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Trabajo Fin de Máster

Cálculo efectivo del modelo de Sullivan de un espacio topológico y su tipo de homotopía racional.

Effective computation of the Sullivan model of a topological space and its rational homotopy type.

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Resumen

El tipo de Homotopía Racional de un espacio topológico es una simplificación de su tipo de homotopía, donde todos los grupos son tensorizados por $\mathbb{Q}$. Pese a esta pérdida de información, estudiar el tipo de homotopía racional tiene la ventaja de ser tratable computacionalmente. Gracias a Sullivan, dado un espacio topológico X que cumple ciertas propiedades, se sabe teóricamente como obtener su tipo de homotopía racional, via la construcción de un álgebra commutativa diferencial graduada, llamada el modelo minimal de Sullivan de X. Este álgebra viene acompañada de un quasi-isomorfismo al álgebra de cocadenas singulares de X, denotada por $C^*(X)$, y nos permite establecer una equivalencia categórica entre tipos de homotopía racional de espacios y clases de isomorfismos de modelos de Sullivan:

$$\left\{\begin{matrix} Tipos\ de\ Homotopía \\ Racional \end{matrix}\right\} \Leftrightarrow \left\{\begin{matrix} Clases\ de\ Isomorfismos \\ de\ Modelos\ de\ Sullivan \end{matrix}\right\}$$

En este trabajo se presenta un método efectivo para el cálculo del modelo minimal de Sullivan de un espacio topológico simplemente conexo, así como una implementación de dicho método en un sistema de álgebra computacional. Para ilustrar dicho métodos, se incluyen algunos ejemplos concretos para diferentes espacios.

Abstract

The rational homotopy type of a topological space is a simplified version of the homotopy type where all homotopy groups are tensored by $\mathbb{Q}$. Despite the lost information, rational homotopy has the advantage of being constructive. Due to Sullivan, for a particular topological space X that satisfies some conditions, it is known theoretically how to obtain its rational homotopy type via the construction of a commutative differential graded algebra, called the Sullivan model of X. This algebra is quasi-isomorphic to the normalized singular cochain algebra of X, $C^*(X)$, and it allows us to establish a categorical equivalence between homotopy types of spaces and isomorphism classes of Sullivan models:

$$\left\{\begin{matrix} Rational\ homotopy \\ types\ of\ spaces \end{matrix}\right\} \Leftrightarrow \left\{\begin{matrix} Isomorphim\ classes\ of \\ minimal\ Sullivan\ models \end{matrix}\right\}$$

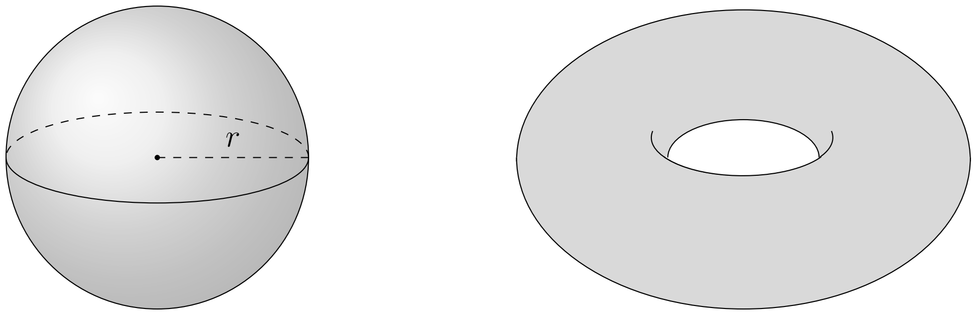
In this work it is presented an effective method to compute the Sullivan minimal model for a simply-connected topological space, and an implementation of such method in a Computer Algebra System. In order to illustrate such method, examples of some computations are included.

A Michel y Jorge.

Contents

1 Introduction

One of the most important aims of Topology is to determine when two topological spaces are equivalent, i.e., if there exists a homeomorphism between them. There is not any methodology to find this equivalence, but mathematicians have other tools to narrow the problem down. It can be reformulated as follow: maybe we can not say whether two topological spaces are equivalent, but if we find some essencial property that defines one space, and this property is not met by the other, then we could say that these two spaces are not equivalent. This kind of properties are called invariants, and one of the main lines of research in topology is to find and characterize these invariants. Algebraic Topology is the field of mathematics that uses algebraic tools to extract properties of topological spaces.



(a) Sphere $\mathbb{S}^2$ (b) Torus $\mathbb{T}^2$

Figure 1.1: Example of two different topological spaces of dimension 2.

In Topology, it is said that two continuous maps are *homotopic* if one can be "continuously deformed" into the other. *Homotopy theory* is the study of continuous maps between topological spaces. The homotopy groups, i.e., the groups of equivalence classes of homotopic continuous maps, is a topological invariant and can be used to determine, for example, if two topological spaces are not topologically the same, i.e., it does not exist an homeomorphism between them.

The mathematical context of the work presented here belongs to the realm of *Rational Homotopy Theory* [3, 4, 6, 8, 9]. As the name suggests, it is a branch of Homotopy Theory, and it begins with the discovery by Sullivan in the 1960's that simply connected topological spaces and continuous maps between them can themselves be rationalized, i.e., given a simply connected space X, it is possible to construct (via rationalization) a space $X_{\mathbb{Q}}$, such that

$$H_*(X_{\mathbb{Q}}) = H_*(X; \mathbb{Q})$$

$$\pi_*(X_{\mathbb{Q}}) = \pi_*(X) \otimes \mathbb{Q},$$

and given a map $f : X \to Y$ between simply connected topological spaces, there exists an induced map $f_{\mathbb{Q}} : X_{\mathbb{Q}} \to Y_{\mathbb{Q}}$. We say that f is called a *rational homotopy equivalence* [6, p. 110-111] if one of the following (equivalent) conditions is satisfied:

- $\pi_*(f) \otimes \mathbb{Q}$ is an isomorphism,
- $H_*(f; \mathbb{Q})$ is an isomorphism,
- $H^*(f; \mathbb{Z}) \otimes \mathbb{Q}$ is an isomorphism,
- The rationalized map $f_{\mathbb{Q}} : X_{\mathbb{Q}} \to Y_{\mathbb{Q}}$ is a weak homotopy equivalence.

The *rational homotopy type* of a topological space X is the homotopy type of $X_{\mathbb{Q}}$, and the *rational homotopy class* of a continuous map between two spaces $f : X \to Y$ is the homotopy class of $f_{\mathbb{Q}} : X_{\mathbb{Q}} \longrightarrow Y_{\mathbb{Q}}$. Rational Homotopy Theory is then the study of properties that depend only on the rational homotopy type of a space or the rational homotopy class of a map. Despite the lost information (see Tables 1.1, 1.2), rational homotopy has the advantage of being remarkably computational.

	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}
$\mathbb{S}^4$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_{12}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_{24} \times \mathbb{Z}_3$	$\mathbb{Z}_5$	$\mathbb{Z}_2$
$\mathbb{S}^5$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{30}$
$\mathbb{S}^6$	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	$\mathbb{Z}$	$\mathbb{Z}_2$

Table 1.1: A subset of the homotopy groups of the spheres $\mathbb{S}^4$, $\mathbb{S}^5$ and $\mathbb{S}^6$.

	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}
$\mathbb{S}^4$	$\mathbb{Z}$	0	0	$\mathbb{Z}$	0	0	0	0	0
$\mathbb{S}^5$	0	$\mathbb{Z}$	0	0	0	0	0	0	0
$\mathbb{S}^6$	0	0	$\mathbb{Z}$	0	0	0	0	$\mathbb{Z}$	0

Table 1.2: A subset of the rational homotopy groups of the spheres $\mathbb{S}^4$, $\mathbb{S}^5$ and $\mathbb{S}^6$. In red color are shown the groups where some information is lost due to rationalization.

