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Markov chain based non-linear fatigue damage accumulation model

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ABSTRACT

Methodology to model fatigue damage accumulation of Fiber-Reinforced Plastics (FRP) composite laminates subjected to variable amplitude spectrum loading is proposed in this paper. Markov chain based Probability Transition Matrices (PTM) are modeled for each of the different block load levels, applying matrix multiplication to combine the PTM's, resulting in a single probability mass function for the full variable amplitude load spectrum. Variable amplitude block load experimental data was used to demonstrate and validate the methodology and compare against other for Fiber-Reinforced Plastics commonly applied linear and non-linear damage accumulation and strength degradation based fatigue damage accumulation models. PTM's modeled directly from experimental data for the different variable amplitude load levels were obtained to calculate the probability of failure for the full load spectrum. When applying Markov chain based probability transition matrices together with matrix multiplications, both load sequence effects as well as non-linear fatigue damage growth behavior can be accurately modeled, resulting that the model predicts failure probabilities very close to the actual experimental results.

1. Introduction

The fatigue damage accumulation evolution in composite structures is very complex and mainly an irreversible phenomenon [1,2], where the damage in the structure under consideration gradually accumulates, which causes stiffness and strength reduction over a period of time and ultimately lead to failure of the local laminates and consequently of the entire structure [3,4]. Therefore, cumulative fatigue damage can be treated as a measure of the degradation in fatigue strength of materials [5].

One of the first papers on predicting fatigue damage accumulation is from Miner [6], whose Linear Damage Rule (LDR) applied together with Stress against N cycles to failure (S–N curve), commonly for different load ratios combined in a Constant Life Diagram (CLD) model [7] has become the main approach for fatigue damage accumulation evaluation and life prediction of composite structures submitted to Variable Amplitude (VA) loads spectrums.

Pinto et al. [8] revealed with experimental data that the fatigue damage accumulation evolution, especially in composite structures, has often a clear non-linear behavior and Rathod et al. [9] showed further highly probabilistic process by nature. It is therefore very important in order to perform a reliable structural fatigue life analysis, that the fatigue damage accumulation models used reflects the damage accumulation evolution of the experimental data from the material and structure under consideration, including non-linear trends, load sequence effects and further that the models can be applied for probabilistic analysis.

In the present paper, a probabilistic fatigue damage accumulation model based on Markov chains is proposed by combining for a VA load spectrum, the Markov chain based Probability Transition Matrices (PTMs) for each of the different load levels by using matrix multiplication, which results in a single PTM of the respective full load spectrum. This model was first introduced by Bogdanoff and Kozin [10] and is therefore for the sake of simplicity called the Bogdanoff Markov chain model (B-model) further through the report. B-models were intended to be used to compute the probability of failure for damage to physical problems from experimental results, but can be also constructed from computational results as shown by Bea et al. [11], which will be the approach followed in the investigation discussed in this section.

Most of the already existing applications in fatigue life analysis using the B-models are related to crack growth in metal structures. Bea and Doblaré [12] demonstrate the advantage of a metal that the crack length a is a quantity that can often be observed. Gansted et al. [13] applied fracture mechanics for the crack growth of metals combined with the Markov chain B-model resulting in Fracture Mechanics Markov chain Fatigue Model (FMF) model.

The B-model can also be used for the damage accumulation process in composite laminates, where the mechanism of accumulation is in general much more difficult to observe. Some early research on the implementation of B-models specific for composite materials

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Definitions and Abbreviations

B-model	Bogdanoff Markov chain model
CA	Constant Amplitude
CD	Cumulative Damage
CDF	Cumulative Distribution Function
CLD	Constant Life Diagram
FMF	Fracture Mechanics Markov chain Fatigue Model
FRP	Fiber-Reinforced Plastic
GFRP	Glass Fiber-Reinforced Plastic
HCF	High Cycle Fatigue
LB	Load Block
LC	Load Case
LCF	Low Cycle Fatigue
LDR	Linear Damage Rule
LRSM	Linear Residual Strength Model
NDR	Non-linear Damage Rule
NRSM	Non-linear Residual Strength Model
PMF	Probability Mass Function
PTM	Probability Transition Matrix
RMSE	Root Mean Square Error
RV	Random input Variable
S–N	Stress against N cycles to failure
UD	Uni Directional
VA	Variable Amplitude

Nomenclature

D	Damage
E	Expectation (mean)
N_f	Number of cycles to failure
UCS	Ultimate Compressive Strength
UTS	Ultimate Tensile Strength
Var	Variance
λ	Weibull scale parameter
ν	Strength degradation parameter
a	Crack length
b	Failure state
k	Weibull shape parameter
r	Ratio of stay to jump probabilities

has been done by Spanos and Rowatt [14] for stiffness loss data of graphite/epoxy laminate tests and Rowatt and Spanos [15] used probabilistic S–N curve curves. Both studies use mainly CA experimental data or even hypothetical data, instead of applying the B-models to VA block loading spectrum's. Further a good comparison is missing how the B-models performs against other fatigue damage accumulation models, which will be part of the current study.

As will be demonstrated in this paper, it is possible to predict the fatigue life of a variable amplitude (block) load spectrum applied on composite laminates, by only using experimental data from the single CA load level fatigue tests for each of the block load levels of a repetitive two block load spectrum.

The key features of the B-model are the use of matrices to represent the damage growth and therefore the ability to model non-linear damage growth using the PTM Markov matrices, further matrix multiplications of the single PTMs models the damage evolution for the different load levels of a full VA load spectrum, taking hereby into account load sequence effects. In this way the damage evolution during an actual test up to failure can be very accurately modeled and predicted as will be demonstrated in this paper with some practical examples in Section 4. Moreover the PTMs matrices can be relatively easy created directly from CA test data of each of the load levels.

This resulting in a relative simple and very accurate method to translate the complex non-linear behavior of composite materials during the material fatigue test to Cumulative Damage (CD) models that can be applied for deterministic but especially also for probabilistic structural fatigue life analysis.

2. Markov chain cumulative damage model

A Markov chain is a type of Markov process, discrete in time and damage states, that satisfies the Markov property, i.e., the memory less property of a stochastic process. This means that the future states of the process (conditional on both past and present states) depends only upon the present state, so not on the sequence of events that preceded it. The probabilities associated with various state changes are called transition probabilities. The process is characterized by a state space, a transition matrix describing the probabilities of particular transitions, and an initial state (or initial distribution) across the state space.

