_j' dA \quad (4.9)$$

4.1.2 Interpolation functions

Shape functions

The displacement field of the extended Reissner-Mindlin theory (including in-plane displacements) for a shell modeled within the framework of a first order shear deformation theory are given by,

$$\begin{aligned} u'(x', y', z') &= u'_0(x', y') + z' \theta_{x'}(x', y') \\ v'(x', y', z') &= v'_0(x', y') + z' \theta_{y'}(x', y') \\ w'(x', y', z') &= w'_0(x', y') \end{aligned} \quad (4.10)$$

Where u'_0, v'_0, w'_0 are the displacements given with reference to the geometric middle plane of the shell. $\theta_{x'}$ and $\theta_{y'}$ are the rotations of the cross section around the x' and y' axes respectively. The rotations are responsible for the basis of the Mindlin plate theory, stating that normals to the neutral plane remain straight but not necessarily normal during deformation.

The generalized displacements represented in terms of interpolation functions are,

$$\begin{aligned}
u'_0(\xi, \eta) &= \sum_{i=1}^4 N_i u_i \\
v'_0(\xi, \eta) &= \sum_{i=1}^4 N_i v_i \\
w'_0(\xi, \eta) &= \sum_{i=1}^4 N_i w_i \\
\theta_{x'}(\xi, \eta) &= \sum_{i=1}^4 N_i \theta_{xi} \\
\theta_{y'}(\xi, \eta) &= \sum_{i=1}^4 N_i \theta_{yi}
\end{aligned} \tag{4.11}$$

Where the shape functions N_i are defined as,

$$N_i = \frac{1}{4}(1 - \xi_i \xi)(1 - \eta_i \eta), \quad i = 1, 2, 3, 4 \tag{4.12}$$

So for each of the 4 nodes the shape functions become,

$$\begin{aligned}
N_1 &= \frac{1}{4}(1 - \xi)(1 - \eta) \\
N_2 &= \frac{1}{4}(1 + \xi)(1 - \eta) \\
N_3 &= \frac{1}{4}(1 + \xi)(1 + \eta) \\
N_4 &= \frac{1}{4}(1 - \xi)(1 + \eta)
\end{aligned} \tag{4.13}$$

In the above set of equations ξ and η are the element natural coordinates.

The shape functions derivatives of N_i with respect to ξ are:

$$\frac{\partial N_1}{\partial \xi} = \frac{\eta}{4} - \frac{1}{4} \tag{4.14}$$

$$\frac{\partial N_2}{\partial \xi} = \frac{1}{4} - \frac{\eta}{4} \tag{4.15}$$

$$\frac{\partial N_3}{\partial \xi} = \frac{\eta}{4} + \frac{1}{4} \tag{4.16}$$

$$\frac{\partial N_4}{\partial \xi} = -\frac{\eta}{4} - \frac{1}{4} \tag{4.17}$$

And the shape functions derivatives of N_i with respect to η are:

$$\frac{\partial N_1}{\partial \eta} = \frac{\xi}{4} - \frac{1}{4} \quad (4.18)$$

$$\frac{\partial N_2}{\partial \eta} = -\frac{\xi}{4} - \frac{1}{4} \quad (4.19)$$

$$\frac{\partial N_3}{\partial \eta} = \frac{\xi}{4} + \frac{1}{4} \quad (4.20)$$

$$\frac{\partial N_4}{\partial \eta} = \frac{1}{4} - \frac{\xi}{4} \quad (4.21)$$

Jacobian matrix transformations

The shape functions derivatives are transformed from a global/local (x, y) to natural (ξ, η) coordinate system using the Jacobian matrix $\mathbf{J}$.

$$\mathbf{J} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \quad (4.22)$$

The Jacobian determinant $\det \mathbf{J}$ is defined by,

$$\det \mathbf{J} = J_{11}J_{22} - J_{12}J_{21} \quad (4.23)$$

Followed by the inverse of the Jacobian $\mathbf{J}^{-1}$, given by,

$$\mathbf{J}^{-1} = \frac{1}{\det \mathbf{J}} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} \quad (4.24)$$

Now using the shape functions of Eq. 4.14 and Eq. 4.14 with the Jacobian matrix $\mathbf{J}$ from Eq. 4.22, the transformation from global/local (x, y) to natural (ξ, η) derivatives of N_i then becomes,

$$\begin{bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{bmatrix} = \mathbf{J} \begin{bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{bmatrix} \quad (4.25)$$

And the reverse transformation from Eq. 4.25 so from natural (ξ, η) to global/local (x, y) derivatives, using the inverse of the Jacobian $\mathbf{J}^{-1}$ from Eq. 4.24, we get,

$$\begin{bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{bmatrix} = \mathbf{J}^{-1} \begin{bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{bmatrix} \quad (4.26)$$

Gauss quadrature

Gauss quadrature is used for solving the integral for the stiffness matrix of the quadrilateral shell element, see Fig. 4.2a,

$$\mathbf{K}'_{eij} = \int_A \mathbf{B}_i'^T \mathbf{D}' \mathbf{B}_j' dA \approx \sum_{qp=1}^n \mathbf{B}_i'^T \mathbf{D}' \mathbf{B}_j' \det \mathbf{J} w_{qp} \quad (4.27)$$

Where qp counts over the Gauss quadrature points and w_{qp} are the Gauss weights for each Gauss-point.

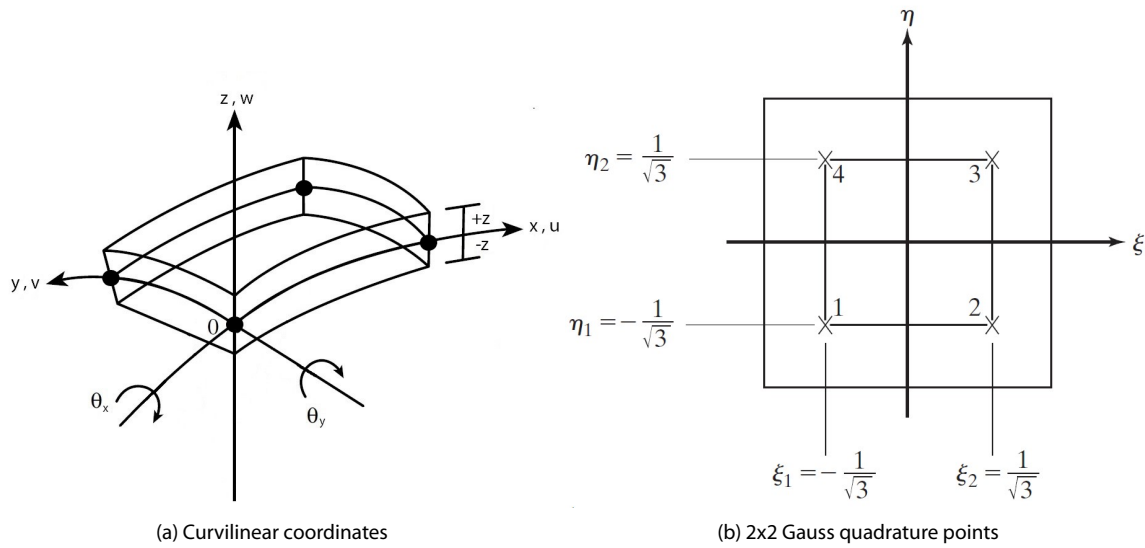


Figure 4.2: Curvilinear coordinates and element natural coordinates

qp	ξ_{qp}	η_{qp}	w_{qp}
1	0	0	2
4	$-\sqrt{1/3}$	$-\sqrt{1/3}$	1
	$\sqrt{1/3}$	$-\sqrt{1/3}$	1
	$\sqrt{1/3}$	$\sqrt{1/3}$	1
	$-\sqrt{1/3}$	$\sqrt{1/3}$	1

Table 4.1: Gauss Quadrature Points qp and weights for 2D quadrilateral elements

For the four-node shell element both Gauss 2x2 so 4 Gauss point integration, as shown in Fig. 4.2b and 1 point reduced integration is applied, the stresses at the four corner nodal points can be determined by the Gauss Quadrature points and weights for 2D quadrilateral elements from Table 4.1,

4.2 Deterministic finite element matrices

4.2.1 K-matrix local element stiffness

The accuracy in modeling composite shells is governed by Reissner-Mindlin's shear-flexible shell theory, which allows transverse shear deformation. The local element stiffness matrix $\mathbf{K}_{ij}^e$ is build up out of the stiffness matrices for bending, shear and membrane part, so we get,

$$\begin{aligned} \mathbf{K}_{eij}' &= \int \int_A^e [(\mathbf{B}_i'^m)^T (\mathbf{B}_i'^b)^T (\mathbf{B}_i'^s)^T] \begin{bmatrix} \mathbf{D}'^m & \mathbf{D}'^{mb} & 0 \\ \mathbf{D}'^{mb} & \mathbf{D}'^b & 0 \\ 0 & 0 & \mathbf{D}'^s \end{bmatrix} \begin{Bmatrix} \mathbf{B}_j'^m \\ \mathbf{B}_j'^b \\ \mathbf{B}_j'^s \end{Bmatrix} dx' dy' \\ &= \mathbf{K}_{ij}'^m + \mathbf{K}_{ij}'^b + \mathbf{K}_{ij}'^s + \mathbf{K}_{ij}'^{mb} + [\mathbf{K}_{ij}'^{mb}]^T \end{aligned} \quad (4.28)$$

Where $\mathbf{B}'^m$, $\mathbf{B}'^b$, $\mathbf{B}'^s$ are respectively the local element strain-displacement matrices for the membrane, bending and shear. $\mathbf{D}'^m$, $\mathbf{D}'^b$, $\mathbf{D}'^s$ are respectively the constitutive-displacement matrices for the membrane, bending and shear and further $\mathbf{D}'^{mb}$ is the membrane-bending coupling matrix.

4.2.2 B-matrix strain displacement

The strain-displacement matrices for the membrane $\mathbf{B}_i^m$, bending $\mathbf{B}_i^b$ and shear $\mathbf{B}_i^s$ for the i th node become,

$$\mathbf{B}_i^m = \begin{bmatrix} \frac{\partial N_i}{\partial x'} & 0 & 0 & 0 & 0 \\ 0 & \frac{\partial N_i}{\partial y'} & 0 & 0 & 0 \\ \frac{\partial N_i}{\partial y'} & \frac{\partial N_i}{\partial x'} & 0 & 0 & 0 \end{bmatrix}_{3 \times 5} \quad (4.29)$$

$$\mathbf{B}_i^b = \begin{bmatrix} 0 & 0 & 0 & \frac{\partial N_i}{\partial x'} & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial N_i}{\partial y'} \\ 0 & 0 & 0 & \frac{\partial N_i}{\partial y'} & \frac{\partial N_i}{\partial x'} \end{bmatrix}_{3 \times 5} \quad (4.30)$$

$$\mathbf{B}_i^s = \begin{bmatrix} 0 & 0 & \frac{\partial N_i}{\partial x'} & -N_i & 0 \\ 0 & 0 & \frac{\partial N_i}{\partial y'} & 0 & -N_i \end{bmatrix}_{2 \times 5} \quad (4.31)$$

The generalized strain-displacement matrix $\mathbf{B}_i$ for the i th node then becomes,

$$\mathbf{B}'_i = \begin{pmatrix} \mathbf{B}_i^{tm} \\ \mathbf{B}_i^{tb} \\ \mathbf{B}_i^{ts} \end{pmatrix} \quad (4.32)$$

The element $\mathbf{B}'_e$ matrix was organized with 8×20 so 20 DOF for the four nodes of the element, but in the global system we have 6 DOF per element node, i.e. 24 DOF per element as the the rotation around the z-axis (the drilling DOF) is not included. As in the global system we need 6 DOF per node so for the problem not to become singular, to the element $\mathbf{B}_e$ matrix extra columns of zeros at location 6, 12, 18 and 24 will be added. The generalized local strain-displacement matrix $\mathbf{B}'_i$ for the i th node then becomes,

$$\begin{bmatrix} \frac{\partial N_i}{\partial x} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\partial N_i}{\partial y} & 0 & 0 & 0 & 0 \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial N_i}{\partial x} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial N_i}{\partial y} & 0 \\ 0 & 0 & 0 & \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} & 0 \\ 0 & 0 & \frac{\partial N_i}{\partial x} & -N_i & 0 & 0 \\ 0 & 0 & \frac{\partial N_i}{\partial y} & 0 & -N_i & 0 \end{bmatrix}$$

↑
Added column with 6th DOF

4.2.3 D-matrix constitutive laminate stiffness matrix

A module was programmed to calculating the local element constitutive matrix $\mathbf{D}'_e$ or also called stress-strain relation matrix.

The $\mathbf{D}'_e$ matrix from Eq. 4.28 is for elements with a composite laminate layup often called the ABD matrix and instead written as,

$$\begin{bmatrix} \mathbf{D}^{tm} & \mathbf{D}^{tmb} & 0 \\ \mathbf{D}^{tmb} & \mathbf{D}^{tb} & 0 \\ 0 & 0 & \mathbf{D}^{ts} \end{bmatrix} = \mathbf{T}^r \begin{bmatrix} \mathbf{A} & \mathbf{B} & 0 \\ \mathbf{B} & \mathbf{D} & 0 \\ 0 & 0 & \mathbf{G} \end{bmatrix} (\mathbf{T}^r)^T \quad (4.33)$$

Where $\mathbf{A}$ is the in-plane membrane or also called extension stiffness matrix, $\mathbf{B}$ is the membrane bending or bending-extension coupling matrix, $\mathbf{D}$ is the bending stiffness matrix and $\mathbf{G}$ represents the shear stiffness matrix. Using constitutive laminate matrix from

Eq. 4.34 and its corresponding lamination parameters from the initial lay-up, applying the so-called CLT [Jones, 1999] we get,

$$\begin{bmatrix} \mathbf{N} \\ \mathbf{M} \\ \mathbf{Q} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} & 0 \\ \mathbf{B} & \mathbf{D} & 0 \\ 0 & 0 & \mathbf{G} \end{bmatrix} \begin{bmatrix} \boldsymbol{\varepsilon} \\ \boldsymbol{\kappa} \\ \boldsymbol{\gamma} \end{bmatrix} \quad (4.34)$$

Or in extended form we get,

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} \quad (4.35)$$

$$\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix} + \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} \quad (4.36)$$

$$\begin{bmatrix} Q_x \\ Q_y \end{bmatrix} = \begin{bmatrix} G_{44} & C_{45} \\ G_{45} & C_{55} \end{bmatrix} \begin{bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} \quad (4.37)$$

As input for the composite laminate stiffness matrix are used the laminate stacking angles (θ), the single ply thicknesses (t_{ply}) and the engineering material constants ($E_{11}, E_{22}, G_{12}, G_{13}, G_{23}$).

A step by step approach will be described that has been used to obtain and the steps programmed to obtain the constitutive element matrix $\mathbf{D}^e$ for the composite laminate layup in the elements.

Step 1) Define the reduced stiffness matrix $\mathbf{Q}$: For the stiffness coefficients Q_{ij} expressed in terms of the engineering material constants follows,

$$\mathbf{Q} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} & 0 & 0 \\ Q_{12} & Q_{22} & Q_{26} & 0 & 0 \\ Q_{16} & Q_{26} & Q_{66} & 0 & 0 \\ 0 & 0 & 0 & Q_{44} & 0 \\ 0 & 0 & 0 & 0 & Q_{55} \end{bmatrix} \quad (4.38)$$

$$\begin{aligned} Q_{11} &= \frac{E_{11}}{(1 - \nu_{12}\nu_{21})} \\ Q_{12} &= \frac{\nu_{21}E_{11}}{(1 - \nu_{12}\nu_{21})} \\ Q_{22} &= \frac{E_{22}}{(1 - \nu_{12}\nu_{21})} \\ Q_{44} &= G_{23} \\ Q_{55} &= G_{13} \\ Q_{66} &= G_{12} \end{aligned} \quad (4.39)$$

Step 2) Construct the transformed reduced stiffness matrix $\bar{\mathbf{Q}}$ for each ply: Using Eq. 4.39, the transformed reduced stiffness coefficients can be defined, according to the first order shear deformation theory [Jones, 1999] and the laminate system from Fig. 4.3,

$$\begin{aligned}
 \bar{Q}_{11} &= Q_{11} \cos^4 \theta + 2(Q_{12} + 2Q_{66}) \sin^2 \theta \cos^2 \theta + Q_{22} \sin^4 \theta \\
 \bar{Q}_{22} &= Q_{11} \sin^4 \theta + 2(Q_{12} + 2Q_{66}) \sin^2 \theta \cos^2 \theta + Q_{22} \cos^4 \theta \\
 \bar{Q}_{12} &= (Q_{11} + Q_{22} - 4Q_{66}) \sin^2 \theta \cos^2 \theta + Q_{12}(\sin^4 \theta + \cos^4 \theta) \\
 \bar{Q}_{66} &= (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66}) \sin^2 \theta \cos^2 \theta + Q_{66}(\sin^4 \theta + \cos^4 \theta) \\
 \bar{Q}_{16} &= (Q_{11} - Q_{12} - 2Q_{66}) \cos^3 \theta \sin \theta - (Q_{22} - Q_{12} - 2Q_{66}) \cos \theta \sin^3 \theta \\
 \bar{Q}_{26} &= (Q_{11} - Q_{12} - 2Q_{66}) \cos \theta \sin^3 \theta - (Q_{22} - Q_{12} - 2Q_{66}) \cos^3 \theta \sin \theta \\
 \bar{Q}_{44} &= Q_{44} \cos^2 \theta + Q_{55} \sin^2 \theta \\
 \bar{Q}_{45} &= (Q_{55} - Q_{44}) \cos \theta \sin \theta \\
 \bar{Q}_{55} &= Q_{55} \cos^2 \theta + Q_{44} \sin^2 \theta
 \end{aligned} \quad (4.40)$$

Where θ is the angle between the local laminate (1, 2, 3) and element (x, y, z) coordinate system, as shown in Fig. 4.3 .

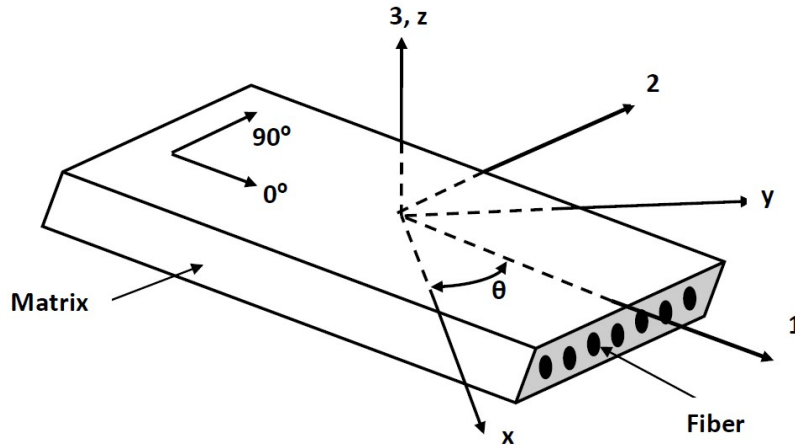


Figure 4.3: Coordinate system (local fiber to global) of composite laminates

With transformed reduced stiffness coefficients of Eq. 4.40, we can then construct the transformed reduced stiffness matrix $\bar{\mathbf{Q}}_{ij}$,

$$\bar{\mathbf{Q}}_{ij} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \quad (4.41)$$

And for the shear we can then construct the transformed reduced stiffness matrix $\bar{\mathbf{Q}}_{lm}$,

$$\bar{\mathbf{Q}}_{lm} = \begin{bmatrix} \bar{Q}_{44} & \bar{Q}_{45} \\ \bar{Q}_{45} & \bar{Q}_{55} \end{bmatrix} \quad (4.42)$$

Although $\bar{Q}_{11}$ and $\bar{Q}_{12}$ are the dominant terms for all fiber orientations, all components of $\bar{\mathbf{Q}}_{ij}$ and $\bar{\mathbf{Q}}_{lm}$ contribute to defining the overall material response characteristics of a laminate.

The components of $\bar{\mathbf{Q}}_{ij}$ in Eq. 4.41 and $\bar{\mathbf{Q}}_{lm}$ in Eq. 4.42 can be expressed in a different more practical form known as the invariant form of $\bar{Q}$ as first introduced by Tsai and Pagano [1968]. The invariant form is both simpler and therefore especially when differentiating to get the sensitivities much more practical to apply.

Using trigonometric identities the reduced in-lane ply stiffnesses in global coordinates can be expressed with the help of the 7^d material invariants U_i . The following definitions are introduced for the invariants:

$$\begin{aligned} U_1 &= \frac{3Q_{11}}{8} + \frac{Q_{12}}{4} + \frac{3Q_{22}}{8} + \frac{Q_{66}}{2} \\ U_2 &= \frac{Q_{11}}{2} - \frac{Q_{22}}{2} \\ U_3 &= \frac{Q_{11}}{8} - \frac{Q_{12}}{4} + \frac{Q_{22}}{8} - \frac{Q_{66}}{2} \\ U_4 &= \frac{Q_{11}}{8} + \frac{3Q_{12}}{4} + \frac{Q_{22}}{8} - \frac{Q_{66}}{2} \\ U_5 &= \frac{Q_{11}}{8} - \frac{Q_{12}}{4} + \frac{Q_{22}}{8} + \frac{Q_{66}}{2} \\ U_6 &= \frac{Q_{44}}{2} + \frac{Q_{55}}{2} \\ U_7 &= \frac{Q_{44}}{2} - \frac{Q_{55}}{2} \end{aligned} \quad (4.43)$$

The explicit form of $\bar{\mathbf{Q}}_{ij}$ can now be expressed as,

$$\begin{aligned} \bar{Q}_{11} &= U_1 + U_2 \cos(2\theta) + U_3 \cos(4\theta) \\ \bar{Q}_{12} &= -U_3 \cos(4\theta) + U_4 \\ \bar{Q}_{16} &= \frac{U_2 \sin(2\theta)}{2} + U_3 \sin(4\theta) \\ \bar{Q}_{22} &= U_1 - U_2 \cos(2\theta) + U_3 \cos(4\theta) \\ \bar{Q}_{26} &= \frac{U_2 \sin(2\theta)}{2} - U_3 \sin(4\theta) \\ \bar{Q}_{66} &= -U_3 \cos(4\theta) + U_5 \end{aligned} \quad (4.44)$$

And for the explicit form for the shear term of $\bar{\mathbf{Q}}_{lm}$ can be expressed as,

$$\begin{aligned}\bar{Q}_{44} &= U_6 + U_7 \cos(2\theta) \\ \bar{Q}_{45} &= -U_7 \sin(2\theta) \\ \bar{Q}_{55} &= U_6 - U_7 \cos(2\theta)\end{aligned}\quad (4.45)$$

Step 3) Construct the A, B, D and C matrices: Using Eq. 4.44 we can then define the separate **A**, **B**, **D** and using Eq. 4.45 we can define **G** matrices. For the in-plane extension stiffness matrix **A**, which influences extensional strains in the laminate, we get,

$$\mathbf{A}_{ij} = \sum_{k=1}^N (\bar{\mathbf{Q}}_{ij})_k (z_k - z_{k-1}) \quad i = 1, 2, 6; j = 1, 2, 6 \quad (4.46)$$

B is the bending-extension coupling matrix, which couples the extensional response to bending response of laminate,

$$\mathbf{B}_{ij} = \frac{1}{2} \sum_{k=1}^N (\bar{\mathbf{Q}}_{ij})_k (z_k^2 - z_{k-1}^2) \quad i = 1, 2, 6; j = 1, 2, 6 \quad (4.47)$$

D is the bending stiffness matrix,

$$\mathbf{D}_{ij} = \frac{1}{3} \sum_{k=1}^N (\bar{\mathbf{Q}}_{ij})_k (z_k^3 - z_{k-1}^3) \quad i = 1, 2, 6; j = 1, 2, 6 \quad (4.48)$$

G represents the shear stiffness matrix by,

$$\mathbf{G}_{lm} = \sum_{k=1}^N (\bar{\mathbf{Q}}_{lm})_k (z_k^3 - z_{k-1}^3) \quad l = 4, 5; m = 4, 5 \quad (4.49)$$

Where, the subscript k denotes the k_{th} layer and the z denotes the laminate thickness coordinate, so the vertical though the thickness position of the ply with respect to the laminate mid plane.

Step 4) Assembly of the ABDG stiffness matrix: So finally for the laminate **ABDG** stiffness matrix $\mathbf{S}_l^{lam}$ that results as output from this module we get:

$$\mathbf{S}_l^{lam} = \begin{bmatrix} \mathbf{A}' & \mathbf{B}' & 0 \\ \mathbf{B}' & \mathbf{D}' & 0 \\ 0 & 0 & \kappa \mathbf{G}' \end{bmatrix}_{8 \times 8} = \left[\begin{array}{ccc|ccc|cc} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} & 0 & 0 \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} & 0 & 0 \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} & 0 & 0 \\ \hline B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} & 0 & 0 \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} & 0 & 0 \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & \kappa G_{44} & \kappa G_{45} \\ 0 & 0 & 0 & 0 & 0 & 0 & \kappa G_{45} & \kappa G_{55} \end{array} \right] \quad (4.50)$$

Where κ is a correction factor to account for non-uniform distribution of shear stresses through the thickness, $\kappa = 5/6$ was used.

Step 5) Transformation to constitutive element matrix $\mathbf{D}$: The transformation matrix $\mathbf{T}^l$ from local laminate (1, 2, 3) to element (x, y, z) coordinate system is defined as.

$$\mathbf{T}_e^l = \begin{bmatrix} c^2 & s^2 & -2sc & 0 & 0 & 0 & 0 & 0 \\ s^2 & c^2 & 2sc & 0 & 0 & 0 & 0 & 0 \\ sc & -sc & c^2 - s^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c^2 & s^2 & -2sc & 0 & 0 \\ 0 & 0 & 0 & s^2 & c^2 & 2sc & 0 & 0 \\ 0 & 0 & 0 & sc & -sc & c^2 - s^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c & -s \\ 0 & 0 & 0 & 0 & 0 & 0 & s & c \end{bmatrix} \quad (4.51)$$

Where $c = \cos(\theta)$ and $s = \sin(\theta)$.

With the local laminate (1, 2, 3) to element (x, y, z) coordinate system transformation matrix $\mathbf{T}_e^l$ defined with Eq. 4.51, as shown in Fig. 4.3, the local element constitutive matrix $\mathbf{D}'_e$ is obtained as,

$$\mathbf{D}'_e = \mathbf{T}_e^l \mathbf{S}^{lam} (\mathbf{T}_e^l)^T \quad (4.52)$$

4.2.4 Local element stiffness matrix assembly

A conventional 4-node shell elements with 5 DOF per node, so 20 DOF per element has been used. As in the global system we need 6 DOF per node so for the problem not to become singular.

