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Three weight Koopman semigroups on Lebesgue spaces

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Abstract

In this paper, we consider three different semiflows $(\phi_t)_{t\geq 0}$, $(\psi_t)_{t\geq 0}$ and $(\varphi_t)_{t\geq 0}$ on the real half-line given by

$$\phi_t(r) := e^{-t}r + 1 - e^{-t},\ \psi_t(r) := \frac{e^t r}{1 + r(e^t - 1)},\ \varphi_t(r) := \frac{(1+e^t)r - 1 + e^t}{(-1+e^t)r + 1 + e^t},$$

for $r, t \geq 0$. These semiflows induce three weight Koopman semigroups, $(T^\gamma_{t,p})_{t>0}$, $(S^\gamma_{t,p})_{t>0}$ and $(R^\gamma_{t,p})_{t>0}$ on the fractional Lebesgue spaces $\mathcal{T}_p^{(\alpha)}(t^\alpha)$, closed subspaces of $L^p(\mathbb{R}^+)$ for some α and $\gamma \geq 0$. We describe spectrum sets, point spectrums and resolvent operators of their infinitesimal generators. Three Cesàro-like operators, defined using the Chen fractional integral,

$$\begin{aligned}
\mathcal{C}_{\mu,\nu} f(r) &:= \frac{1}{|r-1|^{\mu+\nu-1}} \int_{\Gamma_{1,r}} |s-1|^{\mu-1}|r-s|^{\nu-1} f(s)ds, \quad r > 0,\\
\mathfrak{C}^\gamma_{\mu,\nu} f(r) &:= \frac{r^\mu}{|r-1|^{\mu+\nu+\gamma-1}} \int_{\Gamma_{1,r}} \frac{|s-1|^{\mu+\gamma-1}}{s^{\mu+\nu}} |r-s|^{\nu-1} f(s)ds,\ r > 0,\\
\mathbf{C}^\gamma_{\mu,\nu} f(r) &:= 2^\nu \frac{|r+1|^{\mu-\gamma}}{|r-1|^{\mu+\nu-1}} \int_{\Gamma_{1,r}} \frac{|s-1|^{\mu-1}}{|s+1|^{\mu+\nu-\gamma}}\, |r-s|^{\nu-1} f(s)ds,\ r > 0,
\end{aligned}$$

(for certain μ, ν, $\gamma \in \mathbb{R}$ and $\Gamma_{1,r} := (1, r)$ when $r > 1$ and $\Gamma_{1,r} := (r, 1)$ in the case $0 < r < 1$) are subordinated to these C_0-semigroups. These representations allow to obtain their norms and spectrum sets.

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Introduction

The study of composition operators connects fundamental questions about certain operators with elegant classical results from qualitative theory. In dynamic systems, composition operators are often referred to as Koopman operators, in honor of the Franco-American mathematician Bernard O. Koopman. Koopman viewed these operators as a means to integrate classical Hamiltonian mechanics with the theory of Hilbert spaces and their linear transformations [13]. Within this framework, the canonical resolution of the identitysometimes called the "spectrum of the dynamical system" is introduced and described in terms of a one-parameter group of unitary (and composition) operators in Hilbert space.

Let K compact and X locally compact spaces. A mapping $\phi : [0, \infty) \times K \to K$ is called semiflow if for each $t \geq 0$ the mapping ϕ_t given by $\phi_t(x) := \phi(t, x)$ is continuous, $\phi_0(x) = x$ and $\phi_s \circ \phi_t = \phi_{s+t}$ for $s, t \geq 0$. In [16], a general study about positive semigroups in lattice spaces $C(K)$ and $C_0(X)$ is developed.

In this regard, a wealth of literature is available. A comprehensive treatment of Koopman operators in $L^p(\mathbf{X})$, with applications to ergodicity in measure-preserving systems, can be found in [6]. The classic book [16] develops a thorough study of positive semigroups in lattice spaces, and it characterizes continuous flows through semigroups of composition operators and their infinitesimal generators. In a finite measure space, reference [5] presents a characterization of Koopman semigroups on $L^p(\mathbf{X})$ for measure preserving semiflows. Furthermore, it establishes an equivalence between measure-preserving flows on standard probability spaces and continuous flows on compact Borel probability spaces.

On the other hand, the theory of continuous one-parameter semigroups of holomorphic self-maps on the unit disk has garnered significant attention from researchers in recent decades. The excellent monograph [3] offers a thorough and comprehensive overview of the current state of this intriguing field.

Semigroups of holomorphic self-maps on the unit disk, as well as on the complex half-plane or other domains, are closely related to the theory of composition operators. Specifically, each one-parameter semigroup of holomorphic self-maps induces a semigroup of composition operators on certain holomorphic function spaces. This connection enables the transfer of functional analytic properties, such as compactness, cyclicity or spectral properties, to corresponding dynamical questions about semigroups. For further details, see [10, 19, 20].

In this context, an important tool is the continuous Denjoy-Wolff theorem, which identifies fixed points for iterations of holomorphic self-maps of the unit disk (see [3]). Several well-known examples of semigroups of holomorphic semiflows are discussed in the excellent survey [20].

However, a corresponding result to the Denjoy-Wolff theorem for arbitrary semiflows in Lebesgue spaces remains unknown. In this paper, we focus on three specific semiflows $\phi_t, \psi_t, \varphi_t : [0, \infty) \to [0, \infty)$ on the real half-line:

$$\phi_t(r) := e^{-t}r + 1 - e^{-t},\ \psi_t(r) := \frac{e^t r}{1 + r(e^t - 1)},\ \varphi_t(r) := \frac{(1 + e^t)r - 1 + e^t}{(-1 + e^t)r + 1 + e^t},$$

These semiflows originate from semigroups on the unit disk $\mathbb{D}$ and serve as canonical examples of elliptic $(\psi_t)_{t>0}$, parabolic $(\phi_t)_{t>0}$, and hyperbolic $(\varphi_t)_{t>0}$ semigroups on $\mathbb{D}$ (see [3, 20]).

Note that the three semiflows verify

$$\lim_{t\to+\infty} \phi_t(r) = \lim_{t\to+\infty} \psi_t(r) = \lim_{t\to+\infty} \varphi_t(r) = 1, \qquad r \in [0, \infty),$$

that is, the value 1 is an attractive fixed point of these mappings. Moreover, each of the three semiflows has additional fixed points on $\mathbb{R} \cup \{\infty\}$: $(\phi_t)_{t\geq 0}$ and $(\psi_t)_{t>0}$ have an additional fixed point on $\partial([0, \infty])$, with $\lim_{r\to+\infty} \phi_t(r) = +\infty$ and $\psi_t(0) = 0$ for all $t \geq 0$, while $\varphi_t(-1) = -1$.

This article is organized as follows: In Sect. 1, we present some properties of the semiflows $(\phi_t)_{t\geq 0}$, $(\psi_t)_{t\geq 0}$ and $(\varphi_t)_{t\geq 0}$, and we apply Faà di Bruno's formula to compute the n-th derivative of a composition involving these mappings (Lemma 4). Since these semiflows are infinitely differentiable (as in the complex case), it is natural to study the induced Koopman semigroups within certain Sobolev-Lebesgue spaces.

In Sect. 2, we revisit the Lebesgue subspace $\mathcal{T}_p^{(\alpha)}(t^\alpha) \hookrightarrow L^p(\mathbb{R}^+)$ for $p \geq 1$ and $\alpha \geq 0$. These function spaces are of Sobolev type, defined using fractional Weyl derivatives. In case $p = 1$ and $\alpha \in \mathbb{N}$ they were introduced in [2]; while the cases $p = 1$ and $\alpha > 0$ were explored in [9] and $p > 1$ and $\alpha > 0$ in [14, 17]. Although these Sobolev spaces share a similar structure with $L^p(\mathbb{R}^+)$ the associated techniques are considerably more complex.

It is noteworthy that these function spaces align perfectly with the nature of the three semiflows. In Sect. 3, we introduce the weighted Koopman semigroups $(T_{t,p})_{t>0}$, $(S_{t,p})_{t>0}$ and $(R_{t,p})_{t>0}$ defined by

$$\begin{aligned} T_{t,p}f(r) &:= e^{\frac{-t}{p}} f(\phi_t(r)),\\ S^\gamma_{t,p}f(r) &:= e^{\frac{t}{p}}\omega^\gamma_{\psi_t}(r) f(\psi_t(r)),\\ R^\gamma_{t,p}f(r) &:= e^{t(\gamma-\frac{1}{p})}\omega^\gamma_{\varphi_t}(r) f(\varphi_t(r)), \end{aligned}$$

with $r > 0$, $p \geq 1$, $\gamma \geq 0$ and $\phi_t, \psi_t, \varphi_t$ the semiflows given in Definition 1. We will demonstrate that these are uniformly bounded C_0-semigroups on $\mathcal{T}_p^{(\alpha)}(t^\alpha)$ for $\alpha > 0$ in the first case, and on $\mathcal{T}_p^{(n)}(t^n)$ for $n \in \mathbb{N}$ in the second and third cases. We identify their infinitesimal generators, spectra (including point spectra), and resolvent operators. Additionally, the orbits of these C_0-semigroups correspond to the solutions of the Cauchy problems

$$\begin{aligned}
\frac{\partial u(t,r)}{\partial t} &= (1-r)\frac{\partial u(t,r)}{\partial r} - \frac{1}{p}u(t,r),\\
\frac{\partial v(t,r)}{\partial t} &= r(1-r)\frac{\partial v(t,r)}{\partial r} + \left(\frac{1}{p} - \gamma r\right)v(t,r),\\
2\frac{\partial w(t,r)}{\partial t} &= (1-r^2)\frac{\partial w(t,r)}{\partial r} + \left(\gamma - \frac{2}{p} - \gamma r\right)w(t,r),
\end{aligned}$$

for $r, t \geq 0$,

Following some ideas introduced in [14], we define the Cesàro-like operators subordinated to $(T_{t,p})_{t>0}$, $(S^{\gamma}_{t,p})_{t>0}$ and $(R^{\gamma}_{t,p})_{t>0}$. Given $\mu, \nu \in \mathbb{R}$, we consider the integral operators

$$\begin{aligned}
\mathcal{C}_{\mu,\nu}f(r) &:= \frac{1}{|r-1|^{\mu+\nu-1}}\int_{\Gamma_{1,r}} |s-1|^{\mu-1}|r-s|^{\nu-1}f(s)ds, \qquad r>0,\\
\mathfrak{C}^{\gamma}_{\mu,\nu}f(r) &:= \frac{r^{\mu}}{|r-1|^{\mu+\nu+\gamma-1}}\int_{\Gamma_{1,r}} \frac{|s-1|^{\mu+\gamma-1}}{s^{\mu+\nu}}|r-s|^{\nu-1}f(s)ds, \qquad r>0,\\
\mathbf{C}^{\gamma}_{\mu,\nu}f(r) &:= 2^{\nu}\frac{|r+1|^{\mu-\gamma}}{|r-1|^{\mu+\nu-1}}\int_{\Gamma_{1,r}} \frac{|s-1|^{\mu-1}}{|s+1|^{\mu+\nu-\gamma}}|r-s|^{\nu-1}f(s)ds, \quad r>0,
\end{aligned}$$

whenever this integral converges, and $\Gamma_{1,r} := (1,r)$ for $r > 1$ and $\Gamma_{1,r} := (r,1)$ in the case $0 < r < 1$. These three operators are examples of Chen's fractional integral, defined for $\alpha > 0$ and $c \geq 0$ as follows:

$$I^{\alpha}_{c}f(r) := \frac{1}{\Gamma(\alpha)}\int_{\Gamma_{c,r}} |r-s|^{\alpha-1}f(s)ds, \quad r>0,$$

when the path $\Gamma_{c,r}$ is defined by $\Gamma_{c,r} := (c,r)$ for $r > c$ and $\Gamma_{c,r} := (r,c)$ in the case $0 < r < c$, see [18, Section 18.5] and references therein. This fractional integral was introduced in [4] to address certain differential equations with mixed initial conditions. We write $k_{c,\alpha}(t) = \frac{1}{\Gamma(\alpha)}|t-c|^{\alpha-1}$ for $t > 0$, then

$$\begin{aligned}
\mathcal{C}_{\mu,\nu}f &= \frac{\beta(\mu,\nu)}{k_{1,\mu+\nu}}I^{\nu}_{1}(k_{1,\mu}f),\\
\mathfrak{C}^{\gamma}_{\mu,\nu}f &= \frac{\beta(\mu+\gamma,\nu)\beta(\mu+1,\nu)}{\Gamma(\nu)}\frac{k_{0,\mu+1}}{k_{1,\mu+\nu+\gamma}}I^{\nu}_{1}\left(\frac{k_{1,\mu+\gamma}}{k_{0,\mu+\nu+1}}f\right),\\
\mathbf{C}^{\gamma}_{\mu,\nu}f &= 2^{\nu}\frac{\beta(\mu-\gamma+1,\nu)\beta(\mu,\nu)}{\Gamma(\nu)}\frac{k_{-1,\mu-\gamma+1}}{k_{1,\mu+\nu}}I^{\nu}_{1}\left(\frac{k_{1,\mu}}{k_{-1,\mu+\nu-\gamma+1}}f\right).
\end{aligned}$$

The connection between these semigroups and the operators based on Chen's fractional integral is novel and is addressed in this paper (see Theorems 16, 17, 18). Additionally, for $p \geq 1$, we estimate the norms $||\mathcal{C}_{\mu+\frac{1}{p}\gamma,\nu}||$, $||\mathfrak{C}_{\mu-\frac{1}{p},\nu}||$ and

$||\mathbf{C}_{\mu+\frac{1}{p}\gamma,\nu}||$ in Sobolev (and Lebesgue) spaces and demonstrate that

$$\sigma(\mathcal{C}_{\mu+\frac{1}{p},\nu}) = \sigma(\mathfrak{C}^{\gamma}_{\mu-\frac{1}{p},\nu}) = \sigma(\mathbf{C}^{\gamma}_{\mu+\frac{1}{p},\nu}) = \overline{\left\{\beta(\mu+z,\nu) : \Re z \geq 0\right\}},$$

for $\mu, \nu > 0$ and $\gamma \geq \frac{2}{p}$.

Notation. The Gamma and Beta Euler functions are denoted by Γ and β respectively and

$$\Gamma(z) = \int_0^{\infty} t^{z-1}e^{-t}dt, \quad Re(z) > 0; \beta(\mu,\nu) = \int_0^{\infty} e^{-\mu t}(1-e^{-t})^{\nu-1}dt, \quad \mu, \nu > 0.$$

We write C_n for a constant which depends on n and may be different in the next calculation.

1 Semiflows in the real half-line

Definition 1 For $t > 0$, we define the inner maps in the real half-line

$$\begin{aligned} \phi_t(r) &:= e^{-t}r + 1 - e^{-t}, \\ \psi_t(r) &:= \frac{e^t r}{1 + r(e^t - 1)}, \\ \varphi_t(r) &:= \frac{(1+e^t)r - 1 + e^t}{(-1+e^t)r + 1 + e^t}, \end{aligned}$$

for $r > 0$.

Note that $\phi_0(r) = \psi_0(r) = \varphi_0(r) = r$ for $r > 0$. The following proposition collects some basic and interesting properties of these inner maps on $\mathbb{R}^+$ which are, in fact, semiflows.

Proposition 1 *Given the inner maps* $\phi_t, \psi_t, \varphi_t : \mathbb{R}^+ \to \mathbb{R}^+$ *for* $t > 0$. *Then*

(i) $\phi_t(\phi_s) = \phi_{t+s}$, $\psi_t(\psi_s) = \psi_{t+s}$ *and* $\varphi_t(\varphi_s) = \varphi_{t+s}$ *for* $t, s > 0$.

(ii)

$$\phi_t\left(\frac{1}{r}\right) = \frac{1}{\psi_t(r)} \quad \textit{and} \quad \varphi_t\left(\frac{1}{r}\right) = \frac{1}{\varphi_t(r)}, \qquad r > 0.$$

(iii) ϕ_t *and* φ_t *have an unique fixed point,* $r = 1$ *and* ψ_t *has a two fixed points,* $\{0, 1\}$.