The application of the proposed B-model results in a method of applying Markov chains, which takes into account the load sequence effect when several VA load block's are applied in a loads spectrum. First we consider applying two identical LBs. To obtain the probability of change to damage state i to damage state j the algorithm is based on Fig. 1, meaning that the possibilities to jump, under LB_1, from initial state i to intermedium state 1 AND from intermedium state 1 to final state j, under LB_1, OR from initial state i to intermediate state 2 AND from intermediate state 2 to final state j $\cdots$. Since logical AND in probability means to multiply and logical OR means to sum, this algorithm can be mathematically expressed as $\mathbf{P_1} \cdot \mathbf{P_1}$, being $\mathbf{P_1}$ the PTM for LB_1. Now consider that the first LB is LB_1, and after it acts LB_2. In this case it is necessary to consider all the possible jumps from initial state i to all states under LB_1, and then the corresponding jump to every intermedium state to final state j, under LB_2, which means $\mathbf{P_1} \cdot \mathbf{P_2}$.

The basis of this paper is that, as matrix multiplication is non commutative it results in a mathematically consistent non-linear rule. Assume that the damage process in the proposed model is such that only unit-jumps are permitted, as discussed by Bogdanoff and Kozin [10], therefore the basic assumptions can be summarized as the proposed B-model model is a finite state, further a discrete-time embedded Markov process, in which the damage accumulation mechanism is of the unit-jump type. Suppose the initial state of damage is 1 and that failure occurs when state b is reached and further that an item can be in any of the previous states mentioned and in states in between so in any of the states $1, 2, \ldots, b$.

From the theory of Markov chains, $\mathbf{p_t}$ the Probability Mass Function (PMF) of the Random input Variable (RV) at time t is given by the $(1\times b)$ vector,

$$\mathbf{p_t} = \left[p_n(1), p_n(2), p_n(3), \ldots, p_t(b)\right] \tag{1}$$

The initial state of damage at time $t = 0$ the $(1 \times b)$ row vector becomes,

$$\mathbf{p_0} = \left[p_0(1), p_0(2), p_0(3), \ldots, p_0(b)\right] \tag{2}$$

The PMF vector $\mathbf{p_t}$ that defines the probability of staying at each damage level after a given number x of LBs, is completely determined from the PMF of initial compliance $\mathbf{p_0}$ and the PTM which define the severity of each duty cycle. This relationship is modeled as a Markov chain, with b damage states, which follows from the multiplication rule for independent events by,

$$\mathbf{p_t} = \mathbf{p_0}\mathbf{P_n^{\mathit{k}}} = \mathbf{p_0}[\mathbf{P_1}\mathbf{P_2}\ldots\mathbf{P_n}]^k \tag{3}$$

Where $\mathbf{P_n}$ are n different one-step B-model PTM matrices for each LBs and k the amount of load spectrum repetitions. Each of these

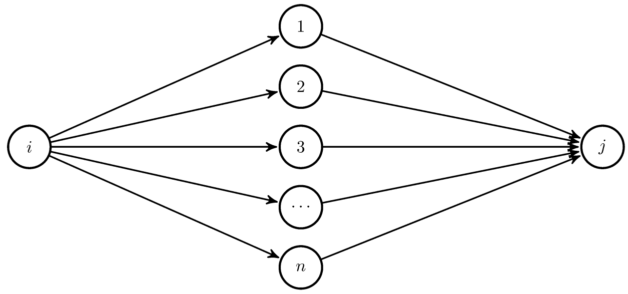

Fig. 1. Chapman–Kolmogorov flow diagram.

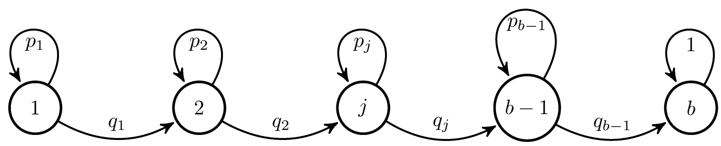

Fig. 2. B-model Markov chain flow diagram.

PTM matrices $\mathbf{P_n}$ is a $b \times b$ unit-jump Markov chain, which has $b-1$ state-dependent transition states of the form,

$$\mathbf{P} = \begin{bmatrix} p_0 & q_0 & 0 & 0 & 0 \\ 0 & p_1 & q_1 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & p_{b-1} & q_{b-1} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{4}$$

Where conditional probability p represents the probability of remaining in the damage level and $q = 1-p$ the probability of jumping to the next adjacent damage level. The PTM only has the diagonal terms consisting of p's and q's, with the ratio of transition probabilities r is $r = p/q$.

The resulting Markov chain is illustrated in Fig. 2, which shows the Markov chain has $b-1$ transient states and one absorbing state b. This can be explained by the cumulative damage process such that the probability for each damage state going back to the previous state is zero.

The transition probability functions of the B-model Markov chain further satisfies the Chapman–Kolmogorov [16] equation, which in terms of PTMs results in the compact form equation.

$$P^{(n)} = P^{(m)} P^{(n-m)} \tag{5}$$

Where $m < r < n$.

Hence the n-step PTM can be found through matrix multiplication, using the Chapman–Kolmogorov equation (5), taking into account the load sequence effects right from the start.

For the experimental data used in this study there are two different load levels for the block loading tests consisting of multiple cycles at a specific constant amplitude load level as illustrated in Fig. 3.

Therefore there are two PTMs, one for a high load/stress level $\mathbf{P_{high}}$ and one for low load/stress level $\mathbf{P_{low}}$, which are combined to get one Load Case (LC),

$$LC = [\mathbf{P_{high}} \cdot \mathbf{P_{low}}] \tag{6}$$

Integrating Eq. (6) in Eq. (3) results in the PMF,

$$\mathbf{p_t} = \mathbf{p_0}[LC]^k = \mathbf{p_0}[\mathbf{P_{high}} \cdot \mathbf{P_{low}}]^k \tag{7}$$

Where k is the total number of loading block repetitions up to a probability of failure of $P_f = 1$

As explained before, a very important parameter of the B-model method is that matrix multiplication can be used to multiply the different PTMs for the different LBs, with has some important benefits such that it takes into account load sequence effects as $\mathbf{P_{high}} \cdot \mathbf{P_{low}} \neq \mathbf{P_{low}} \cdot \mathbf{P_{high}}$.

2.1. Determine PTM parameters

Appropriate PTMs need to be determined which has in this study been done by using the corresponding CA fatigue load test data for each load block of the VA spectrum.

As one can see from the PTM Markov chain of Eq. (4), only two independent parameters need to be evaluated, namely r which defines the ratio of the transition probabilities in each PTM matrix and b which defines the matrix size for each PTM matrix (so it must be an integer). One can find for the cycles to failure N_f of a CA load block the Mean $E[N_f]$ and Variance $Var[N_f]$, which suffice to determine the two parameters r and b of the model using,

$$E[N_f] = (b-1)(1+r) \tag{8}$$

$$Var[N_f] = (b-1)r(1+r) \tag{9}$$

So with Eqs. (8) and (9), the two unknown values r and b can then be calculated with,

$$b = \frac{E[N_f]^2}{E[N_f] + Var[N_f]} + 1 \tag{10}$$

$$r = \frac{E[N_f] - (b-1)}{(b-1)} = \frac{Var[N_f]}{E[N_f]} \tag{11}$$

The various duration of fatigue test cycles for the LBs, leads to different estimated parameters for the PTMs. Parameter b defines the amount of damage states, so a higher b value results in a larger amount of damage states and therefore describes the damage evolution in more detail. On the other hand, the computing time increases by increasing the number of damage states b.