The procedure to get six DOFs at all nodes is:

1. **$\mathbf{B}^e$ matrix:** the local element $\mathbf{B}^e$ matrix has only 8×20 so 20 DOF for the four nodes of the element as the rotation around the z-axis θ_z is not included in the $\mathbf{B}^e$ matrix, so to get to 24 DOF, extra columns of zeros at the 6th DOF locations, so rows and columns 6, 12, 18 and 24.
2. **$\mathbf{K}'^e$ matrix:** insert an fictive arbitrary coefficient K'_{θ_z} the diagonal term of the local stiffness matrix,

$$\mathbf{K}'_{eij} = \begin{bmatrix} \mathbf{K}'_{eij} & 0 \\ 0 & K_{\theta_z} \end{bmatrix} \quad (4.53)$$

The new 24×24 local element stiffness matrix $\mathbf{K}^e$ then becomes,

$$\mathbf{K}'_e = \underset{24 \times 24}{(\mathbf{B}'_e)^T} \underset{24 \times 8}{\mathbf{D}'_e} \underset{8 \times 24}{\mathbf{B}'_e} \quad (4.54)$$

This new local element stiffness matrix $\mathbf{K}^e$ is transformed to global coordinate axes in the standard way and the resulting global equations are not singular.

Reduced integration

The Quadrilateral 4-noded (Q4) with 2×2 full integration presents severe shear locking. The shear strain is zero at the element center only. Therefore, the excess of shear strain can be eliminated by integrating the shear strain at the element center only ($\xi = \eta = 0$). This is simply achieved by using a reduced one point Gauss quadrature for the shear terms in the stiffness matrix. For this purpose the element stiffness matrix is split into different parts as shown in Eq. 4.28, where the different $\mathbf{K}'$ or $\mathbf{B}'$ Matrices are integrated using the Gauss-Legendre quadrature, according to Table 4.2.

This reduced selected integration technique also improves the behavior of 4-noded quadrilaterals of arbitrary shape. It can also be interpreted as a simple procedure to mitigate the excessive influence of the membrane and shear terms in the element stiffness matrix. A disadvantage of reduced integration is that it produces a quadrilateral element that is not geometric-invariant, although it passes the patch test and, therefore, it converges to the exact solution.

Type	Matrix	Gauss Quadrature
Bending	$\mathbf{K}^b, \mathbf{B}^b$	2×2
Membrane	$\mathbf{K}^m, \mathbf{B}^m$	1
Shear	$\mathbf{K}^s, \mathbf{B}^s$	1

Table 4.2: Reduced integration Gauss-Legendre quadrature

4.2.5 Global element stiffness transformation

The shape functions are defined in the plane of the quad shell element, i.e. z-coordinates for the nodes are equal to zero. Once the local element stiffness matrix $\mathbf{K}_l^e$ from Eq.4.9 is identified, the next step is to define the transformation matrix $\mathbf{T}^t$. To describe the components of a vector given in one coordinate system (x_g, y_g) in another coordinate system (x_l, y_l) , we can multiply the vector with the unit vectors spanning the (x, y) system, which corresponds to rotating the vector with the angle between the two systems.

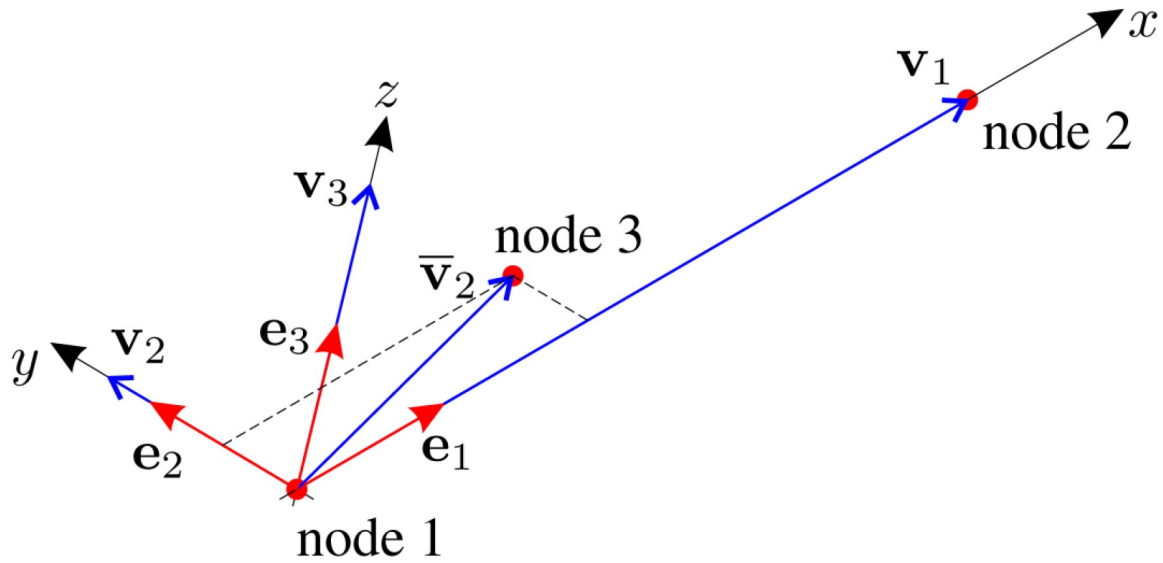


Figure 4.4: Transformation matrix using unit vectors

$$\mathbf{v}_1 = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{bmatrix}, \quad \bar{\mathbf{v}}_2 = \begin{bmatrix} x_3 - x_1 \\ y_3 - y_1 \\ z_3 - z_1 \end{bmatrix}, \quad \mathbf{v}_3 = \mathbf{v}_1 \times \bar{\mathbf{v}}_2, \quad \mathbf{v}_2 = \mathbf{v}_3 \times \mathbf{v}_1 \quad (4.55)$$

Now the unit vectors $\mathbf{e}$ can be defined.

$$\mathbf{e}_1 = \frac{\mathbf{v}_1}{|\mathbf{v}_1|}, \quad \mathbf{e}_2 = \frac{\mathbf{v}_2}{|\mathbf{v}_2|}, \quad \mathbf{e}_3 = \frac{\mathbf{v}_3}{|\mathbf{v}_3|} \quad (4.56)$$

The resulting transformation matrix $\mathbf{L}$ is a orthogonal set of unit vectors placed in the columns.

$$\mathbf{L}_{3 \times 3} = [\mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3] \quad (4.57)$$

So the global coordinates (x_g, y_g) we can get from the local coordinates (x_l, y_l) and the opposite local from global coordinates by.

$$\begin{bmatrix} x_g \\ y_g \\ z_g \end{bmatrix} = \mathbf{L} \begin{bmatrix} x_l \\ y_l \\ z_l \end{bmatrix}, \quad \begin{bmatrix} x_l \\ y_l \\ z_l \end{bmatrix} = \mathbf{L}^T \begin{bmatrix} x_g \\ y_g \\ z_g \end{bmatrix} \quad (4.58)$$

The full element transformation matrix to the global coordinate system $\mathbf{T}^t$ can then be obtained by assembling the $\mathbf{L}^t$ Eq. 4.57 for all the DOFs in an element,

$$\mathbf{T}_g^t = \begin{bmatrix} \mathbf{L}_{3 \times 3} & & & & & \\ & \mathbf{L} & & & & \\ & & \mathbf{L} & & 0 & \\ & & & \mathbf{L} & & \\ & & & & \mathbf{L} & \\ & 0 & & & & \mathbf{L} \\ & & & & & & \mathbf{L} \end{bmatrix} \quad (4.59)$$

With the transformation matrix $\mathbf{T}_g^t$ defined from Eq. 4.59, the global stiffness matrix $\mathbf{K}^e$ for an element in the global coordinate system is obtained from the local element stiffness matrix $\mathbf{K}'^e$ from Eq. 4.28 as,

$$\begin{aligned} \mathbf{K}_e &= (\mathbf{T}^t)^T \mathbf{K}'_e \mathbf{T}^t \\ &= (\mathbf{T}^t)^T \left[\int \int_A (\mathbf{B}'_i)^T \mathbf{D}' \mathbf{B}'_j \right] \mathbf{T}^t dx' dy' = \int \int_A (\mathbf{B}_i)^T \mathbf{D}' \mathbf{B}_j dx' dy' \end{aligned} \quad (4.60)$$

Where $\mathbf{K}'^e$ is the local element stiffness matrix obtained from Eq. 4.9.

The total global structure stiffness matrix $\mathbf{K}$ of the model can be constructed by using the direct stiffness method, by summing the global stiffness matrices $\mathbf{K}^e$ from Eq. 4.60 for each element of the model as,

$$\mathbf{K} = \sum_{e=1}^N \mathbf{K}_e \quad (4.61)$$

Similarly, the global nodal forces $\mathbf{F}$ for each element global nodal force matrix can be summed such that,

$$\mathbf{F} = \sum_{e=1}^N \mathbf{f}_e \quad (4.62)$$

Where $\mathbf{f}_e$ is the column of equivalent global nodal loads obtained by lumping body forces and distributed loads at the model nodes (as well as including concentrated nodal loads).

The total global structure stiffness matrix $\mathbf{K}$, now relates the global nodal forces $\mathbf{F}$ to the global nodal displacements $\mathbf{q}$ for the full model by,

$$\mathbf{Kq} = \mathbf{F} \quad (4.63)$$

Which is the equilibrium equation obtained after discretizing the geometry in a standard FEA in the context of finite element analysis for quasi-static linear problems.

4.3 Deterministic Static Calculations

4.3.1 Nodal displacements

The solution of the system of the global equilibrium equation $\mathbf{K}\mathbf{q} = \mathbf{F}$ from Eq. 4.63 yields the unknown element nodal displacement matrix $\mathbf{q}$.

4.3.2 Element displacements

For the quadrilateral shape element discretized into n -noded finite elements, the displacements and rotations are interpolated within each element as,

$$\mathbf{u}'_e = \begin{Bmatrix} u_0 \\ v_0 \\ w_0 \\ \theta_x \\ \theta_y \end{Bmatrix} = \sum_{i=1}^n \mathbf{N}_i \mathbf{q}_{ei} = [\mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_3, \mathbf{N}_4]_{8 \times 24} \begin{Bmatrix} \mathbf{q}'_{e1} \\ \mathbf{q}'_{e2} \\ \mathbf{q}'_{e3} \\ \mathbf{q}'_{e4} \end{Bmatrix}_{24 \times 1} \quad (4.64)$$

Where nodal displacement vector for each local element node $\mathbf{q}'_{ei}$ is,

$$\mathbf{q}'_{ei} = \begin{Bmatrix} u_{0i} \\ v_{0i} \\ w_{0i} \\ \theta_{xi} \\ \theta_{yi} \end{Bmatrix} \quad (4.65)$$

Further the shape functions matrix $\mathbf{N}_i$ is,

$$\mathbf{N}_i = \begin{bmatrix} N_i & 0 & 0 & 0 & 0 \\ 0 & N_i & 0 & 0 & 0 \\ 0 & 0 & N_i & 0 & 0 \\ 0 & 0 & 0 & N_i & 0 \\ 0 & 0 & 0 & 0 & N_i \end{bmatrix} \quad (4.66)$$

where $N_i(\xi, \eta)$ is the C^0 continuous shape function of node i .

Remember that the 6th DOF result θ_{zi} is obtained due to the fictive arbitrary coefficient K'_{θ_z} added to the stiffness matrix $\mathbf{K}^e$ matrix and therefore can and will not be used as valid analysis result.

4.3.3 Element strain

The strain vector is obtained from the expressions of 3D elasticity theory neglecting the thickness strain ε_z , as it does not contribute to the internal work due to the plane stress assumption ($\sigma_z = 0$).

$$\boldsymbol{\varepsilon} = \begin{Bmatrix} \boldsymbol{\varepsilon}^m \\ 0 \end{Bmatrix} + \begin{Bmatrix} -z\hat{\boldsymbol{\varepsilon}}^b \\ \hat{\boldsymbol{\varepsilon}}^s \end{Bmatrix} = \mathbf{T}^s \hat{\boldsymbol{\varepsilon}} = \mathbf{T}^s \begin{Bmatrix} \hat{\boldsymbol{\varepsilon}}^m \\ \hat{\boldsymbol{\varepsilon}}^b \\ \hat{\boldsymbol{\varepsilon}}^s \end{Bmatrix} \quad (4.67)$$

The transformation of the generalized strains $\hat{\boldsymbol{\varepsilon}}$ to the actual strains $\boldsymbol{\varepsilon}$ at a point is performed via matrix $\mathbf{T}^s$.

$$\mathbf{T}^s_{5 \times 8} = \left[\begin{array}{ccc|ccc|cc} 1 & 0 & 0 & -z' & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -z' & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -z' & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right] \quad (4.68)$$

The generalized local strain vector $\hat{\boldsymbol{\varepsilon}}_e$ can be obtained from the nodal displacement using the derivatives of the shape functions with respect of the natural coordinates (ξ, η) which are grouped in matrix $\mathbf{B}$,

$$\hat{\boldsymbol{\varepsilon}}_e = \mathbf{B}'_e \mathbf{q}'_e = \sum_{i=1}^n \mathbf{B}'_i \mathbf{q}'_i = [\mathbf{B}'_1, \mathbf{B}'_2, \mathbf{B}'_3, \mathbf{B}'_4] \begin{Bmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \\ \mathbf{q}_3 \\ \mathbf{q}_4 \end{Bmatrix} \quad (4.69)$$

24×1

Where $\mathbf{B}'_e$ is the local element strain displacement matrix and $\mathbf{B}'_i$ are the local strain displacement matrices for the i th node element node from Eq. 4.32.

For the local element strains $\boldsymbol{\varepsilon}_e$ we get with Eq. 4.68 and Eq. 4.69,

$$\boldsymbol{\varepsilon}'_e = \mathbf{T}^s \hat{\boldsymbol{\varepsilon}}'_e = \begin{bmatrix} \mathbf{I}_3 & -z\mathbf{I}_3 & \mathbf{0}_2 \\ \mathbf{0}_{2 \times 3} & \mathbf{0}_{2 \times 3} & \mathbf{I}_2 \end{bmatrix} \begin{Bmatrix} \hat{\boldsymbol{\varepsilon}}'^m \\ \hat{\boldsymbol{\varepsilon}}'^b \\ \hat{\boldsymbol{\varepsilon}}'^s \end{Bmatrix} \quad (4.70)$$

8×1

4.3.4 Element stress

The generalized local element stress $\hat{\sigma}^e$ can then be recovered by using Eq. 4.52 and Eq. 4.69,

$$\hat{\sigma}'_e = \mathbf{D}'_{8 \times 8} \hat{\epsilon}'_e = \begin{bmatrix} \mathbf{D}'^{/m} & \mathbf{D}'^{/mb} & 0 \\ \mathbf{D}'^{/mb} & \mathbf{D}'^{/b} & 0 \\ 0 & 0 & \mathbf{D}'^{/s} \end{bmatrix} \begin{Bmatrix} \hat{\epsilon}'^{/m} \\ \dots \\ \hat{\epsilon}'^{/b} \\ \dots \\ \hat{\epsilon}'^{/s} \end{Bmatrix} \quad (4.71)$$

8×1

The local element stress σ^e in the element coordinate system x, y, z can then be recovered by using Eq. 4.68 and Eq. 4.71,

$$\sigma'_e = \mathbf{T}^s_{5 \times 8} \hat{\sigma}'_e = \begin{bmatrix} \mathbf{I}_3 & -z\mathbf{I}_3 & \mathbf{0}_2 \\ \mathbf{0}_{2 \times 3} & \mathbf{0}_{2 \times 3} & \mathbf{I}_2 \end{bmatrix} \begin{Bmatrix} \hat{\sigma}'^{/m} \\ \dots \\ \hat{\sigma}'^{/b} \\ \dots \\ \hat{\sigma}'^{/s} \end{Bmatrix} \quad (4.72)$$

8×1

The local element laminate stress σ'_l in the laminate coordinate system 1, 2, 3 can then be recovered by using Eq. 4.73 and the inverse of the laminate to element coordinate transformation matrix $(\mathbf{T}'_e)^{-1}$ from Eq. 4.51,

$$\sigma'_l = \mathbf{T}^s_{5 \times 8} (\mathbf{T}'_e)^{-1}_{8 \times 8} \hat{\sigma}'_e_{8 \times 1} \quad (4.73)$$

4.4 Probabilistic Finite Element Method

The objective of the PFEM analysis that will be performed is to get as analysis output the sensitivities and variances of the element stress vectors that serve as input for the probabilistic fatigue life analysis using the PCLD damage model the which we get from the stress in terms of their first order moments mean E and variance Var.

To calculate the derivatives, such as from the stiffness matrices, symbolic mathematical computational techniques have been applied, using Python's SymPy module [Meurer et al., 2017].

4.4.1 Taylor series for local element tensors

Discussed in subsection 3.4.1 "PFEM Perturbation using Taylor series" is the development of the First Order Taylor series around the mean value that will be applied for the first order perturbation analysis based PFEM, which enables to obtain the analytical expressions

of the statistical operators mean E and variance Var for the basic static variables displacements, strains and stresses.

From Eq. 3.41 and Eq. 4.70 we can derive the element strain vector mean $\epsilon_e'^0$ and sensitivity $\epsilon_{ei}'^I$ from the generalized local mean $\hat{\epsilon}_e'^0$ and sensitivity $\hat{\epsilon}_{ei}'^I$,

$$\epsilon_e'^0 = \mathbf{T}^s \hat{\epsilon}_e'^0 = \mathbf{T}^s \mathbf{B}_e' \mathbf{q}^0 \quad (4.74)$$

So we get for the first order Taylor series expansion for the local element strain vector of Eq. 3.41,

$$\epsilon_e'^I = \mathbf{T}^s \hat{\epsilon}_e'^I = \mathbf{T}^s \mathbf{B}_e' \mathbf{q}^I \quad (4.75)$$

For the element strains second moments the E and Var of the element strain vector $\epsilon_e(\alpha)$, follow from Eq. 3.43 and Eq. 3.46.

As will be discussed in Chapter 7 "Probabilistic Constant Life Diagram" that for the probabilistic fatigue analysis using a PCLD fatigue model the mean E and variance Var of the element stress vector is required as input. These stress vectors can be obtained from the PFEM analysis for the stress in terms of their second moments mean E and variance Var. Based on Eq. 3.53, the stress tensor is obtained using Eq. 4.71 and Eq. 4.72,

$$\sigma_{ei}'^I = \mathbf{T}^s \hat{\sigma}_e'^I = \mathbf{T}^s \mathbf{D}_e'^0 \epsilon_{ei}'^I + \mathbf{T}^s \mathbf{D}_{ei}'^I \epsilon_e'^0 \quad (4.76)$$

The first order Taylor series around the mean value for the strain vector ϵ_e' with respect to the RV α_i then becomes,

$$\sigma_e'(\alpha) \approx \mathbf{T}^s \left(\mathbf{D}_e' + \sum_{i=1}^n \mathbf{D}_{ei}'^I (\alpha_i - \alpha_i^0) \right) \times \left(\hat{\epsilon}_e'^0 + \sum_{i=1}^n \hat{\epsilon}_{ei}'^I (\alpha_i - \alpha_i^0) \right) \quad (4.77)$$

For local element stress vector σ_e' the estimate of the mean E is derived using Eq. 3.52 and variance using Eq. 3.46 for the case of independent input parameters.

4.5 Static analysis Model validation

A comparison have been made between the 4 node Reissner-Mindlin principle based element implemented in PFEMpy and the same model using a similar element in ABAQUS [Dassault Systèmes Simulia Corp, 2021], the S4R 4-node quad element. Both elements have 5 DOFs and use reduced integration. The S4R, ABAQUS element further uses hourglass control to prevent the hourglass effect.

4.5.1 Model description

Identical flat plate models were be created and meshed in both FEM tools, with dimensions of 500mm by 200mm, build up out of a mesh of 50mm x 50mm shell elements, with the mesh as shown in Fig. 5.4. The layup of each element consists of 4 layers of UD2 UD of each 0.88mm thick [Nijssen et al., 2006b].

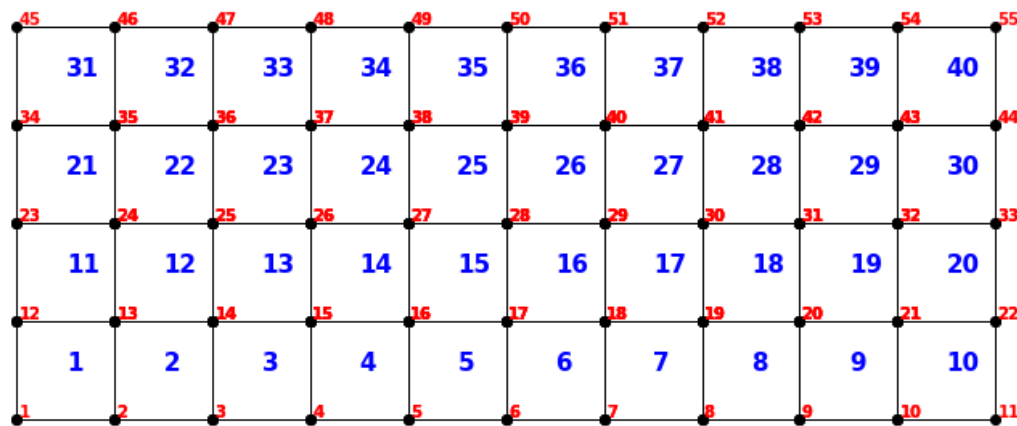


Figure 4.5: FE plate mesh node (red) and element (blue) numbers

4.5.2 Static analysis results

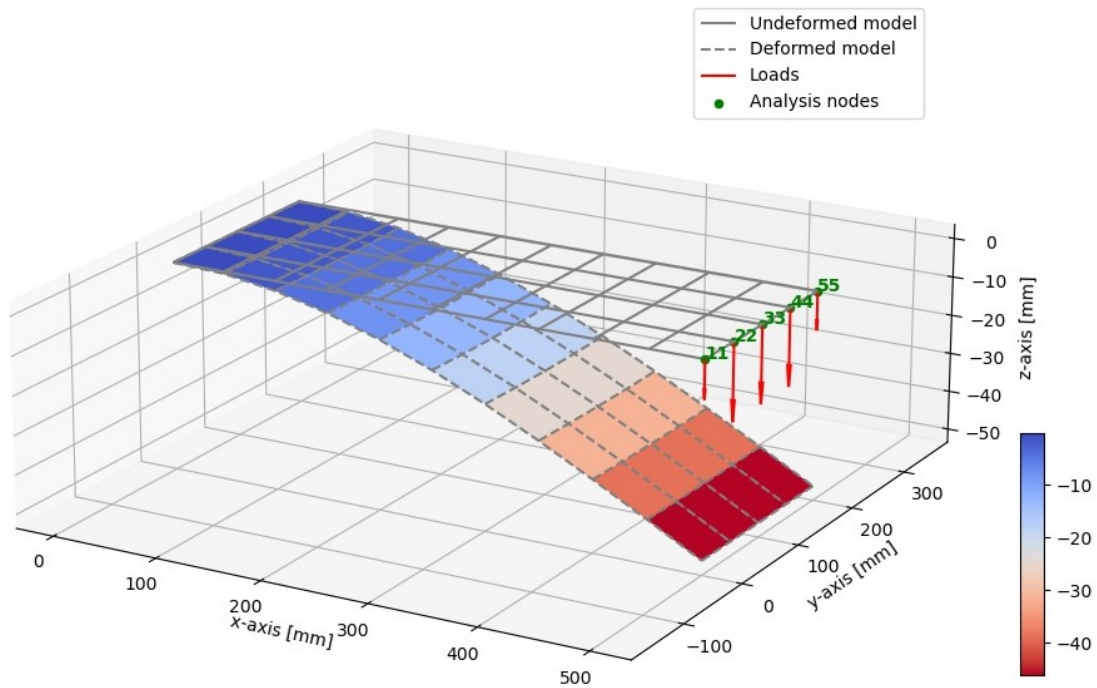
At the first analysis, the BC applied at both models consist of one of the short sides (root side) all the nodes are clamped (all DOFs fixed) and the loads are applied at the opposite side (tip side) divided on the nodes, as shown in Fig. 4.6a,

The resulting nodal displacements of the plate model as shown in Fig. 4.6a.

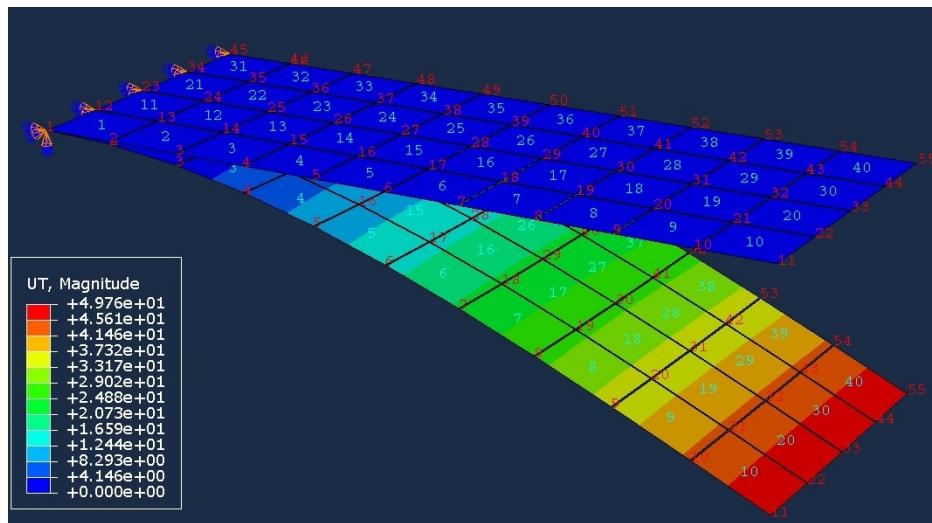
SHELL	q_z nodal displacement [mm]		Error
NODE	PFEMpy	ABAQUS	[%]
11	-49.643	-49.643	0.001 %
22	-49.717	-49.725	0.015 %
33	-49.747	-49.756	0.018%
44	-49.717	-49.725	0.015 %
55	-49.643	-49.643	0.001 %

Table 4.3: Deflections PFEMpy vs ABAQUS

Table 4.3 lists the tip node deflection from the PFEMpy analysis against the validation by ABAQUS. It reveals very small errors between the displacements results from the



(a) PFEMpy Undeformed and deformed shape



(b) ABAQUS Undeformed and deformed shape

Figure 4.6: Analysis composite plate with tip load

PFEMpy and ABAQUS analysis results, with the average error over all the plate tip nodes of only $\approx 0.01\%$ and therefore very similar displacement behavior of both elements.

Table 4.4 lists the tip most element stress from the PFEMpy analysis against the validation by ABAQUS. Also here showing relative small errors between the stress results from the PFEMpy and ABAQUS analysis results, with the average error over all the plate

SHELL ELEMENT	σ'_{11} element stress [MPa]		Error [%]
	PFEMpy	ABAQUS	
10	-1.89	-1.84	2.83 %
20	-2.03	-1.94	4.33 %
30	-2.03	-1.94	4.33 %
40	-1.89	-1.84	2.83 %

Table 4.4: Stress PFEMpy vs ABAQUS

tip nodes of only $\approx 3.5\%$ and therefore very similar stress field behavior of both elements used by each tool/software.

Chapter 5

Probabilistic Wind Turbine Blade Data

In this chapter the analysis input data and models that will be used for the different probabilistic analysis demonstrated in this thesis will be discussed.