(iv) $\phi_t([0,1]) = [1 - e^{-t}, 1]$ *and* $\phi_t([1,\infty]) = [1,\infty)$.

(v) We denote by $\psi_t(\infty) := \lim_{r\to+\infty} \psi_t(r) = \dfrac{e^t}{e^t - 1}$. *Then* $\psi_t([0,1]) = [0,1]$ *and* $\psi_t([1,\infty]) = [1, \psi_t(\infty))$.

(vi) *We define* $\varphi_t(\infty) := \lim_{r\to+\infty} \varphi_t(r) = \dfrac{e^t+1}{e^t-1}$. *Then* $\varphi_t([0,1]) = [\frac{1}{\varphi_t(\infty)}, 1]$ *and* $\varphi_t([1,\infty)) = [1, \varphi_t(\infty))$.

Remark 1 For $b > 0$, we may also consider semiflows

$$\begin{aligned} \phi_{t,b}(r) &:= e^{-t}r + b(1-e^{-t}), \\ \psi_{t,b}(r) &:= \frac{e^t r}{1 + br(e^t-1)}, \\ \varphi_{t,b}(r) &:= b\frac{r(e^t+1)+b(e^t-1)}{r(e^t-1)+b(e^t-1)}, \end{aligned}$$

for $r > 0$. Note that

$$\phi_{t,b} = \tau_b \circ \phi_{t,0} \circ \tau_{-b} \qquad \psi_{t,b}(r) = b\psi_{t,1}(\frac{r}{b}), \qquad \varphi_{t,b}(r) = b\varphi_{t,1}(\frac{r}{b})$$

where $\tau_b(r) := r - b$, $\phi_{t,0}(r) = e^{-tr}$, $\psi_{t,1} = \psi_t$ and $\varphi_{t,1} = \varphi_t$ for $r, t > 0$. Note that

$$\lim_{t\to+\infty} \phi_{t,b}(r) = \lim_{t\to+\infty} \varphi_{t,b}(r) = b, \quad \lim_{t\to+\infty} \psi_t(r) = \frac{1}{b}, \qquad r \in [0,\infty).$$

For a Lebesgue measurable function $f : \mathbb{R}^+ \to \mathbb{C}$, we write by $S = \text{supp}(f)$ the smallest closed subset S of $\mathbb{R}^+$ such that $f = 0$ almost everywhere outside S. Equivalently $\text{supp}(f)$ is the complement of the largest open set on which $f = 0$ almost everywhere. Note that if $\text{supp}(f) \subset [0,1]$ then

$$\text{supp}(f(\phi_t)), \text{supp}(f(\psi_t)), \text{supp}(f(\varphi_t)) \subset [0,1],$$

for $t > 0$. Similarly if $\text{supp}(f) \subset [1,\infty)$ then

$$\text{supp}(f(\phi_t)), \text{supp}(f(\psi_t)), \text{supp}(f(\varphi_t)) \subset [1,\infty), \qquad t > 0.$$

In the case we consider certain subsets of $\mathbb{R}^+$, the semiflows ϕ_t, ψ_t and φ_t may be defined for $t \in \mathbb{R}$ and define flows. The proof of the following proposition is omitted.

Proposition 2 *Let* ϕ_t, ψ_t *and* φ_t *be the functions introduced in Definition 1.*

(i) $\phi_t : [1,\infty) \to [1,\infty)$ *for* $t \in \mathbb{R}$.
(ii) $\psi_t : [0,1] \to [0,1]$ *for* $t \in \mathbb{R}$.
(iii) $\varphi_t : [-1,1] \to [-1,1]$ *for* $t \in \mathbb{R}$.

Now consider $\xi_t : [0,\infty) \to [0,\infty)$ and the weight ω_{ξ_t} defined by $\omega_{\xi_t}(r) := \dfrac{1-\xi_t(r)}{1-r}$ for $r > 0$. Note that

$$\omega_{\phi_t}(r) = e^{-t},$$

$$\omega_{\psi_t}(r) = \frac{1}{1 + r(e^t - 1)},$$
$$\omega_{\varphi_t}(r) = \frac{2}{e^t + 1 + r(e^t - 1)}.$$

Remark 2 Take $(\xi_t)_{t\geq 0}$ a semiflow on $\mathbb{R}^+$ and $(\omega_t)_{t>0}$ a family of continuous functions on $\mathbb{R}^+$. Then $(\omega_t)_{t>0}$ is a semicocycle of $(\xi_t)_{t\geq 0}$ if $\omega_0 = 1$ and $\omega_{t+s} = \omega_t(\omega_s \circ \xi_t)$ for $t, s \geq 0$, see [16, Chapter B-II, section 3]. It is straightforward to show that $(\omega^{\gamma}_{\xi_t})_{t>0}$ is a semicocycle for semiflow $(\xi_t)_{t>0}$ for any $\gamma \geq 0$. Moreover,

$$(\omega^{\gamma}_{\psi_t})^{(n)}(r) = \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)} \frac{(-(e^t-1))^n}{(1+r(e^t-1))^{n+\gamma}} = \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)} (1-e^t)^n \omega^{\gamma+n}_{\psi_t}(r), \quad (1)$$

and

$$(\omega^{\gamma}_{\varphi_t})^{(n)}(r) = \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)} \left(\frac{1-e^t}{2}\right)^n \omega^{\gamma+n}_{\varphi_t}(r), \quad (2)$$

for $n \geq 1$ and $t, r > 0$.

It is easy to check the following proposition and the proof is left to the reader.

Proposition 3 *For $t, r > 0$, the following equalities are satisfied.*

(i) $\psi_t(r) = re^t\,\omega_{\psi_t}(r)$.

(ii)

$$\frac{\partial}{\partial t}(\phi_t(r)) = -e^{-t}r + e^{-t}, \qquad \lim_{t\to 0} \frac{\partial}{\partial t}(\phi_t(r)) = 1 - r,$$
$$\frac{\partial}{\partial r}(\phi_t(r)) = e^{-t}, \qquad \frac{\partial^j}{\partial r^j}(\phi_t(r)) = 0, \qquad j \geq 2.$$

(iii)

$$\frac{\partial}{\partial t}(\psi_t(r)) = \frac{e^t r(1-r)}{(1+r(e^t-1))^2}, \qquad \lim_{t\to 0} \frac{\partial}{\partial t}(\psi_t(r)) = r(1-r),$$

$$\frac{\partial^j}{\partial r^j}(\psi_t(r)) = e^t \frac{(-1)^{j-1} j!(e^t-1)^{j-1}}{(1+r(e^t-1))^{j+1}}, \qquad j \geq 1.$$

(iv)

$$\frac{\partial}{\partial t}(\varphi_t(r)) = \frac{2e^t(1-r^2)}{(r(e^t-1)+1+e^t)^2}, \qquad \lim_{t\to 0} \frac{\partial}{\partial t}(\varphi_t(r)) = \frac{1-r^2}{2}$$

$$\frac{\partial^j}{\partial r^j}(\varphi_t(r)) = 4e^t \frac{(-1)^{j-1} j!(e^t-1)^{j-1}}{(e^t+1+(e^t-1)r)^{j+1}}, \qquad j \geq 1.$$

In the following lemma we check the n-th derivative of the composition functions $f\circ\phi_t$, $f\circ\psi_t$ and $f\circ\varphi_t$. The main tool is the well-known Faà di Bruno's formula for the n-th derivative of a composition of functions $f\circ\xi$ given by

$$\frac{d^n}{dr^n}(f(\xi(r)))=\sum\left(\frac{n!}{m_1!\cdots m_n!}f^{(m_1+\cdots+m_n)}(\xi(r))\prod_{j=1}^{n}\left(\frac{\xi^{(j)}(r)}{j!}\right)^{m_j}\right)$$

where the sum is over all n-tuples of nonnegative integers $(m_1,\cdots,m_n)$ satisfying the following condition

$$m_1+2m_2+\cdots+nm_n=n, \tag{3}$$

and functions f and ξ belong to $C^{(n)}(\mathbb{R}^+)$.

Lemma 4 *Take $f\in C^{(n)}(\mathbb{R}^+)$, $n\in\mathbb{N}$ and the semiflows $(\phi_t)_{t>0}$, $(\psi_t)_{t>0}$ and $(\varphi_t)_{t>0}$. Then*

$$\frac{d^n}{dr^n}(f(\phi_t(r)))=e^{-nt}f^{(n)}(\phi_t)(r),$$
$$\frac{d^n}{dr^n}(f(\psi_t(r)))=\omega^n_{\psi_t}(r)(1-e^t)^n\sum\frac{n!}{m_1!\cdots m_n!}f^{(k)}(\psi_t(r))\omega^k_{\psi_t}(r)\left(\frac{e^t}{1-e^t}\right)^k,$$
$$\frac{d^n}{dr^n}(f(\varphi_t(r)))=\omega^n_{\varphi_t}(r)\left(\frac{1-e^t}{2}\right)^n\sum\frac{n!}{m_1!\cdots m_n!}f^{(k)}(\varphi_t(r))\,\omega^k_{\varphi_t}(r)\left(\frac{2e^t}{1-e^t}\right)^k,$$

where the sum is over all n-tuples of nonnegative integers $(m_1,\cdots,m_n)$ satisfying the condition (3) and $k=m_1+\cdots+m_n$.

Proof We apply the Faà di Bruno's formula for the n-th derivative of a composition of functions and Proposition 3 (iv) and (v) in both cases. As $\phi_t^{(j)}(r)=0$ for $j\geq 2$, we obtain the first identity directly. In the second and third case, note that

$$\begin{aligned}\left(\frac{\psi_t^{(j)}(r)}{j!}\right)^{m_j}&=(e^t-1)^{m_j}\frac{(-(e^t-1))^{m_j(j-1)}}{(1+r(e^t-1))^{m_j(j+1)}}\\&=\left(\frac{-(e^t-1)}{(1+r(e^t-1))}\right)^{jm_j}\left(\frac{-e^t}{(e^t-1)(1+r(e^t-1))}\right)^{m_j},\end{aligned}$$

and then

$$\begin{aligned}\frac{d^n}{dr^n}(f(\psi_t(r)))&=\sum\left(\tfrac{n!}{m_1!\cdots m_n!}f^{(m_1+\cdots+m_n)}(\psi_t(r))\prod_{j=1}^n\left(\tfrac{\psi_t^{(j)}(r)}{j!}\right)^{m_j}\right)\\&=\omega^n_{\psi_t}(r)(1-e^t)^n\sum\tfrac{n!}{m_1!\cdots m_n!}f^{(k)}(\psi_t(r))\omega^k_{\psi_t}(r)\left(\tfrac{e^t}{1-e^t}\right)^k.\end{aligned}$$

Similarly,

$$\left(\frac{\varphi_t^{(j)}(r)}{j!}\right)^{m_j} = \left(\frac{e^t-1}{2}\right)^{jm_j}\left(\frac{2e^t}{e^t-1}\right)^{m_j}\left(\frac{-2}{e^t+1+r(e^t-1)}\right)^{m_j} \left(\frac{-2}{e^t+1+r(e^t-1)}\right)^{jm_j},$$

hence,

$$\begin{aligned}\frac{d^n}{dr^n}(f(\varphi_t(r))) &= \sum\left(\tfrac{n!}{m_1!\cdots m_n!}f^{(m_1+\cdots+m_n)}(\varphi_t(r))\textstyle\prod_{j=1}^n\left(\tfrac{\varphi_t^{(j)}(r)}{j!}\right)^{m_j}\right)\\ &= \omega_{\varphi_t}^n(r)\left(\tfrac{1-e^t}{2}\right)^n\sum\tfrac{n!}{m_1!\cdots m_n!}f^{(k)}(\varphi_t(r))\omega_{\varphi_t}^k(r)\left(\tfrac{2e^t}{1-e^t}\right)^k,\end{aligned}$$

where the sums are over all n-tuples of nonnegative integers $(m_1,\cdots,m_n)$ satisfying the condition (3) and $k=m_1+\cdots+m_n$. ☐

2 Fractional Sobolev spaces defined on $\mathbb{R}^+$

Let $\mathcal{D}_+$ be the class of C^∞-functions with compact support on $[0,\infty)$ and $\mathcal{S}_+$ the Schwartz class on $[0,\infty)$. For a function $f\in\mathcal{S}_+$ and $\alpha>0$, the *Weyl fractional integral* of order α, $W_+^{-\alpha}f$, is defined by

$$W_+^{-\alpha}f(t):=\frac{1}{\Gamma(\alpha)}\int_t^\infty(s-t)^{\alpha-1}f(s)ds,\qquad t\in\mathbb{R}^+.$$

The well-known Hardy inequality states that

$$\int_0^\infty|W_+^{-\alpha}f(t)|^pdt\le\left(\frac{\Gamma(\frac{1}{p})}{\Gamma(\alpha+\frac{1}{p})}\right)^p\int_0^\infty t^{\alpha p}|f(t)|^pdt, \tag{4}$$

for $p\ge1$ and $\alpha\ge0$, see for example [11, pp. 244-245].

The *Weyl fractional derivative* $W_+^\alpha f$ of order α is defined by

$$W_+^\alpha f(t):=(-1)^n\frac{d^n}{dt^n}W_+^{-(n-\alpha)}f(t),\quad t\in\mathbb{R}^+$$

where $n=[\alpha]+1$, and $[\alpha]$ denotes the integer part of α. It is proved that $W_+^{\alpha+\beta}=W_+^\alpha(W_+^\beta)$ for any $\alpha,\beta\in\mathbb{R}$, where $W_+^0=Id$ is the identity operator and $(-1)^nW_+^n=\frac{d^n}{dt^n}$ holds with $n\in\mathbb{N}$, and

$$W_+^\alpha(f_r)=r^\alpha(W_+^\alpha f)_r, \tag{5}$$

where $f_r(s) := f(rs)$ for $r > 0$. If $\mathrm{supp}(f) \subset [0,1]$ then $\mathrm{supp}(W_+^\alpha f) \subset [0,1]$ for $\alpha \in \mathbb{R}$; similarly if $\mathrm{supp}(f) \subset [1,\infty)$ then $\mathrm{supp}(W_+^\alpha f) \subset [1,\infty)$ for $\alpha \in \mathbb{R}$. See more details in [15] and [18].