The main contribution of Sullivan was that he found how to obtain the rational homotopy type of a simply connected topological space X, via the construction of a commutative differential graded algebra, called the Sullivan model of X. This result is summarized in what is known as the main theorem of Rational Homotopy:

Theorem 1.1 (Fundamental theorem of Rational Homotopy)**.** *Let X be a simply-connected topological space of finite type, with Sullivan model $(\Lambda V, d)$. Then, for every $k \geq 0$, the bilinear pairing*

$$< \cdot, \cdot >: V^k \otimes \pi_k(X) \longrightarrow \mathbb{Q},$$

is non-degenerate, and

$$\vartheta_k : V^k \xrightarrow{\simeq} Hom_{\mathbb{Z}}(\pi_k(X), \mathbb{Q}) \simeq \pi_k(X) \otimes \mathbb{Q}$$

is an isomorphism of $\mathbb{Q}$*-vector spaces.*

This algebra allows us to establish a categorical equivalence between rational homotopy types of spaces and isomorphism classes of Sullivan models:

$$\left\{\begin{matrix} Rational\ homotopy \\ types\ of\ spaces \end{matrix}\right\} \Leftrightarrow \left\{\begin{matrix} Isomorphim\ classes\ of \\ minimal\ Sullivan\ models \end{matrix}\right\}.$$

In this work it is presented an effective method to build the minimal Sullivan model of a given topological space X. Furthermore, it is presented a concrete implementation in a Computer Algebra System and examples of different computations.

1.1 Goals

The aim of this work is the analysis, development and implementation of an effective method for the computation of the Sullivan model of a topological space given as a simplicial complex. In particular, the specific goals of this work are:

1. Understand the basis of Rational Homotopy Theory and how is defined the Sullivan model of a topological space.

2. Review the state of the art about previous works on effective methods for the computation of minimal Sullivan algebras.

3. Identify why these methods can not be applied in the context of Sullivan models of topological spaces and propose possible solutions and adaptations.

4. Develop the mathematical tools needed for the computation of such Sullivan models.

5. Implement an effective method in a Computer Algebra System in order to test the algorithm.

1.2 Structure of the document

The document has been organized as follows: Chapter 2 contains all the mathematical background needed for the development and understanding of the presented work. In Chapter 3 is presented the effective method developed for the computation of the Sullivan minimal model of a given topological space. In Chapter 4 are shown some computations for different topological spaces. In Chapter 5 are presented the conclusions and future work. Finally, in Appendix I, as a way to show the complexity of the output for some topological spaces, the generators of degree 3 and their differentials of the minimal model of the K3 surface are listed, and in Appendix II, an example of an element of the algebra $A_{PL}(K)$ for the $K3$ surface is included.

2 Background

In this section are included all the mathematical objects and algorithms previously defined in the literature and needed for the develompent and understanding of the work presented in the next section. For a more extensive description see [3, 4, 6, 7, 8, 10].

2.1 Simplicial Complexes

(For a more detailed description see [5]).

Consider a continuous map $f : (X, x_0) \to (Y, y_0)$ between pointed topological spaces. This map is called a *weak homotopy equivalence* if the induced map

$$\pi_*(f) : \pi_*(X, x_0) \to \pi_*(Y, y_0)$$

is an isomorphism. The spaces X and Y have the same weak homotopy type if there is a finite chain of weak homotopy equivalences

$$X \leftarrow Z_1 \to \cdots \leftarrow Z_n \to Y.$$

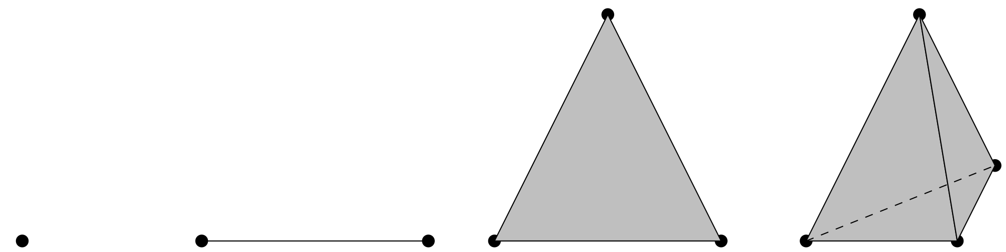
A cellular model or *CW model* for a space Y is a CW complex X together with a weak homotopy equivalence $f : X \to Y$. It is known that every space Y has a CW model and this model is unique up to homotopy equivalence. Two important theorems are in the foundations of Rational Homotopy Theory:

Theorem 2.2 (Whitehead [9, p. 346])**.** *If a map $f : X \to Y$ between connected CW complexes induces isomorphisms $\pi_*(f) : \pi_*(X, x_0) \to \pi_*(Y, y_0)$, $\forall n$, then f is a homotopy equivalence.*

Theorem 2.3 (Whitehead-Serre [6, p. 94])**.** *Let $f : X \to Y$ be a map between simply connected spaces. Then the following assertions are equivalent for a subring R of $\mathbb{Q}$:*

- *$\pi_*(f) \otimes_{\mathbb{Z}} R$ is an isomorphism*
- *$H_*(f, R)$ is an isomorphism*

In this work, instead of using CW models, for computational reasons, these structures are replaced by simplicial complexes. Let us recall that a *simplicial complex* X of dimension n is a construction made up of some building blocks called *simplices*, glued together along common faces, that are simplices of lower dimension (see Figure 2.1).



(a) 0-simplex (b) 1-simplex (c) 2-simplex (d) 3-simplex

Figure 2.1: Building blocks for simplicial complexes.

A simplicial complex can be seen as a collection of simplices satisfying the following properties:

- every *n-simplex* has exactly $n + 1$ faces.
- every face of a simplex of X is in X, and
- the intersection of any two simplices of X, is either a face of each them or the empty set.

These constructions are used to model topological spaces. In this model, an *n-simplex* is homeomorphic to a disc of dimension n, $\mathbb{D}^n$.

Because Rational Homotopy theory is built upon the construction of *CW* models and its singular homology, two important theorems have to be mentioned in order to validate the use of simplicial complexes for the purpose of this work.

The first theorem asserts the existence of a homotopy equivalent simplicial complex Y for every *CW* complex X:

Theorem 2.4 ([9, p. 182]). *Every CW complex X is homotopy equivalent to a simplicial complex Y, which can be chosen to be of the same dimension as X, finite if X is finite, and countable if X is countable.*

The second theorem is important because it allows us to work with the simplicial homology of a given simplicial complex X, as if we were working with the singular homology, object used in [6] for the building of the Rational Homotopy Theory's corpus:

Theorem 2.5 (Equivalence of Simplicial and Singular Homology [9, p. 128]). *Given a simplicial complex, X, the simplicial homology and the singular homology of X are isomorphic.*

Given a simply connected space X, a simplicial complex Y homotopy equivalent to a *CW* model of such space, and combining Theorems 2.3 and 2.5, we can study the rational homotopy type of X through Y.

2.2 The Rational Homotopy Type of a Topological Space

Although the work presented here can be read from a pure algebraic point of view, its motivation comes from Rational Homotopy. For completeness, we now present a brief introduction to the rational homotopy type of a topological space, without getting too deep into technical details. Let us begin with some definitions:

Definition 2.6. A simply-connected topological space X is called *rational* if $\pi_*(X)$ is a $\mathbb{Q}$-vector space. A *rationalization* of a simply-connected space X is a map $\varphi : X \to X_{\mathbb{Q}}$ to a simply-connected rational space $X_{\mathbb{Q}}$ such that φ induces an isomorphism:

$$\pi_*(X) \otimes \mathbb{Q} \xrightarrow{\cong} \pi_*(X_{\mathbb{Q}}).$$

Two important theorems are those related to the existence and uniqueness of the rationalization of a simply-connected space:

Theorem 2.7 (existence, [6] p.109)**.** *For each simply-connected space X there is a relative CW complex $(X_{\mathbb{Q}}, X)$ (with neither 0-cells nor 1-cells) such that the inclusion $\varphi : X \hookrightarrow X_{\mathbb{Q}}$ is a rationalization.*

Theorem 2.8 (uniqueness, [6] p.109)**.** *Let $(X_{\mathbb{Q}}, X)$ be a cellular rationalization and $f : X \to Y$ a continuous map to a simply-connected rational space Y. Then f extends over $X_{\mathbb{Q}}$ to a map $\hat{f} : X_{\mathbb{Q}} \to Y$. This map is unique up to homotopy, i.e., any two extensions of f are homotopic relative to X.*

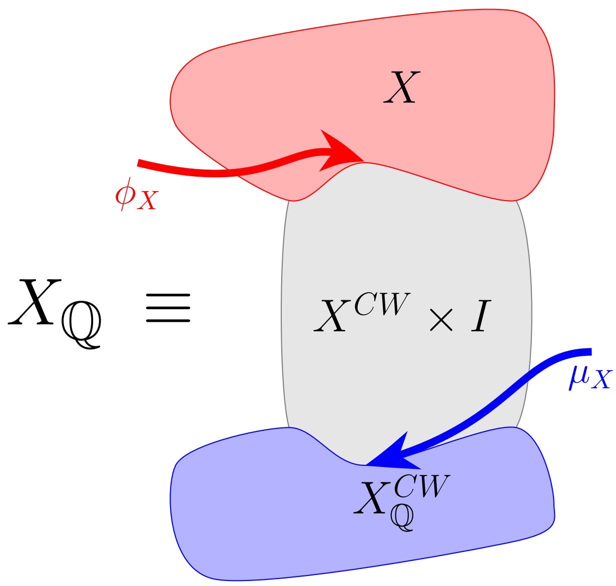


Figure 2.2: Building $X_{\mathbb{Q}}$ as the gluing of X and $X_{\mathbb{Q}}^{CW}$ through the cylinder of X^{CW}.

