



On generic singularities of intersections of ellipsoids: the octahedron

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Abstract

The goal of this work is to study the smoothings of singular coaxial intersections of ellipsoids (where coaxial includes concentric) with generic singularities, with special attention to the three-dimensional case.

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Introduction

In a series of papers, the second named author has extensively studied the intersections of coaxial ellipsoids in \mathbb{R}^n (we will refer to them simply as *intersections* to simplify the text), specially the smooth ones; see [17] and the references therein. In [15, 16], he also studied some singular ones and in this work we continue their study. We are mostly interested in the case of isolated singularities and, more precisely, in the mildest ones (called *generic*), which are the analogue to ordinary double points in complex singularity theory. In future works we will deal with more general singularities.

We will recall how one can express the existence and type of a singularity in terms of the tuples of points defining the equations of the intersection, see (1.1). Also, we

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will recall the definition of the associated polytope of an intersection which in the case of smooth intersections is *simple*.¹

The main goal of this work is to study the deformations of a singular intersection, more precisely, the deformations of its equations giving nearby smooth intersections and less degenerate singular ones. The *smoothings* correspond to the connected components of the space of a deformation consisting only of smooth intersections.

After developing the basic facts about the theory in all dimensions we turn our attention to dimension 3 as we did in [2] for the smooth case. The vertices of a polytope associated with a smooth 3-intersections are simple and each one lies in exactly three coordinate hyperplanes. The generic singularities of 3-dimensional intersections come from vertices of the polytope that lie in four coordinate hyperplanes. There are two types of them, those for which the vertex is simple and those where the vertex lies in 4 faces of the polytope, called a 4-vertex for now. The singularities of the intersections coming from simple or 4-vertices are cones over $\mathbb{S}^0 \times \mathbb{S}^2$ or $\mathbb{S}^1 \times \mathbb{S}^1$, respectively.

Since we will be interested mainly in the simplest cases associated with a given polyhedron, we will study in detail only examples of the second type. We will begin by considering the associated intersection of the simplest one, the quadrangular pyramid. We give a detailed study of the topology of the singular intersection, its smooth part, and its smoothings.

Two other polyhedra are extensively studied in the work: the triangular bipyramid and the octahedron. As for the quadrangular pyramid, we study the topology of their singular intersections, their smooth parts, and their smoothings. The intersection of the triangular bipyramid has 12 singular points and its homology can be computed using [17, Theorem 5.21]: H_1 is trivial and $H_2 \cong \mathbb{Z}^{12}$; actually the intersection is simply connected.

The two possible smoothings are completely described. We pay special attention to the smooth part of the intersection. It can be described as a Galois cover of the complement of a simple link in $(\mathbb{S}^1)^3$, and it admits a complete hyperbolic metric, which turns out to be a topological invariant. This complete hyperbolic manifold has 12 ends and the singular intersection is a compactification obtained by adding one point to each end.

Since the smooth part of the intersection associated with the quadrangular bipyramid is homeomorphic to $(\mathbb{S}^1)^2 \times \mathbb{R}$, a similar property holds: it admits a complete euclidean metric and the singular intersection, the suspension over the torus, is obtained by compactifying the two ends.

The study of the intersection associated with the octahedron is more involved. The singular intersection has 96 singular points. Some data about its topology can be found in Proposition 5.4, namely, its fundamental group is \mathbb{Z}^4 . As for the triangular bipyramid the smooth part of the intersection admits also a complete hyperbolic structure with 96 ends: as before the singular intersection is obtained by compactifying these ends. Finally, we describe the topology of the smoothings, obtaining five 3-manifolds. Three of them are connected sums of products of spheres; one of these connected sums is obtained through two combinatorially distinct simple polytopes. A fourth smoothing

¹ A vertex of a polytope of dimension d is simple if it is contained in exactly d facets and the polytope is simple if all its vertices are simple.

is the product of \mathbb{S}^1 and a surface of genus 17. The last one is a graph manifold in the sense of Waldhausen. There are two families of distinct links with 96 components such that the contraction of each component gives the singular intersection.

An intersection and its polytope are related by a natural orbifold structure in the polytope, where all the faces are mirrors. The orbifold structure of the polytope minus the 4-vertices makes the connection with the geometric structures via the orbifold structures induced by suitable tessellations (hyperbolic for the triangular bipyramid and the octahedron and euclidean for the quadrangular pyramid). These ideas can be extended to other polytopes, e.g., the rhombic dodecahedron. The three hyperbolic structures on the triangular bipyramid, the octahedron, and the rhombic dodecahedron, have a common ancestor in a special tetrahedron, which allows us to compare the volume of the corresponding hyperbolic manifolds, an important topological invariant.

One last word about homeomorphism and diffeomorphism. In the smooth case a deformation of the intersection that does not include singular ones preserves its differentiable type. The same seems to be the case when there are only generic singularities and the deformation preserves them as such. But in the general case a deformation that preserves the singularities and their types may preserve only the topological type of the intersection.

In §1 and §2 we introduce the main notions in the paper, including illustrating examples. Deformations and smoothings are introduced in §3. In §4 we present a detailed description of the smoothings of the intersections associated with the quadrangular pyramid and the triangular bipyramid, including the surgeries related them. The section §5 is devoted to the study of the smoothings of the intersection associated with the octahedron, listing the possible ones, their relations, and the geometric properties. In the last section §6 we continue the exploration of the relationship of the intersections and some complete hyperbolic manifolds and orbifolds.

1 Settings

Let us consider the intersection Z of the points $(x_1, \dots, x_n) \in \mathbb{R}^n$ satisfying

$$\sum_{i=1}^n A_i x_i^2 = 0, \quad \sum_{i=1}^n x_i^2 = 1, \quad \text{with } A_i \in \mathbb{R}^m, \quad (1.1)$$

where the first m equations define a cone and the last one a sphere. To see that such an intersection coincides with an intersection of coaxial (and concentric) ellipsoids, up to a diagonal automorphism of \mathbb{R}^n and permutation of variables, it is enough to add large enough multiples of the last equation to the m homogeneous ones. Reciprocally, any coaxial collection of $m + 1$ ellipsoids with common center at the origin can be put in the above form. The collection of ellipsoids is determined by the n -tuple $\mathcal{A} := (A_1, \dots, A_n) \subset (\mathbb{R}^m)^n$.

Notation 1.1 Up to permutation, we can represent the tuple \mathcal{A} as $(v_1^{m_1}, \dots, v_r^{m_r})$, where $v_1, \dots, v_r \in \mathbb{R}^m$ and the superindices indicate the repetitions of these vectors, i.e., $m_1 + \dots + m_r = n$.

Let $Z_{\geq 0}$ be the intersection of Z with the first orthant $\mathbb{R}_{\geq 0}^n$ of \mathbb{R}^n . The variety Z can be reconstructed from $Z_{\geq 0}$ via the reflections through the coordinate hyperplanes. The intersection $Z_{\geq 0}$ is homeomorphic to the polytope

$$P := \left\{ (r_1, \dots, r_n) \in \mathbb{R}_{\geq 0}^n \mid \sum_{i=1}^n A_i r_i = 0, \sum_{i=1}^n r_i = 1 \right\}. \quad (1.2)$$

The union of the reflections of P through the coordinate hyperplanes is also homeomorphic to Z .

Definition 1.2 An intersection of ellipsoids determined by an n -tuple $\mathcal{A} = (A_1, \dots, A_n)$ in \mathbb{R}^m satisfies the property of *weak hyperbolicity* or (WH)² if the origin is not a convex combination of a subset of $\leq m$ vectors of \mathcal{A} .

This condition is equivalent to the transversality of the equations and therefore implies that Z is smooth, P is a simple polytope and that a small deformation of the n -tuple will give a diffeomorphic intersection and a combinatorially equivalent polytope.

When (WH) holds, the faces of P correspond to its non-empty intersections with the coordinate subspaces and will have the expected dimension given by transversality. In particular, $\dim Z = \dim P = n - m - 1$. For a deeper understanding of the general case we introduce now another way to relate \mathcal{A} and P that includes explicitly the degree of singularity of each face of the polytope.

Definition 1.3 Let $v_1, \dots, v_r \in \mathbb{R}^m$. A convex combination $t_1 v_1 + \dots + t_r v_r$, $t_i \in \mathbb{R}_{\geq 0}$, $t_1 + \dots + t_r = 1$, is *proper* if all the coefficients are positive.

Let us denote $[n] := \{1, \dots, n\}$. For each $J \subset [n]$, we define

$$C_J := \{(r_1, \dots, r_n) \in P \mid r_j > 0 \text{ if } j \in J, r_j = 0 \text{ if } j \notin J\}.$$

We denote by $\mathcal{F}_{\mathcal{A}}$ the set of subsets of $[n]$ such that $C_J \neq \emptyset$. For $J \in \mathcal{F}_{\mathcal{A}}$, let us denote the closure $F_J := \overline{C_J}$.

Lemma 1.4 The faces of the polytope P are $\{F_J \mid J \in \mathcal{F}_{\mathcal{A}}\}$. Moreover, the dimension of the space of convex combinations of the tuple $\mathcal{A}_J := (A_i \mid i \in J)$ is the dimension of the associated face F_J .

Remark 1.5 (WH) is equivalent to the following condition: $\forall J \in \mathcal{F}_{\mathcal{A}} \dim F_J + m + 1 = \#S$. In particular, the vertices of P are associated with sets J of cardinality $m + 1$ for which the origin is a proper convex combination of \mathcal{A}_J in a unique way, i.e., \mathcal{A}_J is affinely independent.

Remark 1.6 A tuple can be deformed by multiplying each A_i by a positive real number c_i . The differential topology of Z is preserved: starting with the equations of Z a

² The concept of weak hyperbolicity (WH) (and its equivalence with the transversality of the intersection) was introduced and named by Chaperon [6] and independently discovered (but not named) in [14, 24] for the case $m = 2$.

diagonal linear change of coordinates in \mathbb{R}^n will give the new homogeneous equations while changing the equation of the sphere into that of an ellipsoid. Then the intersection of the new cone with this ellipsoid is radially diffeomorphic to its intersection with the unit sphere. So we can assume, for example, that all non-zero A_i belong to the unit sphere in \mathbb{R}^m .

Example 1.7 Let us consider $m = 1$ and \mathcal{A} satisfying (WH). Then we can deform \mathcal{A} without changing the topology to have $\mathcal{A} := ((-1)^p, (1)^q)$. Simple algebraic operations show that Z is diffeomorphic to $\mathbb{S}^{p-1} \times \mathbb{S}^{q-1}$ and P is $\Delta_{p-1} \times \Delta_{q-1}$.

Example 1.8 The case $m = 2$ and \mathcal{A} satisfying (WH) can be also easily described. After deformation, $\mathcal{A} = (P_1^{r_1}, \dots, P_{2k+1}^{r_{2k+1}})$, $k > 0$ and $n = r_1 + \dots + r_{2k+1}$, where (P_1, \dots, P_{2k+1}) are equidistributed in the unit circle [14].

If $k = 1$, Z is $\mathbb{S}^{r_1-1} \times \mathbb{S}^{r_2-1} \times \mathbb{S}^{r_3-1}$. If $k > 1$ it is a connected sum of products of spheres. For example, for $k = 2$ and $r_i = 1$, Z is the compact oriented surface of genus 5.

Example 1.9 There is also a complete classification of smooth connected intersections in dimension 2. We obtain the surfaces of genus $g_n = 2^{n-3}(n-4) + 1$ for $n \geq 3$ [17]. The fact that these genera appear as intersection of quadrics was communicated to the second named author by F. Hirzebruch.

We are going now to set three conditions on the intersections which reduce the number of cases that need to be studied. We state them in two versions which we check in each case that they are equivalent.

(N1) There are no redundant equations. In terms of \mathcal{A} , it generates \mathbb{R}^m as a vector space.

Proof The equivalence of the two statements in (N1) is because if the second statement of (N1) does not hold, let $H \subsetneq \mathbb{R}^m$ be the subspace generated by \mathcal{A} , $m' := \dim H < m$. After a linear automorphism of \mathbb{R}^m , we may assume that the last $m - m'$ coordinates of these vectors vanish. Let $\mathcal{A}' \subset \mathbb{R}^{m'}$ be the tuple obtained by forgetting these last coordinates. The intersections associated with \mathcal{A} and \mathcal{A}' coincide and the only difference is the number of equations. \square

From now on, we will assume (N1) holds.

(N2) The intersection Z is not contained in a coordinate hyperplane $x_i = 0$. In terms of tuples, \mathcal{A} is not contained in a closed half-space of \mathbb{R}^m whose boundary is a linear hyperplane.