Lemma 5 *Take $\alpha \in \mathbb{R}$, $t > 0$ and $f \in \mathcal{S}_+$. Then*

(i) For $\alpha \in \mathbb{R}$ and $t > 0$,

$$W_+^\alpha(f(\phi_t))(r) = e^{-t\alpha}\left(W_+^\alpha f\right)(\phi_t(r)), \qquad r > 0.$$

(ii) For $\alpha > 0$, $\gamma > \alpha$, and $t > 0$,

$$\begin{aligned}&W_+^{-\alpha}(\omega_{\psi_t}^\gamma f(\psi_t))(r)\\&= \frac{e^{-t\alpha}}{(\psi_t(\infty))^{\gamma-\alpha-1}\omega_{\psi_t}^{\alpha-1}(r)}\int_{\psi_t(r)}^{\psi_t(\infty)} \frac{(u-\psi_t(r))^{\alpha-1}(\psi_t(\infty)-u)^{\gamma-\alpha-1}}{\Gamma(\alpha)} f(u)du\end{aligned}$$

Proof (i) Take $\alpha > 0$. Then

$$\begin{aligned}W_+^{-\alpha}(f(\phi_t))(r) &= \frac{1}{\Gamma(\alpha)}\int_r^\infty (s-r)^{\alpha-1} f(e^{-t}s + 1 - e^{-t})(s)ds\\&= \frac{e^{t\alpha}}{\Gamma(\alpha)}\int_{e^{-t}r+1-e^{-t}}^\infty (u-(e^{-t}r+1-e^{-t}))^{\alpha-1} f(u)du\\&= e^{t\alpha}\left(W_+^\alpha f\right)(\phi_t(r))\end{aligned}$$

where we change the variable $u = e^{-t}s + 1 - e^{-t}$. Take now $n > \alpha$ and

$$\begin{aligned}W_+^\alpha(f(\phi_t))(r) &= W_+^n\left(W_+^{-(n-\alpha)}(f(\phi_t))\right)(r)\\&= e^{t(n-\alpha)}\left(W_+^n\left(W_+^{-(n-\alpha)}f\right)(\phi_t(r))\right) = e^{-t\alpha}\left(W_+^\alpha f\right)(\phi_t(r))\end{aligned}$$

where we have applied Lemma 4 (i).
(ii) Take $\alpha > 0$ and $\gamma > \alpha$. Then

$$\begin{aligned}W_+^{-\alpha}(\omega_{\psi_t}^\gamma f(\psi_t))(r) &= \int_r^\infty \frac{(s-r)^{\alpha-1}}{\Gamma(\alpha)}\frac{1}{(1+s(e^t-1))^\gamma} f\left(\frac{se^t}{1+s(e^t-1)}\right)ds\\&= \frac{e^{-t(\gamma-1)}}{\Gamma(\alpha)}\int_{\psi_t(r)}^{\psi_t(\infty)}\left(\frac{u}{e^t-u(e^t-1)} - r\right)^{\alpha-1}(e^t-u(e^t-1))^{\gamma-2} f(u)du,\end{aligned}$$

where we change the variable $u = \frac{se^t}{1+s(e^t-1)}$. Note that $u - r(e^t - u(e^t-1)) = (1 + r(e^t-1))(u - \psi_t(r))$ and then

$$
\begin{aligned}
W_+^{-\alpha}(\omega_{\psi_t}^{\gamma} f(\psi_t))(r) &= \frac{e^{-t(\gamma-1)}}{\omega_{\psi_t}^{\alpha-1}(r)} \int_{\psi_t(r)}^{\psi_t(\infty)} \frac{(u-\psi_t(r))^{\alpha-1}\,(e^t-u(e^t-1))^{\gamma-\alpha-1}}{\Gamma(\alpha)} f(u)du \\
&= \frac{e^{-t\alpha}}{(\psi_t(\infty))^{\gamma-\alpha-1}\omega_{\psi_t}^{\alpha-1}(r)} \int_{\psi_t(r)}^{\psi_t(\infty)} \frac{(u-\psi_t(r))^{\alpha-1}\,(\psi_t(\infty)-u)^{\gamma-\alpha-1}}{\Gamma(\alpha)} f(u)du,
\end{aligned}
$$

and we conclude the proof. ☐

We remind a family of subspaces $\mathcal{T}_p^{(\alpha)}(t^\alpha)$ which are contained in $L^p(\mathbb{R}^+)$ and introduced in [14, Definition 2.1] and [17, Section 1.2]. For $\alpha > 0$ let the Banach space $\mathcal{T}_p^{(\alpha)}(t^\alpha)$ be defined as the completion of the Schwartz class $\mathcal{S}_+$ in the norm

$$
||f||_{\alpha,p} := \frac{1}{\Gamma(\alpha+1)} \left(\int_0^\infty |W_+^\alpha f(t)|^p t^{\alpha p} dt \right)^{\frac{1}{p}}.
$$

We understand that $\mathcal{T}_p^{(0)}(t^0) = L^p(\mathbb{R}^+)$ and $||\ \ ||_{0,p} = ||\ \ ||_p$. The case $p = 1$, and $\alpha \in \mathbb{N}$ was introduced in [2] and for $p = 1$, and $\alpha > 0$ in [9, Section 1].

In the next proposition we summarize some results about these family of spaces $\mathcal{T}_p^{(\alpha)}(t^\alpha)$ which were presented in [14, Proposition 2.2].

Proposition 6 *Take $p \geq 1$ and $\beta > \alpha > 0$.*

(i) The operator $D_\alpha : \mathcal{T}_p^{(\alpha)}(t^\alpha) \to L^p(\mathbb{R}^+)$ defined by

$$
D_\alpha f(t) := \frac{t^\alpha}{\Gamma(\alpha+1)} W_+^\alpha f(t), \quad t \in \mathbb{R}^+,\ f \in \mathcal{T}_p^{(\alpha)}(t^\alpha),
$$

is an isometry whose inverse operator $(D_\alpha)^{-1} : L^p(\mathbb{R}^+) \to \mathcal{T}_p^{(\alpha)}(t^\alpha)$ is given by

$$
(D_\alpha)^{-1} f(t) = W_+^{-\alpha}((t^\alpha)^{-1} f)(t), \quad t \in \mathbb{R}^+\ f \in \mathcal{T}_p^{(\alpha)}(t^\alpha).
$$

(ii) $\mathcal{T}_p^{(\beta)}(t^\beta) \hookrightarrow \mathcal{T}_p^{(\alpha)}(t^\alpha) \hookrightarrow L^p(\mathbb{R}^+)$.

(iii) We set $f * g$ the usual convolution product on $\mathbb{R}^+$ defined by

$$
(f * g)(t) := \int_0^t f(t-s)g(s)ds, \qquad t \in \mathbb{R}^+.
$$

Then

$$
\mathcal{T}_p^{(\alpha)}(t^\alpha) * \mathcal{T}_1^{(\alpha)}(t^\alpha) \hookrightarrow \mathcal{T}_p^{(\alpha)}(t^\alpha),
$$

i.e., $\|f * g\|_{\alpha,p} \leq \|f\|_{\alpha,p}\|g\|_{\alpha,1}$ for $f \in \mathcal{T}_p^{(\alpha)}(t^\alpha)$, and $g \in \mathcal{T}_1^{(\alpha)}(t^\alpha)$.

(iv) *If $p > 1$ and p' satisfies $\frac{1}{p} + \frac{1}{p'} = 1$, then the dual of $\mathcal{T}_p^{(\alpha)}(t^\alpha)$ is $\mathcal{T}_{p'}^{(\alpha)}(t^\alpha)$, where the duality is given by*

$$\langle f, g\rangle_\alpha = \frac{1}{\Gamma(\alpha+1)^2}\int_0^\infty W_+^\alpha f(t) W_+^\alpha g(t) t^{2\alpha}dt = \langle D_\alpha f, D_\alpha g\rangle_0$$

for $f \in \mathcal{T}_p^{(\alpha)}(t^\alpha)$ and $g \in \mathcal{T}_{p'}^{(\alpha)}(t^\alpha)$. Note that, in fact,

$$\|f\|_{\alpha,p} = \|D_\alpha f\|_p, \qquad \langle f, g\rangle_\alpha = \langle D_\alpha f, D_\alpha g\rangle_0, \tag{6}$$

for $f \in \mathcal{T}_p^{(\alpha)}(t^\alpha)$, $g \in \mathcal{T}_{p'}^{(\alpha)}(t^\alpha)$.

Example 1 In this example, we consider some functions which belong (or not) to $\mathcal{T}_p^{(\alpha)}(t^\alpha)$ for $p \geq 1, \alpha \geq 0$ and $m \in \mathbb{N}$.

(i) Note that $t^\beta \notin \mathcal{T}_p^{(\alpha)}(t^\alpha)$ for $\beta \in \mathbb{C}$ due to t^β does not belong to $L^p(\mathbb{R}^+)$.

(ii) For $0 < \gamma < \delta$ and $a > 0$ then $W_+^{-\gamma}(a+t)^{-\delta} = \frac{\Gamma(\delta-\gamma)}{\Gamma(\delta)}(t+a)^{\gamma-\delta}$, and

$$W_+^\alpha(a+t)^{-\beta} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)}(t+a)^{-(\alpha+\beta)},$$

for $\alpha, \beta > 0$, see for example [8, p. 201]. Write $f(t) := (a+t)^{-\beta}$ and we conclude functions $(a+t)^{-\beta} \in \mathcal{T}_p^{(\alpha)}(t^\alpha)$ for $\beta > 1/p$ and $a > 0$.

(iii) We define functions $j_c(t) := \dfrac{(1-t)^{c-1}}{\Gamma(c)}\chi_{(0,1)}(t)$ for $t \geq 0$. It is easy to check that $W_+^{-\alpha}(j_c) = j_{\alpha+c}$ for $\alpha > 0$. Then $j_c \in \mathcal{T}_p^{(\alpha)}(t^\alpha)$ if and only if $c > \alpha+1-\frac{1}{p}$. Moreover, $t^\beta(1-t)^c\chi_{(0,1)}(t)$ belongs to $\mathcal{T}_p^{(m)}(t^m)$ for $\Re\beta > \frac{-1}{p}$ and $\Re c > m-\frac{1}{p}$.

(iv) $f(t) = \dfrac{(t-1)^a}{t^b}\chi_{(1,\infty)}(t)$ belongs to $\mathcal{T}_p^{(m)}(t^m)$ for $\Re a > m - \frac{1}{p}$ and $\Re b > \Re a + \frac{1}{p}$.

We introduce in the next definition a decomposition of $\mathcal{T}_p^{(\alpha)}(t^\alpha)$.

Definition 2 Given $\alpha \geq 0$, $p \geq 1$ and the Banach space $\mathcal{T}_p^{(\alpha)}(t^\alpha)$. We introduce the following subspaces

$$\mathcal{T}_0 := \{f \in \mathcal{T}_p^{(\alpha)}(t^\alpha) \mid \operatorname{supp}(f) \subset [0,1]\},$$
$$\mathcal{T}_1 := \{f \in \mathcal{T}_p^{(\alpha)}(t^\alpha) \mid \operatorname{supp}(f) \subset [1,\infty)\}.$$

It is straightforward to check that $\mathcal{T}_0$ and $\mathcal{T}_1$ are closed subspaces and $\mathcal{T}_p^{(\alpha)}(t^\alpha) = \mathcal{T}_0 + \mathcal{T}_1$.

3 Weight Koopman semigroups in the fractional Sobolev spaces

In this section we introduce three uniformly bounded C_0-semigroups $(T_{t,p})_{t>0}$, $(S^\gamma_{t,p})_{t>0}$ and $(R^\gamma_{t,p})_{t>0}$ on the fractional Lebesgue spaces $\mathcal{T}_p^{(\alpha)}(t^\alpha)$. In fact, they are weight Koopman semigroups and their definitions fit perfectly with the Sobolev structure of the $\mathcal{T}_p^{(\alpha)}(t^\alpha)$ space, see Theorems 7, 10 and 13.

Definition 3 Given $p \geq 1$ and a function $f \in \S_+$, we define functions $T_{t,p}f$, $S^\gamma_{t,p}f$ and $R^\gamma_{t,p}f$ by

$$\begin{aligned}
T_{t,p}f(r) &:= e^{\frac{-t}{p}} f(\phi_t(r)) = e^{\frac{-t}{p}} f(e^{-t}r + 1 - e^{-t}),\\
S^\gamma_{t,p}f(r) &:= e^{\frac{t}{p}} \omega^\gamma_{\psi_t}(r) f(\psi_t(r)) = \frac{e^{\frac{t}{p}}}{(1+r(e^t-1))^\gamma} f\left(\frac{e^t r}{1+r(e^t-1)}\right),\\
R^\gamma_{t,p}f(r) &:= e^{t(\gamma-\frac{1}{p})} \omega^\gamma_{\varphi_t}(r) f(\varphi_t(r)) = \frac{2^\gamma\, e^{t(\gamma-\frac{1}{p})}}{(e^t+1+r(e^t-1))^\gamma} f\left(\frac{(1+e^t)r+e^t-1}{e^t+1+r(e^t-1)}\right),
\end{aligned}$$

with $r > 0$, $t \geq 0$, $\gamma \geq 0$. Note that $T_{0,p}f = S^\gamma_{0,p}f = R^\gamma_{0,p}f = f$.

3.1 The weight Koopman C_0-semigroup $(T_{t,p})_{t>0}$

In the next theorem, we show that $(T_{t,p})_{t>0}$ define bounded operators on the Sobolev spaces (and subspaces) introduced in the section 2.

Theorem 7 *Take $p \geq 1$, $\alpha \geq 0$ and $t \geq 0$. Then*

$$\|T_{t,p}f\|_{\alpha,p} \leq \|f\|_{\alpha,p}, \qquad f \in \mathcal{T}_p^{(\alpha)}(t^\alpha).$$

The restriction of $(T_{t,p})_{t\geq 0}$ on the subspace $\mathcal{T}_1$ are extended for $t \in \mathbb{R}$ and

$$\|T_{t,p}f\|_{\alpha,p} \leq \max\{e^{-t\alpha}, 1\}\|f\|_{\alpha,p}, \qquad f \in \mathcal{T}_1, \quad t \in \mathbb{R}.$$

Moreover subspaces $\mathcal{T}_i$ are $(T_{t,p})_{t>0}$-invariant, i.e., $T_{t,p}(\mathcal{T}_i) \subset \mathcal{T}_i$ for $i \in \{0, 1\}$ and

$$\|T_{t,p}f\|_{\alpha,p} \leq e^{-t\alpha}\|f\|_{\alpha,p}, \qquad f \in \mathcal{T}_0, \quad t > 0.$$

Proof By the Lemma 5 (i), $W^\alpha_+\left(T_{t,p}f\right) = e^{-t\alpha}T_{t,p}(W^\alpha_+ f)$ for $f \in \S_+$. Then we get

$$\begin{aligned}
\|T_{t,p}f\|^p_{\alpha,p} &\leq \frac{1}{(\Gamma(\alpha+1))^p}\int_0^\infty |W^\alpha_+(T_{t,p}f)(s)|^p s^{\alpha p} ds\\
&\leq \frac{e^{-t(\alpha p+1)}}{(\Gamma(\alpha+1))^p}\int_0^\infty |W^\alpha_+ f(se^{-t}+1-e^{-t})|^p s^{\alpha p} ds\\
&\leq \frac{e^{-t\alpha p}}{(\Gamma(\alpha+1))^p}\int_{1-e^{-t}}^\infty |W^\alpha_+ f(u)|^p (ue^t - e^t + 1))^{\alpha p} du,
\end{aligned}$$

where we change the variable $s = ue^t - e^t + 1$. For $t \geq 0$, note that $ue^t - e^t + 1 \leq ue^t$ for $u \geq 0$ and $ue^t - e^t + 1 \leq u$ for $u \in [0, 1]$, and we obtain that

$$\|T_{t,p} f\|_{\alpha,p}^p \leq \frac{1}{(\Gamma(\alpha+1))^p} \int_{1-e^{-t}}^{\infty} |W_+^\alpha f(u)|^p u^{\alpha p} du \leq \|f\|_{\alpha,p}^p, \qquad f \in \mathcal{T}_p^{(\alpha)}(t^\alpha),$$
$$\|T_{t,p} f\|_{\alpha,p}^p \leq \frac{e^{-t\alpha p}}{(\Gamma(\alpha+1))^p} \int_{1-e^{-t}}^{1} |W_+^\alpha f(u)|^p u^{\alpha p} du \leq e^{-t\alpha p}\|f\|_{\alpha,p}^p, \qquad f \in \mathcal{T}_0.$$

It is straightforward to check that $\mathcal{T}_i$ are $(T_{t,p})_{t>0}$-invariant for $i \in \{0, 1\}$.