Summing up, in order to rationalize a simply-connected topological space X, we need to build a relative CW complex, $(X_{\mathbb{Q}}, X)$, in such a way that the inclusion is a rationalization. In order to do that, let X^{CW} be the cellular model of X. As X^{CW} is itself a CW complex, it is possible to build a (full CW) rational space $X_{\mathbb{Q}}^{CW}$ where the inclusion $\mu_X : X^{CW} \hookrightarrow X_{\mathbb{Q}}^{CW}$ is a rationalization [6].

Given a simply-connected space X, a CW model of the space (X^{CW}, ϕ_X), and a rationalization of such model $(X_{\mathbb{Q}}^{CW}, \mu_X)$, the rationalization of X is the space

$$X_{\mathbb{Q}} = X \bigcup_{\phi_X} (X^{CW} \times I) \bigcup_{\mu_X} X_{\mathbb{Q}}^{CW},$$

where $(x^{cw}, 0)$ goes to $\phi_X(x^{cw})$ and $(x^{cw}, 1)$ goes to $\mu_X(x^{cw})$ (see Figure 2.2). The inclusion $i : X \to X_{\mathbb{Q}}$ induces isomorphisms in homology with coefficients in $\mathbb{Q}$, and by Theorem 2.3, also it induces the isomorphisms $\pi_*(X_{\mathbb{Q}}) = \pi_*(X) \otimes \mathbb{Q}$.

The *rational homotopy type* of a simply-connected space X is the homotopy type of $X_{\mathbb{Q}}$, and the *rational homotopy class* of a map between simply-connected spaces $f : X \to Y$ is the homotopy class of $f_{\mathbb{Q}} : X_{\mathbb{Q}} \longrightarrow Y_{\mathbb{Q}}$. By Theorem 2.3, we know that rational homotopy groups and rational homology groups are invariants of the rational homotopy type. Also, if two spaces X and Y are CW complexes, so are their rationalizations and, due to Theorem 2.2, we know that the weak homotopy equivalence is a homotopy equivalence in the category of CW complexes. For two given CW complexes, X, Y, we can say that such spaces have the same rational homotopy type if and only if there is a homotopy equivalence between their rationalizations:

$$X_{\mathbb{Q}} \xrightarrow{\cong} Y_{\mathbb{Q}}.$$

In Figure 2.3 are summarized the relations between spaces, maps and their rationalizations:

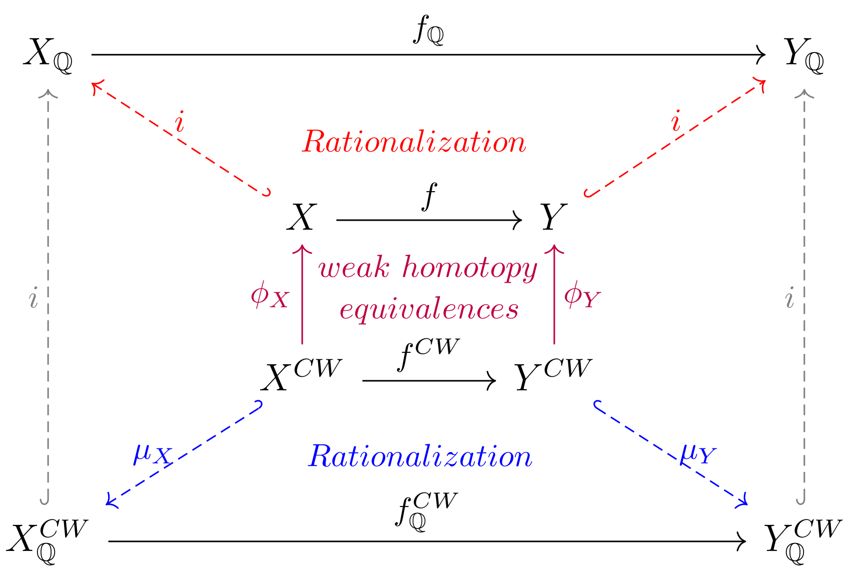


Figure 2.3: Rationalization of spaces and maps.

As it is mentioned at the end of the introduction, the main contribution of Sullivan was to obtain the rational homotopy type of a simply connected topological space X, via the construction of a commutative differential graded algebra, called the Sullivan model of X. This algebra allows us to establish a categorical equivalence between rational homotopy types of spaces and isomorphism classes of Sullivan models:

$$\begin{Bmatrix} \textit{Rational homotopy} \\ \textit{types of spaces} \end{Bmatrix} \Leftrightarrow \begin{Bmatrix} \textit{Isomorphim classes of} \\ \textit{minimal Sullivan models} \end{Bmatrix}.$$

In the next sections are introduced all the algebraic objects needed to define the Sullivan minimal model of a topological space.

2.3 Graded Commutative Differential Algebras

Definition 2.9. A *graded ring* is a ring R where the additive struture is a direct sum of abelian groups

$$R = \bigoplus_i R_i$$

and, for such decomposition, the product satisfies that

$$R_i \cdot R_j \subset R_{i+j}.$$

Remark. Every ring R can be endowed with a trivial graded structure where $R_0 = R$ and $R_i = 0, \forall i \neq 0$. For the rest of the chapter, let R be a commutative ring endowed with the trivial graded structure. Typically, $R = \mathbb{Z}, \mathbb{Q}, \mathbb{R}$.

Definition 2.10. Let M be an R-module. We say that M is *graded* if there exists a family of R-modules $\{M_i\}_{i\in\mathbb{Z}}$ such that $M = \bigoplus_{i\in\mathbb{Z}} M_i$. This decomposition is called a *grading* of M. An element $e \in M$ is said to be *homogeneous* if $e \in M_i$ for some i. In such a case, we say that e has *degree* $|e| = i$.

Example 2.11. Let $M = R\langle e_1, e_2, e_3\rangle$ be a graded R-module, where $|e_1| = 1$ and $|e_2| = |e_3| = 2$. Then, $M = M_1 \oplus M_2$, where $M_1 = R\langle e_1\rangle$ and $M_2 = R\langle e_2, e_3\rangle$.

Definition 2.12. The *tensor product* of two graded R-modules $M = \bigoplus_i M_i$ and $N = \bigoplus_j N_j$ is a graded R-module $M \otimes N$ where

$$(M \otimes N)_k = \bigoplus_{i+j=k} M_i \otimes N_j.$$

The definition of the tensor product of two graded R-modules will be relevant later in the definition of an R-algebra. In particular, it will be important to take into account the case of the tensor product of a graded R-module by itself, $M \otimes M$.

Definition 2.13. A *differential* in a graded R-module $M = \bigoplus_{i\in\mathbb{Z}} M_i$ is a homogeneous linear map $d : M \to M$ of degree k, that is, $d(M_i) \subseteq M_{i+k}$, such that $d^2 = 0$.

In this context, when k is negative we use subscript notation and when k is positive we use superscript notation. Usually, $k \in \{-1, 1\}$. We call the family of pairs $\{(M_i, d_i)\}_{i\in\mathbb{Z}}$ (or $\{(M^i, d^i)\}_{i\in\mathbb{Z}}$) a *complex*, where $d_i : M_i \to M_{i-1}$ is the restriction of d to M_i (analogously, $d^i : M^i \to M^{i+1}$ is the restriction of d to M^i). When $k = -1$ the complex is represented as

$$\cdots \longleftarrow M_{i-1} \xleftarrow{d_i} M_i \xleftarrow{d_{i+1}} M_{i+1} \longleftarrow \cdots ,$$

and when $k = 1$ as

$$\cdots \longrightarrow M^{i-1} \xrightarrow{d^{i-1}} M^i \xrightarrow{d^i} M^{i+1} \longrightarrow \cdots .$$

Example 2.14. Let $M = R\langle e_1, e_2, e_3\rangle$ be the graded R-module defined in the Example 2.11. Define the differential of the generators as:

$$\begin{aligned} d(e_1) &= 0, \\ d(e_2) &= e_1, \\ d(e_3) &= e_1. \end{aligned}$$

Extending by linearity, the differential of any other element of M is defined (note that $d^2 = 0$).