Proof The equivalence of the two statements in (N2) is because if the second statement of (N2) does not hold, after a linear automorphism of \mathbb{R}^m we may assume that the m^{th} -coordinates of the points in \mathcal{A} are ≥ 0 . As (N1) holds, at least one m^{th} -coordinate must be positive, say for A_i ; then $Z \subset \{x_i = 0\}$, and we can see this intersection in \mathbb{R}^{n-1} . \square

From now on, we will assume (N2) holds.

It is obvious that if \mathcal{A} is contained in an open half-space whose boundary is a linear hyperplane of \mathbb{R}^m , then the intersection is empty.

(N3) The intersection Z cuts all the coordinate hyperplanes. For all $i = 1, \dots, n$, the origin is a convex combination of $(A_1, \dots, A_i, \dots, A_n)$.

Proof The equivalence of the two statements in (N3) is because if the second statement of (N3) does not hold, let us assume that whenever the origin can be expressed as a convex combination of \mathcal{A} , the coefficient of A_j does not vanish. With these conditions $Z \cap \{x_j = 0\} = \emptyset$. The projection Z_j of Z in $\{x_j = 0\}$ is also an intersection and $Z \cong Z_j \times \mathbb{S}^0$. \square

From now on, we will assume (N3) holds, unless otherwise stated.

Sometimes we will start from an d -dimensional polytope $P \subset \mathbb{R}^d$ defined by a family \mathcal{I} of n affine inequalities $p_i(t_1, \dots, t_d) \geq 0$, $i = 1, \dots, n$, which define the polytope P . The image of the affine map $(t_1, \dots, t_d) \mapsto (p_1, \dots, p_n)$ intersects the first orthant of \mathbb{R}^m in the image of P and satisfies affine equations that can be put in the standard form (1.2).

When n is the number of facets of P then P gets embedded in \mathbb{R}^n with a facet in each hyperplane we get the best embedding of P , usually known as its *geometric embedding*, that reflects faithfully (and is essentially determined by) the combinatorics of P .

2 Singular intersections: settings

If (WH) does not hold the intersection is not transverse and Z will be singular as an algebraic variety. The easiest case where (WH) does not hold is when $\mathbf{0} \in \mathcal{A}$ and $m > 0$. There is a nice description of the topology of Z and the combinatorics of P in this case in terms of the one given by $\mathcal{A} \setminus \{\mathbf{0}\}$:

Proposition 2.1 ([16, p. 162], [17, Example 5.7]) *Let \mathcal{A}_1 be a tuple in \mathbb{R}^m and let $\mathcal{A}_2 = \mathcal{A}_1 \cup \{\mathbf{0}\}$. Let P_i, Z_i be the polytopes and intersections associated with each tuple. Then, P_2 is the cone over P_1 and Z_2 is the suspension of Z_1 .*

Hence given an intersection Z , its suspension is singular (except if $Z = \mathbb{S}^{n-1}$). If Z is smooth, its suspension has two singular points, each with a neighborhood which is a cone on Z .

For $m = 0$, there is no singular intersection. For $m = 1$, they can be classified since the operation of adding $\mathbf{0}$ is the only one that may induce a singularity.

Example 2.2 Let $m = 1$ and $\mathcal{A} = (-1, 1, 0)$. We have three subsets of S with proper convex combinations: $S_0 := \{\mathbf{0}\}$, yielding a vertex $F_3 := (0, 0, 1)$; $S_1 := \{1, -1\}$ yielding a vertex $F_{12} = (\frac{1}{2}, \frac{1}{2}, 0)$; $S_2 := \mathcal{A}$ yielding the edge F_{123} which coincides with the whole polytope P :

$$P : -r_1 + r_2 + 0 = 0, \quad r_1 + r_2 + r_3 = 1.$$

Note that this is a non-geometric embedding of $[0, 1] \subset \mathbb{R}$. Looking at the numerical invariants there is only a unique singularity at F_3 . The intersection Z is given by

$$Z : -x_1^2 + x_2^2 + 0 = 0, \quad x_1^2 + x_2^2 + x_3^2 = 1,$$

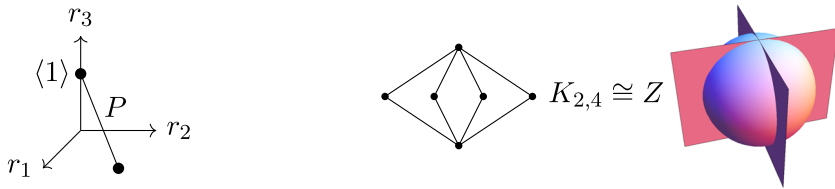


Fig. 1 The polytope P (see Remark 2.7 for the decoration), its reflection on the hyperplanes $K_{2,4}$ homeomorphic to Z

which is the union of two orthogonal meridians of the sphere \mathbb{S}^2 (so it is the suspension of its four points of intersection with the equator), its two singular points are the poles and is homeomorphic to the complete bipartite graph $K_{2,4}$ (Fig. 1).

Example 2.3 This can be generalized as follows. Let $\mathcal{A} := ((-1)^p, (1)^q, (0))$. The polytope P is the cone over $\Delta_{p-1} \times \Delta_{q-1}$ (its intersection with the hyperplane $r_n = 0$, recall Example 1.7 and Proposition 2.1) and apex in the r_n axis, which is its only singular point. In the same way, Z is the suspension of $\mathbb{S}^{p-1} \times \mathbb{S}^{q-1}$.

We may consider also the case with several 0's; the polytope will be obtained by iterated cones and the intersection by iterated suspensions.

Remark 2.4 For $p = q = 2$, we obtain P is the square pyramid and Z is the suspension of the torus. Note that in this case the polytope is not simple since one of the vertices fails to be simple.

Definition 2.5 We say that P is *singular* at F_S of depth $d_{F_S} := \dim F_S - \#S + m + 1$ (or d_S for simplicity) if $d_{F_S} > 0$, i.e., $\dim F_S \geq \#S - m$.

Remark 2.6 Under (WH), a continuous deformation of the points (A_1, \dots, A_n) , as long as the condition (WH) is preserved, does not affect the combinatorics of P nor the differential topology of Z , by Ehresmann type arguments. Without (WH), as far as a deformation is *equisingular*, the topology of Z does not change.

In terms of the polyhedron P , *equisingular* means that the combinatorics of P , including its relative position with respect to the hyperplanes, does not change. In terms of \mathcal{A} , *equisingular* means that the dimension of the spaces of proper convex combinations of the origin in \mathbb{R}^m remain constant. Note that the diffeomorphism class may change.

Remark 2.7 Note that the singularities of a polytope P depend on its relative position with respect to the coordinate hyperplanes. To exhibit the singular faces, we will decorate the faces with their depth. For the sake of simplicity we may omit it whenever the depth is the minimal possible one from the combinatorics of P , i.e., when the face is *geometrically embedded*; in particular for non-singular faces, the depth is zero and it will be usually omitted.

Example 2.8 Let $m = 1$ and $\mathcal{A} = (-1, 1, 0^2)$. We obtain the polytope in Fig. 2, where the edge F_{34} and its two vertices are singular of depth 1. It is not hard to check that Z is homeomorphic to the union of two spheres with a common equator.

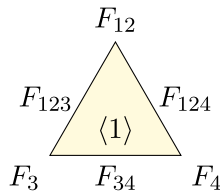


Fig. 2 Triangle with a singular edge

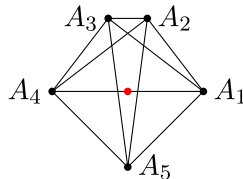


Fig. 3 Only one generic singularity in P

Definition 2.9 Let $v = F_S$ be a vertex of P . We say that P has an *isolated singularity* at v if P is singular at v and not singular at the faces containing v . We say that P has a *generic singularity* at v if moreover $0 = \dim F_S = \#S - m$, i.e. $\#S = m$ and $d_S = 1$.

The tuple $\mathcal{A}_S := (A_j \mid j \in S)$ affinely generates a subspace of dimension $m - 1$ and the origin is in the interior of the convex closure of \mathcal{A}_S .

Example 2.10 Let us consider a pyramid $P \subset \mathbb{R}^{n+1}$ with base an n -gon and such that each face is the intersection with one coordinate hyperplane. For $n \geq 4$ it corresponds to an isolated singularity, only generic if $n = 4$. The intersections in Examples 2.2 and 2.3 are generic singularities.

Let Z be an intersection given by $\mathcal{A} \subset (\mathbb{R}^m)^n$ with a generic singularity. The singularity is given by m points in \mathbb{R}^m which have the origin as a proper convex combination and affinely generate a hyperplane (this is the case of Example 2.3). We may assume that they are A_1, \dots, A_m and that they live in the hyperplane with last coordinate $a_m = 0$. The other points are distributed in each of the two semispaces $a_m > 0$ (p points) and $a_m < 0$ (q points). The local structure is shown in the next result:

Proposition 2.11 ([17, §5.5.3]) *Let $Z \subset \mathbb{R}^n$ be an intersection of dimension $n - m - 1$ defined by an n -tuple \mathcal{A} in \mathbb{R}^m and determining a polytope P . A neighborhood of a generic singular vertex in P is the cone over the product of two simplices $\Delta^{p-1} \times \Delta^{q-1}$ ($n - m - 1 = p + q - 1$). The number of coordinate hyperplanes disjoint from the singular vertex is m . This singular vertex produces 2^m singularities homeomorphic to the cone on a product of spheres $\mathbb{S}^{p-1} \times \mathbb{S}^{q-1}$.*

We end this section with several examples which illustrate the topology.

Example 2.12 For the tuple \mathcal{A} in Fig. 3 we have as P a square with four vertices F_{14} (depth 1), F_{135} , F_{235} , and F_{245} . It is a non-geometric embedding of $[0, 1]^2 \subset \mathbb{R}^2$. The manifold Z is the union of two tori with four points in common.

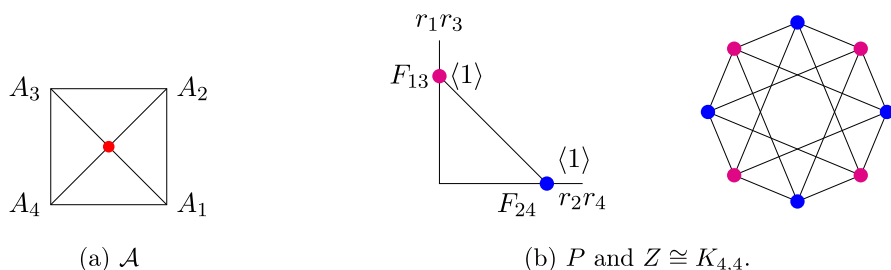


Fig. 4 Two generic singularities

Example 2.13 Figure 4 illustrates an example with two generic singular vertices; the intersection Z is homeomorphic to the complete bipartite graph $K_{4,4}$; each singular vertex is associate to one partition.

We can give a normalized description of all the tuples \mathcal{A} in \mathbb{R}^2 with only generic singularities, compare with Example 1.8. We sketch the ideas of how to reach the description stated in the next proposition. First, we can radially move the points A_i into the unit circle. In the unit circle we have the following allowed deformations. If $A_{i_2} = -A_{i_1}$, we can deform both simultaneously without crossing any other point; if $-A_{i_1} \notin \mathcal{A}$ we can deform A_{i_1} as $A_{i_1,t}$ ($A_{i_1,0} = A_{i_1}$) and keep fixed the other points, if $A_{i_1,t}$ is never antipodal to any A_j , $j \neq i_1$. Let r be the number of generic singularities; there are r couples of antipodal points and we can deform them such that they become equidistributed. Since the singularities are generic these points cannot be repeated. Let A_{i_1}, A_{i_2} be two consecutive distinct points such that $-A_{i_1}, -A_{i_2} \notin \mathcal{A}$ and let $C(A_{i_1}, A_{i_2})$ be the shorter arc between them; if there is no point of \mathcal{A} in the interior of the opposite of $C(A_{i_1}, A_{i_2})$ we can deform A_{i_2} into A_{i_1} . After performing all the possible identifications, the next result is straightforward though its proof is cumbersome.