By the Proposition 1 (i), the map $\phi_t : [1, \infty) \to [1, \infty)$ is defined for $t \in \mathbb{R}$ and then the induced operator $(T_{t,p})_{t\geq 0}$ on $\mathcal{T}_1$ is extended for $t < 0$ using the same formula given in Definition 7. For $t < 0$ and $f \in \mathcal{T}_1$, we have that

$$\begin{aligned}\|T_{t,p} f\|_{\alpha,p}^p &\leq \frac{e^{-t(\alpha p+1)}}{(\Gamma(\alpha+1))^p} \int_1^\infty |W_+^\alpha f(se^{-t} + 1 - e^{-t})|^p s^{\alpha p} ds \\ &\leq \frac{e^{-t\alpha p}}{(\Gamma(\alpha+1))^p} \int_1^\infty |W_+^\alpha f(u)|^p (ue^t - e^t + 1))^{\alpha p} du \\ &\leq e^{-t\alpha p}\|f\|_{\alpha,p}^p,\end{aligned}$$

where we have used the inequality $ue^t - e^t + 1 \leq u$ for $u \in [1, \infty)$ and $t < 0$. □

Theorem 8 *For $p \geq 1$ and $\alpha \geq \mu \geq 0$, the family of operators $(T_{t,p})_{t\geq 0}$ given in Definition 7 is a contractive C_0-semigroup on $\mathcal{T}_p^{(\alpha)}(t^\alpha)$ whose infinitesimal generator A is given by*

$$(Af)(s) := (1 - s)f'(s) - \frac{1}{p} f(s),$$

with domain $D(A) = \{f \in \mathcal{T}_p^{(\alpha)}(t^\alpha) \ |((1 - s)f)' \in \mathcal{T}_p^{(\alpha)}(t^\alpha)\}$, and

$$\{f \in \mathcal{T}_p^{(\alpha)}(t^{\alpha+1}) \ |f' \in \mathcal{T}_p^{(\alpha)}(t^\alpha)\} \subset D(A).$$

Proof By Theorem 7 and Proposition 1 (i) it is clear that $(T_{t,p})_{t\geq 0}$ is a semigroup of contractive operators in $\mathcal{B}(\mathcal{T}_p^{(\alpha)}(t^\alpha))$. To show that $(T_{t,p})_{t\geq 0}$ is strongly continuous it is enough to show that $T_{t,p}g \to g$ on $\mathcal{T}_p^{(\alpha)}(t^\alpha)$ when $t \to 0$ and $g \in \mathcal{S}_+$. For $n \geq \alpha$, note that

$$\begin{aligned}\|T_{t,p}g - g\|_{\alpha,p}^p \leq \|T_{t,p}g - g\|_{n,p}^p &= \frac{1}{(n!)^p} \int_0^\infty |(T_{t,p}g)^{(n)} - g^{(n)}(s)|^p s^{np} ds \\ &= \frac{1}{(n!)^p} \int_0^\infty |e^{-t(\frac{1}{p}+n)} g^{(n)}(se^{-t} + 1 - e^{-t}) - g^{(n)}(s)|^p s^{np} ds \to 0,\end{aligned}$$

by the Dominated Convergence Theorem.

On $\mathcal{T}_p^{(\alpha)}(t^\alpha)$, an easy computation shows that the generator $(A, D(A))$ of $(T_{t,p})_{t\geq 0}$ is given by

$$\begin{aligned} Af(s) &:= \lim_{t\to 0}\frac{T_{t,p}f(s)-f(s)}{t} = \lim_{t\to 0}\frac{e^{\frac{-t}{p}}f(e^{-t}s+1-e^{-t})-f(s)}{t} \\ &= (1-s)f'(s)-\frac{1}{p}f(s), \end{aligned}$$

where $f \in D(A) := \{f \in \mathcal{T}_p^{(\alpha)}(t^\alpha) | Af \in \mathcal{T}_p^{(\alpha)}(t^\alpha)\}$. Note that $(1-s)f' = f + ((1-s)f)'$ and we conclude the equality. In the case that $f \in \mathcal{T}_p^{(\alpha+1)}(t^{\alpha+1})$, then $sf' \in \mathcal{T}_p^{(\alpha)}(t^\alpha)$ and the condition is reduced to impose that $f' \in \mathcal{T}_p^{(\alpha)}(t^\alpha)$. □

Remark 3 Note that $A = \frac{d}{ds} + \Lambda - \frac{1}{p}I$ where $\Lambda f(s) = -sf'(s)$. The operator Λ is the infinitesimal generator of the C_0-group $\mathfrak{T}_{t,p}f(s) := e^{\frac{-t}{p}}f(e^{-t}s)$ considered in [14, Theorem 2.5]. For $b > 0$, we define the C_0-semigroup $(T_{t,p,b})_{t\geq 0}$ by

$$T_{t,p,b}(f)(s) := e^{\frac{-t}{p}}f(\phi_{t,b})(r) = e^{\frac{-t}{p}}f(e^{-t}r+b(1-e^{-t})), \qquad r > 0$$

and in this case the infinitesimal generator is $A_b = b\frac{d}{ds} + \Lambda - \frac{1}{p}I$.

We denote by $\rho(A)$ the resolvent set, $\sigma(A)$ the usual spectrum of the operator A and by $\sigma_p(A)$ the point spectrum of the operator A. In the next proposition, we identify both spectrum sets of the operator A defined in the Banach space $\mathcal{T}_p^{(\alpha)}(t^\alpha)$.

Theorem 9 *For* $1 \leq p < \infty$ *we have*

- *(i)* $\{z \in \mathbb{C}\ |\Re(z) < -\alpha\} \subset \sigma_p(A)$.
- *(ii)* $\sigma(A) = \{z \in \mathbb{C}\ |\Re(z) \leq 0\}$.
- *(iii) The set* $\rho(A) = \mathbb{C}^+$ *and*

$$R(\lambda, A)f(r) = \frac{1}{|r-1|^{\lambda+\frac{1}{p}}}\int_{\Gamma_{1,r}}|s-1|^{\lambda+\frac{1}{p}-1}f(s)ds, \qquad f \in \mathcal{T}_p^{(\alpha)}(t^\alpha), \quad r > 0,$$

for $\lambda \in \mathbb{C}^+$ *and* $\Gamma_{1,r} := (1, r)$ *when* $r > 1$ *and* $\Gamma_{1,r} := (r, 1)$ *in the case* $0 < r < 1$.

Proof (i) Let $\lambda \in \mathbb{C}$ and $f \in \mathcal{T}_p^{(\alpha)}(t^\alpha)$ such that $Af = \lambda f$. Then, f is a solution of the differential equation

$$(1-s)f'(s) = (\lambda + \frac{1}{p})f(s), \qquad s \geq 0.$$

The set of solutions of this linear equation are functions

$$f_\lambda(r) = \frac{c_1}{(r-1)^{\lambda+\frac{1}{p}}}\chi_{(1,\infty)}(r) + c_2(1-r)^{-\lambda-\frac{1}{p}}\chi_{(0,1)}(r), \qquad r \in \mathbb{R}^+\backslash\{1\},$$

for constants c_1, c_2. Note that $f_\lambda \in \mathcal{T}_p^{(\alpha)}(t^\alpha)$ if and only if $c_1 = 0$ and $1 - \frac{1}{p} - \Re\lambda > \alpha + 1 - \frac{1}{p}$, see Example 1. Therefore $\{z \in \mathbb{C} \,|\Re(z) < -\alpha\} \subset \sigma_p(A)$.
(ii) Take $\lambda = \mu + i\delta \in \mathbb{C}$, $-\alpha < \mu < 0$, and $n \in \mathbb{N} \cup \{0\}$ such that $n + \mu < 0 < n + 1 + \mu$. Suppose that $\lambda \in \rho(A)$ and then $R(\lambda, A) \in \mathcal{B}(\mathcal{T}_p^{(\alpha)}(t^\alpha))$. For any $f \in \mathcal{T}_p^{(\alpha)}(t^\alpha)$, we set $g = R(\lambda, A)f \in \mathcal{T}_p^{(\alpha)}(t^\alpha)$ and

$$f(r) = (\lambda - A)g(r) = (\lambda + \frac{1}{p})g(r) - (1 - r)g'(r), \qquad r > 0.$$

Now we consider $f(r) := \dfrac{(r-1)^{\alpha - \frac{1}{p} + 1 - i\delta}}{r^{\mu+\alpha+n+2}}\chi_{(1,\infty)}(r)$. By Example 1 (iv), this function belongs to $\mathcal{T}_p^{(m)}(t^m)$ where $[\alpha] \le m < [\alpha] + 1$ and then also belongs to $\mathcal{T}_p^{(\alpha)}(t^\alpha)$. Solutions of the following differential equation

$$\frac{(r-1)^{\alpha - \frac{1}{p} + 1 - i\delta}}{r^{\mu+\alpha+n+2}} = (\lambda + \frac{1}{p})g(r) - (1 - r)g'(r), \qquad r > 1,$$

are given by

$$g(r) = \frac{c_1}{(r-1)^{\lambda + \frac{1}{p}}} + \frac{1}{(r-1)^{\lambda + \frac{1}{p}}}\int_1^r \frac{(s-1)^{\alpha+\mu}}{s^{\mu+\alpha+n+2}}ds, \qquad r > 1.$$

Note that

$$\int_1^r \frac{(s-1)^{\mu+\alpha}}{s^{\mu+\alpha+n+2}}ds \ge \frac{1}{r^n}\int_1^r (1 - \frac{1}{s})^{\mu+\alpha}\frac{ds}{s^2} = \frac{1}{\mu+\alpha+1}\frac{(r-1)^{\mu+\alpha+1}}{r^{n+\mu+\alpha+1}}, \qquad r > 1,$$

and then

$$\left|\frac{1}{(r-1)^{\lambda+\frac{1}{p}}}\int_1^r \frac{(s-1)^{\alpha+\mu}}{s^{\mu+\alpha+n+2}}ds\right| \ge \frac{1}{\mu+\alpha+1}\frac{(r-1)^{\alpha-\frac{1}{p}+1}}{r^{n+\mu+\alpha+1}}, \qquad r > 1.$$

and we conclude that this function does not belong to $L^p(\mathbb{R}^+)$; $g \notin \mathcal{T}_p^{(\alpha)}(t^\alpha)$ and then $\lambda \in \sigma(A)$.

(iii) By Theorem 7 $\|T_{t,p}f\|_{\alpha,p} \le \|f\|_{\alpha,p}$, for $f \in \mathcal{T}_p^{(\alpha)}(t^\alpha)$ and then $\rho(A) = \mathbb{C}^+$. Take $\lambda \in \mathbb{C}^+$, and

$$\begin{aligned} R(\lambda, A)f(r) &= \int_0^\infty e^{-\lambda t}T_{t,p}f(r)dt = \int_0^\infty e^{-(\lambda+\frac{1}{p})t}f(e^{-t}r + 1 - e^{-t})dt \\ &= \frac{1}{|r-1|^{\lambda+\frac{1}{p}}}\int_{\Gamma_{1,r}} |s-1|^{\lambda+\frac{1}{p}-1}f(s)ds, \qquad f \in \mathcal{T}_p^{(\alpha)}(t^\alpha), \quad r > 0, \end{aligned}$$

where we change the variable $s = e^{-t}r + 1 - e^{-t}$ and $\Gamma_{1,r} = (1, r)$ when $r > 1$ and $\Gamma_{1,r} = (r, 1)$ in the case $0 < r < 1$. Note that

$$R(\lambda, A) f(1) = \lim_{r\to 1} \frac{1}{|r-1|^{\lambda+\frac{1}{p}}} \int_{\Gamma_{1,r}} |s-1|^{\lambda+\frac{1}{p}-1} f(s)ds = \frac{f(1)}{\lambda+\frac{1}{p}},$$

whenever this limit exists. □

Remark 4 Note that the operator $-A$ is sectorial of angle $\pi/2$ in the sense of [12, Chapter 2, Section 2.1] and $0 \notin \sigma_p(A)$.

We consider the subspaces $\mathcal{T}_0$ and $\mathcal{T}_1$ introduced in Definition 2 and the induced operators $A_{\mathcal{T}_0}$ and $A_{\mathcal{T}_1}$, infinitesimal generators of induced C_0-semigroups $(T_{t,p})_{t\geq 0}$ on $\mathcal{T}_0$ and $\mathcal{T}_1$. By Theorems 7 and 9 the following equalities hold

$$\begin{aligned} \sigma(A_{\mathcal{T}_0}) &= \overline{\sigma_p(A_{\mathcal{T}_0})} = \{z \in \mathbb{C} \,|\Re(z) \leq -\alpha\}, \\ \sigma(A_{\mathcal{T}_1}) &= \{z \in \mathbb{C} \mid -\alpha \leq \Re(z) \leq 0\}, \end{aligned}$$

and $\sigma_p(A_{\mathcal{T}_1}) = \emptyset$.

3.2 The weight Koopman C_0-semigroup $(S^{\gamma}_{t,p})_{t>0}$

Now we consider the family of operators $(S^{\gamma}_{t,p})_{t\geq 0}$ introduced in Definition 7. On $L^p(\mathbb{R}^+)$ and $\gamma \geq \frac{2}{p}$, it is a straightforward exercise to check that

$$\|S^{\gamma}_{t,p}\| \leq 1. \tag{7}$$

In the following theorem, we show that the family of operators $(S^{\gamma}_{t,p})_{t\geq 0}$ is uniformly bounded on the spaces $\mathcal{T}^{(m)}_p(t^m)$ for $m \in \mathbb{N} \cup \{0\}$.

Theorem 10 *Take $p \geq 1$, $m \in \mathbb{N} \cup \{0\}$, $\gamma \geq \frac{2}{p}$ and $t \geq 0$. Then*

$$\|S^{\gamma}_{t,p} f\|_{m,p} \leq C_m \|f\|_{m,p}, \qquad f \in \mathcal{T}^{(m)}_p(t^m).$$

Moreover subspaces $\mathcal{T}_i$ are $(S^{\gamma}_{t,p})_{t>0}$-invariant, i.e., $S^{\gamma}_{t,p}(\mathcal{T}_i) \subset \mathcal{T}_i$ for $i \in \{0, 1\}$. The restriction of $(S^{\gamma}_{t,p})_{t\geq 0}$ on the subspace $\mathcal{T}_0$ is extended for $t \in \mathbb{R}$ and

$$\|S^{\gamma}_{t,p} f\|_{m,p} \leq C_m \max\{e^{-t(m+\gamma-\frac{2}{p})}, 1\}\|f\|_{m,p}, \qquad f \in \mathcal{T}_0.$$

The restriction of $(S^{\gamma}_{t,p})_{t\geq 0}$ on the subspace $\mathcal{T}_1$ holds

$$\|S^{\gamma}_{t,p} f\|_{m,p} \leq C_m e^{-t(\gamma-\frac{2}{p})}\|f\|_{m,p}, \qquad f \in \mathcal{T}_1, \quad t > 0,$$

where the constant C_m is independent on t and f.

Proof Consider $m \in \mathbb{N}$ and we apply (1) to get

$$
\begin{aligned}
W_+^m(S_{t,p}^\gamma f)(r) &= (-1)^m e^{\frac{t}{p}} \sum_{n=0}^m \binom{m}{n} (\omega_{\psi_t}^\gamma)^{(m-n)}(r)(f(\psi_t))^{(n)}(r) \\
&= (-1)^m e^{\frac{t}{p}} \sum_{n=0}^m \binom{m}{n} \frac{\Gamma(\gamma+m-n)}{\Gamma(\gamma)} (1-e^t)^{m-n} \omega_{\psi_t}^{m-n+\gamma}(r)(f(\psi_t))^n(r) \\
&= e^{\frac{t}{p}} (e^t-1)^m \omega_{\psi_t}^{m+\gamma}(r) \sum_{n=0}^m \binom{m}{n} \frac{\Gamma(\gamma+m-n)}{\Gamma(\gamma)} \\
&\quad \sum \frac{n!}{m_1! \cdots m_n!} f^{(k)}(\psi_t(r)) \omega_{\psi_t}^k(r) \left(\frac{-e^t}{e^t-1}\right)^k
\end{aligned}
$$

where we have applied the Lemma 4 (ii) and the second sum is over all n-tuples of nonnegative integers $(m_1, \cdots, m_n)$ satisfying the condition (3) and $k = m_1 + \cdots + m_n$. Then we may write

$$
W_+^m(S_{t,p}^\gamma f)(r) = e^{\frac{t}{p}} \sum_{k=0}^m c_{m,k,\gamma} e^{tk} (e^t-1)^{m-k} \omega_t^{m+\gamma+k}(r) f^{(k)}(\psi_t(r))
$$

for certain coefficients $c_{m,k,\gamma}$ independent on f and t. For $p \geq \frac{2}{\gamma}$ and $0 \leq k \leq m$,

$$
\begin{aligned}
\int_0^\infty r^{pm} |\omega_t^{m+\gamma+k}(r) f^{(k)}(\psi_t(r))|^p dr &= \int_0^\infty \frac{e^{-tmp}(\psi_t(r))^{mp}}{(1+r(e^t-1))^{(\gamma+k)p}} |f^{(k)}(\psi_t(r))|^p dr \\
&= e^{-tp(m+\gamma+k)+t} \int_0^{\psi_t(\infty)} (e^t - u(e^t-1))^{p(\gamma+k)-2} u^{mp} |f^{(k)}(u)|^p du \\
&\leq e^{-t(pm+1)} \int_0^{\psi_t(\infty)} u^{mp} |f^{(k)}(u)|^p du \\
&\leq e^{-t(pm+1)} (\psi_t(\infty))^{(m-k)p} \int_0^{\psi_t(\infty)} u^{kp} |f^{(k)}(u)|^p du \\
&= \frac{e^{-t(pk+1)}}{(e^t-1)^{(m-k)p}} \int_0^{\psi_t(\infty)} u^{kp} |f^{(k)}(u)|^p du
\end{aligned}
$$

where we change the variable $u = \psi_t(r)$.