Let $\{(M_i, d_i)\}_{i\in\mathbb{Z}}$ be a complex as above. If no ambiguity arises, we will simply denote it by (M, d). Note that the condition $d^2 = 0$ implies $\operatorname{Im} d_{i+1} \subseteq \operatorname{Ker} d_i$ and both are R-submodules of M_i. Analogously, for complexes with differential of degree 1, $\operatorname{Im} d^{i-1} \subseteq \operatorname{Ker} d^i$. This motivates the following definitions.

Definition 2.15. The ith-*homology* group of a complex (M, d) with differential of degree -1, is defined as the quotient R-module

$$H_i(M, d) := \frac{\operatorname{Ker} d_i}{\operatorname{Im} d_{i+1}}. \tag{1}$$

The elements of $\operatorname{Ker} d_i$ are called (ith-)*cycles* and the elements of $\operatorname{Im} d_{i+1}$ are called (ith-)*boundaries*.

Definition 2.16. The ith-*cohomology* group of a complex (M, d) with differential of degree 1, is defined as the quotient R-module

$$H^i(M, d) := \frac{\operatorname{Ker} d^i}{\operatorname{Im} d^{i-1}}. \tag{2}$$

The elements of $\operatorname{Ker} d^i$ are called (ith-)*cocycles* and the elements of $\operatorname{Im} d_{i-1}$ are called (ith-)*coboundaries*.

Example 2.17. Let us compute the homology groups in Example 2.14. First, as M does not have elements of degree lower than 1 or greater than 2, $H_i = 0, \forall i \notin \{1, 2\}$. In order to obtain H_1, we need to compute the kernel of d_1 and the image of d_2. It is easy to see that $\operatorname{Ker} d_1 = M_1$ and $\operatorname{Im} d_2 = M_1$, so computing the quotient we obtain that $H_1 = 0$.

Let us see what happens with H_2. As there are no elements of degree 3 in M, the image of $d_3 = 0$. On the other hand, the kernel of d_2 is a submodule generated by the element $e_2 - e_3$, so we have that H_2 is generated by the equivalence class of this element. Summarizing, the homology groups of M are:

$$H_i(M, d) = \begin{Bmatrix} R\langle[e_2 - e_3]\rangle & if & i = 2 \\ 0 & & otherwise \end{Bmatrix}.$$

Definition 2.18. A graded *R-algebra* A is a graded R-module, together with an associative linear map of degree zero

$$(A \otimes A)^k \longrightarrow A^k$$
$$x \otimes y \mapsto xy$$

that has an identity $1 \in A^0$. This map is called *product* and satisfies the following property:

$$xy \in A^{i+j},\ \forall x \in A^i,\ \forall y \in A^j.$$

Definition 2.19. A *morphism* $\phi : A \to B$ of graded R-algebras is a linear map of degree zero such that $\phi(xy) = \phi(x)\phi(y)$ and $\phi(1) = 1$. Notice that a morphism of graded R-algebas preserves the degree.

Example 2.20. Let M and N be the graded R-algebras

- $M = R\langle x_1, x_2\rangle$ with $|x_1| = 1, |x_2| = 2$,
- $N = R\langle y_1, y_2, y_3\rangle$ with $|y_1| = 1, |y_2| = |y_3| = 2$.

We can define the morphism φ of graded R-algebras as follows:

$$\varphi : M \longrightarrow N$$
$$x_1 \mapsto y_1$$
$$x_2 \mapsto y_2 - y_3.$$

Definition 2.21. A *graded commutative* algebra is a graded algebra, A, where the product satisfies the property:

$$xy = (-1)^{ij} yx,\ \forall x \in A^i,\ \forall y \in A^j.$$

It is important to remark that this algebra is not commutative, but graded commutative. Also, the property does not work for non-homogeneous elements, i.e when one of the factors is not homogeneous.

Example 2.22. Let $A = \mathbb{Q}\langle a, b, c\rangle$ with $|a|$ and $|c|$ odds and $|b|$ even. Consider now the non-homogeneous element $a + b$ and the products $(a + b) \cdot c$ and $c \cdot (a + b)$. In that case, notice that the commutative law does not work, since

$$\begin{array}{rcr} (a+b)\cdot c & = & a\cdot c + b\cdot c, \\ c\cdot(a+b) & = & -a\cdot c + b\cdot c, \end{array}$$

and it means that, in general

$$(a + b) \cdot c \neq \pm c \cdot (a + b).$$

Definition 2.23. A graded commutative differential algebra (or GCDA), A, over a ring R is a graded R-algebra, $A = \bigoplus_{i=0}^{\infty} A^i$, together with an R-linear map $d_A : A \to A$ that satisfies the following conditions:

- $d_A^2 = 0$,
- $d_A(x) \subseteq A^{i+1}$, $\forall x \in A^i$,
- $d_A(xy) = d_A(x)y + (-1)^i x d_A(y)$, $\forall x \in A^i$, $\forall y \in A$.

From now on, we will asume that $R = \mathbb{Q}$.

2.4 The Sullivan Minimal Model of a GCDA

Definition 2.24. Let $V = \bigoplus_{p\geq 1} V^p$ be a graded $\mathbb{Q}$-vector space. We denote by $\Lambda V = (\bigotimes V)/I$, where $\bigotimes V$ is the tensor algebra of V, and I is the bilateral ideal generated by $\{v \otimes w + (-1)^{ij} w \otimes v \mid v \in V^i,\ w \in V^j\}$, and with the grading induced by the one in V.

Given W a subspace of V, $\Lambda^n W$ will denote the image of $W \otimes \overset{n}{\cdots} \otimes W$ in ΛV, and $\Lambda^{\geq i} V$ will be $\bigoplus_{n\geq i} \Lambda^n V$.

Notation. If $e_1, \ldots, e_n$ are a basis of the vector space V, then $\bigwedge(e_1, \ldots, e_n)$ will denote the algebra ΛV.

Definition 2.25. A GCDA (A, d) is said to be a *Sullivan* algebra if $A \simeq \Lambda V$, for some graded $\mathbb{Q}$-vector space V, that satisfies the following property:

- There exists a filtration
$$V = \bigcup_{k=0}^{\infty} V(k)$$
of graded subspaces
$$V(0) \subseteq V(1) \subseteq V(2) \subseteq \cdots \subseteq V$$
such that
$$d : V(0) \to \mathbb{Q}$$
and
$$d : V(k) \to \Lambda(V(k-1))$$
for $k \geq 1$.

Definition 2.26. A Sullivan algebra is said to be *minimal* if

$$\operatorname{Im} d \subseteq \Lambda^{\geq 2} V, \quad \forall v \in V.$$

Definition 2.27. A Sullivan *i-minimal model* (M, φ) of a GCDA (A, d_A) is a minimal Sullivan algebra (M, d_M) together with an i-quasi-isomorphism:

$$\varphi : M \longrightarrow A,$$

i.e., a morphism φ such that, the induced morphisms

$$\varphi^j : H^j(M) \xrightarrow{\simeq} H^j(A), \tag{3}$$

are isomorphisms for $j \leq i$, and a monomorphism for $j = i + 1$.

Definition 2.28. A Sullivan *minimal model* (M, φ) of a GCDA (A, d_A) is a Sullivan i-minimal model (M, d_M) for all i.

2.5 Effective Method for the Computation of the Minimal Model of a finitely generated GCDA

Some implementations of effective methods for the computation of the minimal model can be found in the literature for the case of finitely generated GCDAs. In [7] an algorithm for the computation of the entire minimal model of a given Sullivan algebra (not necessary minimal) through chain contractions is presented. On the other hand, in [10] the authors present an algorithm for the computation of the minimal model (up to degree i) of a (general) finitely generated GCDA. This algorithm takes as input a finite presentation of a connected GCDA A (i.e. all generators are of positive degree), and it outputs a presentation of its minimal model M up to a given degree, together with the morphism $\varphi : M \to A$. In this work we are going to follow the approach of the latter reference. The adapted method is described in section 3.4.

2.5.1 Limitations of these methods

If the given GCDA A has generators of degree zero, then the homogeneous parts of the algebra could be infinite dimensional. Since this method involves linear algebra computations on the homogenous parts of A, the method does not work for this kind of algebras. The main motivation of this work is to adapt the method to overcome this limitation.