Proposition 2.14 *Let $\mathcal{A} \subset \mathbb{R}^2$ with r generic singularities. Then there exists an equisingular deformation such that*

$$\mathcal{A} = (P_1, \dots, P_{2r}) \cup \bigcup_{i=1}^{2r} \underbrace{(Q_{i,1}^{n_{i,1}}, \dots, Q_{i,m_i}^{n_{i,m_i}})}_{\mathcal{A}_i}$$

where

- (a) (P_1, \dots, P_{2r}) are equidistributed; in particular, $P_j + P_{r+j} = 0$;
- (b) \mathcal{A}_i is an ordered tuple between P_i and P_{i+1} ($P_{2r+1} := P_1$);
- (c) $|m_i - m_{i+r}| \leq 1$;
- (d) between two distinct consecutive points $Q', Q'' \in \mathcal{A}_i$, there exists exactly a unique point $Q \in \mathcal{A}_{i+r}$ such that $-Q \in C(Q', Q'')$; the same statement holds interchanging \mathcal{A}_i and \mathcal{A}_{i+r} .

If $r = 1$, then $m_1, m_2 \geq 1$.

Example 2.15 The connected intersections of dimension 2 with generic singularities are given by n -polygons embedded in \mathbb{R}^{n+r} , $r \leq n$, where r vertices of the polygon are in the intersection of three coordinate hyperplanes. Compare with the smooth case in Example 1.9 where the polygon is geometrically embedded.

3 Deformations and smoothings

Every singular intersection can be deformed slightly and become transversal. In most cases different slight deformations give intersections with different topological and differentiable types. We now study all the possibilities that can arise like this, first by considering all the possible slight deformations and then looking for lower dimensional deformations that include all the possible smooth types as well as the less singular ones that separate them.

Let $\mathcal{A} = (A_1, \dots, A_n) \in (\mathbb{R}^m)^n$ satisfying (N2), (N2); let Z be the intersection of ellipsoids in \mathbb{R}^n they define and P the corresponding polytope. For $\varepsilon > 0$ let U be the product of the balls $\mathbb{B}(A_i; \varepsilon) \subset \mathbb{R}^m$. Each $\mathcal{A}_t \in U$ defines an intersection Z_t and the corresponding polytope P_t .

The open set U admits a stratification \mathcal{S} such that in any stratum the embedded type of P_t and, therefore, also the topological type of Z_t remains constant. The open strata will consist of tuples that satisfy (WH) and the differentiable type of Z_t and the combinatorial type of P_t remain constant. A similar argument happens for the other strata, in general only for the topological type.

Definition 3.1 With the above notation, we say that U is the *domain of a deformation* if \mathcal{A} is in the closure of all the strata and $\forall t \in U$ the conditions (N1) and (N2) hold.

Each open stratum is called a *smoothing* of \mathcal{A} and we will use this same name for the unique type of all the tuples of the stratum and their associated intersections and polytopes, since this cannot cause any confusion. A stratum S where $0 \notin S$ and the Z_t is still singular for $t \in S$ is called a *partial smoothing*.

In many cases we will restrict to a subset $V \subset U$ that contains \mathcal{A} and the conditions about the strata hold for the induced stratification. V is called a *top-versal deformation* if all the topological types of smoothings in U appear in V .

Remark 3.2 Note that even if \mathcal{A} satisfies (N3) it may happen that $\mathcal{A}_t \in U$ do not, i.e., Z is connected and some Z_t are disconnected.

Example 3.3 Let us consider $\mathcal{A} := ((-1)^p, (1)^q, 0) \subset \mathbb{R}$. We can take as U the product of intervals of radius $\varepsilon < \frac{1}{2}$, p around the points -1 , one around the point 0 , and q around the points -1 . There are three strata depending of the value of the parameter t_0 in the interval around 0 :

- (i) If $t_0 < 0$ we get an open smoothing stratum with intersection $\mathbb{S}^p \times \mathbb{S}^{q-1}$ and polytope $\Delta_p \times \Delta_{q-1}$.
- (ii) If $t_0 > 0$ we get an open smoothing stratum with intersection $\mathbb{S}^{p-1} \times \mathbb{S}^q$ and polytope $\Delta_{p-1} \times \Delta_q$.
- (iii) If $t_0 = 0$ we get a closed singular stratum of codimension one with intersection the suspension of $\mathbb{S}^{p-1} \times \mathbb{S}^{q-1}$ and polytope the cone on $\Delta_p \times \Delta_{q-1}$.

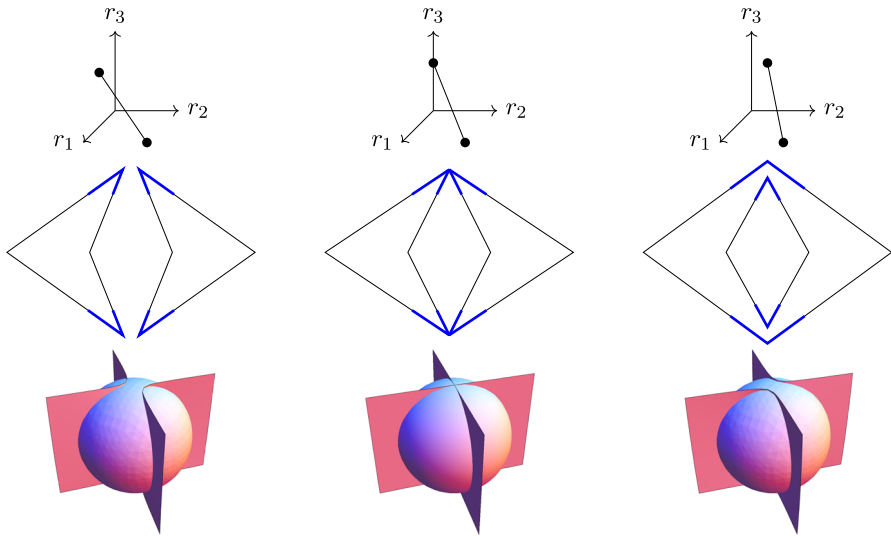


Fig. 5 Smoothings for $p = q = 1$. In the first row, we have the polytopes. In the second row, an abstract representation of the smoothings, emphasizing what replace the singularities. In the third row, the embedded intersections. The middle column is for the singular case, and the other columns for the smoothings

The deformation $V \subset U$ where the parameter t_0 takes all values within its interval and all the other parameters are 0 contains all those topological types and is therefore a top-versal deformation. This situation generalizes to the case with only generic singularities.

The two possible smoothings for $p = q = 1$ are in Fig. 5.

Example 3.4 Example 2.13 has generic singularities in dimension 1. A top-versal deformation has parameters in \mathbb{R}^2 with strata the origin, the four open semiaxes (whose intersections are two copies of the particular case of Example 3.3 ($p = q = 1$)) and the four open quadrants (whose intersections are four copies of \mathbb{S}^1 , i.e. $\mathbb{S}^1 \times \mathbb{S}^0 \times \mathbb{S}^0$).

Observe that here, as in the case $p = q = 1$ of the Example 3.3, all the strata of positive dimension give disconnected intersections and only one topological type due to the symmetries of the examples. We pass from one smoothing to the other by surgeries.

Example 3.5 In dimension 2, the polytope associate to the simplest intersection with generic singularities is given by a triangle Δ_1 with one vertex in the intersection of three coordinate hyperplanes. Figure 6 shows the singular intersection and the two possible smoothings: $\mathbb{S}^1 \times \mathbb{S}^1$ and $\mathbb{S}^0 \times \mathbb{S}^2$ (Fig. 7).

Example 3.6 Let us consider the intersection associated with a triangle Δ_2 with two generic singular vertices, see Fig. 8. The singular variety $Z(\Delta_2)$ is in (e); its smoothings are $\mathbb{S}^0 \times \mathbb{S}^0 \times \mathbb{S}^2$ (a), $\mathbb{S}^0 \times \mathbb{S}^1 \times \mathbb{S}^1$ (b), and \mathcal{F}_5 (c).

There are two partial smoothings, the square C_1 with one generic singular vertex and a triangle $\tilde{\Delta}_2$ disjoint with one coordinate hyperplane. The singular variety $Z(C_1)$

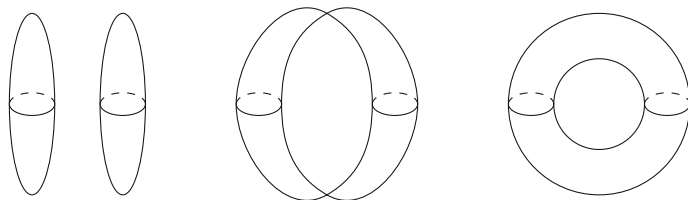


Fig. 6 Smoothings of Example 3.5

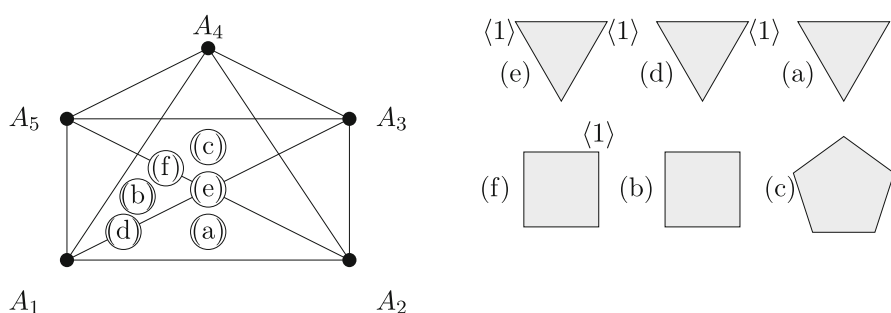


Fig. 7 The left-hand side represents different \mathcal{A} 's depending on whether the origin in \mathbb{R}^2 is. The possibilities are the centers of the circles. The right-hand side represents the polytopes for the intersections in Fig. 8

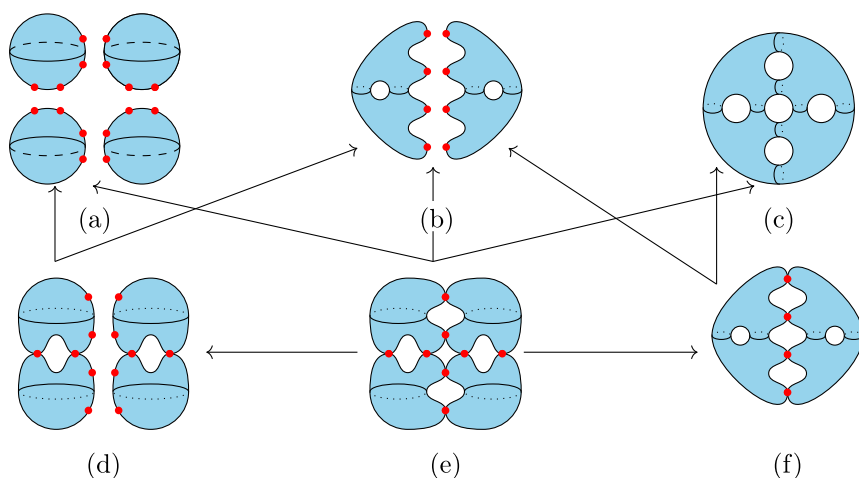


Fig. 8 Intersections associated with a triangle Δ_2 with two generic singular vertices.

is in (f); its smoothings are in (b) and (c). Finally, $Z(\tilde{\Delta}_2) \cong \mathbb{S}^0 \times Z(\Delta_1)$ is in (d) with smoothings in (a) and (c).

Remark 3.7 Let Z be an intersection with only generic singularities. We can define a top-versal deformation of Z from the point of view of polytopes. The effect of this top-versal deformation is to *simplify* the singular vertices. In dimension 1, a singular vertex is in the intersection of two coordinate hyperplanes, and the smoothings consist

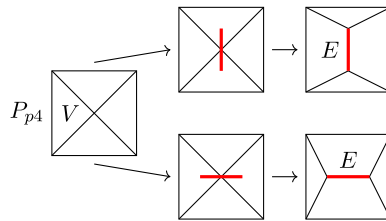


Fig. 9 Smoothing at the 4-vertex V

in moving the vertex to one of those hyperplanes. A similar discussion can be done in dimension 2. In dimension 3 there are two types of singular vertices. A vertex whose link is a quadrangle admits two smoothings as in Fig. 9. If it is a simple vertex in four coordinate hyperplanes, either the vertex moves to the intersection of three such hyperplanes (the polytope does not change and the intersection becomes multiplied by \mathbb{S}^0) or the polytope becomes truncated at this vertex.

The topological types in adjacent open strata (separated by a codimension 1 stratum wall) are related by a move, called *flip* in [5] where they applied it to describe the wall-crossing result in the context of moment-angle manifolds.

Definition 3.8 ([5]) Let P and Q be two simple polytopes of the same dimension q . Let W be a simple polytope of dimension $q + 1$. We say that W is an *elementary cobordism* between P and Q if P and Q are disjoint facets of W and $W \setminus (P \sqcup Q)$ contains a unique vertex v . Then, Q is obtained from P by a *flip* of type (a, b) , where a is the number of edges from v to vertices in P and b is the number of edges from v to vertices in Q .