In order to multiply the n different PTM's P_n as per Eq. (3) with each other the PTM's need to be of the same fixed size and thus have the same b value for each matrix. To achieve this a fixed value for b was chosen for the PTMs.

3. Fatigue damage evolution models for composites

Many efforts have been made to elaborate on non-linear damage accumulation models which include load sequence effects Gao et al. [17], Han et al. [18], Zheng et al. [19], but most of these models were developed for metals and use rather complex formulas with multiple variables to fit the experimental data.

Therefore in order to compare the in this study discussed B-model applied on Glass Fiber-Reinforced Plastic (GFRP) composite material with other fatigue damage evolution models, some models have been

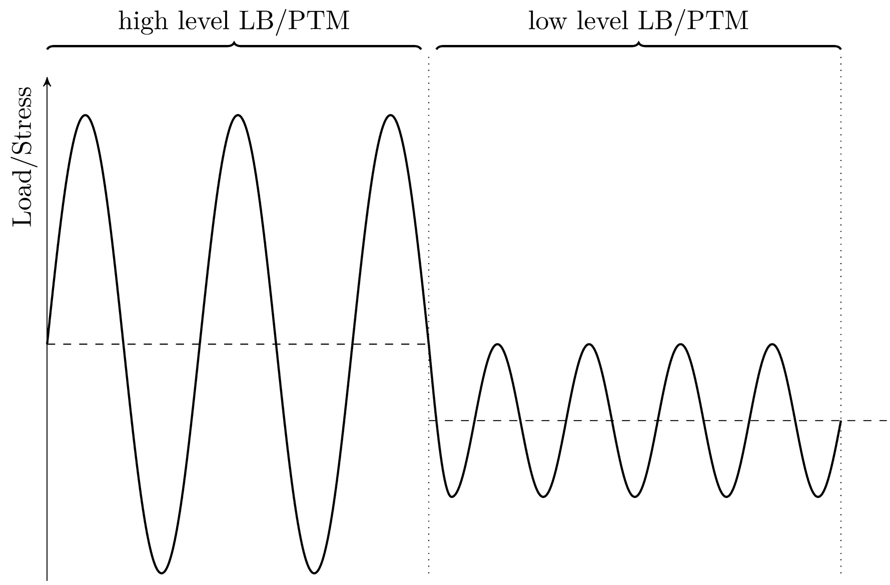

Fig. 3. Definition of load blocks and PTM in LBs.

Table 1
Fatigue damage evolution models.

Model name	Model type	Damage evolution	Sequence effects	Sec.
B-model	Markov chain	Non-linear	Yes	2
LDR	CD	Linear	No	3.1
NDR	CD	Non-linear	No	3.2
LRSM	Residual strength	Linear	Yes	3.3
NRSM	Residual strength	Non-linear	Yes	3.4

selected for comparison that have been studied and applied for GFRP composite materials, further been successful applied to analyze VA load spectrum's and experimentally have been demonstrated in two or more block load test experiments. These selected models to compare with the B-model are listed in Table 1.

3.1. Miner's linear damage rule

The most know and up to today still widely applied damage accumulation rule is the Linear Damage Rule (LDR), The early work on aluminum by Miner [6] resulted in a simple linear damage accumulation rule that was based upon constant amplitude fatigue test results. The basis of this rule is that the damage contribution of each load level is equal to its cycle ratio, which is the number of cycles experienced at that load level divided by the number of constant amplitude cycles to failure at that same load level. The damage contributions of each load level are algebraically added to allow determining an overall damage level.

$$D = \sum_{i=1} \frac{n_i}{N_i} \tag{12}$$

Where D is a quantified damage accumulation parameter previously termed Miner's sum, i is the indexing parameter related to the number of different load levels, n_i is the number of cycles experienced at a σ_i maximum stress level and N_i is the number of constant amplitude cycles to failure for the same stress level σ_i.

Miner's LDR formula, was initially derived for the simple case of constant amplitude cycling for metal components and afterwards extended to the case of more complex fatigue loading and other materials such as glass and carbon reinforced plastics.

Therefore some of the more fundamental shortcomings of applying the LDR as damage accumulation for fatigue life analysis is that it does not predict failure under variable amplitude loading spectrums very well Nijssen [20].

For the VA two block loading case we get for the linear damage accumulation,

$$D = \sum_{i=1}^{k} \frac{n_{high}}{N_{high}} + \frac{n_{low}}{N_{low}} \tag{13}$$

Where k is the total number of two block loading repetitions up to failure $D = 1$.

3.2. Non linear damage rule

In order to improve the accuracy of the LDR, different Non-linear Damage Rules (NDRs) have been proposed to describe the damage accumulation process. A stress independent NDR model for GFRP was developed by Owen and Howe [21] for glass chopped strand-mat/polyester resin specimens subjected to fatigue loading. Their quadratic non-linear damage accumulation model has the form,

$$D = \sum_{i=1}^{k} \left[A \left(\frac{n_i}{N_i} \right) + B \left(\frac{n_i}{N_i} \right)^2 \right] \tag{14}$$

Bond [22] improved the model by replacing the quadratic term in an other variable parameter C resulting in,

$$D = \sum_{i=1}^{k} \left[A \left(\frac{n_i}{N_i} \right) + B \left(\frac{n_i}{N_i} \right)^C \right] \tag{15}$$

Bond and Farrow [23, p. 641] further proposed a metric for calculating the parameters based on initial strength properties UTS and UCS for primarily tensile loading,

$$A = B = |UCS/UTS| \tag{16}$$

$$C = (UTS(|UTS/UCS| + 1) - |UTS|)/UTS \tag{17}$$

So for the VA two block loading case for the Bond and Farrow [23] NDR model we get,

$$D = \sum_{i=1}^{k}\left[\left[A\left(\frac{n_{high}}{N_{high}}\right) + B\left(\frac{n_{high}}{N_{high}}\right)^{C}\right] + \left[A\left(\frac{n_{low}}{N_{low}}\right) + B\left(\frac{n_{low}}{N_{low}}\right)^{C}\right]\right] = 1 \tag{18}$$

3.3. Linear residual strength model

Broutman and Sahu [24] were one of the earliest to develop a Linear Residual Strength Model (LRSM) cumulative damage law founded upon residual strength changes during fatigue life, which accounts for load/stress sequence effects. They examined the tensile fatigue behavior of composite E-glass fiber reinforced epoxy subjected to two block load/stress level VA tests. Their model is based upon a linear loss of strength with cycles of fatigue, as represented by:

$$\sigma_r = \sigma_{ult} - \sum_{i=1}^{k}\left(\sigma_{ult} - \sigma_p^i\right)\left(\frac{n_i}{N_i}\right) \tag{19}$$

Where σ_r is the residual strength, σ_p^i is the maximum peak stress during the fatigue cycle, σ_{ult} is the static strength of the specimen, N_i is the number of constant amplitude cycles to failure at the stress level σ_i, n_i is the number of cycles experienced at stress level σ_i. Broutman and Sahu [24] performed experimental work on two level load block fatigue tests at two different stress level and reported that there were clear load sequence effects on the fatigue lifetime predictions, further that their LRSM prediction results were not the same as the LDR prediction results. For the VA two block loading case Eq. (19) becomes for the LRSM,

$$\left(\frac{1-\sigma_{high}/UTS}{1-\sigma_{low}/UTS}\right)\frac{n_{high}}{N_{high}} + \frac{n_{low}}{N_{low}} \tag{20}$$

Where $\sigma_{high}, \sigma_{low}$ are the cycle peak stress levels for the high and low load block cycles, as defined in Table 3 and N_{high}, N_{low} are the number of LB cycles to failure corresponding to high and low load block respectively as defined in Table 4.