5.1 Composite laminate material data

To perform a probabilistic analysis, sufficient material data has to be available, preferable from a large material coupon testing campaign, to assure well validated material data is used and statistically valid probabilistic material parameters such as the mean E and variance Var can be analyzed.

Property	Distribution	E	Var	CV [%]
E_{11} [MPa]	Normal	39042	1065716	2.64
E_{22} [MPa]	Normal	14077	105487	2.31
G_{12} [MPa]	Normal	4239	9835	2.34
ν_{12}	Normal	0.2906	0.737e-03	9.34
t_{ply} [mm]	Normal	0.88	0.013	11.36

Table 5.1: OPTIMAT UD2 material stiffness properties [Philippidis et al., 2006]

In the OPTIMAT project [Nijssen et al., 2006a], a comprehensive experimental program was performed and the OptiDAT material database [Nijssen et al., 2006b] was created to fully characterize the in-plane mechanical behaviour of Glass/Epoxy material of laminates typical used in WTB. Further probabilist analysis was performed to analyse failure probability under uni-axial and multi-axial static and fatigue loading [Philippidis et al., 2006] and therefore the statistical moments such as the mean E, variance Var and the Coefficient of Variation (CV) were determined as part of the OptiDAT database project.

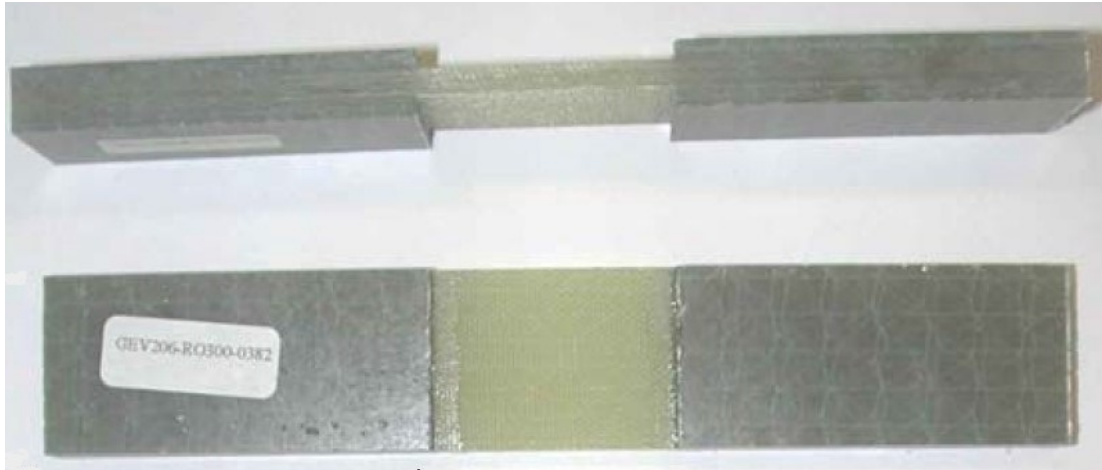


Figure 5.1: UD2 Test coupon UD laminate [Nijssen et al., 2006b]

UD Glass/Epoxy plies, defined in the OPTIMAT material database with material designation UD2 was used, which refers to laminae of 1250 Gram per Square Meter (GSM) E-glass woven mat, with the material (stiffness) properties as defined in Table 5.1.

For the perturbation approach that will be applied in this thesis only the first two statistical moments mean E and the variance Var enter the analysis. Further the CV are given, being the standard deviation divided by the mean expressed as a percentage showing the extent of variability in relation to the E of the population. The distribution type is stated, as this will be used to generate random samples for the MCS analysis, for the defined distribution.

The laminate static strength properties, required as input for the CLD fatigue model were taken from Philippidis et al. [2006] as stated in Table 5.3.

Property	Distribution	E	Var	CV [%]
UTS [MPa]	Normal	776.5	1306.3	4.66
UCS [MPa]	Normal	-521.82	272.27	3.16

Table 5.2: OPTIMAT UD2 static strength properties [Philippidis et al., 2006]

The laminate fatigue properties were determined based on the CA fatigue test data for UD2 from the OptiDAT material database [Nijssen et al., 2006b] CA, as show in Fig. 5.2. The S-N slope m parameter was defined using [ASTM E739-91, 2015], with the results stated in Table 5.3.

Property	Distribution	E	Var	CV [%]
m	Normal	9.28	0.01	1.1

Table 5.3: OPTIMAT UD2 fatigue properties, according to ASTM E739-91 [2015]

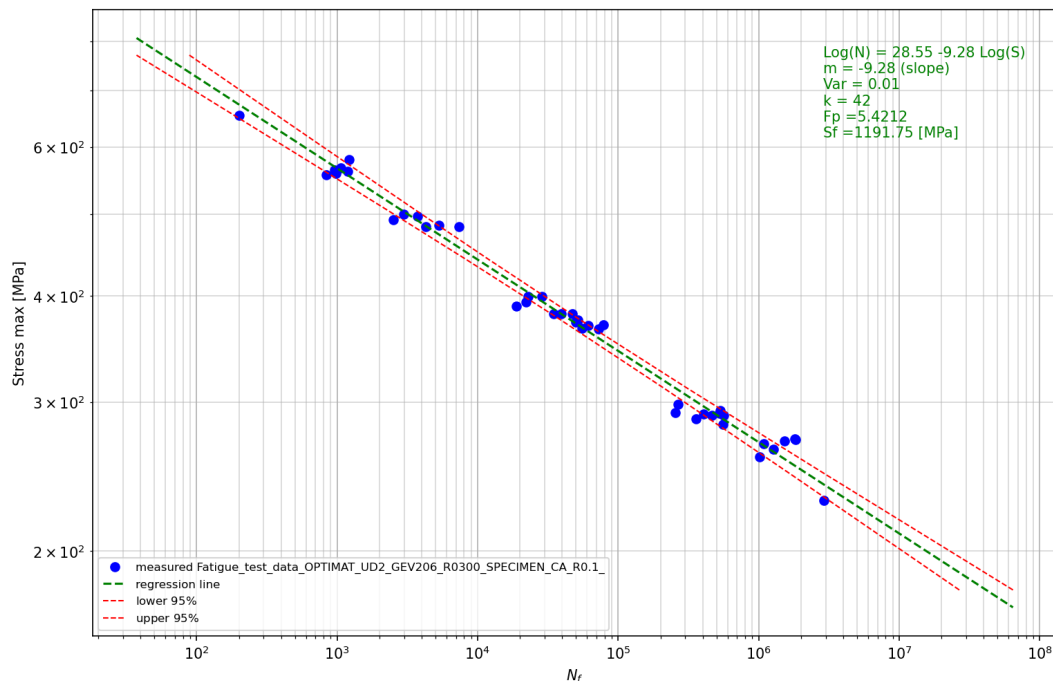


Figure 5.2: UD2 CA fatigue test data [Nijssen et al., 2006b]

5.2 Structural model

Finite element models are typically used for the structural modeling and analysis of a Wind Turbine Blade. Although mainly deterministic parameters are used in FEM, despite the fact that many problems to which they are applied involve uncertainties of considerable degree, in this thesis different probabilistic based studies are performed with a realistic application to a WTB structure.

A WTB structure is typically modeled as a slender thin walled shell structures using a composite layup for the larger part. From the full WTB blade, a SC section was chosen to be investigated in detail, as shown in Fig. 5.3. As discussed earlier in subsection 2.2.1 "Construction And Materials" the SC is (one of) the highest loaded components of a WTB taking up the main lateral bending loads and axial loads. Typically there are two or more SC in each blade shell half to take the normal strain in blade longitudinal direction with in between the SW to take up the shear loads and keep the SC separated from each other.

A full WTB structural FE model easily consists of more than 100.000 elements and/or nodes, which is already for simple static linear analysis quite computational costly to analyze. Therefore in order to simplify the modeling and especially the analysis that will be performed, from a larger blade section containing two SC sections a smaller section will be modeled of one of the SC, to perform the probabilistic analysis. Modeling and analyzing only a smaller section allows the use of a custom made PFEMpy finite element code, as discussed in Chapter 4 "PFEM python code (PFEMpy)", which contain validation by MCS,

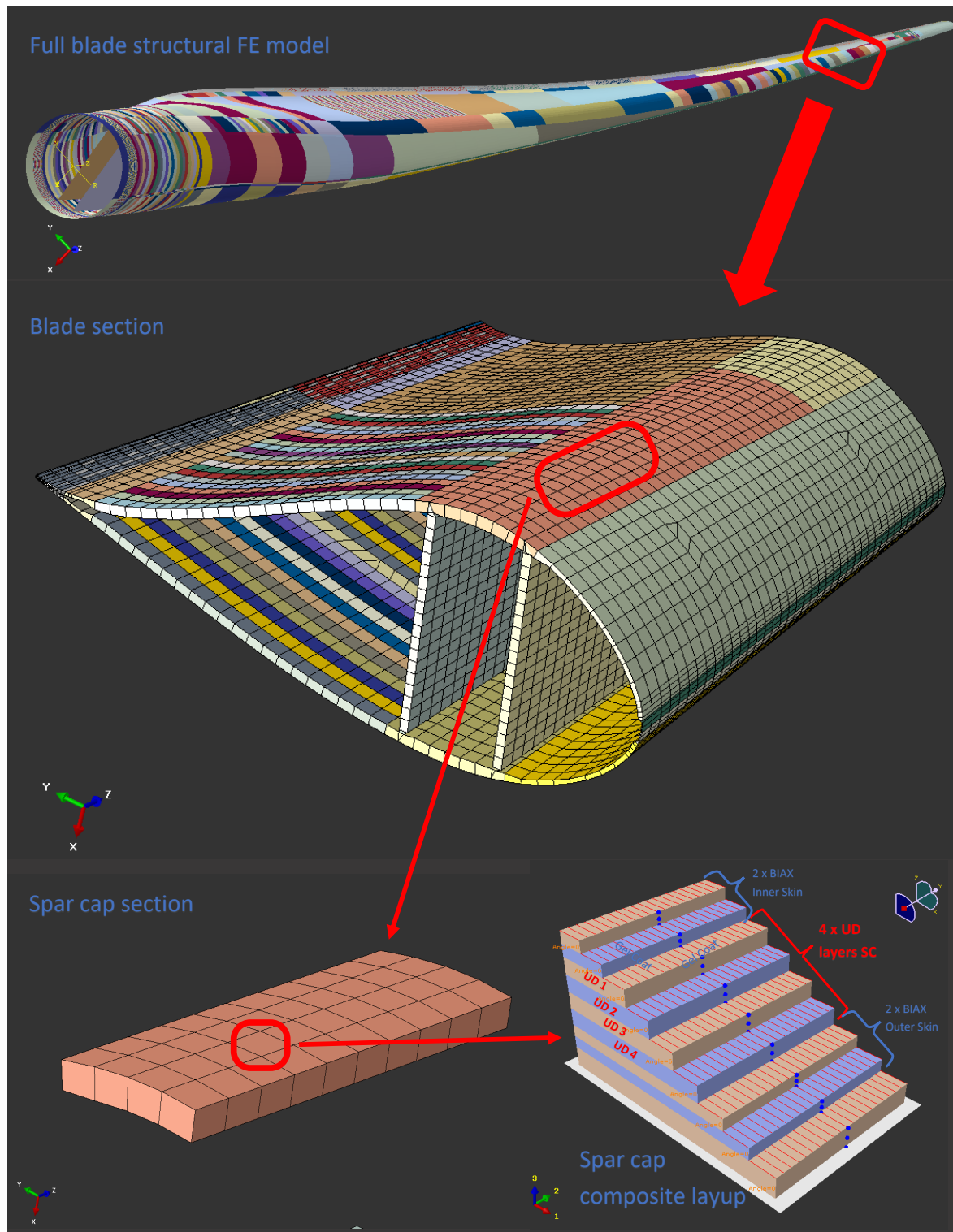


Figure 5.3: Structural FE model of full WTB to detailed SC plate model

which is computational costly and will consume large amount of time to perform on a full WTB or even a blade section, but much less time on a small model containing much less

DOFs and elements. Nonetheless a representative plate model from a part of a SC section, shown at the bottom of Fig. 5.3, will be used for further analysis. The plate model has as dimensions 500mm by 200mm, build up out of a mesh of 50mm x 50mm with 4 node quad shell elements, with the mesh as shown in Fig. 5.4. The layup of each element of the model consists of UD2 1250 GSM E-glass/Epoxy laminate plies, as defined in the OPTIMAT material database Nijssen et al. [2006b], which is a representative UD laminate as used in actual WTBs.

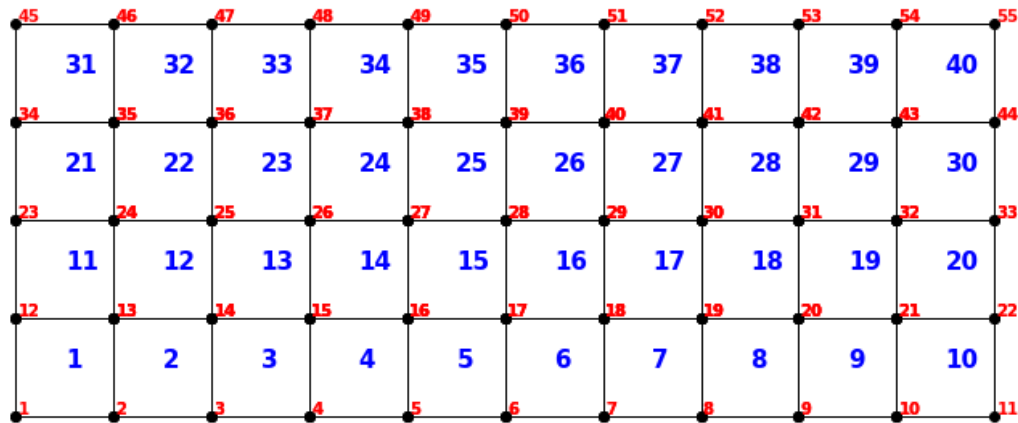


Figure 5.4: FE plate mesh node (red) and element (blue) numbers

For the plate section, instead of the very thick SC laminate, the same layup as the test samples in Philippidis et al. [2006], see Fig. 5.1 was used, so a layup of $[0^\circ]_4$, which means four plies in orientation 0° , of each 0.88mm thick, so with a resulting nominal laminate thickness of $\sim 3.5\text{mm}$, as defined in the OPTIMAT material database Nijssen et al. [2006b].

5.3 Loading and boundary conditions

Loads as nodal forces will be used at different stages and for different types of analysis, such as:

1. **Static deterministic analysis:** to define the mean E of the displacement and stress for the Taylor expansion based PFEM analysis, see subsection 5.3.1 "Static and random load" for the static load definition and section 6.3 "PFEM analysis with MCS validation" for the static analysis performed;
2. **RV for PFEM analysis:** RV are the stochastic character of the external loads on the plate that results consequently in the probability density of the deflection distribution from the analysis model, from which the sensitivity of the RP such as displacement and stress can be analyzed, see subsection 5.3.1 "Static and random load" for

the random load definition and section 6.5 "PFEM with force modulus as RV" for analysis with the nodal force as RV;

3. **Probabilistic fatigue analysis:** Block loads with each variable multiple cycle duration, as defined in subsection 5.3.2 "Fatigue multiple block load spectrum".

5.3.1 Static and random load

In case the load/force is used as RV, the stochastic character of the external load is used and therefore besides a mean E load, also a realistic variance Var load is required, the NEW WHISPER wind turbine specific loading spectrum Bulder et al. [2005], as shown in Fig. 5.5 was used to obtain a realistic variance on a WTB load spectrum. This NEW WHISPER loads spectrum was developed from field test data from different wind turbines, mostly (multi) MW machines.

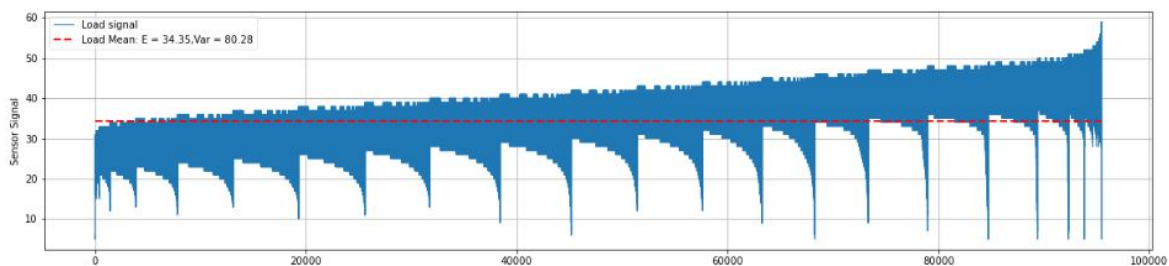


Figure 5.5: NEW WHISPER loading spectrum [Bulder et al., 2005]

As mean load of the load spectrum from Table 5.4, the signal value was used as total force in Newtons, which gave a realistic total nodal force on the structural model used for the PFEM analysis performed in section 6.3 "PFEM analysis with MCS validation".

Property	Distr. Type	E	Var	CV [%]
$F[N]$	Weibull	-34.35	80.28	26.08 %

Table 5.4: Stochastic load properties variables based on NEW WHISPER loading spectrum

This total nodal force was divided on the nodes of the respective loaded edge of the plate model. Using further the variance Var on the NEW WHISPER loads spectrum, a Weibull distribution was applied for the random load distribution to generate the random loads for the MCS, where the amount of random loads depends on the sample size analyzed.

5.3.2 Fatigue multiple block load spectrum

For the probabilistic fatigue life analysis performed with multiple load blocks in this study, three different CA load blocks were defined with each a different duration (amount) of

cycles as illustrated in Fig. 5.6.

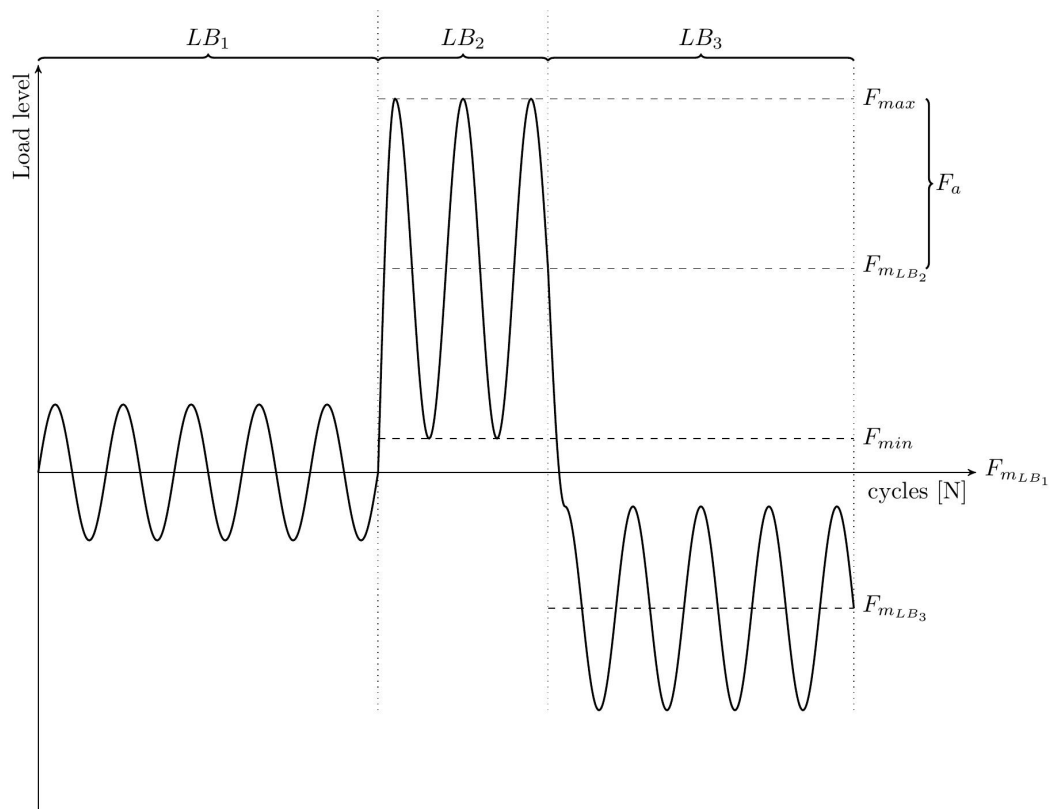


Figure 5.6: Definition of load blocks

The load blocks were as such defined that each load block has besides a different load amplitude and mean also a different load pattern at distinctive load ratio, e.g TT, Tension - Compression loading (TC) and CC Table 5.4. As the effect of the different mean stress levels of the various loading blocks is very critical for the fatigue life of composite materials [Vassilopoulos and Keller, 2011].

Load	[N]	[N]	Load ratio
LB1	100	0	TC
LB2	200	250	TT
LB3	-200	150	CC

Table 5.5: Multiple load block load spectrum properties

For the PFEM analysis as part of the probabilistic fatigue analysis, the force amplitude F_a and mean load F_m of the load blocks from Table 5.5 were applied to the model as the total force in Newtons. This total or load sum was divided on the nodes of the respective loaded edge of the plate model, depending on the analysis.

Chapter 6

Probabilistic FEM analysis on composite plate

6.1 Introduction

In this chapter the application of Probabilistic Finite Element Method using the perturbation method will be demonstrated and validated, as discussed in section 3.4 "Probabilistic finite element method (PFEM)". Applying the perturbation method with first order Taylor series expansions, stochastic randomness is introduced into the system in order to obtain the statistical second moments mean E and Variance Var .

The sensitivities are first computed of the RV with respect to RP, which serve as input for the Taylor series expansions, as shown in Fig. 6.1, followed by the first moments up to order 2, i.e. the mean E and Variance Var , which give the distribution of the response.

The sensitivities are obtained by calculating the derivatives of the structural matrices with respect to the RV. As a by the thesis author custom programmed PFEM code is used [PFEMpy: Rens, 2022], this has been done directly calculating the derivatives of the matrices.

The resulting RP analysis output mean E and the Variance Var of the displacement, strain and stress response from the PFEM analysis is later used as input for probabilistic fatigue life analysis using probabilistic fatigue damage accumulation models.

MCS analysis, as discussed in subsection 3.3.2 "Monte Carlo Simulation", was applied to analyze the sensitivities (gradient based) and statistical first order moments, mean E and Variance Var , in order to validate and compare the PFEM results.

PFEM based methods are applicable solutions when analyzing probabilistic behavior of structural systems with random parameters is required, which combines FEM with statistics and reliability methods and can be applied to linear and non-linear structural mechanics as well as fracture mechanics problems.

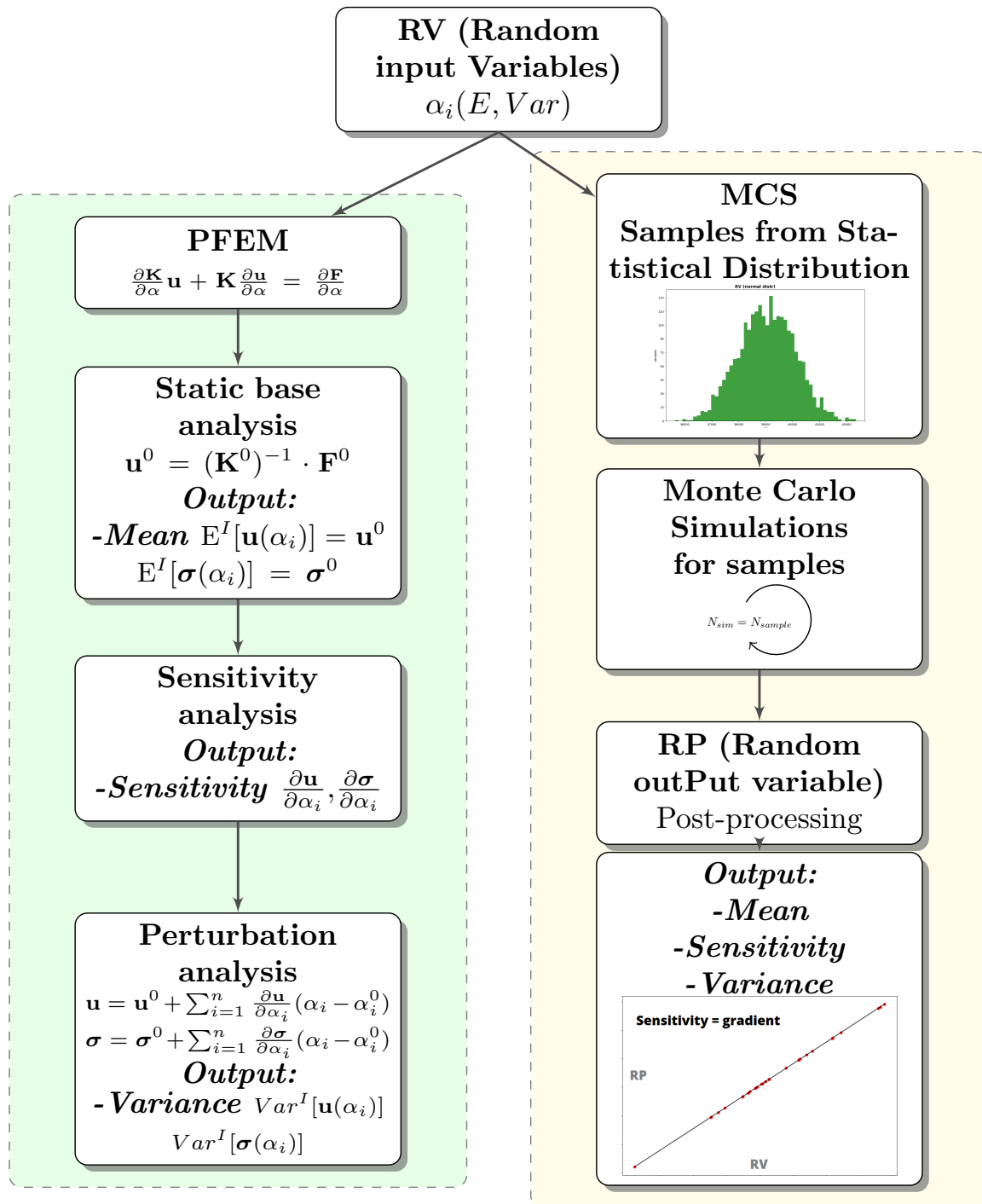


Figure 6.1: PFEM and MCS analysis procedures

6.2 PFEM implementation

The mean E , sensitivity and variance Var of the displacement with respect to the RV of interest were estimated using PFEM by means of a first order Taylor series. To validate the

accuracy of these PFEM analysis and results, the E, sensitivity and Var of the deflection of the plate were calculated by MCS analysis. All analysis were performed and programmed using the custom PFEMpy code as discussed in Chapter 4 "PFEM python code (PFEMpy)", with the model and data as discussed in Chapter 5 "Probabilistic Wind Turbine Blade Data".

The PFEM implementation on the FE for each RV a first order approximation is performed with the following steps,

1. **Perform static base analysis:** Solving the linear static analysis of the plate using the mean values of the RV's, so solving $\mathbf{Ku} = \mathbf{F}$: Using matrix $\mathbf{K}$ and the vector $\mathbf{F}$ evaluated in the mean values of the random variables ($\mathbf{K}^0$ and $\mathbf{F}^0$).