4 3-dimensional smoothings

In dimensions 1 and 2, it is possible to describe all the types of intersections and their smoothings. In higher dimensions this is not possible because there are too many polytopes.

In dimension 3 we will study the smoothings of geometrically embedded polyhedra, namely, some non-simple polytopes $P \subset \mathbb{R}^3$ with vertices lying at no more than 4 faces.

Let ℓ be the number of vertices lying in four faces, called 4-vertices in the sequel. There is a top-versal deformation with parameter space homeomorphic to a neighbourhood of the origin in \mathbb{R}^ℓ , where the 2^ℓ orthants correspond to the smoothings.

For each orthant there are ℓ possible $(2, 2)$ -flips. Note that the automorphism group of the polytope acts on these sets and the number of differentiable types of smoothings may be less than 2^ℓ .

We are going to associate to any such polytope a decorated graph.

Definition 4.1 Let P a 3-dimensional polytope with vertices lying at no more than 4 faces and geometrically embedded. The *decorated graph* $\Gamma(P)$ of P is defined as follows:

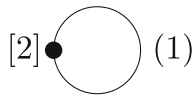


Fig. 10 $\Gamma(P_{p4})$, see Definition 4.1

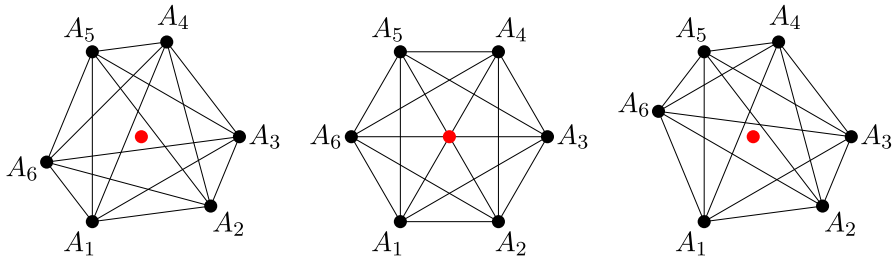


Fig. 11 $\mathcal{A} \subset \mathbb{R}^2$ defining the equations of the bipyramid and the topological types of the smoothings

- The vertices are the orbits of the set of orthants under the action of $(\mathbb{Z}/2)^\ell$.
- The edges are the orbits of the $(2, 2)$ -flips associated with each non-simple vertex.
- The non-loop edges are arbitrarily oriented.
- Each vertex is decorated with $[\alpha]$ where α is the size of the orbit.
- Each non-loop edge is decorated with $(\beta_1, -\beta_2)$, where β_1 is the number of the connecting flips in the direction of the edge and β_2 is the number of the connecting flips in the opposite direction of the edge.
- The loop edges are decorated with (β) where β is the number of flips which do not change the orbit.

Examples of these decorated graphs can be found in Figs. 10 and 14.

4.1 Smoothings of the square pyramid

Consider the square pyramid P_{p4} , the polytope associated with $\mathcal{A} = ((-1)^2, 0, (1)^2)$ having a unique 4-vertex V . Recall that $Z(P_{p4})$ has two singular points, the two apices of the suspension over the torus $\mathbb{S}^1 \times \mathbb{S}^1$, see Proposition 2.1. The neighbourhood of each of these singular points is a cone on the torus.

The smoothings correspond to $\{\mathcal{A}_t \mid t > 0\}$ and $\{\mathcal{A}_t \mid t < 0\}$, where $\mathcal{A}_t := ((-1)^2, t, (1)^2)$, see Fig. 9 for the effect on the polytopes. Observe that a new edge E appears to replace the apex V .

The group of symmetries of P_{p4} is the dihedral group of order 8 and acts transitively on the two smoothings. Hence the corresponding polyhedron P'_{p4} is a triangular prism, see Fig. 9, with 5 faces and 6 vertices. The new edge E is a lateral edge of the prism. The manifold $Z(P'_{p4})$ obtained by reflection on the faces of P'_{p4} is diffeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$ which is the double of a solid torus.

The edge E is contained in two faces, hence under the reflections there are $2^3 = 8$ copies of E in $Z(P'_{p4})$; it is disjoint to exactly one face, and as a consequence they form two circles, $\mathbb{S}^1_1 = \{p_1\} \times \mathbb{S}^1$ and $\mathbb{S}^1_2 = \{p_2\} \times \mathbb{S}^1$, in such a way that $Z(P')$ is

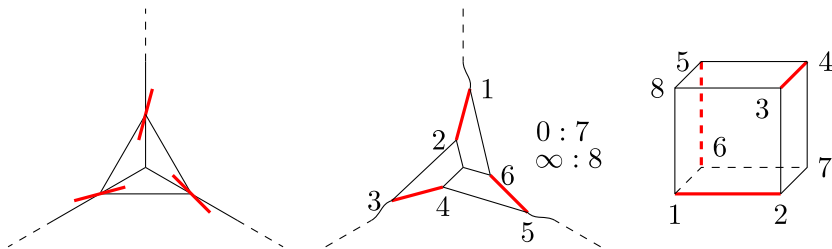


Fig. 12 C-smoothing of the triangular bipyramid. The pictures on the left represent the bipyramid after stereographic projection where the point at infinity is one of the simple vertices

the union by the boundary of the two solid tori which are neighbourhood of these two circles; the solid tori are the product $\mathbb{D}_i \times \mathbb{S}^1$, where \mathbb{D}_i is a closed disk centered at p_i ; both disks cover \mathbb{S}^2 and have common boundary.

The equations of the polytope defined by \mathcal{A}_t in \mathbb{R}^5 are $0 \leq r_i \leq 1$:

$$\begin{cases} r_1 + r_2 + r_3 + r_4 + r_5 = 1 \\ r_1 - r_2 + r_3 - r_4 + r_5 = 0. \end{cases} \quad (4.1)$$

Let us consider the simple polytope $\Delta^2 \times \Delta^2$ of dimension 4 in \mathbb{R}^6 defined by the equations

$$\begin{cases} r_1 + r_3 + \frac{1}{2}r_5 = \frac{1}{2} \\ r_2 + r_4 + \frac{1}{2}r_6 = \frac{1}{2} \end{cases} \Leftrightarrow \begin{cases} r_1 + r_2 + r_3 + r_4 + \frac{1}{2}(r_5 + r_6) = 1 \\ r_1 - r_2 + r_3 - r_4 + \frac{1}{2}(r_5 - r_6) = 0, \end{cases} \quad 0 \leq r_i \leq 1.$$

Let us cut it by a hyperplane $(1-t)r_5 - (1+t)r_6 = 0$, $|t|$ small enough. With the coordinates r_1, \dots, r_4, r'_5 in this hyperplane, where $r'_5 := \frac{r_5+r_6}{2}$, the equations of the cut become those of (4.1).

Remark 4.2 The relation between the 2 triangular prisms obtained by smoothing is a flip of type (2, 2). In a more topological language we pass from one copy to the other by a 0-Dehn surgery on two fibers of the trivial fibration $\mathbb{S}^2 \times \mathbb{S}^1$.

The following result is easy.

Proposition 4.3 Let $\check{Z}(P_{p4})$ be the space obtained from P_{p4} by reflections minus the points coming from the non-simple vertex V , which coincides with the space obtained from P'_{p4} by reflections minus the circles coming from the edge E . This space is homeomorphic to $\mathbb{S}^1 \times \mathbb{S}^1 \times (-1, 1)$. In particular,

- (a) $\pi_1(Z(P_{p4})) = 1$,
- (b) $\pi_1(Z(P'_{p4})) = \mathbb{Z}$,
- (c) $\pi_1(\check{Z}(P_{p4})) = \mathbb{Z}^2$.

4.2 Smoothing of triangular bipyramid

The polytope associated with $\mathcal{A} \subset \mathbb{R}^2$ as in the center of Fig. 11 is the triangular bipyramid BP_3 in \mathbb{R}^6 . There are three singular 4-vertices, each one with two possible smoothings, and therefore there exist $2^3 = 8$ simple polyhedra. The bipyramid is defined by $\mathcal{A} := (A_1, \dots, A_6)$, where $A_j := (\cos \alpha_j, \sin \alpha_j)$, and $\alpha_j := \frac{\pi(j-3)}{3}$.

Note that deforming A_1, A_2, A_4, A_5 to be the vertices of the square of edge 2 centered at the origin, the equation of the triangular bipyramid in \mathbb{R}^6 is

$$\begin{aligned} -r_1 + r_2 + r_3 + r_4 - r_5 - r_6 &= 0 \\ -r_1 - r_2 + r_4 + r_5 &= 0 \\ r_1 + r_2 + r_3 + r_4 + r_5 + r_6 &= 1. \end{aligned}$$

We can define a top-versal smoothing with parameter space $(-\varepsilon, \varepsilon)^2$, for some $0 < \varepsilon \ll 1$, where

$$\mathcal{A}_s := (A_1, A_{2,s}, A_3, A_{4,s}, A_5, A_{6,t}),$$

where $A_{2,s} := (\cos(\alpha_2 + s), \sin(\alpha_2 + s))$, $A_{4,s} := (\cos(\alpha_4 + s), \sin(\alpha_4 + s))$, and $A_{6,t} := (\cos(\alpha_6 + t), \sin(\alpha_6 + t))$, $s, t \in (-\varepsilon, \varepsilon)$. This smoothing contains all the topological types.

We come back to the general smoothing defined by *moving* the 4-vertices in BP_3 , for which we have give $2^3 = 8$ possible smoothings. The group of symmetries of BP_3 (isomorphic to $\mathfrak{S}_3 \times \mathbb{Z}/2$) acts on this space. This group is generated by the symmetry group of the intersection of the two pyramids and the exchange of the apices.

Proposition 4.4 *Under the action of the symmetry group there are two orbits:*

- (i) *The cube C (2 smoothings), see Fig. 12;*
- (ii) *the polyhedron T_2 (6 smoothings), see Fig. 13, which is the book with a pentagonal leaf or, equivalently, the double truncated tetrahedron.*

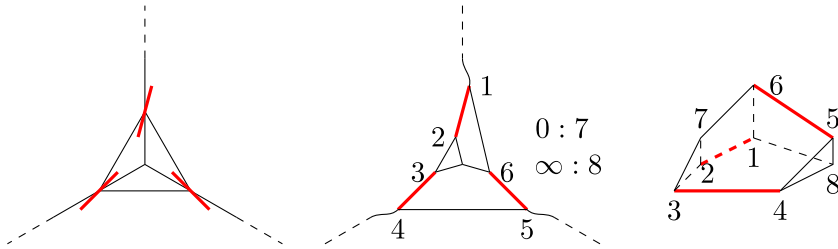
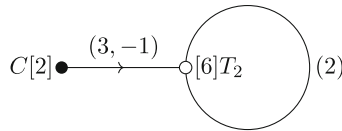
Any (2, 2)-flip of C produces T_2 ; two of the flips in a red edge of T_2 keep it, and the third one passes it to C .

Proof In Fig. 12 (resp. Fig. 13) we have a smoothing symmetric with a 3-cycle in $\mathfrak{S}_3 \subset \text{Aut } BP_3$ (resp. a transposition). The two elements of the orbits of C are the one in Fig. 12 and the one with all the opposite smoothings. From Fig. 13, one can easily check how to find the smoothings for T_2 . \square

The polyhedra have six bidimensional faces, one on each hyperplane $r_i = 0$ and then the two obtained manifolds are connected. Using, e.g. [9], we have that $Z(C) = \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1 = \mathbb{T}^3$.

Observe that the only simple polyhedra with 6 faces and 8 vertices are the cube C and the double truncated tetrahedron T_2 and both are obtained in the smoothing of BP_3 .

To obtain the manifold $Z(T_2)$ we use the following result.


 Fig. 13 T_2 -smoothing of the triangular bipyramid

 Fig. 14 Decorated graph $\Gamma(BP_3)$

Proposition 4.5 ([9, Theorem 2.1], cf. also [5]) *Given a simple polytope P of dimension d , let $P_{\text{sf}}(v)$ be the result of truncating the vertex v and let n be the number of facets. Then $Z(P_{\text{sf}}(v))$ is diffeomorphic to the connected sum*

$$Z(P) \# Z(P) \# (2^{n-d} - 1)(\mathbb{S}^{d-1} \times \mathbb{S}^1).$$

Remark 4.6 Observe that the topology of $Z(P_{\text{sf}}(v))$ does not depend on which vertex of P is truncated. This gives infinitely many examples of different P with the same $Z(P)$, in particular, the result of the operation $P_{\text{sf}}(v)$ on a simplex successively n times is independent of the vertices chosen. Curiously, in any dimension we get the same number of copies of $\mathbb{S}^{d-1} \times \mathbb{S}^1$ since it only depends on $m - d$. This includes the genera of the surfaces associated with the polygons.