In the case that $f \in \mathcal{T}_1$, we apply that $e^t - u(e^t-1) \leq 1$ for $u \geq 1$ to get

$$
\begin{aligned}
&\int_1^\infty r^{pm} |\omega_t^{m+\gamma+k}(r) f^{(k)}(\psi_t(r))|^p dr \\
&\quad = e^{-tp(m+\gamma+k)+t} \int_1^{\psi_t(\infty)} (e^t - u(e^t-1))^{p(\gamma+k)-2} u^{mp} |f^{(k)}(u)|^p du
\end{aligned}
$$

$$
\begin{aligned}
&\leq e^{-tp(m+\gamma+k)+t}(\psi_t(\infty))^{(m-k)p}\int_1^{\psi_t(\infty)} u^{kp}|f^{(k)}(u)|^p du\\
&= \frac{e^{-tp(\gamma+2k)+t}}{(e^t-1)^{(m-k)p}}\int_1^{\psi_t(\infty)} u^{kp}|f^{(k)}(u)|^p du.
\end{aligned}
$$

By the Proposition 1 (ii), the map $\psi_t : [0,1] \to [0,1]$ is defined for $t \in \mathbb{R}$ and then the induced operator $(S^{\gamma}_{t,p})_{t\geq 0}$ on $\mathcal{T}_0$ is extended for $t < 0$ using the same formula given in Definition 7. For $t < 0$ and $f \in \mathcal{T}_0$, we apply that $e^t - u(e^t-1) \leq 1$ for $0 \leq u \leq 1$ and $t < 0$ to get

$$
\begin{aligned}
&\int_0^1 r^{pm}|\omega_t^{m+\gamma+k}(r)f^{(k)}(\psi_t(r))|^p dr\\
&\quad = e^{-tp(m+\gamma+k)+t}\int_0^1 (e^t-u(e^t-1))^{p(\gamma+k)-2}u^{mp}|f^{(k)}(u)|^p du\\
&\quad \leq e^{-tp(m+\gamma+k)+t}\int_0^1 u^{kp}|f^{(k)}(u)|^p du.
\end{aligned}
$$

Finally, we apply the known Jensen's inequality, i.e., for $a_i \geq 0$ and $p \geq 1$,

$$
\left(\sum_{i=1}^{m} a_i\right)^p \leq m^{p-1}\sum_{i=1}^{N} a_i^p,
$$

to get that

$$
\begin{aligned}
\|S^{\gamma}_{t,p}f\|^p_{m,p} &= \frac{1}{(m!)^p}\int_0^{\infty} r^{mp}|W^m_+(S^{\gamma}_{t,p}f)(r)|^p dr\\
&\leq \sum_{k=0}^{m} c^p_{m,k,\gamma}\int_0^{\psi_t(\infty)} u^{kp}|f^{(k)}(u)|^p du \leq C^p_m\|f\|^p_{m,p},
\end{aligned}
$$

for $f \in \mathcal{T}^{(m)}_p(t^m)$. For $f \in \mathcal{T}_1$, we obtain that

$$
\begin{aligned}
\|S^{\gamma}_{t,p}f\|^p_{m,p} &= \frac{1}{(m!)^p}\int_1^{\infty} r^{mp}|W^m_+(S^{\gamma}_{t,p}f)(r)|^p dr\\
&\leq \sum_{k=0}^{m} c^p_{m,k,\gamma}e^{-tp(k+\gamma-2)}\int_1^{\psi_t(\infty)} u^{kp}|f^{(k)}(u)|^p du\\
&\leq C^p_m e^{-tp(\gamma-\frac{2}{p})}\|f\|^p_{m,p}.
\end{aligned}
$$

For $t < 0$ and $f \in \mathcal{T}_0$, we have that

$$\begin{aligned}\|S_{t,p}^{\gamma}f\|_{m,p}^{p} &= \frac{1}{(m!)^{p}}\int_{0}^{1} r^{mp}|W_{+}^{m}(S_{t,p}^{\gamma}f)(r)|^{p}dr\\ &\leq \sum_{k=0}^{m} c_{m,k,\gamma}^{p}e^{-tp(m+\gamma-\frac{2}{p})}\int_{0}^{1} u^{kp}|f^{(k)}(u)|^{p}du\\ &\leq C_{m}^{p}e^{-tp(m+\gamma-\frac{2}{p})}\|f\|_{m,p}^{p},\end{aligned}$$

and we conclude the proof. □

Remark 5 It seems natural to conjecture that

$$\|S_{t,p}^{\gamma}f\|_{m,p}^{p} \leq C_{m}^{p}e^{-tp(m+\gamma-\frac{2}{p})}\|f\|_{m,p}^{p}, \qquad t>0,$$

for $f \in \mathcal{T}_1$ and $m \geq 0$. Note that a similar bound is obtained in Theorem 7 for $f \in \mathcal{T}_0$ and the C_0-semigroup $(T_{t,p})_{t>0}$. The main trouble is to express $W_{+}^{m}(S_{t,p}^{\gamma}f)$ only in terms of $f^{(m)}(\psi_t)$, which is not possible in this case, see Lemma 4.

Theorem 11 *Take $p \geq 1$, $m \in \mathbb{N}\cup\{0\}$, $\gamma \geq \frac{2}{p}$ and $t \geq 0$. Then the family of operators $(S_{t,p}^{\gamma})_{t\geq 0}$ given in Definition 7 is a contractive C_0-semigroup on $\mathcal{T}_p^{(m)}(t^m)$ whose infinitesimal generator B is given by*

$$(Bf)(r) := r(1-r)f'(r) + \left(\frac{1}{p} - \gamma r\right) f(r), \qquad r>0,$$

with domain $D(B) = \{f \in \mathcal{T}_p^{(m)}(t^m)\ |(r(1-r)f)' - (\gamma-2)rf \in \mathcal{T}_p^{(m)}(t^m)\}$, and

$$\{f \in \mathcal{T}_p^{(m+1)}(t^{m+1})\ |rf, r^2f' \in \mathcal{T}_p^{(m)}(t^m)\} \subset D(B).$$

Proof By Theorem 10 and Proposition 1 (i) it is clear that $(S_{t,p}^{\gamma})_{t\geq 0}$ is a semigroup of contractive operators in $\mathcal{B}(\mathcal{T}_p^{(m)}(t^m))$. To show that $(S_{t,p}^{\gamma})_{t\geq 0}$ is strongly continuous it is enough to show that $S_{t,p}^{\gamma}f \to f$ on $\mathcal{T}_p^{(m)}(t^m)$ when $t \to 0$ and $f \in \mathcal{S}_+$. By the proof of Theorem 10, we get that

$$\begin{aligned}&(W_{+}^{m}S_{t,p}^{\gamma}f - W_{+}^{m}f)(r)\\ &= (-1)^m e^{\frac{t}{p}}\sum_{n=0}^{m-1}\binom{m}{n}\frac{\Gamma(\gamma+m-n)}{\Gamma(\gamma)}(1-e^t)^{m-n}\omega_t^{m-n+\gamma}(r)(f(\psi_t))^{(m)}(r)\\ &\quad +(-1)^m e^{\frac{t}{p}}\omega_t^{\gamma}(r)(f(\psi_t))^{(m)}(r) - (-1)^m f^{(m)}(r)\\ &= e^{\frac{t}{p}}\sum_{k=0}^{m-1} c_{m,k,\gamma}e^{tk}(e^t-1)^{m-k}\omega_t^{m+\gamma+k}(r)f^{(k)}(\psi_t(r))\\ &\quad +(-1)^m e^{\frac{t}{p}+tm}\omega_t^{2m+\gamma}(r)f^{(m)}(\psi_t(r)) - (-1)^m f^{(m)}(r)\end{aligned}$$

for $r > 0$. Then

$$\begin{aligned}
&\|S^{\gamma}_{t,p} f - f\|^p_{m,p} \\
&\le e^t \sum_{k=0}^{m-1} c^p_{m,k,\gamma} e^{tkp}(e^t - 1)^{p(m-k)} \int_0^\infty r^{pm} |\omega_t^{m+\gamma+k}(r) f^{(k)}(\psi_t(r))|^p dr \\
&\quad + \frac{1}{(n!)^p} \int_0^\infty r^{mp} |e^{\frac{t}{p}+tm} \omega_t^{2m+\gamma}(r) f^{(m)}(\psi_t(r)) - f^{(m)}(r)|^p dr \\
&\le \sum_{k=0}^{m-1} c^p_{m,k,\gamma} e^{-tp(m-k)}(e^t - 1)^{p(m-k)} \int_0^{\psi_t(\infty)} u^{mp} |f^{(k)}(u)|^p du \\
&\quad + \frac{1}{(n!)^p} \int_0^\infty r^{mp} \left| \frac{e^{\frac{t}{p}+tm}}{(1 + r(e^t - 1))^{2m+\gamma}} f^{(m)}\left(\frac{e^t r}{1 + r(e^t - 1)}\right) - f^{(m)}(r) \right|^p dr
\end{aligned}$$

and $\|S^{\gamma}_{t,p} f - f\|^p_{m,p} \to 0$ when $t \to 0$ by the Dominated Convergence Theorem.

On $\mathcal{T}^{(m)}_p(t^m)$, an easy computation shows that the generator $(B, D(B))$ of $(S^{\gamma}_{t,p})_{t\ge 0}$ is given by

$$\begin{aligned}
Bf(r) &:= \lim_{t\to 0} \frac{S^{\gamma}_{t,p} f(r) - f(r)}{t} = \lim_{t\to 0} \frac{1}{t}\left(\frac{e^{\frac{t}{p}}}{(1 + r(e^t - 1))^{\gamma}} f\left(\frac{e^t r}{1 + r(e^t - 1)}\right) - f(r) \right) \\
&= r(1 - r) f'(r) + (\frac{1}{p} - \gamma r) f(r)
\end{aligned}$$

where $f \in D(B) := \{f \in \mathcal{T}^{(m)}_p(t^m) | Bf \in \mathcal{T}^{(m)}_p(t^m)\}$. To show that $D(B) = \{f \in \mathcal{T}^{(m)}_p(t^m)\ |(r(1 - r)f)' - (\gamma - 2)rf \in \mathcal{T}^{(m)}_p(t^m)\}$, note that

$$(r(1 - r)f)' = r(1 - r)f' + f - 2rf,$$

and we conclude the equality between both sets. In the case that $f \in \mathcal{T}^{(m+1)}_p(t^{m+1})$, then $rf' \in \mathcal{T}^{(m)}_p(t^m)$ and we have proved that

$$\{f \in \mathcal{T}^{(m+1)}_p(t^{m+1})\ | rf, r^2 f' \in \mathcal{T}^{(m)}_p(t^m)\} \subset D(B),$$

for any $p \ge \frac{2}{\gamma}$. $\square$

Theorem 12 *Take $p \ge 1$, $m \in \mathbb{N} \cup \{0\}$, $\gamma \ge \frac{2}{p}$ and the C_0-semigroup $(S^{\gamma}_{t,p})_{t\ge 0}$ whose infinitesimal generator is $(B, D(B))$. Then*

(i) $\{z \in \mathbb{C}\ |\Re(z) < -(m + \gamma - \frac{2}{p})\} \subset \sigma_p(B)$.
(ii) $\sigma(B) = \{z \in \mathbb{C}\ |\Re(z) \le 0\}$.

(iii) The set $\rho(B) = \mathbb{C}^+$ *and*

$$R(\lambda, B)f(r) = \frac{r^{\lambda-\frac{1}{p}}}{|r-1|^{\lambda+\gamma-\frac{1}{p}}} \int_{\Gamma_{1,r}} \frac{|s-1|^{\lambda+\gamma-\frac{1}{p}-1}}{s^{\lambda-\frac{1}{p}+1}} f(s)ds,$$
$$f \in \mathcal{T}_p^{(m)}(t^m), \quad r > 0,$$

for $\lambda \in \mathbb{C}^+$ *and* $\Gamma_{1,r} = (1, r)$ *when* $r > 1$ *and* $\Gamma_{1,r} = (r, 1)$ *in the case* $0 < r < 1$.

Proof (i) Let $\lambda \in \mathbb{C}$ and $f \in \mathcal{T}_p^{(m)}(t^m)$ such that $B(f) = \lambda f$. Then, f is a solution of the differential equation

$$s(1-s)f'(s) = (\lambda - \frac{1}{p} + \gamma s)f(s), \qquad s \geq 0.$$

The set of solutions of this linear equation are functions

$$f_\lambda(r) = \frac{c_1 r^{\lambda-\frac{1}{p}}}{(r-1)^{\lambda-\frac{1}{p}+\gamma}} \chi_{(1,\infty)}(r) + c_2 r^{\lambda-\frac{1}{p}}(1-r)^{-\lambda+\frac{1}{p}-\gamma}\chi_{(0,1)}(r), \qquad r \in \mathbb{R}^+\backslash\{1\},$$

for some constants c_1, c_2. Note that $f_\lambda \in \mathcal{T}_p^{(m)}(t^m)$ if and only if $c_2 = 0$ and $\Re\lambda + m + \gamma - \frac{1}{p} < \frac{1}{p}$, see Remark 1 (iv). Therefore $\{z \in \mathbb{C} \mid \Re(z) < -(m + \gamma - \frac{2}{p})\} \subset \sigma_p(B)$.
(ii) Take $\lambda = \mu + i\delta \in \mathbb{C}$, $-(m + \gamma - \frac{2}{p}) < \mu < 0$, and $n \in \mathbb{N} \cup \{0\}$ such that $n + \mu < 0 < n + 1 + \mu$. Suppose that $\lambda \in \rho(B)$ and then $R(\lambda, B) \in \mathcal{B}(\mathcal{T}_p^{(m)}(t^m))$. For any $f \in \mathcal{T}_p^{(m)}(t^m)$, we set $g = R(\lambda, B)f \in \mathcal{T}_p^{(m)}(t^m)$ and

$$f(r) = (\lambda - B)g(r) = (\lambda - \frac{1}{p} + \gamma r)g(r) - r(1-r)g'(r), \qquad r > 0.$$

Now we consider $f(r) = (1-r)^{m-\frac{1}{p}+1-i\delta} r^{\mu+n+1-\frac{1}{p}} \chi_{(0,1)}(r)$. By Example 1, this function belongs to $\mathcal{T}_p^{(\alpha)}(t^\alpha)$ and solutions of the following differential equation

$$(1-r)^{m-\frac{1}{p}+1-i\delta} r^{\mu+n+1-\frac{1}{p}} = (\lambda - \frac{1}{p} + \gamma r)g(r) - r(1-r)g'(r), \qquad 0 < r < 1,$$

are given by

$$g(r) = \frac{c_1 r^{\lambda-\frac{1}{p}}}{(1-r)^{\lambda-\frac{1}{p}+\gamma}} + \frac{r^{\lambda-\frac{1}{p}}}{(1-r)^{\lambda-\frac{1}{p}+\gamma}} \int_r^1 (1-s)^{\mu+m+\gamma-\frac{2}{p}} s^n ds, \qquad 0 < r < 1.$$

Note that

$$\int_r^1 (1-s)^{\mu+m+\gamma-\frac{2}{p}} s^n ds \geq r^n \int_r^1 (1-s)^{\mu+m+\gamma-\frac{2}{p}} ds = r^n \frac{(1-r)^{\mu+m+\gamma-\frac{2}{p}+1}}{\mu+m+\gamma-\frac{2}{p}+1},$$

for $0 < r < 1$ and then

$$\left| \frac{r^{\lambda-\frac{1}{p}}}{(1-r)^{\lambda-\frac{1}{p}+\gamma}} \int_r^1 (1-s)^{\mu+m+\gamma-\frac{2}{p}} s^n ds \right| \geq \frac{r^{\mu-\frac{1}{p}+n}(1-r)^{m-\frac{1}{p}+1}}{\mu+m+\gamma-\frac{2}{p}+1}, \qquad 0 < r < 1.$$

and we conclude that this function does not belong to $L^p(\mathbb{R}^+)$; $g \notin \mathcal{T}_p^{(m)}(t^m)$ and then $\lambda \in \sigma(B)$.