Output: Mean E values for the displacement $\mathbf{u}_e^0$ for the nodes of each element and element strain ϵ_e^0 and stress σ_e^0

2. **Analyse the sensitivities:** Where the derivatives of the terms in the stiffness matrices for each element $\mathbf{K}^e$ are calculated.

Output: Sensitivities $\frac{\partial RP}{\partial \alpha_i}$ of the RPs with respect to α_i the RVs, with as input RVs the stiffness properties and/or loads and as output RPs the displacement $\frac{\partial \mathbf{u}}{\partial \alpha_i}$, element strain $\frac{\partial \epsilon_e}{\partial \alpha_i}$ and stress $\frac{\partial \sigma_e}{\partial \alpha_i}$ relative to the RV α_i .

3. **Analyse the Variance:** Using first-order Taylor series expansions, applying the sensitivities obtained in the previous step, the variance can be obtained for the output RPs (displacement, strain and stress) by:

$$\text{Var}[\sigma(RP)] = \sum_{i=1}^n \left(\frac{\partial RP}{\partial \alpha_i} \right)^2 \text{Var}(\alpha_i)$$

Output: Variance Var with respect to the nodal displacements, element strain or stress.

6.3 PFEM analysis with MCS validation

The PFEM perturbation analysis are performed using the first order Taylor expansions to analyse the probabilistic response of the composite plate, resulting directly in the sensitivities and the first moments E and Var of the RP with respect to the RV. The resulting sensitivities, mean E and variance Var from the PFEM analysis are validated by MCS, with the steps, as shown in Fig. 6.1, consisting of: (1) select distribution type of the RV input design variables; (2) generation of samples (using Python's numpy random sample generator) for the defined distribution with the E and Var of the RVs ; (3) Perform MCS for the amount of samples generated; (4) Post processing of the data for the RP to obtain the sensitivity from the gradient of the curve fitted through the RP values, followed by analysing the Es and Vars from the same analysis results.

For the validation of the PFEM analysis results performed for the different RV, $N_{sim} = 5000$ simulations were run for the MCS analysis.

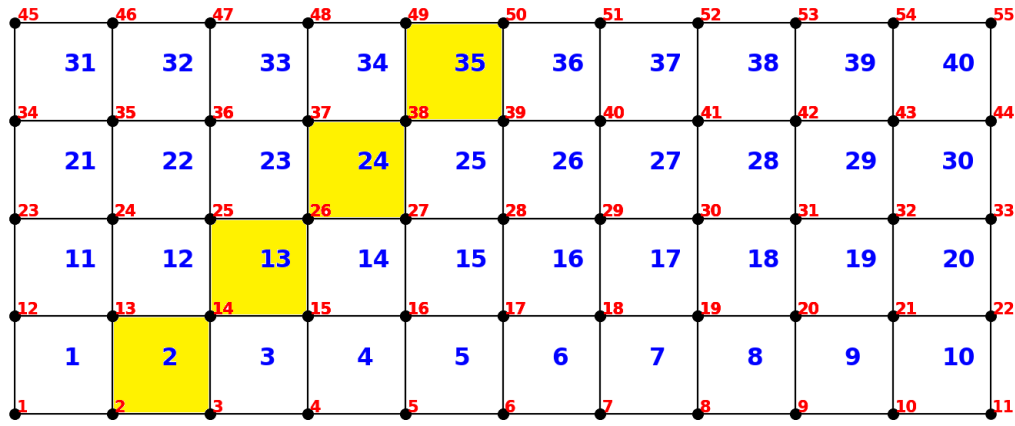


Figure 6.2: PFEM and MCS analysis elements

Elements 2, 13, 24 and 35 are chosen to be analysed, as marked in yellow in Fig. 6.2, as these elements are expected to be in a region of the model where higher stresses are expected. The elements are further chosen in a diagonal line to have sufficient level differences between the stress.

6.4 PFEM with respect to material stiffness properties

The PFEM analysis was performed for the material stiffness properties Young modulus E_{11} and the shear modulus G_{12} as RVs. To obtain the sensitivities of these properties, the following equations had to be solved,

$$\frac{\partial \mathbf{K}}{\partial \alpha_i} \mathbf{u} + \mathbf{K} \frac{\partial \mathbf{u}}{\partial \alpha_i} = 0 \quad (6.1)$$

$$\mathbf{K} \frac{\partial \mathbf{q}}{\partial \alpha_i} = -\frac{\partial \mathbf{K}}{\partial \alpha_i} \mathbf{u} \quad (6.2)$$

$$\frac{\partial \mathbf{q}}{\partial \alpha_i} = -\frac{\partial \mathbf{K}}{\partial \alpha_i} \mathbf{q} \cdot \mathbf{K}^{-1} \quad (6.3)$$

The sensitivity $\frac{\partial \mathbf{K}}{\partial \alpha_i}$ of the global model stiffness matrix $\mathbf{K}$ with respect to the applicable RV α_i , were directly derived in the PFEMpy code calculating the derivatives of the stiffness matrix elements, which was then multiplied with the global nodal displacement $\mathbf{q}$ of the static analysis and then divided by the global stiffness matrix $\mathbf{K}$ by multiplying with the inverse $\mathbf{K}^{-1}$ from Eq. 6.3.

$$E[\mathbf{q}'_e(\alpha)] \approx \mathbf{q}_e'^0 \quad (6.4)$$

With the analyzed mean displacements $\mathbf{q}$, using Eq. 6.4, with Eq. 3.51 the required mean local element stress vector σ'_e can be obtained with,

$$E[\sigma'_e(\alpha_i)] \approx \sigma_e'^0 \approx \mathbf{D}_e'^0 \mathbf{B}_e \mathbf{q}_e'^0 \quad (6.5)$$

With the analyzed sensitivity $\frac{\partial \mathbf{q}}{\partial \alpha_i}$ of the displacement with respect to the applicable RV α_i , the required sensitivity $\frac{\partial \sigma}{\partial \alpha_i}$ of the local element stress vector with respect to the applicable RV α_i can be obtained, using Eq. 3.53, to get,

$$\frac{\partial \sigma'_e}{\partial \alpha_i} = \mathbf{D}_e'^0 \mathbf{B}_e \mathbf{q}_e'^I + \mathbf{D}_{ei}^I \mathbf{B}_e \mathbf{q}_e'^0 \quad (6.6)$$

The Variances Var of the stress with respect to α_i are calculated using Eq. 6.7, applying the earlier obtained sensitivities $\frac{\partial \sigma}{\partial \alpha_i}$ of Eq. 6.6.

$$\text{Var}[\sigma'_e(\alpha_i)] \approx \sum_{i=1}^n \left(\frac{\partial \sigma'_e}{\partial \alpha_i} \right)^2 \text{Var}[\alpha_i] \quad (6.7)$$

6.4.1 PFEM analysis with respect to Young's modulus E_{11}

The boundary conditions applied at the model for this analysis consist of one of the short sides (root side) the nodes have encastre boundary conditions (so zero translation and rotations) and the loads are applied at the opposite side (tip side) divided on the nodes, resulting in the nodal displacements of the plate model as shown in Fig. 6.3.

Table 6.1 lists the mean local element stress with respect to E_{11} obtained from the PFEM analysis against the validation by MCS, revealing very small errors between both analysis results.

SHELL ELEMENT	Mean stress σ'_{11}		Error [%]
	PFEM	MCS	
2	3.2591e+01	3.2592e+01	2.178e-03 %
13	2.7682e+01	2.7682e+01	2.167e-04 %
24	2.4161e+01	2.4161e+01	4.139e-04 %
35	2.0971e+01	2.0971e+01	1.860e-03 %

Table 6.1: Mean stress σ'_{11} with respect to E_{11}

Table 6.2 lists the sensitivity analysis results obtained from the PFEM analysis against the validation by MCS analysis. Showing that the sensitivities of the stress tensor σ with respect to the Young's modulus E_{11} obtained with the PFEM analysis are very low, with further the average error over all the nodes between the PFEM and MCS analysis also very low at ≈ 2.3 %.

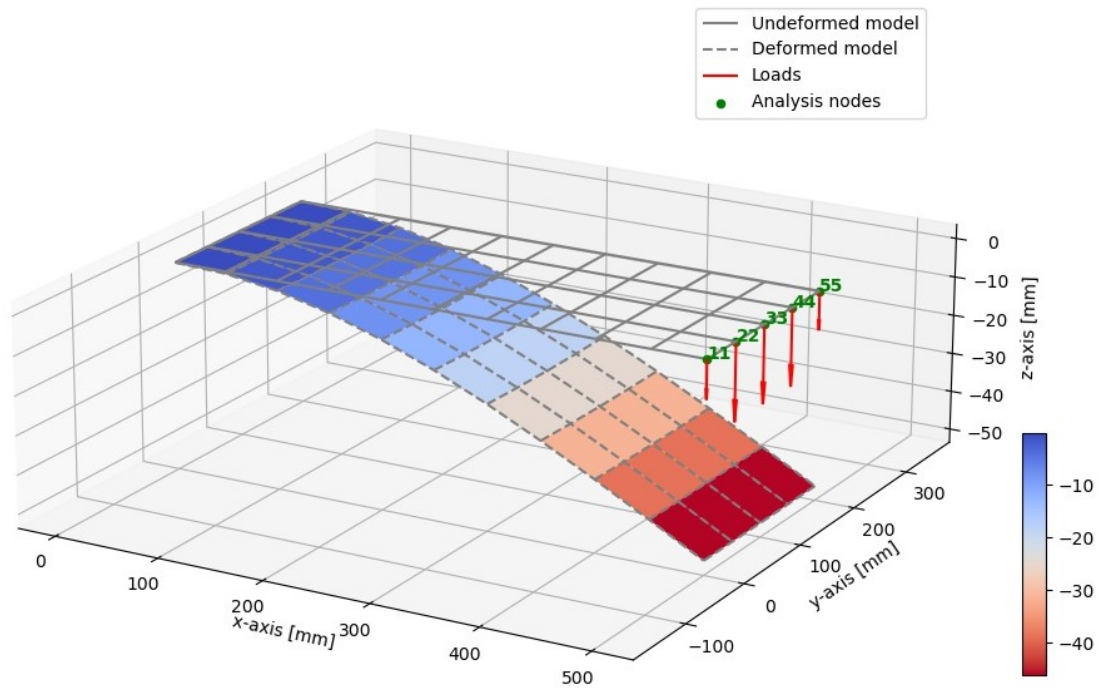


Figure 6.3: Undeformed and deformed shape of composite plate with tip load

SHELL ELEMENT	Sensitivity $\frac{\partial \sigma'_{11}}{\partial E_{11}}$		Error [%]
	PFEM	MCS	
2	1.1311e-05	1.1342e-05	0.27 %
13	1.6566e-06	1.7102e-06	3.24 %
24	3.1150e-06	2.9859e-06	4.04 %
35	2.8110e-06	2.7655e-06	1.61 %

Table 6.2: Sensitivities on stress σ'_{11} with respect to E_{11}

Table 6.3 shows the Variance Var results obtained for the stress from the PFEM analysis against the Variance from the MCS analysis results.

SHELL ELEMENT	Variance σ'_{11}		Error [%]
	PFEM	MCS	
2	1.3635e-04	1.3236e-04	2.93 %
13	2.9246e-06	3.0706e-06	4.99 %
24	1.0341e-05	9.9761e-06	3.53 %
35	8.3208e-06	7.9672e-06	4.25 %

Table 6.3: Variances stress σ'_{11} with respect to E_{11}

The average error of the Var obtained with the PFEM and MCS analysis for $\text{Var}^I[\mathbf{u}(E_{11})]$ for all the elements is $\approx 4.0\%$, which is very good for a first order perturbation analysis method.

6.4.2 PFEM analysis with respect to shear modulus G_{12}

The boundary conditions used here are at the root short side and left long sides (left side) the nodes are clamped and the load is applied divided on the nodes on the opposite sides (tip side and right side), resulting in the nodal displacements as shown in Fig. 6.4.

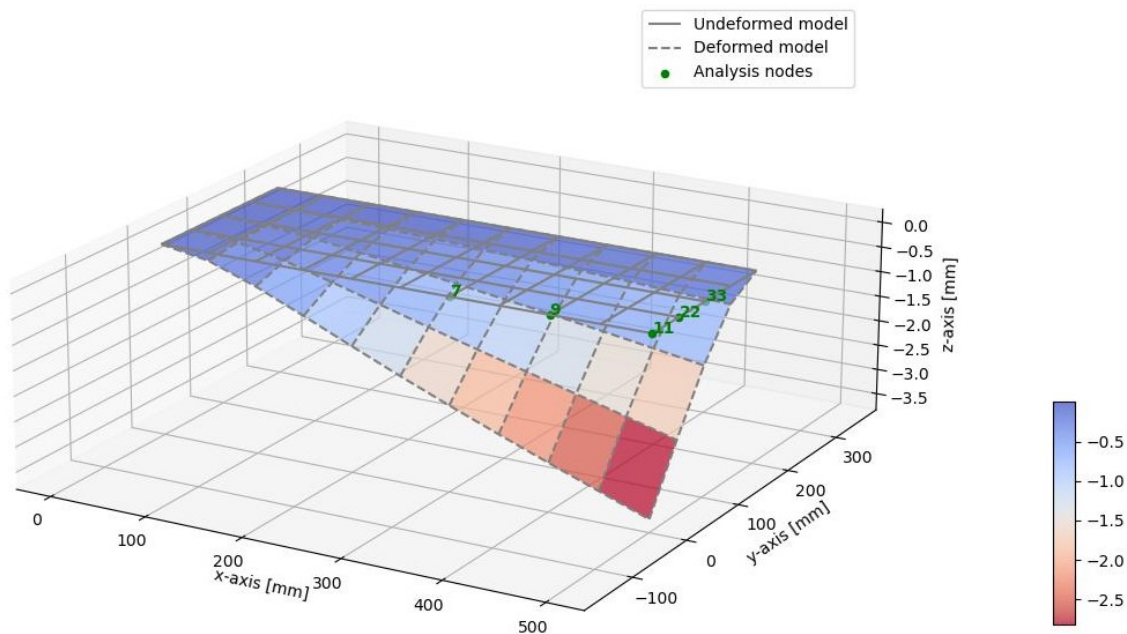


Figure 6.4: Undeformed and deformed shape of composite plate with side and tip load

Table 6.4 lists the mean stress σ'_{11} with respect to G_{12} obtained from the PFEM analysis against the validation by MCS, revealing again very small errors between both analysis results.

SHELL ELEMENT	Mean stress σ'_{11}		Error [%]
	PFEM	MCS	
2	-2.4775e+00	2.4774e+00	3.43e-03 %
13	-9.0230e-01	9.0226e-01	4.82e-03 %
24	-4.6301e-01	4.6299e-01	4.99e-03 %
35	-6.1400e-01	6.1398e-01	2.62e-03 %

Table 6.4: Mean longitudinal stress σ'_{11} with respect to the shear modulus G_{12}

Table 6.5 lists the sensitivity analysis results obtained from the PFEM analysis against the validation by MCS analysis. Showing that the sensitivities of the stress σ with respect to the shear modulus G_{12} obtained with the PFEM analysis are very low, with further the average error between the PFEM and MCS analysis for these elements also very low at only $\approx 0.03\%$.

SHELL ELEMENT	Sensitivity $\frac{\partial \sigma'_{11}}{\partial G_{12}}$		Error [%]
	PFEM	MCS	
2	1.1369e-04	-1.1366e-04	2.4153e-02 %
13	5.8752e-05	-5.8732e-05	3.4154e-02 %
24	3.3926e-05	-3.3911e-05	4.5094e-02 %
35	2.0029e-05	-2.0021e-05	4.3525e-02 %

Table 6.5: Analysis sensitivities on stress σ'_{11} with respect to the shear modulus G_{12}

Table 6.6 show the VarianceVar results obtained from the PFEM analysis against the Variance from the MCS results.

SHELL ELEMENT	Variance σ'_{11}		Error [%]
	PFEM	MCS	
2	1.2711e-04	1.2316e-04	3.11 %
13	3.3948e-05	3.2885e-05	3.13 %
24	1.1320e-05	1.0964e-05	3.15 %
35	3.9455e-06	3.8214e-06	3.15 %

Table 6.6: Analysis variance on stress σ'_{11} with respect to the shear modulus G_{12}

Finally for the RV G_{12} the average error of the Var obtained with the PFEM and MCS analysis for the stress σ'_{11} for all the elements is $\approx 3.13\%$, which again shows a very good correlation for a first order perturbation analysis.

6.5 PFEM with force modulus as RV

The boundary and loading conditions used here are similar as for E_{11} as RV with one of the short sides (root side) the nodes are clamped and the load is applied divided on the nodes on the opposite side (tip side), resulting in the nodal displacements as shown in Fig. 6.3. In this case the acting nodal force modulus $\|F\|$ is considered RV so we have to solve,

$$\mathbf{K} \frac{\partial \mathbf{q}}{\partial \|F\|} = \frac{\partial \mathbf{F}}{\partial \|F\|} \quad (6.8)$$

To obtain the sensitivity for $\|F\|$ as RV a first order approximation is performed by solving the linear static analysis of the plate with $\|F\| = 1$, so Eq. 6.8 becomes,

$$\begin{aligned} \mathbf{K} \frac{\partial \mathbf{q}}{\partial \|F\|} &= 1 \\ \frac{\partial \mathbf{q}}{\partial \|F\|} &= \mathbf{K}^{-1} \cdot 1 \end{aligned} \quad (6.9)$$

With Eq. 6.9 resulting directly in the sensitivity of the nodal displacement $\mathbf{q}$ relative to $\|F\|$, $\frac{\partial \mathbf{q}}{\partial \|F\|}$. For the stress the mean E values were obtained using Eq. 3.55, the sensitivities using Eq. 3.54 and variances Var were obtained using Eq. 6.7.

The mean of the stress component σ'_{11} with respect to $\|F\|$ obtained from the PFEM analysis against the validation by MCS are listed in Table 6.7, revealing again a small average error over all the nodes of ≈ 0.85 between both analysis results.

SHELL ELEMENT	Mean stress σ'_{11} PFEM	MCS	Error [%]
2	-3.2591e+01	-3.2317e+01	0.85 %
13	-2.7682e+01	-2.7449e+01	0.85 %
24	-2.4161e+01	-2.3957e+01	0.85 %
35	-2.0971e+01	-2.0794e+01	0.85 %

Table 6.7: Mean stress σ'_{11} with respect to force modulus $\|F\|$

Table 6.8 shows the sensitivities results obtained from the PFEM analysis against the validation by MCS for the stress σ'_{11} with respect to the RV $\|F\|$. These sensitivities are higher than those obtained for the material (stiffness) properties, showing the higher sensitivity of the nodal force $\|F\|$ with respect to the stress field σ .

SHELL ELEMENT	Sensitivity $\frac{\partial \sigma'_{11}}{\partial \ F\ }$ PFEM	MCS	Error [%]
2	9.4880e-01	9.4880e-01	1.9317e-06 %
13	8.0588e-01	8.0588e-01	1.4106e-06 %
24	7.0338e-01	7.0338e-01	1.3780e-06 %
35	6.1050e-01	6.1050e-01	5.2429e-06 %

Table 6.8: Analysis sensitivities stress σ'_{11} with respect to force modulus $\|F\|$

As shown in Table 6.8, the sensitivities of the stress σ with respect to $\|F\|$, are higher than those obtained for the material properties, showing the high sensitivity of the nodal force $\|F\|$ on the stress field σ . The average error between sensitivity analysis using the PFEM and MCS methods for all the nodes are very low (almost zero), showing the high accuracy of the PFEM analysis.

Applying the perturbation method with a first order Taylor expansion, the Var of the deflection with respect to $\|F\|$ can be calculated using Eq. 6.7 applying the earlier obtained sensitivities $\frac{\partial \sigma}{\partial \|F\|}$ by,

$$\text{Var}^I[\sigma(\|F\|)] = \sum_{i=1}^n \left(\frac{\partial \sigma}{\partial \|F\|} \right)^2 \text{Var}(\|F\|) \quad (6.10)$$

Table 6.9 show the Variance results obtained from the PFEM analysis against the Variance from the MCS analysis results.

SHELL ELEMENT	Variance σ'_{11}		Error [%]
	PFEM	MCS	
2	7.2270e+01	7.1026e+01	1.72 %
13	5.2137e+01	5.1239e+01	1.72 %
24	3.9718e+01	3.9034e+01	1.72 %
35	2.9921e+01	2.9406e+01	1.72 %

Table 6.9: Analysis variance stress σ'_{11} with respect to force modulus $\|F\|$

The average error of the Var obtained with the PFEM and MCS analysis for $\text{Var}(\|F\|)$ for all the nodes is $\approx 1.72\%$, which is in a similar range as for the material properties and very good for a first order perturbation analysis.

6.6 Conclusions

The application of a PFEM method has been developed and presented in this chapter for a laminated composite plate. Partial derivative based sensitivity analysis was applied to obtain the sensitivities of the RV with respect to the RP followed by the perturbation method to obtain the Var. The material mechanical properties and applied loads were considered being stochastic.

A direct comparison has been performed of the analysis results of the PFEM of a composite layup plate with the MCS analysis results, which show very good agreement between the two analysis for the analyzed cases. An average error for the perturbation method obtained Variance of $\approx 2.0\%$ was found, which is relative accurate, considering

only a first order perturbation analysis was performed. One of the main gains of using the PFEM method presented herein is the large minimization of computation time. Although MCS method has proven to be an accurate method to verify the PFEM results, a large drawback is its large computational cost and time. To give an example, the average time to perform the PFEM analysis for all the 5 nodes was less than five seconds (less than one second per node) versus MCS of the same elements took around 20 minutes using for both analysis the same computer.

The sensitivities of the response quantities with respect to the analysis input RV variables are used to compute the mean E and the variance Var of the RP response quantities. To obtain the variance Var , Taylor expansions have been applied which were truncated for higher order moment to only perform a first order perturbation analysis. It has been found that this gives very good results. However in order to obtain smaller errors on the Var of the RP, to apply for accurate probabilistic computations it is desired that the CV, so the factor between the Var and E on the RV is relative small as this influences directly the resulting Var of the RP.

In general the PFEM methods presented in this work are found to be capable of predicting the sensitivity and uncertainty in the response of the system accurately with the least computational effort and therefore will be practical applicable not only for the quantification of uncertainties in complex and large composite structures, but also for further probabilistic analysis such as for example probabilistic fatigue life analysis.

Chapter 7

Probabilistic Constant Life Diagram

7.1 Introduction

In this chapter the application of a probabilistic model PCLD of the deterministic CLD using the perturbation method will be demonstrated and validated. As discussed in section 2.3.1 "Constant Life Diagram", for practical design of composite laminate structures, a family of S-N curves is used combined in a CLD that depict the focus of all stress states that can produce a given fatigue life. With these CLD curves the designer can determine quickly and accurately the effect on lifetime of changes in the stress or strain in a component under design.

The mean E and the Variance Var of the stress response from the PFEM analysis were used as input for the perturbation analysis using Taylor series on the PCLD model. MCS analysis was applied, as discussed in subsection 3.3.2 "Monte Carlo Simulation", to analyze the Variance Var , in order to validate and compare to the PCLD perturbation analysis response results.

7.2 Taylor expansion of Constant Life Diagram

The probabilistic PCLD was developed by perturbation of the variability of the deterministic CLD fatigue model properties around the mean value of these properties. This perturbation method as discussed in subsection 3.3.1 "Perturbation method using Taylor series" will be applied here with Taylor series expansions on the fatigue cycles to failure N_f as output response around the E mean values for the cycles to failure N_f^0 , after which the coefficients in the expansion are identified and evaluated by perturbation analysis. Applying the perturbation method with first order Taylor series expansions, stochastic randomness

is introduced into the system in order to obtain the statistic second moments mean E and Variance Var for the number of cycles to failure N_f can be determined, which are required to construct and apply the cumulative fatigue damage model.

The N_f can be expanded into a Taylor series expansion around the mean value α_i^0 of the random variables α_i as discussed in section 3.3.1 "Taylor series expansion of model", The first order truncated Taylor expansion from Eq. 3.17 can be applied on N_f , which then becomes,

$$N_f(\alpha) = N_f^0 + \sum_{i=1}^{RV} N_{fi}^I (\alpha_i - \mu_{\alpha_i}) + \dots, \quad \text{where} \quad N_{fi}^I = \left. \frac{\partial N_f}{\partial \alpha_i} \right|_{\alpha_i = \alpha_i^0} \quad (7.1)$$

Where N_f^0 is the mean of the cycles to failure and N_{fi}^I the sensitivity of the number of cycles to failure with respect to the corresponding random variable α_i and Random input Variable (RV) of the CLD function.

First the sensitivities are computed of the RV with respect to RP, which serve as input for the Taylor series expansions, followed by the first moments up to order 2, i.e. the mean E and Variance Var , which give the distribution of the cycles to failure N_f as output response RP.

The sensitivities are obtained by calculating the derivatives of the RP which is N_f for the CLD with respect to the output response RVs which are the CLD model parameters. A custom programmed code is used [PFEMpy: Rens, 2022], to compute the Taylor expansions of the PCLD, which has been done by using Python's Sympy Meurer et al. [2017] module to evaluate the derivatives.

With the sensitivities known, the second statistic moments E and Var of the N_f can now be obtained. Using Eq. 3.19 the Expectance E of the cycles to failure N_f can be calculated with,

$$E[N_f] = N_f^0 \quad (7.2)$$

Considering the input parameters of the PCLD independent, based on Eq. 3.22, the first order estimate of the Var for the cycles to failure N_f can be obtained with,

$$Var[N_f(\alpha_i)] \approx \sum_{i=1}^n (N_{fi}^I)^2 Var[\alpha_i], \quad N_{fi}^I = \left. \frac{\partial N_f}{\partial \alpha_i} \right|_{\alpha_i = \alpha_i^0} \quad (7.3)$$

7.2.1 Probabilistic CLD from shifted Goodman model

DNV (former Germanischer Lloyd) proposes a CLD formulation [DNVGLST-0376, 2015], with its peak shifted at equal distance between the UCS and UTS . The highest stress

amplitude cannot always be sustained at a zero mean stress, even if CLD envelopes are approximately described using straight lines as in the symmetric Goodman diagrams, the symmetry axis of the CLD diagram is slightly shifted to the right of the alternating stress axis, due to the different behaviour when loaded/stressed in tension than in compression of glass and carbon fiber composite laminates.