Let Δ be a tetrahedron and let v, w be two vertices. Then,

$$Z(\Delta_{\text{sf}}(v)) \cong Z(\Delta) \# Z(\Delta) \# (2^{4-3} - 1)(\mathbb{S}^2 \times \mathbb{S}^1) \cong \mathbb{S}^2 \times \mathbb{S}^1$$

and

$$Z(T_2) = Z((\Delta_{\text{sf}}(v))_{\text{sf}}(w)) \cong \mathbb{S}^2 \times \mathbb{S}^1 \# \mathbb{S}^2 \times \mathbb{S}^1 \# (2^{5-3} - 1)(\mathbb{S}^2 \times \mathbb{S}^1) \cong 5\mathbb{S}^2 \times \mathbb{S}^1.$$

The manifolds $Z(C)$ and $Z(T_2)$ are tessellated by $2^6 = 64$ copies of C or T_2 respectively. Recall that the $(2, 2)$ -flips are 0-surgeries on the following link. Each one of the three red edges E_i in the cube C replacing a singular vertex of the bipyramid BP_3 give rise to a link $L_i \subset Z(C)$ of 2^2 components, each of these components is made by 2^2 copies of E_i and is one fiber of the 3-dimensional torus. Observe that a tubular neighbourhood of L_i has a *canonical longitude* defined by a parallel fiber or by the intersection with the copies in $Z(C)$ of one of the two faces of C containing E_i . Then 0-surgery in $L_i \subset Z(C)$ produces the manifold $Z(T_2)$. We have proved the following result.

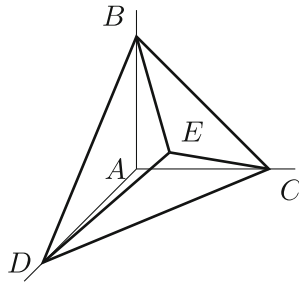


Fig. 15 Bipyramid

Proposition 4.7 *The result of 0-surgery on a link composed by 4 parallel fibers of the 3-dimensional torus \mathbb{T}_3 is the manifold $5\#(\mathbb{S}^2 \times \mathbb{S}^1)$*

Analogously, a 0-surgery in the 4-component link in $Z(T_2)$ defined by the red edge belonging to a pentagonal face in T_2 produces the manifold $Z(C)$, meanwhile 0-surgery in the 4-component link in $Z(T_2)$ defined by the red edge belonging to a square face in T_2 produces again $Z(T_2)$.

The graph $\Gamma(BP_3)$ is in Fig. 14: recall that the weights of the edges refer both to the number of $(2, 2)$ -flips in the polytopes and to the number of equivariant 0-surgeries in the link of the manifold.

4.3 Hyperbolic bipyramid

There exists a finite volume hyperbolic triangular bipyramid (Fig. 15) whose three 4-vertices are ideal points and whose dihedral angles are $\frac{\pi}{2}$. This result follows from Andreev's Theorem ([3], [4, Chap. 6, Theorem 2.8]). We give here a construction of such hyperbolic triangular bipyramid.

Proposition 4.8 *There is a bipyramid BP_H in the three-dimensional hyperbolic space \mathbb{H}^3 where all the 4-vertices are ideal.*

Proof We use the Klein model of the hyperbolic three-dimensional space \mathbb{H}^3 . It consists in the open ball \mathbb{B}^3 of radius 1 in the three-dimensional Euclidean space \mathbb{E}^3 . To every k -dimensional plane $\Pi \subset \mathbb{H}^3$ there corresponds a Euclidean plane $\tilde{\Pi} \subset \mathbb{E}^3$ such that $\Pi = \tilde{\Pi} \cap \mathbb{B}^3$. This model is not conformal and the hyperbolic angle between two planes Π_1 and Π_2 is the Euclidean angle between their boundary circles $\tilde{\Pi}_1 \cap \mathbb{B}^3$ and $\tilde{\Pi}_2 \cap \mathbb{B}^3$.

Let us consider a triangular bipyramid in \mathbb{B}^3 with 4-vertices at the points $D = (1, 0, 0)$, $C = (0, 1, 0)$, and $B = (0, 0, 1)$ and simple vertices at $A = (0, 0, 0)$ and $E = (a, a, a)$ for some $a \in \left(\frac{1}{3}, \frac{\sqrt{3}}{3}\right)$.

It is clear that the dihedral angles at the edges AB , AC and AD are $\frac{\pi}{2}$ because the intersections of the coordinate planes and the unit sphere are circles that meet orthogonally. We can compute a to have also $\frac{\pi}{2}$ dihedral angles at the edges BC , CD , and BD . The edge CD is the intersection $ACD \cap ECD$. The plane $\tilde{\Pi}_{ECD}$ containing

ECD , and hence the points $D = (1, 0, 0)$, $C = (0, 1, 0)$ and $E = (a, a, a)$, is given by the equation

$$ax + ay + (1 - 2a)z = a.$$

The plane $\tilde{\Pi}_{ACD}$ is $z = 0$. The intersection $\tilde{\Pi}_{ACD} \cap \tilde{\Pi}_{ECD}$ is a line intersecting $\partial\mathbb{B}^3$ at the points $(1, 0, 0)$ and $(0, 1, 0)$. The angle between the circles $C_1 = \tilde{\Pi}_{ACD} \cap \partial\mathbb{B}^3$ and $C_2 = \tilde{\Pi}_{ECD} \cap \partial\mathbb{B}^3$ is the dihedral angle in CD . The tangent vector v_1 to C_1 in $D = (1, 0, 0)$ is proportional to the cross product of the normal vector of the plane $\tilde{\Pi}_{ACD}$ and the normal vector to the tangent plane to $\partial\mathbb{B}^3$ at the point $D = (1, 0, 0)$:

$$v_1 = (a, a, 1 - 2a) \times (1, 0, 0) = (1, 1 - 2a, -a)$$

Analogously the tangent vector v_2 to C_2 in $D = (1, 0, 0)$ is the cross product

$$v_2 = (0, 0, 1) \times (1, 0, 0) = (0, 1, 0)$$

Therefore the formula for the scalar product and the condition on the angle to be $\frac{\pi}{2}$ allows us to compute a :

$$0 = \langle v_1, v_2 \rangle = \langle (1, 1 - 2a, -a), (0, 1, 0) \rangle = 1 - 2a = 0 \Rightarrow a = \frac{1}{2}$$

By symmetry, the dihedral angles in BC , and BD is also $\frac{\pi}{2}$. Therefore

$$\begin{aligned}\tilde{\Pi}_{ECD} &: \frac{1}{2}x + \frac{1}{2}y = \frac{1}{2} \Leftrightarrow x + y = 1 \\ \tilde{\Pi}_{EBC} &: \frac{1}{2}y + \frac{1}{2}z = \frac{1}{2} \Leftrightarrow y + z = 1 \\ \tilde{\Pi}_{EBD} &: \frac{1}{2}x + \frac{1}{2}z = \frac{1}{2} \Leftrightarrow x + z = 1.\end{aligned}$$

The angle between $\tilde{\Pi}_{ECD}$ and $\tilde{\Pi}_{EBC}$ at the point $C = (0, 1, 0)$ is computed as before.

$$\begin{aligned}v_{\tilde{\Pi}_{ECD}} &= (1, 1, 0) \times (0, 1, 0) = (0, 0, 1) \\ v_{\tilde{\Pi}_{EBC}} &= (0, 1, 1) \times (0, 1, 0) = (1, 0, 0) \\ \langle v_{\tilde{\Pi}_{ECD}}, v_{\tilde{\Pi}_{EBC}} \rangle &= \langle (0, 0, 1), (1, 0, 0) \rangle = 0\end{aligned}$$

By symmetry again, the angle in all the others edges is also $\frac{\pi}{2}$. \square

There is a tessellation \mathfrak{T}_{BP} by these bipyramids in \mathbb{H}^3 . Let G_{BP} the subgroup of $\text{Aut } \mathbb{H}^3$ generated by the hyperbolic reflections on the six planes containing the faces of BP_H which is the automorphism group of \mathfrak{T}_{BP} . The reflections corresponding to facets sharing an edge commute as they are orthogonal.

4.4 Groups and the bipyramid

The 2-skeleton of the tessellation by bipyramids is formed by hyperbolic planes tessellated by triangles with two ideal vertices, and the 1-skeleton are geodesics. The quotient \mathbb{H}^3/G_{BP} defines a hyperbolic orbifold structure **BP** in BP_H .

Let us consider the natural epimorphism $\omega_{BP} : G_{BP} \rightarrow (\mathbb{Z}/2)^6$. Recall that BP_3 has a natural orbifold coming from the fact that BP_3 is the quotient of $Z(BP_3)$ under the action of $(\mathbb{Z}/2)^6$ by reflections on the coordinate hyperplanes of \mathbb{R}^6 .

The existence of this natural epimorphism and the orbifold structure holds in general for any intersection Z and its associated polytope P : since P is the quotient of Z by an action of $(\mathbb{Z}/2)^n$, it acquires a natural orbifold structure. If a point is contained in k coordinate hyperplanes, its isotropy group is $(\mathbb{Z}/2)^k$. The orbifold fundamental group of P is

$$\pi_1^{\text{orb}}(P) = \langle a_1, \dots, a_n \mid a_i^2 = 1, [a_i, a_j] = 1 \text{ if } P \cap \{x_i = x_j = 0\} \neq \emptyset \rangle,$$

see [11, 12] for the notion of *orbifold fundamental group*. In the case of a geometric embedding of a polytope with n facets in \mathbb{R}^n , the generators are in bijection with the facets and two generators commute if and only if the two facets have non-empty intersection. The case of simple polytopes was studied in [7, Lemma 4.4]; in this case two generators commute if and only if the facets intersect in a codimension 2 face. The fundamental group of Z is the kernel of the natural epimorphism $\pi_1^{\text{orb}}(P) \rightarrow (\mathbb{Z}/2)^n$.

For the case of geometrically embedded polyhedra of dimension 3, let \check{P} be the complement of the non-simple vertices in P and let $\check{Z}(P) \subset Z$ be its preimage under the quotient map. Note that \check{P} inherits a natural structure of orbifold for which

$$\pi_1^{\text{orb}}(\check{P}) = \langle a_1, \dots, a_n \mid a_i^2 = 1, [a_i, a_j] = 1 \text{ if } \check{P} \cap \{x_i = x_j = 0\} \neq \emptyset \rangle.$$

The following result is straightforward from Proposition 4.8.

Corollary 4.9 *The two orbifold structures over **BP** and \check{BP}_3 are isomorphic. In particular, the manifold $\check{Z}(BP_3)$ admits a complete hyperbolic geometric structure with 3×2^2 ends of torus type, $G_{BP} \cong \pi_1^{\text{orb}}(BP_3)$, and $\check{Z}(BP_3) = \ker \omega_{BP}$ the derived subgroup.*

Let us compute some of these fundamental groups. Note that $\check{Z}(BP_3)$ coincides with the 3-torus $Z(C)$ minus the 12 circles coming from three pairwise disjoint edges one in each direction.

- Proposition 4.10** (a) $\pi_1(Z(BP_3)) = 1$,
 (b) $\pi_1(Z(C)) = \mathbb{Z}^3$,
 (c) $\pi_1(Z(T_2)) = \mathbb{F}_5$, the free group with 5 generators,
 (d) $\pi_1(\check{Z}(BP_3))$ is the kernel of the map $G \rightarrow (\mathbb{Z}/2)^3$,

$$G := \langle x, y, z \mid [x, [y, z]] = [y, [z, x]] = [z, [x, y]] = 1 \rangle,$$

where the map sends the generators to generators; the abelianized of this group is \mathbb{Z}^{12} . Actually $\check{Z}(BP_3)$ is a $(\mathbb{Z}/2)^3$ -Galois cover of the complement of three pairwise disjoint factors of \mathbb{T}^3 .

Proof The cases (b) and (c) are trivial. For the case (a), since all the facets of BP_3 intersect, then $\pi_1^{\text{orb}}(BP_3) \cong (\mathbb{Z}/2)^6$ and the kernel is trivial.

