Computing the homology of universal covers via effective homology and discrete vector fields [1]

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Abstract

Effective homology techniques allow us to compute homology groups of a wide family of topological spaces. By the Whitehead tower method, this can also be used to compute higher homotopy groups. However, some of these techniques (in particular, the Whitehead tower) rely on the assumption that the starting space is simply connected. For some applications, this problem could be circumvented by replacing the space by its universal cover, which is a simply connected space that shares the higher homotopy groups of the initial space. In this paper, we formalize a simplicial construction for the universal cover, and represent it as a twisted Cartesian product.

As we show with some examples, the universal cover of a space with effective homology does not necessarily have effective homology in general. We show two independent sufficient conditions that can ensure it: one is based on a nilpotency property of the fundamental group, and the other one on discrete vector fields.

Some examples showing our implementation of these constructions in both SageMath and Kenzo are shown, together with an approach to compute the homology of the universal cover when the group is Abelian even in some cases where there is no effective homology, using the twisted homology of the space.

1. Introduction

The *effective homology* method, implemented in the computer algebra system Kenzo (Dousson et al., 1999), was developed by Francis Sergeraert in (Sergeraert, 1994) with the aim of calculating homology and homotopy groups of complicated topological spaces, allowing in particular to work with spaces of infinite type. Topological spaces are represented in Kenzo by means of *simplicial sets*, a combinatorial structure that generalizes the notion of simplicial complex. The computation of homology groups is done in Kenzo by means of chain equivalences between the initial simplicial set and a finite type chain complex in which the homology can be computed by matrix diagonalization operations. For the computation of homotopy groups, the Whitehead tower method (Whitehead, 1952) is used.

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Most algorithms in Kenzo and, in general, in the effective homology method, require the simplicial sets to be 1-reduced (that is, with only one vertex and the only edge is a degeneration of the vertex). In particular, this condition is necessary for the application of the Kenzo's implementation of the Whitehead tower method. In a previous work (Cuevas-Rozo et al., 2021), a Kenzo interface and an optional package for SageMath (The Sage Developers, 2023) were developed. In that work, we also developed and integrated an algorithm to compute homotopy groups of simply connected simplicial sets that are not necessarily 1-reduced. However, the condition that the simplicial set be simply connected is a necessary condition for the Whitehead tower method (Whitehead, 1952).

A *cover* of a connected topological space X is a topological space Y with a map $f : Y \to X$ such that for every $x \in X$, there exists an open neighborhood U of x such that $f^{-1}(U)$ is a disjoint union of homeomorphic copies of U. If Y is simply connected, then Y is said to be a *universal cover* of Y, and satisfies $\pi_i(Y) \cong \pi_i(X)$ for all $i \geq 2$. The universal cover of a topological space X can be considered as an approach to study X by lifting paths and other geometric objects to the simpler universal cover. This simplifies the resolution of certain topological and geometric issues on X, as they can be streamlined to the examination of the universal cover. In particular, the universal cover can be useful in applying the Whitehead tower method.

In this work, we present algorithms for computing a simplicial model of the universal cover of a space, represented as a twisted Cartesian product, and its effective homology by using homological perturbation theory. The effective homology of the universal cover can be determined with our algorithms when the effective homology of the initial space satisfies one of two independent conditions: one is based on a nilpotency property of the fundamental group, and the other one on discrete vector fields. We also present an approach to compute the homology of the universal cover when the fundamental group is Abelian even in some cases where there is no effective homology, using the twisted homology of the space. Our algorithms have been implemented in the computer algebra systems SageMath (The Sage Developers, 2023) and Kenzo using the Kenzo interface and the optional package for SageMath that we developed in (Cuevas-Rozo et al., 2021). In particular, Algorithm 1 for constructing the universal cover of a simplicial set of finite type with finite fundamental group is implemented In SageMath; Algorithms 2 and 3 for computing the universal cover of simplicial sets of infinite type and their effective homology are implemented in Kenzo. Algorithm 4 for computing the homology of the universal cover in the Abelian case via twisted homology is implemented in SageMath.

This work presents a revised and extended version of our previous conference paper (Marco-Buzunáriz et al., 2023a). In addition to presenting more details, with the necessary preliminaries and proofs of all of our results in Sections 3 and 4 (which were only briefly sketched in (Marco-Buzunáriz et al., 2023a)), we have included two new sections with new results. On the one hand, we have studied the behavior of some algebraic topology constructors and tools with respect to our construction of the universal cover and its effective homology, considering in particular Cartesian products and discrete vector fields. On the other hand, we have also examined the case of Abelian infinite fundamental group, when the universal cover might not have effective homology.

The paper is structured as follows. In the next section, we present preliminary definitions and results about simplicial sets, effective homology and universal covers. In Section 3, our simplicial construction of the universal cover as a twisted Cartesian product is described. Section 4 shows the algorithms for computing the effective homology of universal covers, making use of homological perturbation theory. Then, we explain the implementation of the algorithms and show some didactic examples in Section 5. Cartesian products and discrete vector fields and their relation with universal covers and their effective homology is studied in Section 6, and the case of Abelian infinite fundamental group is considered in Section 7. Finally, we describe the conclusions and possible further work in Section 8.

2. Preliminaries

2.1. Simplicial sets

In this section, we introduce the definition and some basic constructions of simplicial sets, following (May, 1967).

Definition 2.1. Let $\mathcal{D}$ be a category. The category $s\mathcal{D}$ of *simplicial objects in* $\mathcal{D}$ is defined as follows. An object $X \in s\mathcal{D}$ consists of

- for each integer $n \geq 0$, an object $X_n \in \mathcal{D}$;
- for every pair of integers (i, n) such that $0 \leq i \leq n$, *face* and *degeneracy* maps $\partial_i : X_n \to X_{n-1}$ and $s_i : X_n \to X_{n+1}$ (which are morphisms in the category $\mathcal{D}$) satisfying the *simplicial identities*:

$$
\begin{aligned}
\partial_i \partial_j &= \partial_{j-1} \partial_i && \text{if } i < j \\
s_i s_j &= s_{j+1} s_i && \text{if } i \leq j \\
\partial_i s_j &= \begin{cases} s_{j-1} \partial_i & \\ \mathrm{Id} & \\ s_j \partial_{i-1} & \end{cases} && \begin{matrix} \text{if } i < j \\ \text{if } i = j, j+1 \\ \text{if } i > j+1 \end{matrix}
\end{aligned}
$$

Let X and Y be simplicial objects. A *simplicial map* (or *simplicial morphism*) $f : X \to Y$ consists of maps $f_n : X_n \to Y_n$ (which are morphisms in $\mathcal{D}$) that commute with the face and degeneracy operators, that is $f_{n-1} \partial_i = \partial_i f_n$ and $f_{n+1} s_i = s_i f_n$ for all $0 \leq i \leq n$.

If $\mathcal{D}$ is a subcategory of sets (which we will assume from now on), the elements of X_n are called the *n-simplices* of X. In this setting, if $a = \partial_i(b)$, we say that a is the *i'th face of b*. Analogously, we say that $s_i(b)$ is the *i'th degeneration of b*.

Definition 2.2. An n-simplex $x \in X_n$ is *degenerate* if $x = s_j y$ for some $y \in X_{n-1}$ and some $0 \leq j < n$; otherwise, x is called *non-degenerate*. We denote the set of degenerate n-simplices of X by X_n^D and the set of non-degenerate n-simplices of X by X_n^{ND}.

A *simplicial set* is a simplicial object in the category of sets. A *simplicial group* G is a simplicial object in the category of groups; in other words, it is a simplicial set where each G_n is a group and the face and degeneracy operators are group morphisms.

A simplicial set X has a canonically associated chain complex $C_*(X) = (C_n(X), d_n)$, where each chain group $C_n(X)$ is defined as the free $\mathbb{Z}$-module generated by X_n, and the differential $d_n : C_n(X) \to C_{n-1}(X)$ is defined as the alternating sum of faces, $d_n := \sum_{i=0}^{n} (-1)^i \partial_i$. In this work we will assume all the chain complexes associated with simplicial sets to be *normalized* (see (May, 1967, Ch. 5)), which intuitively means that only non-degenerate simplices are considered as generators of the chain groups.

Generalizing simplicial complexes, simplicial sets can be used to represent topological spaces: for every non-degenerate n-simplex in the simplicial set, we take a geometrical n-simplex (that is, the convex hull of $n+1$ points in general position in $\mathbb{R}^n$) and then glue them as follows. The i'th degeneration of a geometrical simplex spanned by $v_0, \dots, v_n$ is the same geometrical object, but considering that the i'th vertex is repeated (that is, the convex hull of $v_0, \dots, v_i, v_i, \dots, v_n$). The i'th face of a geometrical simplex spanned by $v_0, \dots, v_n$ is obtained by removing one of the vertices (that is, the span of $v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_n$). For two simplices σ_1, σ_2 such that $\partial_i \sigma_1 = \sigma_2$, we have an affine map that sends each vertex of the geometrical simplex of $\partial_i \sigma_1$ to the corresponding vertex of the geometrical simplex corresponding to σ_2. We identify the points in $\partial_i \sigma_1$ with the points in σ_2 by this linear map. The resulting topological space is called the *geometrical realization* of the simplicial set. The geometrical realization of a simplicial set X will be denoted by $|X|$. Analogously, given a simplex s, the corresponding geometrical simplex will also be denoted by $|s|$. The *n-skeleton* of a simplicial set X is the set of its simplices of dimension lower or equal than n. Analogously, the n-skeleton of $|X|$ is the subspace formed by points in the geometrical simplices of dimension lower or equal than n.

Example 1. *The* 2*-sphere is the geometrical realization of a simplicial set with only two non-degenerate simplices: one of dimension* 0 *(called v) and one of dimension* 2 *(called t), with $\partial_0 t = \partial_1 t = \partial_2 t = s_0 v$. That is, the three sides of the triangle are attached to the vertex.*

Definition 2.3. The *Cartesian product* $X \times Y$ of two simplicial sets X and Y is the simplicial set whose set of n-simplices is $(X \times Y)_n := X_n \times Y_n$, with coordinate-wise defined face and degeneracy maps: if $(x, y) \in (X \times Y)_n$, then

$$\partial_i(x, y) := (\partial_i x, \partial_i y), \qquad 0 \le i \le n;$$
$$s_i(x, y) := (s_i x, s_i y), \qquad 0 \le i \le n.$$

Example 2. *The unit interval can be represented as a simplicial set X with two vertices (let's denote them v_0, v_1) and one non-degenerate edge e. So the list of* 1*-simplices is formed by e and the degenerations $s_0 v_0, s_0 v_1$, and the list of* 2*-simplices is formed by the degenerations $s_0 s_0 v_0 = s_1 s_0 v_0$, $s_0 e$, $s_1 e$ and $s_0 s_0 v_1 = s_1 s_0 v_1$.*

In the simplicial set $X \times X$, the list of all of 0*-simplices is formed by the* 4 *pairs (v_0, v_0), (v_0, v_1), (v_1, v_0), (v_1, v_1). There would be* 9 *1-simplices formed by pairs of* 1*-simplices of X, but* 4 *of them would be degenerations of the* 0*-simplices; so the* 5 *non-degenerate simplices of $X \times X$ are $(s_0 v_0, e)$, $(s_0 v_1, e)$, $(e, s_0 v_0)$, $(e, s_0 v_1)$ and (e, e). Finally, there are only two non-degenerate* 2*-simplices: $(s_0 e, s_1 e)$ and $(s_1 e, s_0 e)$ (note that they are non-degenerate since they are not of the form $(s_i x, s_i y)$). This construction can be seen in Figure 1.*

Definition 2.4. A *twisting operator* from a simplicial set B to a simplicial group G is a map $\tau : B \to G$ of degree -1, that is a collection of maps $\tau = \{\tau_n : B_n \to G_{n-1}\}_{n \ge 1}$, satisfying the following identities, for any $n \ge 1$ and for any $b \in B_n$:

$$\partial_i(\tau b) = \tau(\partial_i b), \qquad 0 \le i < n-1,$$
$$\partial_{n-1}(\tau b) = \tau(\partial_n b)^{-1} \cdot \tau(\partial_{n-1} b),$$
$$s_i(\tau b) = \tau(s_i b), \qquad 0 \le i \le n-1,$$
$$e_n = \tau(s_n b),$$

where e_n is the identity element of G_n.