The form of this 'shifted peak' but symmetrical CLD is presented in Fig. 7.1. The number of cycles for an arbitrary fatigue cycle, assuming m to be the slope of the S-N curve, is calculated using the the equation,

$$N_{f_{DNV}} = \left(\frac{UTS + |UCS| - |2S_m - UTS + |UCS||}{2S_a} \right)^m \quad (7.4)$$

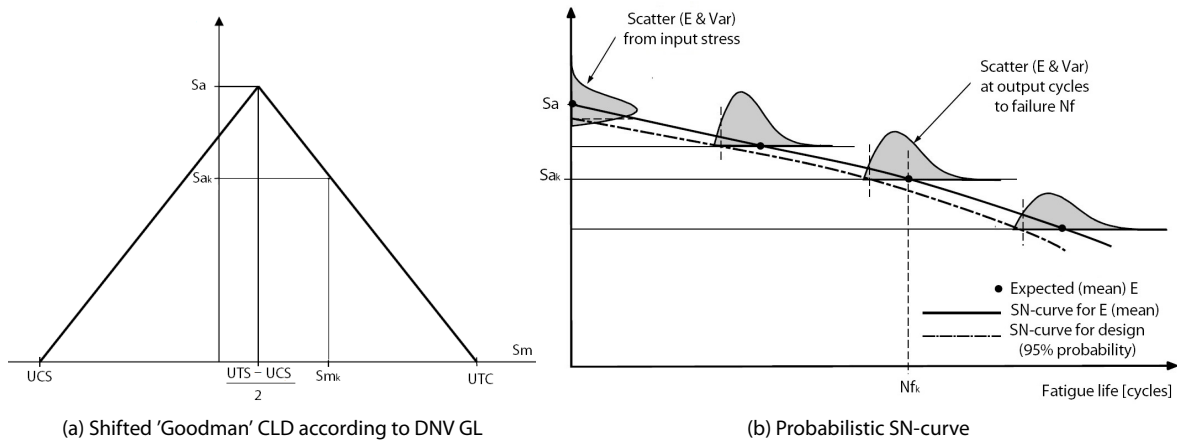


Figure 7.1: CLD shifted 'Goodman' diagram according to DNV GL (DNVGLST-0376 [2015])

Applying the perturbation technique using first order Taylor expansions to the CLD fatigue damage model from Eq. 7.4, we get for the Taylor expansion of Eq. 7.1, applying the random variables of the 'shifted' Goodman CLD of Eq. 7.4.

$$N_f = N_f^0 + \frac{\partial N_f}{\partial S_a} \bigg|_{S_a^0} (S_a - S_a^0) + \frac{\partial N_f}{\partial S_m} \bigg|_{S_m^0} (S_m - S_m^0) + \frac{\partial N_f}{\partial m} \bigg|_{m^0} (m - m^0) + \frac{\partial N_f}{\partial UCS} \bigg|_{UCS^0} (UCS - UCS^0) + \frac{\partial N_f}{\partial UTS} \bigg|_{UTS^0} (UTS - UTS^0) \quad (7.5)$$

The sensitivities $\frac{\partial N_f}{\partial \alpha_i}$ in Eq. 7.5 can be obtained by calculating the partial derivatives of N_f with respect to the different CLD model parameters.

$$\begin{aligned}
\frac{\partial N_f}{\partial S_a} &= -\frac{m \cdot N_{f_{D_{NV}}}}{S_a} \\
\frac{\partial N_f}{\partial S_m} &= \begin{cases} 0 & \text{for: } 2S_m - UTS + |UCS| = 0 \\ -\frac{2m \cdot N_{f_{D_{NV}}} \cdot (2S_m - UTS + |UCS|)}{(UTS + |UCS| - |2S_m - UTS + |UCS||) |2S_m - UTS + |UCS||} & \text{otherwise} \end{cases} \\
\frac{\partial N_f}{\partial m} &= N_{f_{D_{NV}}} \cdot \log \left(\frac{UTS + |UCS| - |2S_m - UTS + |UCS||}{2S_a} \right) \\
\frac{\partial N_f}{\partial UTS} &= \begin{cases} \frac{m \cdot N_{f_{D_{NV}}}}{UTS + |UCS| - |2S_m - UTS + |UCS||} & \text{for: } 2S_m - UTS + |UCS| = 0 \\ \frac{m \cdot N_{f_{D_{NV}}} \cdot (2S_m - UTS + |UCS| + |2S_m - UTS + |UCS||)}{(UTS + |UCS| - |2S_m - UTS + |UCS||) |2S_m - UTS + |UCS||} & \text{otherwise} \end{cases} \\
\frac{\partial N_f}{\partial UCS} &= \begin{cases} 0 & \text{for: } UCS = 0 \wedge 2S_m - UTS = 0 \\ \frac{UCS \cdot m \cdot N_{f_{D_{NV}}}}{(UTS + |UCS| - |2S_m - UTS + |UCS||) |UCS|} & \text{for: } 2S_m - UTS + |UCS| = 0 \\ 0 & \text{for: } UCS = 0 \\ -\frac{UCS \cdot m \cdot N_{f_{D_{NV}}} \cdot (2S_m - UTS + |UCS| - |2S_m - UTS + |UCS||)}{(UTS + |UCS| - |2S_m - UTS + |UCS||) |UCS| |2S_m - UTS + |UCS||} & \text{otherwise} \end{cases} \quad (7.6)
\end{aligned}$$

The second statistic moments E and Var of the N_f can be obtained, applying Eq. 7.2 for the different PCLD parameters, the Expectance E of the cycles to failure N_f can then be calculated with,

$$E[N_f] = \left(\frac{UTS^0 + |UCS^0| - |2S_m^0 - UTS^0 + |UCS^0||}{2S_a^0} \right)^{m^0} \quad (7.7)$$

Using Eq. 7.3 for the different PCLD parameters, the first order estimate of the variance Var for the cycles to failure N_f becomes,

$$\begin{aligned}
\text{Var}[N_f] &= \left(\frac{\partial N_f}{\partial S_a} \right)^2 \cdot \text{Var}(S_a) + \left(\frac{\partial N_f}{\partial S_m} \right)^2 \cdot \text{Var}(S_m) + \left(\frac{\partial N_f}{\partial m} \right)^2 \cdot \text{Var}(m) \\
&\quad + \left(\frac{\partial N_f}{\partial UTS} \right)^2 \cdot \text{Var}(UTS) + \left(\frac{\partial N_f}{\partial UCS} \right)^2 \cdot \text{Var}(UCS) \quad (7.8)
\end{aligned}$$

In order to obtain the mean value for the cycles to failure $E[N_f]$ from Eq. 7.7 and the variance for the cycles to failure $\text{Var}[N_f]$ from Eq. 7.8, the 'shifted' Goodman CLD properties are taken from the input sources as defined in Table 7.1.

The application of the perturbation technique to the 'shifted' Goodman CLD fatigue model from Eq. 7.4 done, performing first a PFEM analysis on the model as defined in section 5.2 "Structural model", for the RV with respect to both the mean stress S_m as well as the stress amplitude S_a of a CA load block, resulting in the mean E and Var of S_m and S_a . Next the sensitivities of the cycles to failure N_f with respect to the CLD parameters

Variable	Source	Method Mean	Method Variance
S_a	PFEM	Static analysis	Perturbation analysis
S_m	PFEM	Static analysis	Perturbation analysis
m	Material properties	Mean test data	Var test data
UTS	Material properties	Mean test data	Var test data
UCS	Material properties	Mean test data	Var test data

Table 7.1: PCLD input data source

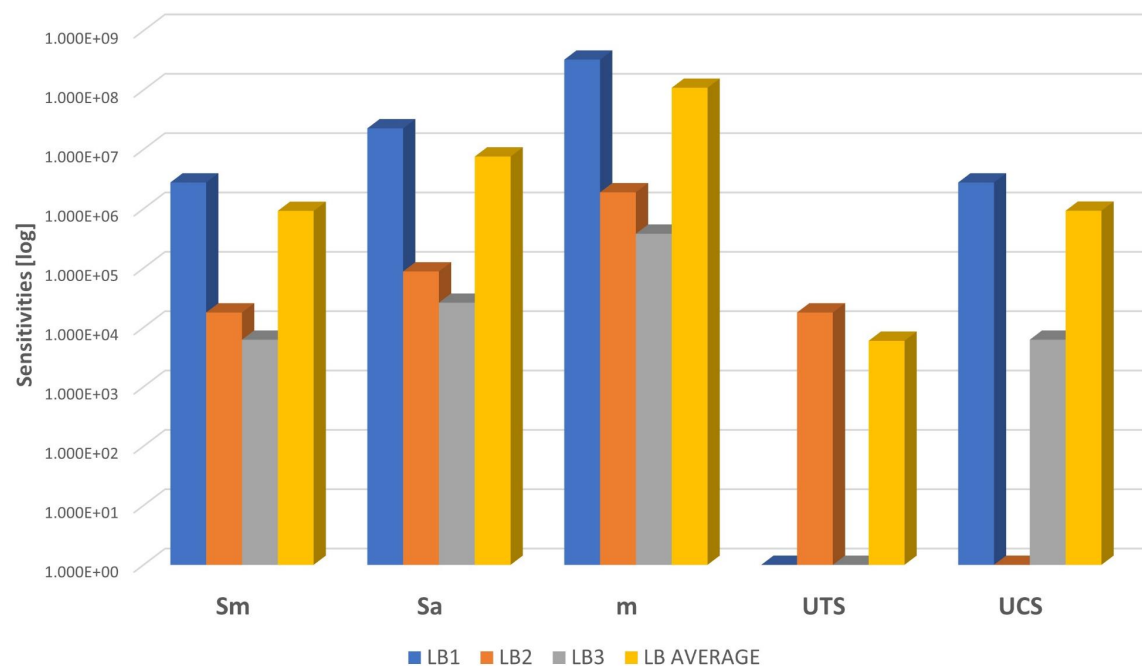


Figure 7.2: PCLD model parameter sensitivities for each load block

were calculated, with the average sensitivities over the nodes for each of the three load block as shown in Fig. 7.2.

Some very important information we can get from the sensitivities shown in Fig. 7.2 is that for the 'shifted' Goodman CLD fatigue model, the S-N curve slope parameter m has the highest sensitivity and therefore the variance on this input parameter has the largest influence on the CLD fatigue model behavior and calculated N_f as output. This shows us that for fatigue verification using the 'shifted' Goodman CLD or also for other similar fatigue model formulations, it is very important to define the S-N curve slope parameter m correctly, which depends for a large part on the amount of test data and the amount of variability or variance this CA fatigue test data inhibits. Further the stress amplitude S_a and afterwards the mean stress S_m parameter which results from the PFEM analysis have a relative large sensitivity and impact both the CLD and PCLD output cycles to failure N_f to

a large extend, therefore showing variability on the structural model behavior by for example the model input parameters variance are taken into account and take an important part in the probabilistic analysis results.

The strength of applying the PCLD model compared to the deterministic determined version of the same CLD model has been demonstrated, by that the PCLD model, takes directly into account the variance on the fatigue test data used for defining the fatigue model properties and the variances on the structural model input parameters. As for the deterministic CLD based fatigue analysis, the test data indirect adjustments and PSFs are applied with 95/95 probability (very) conservative design values and only mean values for the deterministic model behavior. Leading to design analysis results that have much less correlation to the actual test data and its variability and also the variability in the structural model behavior. Using the probabilistic approach applying the PCLD model, both reveals the sensitivities of the fatigue model parameters and takes into account the variability (variance) of the model input parameters.

SHELL ELEM	$E[N_f]$		Error [%]	$\text{Var}[N_f]$		Error [%]	CV_{PCLD} [%]
	PCLD	MCS		PCLD	MCS		
2	7.31E+06	7.69E+06	4.97	6.16E+12	7.03E+12	12.8	33.9
13	3.32E+07	3.52E+07	5.49	1.31E+14	1.570E+14	14.6	34.8
24	1.17E+08	1.24E+08	5.59	1.78E+15	2.03E+15	16.5	35.5
35	4.37E+08	4.64E+08	5.83	2.53E+16	3.05E+16	15.7	36.7

Table 7.2: Analysis Mean $E[N_f]$ and Variance $\text{Var}[N_f]$

For Load Block (LB) 1 of Table 5.5 as load applied, Table 7.2 shows the error on the Mean $E[N_f]$ and variance $\text{Var}^I[N_f(\alpha)]$ obtained between the PFEM and MCS analysis. The error for all the elements is $\approx 5.5\%$ for the mean values and $\approx 15\%$ for the variance, which is a much larger error than for the PFEM analysis, further the CV for the PCLD is $\approx 35\%$ which is also very large.

But it is important to consider that only a first order perturbation analysis is applied where not one but 5 RV with their variances are simultaneous analyzed with the perturbation analysis on the PCLD and RV. Further for the MCS 5 different random values are applied simultaneously for the CLD parameters, to run the the MCS. A future improvement might be to increase to a higher order, such as as second order perturbation analysis and investigate the possible improvement in accuracy of the analysis of the variance values. Of course the possible improvement in accuracy comes at a higher computational cost.

Chapter 8

Markov chain based non-linear fatigue damage accumulation model

8.1 Introduction

In this chapter we are going to propose a cumulative damage model based on Markov chains as developed by Bogdanoff and Kozin [1985], to perform the fatigue damage accumulation calculation for fatigue life analysis. Markov chains, as briefly discussed in subsection 3.2.4 "Markov Chains" have been successfully applied to solve various kinds of engineering and science problems, such as in machine learning, statistical physics, etc. A famous application is from Google, that modeled web-surfing as a Markov chain and then used its stationary distribution to rank the relevance of webpages, so called PageRank.

8.2 Markov Chain Cumulative Damage Model

As using Markov chains as cumulative fatigue damage model, was first proposed by Bogdanoff and Kozin [1985], this model will be therefore called the Bogdanoff model or shortened to B-model further through this thesis.

In the past most of the applications of the B-models as fatigue cumulative damage model are related to crack growth in metal structures, which has the advantage that the crack length a is a quantity that can often be observed. In this chapter we are going to apply the B-model for the damage accumulation process in composite laminates, where the mechanism of accumulation is very complex and not considered.

Some assumptions that we are going to make for the current model proposal is that the damage process is such that only unit-jumps are permitted [Bogdanoff and Kozin, 1985, p. 73], so the basic assumptions can be summarized as:

1. The fatigue process follows a Markovian type of evolution: The Markov property states that given the present, the past and future are conditionally independent, meaning that the evolution of the process is independent of the “past” and only depends on the present. The Markov property greatly simplifies computations of conditional probability, as instead of having to condition on the entire past, we only need to condition on the most recent value.;
2. Time period over which loads act and the damage accumulates is discretized: So measured in units of DC in non-negative integer units of $t = 0, 1, \dots, T$. Working with LBs, one DC are the work of all cycles in a LB, which means the loads inside each DC are considered to be at the same level so of constant severity;
3. Damage can only increase monotonically: The so called one-step B-model, which means it can only increase one discrete damage level in one DC so for each finished LB.

8.2.1 Probability transition matrix

To describe the dynamics of the B-model Markov chain, we need to know the probabilities of moving from any state to any other state, which can be encoded in a matrix, called the Probability Transition Matrix (PTM), whose entry is the probability of going from state p to state q in one step of the chain. Suppose the initial state of damage is p_1 and that failure occurs when state b is reached and further that an item can be in any of the previous states mentioned and in states in between so in any of the states $1, 2, \dots, b$.

The B-model PTM matrix $\mathbf{P}_n$ is a $b \times b$ unit-jump Markov chain, which has $b - 1$ state-dependent transition states of the form,

$$\mathbf{P} = \begin{bmatrix} p_1 & q_1 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ 0 & p_2 & q_2 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & p_3 & q_3 & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & p_j & q_j & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & p_{b-1} & q_{b-1} \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 1 \end{bmatrix} \quad j = 1, 2, \dots, b - 1 \quad (8.1)$$

Where conditional probability p represents the probability of remaining in the damage level and $q = 1 - p$ the probability of jumping to the next adjacent damage level. The B-model PTM only has the diagonal terms consisting of p 's and q 's, with the ratio of transition probabilities r is $r = p/q$.

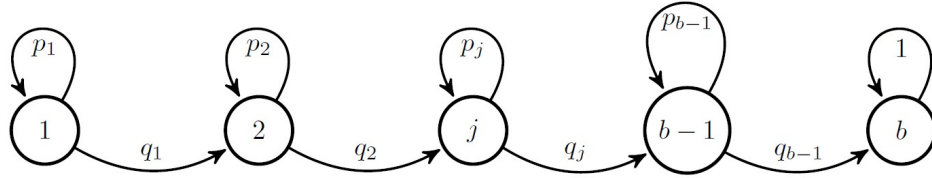


Figure 8.1: B-model Markov chain flow diagram

The resulting Markov chain for the cumulative damage process is illustrated in Fig. 8.1 , which shows the Markov chain has $b - 1$ transient states and one absorbing state b . Once entered the absorbing state b , it cannot be vacated. This can be explained by the cumulative damage process such that the probability for each damage state going back to the previous state is zero.

A basic element of the B-model Markov chain that is proposed here as cumulative damage model are the DC. Typical a load spectrum is or can be divided into Load Block (LB) which contain a fixed amount of cycles. A single repetition of a LB is called a DC. N_{DC} is the number of DCs until failure.

$$N_{DC} = \frac{N_f}{N_{LB}} \quad (8.2)$$

Where it is important that a single DC or the cycles in a LB, must be large enough so damage can accumulate. The loading during the LBs, can be both of constant amplitude or random of nature, where as simplification the state of damage is only considered at the end of each LB. The amount of DC depends on the load level, as to reach the same damage level a higher load level needs fewer DC or LB repetitions than a lower load level.

The first two moments which starts from an initial stage j and reaches the final failure state b , are given by,

$$E[N_{DC}] = \sum_{j=1}^{b-1} \frac{1}{q_j} = \sum_{j=1}^{b-1} (1 + r_j) \quad (8.3)$$

$$Var[N_{DC}] = \sum_{j=1}^{b-1} \frac{1 - q_j}{q_j^2} = \sum_{j=1}^{b-1} r_j (1 + r_j) \quad (8.4)$$

$E[N_{DC}]$ is the Expected value of the variable distribution of the cycles to reach state b and $Var[N_{DC}]$ its Variance,

As one can see from the PTM Markov chain of Eq. 8.1, only two independent parameters need to be evaluated, namely r which defines the ratio of the transition probabilities in each PTM matrix and b which defines the matrix size for each PTM matrix (so it must be an integer). One can find for the number of DC repetitions N_{DC} of a CA load block the Mean $E[N_{DC}]$ and Variance $Var[N_{DC}]$, which suffice to determine the two parameters r and b of the model using,

$$E[N_{DC}] = (b - 1)(1 + r) \quad (8.5)$$

$$Var[N_{DC}] = (b - 1)r(1 + r) \quad (8.6)$$

So with Eq. 8.5 and Eq. 8.6, the two unknown values r and b can then be calculated with,

$$b = \frac{E[N_{DC}]^2}{E[N_{DC}] + Var[N_{DC}]} + 1 \quad (8.7)$$

$$r = \frac{Var[N_{DC}]}{E[N_{DC}]} = \frac{E[N_{DC}] - (b - 1)}{(b - 1)} \quad (8.8)$$

Where $E[N_{DC}]$ the mean number of DC and $Var[N_{DC}]$ the variance on the number of DC, we can obtain for N_{LB} a fixed amount of cycles in a LB.

$$E[N_{DC}] = \frac{E[N_f]}{N_{LB}} \quad (8.9)$$

$$Var[N_{DC}] = \frac{Var[N_f]}{(N_{LB})^2} \quad (8.10)$$

8.2.2 Probability mass function

From the theory of Markov chains, $\mathbf{p}_t$ the Probability Mass Function (PMF) of the RV at time t is given by the $(1 \times b)$ vector,

$$\mathbf{p}_t = [p_n(1), p_n(2), p_n(3), \dots, p_t(b)] \quad (8.11)$$

The initial state of damage at time $t = 0$ the $(1 \times b)$ row vector becomes,

$$\mathbf{p}_0 = [p_0(1), p_0(2), p_0(3), \dots, p_0(b)] \quad (8.12)$$

The PMF vector $\mathbf{p}_t$ that defines the probability of staying at each damage level after a given number x of DCs, is completely determined from the PMF of initial compliance $\mathbf{p}_0$ and the PTM which define the severity of each duty cycle. This relationship is modeled

as a Markov chain, with b damage states, which follows from the multiplication rule for independent events by,

$$\mathbf{p}_t = \mathbf{p}_0 \mathbf{P}_n^k = \mathbf{p}_0 [\mathbf{P}_1 \mathbf{P}_2 \dots \mathbf{P}_n]^k \quad (8.13)$$

Where $\mathbf{P}_n$ are n different one-step PTM matrices for each DCs and k the amount of load spectrum repetitions.

The basis of the proposed method is that, as matrix multiplication is non commutative it results in a mathematically consistent non-linear rule. Assume that the damage process in the proposed model is such that only unit-jumps are permitted, therefore the basic assumptions can be summarized that the proposed B-model model is a finite state, further a discrete-time embedded Markov process, in which the damage accumulation mechanism is of the unit-jump type.

8.2.3 Constructing B-models

The key features of the B-model are the use of matrices to represent the damage growth and therefore the ability to model non-linear damage growth using the PTM Markov matrices, further matrix multiplications of the single PTMs models the damage evolution for the different load levels of a full VA load spectrum, taking hereby into account load sequence effects. In this way the damage evolution during an actual test up to failure can be very accurately modeled and predicted, as the PTMs matrices can be relatively easy created directly from CA test data of each of the load levels. This method has been demonstrated and published as a paper by the thesis author Rens et al. [2022], see section A.1 "Markov chain based non-linear fatigue damage accumulation model".

In order to construct the Markov chain based PTM matrices of the B-model, the E and Var properties of the fatigue cycles to failure N_f of a load block are required. There are two methods to obtain the E and Var properties of the fatigue N_f , based on:

- **Experimental data:** CA fatigue test data with as output N_f is used for each of the load block levels, from this test data the E and Var of N_f are analyzed. This method has been demonstrated and published by Rens et al. [2022], see section A.1 "Markov chain based non-linear fatigue damage accumulation model";
- **Computational models:** Probabilistic structural analysis using PFEM and a PCLD probabilistic fatigue model are used, following the steps on the left (green) side of the flow diagram of Fig. 8.2:
 1. **PFEM analysis:** two PFEM analysis are performed for a RV and the E and Var of the stress amplitude σ_a and mean stress σ_m of the fatigue cycles as output RV.

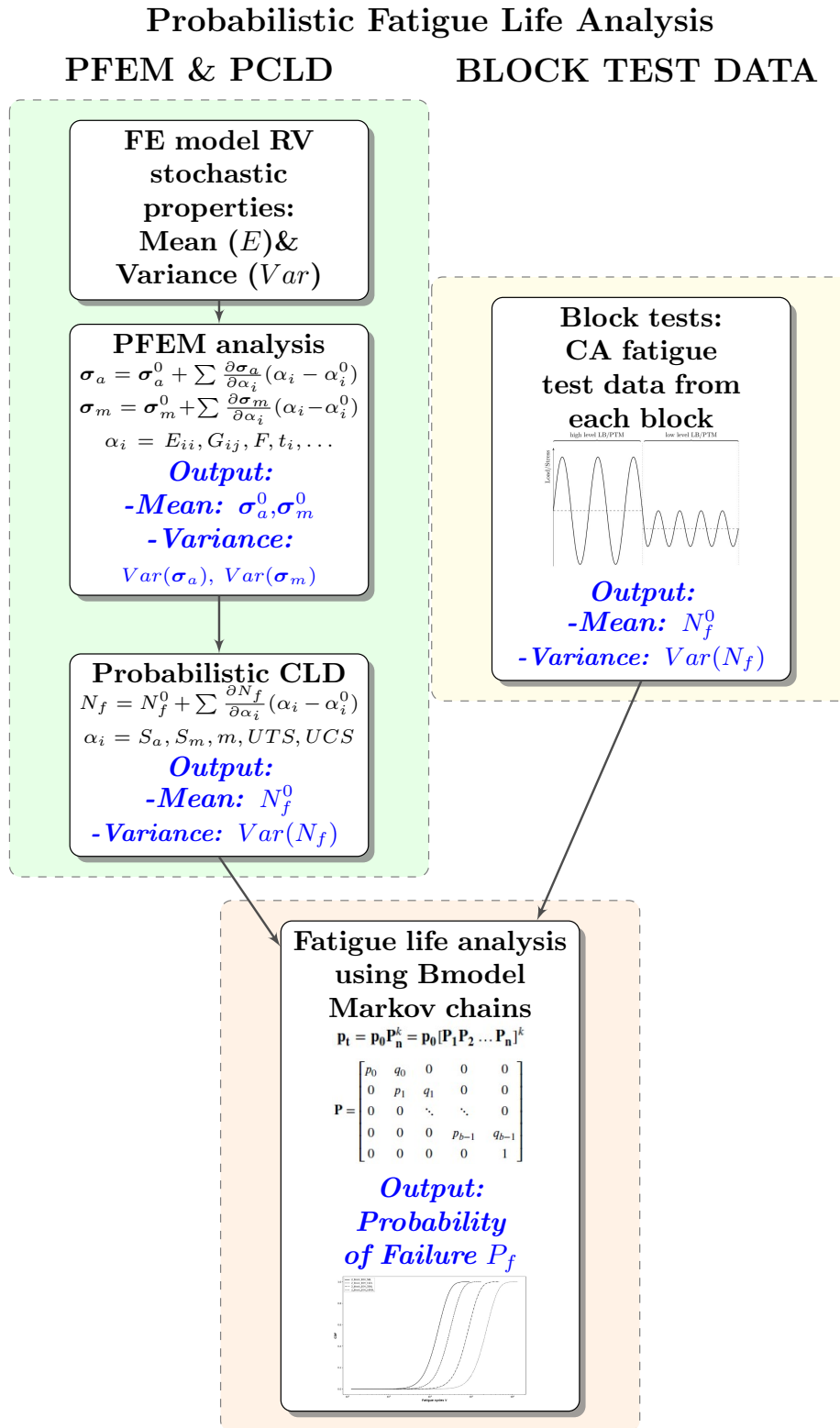


Figure 8.2: Probabilistic fatigue life analysis flow diagram

As output the E and Var of σ_a and σ_m are obtained for the structural model. This PFEM analysis method has been demonstrated in Chapter 6 "Probabilistic FEM analysis on composite plate".

2. **PCLD perturbation analysis:** Using the E and Var of the stress amplitude σ_a and mean stress σ_m of the fatigue cycles obtained in the previous step together with the other PCLD model parameters, resulting in the E and Var of the fatigue cycles to failure N_f . This PCLD analysis method has been demonstrated in Chapter 7 "Probabilistic Constant Life Diagram".

This resulting in a relative simple and very accurate method to translate the complex non-linear behavior of composite materials during the material fatigue test to Cumulative Damage (CD) models that can be applied for deterministic but especially also for probabilistic structural fatigue life analysis.

8.3 B-model parameter investigation

From Eq. 8.7 and Eq. 8.8 we can learn that the value of the PTM parameters b and r , depend on $E[N_{CD}]$ the mean number of DC and $Var[N_{DC}]$ the variance on the number of DC.

Where, $E[N_{CD}]$ depends on N_{LB} the number of cycles in a LB, as N_f stays constant for a fixed load level (fixed load mean and amplitude) constant amplitude LB.

From Eq. 8.10 we can see that $Var[N_{DC}]$ depends largely on $Var[N_f]$, which variance comes from the dispersion of the experimental data or the variance on the cycles to failure resulting from the PCLD fatigue life model.

8.3.1 Load block length vs b PTM matrix size

A parametric study was conducted to find the relation between N_{LB} or $E[N_{CD}]$ and the PTM parameter b . Figure 8.3 shows that with increasing amount of N_{LB} at a constant load/stress level, the actual value of b decreases.