2.6 The Graded Commutative Differential Algebra $A_{PL}(K)$

A simplicial object K with values in a category $\mathfrak{C}$ is a sequence $\{K_n\}_{n\geq 0}$ of objects in $\mathfrak{C}$, together with $\mathfrak{C}$-morphisms:

$$\partial_i : K_n \to K_{n-1},\ 0 \leq i \leq n,$$

$$s_j : K_n \to K_{n+1},\ 0 \leq j \leq n,$$

where the morphism ∂_i (respectively, s_j) is called the i-face (respectively, j-degeneration), and satisfy the identities:

$$\begin{array}{rll} \partial_i\partial_j = & \partial_{j-1}\partial_i, & i < j, \\ s_i s_j = & s_{j+1}s_i, & i \leq j, \\ \partial_i s_j = & \begin{cases} s_{j-1}\partial_i, & i < j, \\ id, & i = j, j+1, \\ s_j\partial_{i-1}, & i > j+1. \end{cases} \end{array} \tag{4}$$

A simplicial set is a simplicial object with values in the category of sets. One way to think of simplicial sets is as simplicial complexes where we allow degenerated simplices (that is, simplices where two or more vertices may coincide). In particular, we will think of simplicial sets as models for topological spaces.

Given a simplicial set K, we will construct a GCDA, denoted by $A_{PL}(K)$. The minimal model of the topological space modeled by K is defined as the minimal model of this algebra. In the next sections are presented all the elements needed for the construction of this algebra.

2.6.1 The cochain algebra $(A_{PL})_n$

First, consider the free graded commutative algebra $\bigwedge(t_0, \ldots, t_n, y_0, \ldots, y_n)$, where the generators t_i have degree 0, and the generators y_j have degree 1. The differential d is given by

$$d(t_i) = y_i,$$

$$d(y_j) = 0.$$

The cochain algebra $(A_{PL})_n$ is defined as the quotient:

$$(A_{PL})_n = \frac{\bigwedge(t_0, \ldots, t_n, y_0, \ldots, y_n)}{\langle \sum t_i - 1, \sum y_j \rangle}. \tag{5}$$

Notice that this algebra is actually isomorphic to $\bigwedge(t_1, \ldots, t_n, y_1, \ldots, y_n)$. That is, the relations allow us to express t_0 in terms of $t_1, \ldots, t_n$ and y_0 in terms of $y_1, \ldots, y_n$. After eliminating t_0, y_0, the expression of each element is unique. However, it is sometimes more convenient to use also t_0, y_0 to write certain formulas, so we will use one expression or the other depending on the context.

2.6.2 The simplicial cochain algebra A_{PL}

We define now a simplicial object in the category of cochain algebras, called the **simplicial cochain algebra** $A_{PL} = \{(A_{PL})_n\}_{n \geq 0}$ as follows:

- For each $n \geq 0$, the cochain algebra $(A_{PL})_n$ is the one defined above.

- The face and degeneration morphisms are the unique cochain algebra morphisms

  $$\partial_i : (A_{PL})_n \to (A_{PL})_{n-1},\ 0 \leq i \leq n,$$

  $$s_j : (A_{PL})_n \to (A_{PL})_{n+1},\ 0 \leq j \leq n,$$

  satisfying

$$\partial_i(t_k) = \begin{cases} t_k, & k < i \\ 0, & k = i \\ t_{k-1}, & k > i \end{cases}, \qquad s_j(t_k) = \begin{cases} t_k, & k < j \\ t_k + t_{k+1}, & k = j \\ t_{k+1}, & k > j \end{cases} \tag{6}$$

The structure of the simplicial cochain algebra A_{PL} can be visualized in figure 2.4, where horizontally is represented the simplicial structure for a given dimension n (subscript), and vertically the graded structure of the algebra (representing the degree p, with superscript). This object has a triangular structure due to the vanishing of the elements when $p > n$.

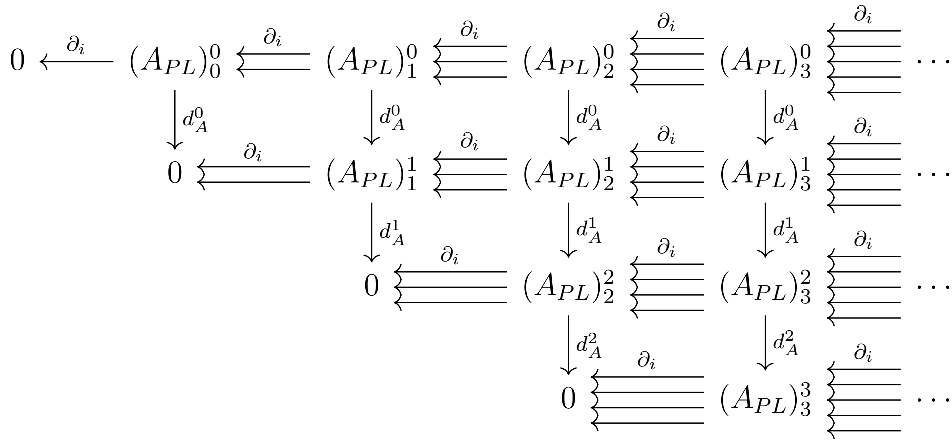


Figure 2.4: The structure of the simplicial cochain algebra $(A_{PL})_n$.

2.6.3 The cochain algebra $A_{PL}(K)$

Let K be a simplicial set, and let $A_{PL} = \{(A_{PL})_n\}_{n\geq 0}$ be the simplicial cochain algebra defined earlier. Then

$$A_{PL}(K) = \bigoplus_{p\geq 0} A^p_{PL}(K)$$

is the cochain algebra defined as follows:

- The homogenous part $A^p_{PL}(K)$ is the set of simplicial set morphisms from K to A^p_{PL}. That is, an element $\Phi \in A^p_{PL}(K)$ is a mapping that assigns to each n-simplex $\sigma \in K_n$, an element $\Phi(\sigma) \in (A^p_{PL})_n$, satisfying

$$\begin{aligned} \Phi(\partial_i(\sigma)) &= \partial_i(\Phi(\sigma)) \\ \Phi(s_j(\sigma)) &= s_j(\Phi(\sigma)) \end{aligned} \tag{7}$$

- Addition, scalar multiplication, product and the differential are induced by the corresponding operations in the algebras $(A_{PL})_n$.

The structure of the cochain algebra $A_{PL}(K)$, at some general dimension n and degree p, can be visualized in figure 2.5. the horizontal direction corresponds to the simplicial structure for a given dimension n (subscript), and the vertical direction corresponds to the graded structure of the algebra (representing the degree p with superscript). As the simplicial cochain algebra $(A_{PL})_n$ is at the ground of this structure, all the elements of $A^p_{PL}(K)_n$ will vanish when $p > n$. This is because it is not possible to assign non-zero polynomial forms of degree greater than the dimension of the simplices.

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \Big\downarrow {\scriptstyle d_A^{p-2}} & & \Big\downarrow {\scriptstyle d_A^{p-2}} & & \Big\downarrow {\scriptstyle d_A^{p-2}} & & \\
\cdots & \overset{\partial_i}{\underset{\leftarrow}{\overset{\leftarrow}{\vdots}}} & A_{PL}^{p-1}(K)_{n-1} & \overset{\partial_i}{\underset{\leftarrow}{\overset{\leftarrow}{\vdots}}} & A_{PL}^{p-1}(K)_{n} & \overset{\partial_i}{\underset{\leftarrow}{\overset{\leftarrow}{\vdots}}} & A_{PL}^{p-1}(K)_{n+1} & \overset{\partial_i}{\underset{\leftarrow}{\overset{\leftarrow}{\vdots}}} & \cdots \\
 & & \Big\downarrow {\scriptstyle d_A^{p-1}} & & \Big\downarrow {\scriptstyle d_A^{p-1}} & & \Big\downarrow {\scriptstyle d_A^{p-1}} & & \\
\cdots & \overset{\partial_i}{\underset{\leftarrow}{\overset{\leftarrow}{\vdots}}} & A_{PL}^{p}(K)_{n-1} & \overset{\partial_i}{\underset{\leftarrow}{\overset{\leftarrow}{\vdots}}} & A_{PL}^{p}(K)_{n} & \overset{\partial_i}{\underset{\leftarrow}{\overset{\leftarrow}{\vdots}}} & A_{PL}^{p}(K)_{n+1} & \overset{\partial_i}{\underset{\leftarrow}{\overset{\leftarrow}{\vdots}}} & \cdots \\
 & & \Big\downarrow {\scriptstyle d_A^{p}} & & \Big\downarrow {\scriptstyle d_A^{p}} & & \Big\downarrow {\scriptstyle d_A^{p}} & & \\
\cdots & \overset{\partial_i}{\underset{\leftarrow}{\overset{\leftarrow}{\vdots}}} & A_{PL}^{p+1}(K)_{n-1} & \overset{\partial_i}{\underset{\leftarrow}{\overset{\leftarrow}{\vdots}}} & A_{PL}^{p+1}(K)_{n} & \overset{\partial_i}{\underset{\leftarrow}{\overset{\leftarrow}{\vdots}}} & A_{PL}^{p+1}(K)_{n+1} & \overset{\partial_i}{\underset{\leftarrow}{\overset{\leftarrow}{\vdots}}} & \cdots \\
 & & \Big\downarrow {\scriptstyle d_A^{p+1}} & & \Big\downarrow {\scriptstyle d_A^{p+1}} & & \Big\downarrow {\scriptstyle d_A^{p+1}} & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

Figure 2.5: The structure of the cochain algebra $A_{PL}(K)$.