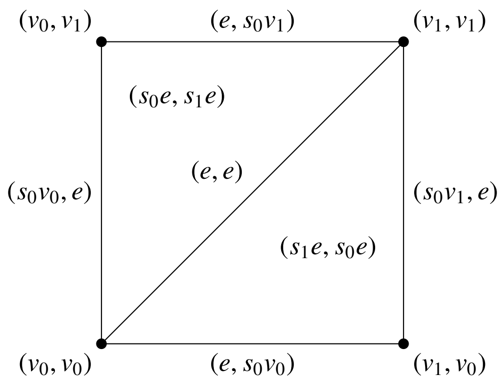

Figure 1: non-degenerate simplices of a product.

We defined twisting operators in a slightly different (but equivalent) way from (May, 1967), in order to agree with the definition implemented in the Kenzo system.

Definition 2.5. Given a simplicial group G, a simplicial set B and a twisting operator $\tau : B \to G$, the *twisted (Cartesian) product* $E(\tau) := G \times_\tau B$ is the simplicial set whose set of n-simplices is $E(\tau)_n = (G \times_\tau B)_n := G_n \times B_n$ and whose face and degeneracy maps are defined in the following way: if $(g, b) \in (G \times_\tau B)_n$, then

$$\begin{aligned} \partial_i(g, b) &:= (\partial_i g, \partial_i b), \qquad 0 \le i < n, \\ \partial_n(g, b) &:= (\tau(b) \cdot \partial_n g, \partial_n b); \\ s_i(g, b) &:= (s_i g, s_i b), \qquad 0 \le i \le n. \end{aligned}$$

It can be easily shown that the identities defining a twisting operator τ are equivalent to the simplicial identities of $G \times_\tau B$.

2.2. *Effective homology and homological perturbation theory*

We now present the main definitions and ideas of the effective homology method, introduced in (Sergeraert, 1994) and explained in depth in (Rubio and Sergeraert, 2002) and (Rubio and Sergeraert, 2006). We will also recall some results in homological perturbation theory that we will need in our work.

Definition 2.6. A *reduction* $\rho \equiv (D_* \Rrightarrow C_*)$ between two chain complexes D_* and C_* is a triple (f, g, h) where:

1. The components f and g are chain complex morphisms $f : \{f_n : D_n \to C_n\}_{n\in\mathbb{Z}}$ and $g = \{g_n : C_n \to D_n\}_{n\in\mathbb{Z}}$;
2. The component h is a homotopy operator $h = \{h_n : D_n \to D_{n+1}\}_{n\in\mathbb{Z}}$ (a graded group homomorphism of degree $+1$);
3. The following relations are satisfied:
   (a) $fg = \mathrm{id}_{C_*}$;

(b) $gf + d_{D_*}h + hd_{D_*} = \mathrm{id}_{D_*}$;

(c) $fh = 0$; $hg = 0$; $hh = 0$.

Remark 1. *These relations show that D_* is the direct sum of C_* and a contractible (and then, acyclic, see (Dold, 1995)) complex. In particular, this implies that the graded homology groups $H_*(D_*)$ and $H_*(C_*)$ are canonically isomorphic.*

Definition 2.7. A *(strong chain) equivalence* $\varepsilon \equiv (C_* \Longleftrightarrow E_*)$ between two complexes C_* and E_* is a triple (D_*, ρ, ρ') where D_* is a chain complex and ρ and ρ' are reductions from D_* over C_* and E_* respectively: $C_* \overset{\rho}{\Leftarrow} D_* \overset{\rho'}{\Rightarrow} E_*$.

Definition 2.8. An *effective* chain complex C_* is a free chain complex (i.e., a chain complex consisting of free $\mathbb{Z}$-modules) where each group C_n is finitely generated, and there is an algorithm that returns a $\mathbb{Z}$-base β_n in each degree n (for details, see Rubio and Sergeraert (2002)).

The homology groups of an effective chain complex C_* can easily be determined by means of some diagonalization algorithms on matrices (see Kaczynski et al. (2004)).

Definition 2.9. An *object with effective homology* is a triple (X, EC_*, ε) where X is an object (e.g. a simplicial set, a topological space) possessing a canonically associated free chain complex $C_*(X)$, EC_* is an effective chain complex and ε is an equivalence between $C_*(X)$ and EC_*, $C_*(X) \overset{\varepsilon}{\Longleftrightarrow} EC_*$.

Given a chain complex C_*, the *trivial reduction* $\mathrm{Id} = (f, g, h) : C_* \Rightarrow C_*$ is the reduction with $f = g = \mathrm{Id}$ and $h = 0$.

In the following proposition, we state a simple result (included in (Rubio and Sergeraert, 2006)) that describes the behavior of reductions with respect to the tensor product (see (Rubio and Sergeraert, 2006, Ch. 5) for the definition of the tensor product of two chain complexes).

Proposition 2.10. *Let $\rho = (f, g, h) : C_* \Rightarrow D_*$ and $\rho' = (f', g', h') : C'_* \Rightarrow D'_*$ be two reductions. Then a reduction $\rho'' := \rho \otimes \rho' = (f'', g'', h'') : C_* \otimes C'_* \Rightarrow D_* \otimes D'_*$ is given by:*

$$f'' := f \otimes f', \qquad g'' := g \otimes g', \qquad h'' := h \otimes \mathrm{Id}_{C'_*} + (gf) \otimes h'.$$

The notion of object with effective homology makes it possible to compute homology groups of complicated spaces by using effective complexes and computing their homology groups. The effective homology method is based on the following idea: Given some topological spaces $X_1, \ldots, X_n$, we consider a topological constructor Φ that produces a new topological space X. For instance, we can consider Φ as the Cartesian product of two simplicial sets X_1 and X_2 or the loop space of a simplicial set X_1. If effective homology versions of the spaces $X_1, \ldots, X_n$ are known, then an effective homology version of the space X can also be built, and this version allows us to compute the homology groups of X, even if X is not of finite type. Considering again our examples, if X_1 and X_2 have effective homology, then specific algorithms have been developed constructing the effective homology of the Cartesian product $X_1 \times X_2$ and the loop space ΩX (for the loop space, an additional hypothesis is needed: X_1 should be 1-reduced). See (Rubio and Sergeraert, 2006) for the algorithms constructing the effective homology of these particular cases. Other constructors for which effective homology have been determined are suspensions, twisted Cartesian products, cones, and pushouts. In this way, starting with objects that are of finite type (and then have *trivial* effective homology) or objects for which a reduction to a finite type object is known (for instance, the Eilenberg–MacLane space $K(\mathbb{Z}, 1)$, for which a reduction to the sphere S^1 can be defined, see (Rubio and Sergeraert, 2006)), one can build complicated objects that have effective homology. This method has been implemented in the Kenzo system (Dousson et al., 1999), a Common Lisp program devoted to Symbolic Computation in Algebraic Topology, which has made it possible to determine homology and homotopy groups of complicated spaces. The computation of homotopy groups of simply connected simplicial sets with effective homology is dealt with in Kenzo by means of the Whitehead tower method (Whitehead, 1952) and requires the construction of a sequence of fibrations involving spaces of infinite types (following the algorithm in (Real, 1996)).

Now, we introduce the main results in homological perturbation theory, which describe how a perturbation (a modification of the differential of a chain complex) transmits through a reduction.

Definition 2.11. Let $C_* = (C_n, d_n)_{n\in\mathbb{Z}}$ be a chain complex. A *perturbation* δ of the differential d is a family of morphisms $\delta = \{\delta_n : C_n \to C_{n-1}\}_{n\in\mathbb{Z}}$ such that the sum $d + \delta$ is again a differential, that is $(d + \delta)^2 = 0$ holds (meaning $(d_{n-1} + \delta_{n-1})(d_n + \delta_n) = 0$, for all $n \in \mathbb{Z}$).

We call $C'_* = (C_n, d_n + \delta_n)_{n\in\mathbb{Z}}$ the *perturbed* chain complex obtained from C_* by introducing the perturbation δ.

Theorem 2.12 (Trivial Perturbation Lemma, TPL)**.** *(Rubio and Sergeraert, 2006) Let $C_* = (C_n, d_{C_n})_{n\in\mathbb{Z}}$ and $D_* = (D_n, d_{D_n})_{n\in\mathbb{Z}}$ be two chain complexes, $\rho = (f, g, h) : C_* \Rightarrow\!\!\!\!\!\Rightarrow D_*$ a reduction, and δ_D a perturbation of the differential d_D. Then a reduction $\rho' = (f', g', h') : C'_* \Rightarrow\!\!\!\!\!\Rightarrow D'_*$ can be built, where:*

1. *$C'_* = (C_*, d_C + g\delta_D f)$ is the perturbed chain complex obtained from C_* by adding the perturbation $g\delta_D f$ to the differential map d_C;*
2. *$D'_* = (D_*, d_D + \delta_D)$ is the perturbed chain complex obtained from D_* by adding the perturbation δ_D to the differential d_D;*
3. *the maps of the new reduction $\rho' = (f', g', h')$ are given by $f' := f$, $g' := g$, $h' := h$.*

Theorem 2.13 (Basic Perturbation Lemma, BPL, Brown (1967))**.** *Let $C_* = (C_n, d_{C_n})_{n\in\mathbb{Z}}$ and $D_* = (D_n, d_{D_n})_{n\in\mathbb{Z}}$ be two chain complexes, $\rho = (f, g, h) : C_* \Rightarrow\!\!\!\!\!\Rightarrow D_*$ a reduction, and δ_C a perturbation of the differential d_C. Suppose that the composition $h\delta_C$ satisfies the following* nilpotency condition*: for every $x \in C_*$ there exists a non-negative integer $m = m(x) \in \mathbb{N}$ such that $(h\delta_C)^m(x) = 0$. Let us consider the operators φ and ψ given by*

$$\varphi := \sum_{i=0}^{\infty} (-1)^i (h\delta_C)^i, \qquad \psi := \sum_{i=0}^{\infty} (-1)^i (\delta_C h)^i.$$

Then a reduction $\rho' = (f', g', h') : C'_* \Rightarrow\!\!\!\!\!\Rightarrow D'_*$ can be built, where:

1. *$C'_* = (C_*, d_C + \delta_C)$ is the perturbed chain complex obtained from C_* by adding the perturbation δ_C to the differential d_C;*
2. *$D'_* = (D_*, d_D + \delta_D)$ is the perturbed chain complex obtained from D_* by adding the perturbation $\delta_D := f\delta_C\varphi g = f\psi\delta_C g$ to the differential d_D;*
3. *the maps of the new reduction $\rho' = (f', g', h')$ are given by*

$$f' := f\psi, \qquad g' := \varphi g, \qquad h' := \varphi h = h\psi$$

where the convergence of the series φ and ψ is guaranteed by the nilpotency condition.

2.3. Universal covers

For completeness, we include here a brief summary of universal covers and their properties. See, for example, (Hatcher, 2002) for a fully detailed explanation.