The parametric study performed reveals the important role of the selection of the discretization step in the time domain, which is defined by the value of the DCs in a LB. A small amount of DC may offer a higher accuracy with respect to time in a linear model.

It is therefore not to choose too many DC for a load block, since b defines the amount of damage states, a higher b value and thus larger PTM size and amount of damage states, describes the damage evolution in more detail. On the other hand, the computing time increases by increasing the number of damage states b . For a more detailed modeling of the damage evolution in a LB, sufficient transitions so matrix size is required.

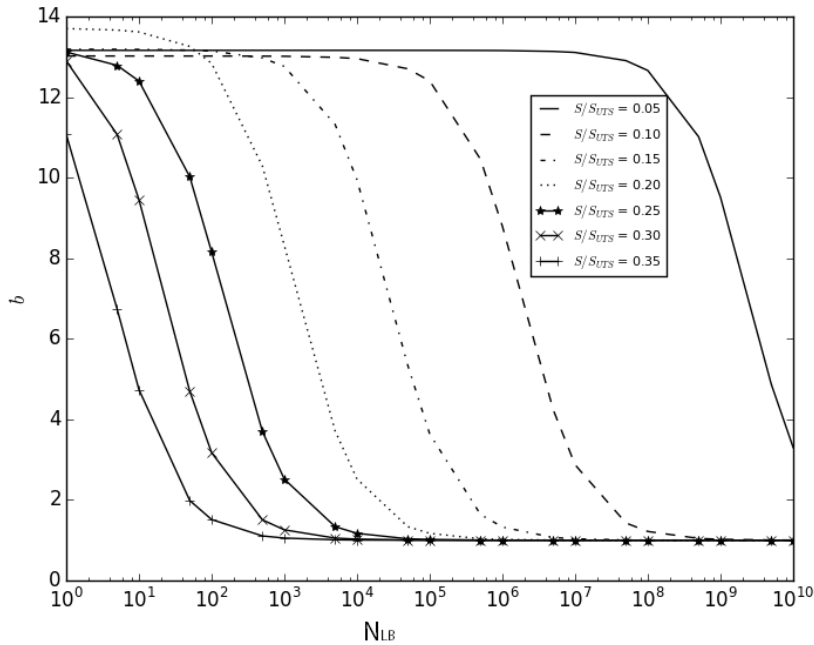


Figure 8.3: PTM duration of a DC vs the b parameter

Most important of all is that in order to multiply the k different PTM matrices from Eq. 8.13 with each other all the PTMs need to be of the same fixed size and thus have the same b value for each matrix. To achieve this an optimum value for b should be chosen. So when accuracy is important ideally the highest possible value for b valid for all PTMs for the different DC. On the other hand when computation time is important a lower possible value for b valid for all PTMs for the different DC should be chosen.

To find the optimal b parameter for all the PTMs that are going to be multiplied with each other, an optimization module was programmed and applied as part of the PFEMpy tool.

8.4 Conclusions

The published paper section A.1 "Markov chain based non-linear fatigue damage accumulation model" demonstrates the practical application of the B-model Markov chains based on experimental data as fatigue damage accumulation model for probabilistic fatigue analysis on composite structures. In this chapter an additional application of the B-model Markov chains based on computational models is discussed that can be applied for design analysis without the need of having experimental data available.

Chapter 9

Probabilistic fatigue life analysis on wind turbine blade

9.1 Introduction

In this last chapter the implementation of the different methods discussed so far in this thesis, such as the PFEM analysis and PCLD models, will be joined to perform a full probabilistic fatigue analysis on a composite structure.

In the previous chapter Chapter 8 "Markov chain based non-linear fatigue damage accumulation model", a method using B-models was proposed to perform the the fatigue damage accumulation calculation, which is the last part of the probabilistic fatigue analysis, that leads to the probability of failure P_f during the fatigue life of a structure. The Markov Chain PTMs were constructed based on experimental data for the different load levels of the load blocks. In this section the B-models will be applied based on computational results, by constructing for a VA load block spectrum the Markov Chain PTMs for each of the different load levels, using perturbation based PFEM analysis and PCLD models, as shown on the left left (green) side of the flow diagram of Fig. 8.2 which is highlighted in Fig. 9.1, in order to get to the second moment statistical values of the cycles to failure for each load block.

9.2 Probabilistic fatigue analysis

9.2.1 Case 1: spar cap longitudinal stiffness E_{11} as RV

In this case the Young's modulus in the laminates principal longitudinal direction E_{11} , will be applied as RV. Variability of the principle stiffness components in a composite structure typically result from different degree of manufacturing and process defects of the

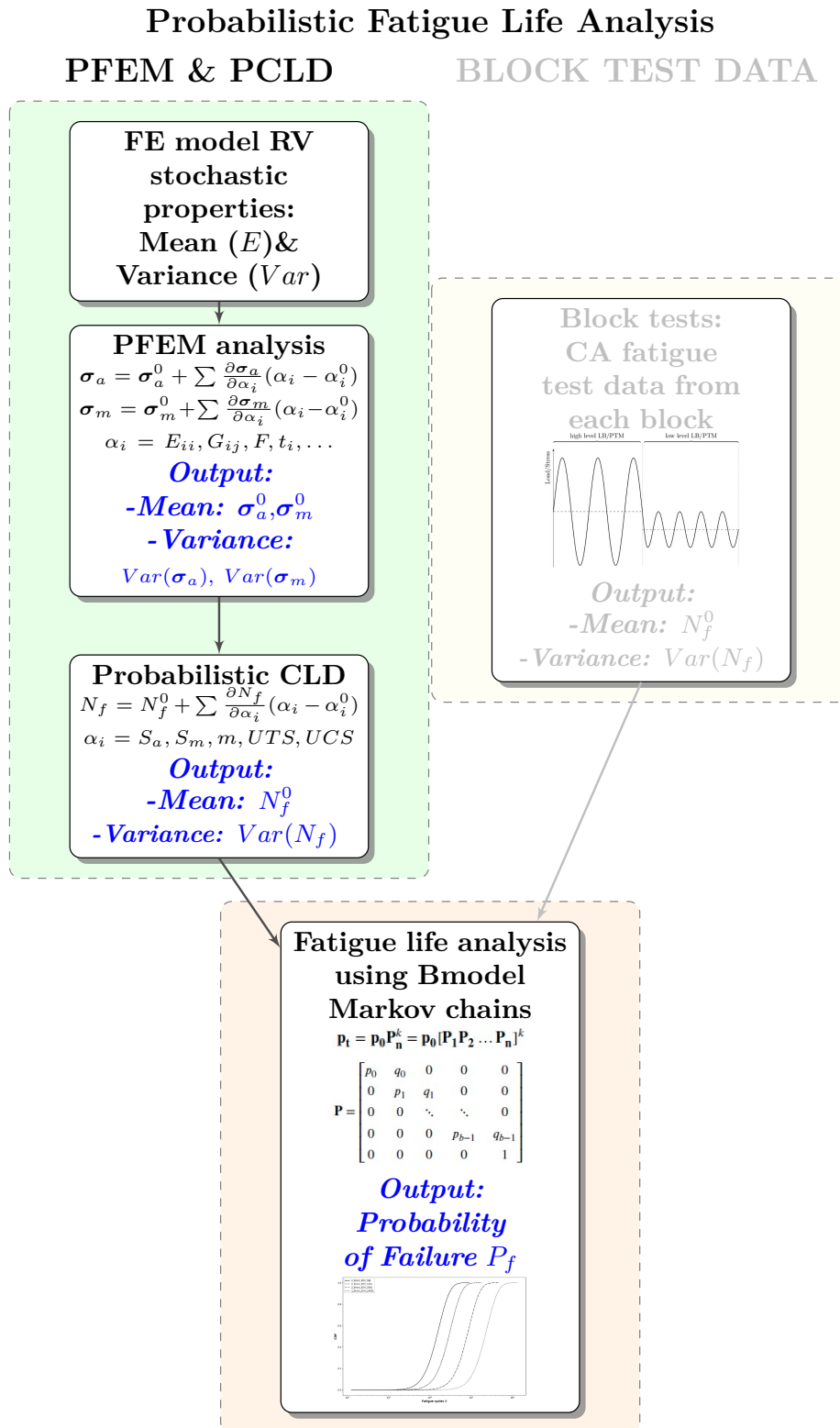


Figure 9.1: Probabilistic fatigue life analysis using PFEM and PCLD models flow diagram

ram materials and the final structure, such as laminate defects (voids, fiber misalignment, etc) which can each cause the actual stiffness and strength of the composite laminates to deviate from the actual design stiffness and strength values. Such laminate defects can often occur due to the manufacturing process, such as during the layup of the laminates in the mould or during the resin infusion of the laminate. As for very large structures such as WTB it is usually impossible to obtain the exact predetermined properties. It is therefore important to investigate how the overall properties and strength of composites are affected by the laminate defects from the design nominal values.

Load	N_{LB}	F_a [N]	F_m [N]
LB1	150	100	0
LB2	25	200	250
LB3	80	-200	150

Table 9.1: Multiple load block load spectrum properties for RV E_{11}

As load spectrum for the probabilistic analysis with E_{11} as RV, the LBs with the load levels and DC repetitions in each block load as defined in Table 9.1 was applied for the analysis, where the defined total plate force was divided the over the nodes of the respective loaded edge of the plate model, as shown in Fig. 6.3.

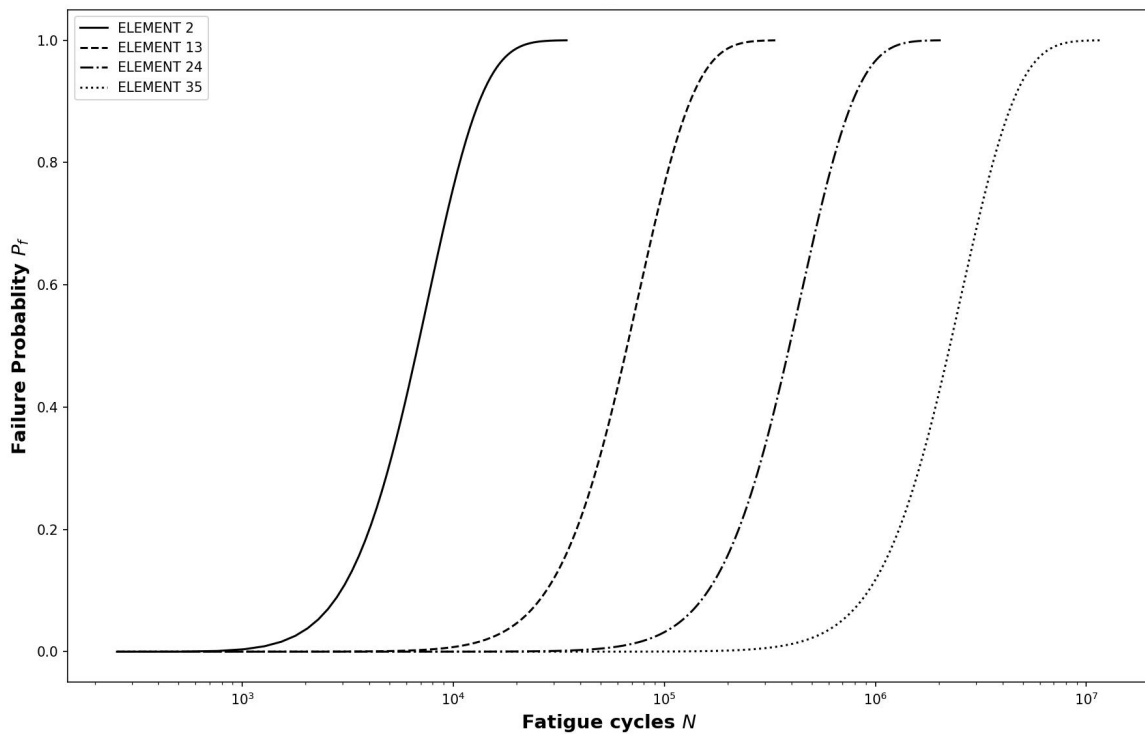


Figure 9.2: Probability of failure CDF for RV E_{11}

The steps of the left side of the flow diagram of Fig. 9.1 were followed:

1. **PFEM analysis:** two PFEM analysis were done for E_{11} as RV and the E and Var of the stress amplitude σ_a and mean stress σ_m of the fatigue cycles as output RV. These analysis were repeated for each of the 3 load blocks of Fig. 5.6, with the amount of DC repetitions of the load cycle as defined in Table 9.1. As output the E and Var of σ_a and σ_m for each load block were obtained, for the plate element locations as defined in Fig. 6.2.
2. **PCLD perturbation analysis:** repeated for each of the 3 load blocks, using the E and Var of the stress amplitude σ_a and mean stress σ_m of the fatigue cycles, obtained in the previous step together with the other PCLD model parameters, resulting in the E and Var of the fatigue cycles to failure N_f for each load block.
3. **B-model cumulative fatigue damage analysis:** Using the the E and Var of the fatigue cycles to failure N_f for each load block, the different PTMs were constructed to join them using the PMFs, in order to calculate the probability of failure (P_f)

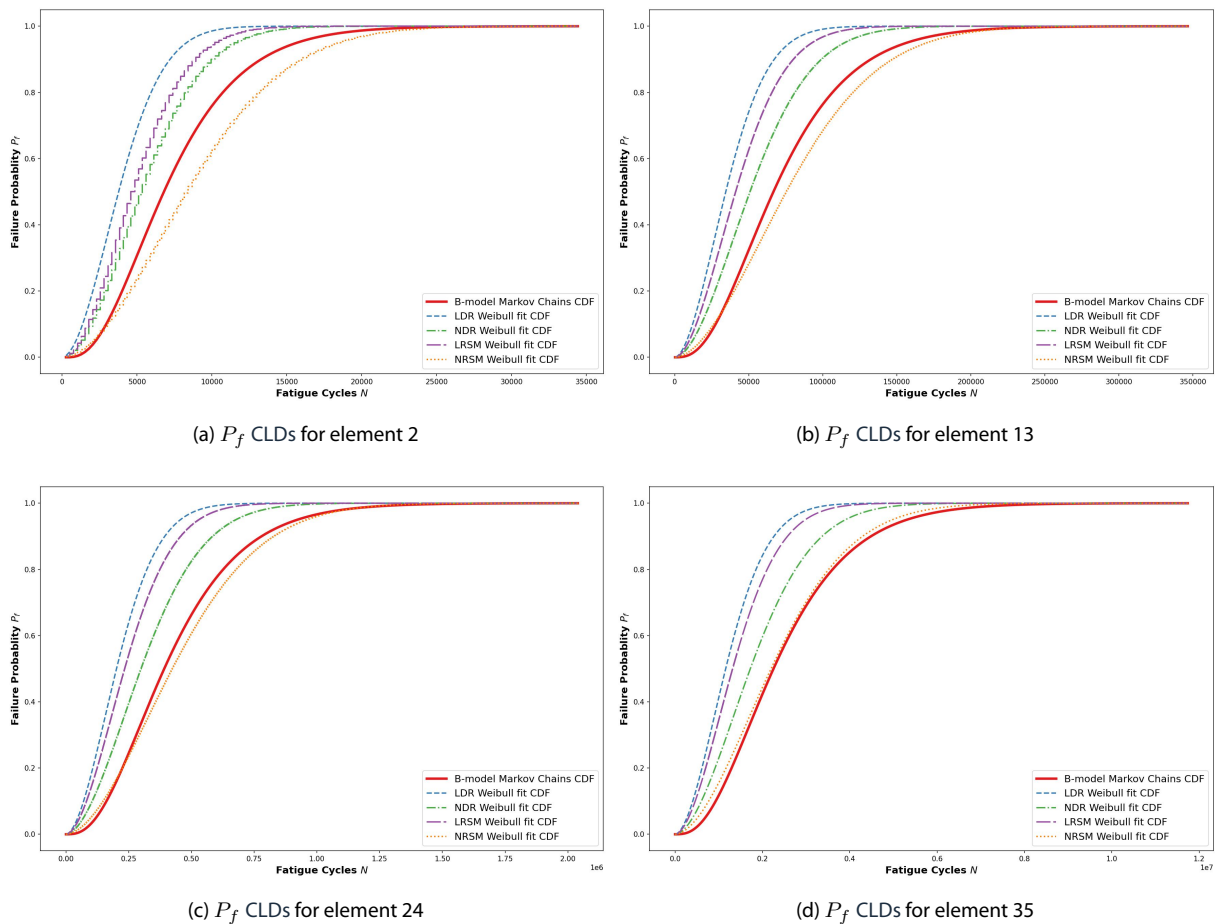


Figure 9.3: Element failure probability curves for RV E_{11}

The result of the different analysis according to the 3 previous described steps is shown in the CDF curves for the probability of failure P_f for each analyzed element of Fig. 9.2, for the repetitive 3 blocks load spectrum, until failure.

The same comparison as was done in Chapter 3 of the published paper, see Appendix A.1, for the B-models analysis based on experimental data, the B-models based on the Probabilistic Constant Life Diagram (PCLD) were also compared with other fatigue damage evolution models that are most commonly applied for composite structures, as shown in Fig. 9.3 for the different analyzed plate model elements.

9.2.2 Case 2: Spar cap stochastic load vector as RV

In this case we are going to investigate the force vector matrix $\mathbf{F}$ as random input variable RV for the probabilistic analysis, which consist of the force module and the variables of the directions of the force vector matrix $\mathbf{F}$.

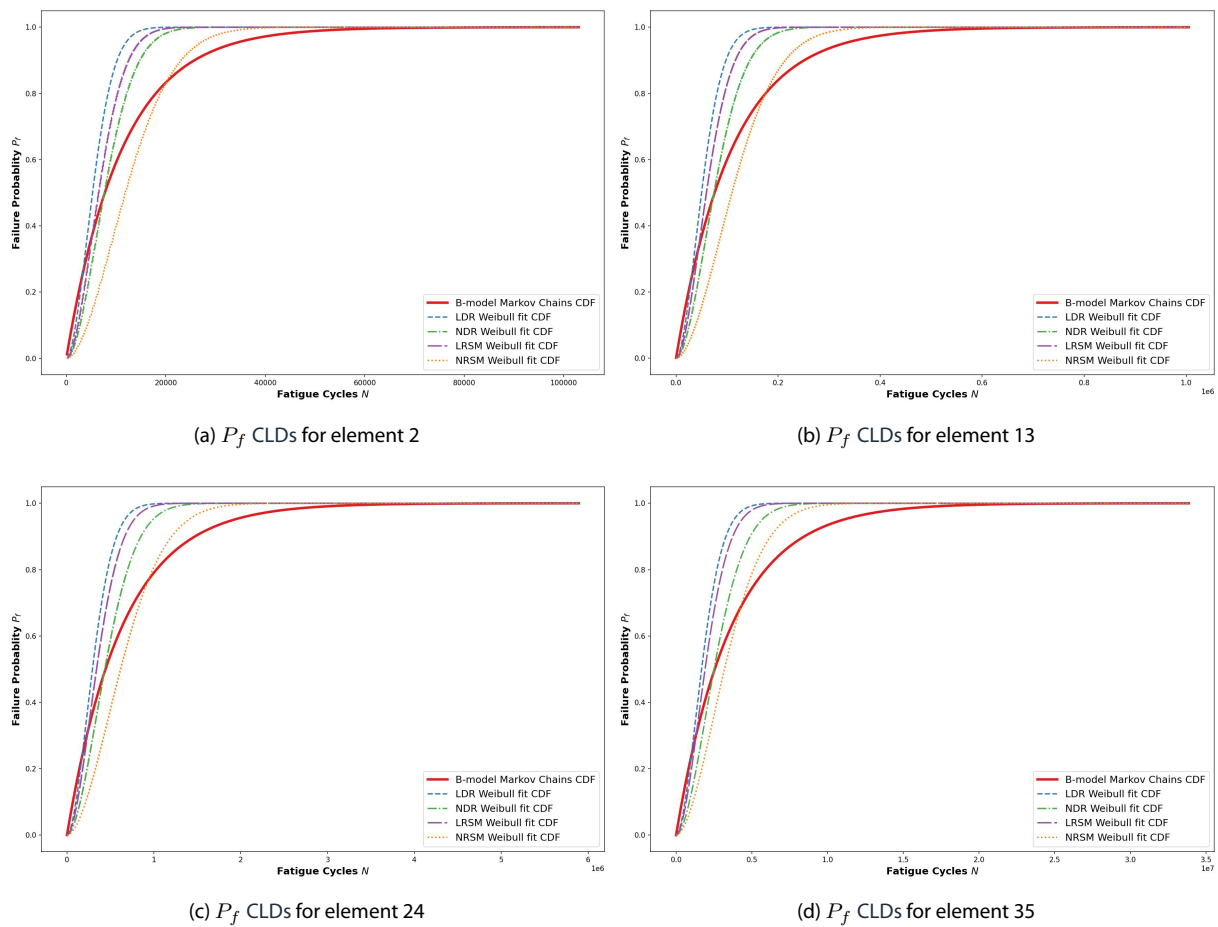
Load	N_{LB}	$F_a [N]$	$F_m [N]$
LB1	100	100	0
LB2	10	200	250
LB3	30	-200	150

Table 9.2: Multiple load block load spectrum properties for RV F

As load spectrum the LBs with the load levels and DC repetitions in each block load as defined in Table 9.2 was applied for the analysis. Further a variance for all loads was assumed of 80.28 N as per Table 5.4. The amplitude force F_a and mean force F_m from Table 9.2 were applied as total nodal plate forces and divided on the nodes of the respective loaded edge of the plate model. The element stress vector was analyzed as random output variable of the PFEM analysis resulting in the σ stress, the sensitivity of the stress with respect to the amplitude and mean force vectors and the σ on the stress results for the amplitude and mean force vectors.

With the obtained expected stress and it's variance for the resulting stress amplitude S_a and stress mean S_m using the PCLD fatigue model, the expected E and Var of the cycles to failure N_f could be determined, which was used to define the B-model PTMs Markov matrices. Also here the B-models this time based on the PCLD were compared with the other fatigue damage evolution models, as shown in Fig. 9.4 for the different analyzed plate model elements.

In subsection 7.2.1 "Probabilistic CLD from shifted Goodman model", sensitivity analysis for the PCLD parameters was performed, where Fig. 7.2 already revealed for PCLD and CLD models that typical the S-N curve slope parameter m has the highest sensitivity and

Figure 9.4: Element failure probability curves for RV F

therefore it's Var the highest impact on the CLD model Random output Parameter mean E and Var results.

Property	Distribution	E	Var	CV [%]
m	Normal	9.54	0.001	0.33

Table 9.3: OPTIMAT UD2 fatigue properties reduced data set, according to ASTM E739-91 [2015]

Besides the S-N slope m parameter of $m = 9.28$ which has a Var = 0.01 as defined in Table 5.3, another S-N slope m parameter was defined. The dataset of UD2 from the OptiDAT material database [Nijssen et al., 2006b], was reduced removing some of the data points furthest from the mean value (green) regression line as show in Fig. 9.5 using [ASTM E739-91, 2015], this results as stated in Table 9.3 in a slightly larger slope m parameter of $m = 9.54$ with a smaller variance Var = 0.001.

For the plate model elements 2, 13, 24 and 35 as shown in Fig. 6.2, two different analysis

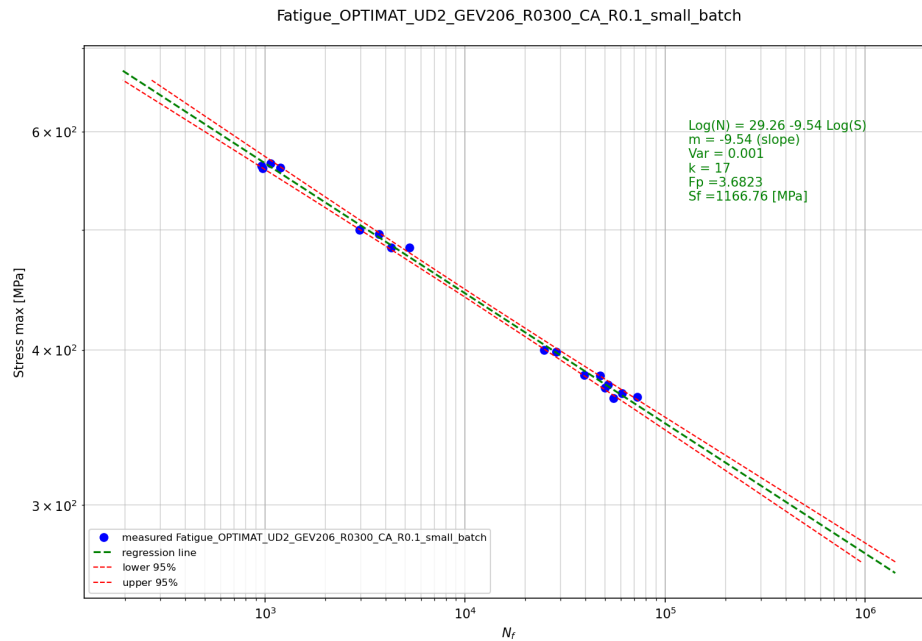


Figure 9.5: UD2 CA fatigue test data [Nijssen et al., 2006b] for reduced data set

were performed with, for the S-N slope m parameter of $m = 9.28$ with $\text{Var} = 0.01$ and the slope m parameter of $m = 9.54$ with $\text{Var} = 0.001$.

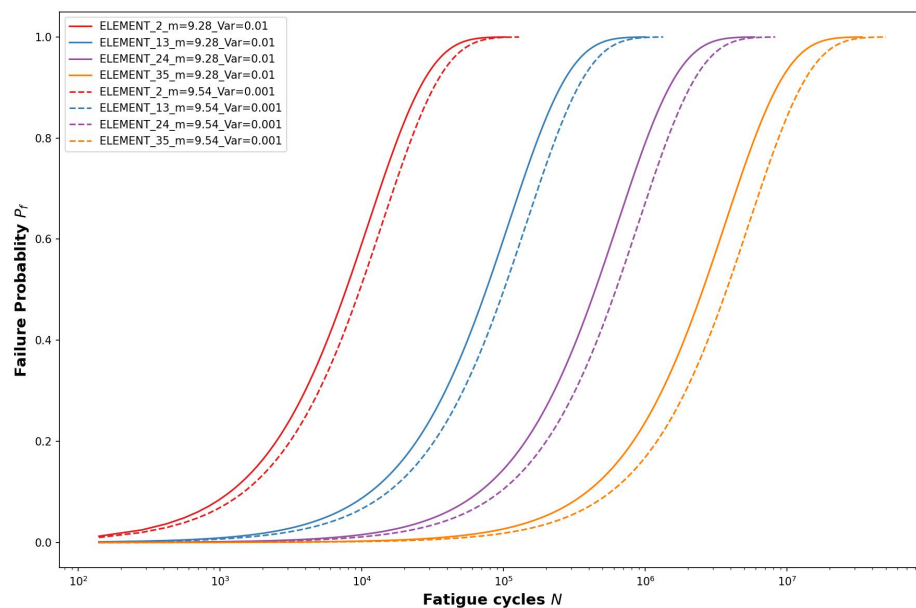


Figure 9.6: Probability of failure CDF for RV F for different slope m values

The result of the impact of a slightly increased slope m parameter and a decrease variance on the slope m parameter are shown in Fig. 9.6, which visualize clearly the trend of similar shape of the CDF curve for the probability of failure, but a reduced probability of

failure for an increased slope m parameter, so for a 'flatter' S-N slope.