(iii) By Theorem 10, we get that $\|S_{t,p}^{\gamma} f\|_{m,p} \leq C_m \|f\|_{m,p}$, for $f \in \mathcal{T}_p^{(m)}(t^m)$ and then $\rho(A) = \mathbb{C}^+$. Take $\lambda \in \mathbb{C}^+$, and $f \in \mathcal{T}_p^{(m)}(t^m)$, we obtain that

$$\begin{aligned} R(\lambda, B) f(r) &= \int_0^\infty e^{-\lambda t} S_{t,p}^{\gamma} f(r) dt = \int_0^\infty \frac{e^{-(\lambda-\frac{1}{p})t}}{(1+r(e^t-1))^\gamma} f\left(\frac{e^t r}{1+r(e^t-1)}\right) dt \\ &= \frac{r^{\lambda-\frac{1}{p}}}{|r-1|^{\lambda-\frac{1}{p}+\gamma}} \int_{\Gamma_{1,r}} \frac{|s-1|^{\lambda-\frac{1}{p}+\gamma-1}}{s^{\lambda-\frac{1}{p}+1}} f(s) ds, \qquad f \in \mathcal{T}_p^{(m)}(t^m), \end{aligned}$$

$r > 0$, where we change the variable $s = \dfrac{e^t r}{1+r(e^t-1)}$. Note that

$$R(\lambda, B) f(1) = \lim_{r \to 1} \frac{r^{\lambda-\frac{1}{p}}}{|r-1|^{\lambda-\frac{1}{p}+\gamma}} \int_{\Gamma_{1,r}} \frac{|s-1|^{\lambda-\frac{1}{p}+\gamma-1}}{s^{\lambda-\frac{1}{p}+1}} f(s) ds = \frac{f(1)}{\lambda-\frac{1}{p}+\gamma},$$

whenever this limit exists. □

Remark 6 We consider the subspaces $\mathcal{T}_0$ and $\mathcal{T}_1$ introduced in Definition 2 and the operators $B_{\mathcal{T}_0}$ and $B_{\mathcal{T}_1}$, infinitesimal generators of induced C_0-semigroups $(S_{t,p}^{\gamma})_{t\geq 0}$ on $\mathcal{T}_0$ and $\mathcal{T}_1$. By Theorems 10 and 12, the following equalities hold

$$\sigma(B_{\mathcal{T}_0}) = \left\{ z \in \mathbb{C} \mid -\left(m+\gamma-\frac{2}{p}\right) \leq \Re(z) \leq 0 \right\},$$
$$\left\{ z \in \mathbb{C} \mid \Re(z) \leq -\left(m+\gamma-\frac{2}{p}\right) \right\} \subset \sigma(B_{\mathcal{T}_1}) \subset \left\{ z \in \mathbb{C} \mid \Re(z) \leq -\left(\gamma-\frac{2}{p}\right) \right\}.$$

In fact, we conjecture that $\sigma(B_{\mathcal{T}_1}) = \{z \in \mathbb{C} \mid \Re(z) \leq -(m+\gamma-\frac{2}{p})\}$, see Remark 5.

3.3 The weight Koopman C_0-semigroup $(R^{\gamma}_{t,p})_{t>0}$

Now we consider the family of operators $(R^{\gamma}_{t,p})_{t\geq 0}$ introduced in Definition 7. In the following theorem, we show that the family of operators $(R^{\gamma}_{t,p})_{t\geq 0}$ is uniformly bounded on the spaces $\mathcal{T}_p^{(m)}(t^m)$ for $m \in \mathbb{N}\cup\{0\}$.

Theorem 13 *Take* $p \geq 1, m \in \mathbb{N}\cup\{0\}$, $\gamma \geq \frac{2}{p}$ *and* $t \geq 0$. *Then for* $f \in \mathcal{T}_p^{(m)}(t^m)$

$$\|R^{\gamma}_{t,p} f\|_{m,p} \leq C_m \|f\|_{m,p}, \qquad t > 0.$$

For $m = 0$, $\|R^{\gamma}_{t,p} f\|_p \leq 2^{\gamma-\frac{2}{p}}\|f\|_p$, *for* $f \in L^p(\mathbb{R}^+)$.

Proof Let $t \geq 0$. First we consider the case $f \in L^p(\mathbb{R}^+)$, we get

$$\begin{aligned}
\int_0^\infty |R^{\gamma}_{t,p} f(r)|^p dr &= \int_0^\infty e^{tp(\gamma-\frac{1}{p})}\left|\frac{1-\varphi_t(r)}{1-r}\right|^{\gamma p} |f(\varphi_t(r))|^p\, dr\\
&= \int_0^\infty e^{tp(\gamma-\frac{1}{p})}\left|\frac{1-\varphi_t(r)}{1-r}\right|^{\gamma p} \left|f\left(\frac{(1+e^t)r+e^t-1}{e^t+1+r(e^t-1)}\right)\right|^p dr\\
&= \int_{\frac{1}{\varphi_t(\infty)}}^{\varphi_t(\infty)} e^{tp(\gamma-\frac{1}{p})}\left(\frac{e^t+1-u(e^t-1)}{2e^t}\right)^{\gamma p}\frac{4e^t}{(u(e^t-1)-(1+e^t))^2}|f(u)|^p du\\
&\leq \int_{\frac{1}{\varphi_t(\infty)}}^{\varphi_t(\infty)} \left(\frac{4e^t}{e^t+1}\right)^{\gamma p-2} 2^{2-\gamma p}\,|f(u)|^p du \leq 2^{\gamma p-2}||f||_p^p,
\end{aligned}$$

where we change the variable $u = \varphi_t(r)$.

Now, consider $m \in \mathbb{N}$ and we apply (2) to get

$$\begin{aligned}
W_+^m(R^{\gamma}_{t,p} f)(r) &= (-1)^m e^{t(\gamma-\frac{1}{p})}\sum_{n=0}^m \binom{m}{n}(\omega^{\gamma}_{\varphi_t})^{(m-n)}(r)(f(\varphi_t))^{(n)}(r)\\
&= (-1)^m e^{t(\gamma-\frac{1}{p})}\sum_{n=0}^m \binom{m}{n}\frac{\Gamma(\gamma+m-n)}{\Gamma(\gamma)}\left(\frac{1-e^t}{2}\right)^{m-n}\omega^{m-n+\gamma}_{\varphi_t}(r)(f(\varphi_t))^{(n)}(r)\\
&= (-1)^m e^{t(\gamma-\frac{1}{p})}\omega^{m+\gamma}_{\varphi_t}(r)\sum_{n=0}^m \binom{m}{n}\frac{\Gamma(\gamma+m-n)}{\Gamma(\gamma)}\left(\frac{1-e^t}{2}\right)^{m-n}\omega^{-n}_{\varphi_t}(r)\\
&\quad \omega^{n}_{\varphi_t}(r)\left(\frac{1-e^t}{2}\right)^{n}\sum\frac{n!}{m_1!\cdots m_n!}f^{(k)}(\varphi_t(r))\omega^{k}_{\varphi_t}(r)\left(\frac{2e^t}{1-e^t}\right)^{k}\\
&= e^{t(\gamma-\frac{1}{p})}\left(\frac{e^t-1}{2}\right)^{m}\omega^{m+\gamma}_{\varphi_t}(r)\sum_{n=0}^m \binom{m}{n}\frac{\Gamma(\gamma+m-n)}{\Gamma(\gamma)}\\
&\quad \sum\frac{n!}{m_1!\cdots m_n!}f^{(k)}(\varphi_t(r))\omega^{k}_{\varphi_t}(r)\left(\frac{2e^t}{1-e^t}\right)^{k}
\end{aligned}$$

where we applied the Lemma 4 and the second sum is over all n-tuples of nonnegative integers $(m_1, \dots, m_n)$ satisfying the condition (3) and $k = m_1 + \cdots + m_n$. We write

$$W_+^m(R^\gamma_{t,p}f)(r) = e^{t(\gamma-\frac{1}{p})}\sum_{k=0}^{m} d_{m,k,\gamma}\, e^{tk}\,(e^t-1)^{m-k}\omega_{\varphi_t}^{m+\gamma+k}(r) f^{(k)}(\varphi_t(r))$$

for certain coefficients $d_{m,k,\gamma}$ independent on f and t. For $p \geq \frac{2}{\gamma}$, $0 \leq k \leq m$ and making the variable change $u = \varphi_t(r)$ we obtain

$$\begin{aligned}
&\int_0^\infty r^{pm}|\omega_{\varphi_t}(r)|^{p(m+\gamma+k)}\,|f^{(k)}(\varphi_t(r))|^p\,dr\\
&= \int_0^\infty r^{pm}\left(\frac{2}{e^t+1+r(e^t-1)}\right)^{p(m+\gamma+k)}\left|f^{(k)}\left(\frac{(1+e^t)r+e^t-1}{(e^t-1)r+1+e^t}\right)\right|^p dr\\
&= \frac{1}{2^{p(m+\gamma+k)-2}}\int_{\frac{1}{\varphi_t(\infty)}}^{\varphi_t(\infty)}(u(1+e^t)+1-e^t)^{pm}\,\frac{(1+e^t-u(e^t-1))^{p(\gamma+k)-2}}{e^{pt(m+\gamma+k)-t}}\\
&\quad |f^{(k)}(u)|^p du\\
&\leq \frac{1}{2^{p(m+\gamma+k)-2}}\int_{\frac{1}{\varphi_t(\infty)}}^{\varphi_t(\infty)}(u(1+e^t))^{pk}\left(\frac{4e^t}{e^t-1}\right)^{p(m-k)}\left(\frac{4e^t}{e^t+1}\right)^{p(k+\gamma)-2}\\
&\quad \frac{|f^{(k)}(u)|^p}{e^{pt(m+\gamma+k)-t}}du\\
&= 2^{p(m+\gamma-k)-2}\left(\frac{1+e^t}{e^t}\right)^{pk}\frac{e^{tp(-k-\gamma)+t}}{(e^t-1)^{p(m-k)}}\int_{\frac{1}{\varphi_t(\infty)}}^{\varphi_t(\infty)}u^{pk}\,|f^{(k)}(u)|^p du\\
&\leq 2^{p(m+\gamma-k)-2}2^{pk}\frac{e^{tp(-k-\gamma)+t}}{(e^t-1)^{p(m-k)}}\int_{\frac{1}{\varphi_t(\infty)}}^{\varphi_t(\infty)}u^{pk}\,|f^{(k)}(u)|^p du\\
&\leq 2^{p(m+\gamma)-2}\frac{e^{tp(-k-\gamma+\frac{1}{p})}}{(e^t-1)^{p(m-k)}}||f||^p_{k,p}
\end{aligned}$$

and we conclude the proof. □

Remark In the case of $(R^\gamma_{t,p})_{t\geq 0}$, we may also consider its restriction on the closed subspaces $\mathcal{T}_0$ and $\mathcal{T}_1$. However in these spaces, we can not improve the norm bounds or to extend into a C_0-group, as in previous subsections.

Theorem 14 *Take $p \geq 1$ and $\gamma \geq \frac{2}{p}$. Then the family of operators $(R^\gamma_{t,p})_{t\geq 0}$ is a C_0-semigroup on $\mathcal{T}_p^{(m)}(t^m)$ whose infinitesimal generator C is given by*

$$(Cf)(r) := \frac{1}{2}\left(\gamma - \frac{2}{p} - \gamma r\right) f(r) + \frac{1}{2}(1-r^2)f'(r)$$

with domain $D(C) = \{f \in \mathcal{T}_p^{(m)}(t^m) \mid ((1-r^2)f)' - (\gamma-2)rf \in \mathcal{T}_p^{(m)}(t^m)\}$, and $\{f \in \mathcal{T}_p^{(m+1)}(t^{m+1}) \mid f' \in \mathcal{T}_p^{(m)}(t^m))\} \subset D(C)$.

Proof Using Theorem 13 and Proposition 1 (i) it is clear that $(R_{t,p}^{\gamma})_{t\geq 0}$ is a semigroup of operators in $\mathcal{B}(\mathcal{T}_p^{(m)}(t^m))$. To show that $(R_{t,p}^{\gamma})_{t\geq 0}$ is strongly continuous we verify that $R_{t,p}^{\gamma} \to I$ strongly on $\mathcal{T}_p^{(m)}(t^m)$ when $t \to 0$.

For $r > 0$ and $f \in \mathcal{T}_p^{(m)}(t^m)$, from the proof of Theorem 13, we get that

$$\begin{aligned}(W_+^m R_{t,p}^{\gamma} f - W_+^m f)(r) &= e^{t(\gamma-\frac{1}{p})}\sum_{k=0}^{m-1} d_{m,k,\gamma}\, e^{tk}\, (e^t-1)^{m-k}\omega_{\varphi_t}^{m+\gamma+k}(r) f^{(k)}(\varphi_t(r))\\ &\quad + (-1)^m e^{t(m+\gamma-\frac{1}{p})}\,\omega_{\varphi_t}^{2m+\gamma}(r) f^{(m)}(\varphi_t(r)) - (-1)^m f^{(m)}(r).\end{aligned}$$

Then $\|R_{t,p}^{\gamma} f - f\|_{m,p}^p$

$$\begin{aligned}&\leq e^{tp(\gamma-\frac{1}{p})}\sum_{k=0}^{m-1} d_{m,k,\gamma}^p\, e^{tpk}\, (e^t-1)^{p(m-k)} \int_0^\infty r^{pm}|\omega_{\varphi_t}^{m+\gamma+k}(r) f^{(k)}(\varphi_t(r))|^p dr\\ &+ \frac{1}{(n!)^p}\int_0^\infty r^{pm}|e^{t(m+\gamma-\frac{1}{p})}\omega_{\varphi_t}^{2m+\gamma}(r) f^{(m)}(\varphi_t(r)) - f^{(m)}(r)|^p\, dr\\ &\leq \frac{e^{-pt(m+\gamma+k)-t}}{2^{p(m+\gamma+k)-2}}\int_{\frac{1}{\varphi_t(\infty)}}^{\varphi_t(\infty)} (1+e^t)^{pm}\left(u-\frac{e^t-1}{e^t+1}\right)^{pm}\left(\frac{4e^t}{e^t+1}\right)^{p(k+\gamma)-2}|f^{(k)}(u)|^p du\\ &\quad + \frac{1}{(n!)^p}\int_0^\infty r^{pm}|e^{t(m+\gamma-\frac{1}{p})}\omega_{\varphi_t}^{2m+\gamma}(r) f^{(m)}(\varphi_t(r)) - f^{(m)}(r)|^p\, dr\\ &\quad \leq \frac{e^{-pt(\gamma+k)-t}}{2^{p(\gamma+k)-2}}\int_{\frac{1}{\varphi_t(\infty)}}^{\varphi_t(\infty)} u^{pm}\,|f^{(k)}(u)|^p du\\ &\quad + \frac{1}{(n!)^p}\int_0^\infty r^{pm}|e^{t(m+\gamma-\frac{1}{p})}\omega_{\varphi_t}^{2m+\gamma}(r) f^{(m)}(\varphi_t(r)) - f^{(m)}(r)|^p\, dr,\end{aligned}$$

since $\varphi_t(r) \to r$ and $\omega_{\varphi_t}(r) \to 1$ when $t \to 0$, by the Dominated Convergence Theorem we obtain that $\|R_{t,p}^{\gamma} f - f\|_{m,p}^p \to 0$ if $t \to 0$.