For (d), we may use $\pi_1^{\text{orb}}(\check{BP}_3)$, but we are going to use an alternative construction: $\check{Z}(BP_3)$ is a $(\mathbb{Z}/2)^3$ -Galois cover of the complement of three pairwise disjoint factors of \mathbb{T}^3 , whose fundamental group is G . \square

We finish the discussion with the notion of *small cover*, see [7], i.e., the *smallest* cover of BP_3 which is a manifold (in this case a complete hyperbolic manifold). This small cover is obtained from an epimorphism $\pi_1^{\text{orb}}(BP_3) \rightarrow (\mathbb{Z}/2)^2$, sending a generator of associated with a facets to a non-trivial element of $(\mathbb{Z}/2)^2$, such that the images of generators associated with intersecting faces are distinct. This small cover has 3 ends of torus type. It is not hard to check that the fundamental group of this small cover is isomorphic to

$$\langle a_1, \dots, a_4 \mid a_1^2 = [a_2, a_1 \cdot a_3] = [a_3, a_1 \cdot a_4] = [a_4, a_1 \cdot a_2] = 1 \rangle.$$

5 The octahedron variety

The octahedron is a non-simple polyhedron with only 4-vertices. The study of its smoothings is richer than in the previous examples and it deserves its own section. As for the other ones, there are models of this polyhedron in \mathbb{R}^8 where the faces are the intersection with the coordinate hyperplanes and, in particular, the faces are pairwise orthogonal. In the following section we give another presentation of an octahedron with orthogonal faces in the hyperbolic space.

5.1 The octahedron P_O

We can construct a geometric embedding $P_O \subset \mathbb{R}^8$ of the octahedron given by the equations $r_i \geq 0$, $\sum_{i=1}^8 r_i = 1$, and

$$\begin{aligned} r_1 - r_5 - r_6 - r_7 + 2r_8 &= 0 & r_2 - r_5 - r_6 + r_8 &= 0 \\ r_3 - r_5 - r_7 + r_8 &= 0 & r_4 - r_6 - r_7 + r_8 &= 0; \end{aligned}$$

hence,

$$\mathcal{A} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

We can construct the intersection of ellipsoids $Z(P_O)$ which is a 3-dimensional variety with singularities. In the sequel we study the smoothings of this singular variety. We

are also interested in the smooth part $\check{Z}(P_O)$, which is topologically obtained as the union of the reflections of \check{P}_O , the octahedron minus its six vertices. Since each vertex is disjoint to 4 facets $Z(P_O) \setminus \check{Z}(P_O)$ consists of $96 = 2^4 \times 6$ points; all the smoothings will admit a submanifold homeomorphic to $\check{Z}(P_O)$ which will be the complement of a link with 96 components, and we connect the different smoothings by 0-surgeries of sub-links with 16 components.

5.2 Hyperbolic geometry of the octahedron orbifold

As in §4.3, there exists a hyperbolic ideal octahedron of finite volume with $\frac{\pi}{2}$ dihedral angles. In fact, it is known, [22, pg. 217], that the only regular polyhedra with a hyperbolic structure with $\frac{\pi}{2}$ dihedral angles are the dodecahedron and the octahedron. The dodecahedron is a bounded hyperbolic polyhedron and the octahedron is ideal with finite volume. The hyperbolic dodecahedron was used in [2] to study a smooth intersection of quadrics but the octahedron correspond to a singular intersection of ellipsoids. The next result is a explicit construction of the hyperbolic structure in the octahedron.

Proposition 5.1 *There is a regular octahedron O_H in the three-dimensional hyperbolic space \mathbb{H}^3 where all the vertices are ideal, hence, O_H is homeomorphic to \check{P}_O .*

Proof Consider the regular octahedron P_O with vertices $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, $(0, 0, \pm 1)$ sitting in a ball \mathbb{B}_R^3 of radius R centered at the origin. This octahedron is symmetric respect to the coordinate planes. The interior of the ball \mathbb{B}_R^3 is taken as the Klein model, \mathbb{H}^3 , for the hyperbolic space. We want to choose R such that every dihedral angle defined by two adjacent faces in P_O is $\frac{\pi}{2}$. The dihedral angle between two intersecting planes in the Klein model is the same as the Euclidean angle between their bounding circles in $\partial\mathbb{B}_R^3$.

Let us compute the dihedral angle between the face ABC and the face ACE . The vectors $(1, 1, 1)$ and $(1, -1, 1)$ are respectively perpendicular to the planes ABC and ACE . The bounding circles $ABC \cap \partial\mathbb{B}_R^3$ and $ACE \cap \partial\mathbb{B}_R^3$ intersects in the point $(x_0, 0, z_0)$, where $x_0 + z_0 = 1$ and $x_0^2 + z_0^2 = R^2$. The vector $(x_0, 0, z_0)$ is perpendicular to the sphere $\partial\mathbb{B}_R^3$ at $(x_0, 0, z_0)$, so that $(x_0, 0, z_0) \times (1, 1, 1) = (-z_0, z_0 - x_0, x_0)$ is tangent to the bounding circle $ABC \cap \partial\mathbb{B}_R^3$. Similarly, the vector $(x_0, 0, z_0) \times (1, -1, 1) = (z_0, z_0 - x_0, -x_0)$ is tangent to the bounding circle $ACE \cap \partial\mathbb{B}_R^3$. Thus, R should be chosen so that

$$\begin{aligned} \langle (-z_0, z_0 - x_0, x_0), (z_0, z_0 - x_0, -x_0) \rangle &= -2x_0z_0 = 0 \\ x_0 + z_0 &= 1 \\ x_0^2 + z_0^2 &= R^2. \end{aligned}$$

The first two equations have two solutions for (x_0, z_0) , $\{(0, 1), (1, 0)\}$, the two intersecting points of the bounding circles. In both cases, the third equation implies $R = 1$. Hence we have proved that the hyperbolic regular octahedron with right dihedral angles has ideal vertices and it is inscribed in $\partial\mathbb{B}_1^3$. This regular octahedron O_H is homeomorphic to \check{P}_O . \square

Face 1: ABC
 Face 2: ABD
 Face 3: ADE
 Face 4: ACE
 Face 5: BDF
 Face 6: BCF
 Face 7: CEF
 Face 8: DEF

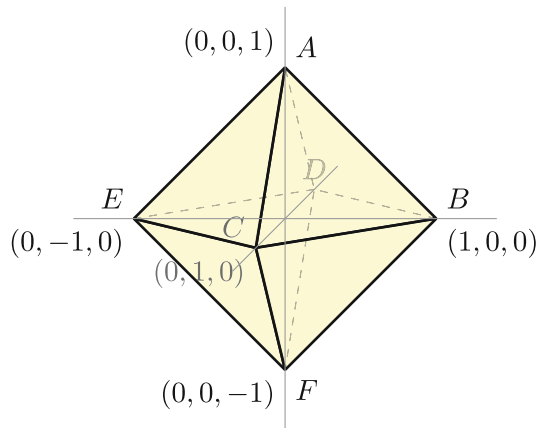


Fig. 16 Octahedron

The faces of O_H are contained in hyperbolic planes and the edges are geodesics. In fact, the group G_O generated by reflection on the eight planes containing the faces acts on \mathbb{H}^3 and it produces a regular tessellation $\mathfrak{T}_{\mathfrak{P}_8}$ made up by regular octahedra. The reflections corresponding to facets sharing an edge commute as they are orthogonal.

5.3 Groups and octahedron

The 2-skeleton of the tessellation by octahedra is formed by hyperbolic planes tessellated by ideal triangles. The 1-skeleton is formed by geodesics and all the vertices are ideal points. The quotient \mathbb{H}^3/G_O defines a hyperbolic orbifold structure \mathbf{O} in O_H . There is a natural epimorphism $\omega_H : G_O \twoheadrightarrow (\mathbb{Z}/2)^8$. On the other side, \check{P}_O has a natural orbifold structure from the quotient $\rho_O : \check{Z}(P_O) \rightarrow \check{P}_O$, which is associated with the monodromy $\omega_O : \pi_1^{\text{orb}}(\check{P}_O) \rightarrow (\mathbb{Z}/2)^8$ of ρ_O . The following result comes from Proposition 5.1.

Corollary 5.2 *The two orbifold structures over O_H and \check{P}_O are isomorphic. In particular, $\check{Z}(P_O)$ admits a complete hyperbolic geometric structure, $G \cong \pi_1^{\text{orb}}(\check{P}_O)$, and $\check{Z}(P_O) = \ker \omega$ the derived subgroup.*

The group $\pi_1^{\text{orb}}(\check{P}_O)$ has 8 generators x_1, \dots, x_8 , labelled as the facets in Fig. 16 with the relations $x_i^2 = 1$, $i = 1, \dots, 8$, and

$$[x_i, x_j] = 1, \quad (i, j) \in \{(1, 2), (2, 3), (3, 4), (4, 1), (5, 6), (6, 7), (7, 8), (8, 5), (1, 6), (2, 5), (3, 8), (4, 7)\} \quad (5.1)$$

corresponding to the pairs of facets having an edge in common. The group $\pi_1(\check{Z}(P_O))$ is the derived subgroup of $\pi_1^{\text{orb}}(\check{P}_O)$ and its abelianization is \mathbb{Z}^{100} .

The assertion about $\pi_1(\check{Z}(P_O))$ follows from the general arguments in the proof of Proposition 4.10. A presentation of $\pi_1(\check{Z}(P_O))$ and the rank of its abelianization

can be found in <https://github.com/enriqueartal/SingularQuadricIntersections>. There are several intermediate orbifold covers defined by epimorphisms $\tau : \pi_1(\check{Z}(P_O)) \rightarrow (\mathbb{Z}/2)^m$, $1 \leq m \leq 8$, between $\check{Z}(P_O)$ and \check{P}_0 which are interesting.

To check the *orbifold points* of these covers we must study how they behave for the different isotropy groups:

- (I0) The isotropy group of a point p in the interior of P_O is trivial, P_O is a manifold around p and no condition is needed for the covering to be a manifold around its preimages.
- (I1) The isotropy group of a point p in the interior of a facet of P_O is $\mathbb{Z}/2$ and is generated by some x_i . Hence the covering associated with τ is a manifold around the preimages of p if $\tau(x_i) \neq 0$.
- (I2) The isotropy group of a point p in the interior of an edge of P_O is $(\mathbb{Z}/2)^2$ and is generated by some x_i, x_j . Hence the covering associated with τ is a manifold around the preimages of p if $\tau(\{x_i, x_j\})$ is a subgroup isomorphic to $(\mathbb{Z}/2)^2$.
- (I3) There is no point whose isotropy group $(\mathbb{Z}/2)^3$ as there is no simple vertex.
- (I4) Though there is no point whose isotropy group $(\mathbb{Z}/2)^4$, this is the case if we consider P_O instead of \check{P}_0 . In this case p is a vertex, V is a closed regular neighborhood of p in P_O , then the orbifold fundamental group of the punctured neighbourhood $\check{V} := V \setminus \{p\}$ is

$$\begin{aligned} G_{i,j,k,l} &:= \langle x_i, x_j, x_k, x_l; x_i^2, x_j^2, x_k^2, x_l^2, (x_i x_j)^2, \\ &\quad (x_j x_k)^2, (x_k x_l)^2, (x_l x_i)^2 \rangle, (i, j, k, l) \\ &\in \{(1, 2, 3, 4), (5, 6, 7, 8), (1, 2, 5, 6), (3, 4, 7, 8), (1, 4, 6, 7), (2, 3, 5, 8)\}. \end{aligned}$$

The derived subgroup is isomorphic \mathbb{Z}^2 . Let us study the connected components \check{W} of the preimage of this neighbourhood in the cover induced by τ .

- (V1) If $\tau(G_{i,j,k,l})$ is $\mathbb{Z}/2$ and the condition in (I1) holds for i, j, k, l , then the boundary of \check{W} is homeomorphic to the orbifold \mathbb{S}_{2222}^2 .
- (V2) If $\tau(G_{i,j,k,l})$ is $(\mathbb{Z}/2)^2$ and the condition in (I2) holds for $(i, j), (j, k), (k, l), (l, i)$, then the boundary of \check{W} is homeomorphic to a 2-torus.
- (V3) If $\tau(G_{i,j,k,l})$ is $(\mathbb{Z}/2)^4$, then the boundary of \check{W} is homeomorphic to a 2-torus.