A continuous map $f : \tilde{X} \to X$ between two path-connected spaces is said to be a *covering map* if for every $x \in X$, there exists a neighborhood U such that $f^{-1}(U)$ is a disjoint union of homeomorphic copies of U (and the restriction of f to each such copy is in fact a homeomorphism). In this situation, $\tilde{X}$ is said to be a *cover* of X.

A relevant property of covering maps is that they have *unique elevation of paths*: given a point $\tilde{x}_0 \in \tilde{X}$ and a path $\gamma : I \to X$ starting at $x_0 = f(\tilde{x}_0)$, there exists a unique lift $\gamma_{x_0} : I \to \tilde{X}$ starting at x_0. This implies that the induced map $f^* : \pi_1(\tilde{X}) \to \pi_1(X)$ is injective, so we can see the fundamental group of $\tilde{X}$ as a subgroup of the fundamental group of X. If this subgroup is normal, the cover is said to be *regular* (or *Galois*). In this case, the base space X is a quotient of $\tilde{X}$ by the group of so-called *deck transformations*. This group is isomorphic to the quotient of $\pi_1(X)$ by $\pi_1(\tilde{X})$.

If $\tilde{X}$ is simply connected, it is said to be a *universal cover* of X. Every connected, locally path connected and semilocally 1-connected space has a universal cover (and, in fact, only one, in the sense that two such covers would be homeomorphic). That is, such spaces are the quotient of a certain simply connected space by the action of some group.

Since covering maps are locally trivial fibrations, with discrete fiber, we can apply the long exact sequence of homotopy groups of these fibrations to get isomorphisms $\pi_i(\tilde{X}) \to \pi_i(X)$ for $i \geq 2$. That is, the higher homotopy groups of a space coincide with the ones of its universal cover.

As we have said before, one of the possible applications of effective homology is to compute the homotopy groups of spaces through the Whitehead tower method. This requires the spaces to be simply connected, but we can satisfy that condition if we compute the universal cover of the space. In fact, this can be seen as a first step in the Whitehead tower method.

3. A simplicial construction for universal covers

A simplicial set is said to be *connected* if its geometrical realization is connected. An equivalent property is that for every two 0-simplices v, w there exists a sequence of 1-simplices $(e_0, \dots, e_n)$ such that $\partial_1(e_0) = v$, $\partial_0(e_n) = w$, and $\partial_0(e_i) = \partial_1(e_{i+1})$.

Given a connected simplicial set X, a presentation of its fundamental group can be found as follows:

1. Choose a maximal tree T in the 1-skeleton of X [2].
2. Take a generator g_e for each edge e that is not in T.
3. For each non-degenerate 2-simplex σ in X, add a relation given by $g_{\partial_2\sigma} g_{\partial_0\sigma} g^{-1}_{\partial_1\sigma}$ (assuming that the edges in T correspond to the trivial element). This relation represents the fact that a closed path that follows the boundary of a triangle can be retracted to a constant path.

[2] The 1-skeleton of a simplicial set can be seen as a graph where the vertices are the 0-simplices, and the edges are the nondegenerate 1-simplices.

The idea behind this construction is the following: every loop can be isotoped to a path that follows the 1-skeleton (which is a graph), so we can represent the elements of the fundamental group as a sequence of vertices where two consecutive ones are joined by an edge. Since T is a maximal tree, given any vertex v_i, there is a unique such path p_i joining the base point to v_i. Now, given any edge e_i joining vertices v_j and v_k, the concatenation of p_j, e_i and p_k reversed gives us an element of the fundamental group. If e is inside T, this path is trivial in the fundamental group, so we only need the edges outside T to generate the fundamental group. Figure 2 shows how the triangles induce the given relations. Every homotopy between two paths can be deformed to the 2-skeleton, so every relation in the fundamental group is consequence of the relations given by the 2-simplices.

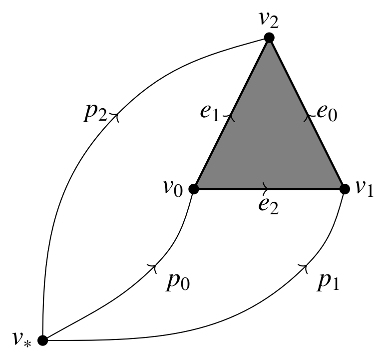

Figure 2: Relation given by a 2-simplex

The result is a presentation of $G := \pi_1(|X|, v_*)$, where v_* is the point corresponding to a 0-simplex. Note that we also get a map τ' that sends edges to elements of the group (assigning the edges in T to the trivial element, and each edge e not in T to the corresponding generator g_e). This assignment can be lifted to higher-dimensional simplices by taking the first face (the simplex s will be sent to the same group element as $\partial_0(s)$, that is $\tau'(s) = \tau'((\partial_0^d)(s))$ when s is of dimension $d + 1$). In order to work effectively with this presentation, we need to be able to solve the word problem in this presentation (which cannot be ensured in the general case, but is solved for several families of groups, including the cases of finite, free, Abelian, polycyclic or simple groups, see for example Mostowski (1966) and Moede and Neumann-Brosig (2022)).

Now consider a multiplicative group H and a surjective group morphism $\chi : \pi_1(|X|, v_*) \to H$. We define a map

$$\tau : X \to H$$

given by the following recursive rules:

- $\forall v \in X_0, \tau(v) = 1$
- $\forall e \in T, \tau(e) = 1$
- $\forall v \in X_0, \tau(s_0 v) = 1$
- $\forall e \in X_1 \setminus T, \tau(e) = \chi(\tau'(e))$
- $\forall \sigma \in X_n$ with $n > 1$, $\tau(\sigma) = \tau(\partial_0 \sigma)$

Lemma 3.1. *With the previous assignation rules, for each $\sigma \in X_n$ with $n \geq 2$, the following equality holds:*

$$\tau(\partial_{n-1}\sigma) = \tau(\partial_n\sigma) \cdot \tau(\partial_0\sigma).$$

Proof. We prove it by induction on n.

For $n = 2$, it is a direct consequence of the relations in the presentation of the group.

Now assume that it is true for simplices of dimension up to $n-1$, and take a simplex σ of dimension n. Applying the recursive definition of τ, we get:

$$\begin{array}{rcl}
\tau(\partial_{n-1}\sigma) & = & \tau(\partial_0\partial_{n-1}\sigma) \\
& = & \tau(\partial_{n-2}\partial_0\sigma) \\
& = & \tau(\partial_{n-1}\partial_0\sigma) \cdot \tau(\partial_0\partial_0\sigma) \\
& = & \tau(\partial_0\partial_n\sigma) \cdot \tau(\partial_0\partial_0\sigma) \\
& = & \tau(\partial_n\sigma) \cdot \tau(\partial_0\sigma)
\end{array}$$

□

Lemma 3.2. *For each $\sigma \in X_n$ with $n \geq 2$, and $0 \leq i < n-1$, $\tau(\sigma) = \tau(\partial_i\sigma)$.*

Proof. By definition $\tau(\partial_i\sigma) = \tau(\partial_0\partial_i\sigma)$. Iterating this i times, we get $\tau(\partial_0 \ldots \partial_0\partial_i\sigma)$ which, by the properties of the face maps in a simplicial set, equals $\tau(\partial_0\partial_0 \ldots \partial_0\sigma)$. Again, iterating the definition $i+1$ times, this is equal to $\tau(\sigma)$. □

We will now construct a new simplicial set $\tilde{X}$ with the following rules:

- The sets of simplices are $\tilde{X}_n = H \times X_n$
- The degeneracy maps are $\tilde{s}_i(h, \sigma) = (h, s_i\sigma)$
- The face maps for an n-dimensional simplex (h, σ) are given by:
  - $\tilde{\partial}_i(h, \sigma) = (h, \partial_i\sigma)$, for $i < n$
  - $\tilde{\partial}_n(h, \sigma) = (h \cdot \tau(\sigma)^{-1}, \partial_n\sigma)$

Lemma 3.3. *The maps $\{\tilde{s}_i\}$ and $\{\tilde{\partial}_i\}$ define a structure of simplicial set in $\tilde{X}$.*

Proof. The properties of the compositions of the degeneracies are a consequence of the ones in X. To check the rule for composition of faces, we distinguish cases for an n-dimensional simplex.

For $n \geq 2$, we have

$$\begin{array}{c}
\tilde{\partial}_{n-1}\tilde{\partial}_n(h, \sigma) = \tilde{\partial}_{n-1}(h \cdot \tau(\sigma)^{-1}, \partial_n\sigma) \\
= (h \cdot \tau(\sigma)^{-1} \cdot \tau(\partial_n\sigma)^{-1}, \partial_{n-1}\partial_n\sigma) \\
= (h \cdot \tau(\partial_0\sigma)^{-1} \cdot \tau(\partial_n\sigma)^{-1}, \partial_{n-1}\partial_{n-1}\sigma) \\
= (h \cdot \tau(\partial_{n-1}\sigma)^{-1}, \partial_{n-1}\partial_{n-1}\sigma) \\
= \tilde{\partial}_{n-1}(h, \partial_{n-1}\sigma) = \tilde{\partial}_{n-1}\tilde{\partial}_{n-1}(h, \sigma)
\end{array}$$

If $i < n-1$,

$$
\begin{aligned}
\tilde{\partial}_i\tilde{\partial}_n(h,\sigma) &= \tilde{\partial}_i(h\cdot\tau(\sigma)^{-1},\partial_n\sigma)\\
&= (h\cdot\tau(\sigma)^{-1},\partial_i\partial_n\sigma)\\
&= (h\cdot\tau(\sigma)^{-1},\partial_{n-1}\partial_i\sigma)\\
&= (h\cdot\tau(\partial_i\sigma)^{-1},\partial_{n-1}\partial_i\sigma)\\
&= \tilde{\partial}_{n-1}(h,\partial_i\sigma)\\
&= \tilde{\partial}_{n-1}\tilde{\partial}_i(h,\sigma)
\end{aligned}
$$

For $i < j < n$, the relations between ∂_i and ∂_j transfer directly to $\tilde{\partial}_i$ and $\tilde{\partial}_j$. □

As a direct consequence of the definition of $\tilde{\partial}_i$, we get the following result.

Lemma 3.4. *The map*

$$
\begin{aligned}
\pi : \tilde{X} &\to X\\
(h,\sigma) &\mapsto \sigma
\end{aligned}
$$

is a simplicial map.

This implies that π induces a continuous map $|\tilde{X}| \to |X|$.
We will now see that it is indeed a covering map. To do so, we need some preliminary results.

Lemma 3.5. *Let σ_0, σ_1 be simplices in X such that $\partial_{a_1}\cdots\partial_{a_m}\sigma_1 = \sigma_0$ for some $a_1 < \cdots < a_m$. Then for every $g \in H$, there exists a unique $g' \in H$ such that $\tilde{\partial}_{a_1}\cdots\tilde{\partial}_{a_m}(g',\sigma_1) = (g,\sigma_0)$.*

Proof. Applying the rules that define $\tilde{\partial}_i$, we get that $\tilde{\partial}_{a_1}\cdots\tilde{\partial}_{a_m}(1,\sigma_1) = (h,\partial_{a_1}\cdots\partial_{a_m}\sigma_1) = (h,\sigma_0)$ for some $h \in H$. Then, the equation

$$\tilde{\partial}_{a_1}\cdots\tilde{\partial}_{a_m}(g',\sigma_1) = (g,\sigma_0)$$

is equivalent to

$$(g'h,\sigma_0) = (g,\sigma_0)$$

and the only solution is $g' = gh^{-1}$. □

Notice that, by the way the degeneration maps are defined in $\tilde{X}$, this result can be trivially generalized to the case where we have a sequence of both faces and degenerations.
In the situation of the previous lemma, we will say that σ_1 is *incident* to σ_0.