On $\mathcal{T}_p^{(m)}(t^m)$, a direct calculation shows that

$$\begin{aligned}(Cf)(r) &= \lim_{t\to 0}\frac{R_{t,p}^{\gamma} f(r) - f(r)}{t} = \lim_{t\to 0}\frac{e^{t(\gamma-\frac{1}{p})}\omega_{\varphi_t}^{\gamma}(r) f(\varphi_t(r)) - f(r)}{t}\\ &= \lim_{t\to 0} e^{t(\gamma-\frac{1}{p})}\omega_{\varphi_t}^{\gamma}(r)\Bigg[(\gamma-\frac{1}{p} f(\varphi_t(r)) + \omega_{\varphi_t}^{-1}(r)\frac{-2\gamma e^t(r+1)}{((e^t-1)r+e^t+1)^2}\, f(\varphi_t(r))\\ &\quad + f'(\varphi_t(r))\frac{2e^t(1-r^2)}{(e^t+1+r(e^t-1))^2}\Bigg]\\ &= \frac{1}{2}\left(\gamma-\frac{2}{p}-\gamma r\right) f(r) + \frac{1}{2}(1-r^2) f'(r).\end{aligned}$$

In a similar way to the proof of the Theorem 11, we obtain that

$$D(C) = \{f \in \mathcal{T}_p^{(m)}(t^m) \mid ((1-r^2)f)' - (\gamma-2)rf \in \mathcal{T}_p^{(m)}(t^m)\},$$

and $\{f \in \mathcal{T}_p^{(m+1)}(t^{m+1}) \mid f' \in \mathcal{T}_p^{(m)}(t^m))\} \subset D(C)$. □

Theorem 15 *For* $1 \leq p < \infty$, $m \in \mathbb{N} \cup \{0\}$, $p \geq \frac{2}{\gamma}$ *and the* C_0*-semigroup* $(R^{\gamma}_{t,p})_{t\geq 0}$ *whose infinitesimal generator is* $(C, D(C))$*. Then*

- *(i)* $\{z \in \mathbb{C} \mid \Re z < -m\} \subset \sigma_p(C)$.
- *(ii)* $\sigma(C) = \{z \in \mathbb{C} \mid \Re(z) \leq 0\}$.
- *(iii) The set* $\rho(C) = \mathbb{C}^+$ *and*

$$R(\lambda, C) f(r) =$$
$$2\frac{|r+1|^{\lambda-\gamma+\frac{1}{p}}}{|r-1|^{\lambda+\frac{1}{p}}} \int_{\Gamma_{1,r}} |s+1|^{-\lambda+\gamma-\frac{1}{p}-1} |s-1|^{\lambda+\frac{1}{p}-1} f(s)ds, \quad f \in \mathcal{T}^{(m)}_p(t^m),$$

for $r > 0$, $\lambda \in \mathbb{C}^+$ *and* $\Gamma_{1,r} = (1, r)$ *if* $r > 1$ *and* $\Gamma_{1,r} = (r, 1)$ *when* $0 < r < 1$.

Proof (i) Let $\lambda \in \mathbb{C}$ and $f \in \mathcal{T}^m_p(t^m)$ such that $Cf = \lambda f$. Then, f is a solution of the differential equation

$$(1 - r^2) f'(r) = (a + br) f(r), \quad r > 0,$$

where $a = -\gamma + \frac{2}{p} + 2\lambda$, $b = \gamma$. The solutions of this equations are functions

$$\begin{aligned} f_\lambda(r) &= c_1 (1+r)^{\frac{a-b}{2}} (1-r)^{-\frac{a+b}{2}} \chi_{(0,1)} + c_2 (1+r)^{\frac{a-b}{2}} (r-1)^{-\frac{a+b}{2}} \chi_{(1,\infty)} \\ &= c_1 \frac{(1+r)^{-\gamma+\frac{1}{p}+\lambda}}{(1-r)^{\frac{1}{p}+\lambda}} \chi_{(0,1)} + c_2 \frac{(1+r)^{-\gamma+\frac{1}{p}+\lambda}}{(r-1)^{\frac{1}{p}+\lambda}} \chi_{(1,\infty)} \end{aligned}$$

for constants c_1, c_2. From Example 1 we obtain that $f_\lambda \in \mathcal{T}^m_p(t^m)$ if and only if

- $c_2 = 0$ and $m + \frac{1}{p} + \Re\lambda < \frac{1}{p}$, it is, $\Re\lambda < -m$. or
- $c_1 = 0$; in $r = 1$, we have $\Re\lambda < -m$; and in $r = \infty$, $-\gamma p + 1 + \Re\lambda p - (1 + \Re\lambda p) < -1$, it is $\gamma > \frac{1}{p}$.

(ii) $m \in \mathbb{N}$ and $\lambda \in \mathbb{C}$ such that $-m < \Re\lambda < 0$. We choose n with $\Re\lambda - n - 1 < -m \leq \Re\lambda + n$ and write

$$f(r) = \frac{(1+r)^{\lambda-\gamma+\frac{1}{p}-n-1}}{(r-1)^{\lambda+\frac{1}{p}-n-1}} \in \mathcal{T}^m_p(t^m).$$

If $f = (\lambda - C)g$ is not zero then $g = (\lambda - C)^{-1} f \in \mathcal{T}^m_p(t^m)$ and $g \neq 0$. The equation

$$2f = (2\lambda - \gamma + \frac{2}{p} + \gamma r)\, g - (1 - r^2) g',$$

has solution

$$g(r) = g^*(r) \left(M - \int_1^r \frac{2f(s)}{g^*(s)(s^2-1)} ds \right) \quad \text{for } r > 1,$$

where $g^*(r) = \dfrac{(1+r)^{-\gamma+\frac{1}{p}+\lambda}}{(r-1)^{\frac{1}{p}+\lambda}}\chi_{(1,+\infty)}$ and M a constant. By a direct calculation

$$\frac{f(s)}{g^*(s)(s^2-1)} = \frac{(1+s)^{-\gamma+\frac{1}{p}+\lambda-n-1}}{(s-1)^{\frac{1}{p}+\lambda-n-1}}\frac{(s-1)^{\frac{1}{p}+\lambda}}{(1+s)^{-\gamma+\frac{1}{p}+\lambda}(s^2-1)}$$
$$= \frac{(1+s)^{-n-1}}{(s-1)^{-n-1}(s-1)(s+1)} = \frac{(s-1)^n}{(s+1)^{n+2}}.$$

Hence

$$\int_1^\infty \frac{2f(s)}{g^*(s)(s^2-1)}ds = 2\int_1^\infty \frac{(s-1)^n}{(s+1)^{n+2}}dr = 2\int_0^\infty \frac{x^n}{(x+2)^{n+2}}dx$$
$$= 2\frac{n}{n+1}\int_0^\infty \frac{x^{n-1}}{(x+2)^{n+1}}dx = \frac{n}{n+1}\int_0^\infty \frac{\left(\frac{x}{2}\right)^{n-1}}{\left(\frac{x}{2}+1\right)^{n+1}}d\frac{x}{2}$$
$$= \frac{n}{n+1}\int_0^\infty \frac{v^{n-1}}{(v+1)^{n+1}}dv = \frac{n}{n+1}\frac{\Gamma(n)\Gamma(2)}{\Gamma(n+2)} = \frac{1}{(n+1)^2}$$

and

$$\left|M - \int_1^\infty \frac{2f(s)}{g^*(s)(s^2-1)}dr\right| \geq \left||M| - \int_1^\infty \frac{2f(s)}{g^*(s)(s^2-1)}ds\right| \geq \left||M| - \frac{1}{(n+1)^2}\right| \geq K,$$

taking $\left||M| - \frac{1}{(n+1)^2}\right| > 0$ and K a positive constant. Hence $|g| \geq K|g^*|$, and we conclude that this function does not belong to $L^p(\mathbb{R}^+)$, $g \notin \mathcal{T}_p^m(t^m)$ and $\lambda \in \sigma(C)$.

(iii) Take $\lambda \in \mathbb{C}^+$,

$$R(\lambda, C)f(r) = \int_0^\infty e^{-\lambda t} R^\gamma_{t,p} f(r)dt$$
$$= 2^\gamma \int_0^\infty e^{-\lambda t}\frac{e^{t(\gamma-\frac{1}{p})}}{(e^t+1+r(e^t-1))^\gamma} f\left(\frac{(1+e^t)r+e^t-1}{e^t+1+r(e^t-1)}\right)dt$$

with the change of variable $s = \dfrac{(1+e^t)r+e^t-1}{e^t+1+r(e^t-1)}$, we obtain that

$$ds = -\frac{1}{2}(s+1)(s-1)\,dt,\ \ e^t = \frac{(r-1)(s+1)}{(s-1)(r+1)},\ \text{and}\ \ e^t+1+r(e^t-1) = 2\frac{r-1}{s-1}.$$

Replacing in the integral

$$\begin{aligned} R(\lambda, C)f(r) &= 2^{\gamma}\int_0^{\infty} e^{-\lambda t}\frac{e^{t(\gamma-\frac{1}{p})}}{(e^t+1+r(e^t-1))^{\gamma}} f\left(\frac{(1+e^t)r+e^t-1}{e^t+1+r(e^t-1)}\right)dt \\ &= 2^{\gamma}\int_{\Gamma_{1,r}} \left|\frac{(r-1)(s+1)}{(s-1)(r+1)}\right|^{-\lambda+\gamma-\frac{1}{p}} \frac{1}{2^{\gamma}}\left|\frac{s-1}{r-1}\right|^{\gamma}\frac{2}{|s-1|\,|s+1|}f(s)ds \\ &= 2\frac{|r+1|^{\lambda-\gamma+\frac{1}{p}}}{|r-1|^{\lambda+\frac{1}{p}}}\int_{\Gamma_{1,r}} |s+1|^{-\lambda+\gamma-\frac{1}{p}-1}|s-1|^{\lambda+\frac{1}{p}-1}f(s)ds. \end{aligned}$$

Note that

$$\begin{aligned} R(\lambda, C)f(1) &= \lim_{r\to 1} 2\frac{|r+1|^{\lambda-\gamma+\frac{1}{p}}}{|r-1|^{\lambda+\frac{1}{p}}}\int_{\Gamma_{1,r}} |s+1|^{-\lambda+\gamma-\frac{1}{p}-1}|s-1|^{\lambda+\frac{1}{p}-1}f(s)ds \\ &= \frac{f(1)}{\lambda+\frac{1}{p}} \end{aligned}$$

whenever this limit exists. □

4 Cesàro-like operators subordinated to weight Koopman semigroups

Let $(X, \|\ \ \|)$ be a Banach space and $T = (T_t)_{t>0}$ a uniformly bounded C_0-semigroup on $(X, \|\ \ \|)$. Then we may define the bounded operator

$$C_{\mu,\nu}(x) := \int_0^{\infty} e^{-\mu t}(1-e^{-t})^{\nu-1}T_t(x)dt, \qquad x \in X,$$

for $\mu, \nu > 0$. This point of view has been followed in [1] and [14] for concrete C_0-semigroups in particular function spaces $(X, \|\ \ \|)$. In this section, we identify the operator $C_{\mu,\nu}$ for the three weight Koopman semigroups $(T_{t,p})_{t>0}$, $(S_{t,p})_{t>0}$ and $(R_{t,p})_{t>0}$ studied in the section above. These integral operators are connected with the Chen fractional integral, as we have commented in the Introduction. We estimate norms of these Cesàro-like operators and identify their spectrum sets.

4.1 Cesàro-like operators subordinated to $(T_{t,p})_{t>0}$

Given $\mu, \nu \in \mathbb{R}$, we consider the integral operator

$$\mathcal{C}_{\mu,\nu}f(r) := \frac{1}{|r-1|^{\mu+\nu-1}}\int_{\Gamma_{1,r}} |s-1|^{\mu-1}|r-s|^{\nu-1}f(s)ds, \qquad r > 0,$$

whenever this integral converges and $\Gamma_{1,r} := (1, r)$ for $r > 1$ and $\Gamma_{1,r} := (r, 1)$ in the case $0 < r < 1$.

Theorem 16 *For $p \geq 1$, $\mu, \nu > 0$,* *(i) If* $f \in \mathcal{T}_p^{(\alpha)}(t^\alpha)$*, then*

$$\mathcal{C}_{\mu+\frac{1}{p},\nu} f(r) = \int_0^\infty e^{-\mu t}(1-e^{-t})^{\nu-1} T_{t,p} f(r) dt, \quad r \geq 0, \tag{8}$$

where the semigroup $(T_{t,p})_{t\geq 0}$ *is given in Definition in 7.*

(ii) The operator $\mathcal{C}_{\mu+\frac{1}{p},\nu}$ *is a bounded operator on* $\mathcal{T}_p^{(\alpha)}(t^\alpha)$ *for* $\alpha \geq 0$ *and*

$$\beta(\mu+\alpha,\nu) \leq ||\mathcal{C}_{\mu+\frac{1}{p},\nu}|| \leq \beta(\mu,\nu).$$

In the particular case of $\alpha = 0$*, on* $L^p(\mathbb{R}^+)$*, the equality*

$$||\mathcal{C}_{\mu+\frac{1}{p},\nu}|| = \beta(\mu,\nu) \quad \textit{holds.}$$

(iii) The spectrum of $\mathcal{C}_{\mu+\frac{1}{p},\nu}$ *on* $\mathcal{B}(\mathcal{T}_p^{(\alpha)}(t^\alpha))$ *is the set*

$$\sigma(\mathcal{C}_{\mu+\frac{1}{p},\nu}) = \overline{\left\{\beta(\mu+z,\nu) : \Re z \geq 0\right\}}.$$

Proof (i) Since $\|T_{t,p}\| \leq 1$, (Theorem 7), we take $\mu, \nu > 0$ to get that

$$\begin{aligned}
&\int_0^\infty e^{-\mu t}(1-e^{-t})^{\nu-1} T_{t,p} f(r) dt \\
&= \int_0^\infty e^{-(\mu+\frac{1}{p})t}(1-e^{-t})^{\nu-1} f(e^{-t}r + 1 - e^{-t}) dt \\
&= \frac{1}{|r-1|^{\mu+\frac{1}{p}+\nu-1}} \int_{\Gamma_{1,r}} |s-1|^{\mu+\frac{1}{p}-1} |r-s|^{\nu-1} f(s) ds = \mathcal{C}_{\mu+\frac{1}{p},\nu} f(r),
\end{aligned}$$

where we change of variable $s = e^{-t}r + 1 - e^{-t}$ and the equality (8) is proved.
(ii) By the item (i), the operator $\mathcal{C}_{\mu+\frac{1}{p},\nu} f$ is well defined and is a bounded operator on $\mathcal{T}_p^{(\alpha)}(t^\alpha)$ for $p \geq 1$: taken $f \in \mathcal{T}_p^{(\alpha)}(t^\alpha)$, then

$$||\mathcal{C}_{\mu+\frac{1}{p},\nu} f||_{\alpha,p} \leq \int_0^\infty (1-e^{-t})^{\nu-1} e^{-\mu t} ||T_{t,p} f||_{\alpha,p} dt \leq \beta(\mu,\nu) ||f||_{\alpha,p}.$$

Now, we consider the eigenfunctions $f_\lambda(s) = (1-r)^{-\lambda-\frac{1}{p}} \chi_{(0,1)}(r)$ which belong to $\mathcal{T}_p^{(\alpha)}(t^\alpha)$ in the case $\lambda < -\alpha$ (see Theorem 9 (i)). Then

$$\mathcal{C}_{\mu+\frac{1}{p},\nu} f_\lambda(r) = \int_0^\infty e^{-(\mu-\lambda)t}(1-e^{-t})^{\nu-1} dt f_\lambda(r) = \beta(\mu-\lambda,\nu) f_\lambda(r),$$

for any $\lambda < -\alpha$, and we conclude that $\beta(\mu+\alpha,\nu) \leq ||\mathcal{C}_{\mu+\frac{1}{p},\nu}||$. On $L^p(\mathbb{R}^+)$, we obtain the equality $||\mathcal{C}_{\mu+\frac{1}{p},\nu}|| = \beta(\mu,\nu)$.