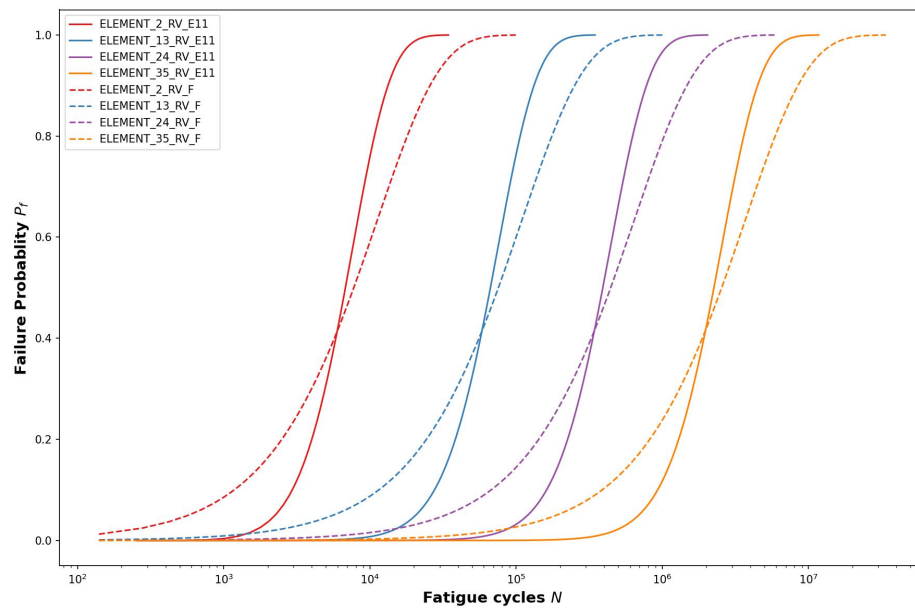


Figure 9.7: Probability of failure CDF for RV E_{11} and F

As last a comparison has been made of the CDF curves for the probability of failure for the RV E_{11} , against the RV F . What can be observed is the large difference in the toe between the curve for E_{11} and the curve for F as RV. For RV F the damage is accumulating much faster and at an earlier stage than for RV E_{11} up to a $P_f \approx 0.4$ where at the shoulder the damage accumulation for F is slower than for E_{11} .

Conclusions

Chapter 10

General Conclusions, Recommendations and Future Works

This PhD thesis contributes to the development of new novel methods in order to analyze and obtain predictions for probabilistic fatigue life analysis of composite structures of wind turbine blades under stochastic fatigue load conditions using finite element method based probabilistic methods.

In doing so, as a first step in this process, a description of the state of the art on deterministic methods to analyze the fatigue life of a composite Wind Turbine Blade was discussed in Chapter 2, investigating drawbacks and limits of the low accuracy level of the still very commonly applied Linear Damage Rule (LDR) fatigue accumulation damage rule. Followed by an introduction into probabilistic analysis methods for composite structures given in Chapter 3.

In order to perform the probabilistic based FEM analysis, a custom FEM code was developed, discussed in Chapter 4, using symbolic computation methods to fast derive the partial derivatives required for the sensitivity analysis. The sensitivities need to be obtained to add stochastic randomness to the system using perturbation based Taylor series expansions in order to analyze the variances for the random output variables of the Probabilistic Finite Element Method (PFEM) analysis.

The outcome of the research performed during this thesis will be discussed in relation to the existing knowledge and methods from the literature. The systematic description of the implementation of probabilistic fatigue life analysis on a composite Wind Turbine Blade (WTB) in this case for a composite laminate of UD glass fiber with epoxy matrix subjected to axial fatigue loading, has been pursued. The conclusions on the published paper and chapters of this thesis will be summarized and discuss followed by recommendations for future research.

10.1 Summary of Results

In this section summaries of the main topics and results and conclusions are highlighted, but details should be read in the chapters and attached paper.

10.1.1 PFEM

In this part of the thesis a novel perturbation based Probabilistic Finite Element Method PFEM was presented based Taylor series expansions. For this first a custom FEM code was developed as discussed in Chapter 4, to perform the probabilistic FEM analysis, using symbolic computation to fast derive the partial derivatives required for the sensitivity analysis. Partial derivative based sensitivity analysis was applied to obtain the sensitivities of the Random input Variable (RV) with respect to the Random output Parameter (RP) followed by the perturbation method to obtain the Variances Vars. The material mechanical properties and applied loads were considered being stochastic. A direct comparison has been performed of the analysis results of the PFEM of a composite layup plate with the Monte Carlo Simulation (MCS) analysis results. It was demonstrated that the MCS is a computationally much more costly method but a good technique to verify the PFEM perturbation analysis results which show very good agreement between the two analysis for the analyzed cases. Taylor expansions have been applied, truncated to only perform a first order perturbation analysis, to obtain the σ . It has been found that this gives good results, especially for small Coefficient of Variation (CV) between the Var and σ of the RV. For larger CVs the error increases, where it should be investigated if a higher order than first order perturbation analysis decrease the error to improve the results.

10.1.2 B-model Markov chain cumulative fatigue damage model

A Markov chains based Bogdanoff Markov chain model (B-model) cumulative fatigue damage model has been presented through this thesis. The novel approach to model cumulative fatigue damage for composite laminates were first presented in Paper section A.1 "Markov chain based non-linear fatigue damage accumulation model", where it has been demonstrated that the B-model Markov chain models can both be defined using experimental data as well as computational models and for both methods it is possible to predict accurately the fatigue life of composite laminates subjected to variable amplitude load spectrum's.

It shows that these models can be practically applied for design analysis to evaluate the probabilistic damage evolution of composite structures with materials characterized by Probabilistic Constant Life Diagram (PCLD) fatigue models. The damage evolution probability at any loading step and the probability of failure at any damage state can be analyzed

by this novel method. Multiple Probability Transition Matrix (PTM) can be used and combined to represent the different loading situations that can occur during a WTB fatigue design life. The B-model Markov chain models show not to have some of the important restriction or errors of "classic" fatigue analysis methods, like the load sequence effect error. Further non-linear damage accumulation can be modeled using the B-model Markov chain models.

The B-model results need still to be validated against more complex load spectra that are representative for actual wind turbine blade fatigue loads, as used for their fatigue life analysis and design.

To build the one step B-model, it is necessary to analyze the mean and variance of N_f for the different load blocks from a fatigue spectrum load block experimental data or from a design load spectrum. The variance on the fatigue life, $\text{Var}(N_f)$, can be approximated by means of first order Taylor series obtaining an acceptable CV. All this procedure can be an alternative with a lower computational cost than a MCS simulation.

Finally, this thesis provides validated methods to perform probabilistic fatigue life analysis for WTB and other composite structures subjected to stochastic load conditions during its (design) lifetime .

10.2 Original developments in this thesis

While there is still a huge amount of work to be performed on probabilistic analysis, at the end of this work it has been possible to contribute to the actual state of the art at least on the following aspects points:

- Using Probability Transition Matrices to model non-linear fatigue damage evolution of variable amplitude block load spectrum's.
- The B-model Probability Transition Matrices can be directly obtained from constant amplitude test data as well as from probabilistic fatigue models in the form of fatigue S-N curves and/or Constant Life Diagram (CLD).
- Matrix multiplication for the different load level Probability Transition Matrices takes into account load sequence effects.
- Fatigue life analysis results in a full Cumulative Distribution Function (CDF) probability curve of the fatigue damage evolution instead of only a single damage value as resulting from Linear Damage Rule (LDR).
- The practical application of a Probabilistic Finite Element Method (PFEM) method has been developed for a laminated composite plate.

Parts of the developments have been published and presented in:

Published research article:

- Rens, Bea-Bragado, and Bea [2022], [Markov chain based non-linear fatigue damage accumulation model](#). Probabilistic Engineering Mechanics Journal (ScienceDirect, Elsevier B.V.), Volume 68, April 30, 2022, Article 103250. (JCR Journal, Journal Impact Factor of 2.954, CiteScore 4.7, Quartile Q1)

Publication at international congresses:

- Rens and Bea [2019], PFEM Analysis On Fibre-Reinforced Plastic Composite Structures. MECHCOMP2019 5th International Conference on Mechanics of Composites, Lisbon (Portugal), July 4th, 2019, Abstract 15562.
- Rens [2019], Probabilistic Fatigue Life Analysis Composite Wind Turbine Blade Structures, Wind Turbine Blade Manufacture 2019 Conference, Dusseldorf (Germany), 9-11 December 2019

Publication at national congress:

- Rens [2022], Glass pultrusion blade design and probabilistic fatigue life analysis, 6th Annual Wind Blade Materials Development Forum, Barcelona, 22 September 2022

Final year project thesis director:

- Ignacio Bea Bragado, Non-linear Fatigue Damage Evolution Models for Composites, University of Zaragoza, Zaragoza, 2019

10.3 Potential benefits for wind turbine blade designs

Lower blade mass and cost: Substantial blade mass and cost reduction can be achieved by using statistical distribution of measured material properties instead of the typical for deterministic design and analysis applied conservative Partial Safety Factors PSF on top of already very conservative 95% lower bound confidence corrected material strength allowable and fatigue parameter values, as required per current design standards and guidelines.

More effective blade design: Sensitivity analysis where the sensitivities of the analysis input data (loads, material properties, etc.) with respect to the output data from a FE analysis, reveal both which design parameters have the highest impact/sensitivity on the

analysis output, so on which to best focus in order to blade improvements and optimization

Blade Operation and Maintenance: Detailed probabilities of failure are known during the blades design lifetime (and beyond), therefore maintenance and possible replacement can be planned much more efficiently, which results in increased reliability, reduced amount of failures and therefore a reduction in Operation and Maintenance cost. Further probabilistic analysis can in a more detail scale be applied to separate blade components and sub-systems (root joint, adhesive bond lines, spar caps, lightning protection system) during lifetime, resulting in probabilities of failure for these components/systems, and optimize the maintenance and replacement of these separate parts.

10.4 Limitations of Work

A question that often appears after reviewing probabilistic analysis methods and results is how it can replace deterministic based design and analysis most effectively?:

- Which PSF applied during deterministic based design and certification analysis be (partly) be replaced by probabilistic analysis and how will this affect the design reliability? Moreover which deterministic design PSF can most likely not be replaced by probabilistic analysis and design?
- Which is the best method to apply failure probabilities distributions for design analysis and how to compare to deterministic single value safety factor?
- If certain design and analysis input parameters (both their mean and variances) were chosen differently, how would these affect the probabilistic analysis results?
- How would probabilistic based design and analysis results change if calculated loads according to design standards are replaced with loads from actual on-site measurements?

10.5 Future work

For the new methodologies introduced on probabilistic analysis, diverse continuous improvements and applications can be identified on the continuity of this thesis topic.

For the PFEM analysis, in order to achieve even a higher accuracy and smaller error on the Variance Var of the random output variable RP, the Taylor expansions can be extended to incorporate a second or even higher order Taylor expansions for the perturbation analysis which might give improvement in accuracy, but on the other hand most likely for

each increase in order a much higher complexity and computational cost and therefore analysis time penalty. Further using co-variances simultaneous multiple random input variables can be investigated and the sensitivity on the different random output variables can be determined.

The proposed PFEM method can be implemented for a variety of probabilistic problems with different nature of random variables, where interesting future works can be proposed, applying the PFEM method to more complex structures and loading conditions. Further a cooperation with one or more of the larger commercial FEM software developers such as ANSYS [ANSYS Inc., 2022] and/or ABAQUS [Dassault Systèmes Simulia Corp, 2022] would enable the use of the PFEM analysis method for practical FEA applications and availability to different industries.

Regarding the B-model models there are multiple improvements that can be investigated and applied. First of all future efforts should be done in order to develop a methods to apply the Markov chain based damage accumulation models for the typical stochastic time series loads applied for consisting of many different load scenarios and load block at many different load levels so probabilistic analysis on a full wind turbine blade can be performed in a reasonable time span.

Also some attention should be paid to the stress redistribution caused by the growth of fatigue damage in composite laminates which influences the stiffness of the composite plate that causes a stress redistribution that can modify the life prediction. This is an important feature that should be included in future improvements.

Furthermore there is still a lot of work that can be done on the probabilistic versions of the probabilistic S-N curves and combined for different load ratios in PCLD models, which form an important part of the probabilistic analysis to translate the critical fatigue stress and it's distribution (variance) resulting from the PFEM analysis to fatigue cycles to failure with it's probabilistic distribution. Firstly, some of the parameters of the probabilistic fatigue PCLD model, have been determined using first order Taylor expansions (the variance of the constitutive variables of the fatigue properties). Additional experimental data should be obtained in order to use actual measured variance values on the material fatigue parameters.

Appendix

Appendix A

Publication

A.1 Markov chain based non-linear fatigue damage accumulation model



Contents lists available at ScienceDirect

Probabilistic Engineering Mechanics

journal homepage: www.elsevier.com/locate/probengmech

Markov chain based non-linear fatigue damage accumulation model

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ABSTRACT

Methodology to model fatigue damage accumulation of Fiber-Reinforced Plastics (FRP) composite laminates subjected to variable amplitude spectrum loading is proposed in this paper. Markov chain based Probability Transition Matrices (PTM) are modeled for each of the different block load levels, applying matrix multiplication to combine the PTM's, resulting in a single probability mass function for the full variable amplitude load spectrum. Variable amplitude block load experimental data was used to demonstrate and validate the methodology and compare against other for Fiber-Reinforced Plastics commonly applied linear and non-linear damage accumulation and strength degradation based fatigue damage accumulation models. PTM's modeled directly from experimental data for the different variable amplitude load levels were obtained to calculate the probability of failure for the full load spectrum. When applying Markov chain based probability transition matrices together with matrix multiplications, both load sequence effects as well as non-linear fatigue damage growth behavior can be accurately modeled, resulting that the model predicts failure probabilities very close to the actual experimental results.

1. Introduction

The fatigue damage accumulation evolution in composite structures is very complex and mainly an irreversible phenomenon [1,2], where the damage in the structure under consideration gradually accumulates, which causes stiffness and strength reduction over a period of time and ultimately lead to failure of the local laminates and consequently of the entire structure [3,4]. Therefore, cumulative fatigue damage can be treated as a measure of the degradation in fatigue strength of materials [5].

One of the first papers on predicting fatigue damage accumulation is from Miner [6], whose Linear Damage Rule (LDR) applied together with Stress against N cycles to failure (S–N curve), commonly for different load ratios combined in a Constant Life Diagram (CLD) model [7] has become the main approach for fatigue damage accumulation evaluation and life prediction of composite structures submitted to Variable Amplitude (VA) loads spectrums.

Pinto et al. [8] revealed with experimental data that the fatigue damage accumulation evolution, especially in composite structures, has often a clear non-linear behavior and Rathod et al. [9] showed further highly probabilistic process by nature. It is therefore very important in order to perform a reliable structural fatigue life analysis, that the fatigue damage accumulation models used reflects the damage accumulation evolution of the experimental data from the material and structure under consideration, including non-linear trends,

load sequence effects and further that the models can be applied for probabilistic analysis.

In the present paper, a probabilistic fatigue damage accumulation model based on Markov chains is proposed by combining for a VA load spectrum, the Markov chain based Probability Transition Matrices (PTMs) for each of the different load levels by using matrix multiplication, which results in a single PTM of the respective full load spectrum. This model was first introduced by Bogdanoff and Kozin [10] and is therefore for the sake of simplicity called the Bogdanoff Markov chain model (B-model) further through the report. B-models were intended to be used to compute the probability of failure for damage to physical problems from experimental results, but can be also constructed from computational results as shown by Bea et al. [11], which will be the approach followed in the investigation discussed in this section.

Most of the already existing applications in fatigue life analysis using the B-models are related to crack growth in metal structures. Bea and Doblaré [12] demonstrate the advantage of a metal that the crack length a is a quantity that can often be observed. Gansted et al. [13] applied fracture mechanics for the crack growth of metals combined with the Markov chain B-model resulting in Fracture Mechanics Markov chain Fatigue Model (FMF) model.

The B-model can also be used for the damage accumulation process in composite laminates, where the mechanism of accumulation is in general much more difficult to observe. Some early research on the implementation of B-models specific for composite materials

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(<http://creativecommons.org/licenses/by-nc-nd/4.0/>).

Definitions and Abbreviations

B-model	Bogdanoff Markov chain model
CA	Constant Amplitude
CD	Cumulative Damage
CDF	Cumulative Distribution Function
CLD	Constant Life Diagram
FMF	Fracture Mechanics Markov chain Fatigue Model
FRP	Fiber-Reinforced Plastic
GFRP	Glass Fiber-Reinforced Plastic
HCF	High Cycle Fatigue
LB	Load Block
LC	Load Case
LCF	Low Cycle Fatigue
LDR	Linear Damage Rule
LRSM	Linear Residual Strength Model
NDR	Non-linear Damage Rule
NRSM	Non-linear Residual Strength Model
PMF	Probability Mass Function
PTM	Probability Transition Matrix
RMSE	Root Mean Square Error
RV	Random input Variable
S-N	Stress against N cycles to failure
UD	Uni Directional
VA	Variable Amplitude

Nomenclature

D	Damage
E	Expectation (mean)
N_f	Number of cycles to failure
UCS	Ultimate Compressive Strength
UTS	Ultimate Tensile Strength
Var	Variance
λ	Weibull scale parameter
ν	Strength degradation parameter
a	Crack length
b	Failure state
k	Weibull shape parameter
r	Ratio of stay to jump probabilities

has been done by Spanos and Rowatt [14] for stiffness loss data of graphite/epoxy laminate tests and Rowatt and Spanos [15] used probabilistic S-N curve curves. Both studies use mainly CA experimental data or even hypothetical data, instead of applying the B-models to VA block loading spectrum's. Further a good comparison is missing how the B-models performs against other fatigue damage accumulation models, which will be part of the current study.

As will be demonstrated in this paper, it is possible to predict the fatigue life of a variable amplitude (block) load spectrum applied on composite laminates, by only using experimental data from the single CA load level fatigue tests for each of the block load levels of a repetitive two block load spectrum.

The key features of the B-model are the use of matrices to represent the damage growth and therefore the ability to model non-linear damage growth using the PTM Markov matrices, further matrix multiplications of the single PTMs models the damage evolution for the different load levels of a full VA load spectrum, taking hereby into account load sequence effects. In this way the damage evolution during an actual test up to failure can be very accurately modeled and predicted as will be demonstrated in this paper with some practical examples in

Section 4. Moreover the PTMs matrices can be relatively easy created directly from CA test data of each of the load levels.

This resulting in a relative simple and very accurate method to translate the complex non-linear behavior of composite materials during the material fatigue test to Cumulative Damage (CD) models that can be applied for deterministic but especially also for probabilistic structural fatigue life analysis.

2. Markov chain cumulative damage model

A Markov chain is a type of Markov process, discrete in time and damage states, that satisfies the Markov property, i.e., the memory less property of a stochastic process. This means that the future states of the process (conditional on both past and present states) depends only upon the present state, so not on the sequence of events that preceded it. The probabilities associated with various state changes are called transition probabilities. The process is characterized by a state space, a transition matrix describing the probabilities of particular transitions, and an initial state (or initial distribution) across the state space.

The application of the proposed B-model results in a method of applying Markov chains, which takes into account the load sequence effect when several VA load block's are applied in a loads spectrum. First we consider applying two identical LBs. To obtain the probability of change to damage state i to damage state j the algorithm is based on Fig. 1, meaning that the possibilities to jump, under LB_1 , from initial state i to intermedium state 1 AND from intermedium state 1 to final state j , under LB_1 , OR from initial state i to intermediate state 2 AND from intermediate state 2 to final state j ... Since logical AND in probability means to multiply and logical OR means to sum, this algorithm can be mathematically expressed as $P_1 \cdot P_1$, being P_1 the PTM for LB_1 . Now consider that the first LB is LB_1 , and after it acts LB_2 . In this case it is necessary to consider all the possible jumps from initial state i to all states under LB_1 , and then the corresponding jump to every intermedium state to final state j , under LB_2 , which means $P_1 \cdot P_2$.

The basis of this paper is that, as matrix multiplication is non commutative it results in a mathematically consistent non-linear rule. Assume that the damage process in the proposed model is such that only unit-jumps are permitted, as discussed by Bogdanoff and Kozin [10], therefore the basic assumptions can be summarized as the proposed B-model model is a finite state, further a discrete-time embedded Markov process, in which the damage accumulation mechanism is of the unit-jump type. Suppose the initial state of damage is 1 and that failure occurs when state b is reached and further that an item can be in any of the previous states mentioned and in states in between so in any of the states $1, 2, \dots, b$.

From the theory of Markov chains, p_t the Probability Mass Function (PMF) of the Random input Variable (RV) at time t is given by the $(1 \times b)$ vector,

$$p_t = [p_n(1), p_n(2), p_n(3), \dots, p_t(b)] \quad (1)$$

The initial state of damage at time $t = 0$ the $(1 \times b)$ row vector becomes,

$$p_0 = [p_0(1), p_0(2), p_0(3), \dots, p_0(b)] \quad (2)$$

The PMF vector p_t that defines the probability of staying at each damage level after a given number x of LBs, is completely determined from the PMF of initial compliance p_0 and the PTM which define the severity of each duty cycle. This relationship is modeled as a Markov chain, with b damage states, which follows from the multiplication rule for independent events by,

$$p_t = p_0 P_n^k = p_0 [P_1 P_2 \dots P_n]^k \quad (3)$$

Where P_n are n different one-step B-model PTM matrices for each LBs and k the amount of load spectrum repetitions. Each of these

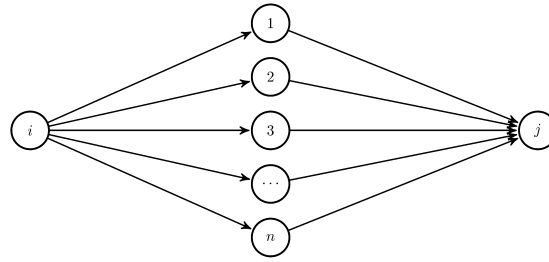


Fig. 1. Chapman-Kolmogorov flow diagram.

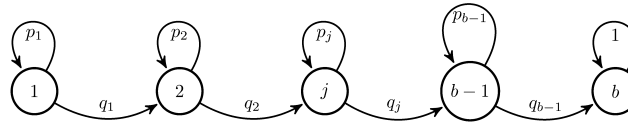


Fig. 2. B-model Markov chain flow diagram.

PTM matrices $\mathbf{P}_n$ is a $b \times b$ unit-jump Markov chain, which has $b - 1$ state-dependent transition states of the form,

$$\mathbf{P} = \begin{bmatrix} p_0 & q_0 & 0 & 0 & 0 \\ 0 & p_1 & q_1 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & p_{b-1} & q_{b-1} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (4)$$

Where conditional probability p represents the probability of remaining in the damage level and $q = 1 - p$ the probability of jumping to the next adjacent damage level. The PTM only has the diagonal terms consisting of p 's and q 's, with the ratio of transition probabilities r is $r = p/q$.

The resulting Markov chain is illustrated in Fig. 2, which shows the Markov chain has $b - 1$ transient states and one absorbing state b . This can be explained by the cumulative damage process such that the probability for each damage state going back to the previous state is zero.

The transition probability functions of the B-model Markov chain further satisfies the Chapman-Kolmogorov [16] equation, which in terms of PTMs results in the compact form equation.

$$\mathbf{P}^{(n)} = \mathbf{P}^{(m)} \mathbf{P}^{(n-m)} \quad (5)$$

Where $m < r < n$.

Hence the n -step PTM can be found through matrix multiplication, using the Chapman-Kolmogorov equation (5), taking into account the load sequence effects right from the start.

For the experimental data used in this study there are two different load levels for the block loading tests consisting of multiple cycles at a specific constant amplitude load level as illustrated in Fig. 3.

Therefore there are two PTMs, one for a high load/stress level $\mathbf{P}_{\text{high}}$ and one for low load/stress level $\mathbf{P}_{\text{low}}$, which are combined to get one Load Case (LC),

$$LC = [\mathbf{P}_{\text{high}} \cdot \mathbf{P}_{\text{low}}] \quad (6)$$

Integrating Eq. (6) in Eq. (3) results in the PMF,

$$\mathbf{p}_t = \mathbf{p}_0 [LC]^k = \mathbf{p}_0 [\mathbf{P}_{\text{high}} \cdot \mathbf{P}_{\text{low}}]^k \quad (7)$$

Where k is the total number of loading block repetitions up to a probability of failure of $P_f = 1$

As explained before, a very important parameter of the B-model method is that matrix multiplication can be used to multiply the different PTMs for the different LBs, with has some important benefits such that it takes into account load sequence effects as $\mathbf{P}_{\text{high}} \cdot \mathbf{P}_{\text{low}} \neq \mathbf{P}_{\text{low}} \cdot \mathbf{P}_{\text{high}}$.

2.1. Determine PTM parameters

Appropriate PTMs need to be determined which has in this study been done by using the corresponding CA fatigue load test data for each load block of the VA spectrum.

As one can see from the PTM Markov chain of Eq. (4), only two independent parameters need to be evaluated, namely r which defines the ratio of the transition probabilities in each PTM matrix and b which defines the matrix size for each PTM matrix (so it must be an integer). One can find for the cycles to failure N_f of a CA load block the Mean $E[N_f]$ and Variance $Var[N_f]$, which suffice to determine the two parameters r and b of the model using,

$$E[N_f] = (b - 1)(1 + r) \quad (8)$$

$$Var[N_f] = (b - 1)r(1 + r) \quad (9)$$

So with Eqs. (8) and (9), the two unknown values r and b can then be calculated with,

$$b = \frac{E[N_f]^2}{E[N_f] + Var[N_f]} + 1 \quad (10)$$

$$r = \frac{E[N_f] - (b - 1)}{(b - 1)} = \frac{Var[N_f]}{E[N_f]} \quad (11)$$

The various duration of fatigue test cycles for the LBs, leads to different estimated parameters for the PTMs. Parameter b defines the amount of damage states, so a higher b value results in a larger amount of damage states and therefore describes the damage evolution in more detail. On the other hand, the computing time increases by increasing the number of damage states b .

In order to multiply the n different PTM's P_n as per Eq. (3) with each other the PTM's need to be of the same fixed size and thus have the same b value for each matrix. To achieve this a fixed value for b was chosen for the PTMs.

3. Fatigue damage evolution models for composites

Many efforts have been made to elaborate on non-linear damage accumulation models which include load sequence effects Gao et al. [17], Han et al. [18], Zheng et al. [19], but most of these models were developed for metals and use rather complex formulas with multiple variables to fit the experimental data.