3.4. Non-linear residual strength model

Reifsnider and Stinchcomb [25] proposed a Non-linear Residual Strength Model (NRSM) based on Eq. (20), but adding a non-linear parameter ν. For the two block loading case, the Reifsnider and Stinchcomb NRSM becomes,

$$\left(\frac{1-\sigma_{high}/UTS}{1-\sigma_{low}/UTS}\right)^{\frac{1}{\nu}}\frac{n_{high}}{N_{high}} + \frac{n_{low}}{N_{low}} = 1 \tag{21}$$

Where ν is the non-linear strength degradation parameter, Eq. (20) is identical to Eq. (21) if $\nu = 1$.

3.5. Weibull fit

To compare the B-model probability of failure curves, Weibull Cumulative Distribution Function (CDF) curves have been modeled fitted to both directly to the experimental data points as well as to the fatigue cumulative damage results when applying the different above described fatigue damage accumulation models.

The CDF curve for the applied two parameter Weibull distribution Weibull [26] can be defined as,

$$F(x; k, \lambda) = 1 - e^{-(x/\lambda)^k} \tag{22}$$

Where F denotes the cumulative failure probability. The only input variables required to derive the Weibull parameters k and λ are the mean E and variance Var of the VA test data (cycles to failure N_f).

Table 2
Fatigue damage evolution model parameters.

Model name	Model parameters			
	A	B	C	ν
NDR	0.74	0.74	1.36	
NRSM				0.5

The shape k parameters were be obtained using the efficiency Var/E^2 from the first two moments E and Var [27],

$$\frac{Var}{E^2} = \frac{2k}{B\left(\frac{1}{k}, \frac{1}{k}\right)} - 1 \tag{23}$$

Followed by the scale λ parameter of Eq. (22), which can then be obtained using the first moment E [28] by,

$$E(X) = \lambda\Gamma\left(1 + \frac{1}{k}\right) \tag{24}$$

Using Eqs. (23) and (24) the λ and k were numerical obtained, solving the nonlinear equation system using SciPy [29] and SymPy [30].

4. Validation using experimental data

The B-model presented in Section 2 was validated against experimental data for Fiber-Reinforced Plastic (FRP) laminates as described in Section 4.1 and compared with other fatigue damage evolution models as discussed in Section 3.

Section 4.1.1 describes and Section 4.1.2 shows the results of how the single PTM are defined for the high and the low test load levels, followed by the matrix multiplication of the PTMs that results in the final test spectrum PMFs, which has been used use to predict the data of the full two block load test spectrum, up till failure.

4.1. Experimental data composite laminate

Fatigue test data from The Sandia National Laboratories and Montana State University DOE/MSU Composite Material Fatigue Database [31] has been used, as it contains an extensive range of fatigue data, from both repeated block testing of subsequent high load level High Cycle Fatigue (HCF) and lower load level Low Cycle Fatigue (LCF) load blocks as well as the corresponding CA fatigue load test data for each load block and the static extreme test data.

Test data was taken from a detailed study of spectrum loading effects using a typical tri-axial laminate from the database, material named $DD5$. This laminate has the layup configuration $(0/\pm 45/0)_S$, with a fiber volume fraction of $V_f = 38\%$, The 0° plies are D155 stitched Uni Directional (UD) fabric, the ±45° plies are DB120 stitched fabrics from manufacturer Owens Coming, and the resin used for the laminate was an orthopolyester resin matrix (CoRezyn 63-AX-051 by Interplastics Corp.). The test methodologies for a series of tests performed with DD5 specimens using simple two load blocks of different duration's and loads, repeated to failure are described in detail in Wahl et al. [32].

For the nonlinear fatigue damage evolution models from Section 3 and Table 1, in Table 2 are listed for the DD5 material laminate the missing model parameters.

In Table 2, the NDR model parameter A and B were calculated with Eq. (16) and parameter C was calculated with Eq. (17), using the static strength properties for DD5 UTS = 724 MPa and UCS = −534 MPa. Further for the NRSM model the strength degradation parameter ν, was tuned to $\nu = 0.5$ as this gave the best fit overall for all the block load tests to the test data Weibull CDF.

Fig. 4 shows for each two load level cycle block combination test the amount and dispersion of the test data points.

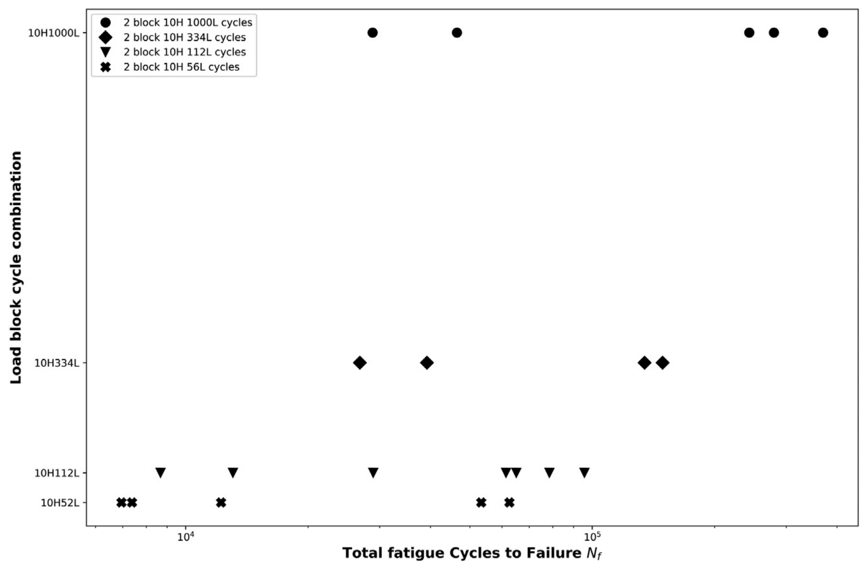

Fig. 4. 2 block test result data ranked per low load block cycles.