Lemma 3.6. *Let σ, σ_0 be non-degenerate incident simplices such that $\partial_{a_1}\cdots\partial_{a_m}\sigma = s_{b_1}\cdots s_{b_{l+1}}\sigma_0$, and $\partial_i\partial_{a_1}\cdots\partial_{a_m}\sigma = s_{c_1}\cdots s_{c_l}\sigma_0$ for some i, $a_1\ldots,a_m$, $b_1\ldots,b_{l+1}$ and some $c_1,\ldots,c_l$ (that is, there is a face F of σ that is a degeneration of σ_0, and so is a face of F).*

If $\tilde{\partial}_{a_1}\cdots\tilde{\partial}_{a_m}(g,\sigma) = \tilde{s}_{b_1}\cdots\tilde{s}_{b_{l+1}}(h,\sigma_0)$ and $\tilde{\partial}_i\tilde{\partial}_{a_1}\cdots\tilde{\partial}_{a_m}(g',\sigma) = \tilde{s}_{c_1}\cdots\tilde{s}_{c_l}(h,\sigma_0)$, then g and g' must be equal.

Proof. Let be d the dimension of σ_0. If $i < d + l + 1$, then clearly

$$
\begin{aligned}
\tilde{\partial}_i\tilde{\partial}_{a_1}\cdots\tilde{\partial}_{a_m}(g',\sigma) &= \tilde{\partial}_i(g'h',\partial_{a_1}\cdots\partial_{a_m}\sigma)\\
&= (g'h',\partial_i\partial_{a_1}\cdots\partial_{a_m}\sigma)
\end{aligned}
$$

for some h' that depends only on $a_1\ldots,a_m$, σ and τ; whereas

$$\tilde{\partial}_{a_1} \cdots \tilde{\partial}_{a_m}(g, \sigma) = (gh', \partial_{a_1} \cdots \partial_{a_m}\sigma).$$

with the same h' as before. If both are degenerations of the same simplex, we have that $g' = g$ (since the degeneration maps in $\tilde{X}$ don't affect the first coordinate).

Let us consider the case where $i = d + l + 1$. Without loss of generality, we can assume that $b_1 > \cdots > b_{l+1}$. Looking at the dimensions of the simplices, necessarily $b_j \leq d + l + 1 - j$; so $b_1 \leq d + l$.

If $b_1 < d + l$, we have that

$$\begin{aligned}\partial_{d+l+1} s_{b_1} \cdots s_{b_{l+1}} \sigma_0 &= s_{b_1} \partial_{d+l} s_{b_2} \cdots s_{b_{l+1}} \sigma_0 \\ &= s_{b_1} \cdots s_{b_{l+1}} \partial_d \sigma_0\end{aligned}$$

So we would have

$$s_{c_1} \cdots s_{c_l} \sigma_0 = s_{b_1} \cdots s_{b_{l+1}} (\partial_d \sigma_0)$$

and hence

$$\begin{aligned}\sigma_0 &= \partial_{c_l} \cdots \partial_{c_1} s_{c_1} \cdots s_{c_l} \sigma_0 \\ &= \partial_{c_l} \cdots \partial_{c_1} s_{b_1} \cdots s_{b_{l+1}} (\partial_d \sigma_0)\end{aligned}$$

Using the relations between the face and degeneration maps, we can express the last term as a degeneration (we can "move left" the degeneration maps unless they cancel with one face map, but there are not enough face maps to cancel all degenerations). That contradicts the assumption that σ_0 is non-degenerate. Since this cannot happen, necessarily b_1 must be equal to $d + l$.

In that case,

$$\begin{aligned}\tilde{\partial}_{d+l+1} \tilde{\partial}_{a_1} \cdots \tilde{\partial}_{a_m}(g', \sigma) &= \tilde{\partial}_{d+l+1} \tilde{s}_{d+l} \tilde{s}_{b_2} \cdots \tilde{s}_{b_{l+1}}(h, \sigma_0) \\ &= \tilde{s}_{b_2} \cdots \tilde{s}_{b_{l+1}}(h, \sigma_0) \\ &= \tilde{\partial}_{d+l} \tilde{s}_{d+l} \tilde{s}_{b_2} \cdots \tilde{s}_{b_{l+1}}(h, \sigma_0) \\ &= \tilde{\partial}_{d+l} \tilde{\partial}_{a_1} \cdots \tilde{\partial}_{a_m}(g', \sigma)\end{aligned}$$

and we can apply the previous case. □

Proposition 3.7. *The map induced by π is a covering map.*

Proof. Let $x \in |X|$, consider the smallest n such that x lives in the n-skeleton of $|X|$. There is a unique non-degenerate n-dimensional simplex σ_x such that $x \in |\sigma_x|$, and moreover, x lives in the interior of $|\sigma_x|$. For each $g \in H$, there is exactly one point $x_g \in |\pi|^{-1}(x)$ in $|(g, \sigma_x)|$[3].

Recall that, for every non-degenerate simplex σ in X, we have a continuous map

$$g_\sigma : |\sigma| \to |X|$$

[3] $|\pi|$ is the map between the geometric realizations induced by π

that is a homeomorphism on the image when restricted to the interior. Moreover, the images of these maps cover all $|X|$, and the topology of $|X|$ is the one induced by this family of maps.

For any non-degenerate simplex σ of dimension $m > n$, the set $g_\sigma^{-1}(x)$ is nonempty if and only if σ is incident to σ_x. In that case, $g_\sigma^{-1}(x)$ is contained in the boundary of $|\sigma|$. If we take a k-dimensional face of σ (that is, a simplex of the form $f = \partial_{i_1} \cdots \partial_{i_j}\sigma$), such that f is a degeneration of σ_x, the restriction of g_σ to the interior of $|f|$ sends the interior of $|f|$ (which is a simplex of dimension $m - j$) to the interior of $|\sigma_x|$ (which has dimension n) as a surjective affine map. So the intersection of the interior of $|f|$ with $g_\sigma^{-1}(x)$ is the interior of a polytope P_f of dimension $m - j - n$. If f is not a degeneration of σ_x, then this intersection is empty.

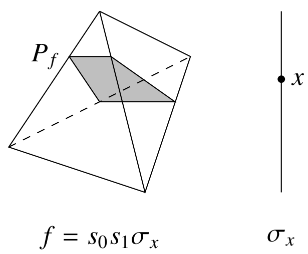

Figure 3: Example of the the polytope P_f.

If we take a face of f, $f' = \partial_j f$, the corresponding polytope $P_{f'}$ is exactly the intersection of P_f with f'. That is, $g_\sigma^{-1}(x)$ is a polytopal complex $K_{x,\sigma}$, where the incidences between polytopes correspond to the incidences between the corresponding faces. Let $\{K_{x,\sigma}^1, \dots, K_{x,\sigma}^{n_{x,\sigma}}\}$ be the connected components of $K_{x,\sigma}$. If we fix two points $y, y' \in K_{x,\sigma}^j$, belonging to faces f, f' respectively, there must be a chain of faces $f = f_0, f_1, \dots, f_s = f'$ such that each one is incident to the next. Now, if $y_h = x_g = y'_{h'}$ for some $h, h' \in H$, applying Lemma 3.6 to the sequence, we obtain that $h = h'$. That is, the points on the same connected component that are mapped to x_g have the same associated element of the group.

Finally, consider U a neighborhood of x that is small enough to ensure that, for every non-degenerate simplex σ, $U \cap |\sigma| = U_{x,\sigma}^1 \cup \cdots \cup U_{x,\sigma}^{n_{x,\sigma}}$, where $U_{x,\sigma}^k$ is a neighborhood of $K_{x,\sigma}^k$, and the unions are disjoint. By the previous remark, there will be $h_{x,\sigma}^1, \dots, h_{x_\sigma}^{n_{x,\sigma}} \in H$ such that the corresponding $(h_{x,\sigma}^k, U_{x,\sigma}^k)$ glued together form a neighborhood of x_g; and moreover, this gluing will mimic the one of $U_{x,\sigma}^k$ (because the simplicial map respects faces), so this neighborhood will be homeomorphic to U. □

The previous results ensure us that we can construct covers of a simplicial set. The following result shows that a suitable choice of H and χ allows us to obtain the universal cover:

Proposition 3.8. *If $\chi : \pi_1(|X|, v_*) \to H$ is an isomorphism, then $|\tilde{X}|$ is the universal cover of $|X|$ and π induces the corresponding covering map.*

Proof. We just have to show that the induced map in the fundamental group is trivial. An element of $\pi_1(|\tilde{X}|, (v_*, 1))$ can be represented by a sequence of edges $(g_0, e_0), \dots, (g_n, e_n)$, such that $\tilde{\partial}_0(g_i, e_i) = \tilde{\partial}_1(g_{i+1}, e_{i+1})$. The formulas that define $\tilde{X}$ ensure that $g_{i+1} = g_i \cdot \tau(e_i)$. So, since the path must be closed, we get $\tau(e_0) \cdots \tau(e_n) = 1$. Since this is exactly the element of $\pi_1(|X|, v_*)$ represented by the path $e_0, \dots, e_n$, this means that the image of the path by π is trivial in $\pi_1(|X|, v_*)$. □

In fact, it is easy to adapt the previous proof to see that, in general, we get the regular cover that corresponds to the group representation χ. These results lead to Algorithm 1.

Algorithm 1: Cover of a surjective group morphism on the fundamental group

Input:

- a connected simplicial set X,
- a multiplicative group H,
- a surjective group morphism $\chi : \pi_1(|X|, v_*) \to H$.

Output: a simplicial set $\tilde{X}$ whose geometrical realization is the cover of $|X|$ corresponding to χ.

1 Define the map $\tau : X \to H$ by the recursive rules explained at the beginning of this section.
2 Define the set of simplices of $\tilde{X}$ as $\tilde{X}_n = H \times X_n$.
3 Define the degeneracy operators as $\tilde{s}_i(h, \sigma) = (h, s_i\sigma)$ for $h \in H$ and $\sigma \in X$.
4 Define the face maps, for an n-dimensional simplex (h, σ), by $\tilde{\partial}_i(h, \sigma) = (h, \partial_i\sigma)$, for $i < n$, and $\tilde{\partial}_n(h, \sigma) = (h \cdot \tau(\sigma)^{-1}, \partial_n\sigma)$.

return $\tilde{X}$

Typically, in Algorithm 1, the group H will be $\pi_1(|X|, v_*)$ itself, or some group constructed explicitly as a quotient of it (like its abelianization, or its image by a finite representation). As mentioned above, if $H = \pi_1(|X|, v_*)$ (and χ is an isomorphism), then the output $\tilde{X}$ is a simplicial model for the universal cover of X.

Observe that the map τ described before can also be seen as a twisting operator $\tau : X \to K(H, 0)$, where $K(H, 0)$ is the simplicial set defined as $K(H, 0)_n = H$ for every n and all face and degeneracy maps equals to the identity map (this simplicial set is an Eilenberg–MacLane space with $\pi_0(K(H, 0), 1) = H$ and $\pi_i(K(H, 0), 1) = 0$ for all $i > 0$). In fact, any map $\tau : X_1^{ND} \to H$ (where X_1^{ND} is the set of non-degenerate edges of X) satisfying $\tau(\partial_1 x) = \tau(\partial_2 x) \cdot \tau(\partial_0 x)$ for all $x \in X_2^{ND}$ can be extended to a map $\tau : X \to K(H, 0)$ as described at the beginning of this section (satisfying the properties of the twisting operator). Then, the simplicial set $\tilde{X}$ is isomorphic to the twisted Cartesian product defined by $\tau : X \to K(H, 0)$, following Definition 2.5. We obtain in this way Algorithm 2.