Therefore in order to compare the in this study discussed B-model applied on Glass Fiber-Reinforced Plastic (GFRP) composite material with other fatigue damage evolution models, some models have been

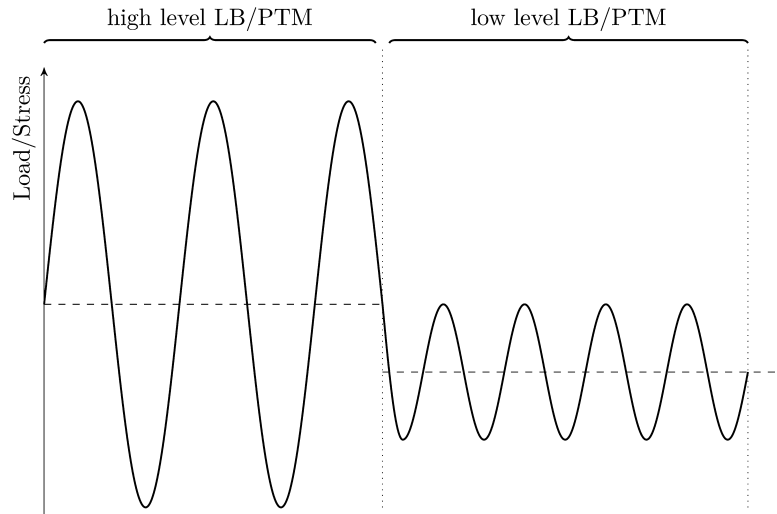


Fig. 3. Definition of load blocks and PTM in LBs.

Table 1
Fatigue damage evolution models.

Model name	Model type	Damage evolution	Sequence effects	Sec.
B-model	Markov chain	Non-linear	Yes	2
LDR	CD	Linear	No	3.1
NDR	CD	Non-linear	No	3.2
LRSM	Residual strength	Linear	Yes	3.3
NRSM	Residual strength	Non-linear	Yes	3.4

selected for comparison that have been studied and applied for GFRP composite materials, further been successful applied to analyze VA load spectrum's and experimentally have been demonstrated in two or more block load test experiments. These selected models to compare with the B-model are listed in Table 1.

3.1. Miner's linear damage rule

The most know and up to today still widely applied damage accumulation rule is the Linear Damage Rule (LDR), The early work on aluminum by Miner [6] resulted in a simple linear damage accumulation rule that was based upon constant amplitude fatigue test results. The basis of this rule is that the damage contribution of each load level is equal to its cycle ratio, which is the number of cycles experienced at that load level divided by the number of constant amplitude cycles to failure at that same load level. The damage contributions of each load level are algebraically added to allow determining an overall damage level.

$$D = \sum_{i=1}^k \frac{n_i}{N_i} \quad (12)$$

Where D is a quantified damage accumulation parameter previously termed Miner's sum, i is the indexing parameter related to the number of different load levels, n_i is the number of cycles experienced at a σ_i maximum stress level and N_i is the number of constant amplitude cycles to failure for the same stress level σ_i .

Miner's LDR formula, was initially derived for the simple case of constant amplitude cycling for metal components and afterwards extended to the case of more complex fatigue loading and other materials such as glass and carbon reinforced plastics.

Therefore some of the more fundamental shortcomings of applying the LDR as damage accumulation for fatigue life analysis is that it does not predict failure under variable amplitude loading spectrums very well Nijssen [20].

For the VA two block loading case we get for the linear damage accumulation,

$$D = \sum_{i=1}^k \frac{n_{high}}{N_{high}} + \frac{n_{low}}{N_{low}} \quad (13)$$

Where k is the total number of two block loading repetitions up to failure $D = 1$.

3.2. Non linear damage rule

In order to improve the accuracy of the LDR, different Non-linear Damage Rules (NDRs) have been proposed to describe the damage accumulation process. A stress independent NDR model for GFRP was developed by Owen and Howe [21] for glass chopped strand-mat/polyester resin specimens subjected to fatigue loading. Their quadratic non-linear damage accumulation model has the form,

$$D = \sum_{i=1}^k \left[A \left(\frac{n_i}{N_i} \right) + B \left(\frac{n_i}{N_i} \right)^2 \right] \quad (14)$$

Bond [22] improved the model by replacing the quadratic term in an other variable parameter C resulting in,

$$D = \sum_{i=1}^k \left[A \left(\frac{n_i}{N_i} \right) + B \left(\frac{n_i}{N_i} \right)^C \right] \quad (15)$$

Bond and Farrow [23, p. 641] further proposed a metric for calculating the parameters based on initial strength properties UTS and UCS for primarily tensile loading,

$$A = B = |UCS/UTS| \quad (16)$$

$$C = (UTS(|UTS/UCS| + 1) - |UTS|)/UTS \quad (17)$$

So for the VA two block loading case for the Bond and Farrow [23] NDR model we get,

$$D = \sum_{i=1}^k \left[\left[A \left(\frac{n_{high}}{N_{high}} \right) + B \left(\frac{n_{high}}{N_{high}} \right)^C \right] + \left[A \left(\frac{n_{low}}{N_{low}} \right) + B \left(\frac{n_{low}}{N_{low}} \right)^C \right] \right] = 1 \quad (18)$$

3.3. Linear residual strength model

Broutman and Sahu [24] were one of the earliest to develop a Linear Residual Strength Model (LRSM) cumulative damage law founded upon residual strength changes during fatigue life, which accounts for load/stress sequence effects. They examined the tensile fatigue behavior of composite E-glass fiber reinforced epoxy subjected to two block load/stress level VA tests. Their model is based upon a linear loss of strength with cycles of fatigue, as represented by:

$$\sigma_r = \sigma_{ult} - \sum_{i=1}^k (\sigma_{ult} - \sigma_p^i) \left(\frac{n_i}{N_i} \right) \quad (19)$$

Where σ_r is the residual strength, σ_p^i is the maximum peak stress during the fatigue cycle, σ_{ult} is the static strength of the specimen, N_i is the number of constant amplitude cycles to failure at the stress level σ_i , n_i is the number of cycles experienced at stress level σ_i . Broutman and Sahu [24] performed experimental work on two level load block fatigue tests at two different stress level and reported that there were clear load sequence effects on the fatigue lifetime predictions, further that their LRSM prediction results were not the same as the LDR prediction results. For the VA two block loading case Eq. (19) becomes for the LRSM,

$$\left(\frac{1 - \sigma_{high}/UTS}{1 - \sigma_{low}/UTS} \right) \frac{n_{high}}{N_{high}} + \frac{n_{low}}{N_{low}} \quad (20)$$

Where $\sigma_{high}, \sigma_{low}$ are the cycle peak stress levels for the high and low load block cycles, as defined in Table 3 and N_{high}, N_{low} are the number of LB cycles to failure corresponding to high and low load block respectively as defined in Table 4.

3.4. Non-linear residual strength model

Reifsnider and Stinchcomb [25] proposed a Non-linear Residual Strength Model (NRSRM) based on Eq. (20), but adding a non-linear parameter ν . For the two block loading case, the Reifsnider and Stinchcomb NRSRM becomes,

$$\left(\frac{1 - \sigma_{high}/UTS}{1 - \sigma_{low}/UTS} \right)^{\frac{1}{\nu}} \frac{n_{high}}{N_{high}} + \frac{n_{low}}{N_{low}} = 1 \quad (21)$$

Where ν is the non-linear strength degradation parameter, Eq. (20) is identical to Eq. (21) if $\nu = 1$.

3.5. Weibull fit

To compare the B-model probability of failure curves, Weibull Cumulative Distribution Function (CDF) curves have been modeled fitted to both directly to the experimental data points as well as to the fatigue cumulative damage results when applying the different above described fatigue damage accumulation models.

The CDF curve for the applied two parameter Weibull distribution Weibull [26] can be defined as,

$$F(x; k, \lambda) = 1 - e^{-(x/\lambda)^k} \quad (22)$$

Where F denotes the cumulative failure probability. The only input variables required to derive the Weibull parameters k and λ are the mean E and variance Var of the VA test data (cycles to failure N_f).

Table 2

Fatigue damage evolution model parameters.

Model name	Model parameters			
	A	B	C	ν
NDR	0.74	0.74	1.36	
NRSRM				0.5

The shape k parameters were be obtained using the efficiency Var/E^2 from the first two moments E and Var [27],

$$\frac{Var}{E^2} = \frac{2k}{B\left(\frac{1}{k}, \frac{1}{k}\right)} - 1 \quad (23)$$

Followed by the scale λ parameter of Eq. (22), which can then be obtained using the first moment E [28] by,

$$E(X) = \lambda \Gamma\left(1 + \frac{1}{k}\right) \quad (24)$$

Using Eqs. (23) and (24) the λ and k were numerical obtained, solving the nonlinear equation system using SciPy [29] and SymPy [30].

4. Validation using experimental data

The B-model presented in Section 2 was validated against experimental data for Fiber-Reinforced Plastic (FRP) laminates as described in Section 4.1 and compared with other fatigue damage evolution models as discussed in Section 3.

Section 4.1.1 describes and Section 4.1.2 shows the results of how the single PTM are defined for the high and the low test load levels, followed by the matrix multiplication of the PTMs that results in the final test spectrum PMFs, which has been used use to predict the data of the full two block load test spectrum, up till failure.

4.1. Experimental data composite laminate

Fatigue test data from The Sandia National Laboratories and Montana State University DOE/MSU Composite Material Fatigue Database [31] has been used, as it contains an extensive range of fatigue data, from both repeated block testing of subsequent high load level High Cycle Fatigue (HCF) and lower load level Low Cycle Fatigue (LCF) load blocks as well as the corresponding CA fatigue load test data for each load block and the static extreme test data.

Test data was taken from a detailed study of spectrum loading effects using a typical tri-axial laminate from the database, material named DD5. This laminate has the layup configuration $(0/\pm 45/0)_S$, with a fiber volume fraction of $V_f = 38\%$. The 0° plies are D155 stitched Uni Directional (UD) fabric, the $\pm 45^\circ$ plies are DB120 stitched fabrics from manufacturer Owens Corning, and the resin used for the laminate was an orthopolyester resin matrix (CoRezyn 63-AX-051 by Interplas-tics Corp.). The test methodologies for a series of tests performed with DD5 specimens using simple two load blocks of different duration's and loads, repeated to failure are described in detail in Wahl et al. [32].

For the nonlinear fatigue damage evolution models from Section 3 and Table 1, in Table 2 are listed for the DD5 material laminate the missing model parameters.

In Table 2, the NDR model parameter A and B were calculated with Eq. (16) and parameter C was calculated with Eq. (17), using the static strength properties for DD5 $UTS = 724$ MPa and $UCS = -534$ MPa. Further for the NRSRM model the strength degradation parameter ν , was tuned to $\nu = 0.5$ as this gave the best fit overall for all the block load tests to the test data Weibull CDF.

Fig. 4 shows for each two load level cycle block combination test the amount and dispersion of the test data points.

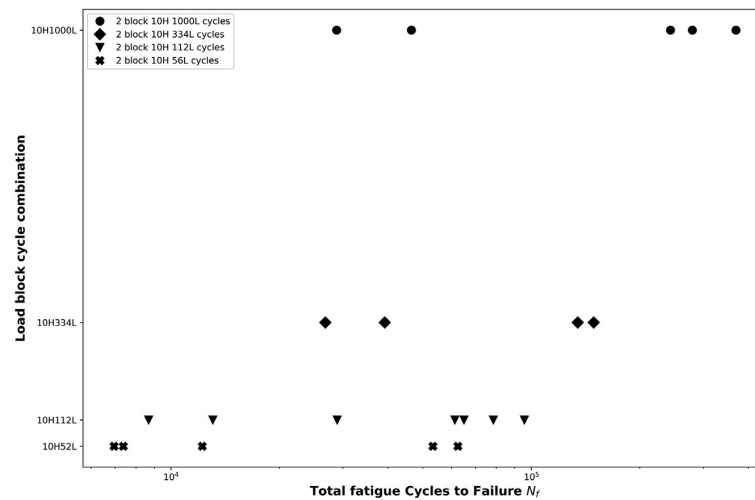


Fig. 4. 2 block test result data ranked per low load block cycles.

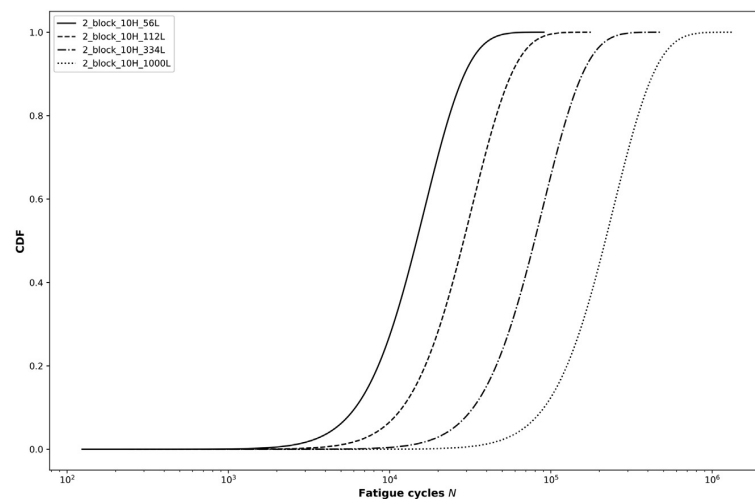


Fig. 5. B-model CDF's for number of cycles.

Table 3
Fatigue CA test data.

Load level	R-ratio	σ_{max} [MPa]	σ_{min} [MPa]	E_{N_f}	Var_{N_f}
High	0.1	414	41.4	2672	2.3112E+6
Low	0.1	241	24.1	2457832	2.0974E+12

4.1.1. Example analysis using two PTM's for a high and low load level

A B-model Markov chain analysis was performed with a combination of “high” and “low” load level LBs. A first step in the analysis was to construct the single PTMs for the two different load block levels P_{high} and P_{low} .

The PTM parameters were analyzed using Eq. (10) to calculate parameter b and Eq. (11) to calculate parameter r , for both using the mean $E[N_f]$ and variance $Var[N_f]$ values, as listed in Table 3, which were obtained from the CA load experimental data [31].

The b value which defines the matrix size was kept at a constant value, in order to multiply the two different PTMs, P_{high} and P_{low} . The

number of cycles in a LB was given by the cycles of a test load block for each PTM, as listed in Table 4.

The next step in the analysis was to multiply the P_{high} and P_{low} , which resulted in a single PTM for one high and low block LC, so for example test load “2 block 10H 56L” the PTM is for the 10 “high” and 56 “low” cycles, so the PMF of Eq. (7) becomes,

$$P_{10H56L} = P_0 [P_{10H} \cdot P_{56L}]^k \quad (25)$$

Finally the evolutionary probability of failure was calculated for the combined “high” and “low” PTM was repeatedly multiplied according to Eq. (25) until a failure probability of $P_f = 1$ was reached, resulting for each of the two block load sequence combinations in the B-model CDF curves as shown in Fig. 5.

4.1.2. Results

A comparison have been made of the B-model CDF curves against the Weibull CDF fit of the test result data and the fatigue damage evolution models listed in Table 1 for each of the 2 block load level sequence tests.

Table 4
Single PTM data.

PTM name	LB cycles	LB statistics		PTM param	
		E	Var	b	r
10H	10	267	23112	4	87
56L	56	43890	6.6881E+10	4	15234
112L	112	21945	1.6720E+8	4	7619
334L	334	7359	1.8801E+7	4	2555
1000L	1000	2458	2.0974E+6	4	853

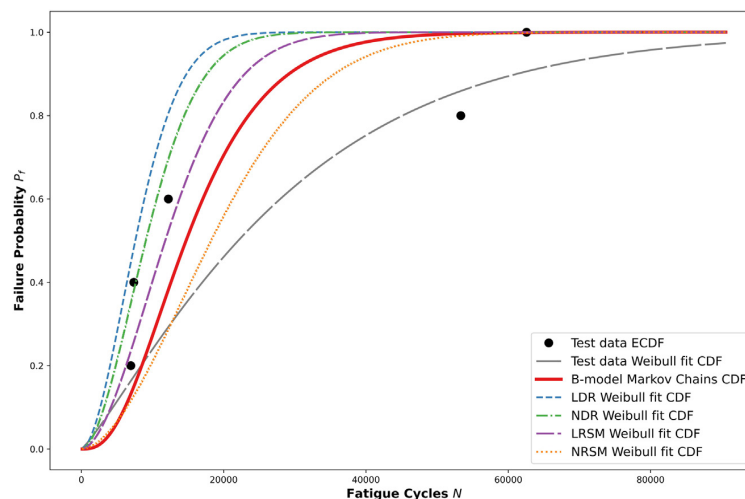


Fig. 6. Fatigue life for repeated 2 block sequence of 10 high and 56 low cycles.

It is on the forehand important to clarify again that in this study as input for the B-model analysis only data has been used from separate high and low load level CA fatigue test data and not the combined two block load spectrum tests data. On the other hand for the other fatigue damage accumulation and strength reduction models used to compare the B-model the VA two block load level spectrum tests data was directly used.

2 block 10H 56L Test: For this case, as shown in Fig. 6, five fatigue tests results were available. For a low failure probabilities the B-model predicts a somewhat larger fatigue life than the test data Weibull CDF curve, which is also the case for the NRSM model. The LDR, NDR and LRSM models show in general shorter fatigue life at equal failure probability levels compared to both the test data Weibull and B-model curve.

2 block 10H 112L Test: As shown in Fig. 7, only a close fit at the toe of the CDF curve from test data Weibull distribution compared to the curve from the B-model was achieved. This was mainly caused by the large variance (largest from all block test used in this paper) of the two block test data results, as shown in Table 4, which makes it more complicated and less accurate make a good test data Weibull curve fit.

2 block 10H 334L Test shown in Fig. 8, with only four fatigue test results available, the matching between the test data Weibull curve and B-model curve are very close for the entire failure probability range. The predicted fatigue life resulting from the B-model is slightly higher than for the test data Weibull fit, up to a failure probability of ≈ 0.5 . This test data result set has a relative small variance or spread compared to the other tests, as shown in Fig. 4 and Table 4, which results in the close fit of the B-model and test data Weibull CDF and hereby showing a better performance of the B-model for smaller variances.

2 block 10H 1000L Test Fig. 9 shows that the predicted fatigue life from B-model is higher than the test data Weibull CDF. In this case the load sequence is significantly different than the previous ones because

only 10 high load level cycles are applied, and after that 1000 low load cycles. More than a combination of block the high cycle blocks causes peak overloads acting on the much larger low load cycle blocks. In this case a different damage mode may appear in the material, which are not cover for by the block load sequence effects of the B-model.

In general more fatigue tests results are desirable for each load block combination test, in order to obtain a good comparison of the data and to make a better Weibull fit of the test data points. But overall the predicted fatigue life probability distribution from the B-model fits very well to the actual two block loading test behavior from the experimental data.

To validate and compare the performance of each fatigue model with respect to the experimental data points, the Root Mean Square Error was calculated, which gives a good comparison of the fit of the different fatigue evolution models, with respect to their prediction errors.

$$RMSE = \sqrt{\frac{\sum_{i=1}^n (\hat{y}_i - y_m)^2}{n}} \quad (26)$$

Where $\hat{y}_i$ are the test data points, y_m the corresponding model points for $\hat{x}_i$, n the amount of data points.

Fig. 10 shows that the B-model has overall for all the block load sequence tests the lowest RMSE and therefore shows the better fit to the test data points out of all the fatigue evolution models.

Further Fig. 10 shows that out of all the fatigue evolution models the B-model has on average for all the block load tests the lowest Root Mean Square Error which means the best model fit to the actual block test data points, which therefore results in the most accurate fatigue damage evolution model. The Non-linear Residual Strength Model has after the B-model the best model fit to the test data points with also an overall relative low RMSE followed by the Linear Residual Strength Model which has low RMSE at two tests but much higher fitting errors for the other two tests. Both the Linear Damage Rule as well as the

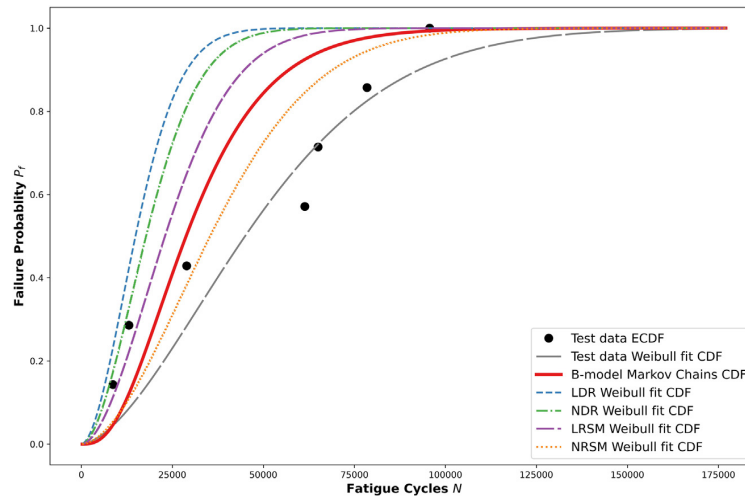


Fig. 7. Fatigue life for repeated 2 block sequence of 10 high and 112 low cycles.

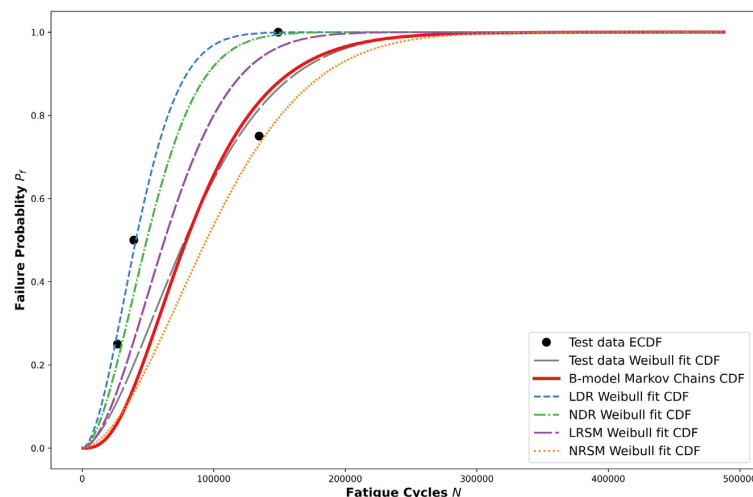


Fig. 8. Fatigue life for repeated 2 block sequence of 10 high and 334 low cycles.

Non-linear Damage Rule have overall higher Root Mean Square Error and therefore show to be the worst model fit to the actual block test data points and therefore less accurate and realistic fatigue damage evolution models.

5. Conclusions

The systematic description for the implementation of the B-model Markov chains as fatigue damage accumulation model for probabilistic fatigue analysis on composite structures in this case for a tri-axial glass fiber laminate with polyester resin matrix subjected to axial fatigue loading, has been pursued. Different PTMs representing the different block load levels were obtained directly from experimental data, these PTMs then were multiplied using matrix multiplication to obtain for each block load spectrum of the variable amplitude two block load test a final PMF from which the CDF of the block load combinations have been obtained, representing the cumulative damage of the tests.

Some of the key features of the B-model are that it takes into account both load sequence effects as well as non-linear fatigue damage

growth and therefore does not have some of the important restriction or errors of other especially linear damage evolution models.

Some of the benefits of the B-model are that the required PTM Markov chains can be obtained using different methods:

- Directly created from the constant amplitude load block experimental data, which was the method applied in this paper and demonstrated to be a relative simple and very accurate method to translate the behavior from the material fatigue test directly to the fatigue damage accumulation model.
- Using probabilistic S-N curves or combined for different load ratios in probabilistic constant life diagram.
- Using other damage accumulation metrics such as strength or stiffness reduction models.

Further the B-model analysis outcome gives the probability of failure against fatigue life cycles as CDF curves, instead of commonly a single damage value or fatigue reserve factor typically resulting from the deterministic fatigue damage accumulation models. Therefore the damage evolution probability at any fatigue life loading stage and the

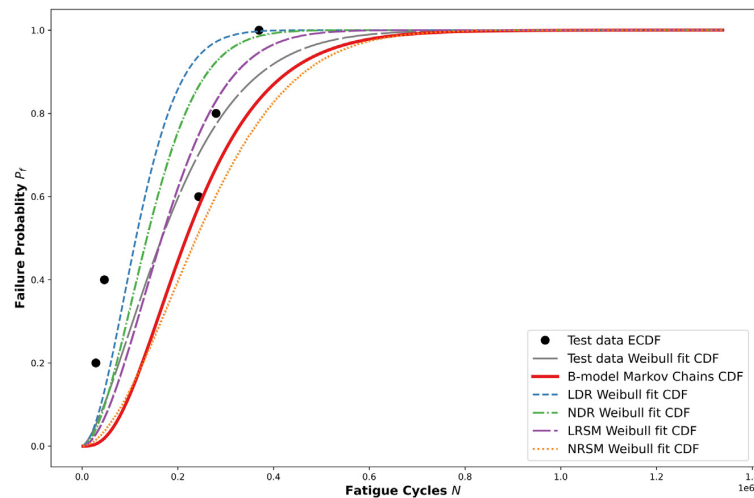


Fig. 9. Fatigue life for repeated 2 block sequence of 10 high and 1000 low cycles.

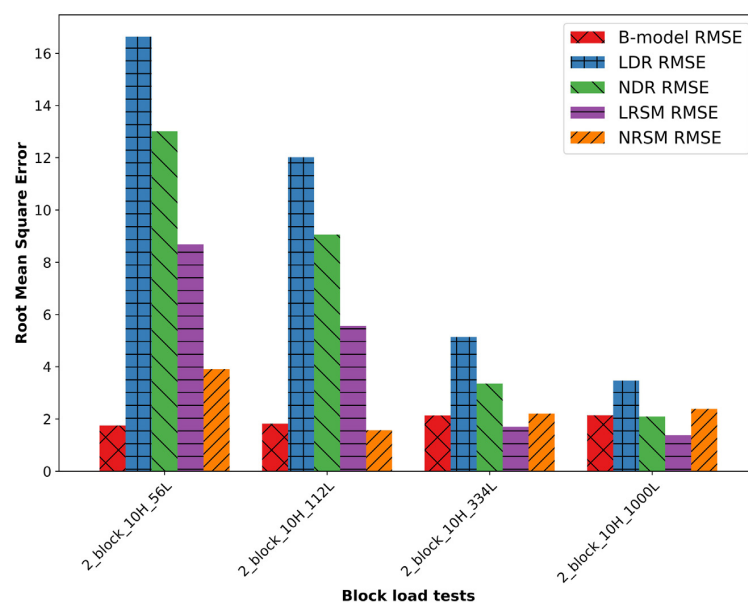


Fig. 10. RMSE fatigue evolution models against test data points.

probability of failure at any damage state can be analyzed by the method presented in this paper.

It has been demonstrated in this paper that using the B-model it is possible to predict accurately the fatigue life of composite laminates subjected to variable amplitude load spectrum.

The validation results and correlation to other models, show the very good fit of the B-model to the variable amplitude two block load test data with low root mean square errors, proving the very good performance of the B-model compared with other commonly used cumulative fatigue damage evolution models.

Some drawbacks of the B-model method are its sensitivity to rather large variance values. Further the B-model results need still to be validated against more complex loads spectra that are representative for actual more complex stochastic structural load spectrum's, where multiple PTMs can be used and combined to represent the different

loading situations that can occur during a composite structural design life.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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