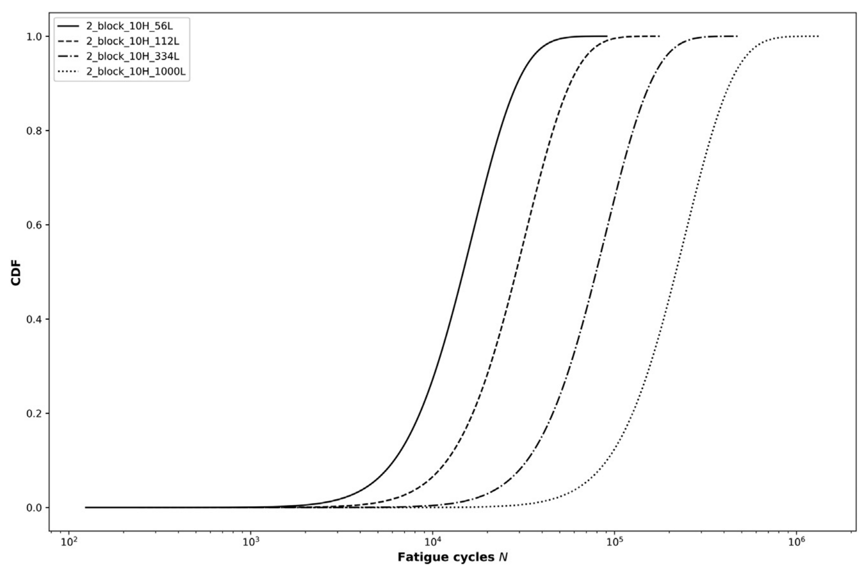

Fig. 5. B-model CDF's for number of cycles.

Table 3
Fatigue CA test data.

Load level	R-ratio	σ_{max} [MPa]	σ_{min} [MPa]	E_{N_f}	Var_{N_f}
High	0.1	414	41.4	2672	2.3112E+6
Low	0.1	241	24.1	2457832	2.0974E+12

4.1.1. Example analysis using two PTM's for a high and low load level

A B-model Markov chain analysis was performed with a combination of "high" and "low" load level LBs. A first step in the analysis was to construct the single PTMs for the two different load block levels $\mathbf{P_{high}}$ and $\mathbf{P_{low}}$.

The PTM parameters were analyzed using Eq. (10) to calculate parameter b and Eq. (11) to calculate parameter r, for both using the mean $E[N_f]$ and variance $Var[N_f]$ values, as listed in Table 3, which were obtained from the CA load experimental data [31].

The b value which defines the matrix size was kept at a constant value, in order to multiply the two different PTMs, $\mathbf{P_{high}}$ and $\mathbf{P_{low}}$. The number of cycles in a LB was given by the cycles of a test load block for each PTM, as listed in Table 4.

The next step in the analysis was to multiply the $\mathbf{P_{high}}$ and $\mathbf{P_{low}}$, which resulted in a single PTM for one high and low block LC, so for example test load "2 block 10H 56L" the PTM is for the 10 "high" and 56 "low" cycles, so the PMF of Eq. (7) becomes,

$$\mathbf{p_{10H56L}} = \mathbf{p_0}[\mathbf{P_{10H}} \cdot \mathbf{P_{56L}}]^k \tag{25}$$

Finally the evolutionary probability of failure was calculated for the combined "high" and "low" PTM was repeatedly multiplied according to Eq. (25) until a failure probability of $P_f = 1$ was reached, resulting for each of the two block load sequence combinations in the B-model CDF curves as shown in Fig. 5.

4.1.2. Results

A comparison have been made of the B-model CDF curves against the Weibull CDF fit of the test result data and the fatigue damage evolution models listed in Table 1 for each of the 2 block load level sequence tests.

Table 4
Single PTM data.

PTM name	LB cycles	LB statistics		PTM param	
		E	Var	b	r
10H	10	267	23112	4	87
56L	56	43890	6.6881E+10	4	15234
112L	112	21945	1.6720E+8	4	7619
334L	334	7359	1.8801E+7	4	2555
1000L	1000	2458	2.0974E+6	4	853

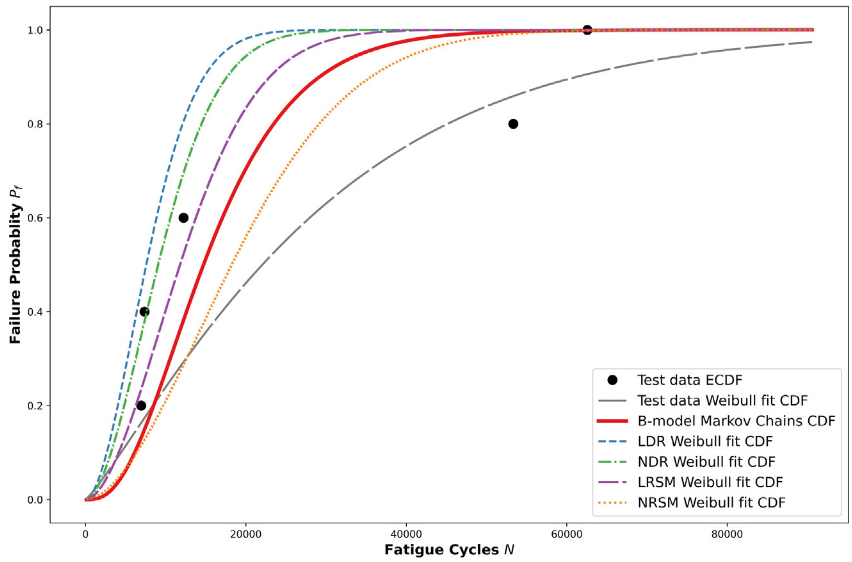

Fig. 6. Fatigue life for repeated 2 block sequence of 10 high and 56 low cycles.

It is on the forehand important to clarify again that in this study as input for the B-model analysis only data has been used from separate high and low load level CA fatigue test data and not the combined two block load spectrum tests data. On the other hand for the other fatigue damage accumulation and strength reduction models used to compare the B-model the VA two block load level spectrum tests data was directly used.

2 block 10H 56L Test: For this case, as shown in Fig. 6, five fatigue tests results were available. For a low failure probabilities the B-model predicts a somewhat larger fatigue life than the test data Weibull CDF curve, which is also the case for the NRSM model. The LDR, NDR and LRSM models show in general shorter fatigue life at equal failure probability levels compared to both the test data Weibull and B-model curve.

2 block 10H 112L Test: As shown in Fig. 7, only a close fit at the toe of the CDF curve from test data Weibull distribution compared to the curve from the B-model was achieved. This was mainly caused by the large variance (largest from all block test used in this paper) of the two block test data results, as shown in Table 4, which makes it more complicated and less accurate make a good test data Weibull curve fit.