Remark 2. *Since we are specially interested in universal covers, we described Algorithm 2 for this particular case. For other regular covers, it could be adapted in a similar way as Algorithm 1.*

With this ingredient, we can implement the effective homology of universal covers in Kenzo as a twisted Cartesian product (which in particular, could allow us to deal with simplicial sets of infinite type), using the homological perturbation theorems as will be explained in the next section.

4. An algorithm computing the effective homology of the simplicial universal cover

Using the simplicial version of the universal cover of a simplicial set as a twisted Cartesian product presented in Algorithm 2, in this section we explain how to determine the effective

Algorithm 2: Universal cover of a simplicial set as a twisted Cartesian product

Input:

- a connected simplicial set X,
- the fundamental group of X, $\pi_1(|X|, v_*) = H$,
- a map defined on the non-degenerate edges of X, $\tau : X_1^{ND} \to H$, such that $\tau(\partial_1 x) = \tau(\partial_2 x) \cdot \tau(\partial_0 x)$ for all $x \in X_2^{ND}$.

Output: a twisting operator $\tau : X \to K(H,0)$ that defines the simplicial model of the universal cover of X, expressed as a twisted Cartesian product $K(H,0) \times_\tau X$.

Define the map $\tau' : X \to K(H,0)$ induced by $\tau : X \to H$ (defined by the recursive rules explained at the beginning of this section) and the relation $K(H,0)_n = H$ for all $n \geq 0$.

Construct the simplicial set K as the twisted Cartesian product associated to τ' as described in Definition 2.5.

return K

homology of universal covers by applying the homological perturbation theory.

Let X be a connected simplicial set (not necessarily of finite type) with fundamental group $\pi_1(|X|, v_*) = H$, and $\tau : X_1^{ND} \to H$ satisfying the assumptions on the inputs of Algorithm 2. Let us also suppose now that X has effective homology given by two reductions $C_*(X) \Lleftarrow DX_* \Rrightarrow EX_*$, where EX_* is a chain complex of finite type (effective).

Proposition 4.1. *Under the previous assumptions, three reductions as in the following diagram are obtained:*

$$\begin{array}{ccc} C_*(K(H,0) \times X) & C_*(K(H,0)) \otimes DX_* & \\ \Downarrow \rho_1 & \swarrow \rho_2 \quad \searrow \rho_3 & \\ C_*(K(H,0)) \otimes C_*(X) & & C_*(K(H,0)) \otimes EX_* \end{array} \tag{1}$$

Proof. These reductions are the ingredients of the algorithm for computing the effective homology of the Cartesian product of two simplicial sets described, for example, in (Rubio and Sergeraert, 2006). More specifically, the first reduction on the left, ρ_1, is the Eilenberg–Zilber reduction (Eilenberg and Zilber, 1953), defined from the chain complex associated with a Cartesian product of two simplicial sets (in our case, $K(H,0)$ and X) to the tensor product of the chain complexes of the two factors. The reductions ρ_2 and ρ_3 are constructed, respectively, as the tensor product of the trivial reduction $C_*(K(H,0)) \Rrightarrow C_*(K(H,0))$ with the two reductions of the effective homology of X, $DX_* \Rrightarrow C_*(X)$ and $DX_* \Rrightarrow EX_*$ (following the formulas presented in Proposition 2.10). □

Now, we try to apply the homological perturbation theory to the reductions in (1), by considering the twisted Cartesian product $K(H,0) \times_\tau X$, where τ is the twisting operator induced by the map $\tau : X_1^{ND} \to H$ as explained before. This means that the differential of $C_*(K(H,0) \times_\tau X)$ is a perturbed version of the differential of $C_*(K(H,0) \times X)$, where the perturbation is given, for any $(k, x) \in K(H,0)_n \times X_n$, by

$$\delta(k, x) := (-1)^n \left[(\tau(x) \cdot \partial_n k, \partial_n x) - (\partial_n k, \partial_n x) \right]. \tag{2}$$

We begin by applying the BPL to the reduction ρ_1, obtaining a new reduction described in the following result.

Lemma 4.2. *Under the previous assumptions, a reduction $\hat{\rho}_1 : C_*(K(H,0)\times_\tau X) \Rrightarrow C_*(K(H,0))\otimes_t C_*(X)$ is obtained, where $C_*(K(H,0)) \otimes_t C_*(X)$ is the perturbed chain complex obtained from $C_*(K(H,0)) \otimes C_*(X)$ by adding a new perturbation δ' obtained by the BPL.*

Proof. This is a direct consequence of a major classical result, known as the twisted Eilenberg–Zilber theorem (Brown, 1959), and is obtained by applying the Basic Perturbation Lemma to the Eilenberg–Zilber reduction. □

Now, using the Trivial Perturbation Lemma, we obtain the following result.

Lemma 4.3. *Under the previous assumptions, a reduction $\hat{\rho}_2 : C_*(K(H,0))\otimes_t DX_* \Rrightarrow C_*(K(H,0))\otimes_t C_*(X)$ is obtained, where $C_*(K(H,0))\otimes_t DX_*$ is a chain complex with the same underlying graded module as $C_*(K(H,0)) \otimes DX_*$ and the differential map is obtained by adding a new perturbation δ'' obtained by the TPL.*

Proof. Apply the Trivial Perturbation Lemma to the reduction ρ_2 in (1) with the perturbation δ'. □

A last step is necessary for the construction of the effective homology of $K(H,0) \times_\tau X$: we need to apply the BPL to the reduction $\rho_3 = (f_3, g_3, h_3) : C_*(K(H,0))\otimes DX_* \Rrightarrow C_*(K(H,0))\otimes EX_*$ (with the perturbation δ''). This is the most difficult part of the process, since to apply the BPL one must verify that the nilpotency condition of the composition $h_3\delta''$ is satisfied. In (Rubio and Sergeraert, 2006), a proof of this condition is given when the base space of the fibration is 1-reduced, but this is not our case (the base space is X and, in general, it is not simply connected; in fact, if X is simply connected, then it is its own universal cover). Therefore, a different proof of the nilpotency condition must be developed. To do this, we use the following lemmas.

Lemma 4.4. *Let X be a simplicial set, $\pi_1(|X|, v_*) = H$ its fundamental group, $\tau : X_1^{ND} \to H$ such that $\tau(\partial_1 x) = \tau(\partial_2 x)\cdot\tau(\partial_0 x)$ for all $x \in X_2^{ND}$, and $K(H,0) \times_\tau X$ as defined in Algorithm 2. The twisted Eilenberg–Zilber reduction $\hat{\rho}_1 = (\hat{f}_1, \hat{g}_1, \hat{h}_1) : C_*(K(H,0) \times_\tau X) \Rrightarrow C_*(K(H,0)) \otimes_t C_*(X)$ is in this case an isomorphism and the new perturbation δ' on $C_*(K(H,0)) \otimes_t C_*(X)$ is given, for $(k \otimes x) \in (C_*(K(H,0)) \otimes_t C_*(X))_n$, by:*

$$\delta'(k \otimes x) := (-1)^n[(\tau(x)\cdot k) \otimes \partial_n x - k \otimes \partial_n x].$$

Proof. Since $K(H,0)$ is defined as $K(H,0)_n = H$ for all $n \geq 0$ and all face and degeneracy maps are identity maps, all simplices in $K(H,0)$ are degenerate except those in dimension 0. Then, non-degenerate n-simplices of $K(H,0) \times_\tau X$ are pairs $(s_{n-1}\cdots s_0 k, x)$ with $k \in H$ and x a non-degenerate n-simplex of X, and the generators of $C_*(K(H,0)) \otimes_t C_*(X)$ are elements $k \otimes x$ with $k \in H$ and x a non-degenerate n-simplex of X. In this way, the sets $(C_*(K(H,0)) \otimes_t C_*(X))_n = C_0(K(H,0)) \otimes C_n(X)$ and $C_n(K(H,0) \times X)$ are canonically isomorphic. It is also easy to observe that the differential map of both chain complexes $C_*(K(H,0) \times_\tau X)$ and $C_*(K(H,0)) \otimes_t C_*(X)$ are the same, the maps $\hat{f}_1$ and $\hat{g}_1$ of the Eilenberg–Zilber reduction are in this case the canonical isomorphisms $\hat{f}_1(s_{n-1}\cdots s_0 k, x) = k \otimes x$ and $\hat{g}_1(k \otimes x) = (s_{n-1}\cdots s_0 k, x)$ for $k \in H$ and $x \in X_n$, and $\hat{h}_1$ is the null map. Finally, the perturbation δ' defined on $C_*(K(H,0)) \otimes C_*(X)$ is obtained by applying the BLP to the Eilenberg–Zilber reduction and the perturbation δ, and is given by:

$$\begin{aligned}\delta'(k\otimes x) &= \hat{f}_1\delta\hat{g}_1(k\otimes x) = \hat{f}_1\delta(s_{n-1}\cdots s_0 k, x)\\ &= \hat{f}_1((-1)^n((\tau(x)\cdot s_{n-2}\cdots s_0 k, \partial_n x) - (s_{n-2}\cdots s_0 k, \partial_n x)))\\ &= (-1)^n((\tau(x)\cdot k)\otimes \partial_n x - k\otimes \partial_n x).\end{aligned}$$

□

Lemma 4.5. *Let X be a simplicial set, $\pi_1(|X|, v_*) = H$ its fundamental group, $\tau : X_1^{ND} \to H$ such that $\tau(\partial_1 x) = \tau(\partial_2 x)\cdot\tau(\partial_0 x)$ for all $x \in X_2^{ND}$, and $K(H,0)\times_\tau X$ as defined in Algorithm 2. Let us suppose that X has effective homology given by two reductions $\rho_1^X = (f_1^X, g_1^X, h_1^X) : DX_* \Rightarrow C_*(X)$ and $\rho_2^X = (f_2^X, g_2^X, h_2^X) : DX_* \Rightarrow EX_*$. Then, the perturbation δ'' induced on the chain complex $C_*(K(H,0))\otimes DX_*$ is given, for $(k\otimes x) \in (C_*(K(H,0))\otimes_t DX_*)_n$, by:*

$$\delta''(k\otimes x) = (-1)^n[(\tau(f_1^X(x))\cdot k)\otimes g_1^X\partial_n f_1^X(x) - k\otimes g_1^X\partial_n f_1^X(x)]$$

where $\partial_n : C_n(X) \to C_{n-1}(X)$ and $\tau : C_n(X) \to H$ are obtained by extending by linearity the face ∂_n of the simplicial set X and the map $\tau : X \to H$.

Proof. The formula of the perturbation δ'' is directly obtained by applying the TPL to the reduction $\rho_2 : C_*(K(H,0))\otimes DX_* \Rightarrow C_*(K(H,0))\otimes C_*(X)$ and the formula for the perturbation δ' obtained in Lemma 4.4. □

Lemma 4.6. *Let X be a simplicial set, $\pi_1(|X|, v_*) = H$ its fundamental group, $\tau : X_1^{ND} \to H$ such that $\tau(\partial_1 x) = \tau(\partial_2 x)\cdot\tau(\partial_0 x)$ for all $x \in X_2^{ND}$, and $K(H,0)\times_\tau X$ as defined in Algorithm 2. Let us suppose that X has effective homology given by two reductions $\rho_1^X = (f_1^X, g_1^X, h_1^X) : DX_* \Rightarrow C_*(X)$ and $\rho_2^X = (f_2^X, g_2^X, h_2^X) : DX_* \Rightarrow EX_*$. Let $\rho_3 = (f_3, g_3, h_3)$ be the right reduction in diagram (1). Then, the composition $h_3\delta''$ is given, for $(k\otimes x) \in (C_*(K(H,0))\otimes_t DX_*)_n$, by:*

$$h_3\delta''(k\otimes x) = (-1)^n[(\tau(f_1^X(x))\cdot k)\otimes h_2^X g_1^X\partial_n f_1^X(x) - k\otimes h_2^X g_1^X\partial_n f_1^X(x)].$$

Proof. Taking into account Proposition 2.10 and the fact that in the trivial reduction $C_*(K(H,0)) \Rightarrow C_*(K(H,0))$ the map h is null, we obtain that h_3 is defined, for $(k\otimes x) \in (C_*(K(H,0))\otimes_t DX_*)_n$, as:

$$h_3(k\otimes x) = (\mathrm{Id}\otimes h_2^X)(k\otimes x) = k\otimes h_2^X(x)$$

The result is easily obtained combining this formula with the expression of the perturbation δ'' of Lemma 4.5. □

Let us recall that, to apply the BPL to the reduction ρ_3 in diagram (1), we need the composition $h_3\delta''$ to be locally nilpotent, that is, for every $y \in C_*(K(H,0))\otimes_t DX_*$ there exists a nonnegative integer $m = m(y) \in \mathbb{N}$ such that $(h_3\delta'')^m(y) = 0$. Using the description of the composition $h_3\delta''$ in Lemma 4.6, we obtain the following corollaries.