3 Effective method for the computation of the Sullivan model of a topological space

Here we will see how to adapt the algorithm mentioned in section 2.5 to the case of the algebra $A_{PL}(K)$ for a finite simplicial set K (a simplicial set is said finite if it has a finite number of non degenerated simplices). Note that the method as it is cannot be used in this case because each homogenous part has infinite dimension due to the nature of the underlying cochain algebra $(A_{PL})_n$ that has generators t_i of degree 0.

For a given simplicial complex K, there exists a quasi-isomorphism of cochain complexes $\oint : A_{PL}(K) \to C^*(K)$ that will be explained in section 3.1. Figure 3.1 summarizes the relationships between these objects and the Sullivan model (M, φ). The precise definition of these maps can be found in [6]. In order to adapt the method for the computation of the minimal Sullivan model, we need to define sections for the maps $\oint$ and d_A. The main contribution of this work is developed here.

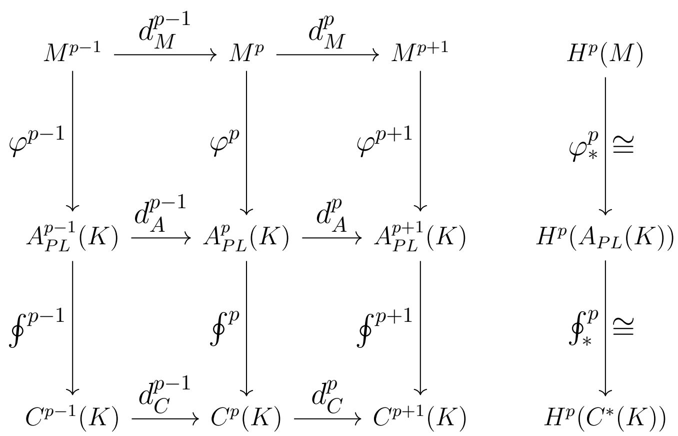


Figure 3.1: Diagram with the structures and maps involved in the computation of the Sullivan model of $A_{PL}(K)$.

On the one hand, as it is not possible to work directly with the cohomology of $A_{PL}(K)$, we will make use of the simplicial cohomology in order to find representatives of the cohomology generators of $A_{PL}(K)$. In 3.2 we describe a section of $\oint^p$ that allows us to find and element of $A_{PL}(K)$ whose integral is a given cochain (see figure 3.2). It follows that $\oint^p \circ \Theta^p = Id_{C^p}$.

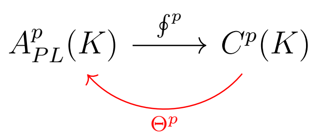


Figure 3.2: A section for the integral map.

On the other hand, we will also need to find a $(p-1)$-degree element of $A_{PL}(K)$ whose differential is a given exact element of degree p. In 3.3 we describe the section of the differential used to find this element (see figure 3.3). It follows that $d_A^{p-1} \circ d_{sec}^{p-1} = Id_{A^p}$.

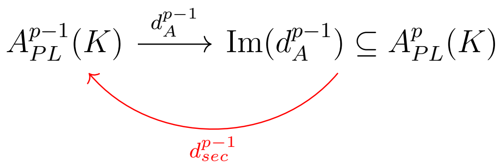


Figure 3.3: A section for the differential map of $A_{PL}(K)$.

3.1 The integration map $\oint$

We define a linear map:

$$\int_n : (A_{PL})_n^n \to \mathbb{Q}$$

by setting

$$\int_n t_1^{k_1} \dots t_n^{k_n} y_1 \dots y_n = \int_0^1 dt_1 \cdots \int_0^{1-\sum_1^{n-1} t_i} t_1^{k_1} \dots t_n^{k_n} dt_n = \frac{k_1! k_2! \dots k_n!}{(k_1 + \cdots + k_n + n)!} \quad (8)$$

This linear map is not injective, but there is a section $\theta_n : \mathbb{Q} \to (A_{PL})_n^n$ given by

$$\theta_n(c) = c \cdot (n!) \cdot y_1 \dots y_n. \quad (9)$$

It is easy to see that $(\int_n \circ \theta_n) = id$.

3.1.1 Integration map in $A_{PL}(K)$

We can use this map in $A_{PL}(K)$ to construct a natural quasi-isomorphism of cochain complexes

$$\oint : A_{PL}(K) \to C^*(K)$$

as follows. For a homogeneous element $\Phi_n^n \in A_{PL}(K)_n^n$, its integral will be an element of $C^n(K)$, given by

$$\left(\oint (\Phi_n^n) \right)(\sigma) = \int_n \Phi_n^n(\sigma) \quad (10)$$

for every $\sigma \in K_n$. For homogeneous elements $\Phi_m^n \in A_{PL}(K)_m^n$ with $n \neq m$, we define $\left(\oint(\Phi_m^n)\right)(\sigma) = 0$. Then we extend to all $A_{PL}(K)$ by linearity.

$$\begin{array}{ccc} A_{PL}^{p-1}(K) & \xrightarrow{d_A^{p-1}} & A_{PL}^p(K) \\ \downarrow \oint^{p-1} & & \downarrow \oint^p \\ C^{p-1}(K) & \xrightarrow{d_C^{p-1}} & C^p(K) \end{array}$$

Figure 3.4: The commutative diagram between $A_{PL}(K)$ and $C^*(K)$.

3.2 Lift a cochain

Now we will define a linear map

$$\Theta : C^*(K) \to A^*_{PL}(K)$$

satifying

$$\oint \circ \Theta = id.$$

In order to define this map, we need some auxiliary functions.

3.2.1 Propagate a simplex through the simplicial structure

Let $I = (i_1 < \cdots < i_k)$ be a strictly increasing sequence of k natural numbers between 0 and $n+k$. Let $J^I = (j_0 < \ldots < j_n)$ be the result of eliminating the elements of I from $(0, \ldots, n+k)$.

Consider the following ring morphisms

$$\begin{array}{rccc} \Upsilon_I : & \mathbb{Q}[t_0, \ldots, t_n] & \longrightarrow & \mathbb{Q}[t_0, \ldots, t_{n+k}] \\ & t_i & \mapsto & t_{j_i} \end{array} \tag{11}$$

and

$$\begin{array}{rcc} \overline{\partial}_I : (A_{PL})_{n+k} & \longrightarrow & (A_{PL})_n \\ f & \mapsto & (\partial_{i_1} \circ \cdots \circ \partial_{i_k})(f) \end{array} \tag{12}$$

Note that any sequence of k face maps is of this form.

For each element $p = q(t_0, \ldots, t_n) \cdot y_1 \cdots y_n \in (A_{PL})^n_n$, define

$$\Gamma^{n,n+k}_I(p) := \Upsilon_I(q) \sum_{l=0}^{n} (-1)^l t_{j_l} (y_{j_0} \ldots \widehat{y_{j_l}} \ldots y_{j_n}) \tag{13}$$

and extend it to all $(A_{PL})^n_n$ by linearity.

The following results can be proven by direct computation (the details are left to the reader):

Lemma 3.29. *The linear map*

$$\Gamma^{n,n+k}_I : (A_{PL})^n_n \to (A_{PL})^n_{n+k}$$

is a section of $\overline{\partial}_I$.