2 block 10H 334L Test shown in Fig. 8, with only four fatigue test results available, the matching between the test data Weibull curve and B-model curve are very close for the entire failure probability range. The predicted fatigue life resulting from the B-model is slightly higher than for the test data Weibull fit, up to a failure probability of $\approx$ 0.5. This test data result set has a relative small variance or spread compared to the other tests, as shown in Fig. 4 and Table 4, which results in the close fit of the B-model and test data Weibull CDF and hereby showing a better performance of the B-model for smaller variances.

2 block 10H 1000L Test Fig. 9 shows that the predicted fatigue life from B-model is higher than the test data Weibull CDF. In this case the load sequence is significantly different than the previous ones because only 10 high load level cycles are applied, and after that 1000 low load cycles. More than a combination of block the high cycle blocks causes peak overloads acting on the much larger low load cycle blocks. In this case a different damage mode may appear in the material, which are not cover for by the block load sequence effects of the B-model.

In general more fatigue tests results are desirable for each load block combination test, in order to obtain a good comparison of the data and to make a better Weibull fit of the test data points. But overall the predicted fatigue life probability distribution from the B-model fits very well to the actual two block loading test behavior from the experimental data.

To validate and compare the performance of each fatigue model with respect to the experimental data points, the Root Mean Square Error was calculated, which gives a good comparison of the fit of the different fatigue evolution models, with respect to their prediction errors.

$$\mathrm{RMSE} = \sqrt{\frac{\sum_{t=1}^{n}(\hat{y}_t - y_m)^2}{n}} \tag{26}$$

Where $\hat{y}_t$ are the test data points, y_m the corresponding model points for $\hat{x}_t$, n the amount of data points.

Fig. 10 shows that the B-model has overall for all the block load sequence tests the lowest RMSE and therefore shows the better fit to the test data points out of all the fatigue evolution models.

Further Fig. 10 shows that out of all the fatigue evolution models the B-model has on average for all the block load tests the lowest Root Mean Square Error which means the best model fit to the actual block test data points, which therefore results in the most accurate fatigue damage evolution model. The Non-linear Residual Strength Model has after the B-model the best model fit to the test data points with also an overall relative low RMSE followed by the Linear Residual Strength Model which has low RMSE at two tests but much higher fitting errors for the other two tests. Both the Linear Damage Rule as well as the

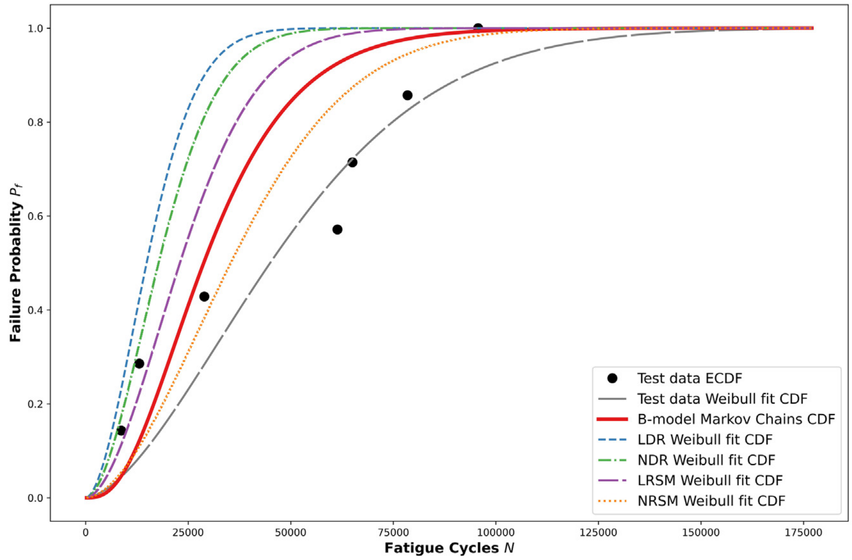

Fig. 7. Fatigue life for repeated 2 block sequence of 10 high and 112 low cycles.

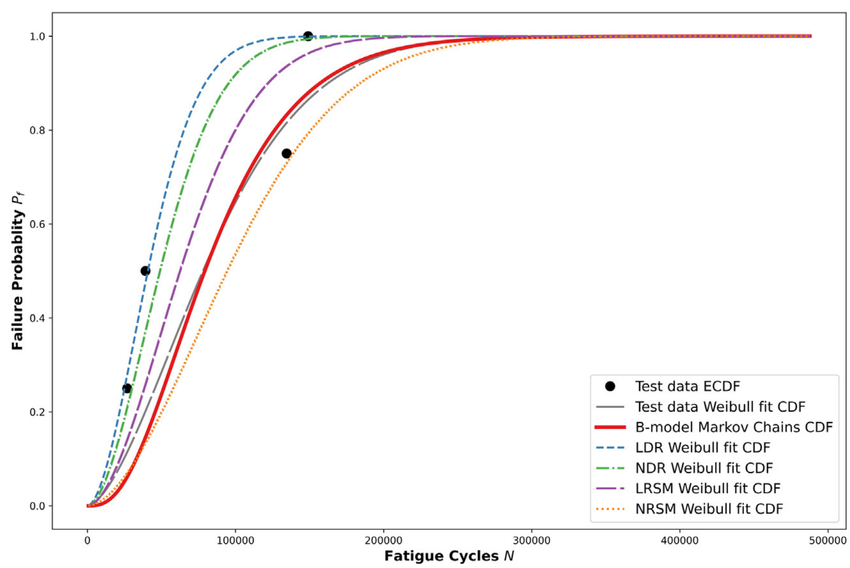

Fig. 8. Fatigue life for repeated 2 block sequence of 10 high and 334 low cycles.

Non-linear Damage Rule have overall higher Root Mean Square Error and therefore show to be the worst model fit to the actual block test data points and therefore less accurate and realistic fatigue damage evolution models.

5. Conclusions

The systematic description for the implementation of the B-model Markov chains as fatigue damage accumulation model for probabilistic fatigue analysis on composite structures in this case for a tri-axial glass fiber laminate with polyester resin matrix subjected to axial fatigue loading, has been pursued. Different PTMs representing the different block load levels were obtained directly from experimental data, these PTMs then were multiplied using matrix multiplication to obtain for each block load spectrum of the variable amplitude two block load test a final PMF from which the CDF of the block load combinations have been obtained, representing the cumulative damage of the tests.

Some of the key features of the B-model are that it takes into account both load sequence effects as well as non-linear fatigue damage growth and therefore does not have some of the important restriction or errors of other especially linear damage evolution models.

Some of the benefits of the B-model are that the required PTM Markov chains can be obtained using different methods:

- Directly created from the constant amplitude load block experimental data, which was the method applied in this paper and demonstrated to be a relative simple and very accurate method to translate the behavior from the material fatigue test directly to the fatigue damage accumulation model.
- Using probabilistic S–N curves or combined for different load ratios in probabilistic constant life diagram.
- Using other damage accumulation metrics such as strength or stiffness reduction models.