Corollary 4.7. *Let X be a simplicial set, $\pi_1(|X|, v_*) = H$ its fundamental group, $\tau : X_1^{ND} \to H$ such that $\tau(\partial_1 x) = \tau(\partial_2 x)\cdot\tau(\partial_0 x)$ for all $x \in X_2^{ND}$, and $K(H,0)\times_\tau X$ as defined in Algorithm 2. Let us suppose that X has effective homology given by two reductions $\rho_1^X = (f_1^X, g_1^X, h_1^X) : DX_* \Rightarrow C_*(X)$ and $\rho_2^X = (f_2^X, g_2^X, h_2^X) : DX_* \Rightarrow EX_*$ and assume that the composition $h_2^X g_1^X\partial_n f_1^X$ satisfies that, for every element $y \in DX_n$, there exists some natural number m with $(h_2^X g_1^X\partial_n f_1^X)^m(y) = 0$. Then, we can apply the BPL to the reduction ρ_3 of diagram (1).*

Corollary 4.8. *Under the previous assumptions, a reduction $\hat{\rho}_3 : C_*(K(H,0)) \otimes_t DX_* \Rightarrow C_*(K(H,0)) \otimes_t EX_*$ is obtained, where $C_*(K(H,0)) \otimes_t EX_*$ is a chain complex with the same underlying graded module as $C_*(K(H,0)) \otimes EX_*$ and the differential map is obtained by adding a new perturbation δ''' by applying the BPL.*

Combining Lemmas 4.2 and 4.3 and Corollary 4.8, we obtain the following diagram of reductions.

$$\begin{array}{ccccc} C_*(K(H,0) \times_\tau X) & & C_*(K(H,0)) \otimes_t DX_* & & \\ \Downarrow \hat{\rho}_1 & \swarrow\!\!\swarrow \hat{\rho}_2 & & \searrow\!\!\searrow \hat{\rho}_3 & \\ C_*(K(H,0)) \otimes_t C_*(X) & & & & C_*(K(H,0)) \otimes_t EX_* \end{array} \tag{3}$$

where we recall that the reductions $\hat{\rho}_1$, $\hat{\rho}_2$, and $\hat{\rho}_3$ have been obtained by applying, respectively, the BPL, TPL and BPL to the reductions ρ_1, ρ_2, and ρ_3 of diagram (1).

Finally, we observe that, if the group $\pi_1(|X|, v_*) = H$ is finite, then the simplicial group $K(H,0)$ is of finite type (effective) and therefore the chain complex $C_*(K(H,0)) \otimes_t EX_*$ is also effective. Combining this with the previous results, we obtain the main result of this section: the effective homology of the universal cover of a simplicial set.

Theorem 4.9. *Let X be a simplicial set, $\pi_1(|X|, v_*) = H$ its fundamental group be a finite group, $\tau : X_1^{ND} \to H$ such that $\tau(\partial_1 x) = \tau(\partial_2 x) \cdot \tau(\partial_0 x)$ for all $x \in X_2^{ND}$, and $K(H,0) \times_\tau X$ as defined in Algorithm 2. Let us suppose that X has effective homology given by two reductions $\rho_1^X = (f_1^X, g_1^X, h_1^X) : DX_* \Rightarrow C_*(X)$ and $\rho_2^X = (f_2^X, g_2^X, h_2^X) : DX_* \Rightarrow EX_*$ and assume that the composition $h_2^X g_1^X \partial_n f_1^X$ satisfies that, for every element $y \in DX_n$, there exists some natural number m with $(h_2^X g_1^X \partial_n f_1^X)^m(y) = 0$. Then, an equivalence $K(H,0) \times X \Longleftarrow\!\!\Rightarrow EX'_*$ is obtained, where EX'_* is an effective chain complex.*

Proof. We consider the composition of the reductions of diagram (3) and observe that the chain complex $EX'_* = C_*(K(H,0)) \otimes_t EX_*$ has a finite number of generators in each degree. □

The proof of this theorem (by means of the previous results) leads to the construction of Algorithm 3.

This algorithm can be combined with the Whitehead tower method for computing homotopy groups (implemented in Kenzo for 1-reduced simplicial sets with effective homology and then enhanced in (Cuevas-Rozo et al., 2021) to deal with simply connected simplicial sets), taking into account that the universal cover $\tilde{X}$ satisfies $\pi_i(\tilde{X}) \cong \pi_i(X)$ for $i \geq 2$. Given a non-simply connected simplicial set X, we determine its universal cover (which is simply connected) and its effective homology thanks to Algorithms 2 and 3, and we apply the algorithm for computing its homotopy groups implemented in (Cuevas-Rozo et al., 2021).

Example 3. *Let us see an example to showcase that the finiteness condition of the fundamental group is not superfluous. Consider the simplicial set X with the following non-degenerate simplices:*

- *A 0-dimensional simplex v.*
- *A 1-dimensional simplex e, with $\partial_0(e) = \partial_1(e) = v$.*
- *A 2-dimensional simplex t, with $\partial_0(t) = \partial_1(t) = \partial_2(t) = s_0(v)$.*

The corresponding topological space $|X|$ is $\mathbb{S}^1 \vee \mathbb{S}^2$; therefore, the fundamental group is $\mathbb{Z}$, and the non-trivial homology groups are $H_1(X) \cong H_2(X) \cong \mathbb{Z}$. However, the universal cover $\tilde{X}$ is a real line with one copy of $\mathbb{S}^2$ attached to each integer point. This space is simply connected, but $H_2(\tilde{X})$ is the direct sum of infinitely many copies of $\mathbb{Z}$. Hence, $\tilde{X}$ cannot have effective homology, because it has a homology group that is not finitely generated.

5. Implementation and didactic examples

Algorithm 1 has been implemented in SageMath (The Sage Developers, 2023). It is valid only for simplicial sets of finite type (with a finite number of non-degenerate simplices) and finite fundamental group. Our implementation is already available in SageMath10.0.

We illustrate its usage with an example. We start by creating a simplicial set with finite fundamental group. In this case, we take the complex corresponding to the usual presentation of the symmetric group on 3 elements, and take its product with the projective space (note that this simplicial set depends on the specific presentation of the initial group, not on the group itself).

```
sage: G = SymmetricGroup(3).as_finitely_presented_group()
sage: G
Finitely presented group < a, b | b^2, a^3, (a*b)^2 >
sage: C = simplicial_sets.PresentationComplex(G)
sage: RP3 = simplicial_sets.RealProjectiveSpace(3)
sage: S = C.product(RP3) ; S
Simplicial set with 12 non-degenerate simplices x
RP^3
sage: S.fundamental_group()
Finitely presented group < e0, e5, e9 | e0^2, e9^2,
e5^3, e0*e5*e0*e5^-1, (e5*e9)^2, (e9*e0)^2,
(e5*e0*e9)^2 >
sage: S.fundamental_group().cardinality()
12
```

The method `universal_cover` automatically computes a presentation of the fundamental group, and uses the identity as the surjective morphism in Algorithm 1. In this example, creating its universal cover takes 22 seconds in an Intel Core i7-10700, using 260MB of RAM:

```
sage: SC = S.universal_cover()
sage: SC
Simplicial set with 4176 non-degenerate simplices
```

We can check that it is indeed simply connected:

```
sage: SC.fundamental_group()
Finitely presented group <  |  >
```

And now we can compute its usual topological invariants, as any other simplicial set, and compare them with the base space (this computation takes about a minute to complete):

```
sage: [SC.homology(i) for i in range(6)]
[0, 0, Z^11, Z, 0, Z^11]
sage: [S.homology(i) for i in range(6)]
[0, C2 x C2, Z x C2, Z x C2 x C2, C2, Z]
```

Using the interface with Kenzo, we can apply the Whitehead tower method to compute also the higher homotopy groups of the cover (which coincide with the corresponding homotopy groups of the base space):

```
sage: from sage.interfaces.kenzo import KFiniteSimplicialSet
sage: KSC = KFiniteSimplicialSet(SC)
sage: KSC.homotopy_group(2)
Multiplicative Abelian group isomorphic to Z x Z x Z x Z x Z x Z x
Z x Z x Z x Z x Z
```

If we want to see the group it internally computed and the τ map, we can get it with the `_universal_cover_dict` method.

```
sage: P, tau = S._universal_cover_dict()
sage: P
Finitely presented group < e0, e1, e2, e3, e4, e5, e6, e7, e8, e9, e10 | e8*e1^-1*e10,
sage: tau
{(s_0 Delta^0, f): e0,
 (b^-1, s_0 1): e1,
 (b^-1, f): e2,
 (b, s_0 1): e3,
 (b, f): e4,
 (a^-1, s_0 1): e5,
 (a^-1, f): e6,
 (a, s_0 1): e7,
 (a, f): e8,
 (Delta^1, s_0 1): e9,
 (Delta^1, f): e10}
```

If we want to construct a cover using some quotient of the fundamental group, we need to give a dictionary providing the corresponding τ (sage will check that it corresponds to a valid morphism from the fundamental group). One way to do it is to explicitly construct a quotient of the fundamental group and apply it to the values used in the universal cover. For example, we can construct the maximal Abelian cover like that:

```
sage: ab = P.abelianization_map()
sage: ab
Group morphism:
  From: Finitely presented group < e0, e1, e2, e3, e4, e5, e6, e7, e8, e9, e10 | e8*e1
  To:   Finitely presented group < F1, F2 | F1^2, F2^2, (F2*F1)^2 >
sage: S.cover({e : ab(h) for e, h in tau.items()})
Simplicial set with 1392 non-degenerate simplices
sage: AC = S.cover({e : ab(h) for e, h in tau.items()})
sage: AC
Simplicial set with 1392 non-degenerate simplices
sage: AC.fundamental_group()
Finitely presented group < e0 | e0^3 >
sage: AC.homology(2)
Z x Z x Z
```

Algorithms 2 and 3 have been implemented as functions in the Kenzo system (the code can be found at (Marco-Buzunáriz et al., 2023b)). As said before, one of the main advantages of the use of this system is that it allows one to work with spaces of infinite type and, moreover, it allows one to use the effective homology theory. In this way, it is possible to determine a simplicial model of the universal cover of simplicial sets of infinite type (with finite fundamental group) and determine its homology and homotopy groups.