Proposition 5.3 *Let M_τ be the total space of the orbifold cover associated with $\tau : \pi_1(\check{Z}(P_O)) \rightarrow (\mathbb{Z}/2)^m$.*

- (a) *If $m = 1$ and (II) holds, then M_τ is the double of P_O as a complete hyperbolic orbifold homeomorphic to the sphere \mathbb{S}^3 minus 6 points.*
 - (i) *There are six cusps, coming from the vertices.*
 - (ii) *There are 12 geodesics coming from the 1-skeleton of the octahedron for which the angle around the edges is π .*
 - (iii) *The boundary of a neighbourhood of every cusp is the orbifold \mathbb{S}_{2222}^2 , a two dimensional sphere with 4 cone points with isotropy the cyclic group of order 2.*

(b) If τ is defined as

$$\begin{aligned}\pi_1(\check{Z}(P_O)) &\longrightarrow (\mathbb{Z}/2)^2 \\ x_1, x_3, x_6, x_8 &\longmapsto e_1 \\ x_2, x_4, x_5, x_7 &\longmapsto e_2,\end{aligned}$$

then M_τ is a complete hyperbolic manifold with 6 ends which are of torus type and $\pi_1(M_\tau)$ is isomorphic to $\langle a_1, \dots, a_6 \mid [a_1, a_2] = [a_3, a_4] = [a_5, a_4 a_1] = [a_6, a_2 a_3] = 1, a_4 a_6 a_5 a_3 = a_5 a_4 a_3 a_6 \rangle$.

The hyperbolic manifold $\check{Z}(P_O)$ has 96 cusps. The variety $Z(P_O)$ is obtained adding points in the places of the cusps. Following the ideas in the proof of Proposition 4.10 and computations with `Sagemath` we get some topological information on $Z(P_O)$.

Proposition 5.4 *The orbifold fundamental group of P_O has the generators and relations of $\pi_1^{orb}(\check{P}_O)$ plus the relations*

$$\begin{aligned}[x_i, x_j] = 1, \quad (i, j) \in \{ &(1, 3), (2, 4), (5, 7), (6, 8), (1, 7), (4, 6) \\ &(1, 5), (2, 6), (3, 7), (4, 8), (3, 8), (4, 7), (3, 5), (2, 8) \}.\end{aligned}\quad (5.2)$$

and the fundamental group of $Z(P_O)$ is isomorphic to \mathbb{Z}^4 .

5.4 The smoothings of the intersection associated with the octahedron

It is useful to work with a compact manifold Z_T which has the homotopy type of $\check{Z}(P_O)$ which is obtained by removing conic neighborhoods of the singular points of $Z(P_O)$. This is manifold whose boundary is composed by 96 tori. Actually, since these tori are the boundary of neighborhoods of generic singularities for each boundary component we have a natural product decomposition $\mathbb{S}^1 \times \mathbb{S}^1$ on each boundary component. Hence we can obtain a smooth manifold by adding a solid torus $\mathbb{D}^2 \times \mathbb{S}^1$ along each boundary component and compatible with the product structure. It corresponds to compactify $\check{Z}(P_O)$ by \mathbb{S}^1 's instead of points. There are two possible ways to add this solid tori at each boundary component. Since the cusps are related by the monodromy action of the covering, the boundary components come in 6 packages of 24 tori and then we have $64 = 2^6$ natural choices, which coincide with the possible smoothings of $Z(P_O)$.

Since the symmetry group of P_O acts on these choices, as in the previous sections they are distributed in orbits and we will have to study less cases. Each choice corresponds to a polyhedron with 8 faces (coming from the faces of the octahedron) $12 + 6$ edges (coming from the edges of the octahedron and from its vertices) and $2 \cdot 6$ vertices (each vertex of the octahedron is doubled).

Proposition 5.5 *There are exactly 14 simply connected simple polyhedra of dimension 3 with exactly 8 faces, 18 edges, and 12 vertices [8, 18]. Up to permutation, only for six of them it is possible to contract six pairwise disjoint edges to obtain an octahedron, but one of them in two distinct ways.*

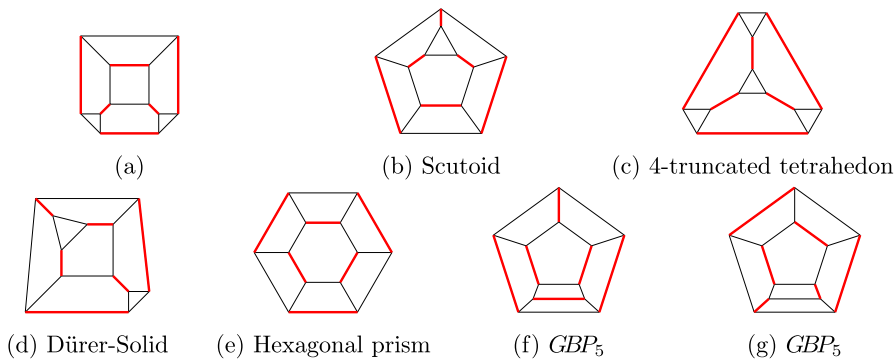


Fig. 17 Smoothings of the octahedron

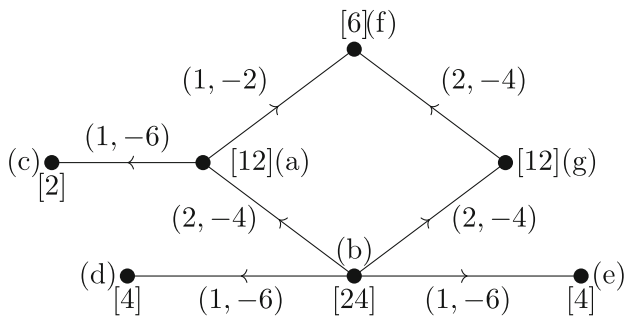


Fig. 18 Graph $\Gamma(P_O)$

The second part of this proposition is related with the result we are interested in: up to automorphism there are exactly seven distinct topological types of smoothings. There is an apparent contradiction between these two statements. As we can see in Fig. 17, one of the smoothings corresponds to the same polyhedron but there are two distinct ways of contracting onto a octahedron. The details can be found in <https://github.com/enriqueartal/SingularQuadricIntersections>.

Theorem 5.6 *The pairs composed by the polyhedra and the sets of red edges in Fig. 17 are the seven topological types of smoothings of the octahedron. The size of the orbits of each type are: 12 in (a), 24 in (b), 2 in (c), 4 in (e), 4 in (d), 6 in (f), 12 in (g).*

Some of these polyhedra have their own names: the polyhedra (b) is known as *Scutoid* [10]; the polyhedra (f) or (g) GBP_5 is the *Gyrobipentaprism*, that is the result of pasting together two pentagonal prism along a lateral face by a $\frac{\pi}{2}$ -turn; and the polyhedra (d) is the *Dürer-Solid*, the solid depicted in an engraving entitled *Melencolia I* by Albrecht Dürer in 1514. The graph $\Gamma(P_O)$ is in the Fig. 18. Their stereographic projections are in Fig. 19.

In the following sections, we describe the topology of all these smoothings. With the help of *Sagemath* [21], we describe some fundamental subgroups. The code that allowed us to find it can be found in <https://github.com/enriqueartal/>

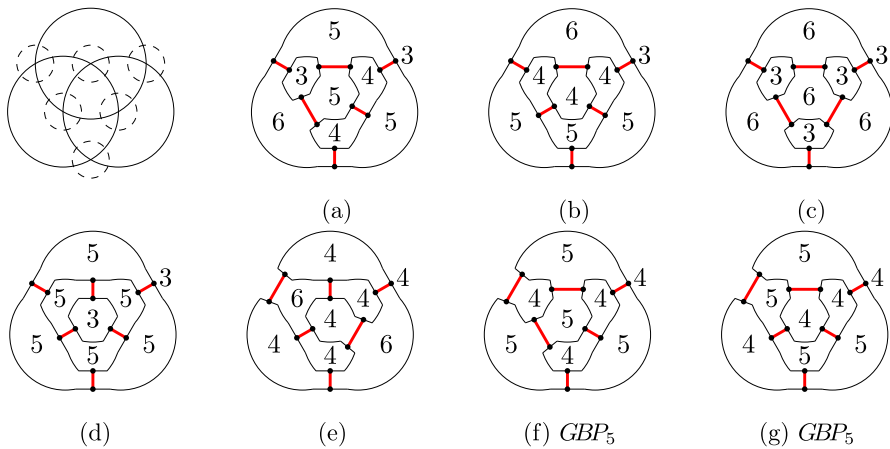


Fig. 19 Stereographic presentation

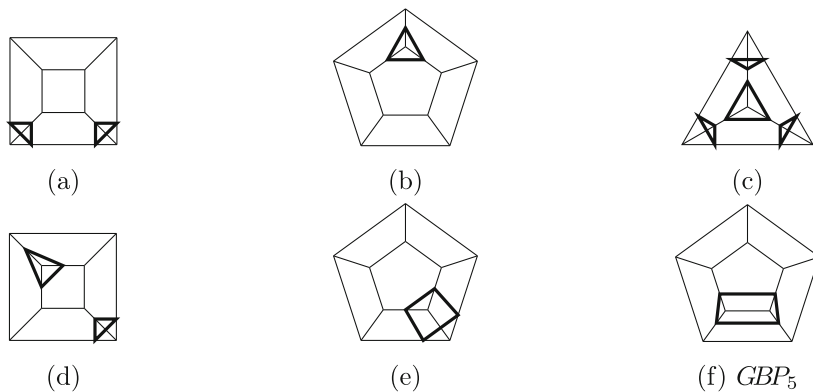


Fig. 20 Some truncations of simple polyhedra

[SingularQuadricIntersections](#); it can be executed in a local installation of Sagemath or using Binder [13].

Proposition 5.7 *The fundamental group of $Z(P_O)$ is free abelian of rank 4. The fundamental group of the manifold Z_T is the derived of G' in (5.1) and its abelianization is free of rank 100.*

5.5 Topology of the manifolds obtained in cases (a)–(e)

The topology can be described in most cases using the fact that the resulting polyhedra are truncations of simpler polyhedra (see Fig. 20). These polyhedra are connected by flips of type (2, 2). We are going to relate them with the distinct smoothings of the octahedron.

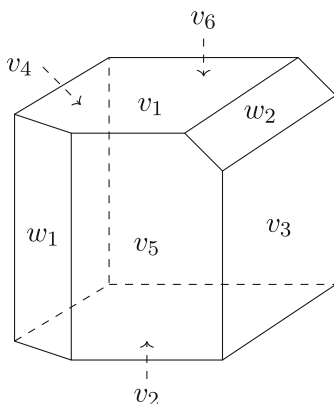


Fig. 21 Double bevelling of a cube

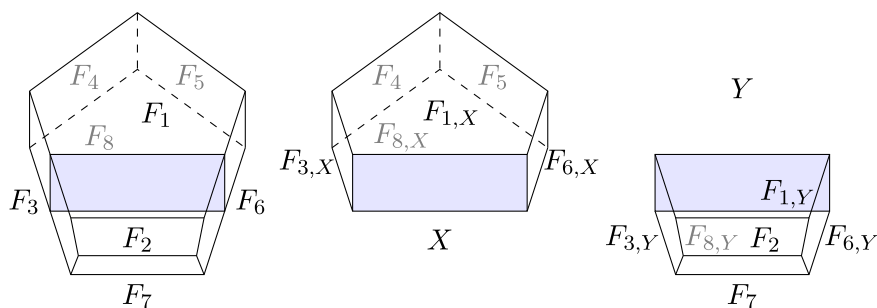


Fig. 22 The polytope GBP_5 in Fig. 17(f) as union of X and Y

The cases (a), (b), (c), and (d) are obtained from well-known polyhedra by truncating vertices, while (f), (g), and (e) are the *bevellings*³ of the pentagonal prism by cutting along a horizontal, for (f) subref, (g), or a vertical edge, for (e).

They can be also described as the double bevellings of the cube; the common case of (f) and (g) is the bevelling on two non-parallel disjoint edges and is the prototype of a polytope whose *moment-angle manifold* (the intersection of ellipsoids obtained by duplicating every A_i and whose quotient by a torus gives the same polytope) has non-trivial Massey products, and it is plausible that the same is true for $Z(GBP)$, see [20, p. 26], where the Fig. 21 appears. In particular, it is not a connected sum of elementary manifolds, as the rest of them. The topology of this intersection will be described in the next section.

We now describe all the other cases. Let us denote by $\mathcal{S}_{g,n}$ the closed orientable surface of genus g with n boundary components (for short, $\mathcal{S}_g = \mathcal{S}_{g,0}$). Since (e) is an hexagonal prism, then the corresponding intersection is diffeomorphic to $\mathcal{S}_{17} \times \mathbb{S}^1$.

For the remaining cases we use Proposition 4.5 for truncations of simple polytopes.