Further the B-model analysis outcome gives the probability of failure against fatigue life cycles as CDF curves, instead of commonly a single damage value or fatigue reserve factor typically resulting from the deterministic fatigue damage accumulation models. Therefore the damage evolution probability at any fatigue life loading stage and the

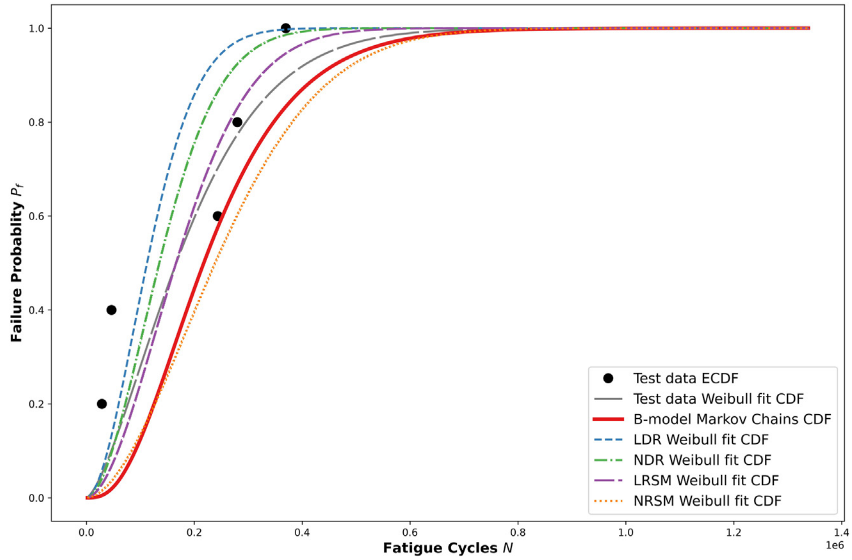

Fig. 9. Fatigue life for repeated 2 block sequence of 10 high and 1000 low cycles.

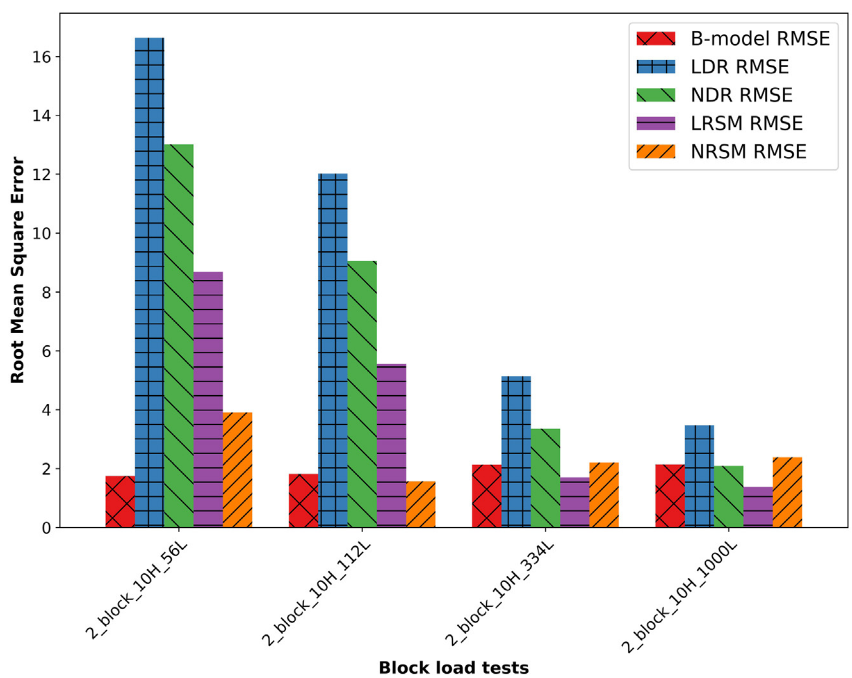

Fig. 10. RMSE fatigue evolution models against test data points.

probability of failure at any damage state can be analyzed by the method presented in this paper.

It has been demonstrated in this paper that using the B-model it is possible to predict accurately the fatigue life of composite laminates subjected to variable amplitude load spectrum.

The validation results and correlation to other models, show the very good fit of the B-model to the variable amplitude two block load test data with low root mean square errors, proving the very good performance of the B-model compared with other commonly used cumulative fatigue damage evolution models.

Some drawbacks of the B-model method are its sensitivity to rather large variance values. Further the B-model results need still to be validated against more complex loads spectra that are representative for actual more complex stochastic structural load spectrum's, where multiple PTMs can be used and combined to represent the different loading situations that can occur during a composite structural design life.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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References

[1] A. Vassilopoulos, T. Keller, Fatigue of Fiber-reinforced Composites, First ed., Springer-Verlag London Limited, 2011, pp. 1–238, http://dx.doi.org/10.1007/978-1-84996-181-3.

[2] B. Harris, Engineering Composite Materials, Second Edition, IOM Communications Ltd., 1999.

[3] R. Talreja, J. Varna, Fatigue damage mechanisms, in: Modeling Damage, Fatigue and Failure of Composite Materials, First Edition, Woodhead Publishing, 2015, pp. 25–40, http://dx.doi.org/10.1016/C2013-0-16521-X, URL https://www.sciencedirect.com/book/9781782422860/modeling-damage-fatigue-and-failure-of-composite-materials.

[4] M. Salkind, Fatigue of composites, in: Composite Materials: Testing and Design (Second Conference), STP497, ASTM American Society for Testing and Materials, 1972, pp. 143–169, http://dx.doi.org/10.1520/STP27746S.

[5] R. Link, L. Yang, A. Fatemi, Cumulative fatigue damage mechanisms and quantifying parameters: A literature review, J. Test. Eval.- J TEST EVAL 26 (1998) http://dx.doi.org/10.1520/JTE11978J.

[6] M.A. Miner, Cumulative damage in fatigue, Int. J. Fatigue 12 (1945) A159–A164, 10.1115/1.4009458.

[7] A. Vassilopoulos, T. Keller, 4 - modeling the fatigue behavior of fiber-reinforced composite materials under constant amplitude loading, in: Fatigue of Fiber-Reinforced Composites, First Edition, Springer-Verlag London Limited, 2011, pp. 87–132, http://dx.doi.org/10.1007/978-1-84996-181-3_4.

[8] J. Pinto, J. Pujol, C. Cimini, Probabilistic cumulative damage model to estimate fatigue life, Fatigue Fract. Eng. Mater. Struct. 37 (2014) http://dx.doi.org/10.1111/ffe.12087.

[9] V. Rathod, O.P. Yadav, A. Rathore, R. Jain, Probabilistic modeling of fatigue damage accumulation for reliability prediction, Int. J. Qual. Stat. Reliab. (2011) http://dx.doi.org/10.1155/2011/718901.

[10] J. Bogdanoff, J. Kozin, Probabilistic Models of Cumulative Damage, John Wiley and Sons, 1985.