(iii) We consider the Hille-Phillips functional calculus $\mathcal{L}(\cdot)(-A) : L^1(\mathbb{R}^+) \to \mathcal{B}(X)$ given by

$$\mathcal{L}(h)(-A)f = \int_0^\infty h(t)T_{t,p}f\,dt, \quad h \in L^1(\mathbb{R}_+),\ f \in \mathcal{T}_p^{(\alpha)}(t^\alpha).$$

By the item (i), we have that $\mathcal{C}_{\mu+\frac{1}{p},\nu}f = \mathcal{L}(h_{\mu,\nu})(-A)$, where $h_{\mu,\nu}(t) := e^{-\mu t}(1-e^{-t})^{\nu-1}$ for $t > 0$. Observe that $h_{\mu,\nu} \in L^1(\mathbb{R}^+)$, and

$$\mathcal{L}(h_{\mu,\nu})(z) = \beta(\mu+z,\nu) := g_{\mu,\nu}(z), \quad z \in \overline{\mathbb{C}_+},\ \mu,\nu > 0,$$

where $\mathcal{L}$ denotes the Laplace transform, and $\mathcal{L}(h_{\mu,\nu}) \in C_0(\mathbb{R}_+) \cap \mathcal{H}_0(\mathbb{C}_+)$.

Since the function Γ is a meromorphic function with poles in $\{0,-1,-2,\cdots\}$, we may extend the function $g_{\mu,\nu}$ in an holomorphic way to $\mathbb{C}\backslash(-\infty,0)$. As $g_{\mu,\nu}(0) = \beta(\mu,\nu)$ and $g_{\mu,\nu}$ is holomorphic in a neighborhood of 0, then $g_{\mu,\nu}$ has finite polynomial limit at 0. Also, by [7],$\lim_{|z|\to\infty} g_{\mu,\nu}(z) = 0$ with $z \in \mathbb{C}\backslash(-\infty,0)$ so $g_{\mu,\nu}$ has finite polynomial limit at ∞. Using [12, Lemma 2.2.3] we have that $g_{\mu,\nu} \in \mathcal{E}_{\varphi_0}$, where $\mathcal{E}_{\varphi_0}$ denotes the extended Dunford-Riesz class. Therefore, since $-A$ is a sectorial operator of angle $\frac{\pi}{2}$ and A is injective ($0 \notin \sigma_p(A)$) we can apply the spectral mapping theorem [12, Theorem 2.7.8] and we get

$$\sigma(\mathcal{C}_{\mu+\frac{1}{p},\nu}) = \sigma(g_{\mu,\nu}(-A)) = \overline{g_{\mu,\nu}(\sigma(-A))} = \overline{\left\{\beta(\mu+z,\nu) : \Re z \geq 0\right\}},$$

and we finish the proof. □

Remark 7 An natural extension of Theorem 16 holds for $\Re\mu, \Re\nu > 0$, due to the function $h_{\mu,\nu}(t) := e^{-\mu t}(1-e^{-t})^{\nu-1}$ belongs to $L^1(\mathbb{R}^+)$ for $\Re\mu, \Re\nu > 0$.

In the Fig. 1, we represent certain spectrum sets of $\sigma(\mathcal{C}_{\mu+\frac{1}{p},\nu})$ for $p \geq 1$.

4.2 Cesàro-like operators subordinated to $(S^\gamma_{t,p})_{t>0}$

Given $\mu,\nu,\gamma \in \mathbb{R}$, we consider the integral operator

$$\mathfrak{C}^\gamma_{\mu,\nu}f(r) := \frac{r^\mu}{|r-1|^{\mu+\nu+\gamma-1}} \int_{\Gamma_{1,r}} \frac{|s-1|^{\mu+\gamma-1}}{s^{\mu+\nu}} |r-s|^{\nu-1} f(s)ds, \qquad r > 0,$$

whenever this integral converges and $\Gamma_{1,r} := (1,r)$ for $r > 1$ and $\Gamma_{1,r} := (r,1)$ in the case $0 < r < 1$.

Theorem 17 *Take $p \geq 1$, $\mu,\nu > 0$, $\gamma \geq \frac{2}{p}$ and the semigroup $(S^\gamma_{t,p})_{t\geq 0}$ given in Definition in 7.*

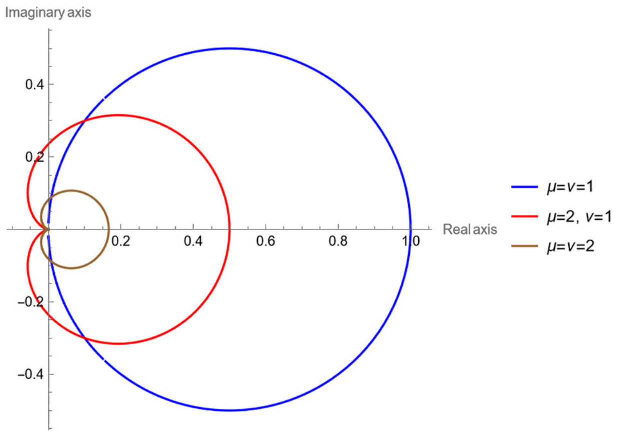

Fig. 1 The boundary of spectrum sets $\sigma(\mathcal{C}_{\mu+\frac{1}{p},\nu})$

(i) *If* $f \in \mathcal{T}_p^{(m)}(t^m)$*, then*

$$\mathfrak{C}^{\gamma}_{\mu-\frac{1}{p},\nu} f(r) = \int_0^\infty e^{-\mu t}(1-e^{-t})^{\nu-1} S^{\gamma}_{t,p} f(r)dt, \quad r \geq 0, \tag{9}$$

(ii) *The operator* $\mathfrak{C}^{\gamma}_{\mu-\frac{1}{p},\nu}$ *is a bounded operator on* $\mathcal{T}_p^{(m)}(t^m)$ *for* $m \in \mathbb{N} \cup \{0\}$ *and*

$$\beta(\mu + m + \gamma - \frac{2}{p}, \nu) \leq ||\mathfrak{C}^{\gamma}_{\mu-\frac{1}{p},\nu}|| \leq \sup_{t\geq 0} \|S^{\gamma}_{t,p}\| \beta(\mu,\nu).$$

On $L^p(\mathbb{R}^+)$*, and* $\gamma = \frac{2}{p}$*, the equality* $||\mathfrak{C}^{\frac{2}{p}}_{\mu-\frac{1}{p},\nu}|| = \beta(\mu,\nu)$ *holds.*

(iii) *The spectrum of* $\mathfrak{C}^{\gamma}_{\mu-\frac{1}{p},\nu}$ *on* $\mathcal{B}(\mathcal{T}_p^{(m)}(t^m))$ *is the set*

$$\sigma(\mathfrak{C}^{\gamma}_{\mu-\frac{1}{p},\nu}) = \overline{\left\{\beta(\mu+z,\nu) : \Re z \geq 0\right\}},$$

Proof (i) Since $\sup_{t\geq 0} \|S^{\gamma}_{t,p}\| < +\infty$, (Theorem 10), we take $\mu, \nu > 0$ to get that

$$\begin{aligned}
&\int_0^\infty e^{-\mu t}(1-e^{-t})^{\nu-1} S^{\gamma}_{t,p} f(r)dt \\
&= \int_0^\infty e^{-(\mu-\frac{1}{p})t} \frac{(1-e^{-t})^{\nu-1}}{(1+r(e^t-1))^{\gamma}} f\left(\frac{e^t r}{1+r(e^t-1)}\right) dt \\
&= \frac{r^{\mu-\frac{1}{p}}}{|r-1|^{\mu-\frac{1}{p}+\nu+\gamma-1}} \int_{\Gamma_{1,r}} |s-1|^{\mu-\frac{1}{p}+\gamma-1} \frac{|r-s|^{\nu-1}}{s^{\mu-\frac{1}{p}+\nu}} f(s)ds = \mathfrak{C}^{\gamma}_{\mu-\frac{1}{p},\nu} f(r),
\end{aligned}$$

where we change of variable $s = \frac{e^t r}{1+r(e^t-1)}$ and the equality (9) is proved.
(ii) By the first item, the operator $\mathfrak{C}^{\gamma}_{\mu-\frac{1}{p},\nu} f$ is well defined and is a bounded operator on $\mathcal{T}_p^{(m)}(t^m)$ for $p \geq 1$: taken $f \in \mathcal{T}_p^{(m)}(t^m)$, then

$$||\mathfrak{C}^{\gamma}_{\mu-\frac{1}{p},\nu} f||_{m,p} \leq \int_0^\infty (1-e^{-t})^{\nu-1} e^{-\mu t} ||S^{\gamma}_{t,p} f||_{m,p} dt \leq \sup_{t\geq 0} \|S^{\gamma}_{t,p}\| \beta(\mu,\nu) ||f||_{m,p}.$$

Now we take the eigenfunctions $f_\lambda(r) = \frac{r^{\lambda-\frac{1}{p}}}{(r-1)^{\lambda-\frac{1}{p}+\gamma}} \chi_{(1,\infty)}(r)$ which belongs to $\mathcal{T}_p^{(m)}(t^m)$ for $\lambda < -(m+\gamma-\frac{2}{p})$ (Theorem 12 (i)). Then

$$\mathfrak{C}^{\gamma}_{\mu-\frac{1}{p},\nu} f_\lambda(r) = \int_0^\infty e^{-(\mu-\lambda)t}(1-e^{-t})^{\nu-1} dt f_\lambda(r) = \beta(\mu-\lambda,\nu) f_\lambda(r), \quad r \geq 0,$$

for $\lambda < -(m+\gamma-\frac{2}{p})$. Then we conclude that $\beta(\mu+m+\gamma-\frac{2}{p},\nu) \leq ||\mathfrak{C}^{\gamma}_{\mu-\frac{1}{p},\nu}||$.

On $L^p(\mathbb{R}^+)$, we have that $\|S^{\gamma}_{t,p}\| \leq 1$ for $t \geq 0$ and $p \geq 1$, see (7), and

$$\beta(\mu+\gamma-\frac{2}{p},\nu) \leq ||\mathfrak{C}^{\gamma}_{\mu-\frac{1}{p},\nu}|| \leq \beta(\mu,\nu).$$

For $\gamma = \frac{2}{p}$, we obtain $||\mathfrak{C}^{\frac{2}{p}}_{\mu-\frac{1}{p},\nu}|| = \beta(\mu,\nu)$.
(iii) The proof of this item runs similar ideas as in Theorem 16 (iii). ◻

4.3 Cesàro-like operators subordinated to $(R^{\gamma}_{t,p})_{t>0}$

Given $\mu, \nu \in \mathbb{R}$, we consider the integral operator

$$\begin{aligned}\mathbf{C}^{\gamma}_{\mu,\nu} f(r) &:= 2^\nu \frac{|r+1|^{\mu-\gamma}}{|r-1|^{\mu+\nu-1}} \int_{\Gamma_{1,r}} \frac{|s-1|^{\mu-1}}{|s+1|^{\mu+\nu-\gamma}} |r-s|^{\nu-1} f(s) ds, \qquad r > 0,\\ &= \frac{|\frac{r}{2}+\frac{1}{2}|^{\mu-\gamma}}{|r-1|^{\mu+\nu-1}} \int_{\Gamma_{1,r}} \frac{|s-1|^{\mu-1}}{|\frac{s}{2}+\frac{1}{2}|^{\mu+\nu-\gamma}} |r-s|^{\nu-1} f(s) ds\end{aligned}$$

whenever this integral converges and $\Gamma_{1,r} := (1,r)$ for $r > 1$ and $\Gamma_{1,r} := (r,1)$ in the case $0 < r < 1$.

Theorem 18 *Take $p \geq 1$, $\mu,\nu > 0$, $\gamma \geq \frac{2}{p}$ and the semigroup $(R^{\gamma}_{t,p})_{t\geq 0}$ given in Definition in 7.*

(i) If $f \in \mathcal{T}_p^{(m)}(t^m)$, then

$$\mathbf{C}^{\gamma}_{\mu+\frac{1}{p},\nu} f(r) = \int_0^\infty e^{-\mu t}(1-e^{-t})^{\nu-1} R^{\gamma}_{t,p} f(r) dt, \quad r \geq 0.$$

(ii) The operator $\mathbf{C}^{\gamma}_{\mu+\frac{1}{p},\nu}$ *is a bounded operator on* $\mathcal{T}_p^{(m)}(t^m)$ *for* $m \in \mathbb{N} \cup \{0\}$ *and*

$$\beta(\mu+m,\nu) \leq ||\mathfrak{C}^{\gamma}_{\mu+\frac{1}{p},\nu}|| \leq \sup_{t\geq 0} \|R^{\gamma}_{t,p}\| \beta(\mu,\nu).$$

On $L^p(\mathbb{R}^+)$*, and* $\gamma = \frac{2}{p}$*, the equality* $||\mathbf{C}^{\frac{2}{p}}_{\mu+\frac{1}{p},\nu}|| = \beta(\mu,\nu)$ *holds.*

(iii) The spectrum of $\mathbf{C}^{\gamma}_{\mu+\frac{1}{p},\nu}$ *on* $\mathcal{B}(\mathcal{T}_p^{(m)}(t^m))$ *is the set*

$$\sigma(\mathbf{C}^{\gamma}_{\mu+\frac{1}{p},\nu}) = \overline{\left\{\beta(\mu+z,\nu) : \Re z \geq 0\right\}},$$

Proof (i) For $\mu, \nu > 0$, we have that

$$\begin{aligned}
&\int_0^\infty e^{-\mu t}(1-e^{-t})^{\nu-1} R^{\gamma}_{t,p} f(r) dt \\
&= 2^{\gamma} \int_0^\infty e^{-\mu t}(1-e^{-t})^{\nu-1} \frac{e^{t(\gamma-\frac{1}{p})}}{(e^t+1+r(e^t-1))^{\gamma}} f\left(\frac{(1+e^t)r+e^t-1}{e^t+1+r(e^t-1)}\right) dt \\
&= 2^{\nu} \frac{|r+1|^{\mu+\frac{1}{p}-\gamma}}{|r-1|^{\mu+\frac{1}{p}+\nu-1}} \int_{\Gamma_{1,r}} \frac{|s-1|^{\mu+\frac{1}{p}-1}}{|s+1|^{\mu+\frac{1}{p}+\nu-\gamma}} |r-s|^{\nu-1} f(s) ds = \mathbf{C}^{\gamma}_{\mu+\frac{1}{p},\nu} f(r)
\end{aligned}$$

(ii) By the first item, the operator $\mathbf{C}^{\gamma}_{\mu+\frac{1}{p},\nu} f$ is well defined and is a bounded operator on $\mathcal{T}_p^{(m)}(t^m)$ for $p \geq 1$: taken $f \in \mathcal{T}_p^{(m)}(t^m)$, then

$$||\mathbf{C}^{\gamma}_{\mu+\frac{1}{p},\nu} f||_{m,p} \leq \sup_{t\geq 0} \|R^{\gamma}_{t,p}\| \beta(\mu,\nu) ||f||_{m,p}.$$

Now we take the eigenfunctions $f_\lambda(r) = \frac{(1+r)^{-\gamma+\frac{1}{p}+\lambda}}{(1-r)^{\frac{1}{p}+\lambda}} \chi_{(0,1)}$ which belongs to $\mathcal{T}_p^{(m)}(t^m)$ for $\lambda < -m$ (Theorem 15). Then

$$\mathfrak{C}^{\gamma}_{\mu-\frac{1}{p},\nu} f_\lambda(r) = \int_0^\infty e^{-(\mu-\lambda)t}(1-e^{-t})^{\nu-1} dt f_\lambda(r) = \beta(\mu-\lambda,\nu) f_\lambda(r), \quad r \geq 0,$$

for $\lambda < -m$. Then we conclude that $\beta(\mu+m,\nu) \leq ||\mathfrak{C}^{\gamma}_{\mu-\frac{1}{p},\nu}||$. On $L^p(\mathbb{R}^+)$ and $\gamma = \frac{2}{p}$. we obtain the equality $||\mathbf{C}^{\frac{2}{p}}_{\mu+\frac{1}{p},\nu}|| = \beta(\mu,\nu)$ see Theorem 13.

(iii) The proof of this item runs similar ideas than in Theorem 16 □

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Declarations

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