To illustrate our programs and the power of the effective homology theory in our problem, we consider as a didactic example the following simplicial set of infinite type: we build in Kenzo the

Cartesian product of the projective plane with the "semiline" divided in intervals. The projective plane $\mathbb{R}P$ is given by non-degenerate simplices $\mathbb{R}P_0 = \{v\}$, $\mathbb{R}P_1^{ND} = \{a\}$ and $\mathbb{R}P_2^{ND} = \{t\}$, and faces $\partial_0 a = \partial_1 a = v$, $\partial_0 t = \partial_2 t = a$ and $\partial_1 t = s_0 v$; it is a simplicial set of finite type. The "semiline" is represented as a simplicial set A with non-degenerate simplices given by $A_0 = \{n | n \in \mathbb{N}\}$, $A_1^{ND} = \{[n, n+1] | n \in \mathbb{N}\}$ and $\partial_0([n, n+1]) = n+1$, $\partial_1([n, n+1]) = n$. The simplicial set A has an infinite number of non-degenerate simplices, but we can construct its effective homology in an explicit way as follows. We construct a reduction ρ_2^A:

$$h_2^A \circlearrowright C_*(A) \underset{g_2^A}{\overset{f_2^A}{\rightleftarrows}} C_*(*)$$

where $C_*(*)$ is a chain complex with only one generator $*$ in degree 0, f_2^A is defined by $f_2^A(x) = *$ if $x \in A_0$ and $f_2^A(x) = 0$ for all $x \in C_n(A)$ with $n > 0$, g_2^A is given by $g_2^A(*) = 0 \in A_0$ and $h_2^A(n) = [0,1] + [1,2] + \cdots + [n-1, n]$ (and $h_2^A(x) = 0$ for $x \in C_n(A)$ with $n > 0$). It is easy to verify that these maps satisfy the properties of reduction. The left reduction in the effective homology of A_* is the identity reduction $\rho_1^A = \mathrm{Id} : C_*(A) \Lleftarrow C_*(X)$.

Now, the Cartesian product $X = \mathbb{R}P \times A$ is also a simplicial set with effective homology (it is built automatically by Kenzo with the same idea as the reductions in diagram (1)). Moreover, since A is contractible, its fundamental group is equal to the fundamental group of $\mathbb{R}P$, that is $\pi_1(\mathbb{R}P) = \mathbb{Z}/2\mathbb{Z}$.

To apply Algorithm 3 and construct the universal cover of X with effective homology, it is also necessary to define a map $\tau : X_1^{ND} \to \mathbb{Z}/2\mathbb{Z}$, and we can do it in the following way: let e be an edge in X_1^{ND}, $e = (y, z)$ with $y \in \mathbb{R}P$ and $z \in A$. If $y = a$ (the unique edge in $\mathbb{R}P$), then we define $\tau(e) = 1$; if $y \neq a$, then we define $\tau(e) = 0$. Finally, it is not difficult to verify that the effective homology of X satisfies the condition of Algorithm 3 (that is the necessary hypothesis of Corollary 4.7), that is, the composition $h_2^X g_1^X \partial_n f_1^X$ satisfies the nilpotency condition.

Applying Algorithm 3, we can construct in Kenzo the simplicial model for the universal cover of X with its effective homology as follows. To this aim, we build the Cartesian product of the projective plane and the semiline (we omit the construction of the simplicial sets `proj-plane` and `semiline`) and we store it in the variable `X`. Then, we define the map $\tau : X_1^{ND} \to \mathbb{Z}/2\mathbb{Z}$ as a function that receives an edge and returns an element of the group (in this case, 0 or 1) and we store it in `X-twop-edges`. Finally, we call the function `universal-cover`:

```
> (setf X (crts-prdc proj-plane semiline))
[K21 Simplicial-Set]
> (setf X-twop-edges
  #'(lambda (edge)
      (with-crpr (dgop1 gmsm1 dgop2 gmsm2) edge
        (if (and (eq 0 dgop1) (eq gmsm1 'a))
            1 0))))
#<Interpreted Function (unnamed) @ #x20e50d82>
> (setf X-univ-cover (universal-cover X (cyclicgroup 2) X-twop-edges))
[K45 Simplicial-Set]
```

The inputs of the function `universal-cover`, which implements Algorithm 3, are a simplicial set (in this case, X), the fundamental group H (here, the cyclic group $\mathbb{Z}/2\mathbb{Z}$ constructed by means of the function `cyclicgroup`), and a function describing the map $\tau : X_1^{ND} \to H$ (in this case, the function `X-twop-edges` previously defined over the non-degenerate simplices of X as explained before). The effective homology of X, which is the fourth input of Algorithm 3, is not necessary to be explicitly given since Kenzo computes it automatically and stores it in one of the slots of X.

As said before, the effective homology allows us to determine its homology and homotopy groups. For instance, we compute the homotopy group of dimension 5:

```
> (homotopy X-univ-cover 5)
Homotopy in dimension 5 :
Component Z/2Z
```

which indeed is the correct result, since $\pi_5(\mathbb{S}^2) = \mathbb{Z}/2\mathbb{Z}$.

6. Study of some algebraic topology constructors and tools

6.1. Cartesian products

Given two connected simplicial sets X and Y, the universal cover of the Cartesian product $X \times Y$ is well known to be isomorphic to the Cartesian product of the universal covers of the factors. If X and Y satisfy the hypotheses of Algorithm 3, then we can compute the effective homology of the universal covers $\tilde{X}$ and $\tilde{Y}$. Applying then the algorithm for computing the effective homology of a Cartesian product, we would obtain the effective homology of the universal cover of $X \times Y$.

6.2. Discrete vector fields

Discrete vector fields (an important idea of Robin Forman's Discrete Morse Theory (Forman, 1998)) are a tool allowing one to obtain the effective homology of complicated spaces in some interesting situations. For example, a discrete vector field can be defined producing the well-known Eilenberg–Zilber theorem (Eilenberg and Zilber, 1953), which, as seen before, provides a reduction from the chain complex of the Cartesian product of two simplicial sets to the tensor product of the chain complexes of the factors. Moreover, the same vector field is valid for the case of twisted Cartesian products, described in the twisted Eilenberg–Zilber theorem (Brown, 1959). See (Romero and Sergeraert, 2010) for the definition of these vector fields and for details of the following definitions.

Let $C_* = (C_n, d_n)$ be a free chain complex with distinguished $\mathbb{Z}$-bases $B_n \subseteq C_n$, whose elements we call *n-cells*.

Definition 6.1. A *discrete vector field V* on C_* is a collection of pairs of cells $V = \{(x_k; y_k)\}_{k \in K}$ such that:

- Every x_k is an element of some B_n, in which case $y_k \in B_{n+1}$. The degree n depends on k and in general is not constant.
- Each component $x_k \in B_n$ is a *regular face* of the corresponding $y_k \in B_{n+1}$ (that is, the coefficient of x_k in $d_{n+1}(y_k)$ is $+1$ or -1).
- Each generator (cell) of C_* appears at most once in V.

Definition 6.2. A pair $(x_j; y_j)$ of V is called a *vector*. The cells x_j and y_j are called respectively a *source cell* and a *target cell*. A cell $x \in B_n$ that does not appear in the discrete vector field V is called a *critical cell*.

Definition 6.3. Given a discrete vector field V, a *V-path π of degree n and length m* is a sequence $\pi = \{(x_{j_k}; y_{j_k})\}_{0 \le k < m}$ such that:

- Every pair $(x_{j_k}; y_{j_k})$ is a vector of V and y_{j_k} is an n-cell.

- For every $0 < k < m$, the component x_{j_k} is a *face* of $y_{j_{k-1}}$ (meaning that the coefficient of x_{j_k} in $d_n(y_{j_{k-1}})$ is non-null), non necessarily regular but different from $x_{j_{k-1}}$.

Definition 6.4. A discrete vector field V is called *admissible* if, for every $n \in \mathbb{Z}$, a function $\lambda_n : B_n \to \mathbb{N}$ is provided such that the length of every V-path starting from $x \in B_n$ is bounded by $\lambda_n(x)$.

The following result, due to Forman (Forman, 1998, § 8), has been generalized in (Romero and Sergeraert, 2010) to the case of chain complexes not necessarily of finite type.

Theorem 6.5. *(Forman (1998); Romero and Sergeraert (2010)) Let $C_* = (C_n, d_n, B_n)$ be a free chain complex and $V = \{(x_k; y_k)\}_{k \in K}$ be an admissible discrete vector field on C_*. Then the vector field V defines a canonical reduction $\rho = (f, g, h) : (C_n, d_n) \Rrightarrow (C_n^c, d_n')$ where C_n^c is the free $\mathbb{Z}$-module generated by the critical n-cells and d_n' is an appropriate differential canonically defined from C_* and V.*

Let us suppose now that X is a simplicial set, $\pi_1(|X|, v_*) = H$ its fundamental group and V a vector field defined on the associated normalized chain complex, $C_*^N(X)$ (the n-cells are in this case the non-degenerate n-simplices of X). The vector field V induces a vector field $\tilde{V}$ on the (normalized chain complex of the) universal cover $\tilde{X}$ taking as vectors of $\tilde{V}$, for each element $h \in H$):

- the pairs $((h, x), (h, y))$ for each vector (x, y) of V such that $x \in X_n$ and $x \neq \partial_{n+1} y$

- and the pairs $((h, x), (h \cdot \tau(y), y))$ for each vector (x, y) in V such that $x \in X_n$ and $x = \partial_{n+1} y$.

Proposition 6.6. *If V is an admissible vector field on a simplicial set X, then $\tilde{V}$ is an admissible vector field on the universal cover $\tilde{X}$.*

Proof. First of all, we prove that $\tilde{V}$ satisfies the properties of Definition 6.1. The first and third one are trivial. For the second one, we have two cases:

- Given a vector (x, y) of V such that $x \in X_n$ and $x \neq \partial_{n+1} y$, one has that $x = \partial_j y$ for $0 \leq j < n + 1$. Then, the corresponding vectors in $\tilde{V}$, that is, the pairs $((h, x), (h, y))$ for $h \in H$ satisfy $\partial_j(h, y) = (h, x)$ and the coefficient of (h, x) in $\tilde{d}_{n+1}(h, y)$ is equal to the coefficient of x in $d_{n+1}(y)$, so that (h, x) is a regular face of its associated target cell (h, y).

- Given a vector (x, y) of V such that $x \in X_n$ and $x = \partial_{n+1} y$, then, for all $h \in H$, $\partial_{n+1}(h \cdot \tau(y), y) = (h \cdot \tau(y) \cdot \tau^{-1}(y), \partial_{n+1} y) = (h, x)$. Moreover, if there are other faces $\partial_j(h \cdot \tau(y), y) = (h, x)$ with $0 \leq j < n + 1$, then $\partial_j y = x$, which implies that the coefficient of (h, x) in $\tilde{d}_{n+1}(h \cdot \tau(y), y)$ is again $+1$ or -1 and therefore (h, x) is a regular face of the associated target simplex $(h \cdot \tau(y), y)$.

To prove that $\tilde{V}$ is admissible, it suffices to define $\tilde{\lambda}_n : \tilde{X}_n \to \mathbb{N}$ as $\tilde{\lambda}_n(h, x) = \lambda_n(x)$, where λ_n is the function proving the admissibility of the vector field V.

□

Corollary 6.7. *Let X be a connected simplicial set, $\pi_1(|X|, v_*) = H$ its fundamental group, $\tau : X_1^{ND} \to H$ such that $\tau(\partial_1 x) = \tau(\partial_2 x) \cdot \tau(\partial_0 x)$ for all $x \in X_2^{ND}$, and $K(H, 0) \times_\tau X$ as defined in Algorithm 2. Let V be an admissible discrete vector field on X such that the set X_n^c of n-critical cells is finite for every dimension n. If H is finite, a reduction $\rho : C_*(\tilde{X}) \to C_*(\tilde{X}^c)$ can be built, where $C_*(\tilde{X}^c)$ is an effective chain complex.*

A particular case of this situation is obtained for the Eilenberg–MacLane spaces $K(\mathbb{Z}, 1)$ and $K(\mathbb{Z}/m\mathbb{Z}, 1)$, for $m \geq 2$. In both cases, admissible discrete vector fields can be defined on the associated chain complexes, producing a reduction to an effective chain complex (see (Romero and Sergeraert, 2010) and (Romero, 2010) respectively). Thanks to our previous results, these vector fields can be used to define vector fields on the associated universal covers. For $K(\mathbb{Z}/m\mathbb{Z}, 1)$, this provides the effective homology of its universal cover. The case of $K(\mathbb{Z}, 1)$ (more concretely, the case of infinite fundamental group) will be studied in Section 7.