[11] J. Bea, M. Doblaré, L. Gracia, Evaluation of the probability distribution of crack propagation life in metal fatigue by means of probabilistic finite element method and B-models, Eng. Fract. Mech. 63 (6) (1999) 675–711, http://dx.doi.org/10.1016/S0013-7944(99)00053-3, URL https://www.sciencedirect.com/science/article/pii/S0013794499000533.

[12] J. Bea, M. Doblaré, Enhanced B-PFEM model for fatigue life prediction of metals during crack propagation, Comput. Mater. Sci. 25 (2002) 14–33.

[13] L. Gansted, R. Brincker, L.P. Hansen, The fracture mechanical markov chain fatigue model compared with empirical data, Fract. Dyn. (1994) R9431, URL: https://vbn.aau.dk/ws/portalfiles/portal/204441130/The_Fracture_Mechanical_Markov_Chain_Fatigue_Model_Compared_with_Empirical_Data.pdf.

[14] P.D. Spanos, J.D. Rowatt, A Probalistic Model for the Accumulation of Fatigue Damage in Composite Laminates, 1994, pp. 494–503, http://dx.doi.org/10.1007/978-3-642-85092-9_32.

[15] J. Rowatt, P. Spanos, Markov chain models for life prediction of composite laminates, Struct. Saf. 20 (2) (1998) 117–135, http://dx.doi.org/10.1016/S0167-4730(97)00025-8, URL https://www.sciencedirect.com/science/article/pii/S0167473097000258.

[16] A. Papoulis, S.U. Pillai, Probability, Random Variables, and Stochastic Processes, Fourth, McGraw Hill, Boston, 2002, URL http://www.worldcat.org/search?qt=worldcat_org_all&q=0071226613.

[17] H. Gao, H.-Z. Huang, S.-P. Zhu, Y.-F. Li, R. Yuan, A modified nonlinear damage accumulation model for fatigue life prediction considering load interaction effects, Sci. World J. 2014 (2014) 164378, http://dx.doi.org/10.1155/2014/164378.

[18] L. Han, D. Huang, X. Yan, C. Chen, X. Zhang, M. Qi, Combined high and low cycle fatigue life prediction model based on damage mechanics and its application in determining the aluminized location of turbine blade, Int. J. Fatigue 127 (2019) http://dx.doi.org/10.1016/j.ijfatigue.2019.05.022.

[19] X. Zheng, C. Engler-Pinto, X. Su, H. Cui, W. Wen, Modeling of fatigue damage under superimposed high-cycle and low-cycle fatigue loading for a cast aluminum alloy, Mater. Sci. Eng. 560 (2013) 792–801, http://dx.doi.org/10.1016/j.msea.2012.10.037.

[20] R. Nijssen, Fatigue Life Prediction and Strength Degradation of Wind Turbine Rotor Blade Composites, KC-WMC, 2006.

[21] M.J. Owen, R.J. Howe, The accumulation of damage in a glass-reinforced plastic under tensile and fatigue loading, J. Phys. D: Appl. Phys. 5 (9) (1972) 1637–1649, http://dx.doi.org/10.1088/0022-3727/5/9/319.

[22] I. Bond, Fatigue life prediction for GRP subjected to variable amplitude loading, Composites A 30 (8) (1999) 961–970, http://dx.doi.org/10.1016/S1359-835X(99)00011-1, URL https://www.sciencedirect.com/science/article/pii/S1359835X99000111.

[23] I. Bond, I. Farrow, Fatigue life prediction under complex loading for XAS/914 cfrp incorporating a mechanical fastener, Int. J. Fatigue 22 (8) (2000) 633–644, http://dx.doi.org/10.1016/S0142-1123(00)00050-5, URL https://www.sciencedirect.com/science/article/pii/S0142112300000505.

[24] L.J. Broutman, S. Sahu, A new theory to predict cumulative fatigue damage in fiberglass reinforced plastics, in: Composite Materials: Testing and Design (Second Conference), STP497, ASTM American Society for Testing and Materials, 1972, pp. 170–188, http://dx.doi.org/10.1520/STP27746S.

[25] K.L. Reifsnider, W. Stinchcomb, A critical-element model of the residual strength and life of fatigue-loaded composite coupons, in: Composite Materials: Fatigue and Fracture, STP907, ASTM International, 1986, pp. 298–313.

[26] W. Weibull, A statistical distribution function of wide applicability, J. Appl. Mech. 18 (1951) 293–297.

[27] R. P. McEwen, B. R. Parresol, Moment expressions and summary statistics for the complete and truncated Weibull distribution, Comm. Statist. Theory Methods 20 (4) (1991) 1361–1372, http://dx.doi.org/10.1080/03610929108830570.

[28] I. Miller, R. Johnson, J. Freund, R. Johnson, Miller & Freund's Probability and Statistics for Engineers, Prentice Hall International, 2000.

[29] P. Virtanen, R. Gommers, T.E. Oliphant, M. Haberland, T. Reddy, D. Cournapeau, E. Burovski, P. Peterson, W. Weckesser, J. Bright, S.J. van der Walt, M. Brett, J. Wilson, K.J. Millman, N. Mayorov, A.R.J. Nelson, E. Jones, R. Kern, E. Larson, C.J. Carey, İ. Polat, Y. Feng, E.W. Moore, J. VanderPlas, D. Laxalde, J. Perktold, R. Cimrman, I. Henriksen, E.A. Quintero, C.R. Harris, A.M. Archibald, A.H. Ribeiro, F. Pedregosa, P. van Mulbregt, SciPy 1.0 Contributors, Scipy 1.0: fundamental algorithms for scientific computing in python, Nature Methods 17 (2020) 261–272, http://dx.doi.org/10.1038/s41592-019-0686-2.

[30] A. Meurer, C.P. Smith, M. Paprocki, O. Čertík, S.B. Kirpichev, M. Rocklin, A. Kumar, S. Ivanov, J.K. Moore, S. Singh, T. Rathnayake, S. Vig, B.E. Granger, R.P. Muller, F. Bonazzi, H. Gupta, S. Vats, F. Johansson, F. Pedregosa, M.J. Curry, A.R. Terrel, v. Roučka, A. Saboo, I. Fernando, S. Kulal, R. Cimrman, A. Scopatz, Sympy: symbolic computing in python, PeerJ Comput. Sci. 3 (2017) e103, http://dx.doi.org/10.7717/peerj-cs.103.

[31] J. Mandell, D. Samborsky, DOE/MSU Composite Material Fatigue Database, Sandia National Laboratories, 2019, URL https://energy.sandia.gov/programs/renewable-energy/wind-power/blade-reliability/mhk-materials-database/.

[32] N. Wahl, M. J.F., D. Samborsky, Spectrum Fatigue Lifetime and Residual Strength for Fiberglass Laminates, Technical Report SAND2002-0546, 29, Sandia National Laboratories, Albuquerque, NM, 2002, pp. 342–351.