Note that (a) and (d) are different double truncations of a cube (there are 3 possible ones). Hence the two intersections are diffeomorphic to $4(\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1) \# 29(\mathbb{S}^2 \times \mathbb{S}^1)$.

³ In Spanish, the term *biselados* was suggested by Alberto Verjovsky.

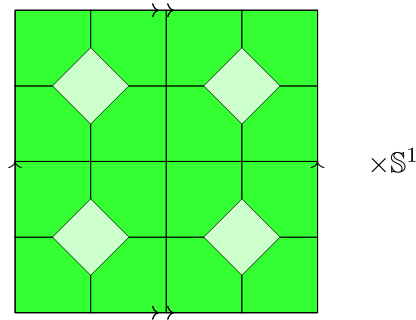


Fig. 23 The manifold associated with a pentagonal prism with a non-mirroring lateral face

The case (b) is a truncation of the pentagonal prism, so the intersection is diffeomorphic to $2(\mathcal{S}_5 \times \mathbb{S}^1) \# 15(\mathbb{S}^2 \times \mathbb{S}^1)$. Finally, (c) is obtained by truncating the four vertices of the tetrahedron so its intersection is diffeomorphic to $49(\mathbb{S}^2 \times \mathbb{S}^1)$.

5.6 The manifold Z associated with the Gyrobipentaprism

We devote this section to the manifold Z associated with polyhedra Gyrobipentaprism GBP_5 in Fig. 17(f), which has not a straightforward description as for the other cases. Following the same ideas of the truncation we cut the polytope GBP_5 along a rectangle as in Fig. 22 obtaining two new polytopes X and Y . We denote the faces of GBP_5 , X , Y as in this figure. Both X and Y are pentagonal prisms. To construct Z , we must perform reflections on all the faces except the ones coming from the cutting rectangle. We obtain two 3-manifolds with four components in the boundary, all of them tori. The manifold Z is obtained by suitably gluing four copies of both manifolds along the boundary components. The precise description of the gluing and the final result occupies the rest of this section.

Let us denote the faces in Fig. 22. The faces $F_1, F_{1,X}, F_{1,Y}$ (resp. $F_1, F_{1,X}, F_{1,Y}$) are the upper (resp. lower) bases. The face F_2 (resp. F_7) is the upper (resp. lower) face of the bevelling. The faces $F_3, F_{3,X}, F_{3,Y}$ are in the front left-hand side. The faces F_4, F_5 are the backward lateral faces.

Since both polytopes are *equal*, we describe them together. Consider a pentagonal prism where we mirror along all faces but one of the lateral ones. We consider several steps:

- (T1) Reflection along the four lateral faces that are mirrors. The result is the product of the reflection of a pentagon in four sides by the interval I . We can see the result in Fig. 23, i.e., a torus with four holes by interval: $\mathcal{S}_{1,4} \times I$.
- (T2) Reflecting on the basis faces. We obtain $\mathcal{S}_{1,4} \times \mathbb{S}^1$, a trivial \mathbb{S}^1 -fibre bundle.
- (T3) The boundary of this manifold consists of 4 tori corresponding to the non-mirroring lateral face of the original pentagonal prism.
- (T4) With this process we denote by X_e the manifold obtained by reflection of X in the lateral faces $F_{3,X}, F_4, F_5, F_{6,X}$, and in the bases $F_{1,X}$ and $F_{8,X}$.

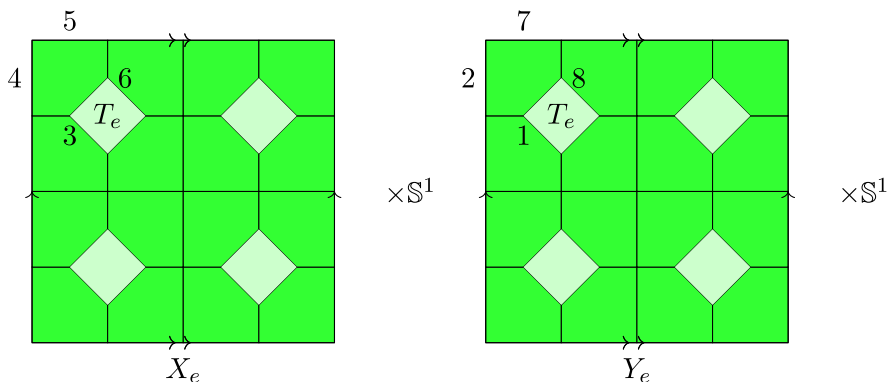


Fig. 24 The manifolds X_e and Y_e

- (T5) Analogously, let Y_e the manifold obtained by reflection of Y in the lateral faces $F_{1,Y}$, F_2 , F_7 , $F_{8,Y}$, and in the bases $F_{3,Y}$ and $F_{6,Y}$.
- (T6) The result of the two previous statements can be seen in Fig. 24.
- (T7) The manifold Z_X (associated with X with one missing reflection) is obtained by reflecting along the faces F_2 , F_7 which are the faces of Y disjoint to X . We obtain four diffeomorphic connected components labelled by the reflections, $Z_X = X_e \cup X_2 \cup X_7 \cup X_{27}$.
- (T8) Analogously, the manifold Z_Y is obtained by reflecting along the faces F_4 , F_5 . We obtain four diffeomorphic connected components labelled by the reflections, $Z_Y = Y_e \cup Y_4 \cup Y_5 \cup Y_{45}$.
- (T9) Let us describe the boundaries of these manifolds using the labellings of the reflections:

$$\begin{aligned}
 \partial X_e &= T_e \cup T_4 \cup T_5 \cup T_{45} \\
 \partial X_2 &= T_2 \cup T_{24} \cup T_{25} \cup T_{245} \\
 \partial X_7 &= T_7 \cup T_{47} \cup T_{57} \cup T_{457} \\
 \partial X_{27} &= T_{27} \cup T_{247} \cup T_{257} \cup T_{2457} \\
 \partial Y_e &= T_e \cup T_2 \cup T_7 \cup T_{27} \\
 \partial Y_4 &= T_4 \cup T_{24} \cup T_{47} \cup T_{247} \\
 \partial Y_5 &= T_5 \cup T_{25} \cup T_{57} \cup T_{257} \\
 \partial Y_{45} &= T_{45} \cup T_{245} \cup T_{457} \cup T_{2457}
 \end{aligned}$$

- (T10) Each manifold is a trivial fibration. To get Z we must glue them along the boundary components, exchanging sections and fibers. This is how Waldhausen graph manifolds are constructed using Neumann's plumbing, see [19, 23]

Let us describe the manifold Z . Consider the complete bipartite graph $K_{4,4}$. We associate to each vertex an oriented \mathbb{S}^1 -bundle with base a 2-torus and Euler number 0, i.e., a manifold diffeomorphic to \mathbb{T}^3 . On each fiber bundle we take out an open fibered

solid torus for each edge. Then, we glue along the tori associated with each edge interchanging fiber and section.

This description determines completely Z and it is possible to give a presentation of its fundamental group, see e.g. [1]. For example the homology is free of rank 31.

There is an alternative description of this manifold which connects it to complex geometry. Let us consider an elliptic curve E (a compact complex manifold of genus 1), and take four points $p_1, \dots, p_4 \in E$. In the projective surface $W := E \times E$ consider the curve

$$D := \bigcup_{i=1}^4 E \times \{p_i\} \cup \bigcup_{i=1}^4 \{p_i\} \times E -$$

Let us take regular neighbourhood of D in E constructed as a union of tubular neighbourhoods of the irreducible components of D , which at the double points looks like a polydisk in \mathbb{C}^2 . Then, Z is homeomorphic to the boundary of this neighbourhood.

Can we say something about the link induced by the red edges in Fig. 17f, g?

6 More hyperbolic relations

6.1 Hyperbolic relations between $\check{Z}(BP_3)$ and $\check{Z}(P_O)$

There is a hyperbolic structure T_H in the tetrahedron $ABCD$ in Fig. 15 with three ideal points B , C and D . In this hyperbolic structure the dihedral angle at the edges AB , AC , and AD is $\frac{\pi}{2}$, but at BC , CD , and BD the dihedral angle is $\frac{\pi}{4}$. This results follows from the hyperbolic structure given to the triangular bipyramid in Proposition 4.8.

The tetrahedron T_H has a orbifold hyperbolic structure with mirror faces and angles $\frac{\pi}{n}$, $n = 2, 4$. Therefore it defines a tessellation \mathfrak{T}_T in the hyperbolic space.

Theorem 6.1 *The tessellations \mathfrak{T}_{BP} and \mathfrak{T}_{P_H} are subtessellations in \mathfrak{T}_T . The volume of $\check{Z}(P_O)$ is a multiple of the volume of $\check{Z}(BP_3)$.*

$$\text{Vol}(\check{Z}(P_O)) = 2^4 \text{Vol}(\check{Z}(BP_3)).$$

Proof The bipyramid BP_H (Fig. 15) can be seen as the union of two tiles, the tetrahedron $ABCD$ and its reflection on the face BCD . The octahedron P_H is the union of eight tiles, the tetrahedron $ABCD$ and its reflection on the faces ABC , ACD and ABD . Then

$$\begin{aligned} \text{Vol}(\check{Z}(BP_3)) &= 2^6 \text{Vol}(BP_H) = 2^{6+1} \text{Vol}(T_H) \\ \text{Vol}(\check{Z}(P_O)) &= 2^8 \text{Vol}(P_O) = 2^{8+3} \text{Vol}(T_H) \end{aligned}$$

implies

$$\text{Vol}(\check{Z}(P_O)) = 2^4 \text{Vol}(\check{Z}(BP_3)).$$

□

6.2 The rhombic dodecahedron

There is another interesting hyperbolic polyhedron whose associated tessellation is a subtessellations in \mathfrak{T}_T . Consider the following hyperbolic rhombic dodecahedron RD_H in the Klein model of the hyperbolic 3-space \mathbb{H}^3 : The six 4-vertices are the ideal points $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$ and $(0, 0, \pm 1)$ (the vertices of the octahedron), the eight simple vertices are the points $(\pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2})$. It is obtained by reflection of the triangular bipyramid $ABCDE$ at its three facets ABC , ACD and ABD (Fig. 15), that is, reflection at the three coordinate planes. The edges of RD_H are BE , CE , DE and their images by the reflections. The dihedral angle at each edge BE is the same as the angle in the triangular bipyramid, that is $\frac{\pi}{2}$ as was proved in Proposition 4.8, and, by symmetry, all the others dihedral angles are also $\frac{\pi}{2}$. Observe that the group of symmetries of RD_H is the same that the one of the octahedron.

The tessellation \mathfrak{T}_{RD} in \mathbb{H}^3 defined by RD_H is a subtessellations of \mathfrak{T}_T . The polyhedron RD_H is made up of 8 triangular bipyramid BP_H , 8 tiles of \mathfrak{T}_{BP} , then by 16 tiles of \mathfrak{T}_T .

$$\text{Vol}(RD_H) = 8 \text{Vol}(BP_H) = 16 \text{Vol}(T_H)$$

The group G_{RD} , subgroup of $\text{Aut } \mathbb{H}^3$ generated by the hyperbolic reflections on the twelve planes containing the faces of RD_H , is the automorphism group of \mathfrak{T}_{BP} . The quotient \mathbb{H}^3/G_{RD} defines a hyperbolic orbifold structure \mathbf{RD} in RD_H .

There is a geometric embedding of the rhombic dodecahedron P_{RD} in \mathbb{R}^{12} . This is the polytope associated with an intersection of ellipsoids with 6 generic isolated singularities. The manifold $Z(RD)$ has 6×2^8 singular points. Recall that \check{P}_{RD} , complement of the non-simple vertices in P_{RD} , has an orbifold structure. This orbifold structure is isomorphic to the orbifold structure \mathbf{RD} . Then $\check{Z}(RD)$ has a complete hyperbolic structure with 6×2^8 ends of torus type and $G_{RD} \cong \pi_1^{\text{orb}}(\check{P}_{RD})$.

$$\text{Vol}(\check{Z}(RD)) = 2^{12} \text{Vol}(RD_H) = 2^{15} \text{Vol}(BP_H) = 2^{16} \text{Vol}(T_H)$$

$$\text{Vol}(\check{Z}(RD)) = 2^9 \text{Vol}(\check{Z}(BP_3)) = 2^5 \text{Vol}(\check{Z}(P_O))$$

The smoothings of the manifold $Z(RD)$ can be studied. In fact the smoothings of the rhombic dodecahedron, having the same symmetry group that the octahedra, reduces to seven topological types. One of them is the hyperbolic dodecahedron studied in [2].

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