Lemma 3.30. *Let I' be another strictly increasing sequence of k natural numbers between 0 and $n+k$ different from I. Then the composition map $\bar{\partial}_{I'} \circ \Gamma^{n,n+k}_I$ is the zero map.*

Lemma 3.31. *Let $I' = (i_1 < \cdots < \widehat{i_l} < \ldots i_k)$ be the result of eliminating one entry in I, then $\Gamma^{n,n+k-1}_{I'} = \partial_{i_l} \circ \Gamma^{n,n+k}_I$.*

These maps will be called *propagation* maps.
Now take a degree n cochain $\phi : K_n \to \mathbb{Q}$ that maps the nondegenerate simplex σ to the number c, and the rest of the nondegenerate simplices to zero. We define $\Theta(\phi)$ as follows:

$$\Theta(\phi)(S) := \begin{cases} \theta_n(c) & \sigma = S \\ \Gamma_I^{n,dim(S)}(\theta_n(c)) & \sigma = \bar{\partial}_I(S) \\ 0 & \text{otherwise} \end{cases} \tag{14}$$

and extend it to $C^*(K)$ by linearity.
By the previous lemmas, it can checked that $\Theta(\phi) \in A_{PL}(K)$ and that $\oint \circ \Theta = id$.

3.3 Find a preimage by the differential of an exact element of $A_{PL}(K)$

Consider an element $\Phi \in A_{PL}(K)^p$ whose differential is zero and it has associated the trivial class in cohomology. In this section we show how to obtain an element $\Phi' \in A_{PL}(K)^{p-1}$ where $d(\Phi') = \Phi$.

Although $A_{PL}(K)$ is not finitely generated, in order to compute a preimage by the differential of Φ, we are going to see that it suffices to restrict the problem to a finite dimensional subspace of $A_{PL}(K)$.

3.3.1 Primitive basis of $\Phi(\sigma)$

Let $\sigma \in K_n$ and let $\Phi(\sigma) \in (A_{PL})^p_n$ be a polynomial where $n \geq p$. Then $\Phi(\sigma) = \sum_k q_k$ where the q_k are monomials of the form

$$q_k = c_k \cdot t_1^{l_1} \dots t_n^{l_n} y_{i_1} \dots y_{i_p} \tag{15}$$

with $i_j \in \{1, \dots, n\}$, $c_k \in \mathbb{Q}$. Notice that, after expressing t_0 and y_0 in terms of the rest of variables, these expressions are unique.

For each monomial q_k and each i_j, we define the monomials

$$\xi_{i_j}(q_k) := y_{i_1} \dots \widehat{y}_{i_j} \dots y_{i_p}, \tag{16}$$

$$\Psi_{i_j}(q_k) := t_1^{l_1} \dots t_{i_j}^{l_{i_j}+1} \dots t_n^{l_n} \xi_{i_j}(g_k), \tag{17}$$

and the primitive basis of a monomial as the set of terms

$$\Psi(q_k) := \{\Psi_{i_j}(q_k)\}_{j=1,\dots,p} \cup \{\xi_{i_j}(q_k)\}_{j=1,\dots,p}. \tag{18}$$

Definition 3.32. The primitive basis of $\Phi(\sigma)$ is the set of pairs

$$\Psi(\Phi(\sigma)) = \bigcup_k (\Psi(q_k), \sigma) \tag{19}$$

Here we interpret the element (q, σ) as the map $K \to A_{PL}$ that sends σ to q and the rest of the simplices to zero.

Definition 3.33. The primitive basis of $\Phi \in A_{PL}(K)$ is the set

$$\Psi(\Phi) = \bigcup_{\sigma \in K} \Psi(\Phi(\sigma)) \tag{20}$$

3.3.2 Differential basis of $\Phi(\sigma)$

Let $m \in (A_{PL})_n^{p-1}$ be a monomial of degree $p-1$, and let $\eta(m)$ be the set of monomials of degree p that appear in $d(m)$.

Definition 3.34. The differential basis of Φ is the set of terms

$$\Omega(\Phi) = \bigcup_{\sigma \in K} \{(a, \sigma) \mid a \in \eta(\Psi(\Phi(\sigma)))\} \tag{21}$$

That is, $\Omega(\Phi)$ contains the terms needed to express the differential of all elements in $\Psi(\Phi)$. By construction, Φ lives in the vector space V_Φ^f spanned by $\Omega(\Phi)$.

The differential induces a linear map from the vector space V_Φ^i spanned by $\Psi(\Phi)$ to V_Φ^f. Since they are both finite dimensional vector spaces, finding the preimage of an element can be done by solving a system of linear equations.

However, in general, the elements of V_Φ^i do not live in $A_{PL}(K)$ because we cannot ensure the compatibility with the face maps. Let us see how to fix this problem.

3.3.3 Restrictions of face maps

We are looking for an element $\chi \in A_{PL}(K)$ such that $d(\chi) = \Phi$. We can express the element we are looking for as a linear combination

$$\chi = \sum_{\Lambda \in \Psi(\Phi)} a_\Lambda \cdot \Lambda. \tag{22}$$

By construction of $\Omega(\Phi)$, the element $d(\chi)$ will be a certain linear combination of the elements of $\Omega(\Phi)$, where the coefficients depend linearly on the a_Λ's. So the condition $d(\chi) = \Phi$ will be given by an equality for each basis element.

Now the condition for $\chi \in A_{PL}(K)$ is the compatibility with the face maps. That is, it must satisfy

$$(\partial_i \chi)(\sigma) = \chi(\partial_i \sigma) \tag{23}$$

for every $\sigma \in K$. Again, this will give us a finite set of linear equations on the a_Λ's. So by solving the complete system of linear equations we obtain the desired element χ. Since we are assuming that Φ represents a trivial cohomology class, a solution to this system of equations is granted to exist.

3.3.4 An example of finding a preimage for a given exact element of $A_{PL}(K)$

In order to illustrate how this method works, let us begin with the an example. Consider the simplicial complex K built with two 2-simplices glued together as it is represented in figure 3.5. This simplicial complex models a disk $\mathbb{D}^2$:

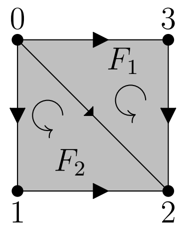


Figure 3.5: Example of a simplicial complex of dimension 2 with 2 facets.

The space is contractible, so we know that it has no cohomology for degrees greater than 0 (it has just one connected component so $H^0 = \mathbb{Z}$). This means that every closed form in $A_{PL}(K)$ is exact. Using that, consider the element of $A^2_{PL}(K)$:

$$\begin{array}{ccc} \varphi^2_2 : K_2 & \longrightarrow & (A_{PL})^2_2 \\ F_1 & \mapsto & y_1 y_2 \\ F_2 & \mapsto & y_1 y_2 \end{array}$$

which assing a polynomial for each facet of K and the polynomial 0 for each of its faces. In order to find an element $\omega \in A^1_{PL}(K)$ so that $d(\omega) = \varphi$, we need to solve the following system of linear equations:

$$\underbrace{\begin{bmatrix} d_{1,1} & d_{1,2} & \cdots & d_{1,n} \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ d_{m,1} & d_{m,2} & \cdots & d_{m,n} \end{bmatrix}}_{\textbf{Diff matrix}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{\textbf{var matrix}} = \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix}}_{\textbf{constant matrix}}$$

The main difficult is to construct the differential matrix, because it is a map, $d^{p-1} : A^{p-1}_{PL}(K) \to A^p_{PL}(K)$, between vector spaces of infinite dimensions. In order to solve this problem, we restrict such spaces to a finite subspaces determined by the generators that appear in the given element $\varphi(K)$. The variables of such system are the coefficients of the generators of the vector space $A^{p-1}_{PL}(K)$, the constant matrix is obtained by the polynomials assigned to the given element and it is formed by the coefficients of such elements using the basis the generators of the vector space $A^p_{PL}(K)$ that appear in such polynomials, plus a collection of zeros used for the simplicial restrictions. In figure 3.7 it is represented the structure of the differential matrix build with this method.