Discrete vector fields can also be used to compute the effective homology of twisted Cartesian products (Romero and Sergeraert, 2010) and classifying spaces (Delgado et al., 2022). If the initial spaces are effective (of finite type), then these vector fields combined with our Corollary 6.7 provide the effective homology of the universal cover. If some of the factor of the Cartesian product or the initial space in the classifying space is of infinite type, then other reductions appear in the effective homology of the new spaces and these reductions should be studied to see if they satisfy the hypotheses of Algorithm 3.

7. The case of Abelian infinite fundamental group

As we saw in Example 3, when the fundamental group is not finite, the universal cover might not have effective homology. However, in some cases its homology groups can be computed nevertheless.

As before, let X be a connected simplicial set, and G its fundamental group. For any $n \in \mathbb{N}$, the chain group $C_n(X)$ is freely generated by the set X_n^{ND} of non-degenerate simplices of dimension n. By construction of $\tilde{X}$, $C_n(\tilde{X})$ is freely generated by $G \times X_n^{ND}$. That is, the Abelian group structure of $C_n(\tilde{X})$ is isomorphic to $\mathbb{Z}[G] \otimes C_n(X)$.

Moreover, it is easy to check that the boundary map

$$\begin{array}{rccc} \tilde{d}_n: & C_n(\tilde{X}) & \longrightarrow & C_{n-1}(\tilde{X}) \\ & (g, \sigma) & \mapsto & \sum_{i=0}^{n-1} (-1)^i (g, \partial_i \sigma) + (-1)^n (g \cdot \tau(\sigma)^{-1}, \partial_n \sigma) \end{array}$$

is a morphism of $\mathbb{Z}[G]$-modules. So, the homology groups of $\tilde{X}$ will be, in fact, the homology groups of the chain complex of $\mathbb{Z}[G]$-modules

$$\cdots \xleftarrow{\tilde{d}_{n-1}} C_{n-1}(\tilde{X}) \xleftarrow{\tilde{d}_n} C_n(\tilde{X}) \xleftarrow{\tilde{d}_{n+1}} C_{n+1}(\tilde{X}) \xleftarrow{\tilde{d}_{n+2}} \cdots$$

If the simplicial set X is effective itself, this chain complex can be computed explicitly; and if the group G is Abelian, the group algebra $\mathbb{Z}[G]$ is isomorphic to a quotient of a polynomial ring. In this case, a generating set of $\ker(\tilde{d}_n)$ will be given by the syzygies of the images of the generators (these syzygies can be computed using Gröbner basis methods over modules). The module $\ker(\tilde{d}_n)$ will be a quotient of the free module generated by those generators, where the relations are again the syzygies between the generators. Adding the images of the generators of $C_{n+1}(\tilde{X})$, expressed in the generators of $\ker(\tilde{d}_n)$, we obtain presentation matrices for $H_n(\tilde{X})$ as $\mathbb{Z}[G]$-modules. The detailed steps to follow are presented in Algorithm 4.

We implemented this approach (see (Miguel Marco, 2023)) and was included in version 10.3 of SageMath.

Here we present an example that showcases how this approach can handle Example 3:

```
sage: X = simplicial_sets.Sphere(1).wedge(simplicial_sets.Sphere(2))
sage: X.twisted_homology(1)
```

```
Quotient module by Submodule of Ambient free module of rank 0
over the integral domain Multivariate Polynomial Ring in f1, f1inv
over Integer Ring
Generated by the rows of the matrix:
[]
sage: X.twisted_homology(2)
Quotient module by Submodule of Ambient free module of rank 1
over the integral domain Multivariate Polynomial Ring in f1, f1inv
over Integer Ring
Generated by the rows of the matrix:
[f1*f1inv - 1]
```

Notice how $H_1(\tilde{X})$ is trivial (it lives inside a module of rank 0), whereas $H_2(\tilde{X})$ is isomorphic to $\mathbb{Z}[t, t^{-1}]$; that is, as an Abelian group, it is freely generated by one generator for each integer.

8. Conclusions and further work

This paper presents contributions in order to perform new computations in algebraic topology. So far, SageMath allows one to compute homology groups of chain complexes (and simplicial sets) of finite type. Furthermore, Kenzo (and its interface in SageMath) allows one to compute homology and homotopy groups for certain types of infinite spaces by means of effective homology, being the only software able to perform this kind of computation. The computation of homotopy groups requires the simplicial sets to be simply connected. This work aims to generalize the computational capability of both programs, allowing one to perform such computations involving simplicial sets that are not simply connected.

For this task, we rely on the idea of universal cover of a space, which is another simply connected space, with a covering map. This universal cover satisfies that the higher homotopy groups coincide with those of the initial space. For this purpose, we have first developed a simplicial version of the universal cover, which is represented by means of twisted Cartesian products. Then, we have developed an algorithm for computing the effective homology of the simplicial universal cover, based on the application of the homological perturbation theory. The algorithms have been implemented in both SageMath and Kenzo.

As a future work, our intention is to take advantage of the existing interface between SageMath and Kenzo (Cuevas-Rozo et al., 2021), so that all the calculations that Kenzo is able to perform thanks to this work (mainly, homotopy group calculations through the universal cover), can also be performed directly in SageMath in a transparent manner. For this case, the connection between both programs is not trivial and requires a deep study of the possibilities: to build the universal cover in Kenzo, we need a function from the edges to the fundamental group. This will require to develop a deeper integration between Kenzo and SageMath, that could allow to call SageMath functions from Kenzo.

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Algorithm 3: Effective homology of the universal cover

Input:

- a connected simplicial set X,
- the fundamental group of X, $\pi_1(|X|, v_*) = H$, with a finite number of elements,
- a map defined on the non-degenerate edges of X, $\tau : X_1^{ND} \to H$, such that $\tau(\partial_1 x) = \tau(\partial_2 x) \cdot \tau(\partial_0 x)$ for all $x \in X_2^{ND}$,
- the effective homology of X given by two reductions $\rho_1^X = (f_1^X, g_1^X, h_1^X) : DX_* \Rrightarrow C_*(X)$ and $\rho_2^X = (f_2^X, g_2^X, h_2^X) : DX_* \Rrightarrow EX_*$ such that the composition $h_2^X g_1^X \partial_n f_1^X$ satisfies the nilpotency condition.

Output: the effective homology of the simplicial model of the universal cover of X, $K(H,0) \times_\tau X$.

1 Construct the Cartesian product $K(H,0) \times X$.

2 Construct the Eilenberg–Zilber reduction $\rho_1 : C_*(K(H,0) \times X) \Rrightarrow C_*(K(H,0)) \otimes C_*(X)$, which in this case has been proven to be an isomorphism.

3 Construct the reductions $\rho_2 : C_*(K(H,0)) \otimes DX_* \Rrightarrow C_*(K(H,0)) \otimes C_*(X)$ and $\rho_3 : C_*(K(H,0)) \otimes DX_* \Rrightarrow C_*(K(H,0)) \otimes EX_*$ as the tensor product of the trivial reduction $C_*(K(H,0)) \Rrightarrow C_*(K(H,0))$ with the two reductions of the effective homology of X, $DX_* \Rrightarrow C_*(X)$ and $DX_* \Rrightarrow EX_*$ respectively (following the formulas presented in Proposition 2.10).

4 Apply the Basic Perturbation Lemma (Theorem 2.13) to the reduction ρ_1 with the perturbation δ corresponding to the twisting operator τ, obtaining a new reduction $\rho_1' : C_*(K(H,0) \times_\tau X) \Rrightarrow C_*(K(H,0)) \otimes_t C_*(X)$ and a perturbation δ' on $C_*(K(H,0)) \otimes C_*(X)$.

5 Apply the Trivial Perturbation Lemma (Theorem 2.12) to the reduction ρ_2 with the perturbation δ', obtaining a new reduction $\rho_2' : C_*(K(H,0)) \otimes_t DX_* \Rrightarrow C_*(K(H,0)) \otimes_t C_*(X)$ and a perturbation δ'' on $C_*(K(H,0)) \otimes_t DX_*$.

6 Apply the Basic Perturbation Lemma (Theorem 2.13) to the reduction ρ_3 with the perturbation δ'', obtaining a new reduction $\rho_3' : C_*(K(H,0)) \otimes_t DX_* \Rrightarrow C_*(K(H,0)) \otimes_t EX_*$.

return Composition of the reductions ρ_1', ρ_2', and ρ_3' (see diagram (3))

Algorithm 4: Twisted homology of a simplicial set with Abelian fundamental group.

Input:

- an effective connected simplicial set X with Abelian fundamental group G,
- a positive integer n,

Output: a presentation matrix of $H_n(\tilde{X})$ as $\mathbb{Z}[G]$-module.

Express the group G as direct sum of cyclic groups: $G = \mathbb{Z}/t_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/t_l\mathbb{Z} \oplus \mathbb{Z}^f$

Take the polynomial ring $R = \mathbb{Z}[x_1, \bar{x}_1, \dots, x_{l+f}, \bar{x}_{l+f}]$ and the ideal $I = (x_1\bar{x}_1 - 1, \dots, x_{l+f}\bar{x}_{l+f} - 1, x_1^{t_1} - 1, \dots, x_l^{t_l} - 1)$. Consider the isomorphism $\nu : \mathbb{Z}[G] \to R/I$.

Construct ordered lists $L_{n-1} = (\sigma_1^{n-1}, \dots \sigma_{m_{n-1}}^{n-1}), L_n = (\sigma_1^n, \dots \sigma_{m_n}^n), L_{n+1} = (\sigma_1^{n-1}, \dots \sigma_{m_{n+1}}^{n+1})$ with the non-degenerate simplices of dimensions $n-1, n$ and $n+1$.

Construct a $m_{n-1} \times m_n$ matrix M^n, and a $m_n \times m_{n+1}$ matrix M^{n+1}, both initialized with zero entries.

For each σ_j^n, and for each $0 \leq d < n$, if $\partial_d(\sigma_j^n) = \sigma_i^{n-1}$, increase $M_{i,j}^n$ by $(-1)^d$. If $\partial_n(\sigma_j^n) = \sigma_k^{n-1}$, increase $M_{k,j}^n$ by $(-1)^n \nu(\tau(\sigma_j^n)^{-1})$.

For each σ_j^{n+1}, and for each $0 \leq d < n+1$, if $\partial_d(\sigma_j^{n+1}) = \sigma_i^n$, increase $M_{i,j}^{n+1}$ by $(-1)^d$. If $\partial_{n+1}(\sigma_j^{n+1}) = \sigma_k^n$, increase $M_{k,j}^{n+1}$ by $(-1)^n \nu(\tau(\sigma_j^n)^{-1})$.

Compute the syzygy matrix N between the columns of M^n (that is, the columns of N generate the right kernel of M^n.

For each column c_i of M^{n+1}, express c_i as a linear combination of the rows of N obtaining a matrix R_1 such that $M^{n+1} = NR_1$.

Compute the syzygy matrix between the columns of N, obtaining a matrix R_2 whose columns generate the right kernel of N.

Output the matrix obtained by stacking R_1^T and R_2^T.