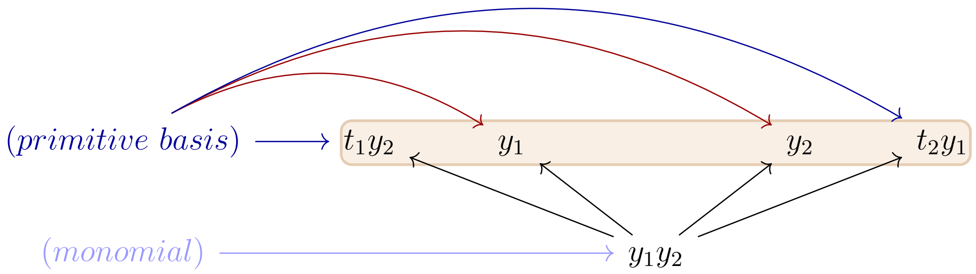


Figure 3.6: Example of the primitive basis of a monomial $p \in (A_{PL})^2$.

As we mentioned above, it is necessary to determine the vector subspaces where the differential is defined. In order to do so, for each monomial that appears in every polynomial $\varphi(\sigma)$, $\sigma \in K$, we compute all possible monomials in the preimage of this element, and due to the relations of $(A_{PL})_n$, we add as extra terms, the basis of the kernel of the differential without variables t_i. We do it for each simplex, so the result will be a list of monomials:

$$\begin{aligned}
(0,1,2):&\quad [y_2, y_1, t_2y_2, t_1y_2, t_2y_1, t_1y_1, t_1t_2y_2, t_1t_2y_1],\\
(1,2):&\quad [y_1, t_1y_1, t_1^2y_1],\\
(0,2):&\quad [y_1, t_1y_1],\\
(0,1):&\quad [y_1, t_1y_1],\\
(0,2,3):&\quad [y_2, y_1, t_2y_2, t_1y_2, t_2y_1, t_1y_1, t_1t_2y_2, t_1t_2y_1],\\
(2,3):&\quad [y_1, t_1y_1, t_1^2y_1],\\
(0,3):&\quad [y_1, t_1y_1]
\end{aligned}$$

In order to determine the vector subspace of $A^p_{PL}(K)$, as we have a list of monomials that forms the primitive basis for each simplex, we differentiate each monomial and the images of such elements will form the basis of the differential for each simplex:

$$\begin{aligned}
(0,1,2):&\quad [y_1y_2, t_2y_1y_2, t_1y_1y_2],\\
(0,2,3):&\quad [y_1y_2, t_2y_1y_2, t_1y_1y_2],
\end{aligned}$$

In our example, the differential matrix block for the simplex $(0,1,2)$ is:

$$\begin{array}{c} \\ y_1y_2 \\ t_2y_1y_2 \\ t_1y_1y_2 \end{array}
\begin{array}{c}
\begin{array}{cccccccc} y_2 & y_1 & t_2y_2 & t_1y_2 & t_2y_1 & t_1y_1 & t_1t_2y_2 & t_1t_2y_1 \end{array} \\
\begin{pmatrix} 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}
\end{array}$$

Using the differential basis, we can obtain the constant vector formed by the coeficients of each polynomial assigned to each simplex. In our example:

$$(1,0,0,1,0,0)$$

This would be enough for the computation of a preimage by the differential without simplicial restrictions. The last part of the construction will be to add, for each pair of simplices, the face restrictions consisting of a set of homogeneous equations forcing the system to fulfill the required conditions. In our example, the size of the block matrix corresponding for such equations is 14×28.

Now we are able to build the constant matrix consisting of the differential vector plus a set of zeros corresponding to the homogeneous equations related to simplicial restrictions:

$$(\underbrace{1,0,0,}_{Dif_{F_1}}\underbrace{1,0,0,}_{Dif_{F_2}}\underbrace{0,0,0,0,0,0,0,0,0,0,0,0,0,0}_{Face\ restrictions})$$

In our example, the dimension of the matrix for the whole linear system is 20×28.

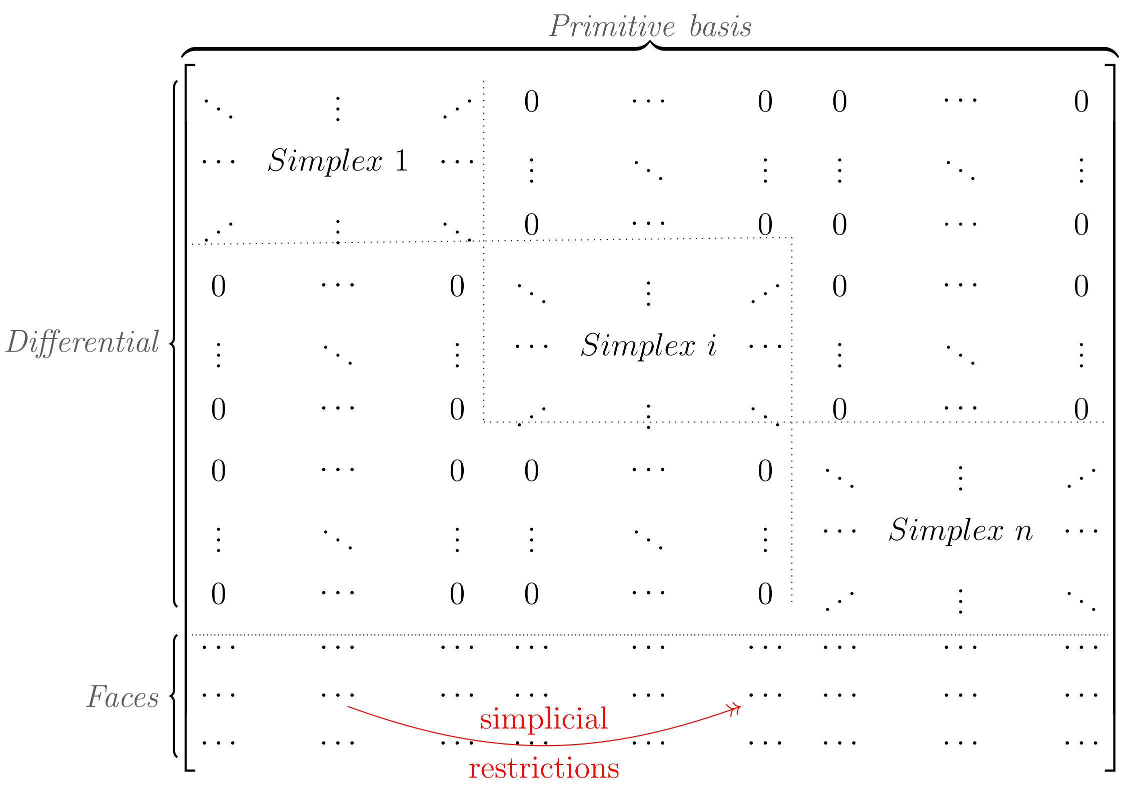


Figure 3.7: Matrix representation of the linear system used to find a preimage by the differential.

Solving the system, we obtain a solution expressed in terms of the primitive basis defined above

$$\left(\frac{1}{2},1,0,\frac{1}{2},-\frac{1}{2},0,0,0,0,\frac{1}{2},0,\frac{1}{2},-\frac{1}{2},0,0,0,1,0,\frac{1}{2},0,0,0,0,0,0,0,0,0\right)$$

that expressed in terms of polynomial is ($\omega \in A^1_{PL}(K)$):

$$\begin{aligned}
\varphi^1_2(F_1) &= -\tfrac{1}{2}t_2y_1 + \tfrac{1}{2}t_1y_2 + \tfrac{1}{2}y_1\\
\varphi^1_2(F_2) &= -\tfrac{1}{2}t_2y_1 + \tfrac{1}{2}t_1y_2 + y_1 + \tfrac{1}{2}y_2\\
\varphi^1_2([0,1]) &= y_1\\
\varphi^1_2([0,2]) &= \tfrac{1}{2}y_1
\end{aligned}$$

3.4 Algorithm

Now we have the ingredients to give the complete algorithm. The input will be a simplicial set of finite type K. The minimal model M will be constructed by adding generators, and for each generator we stablish also its differential and image by the map.

So the output will be a list of triplets $(x^d_i, d(x^d_i), \varphi(x^d_i))$, where

- x^d_i is a generator of M of degree d.
- $d(x^d_i)$ is a polynomial on the previous generators.
- $\varphi(x^d_i)$ is an element of $(A_{PL}(K))^d$.

This data will determine the free GCDA M with differential d, and a quasi-isomorphism ϕ.

The algorithm works degree by degree. So we will obtain an increasing sequence of i-minimal models. To simplify notation, we will refer to the model obtained so far in each moment as M.

1. Let $k_0 > 0$ be the smallest degree for which $H^{k_0}(K)$ is not trivial.

2. Take a basis $[a^{k_0}_0], \dots, [a^{k_0}_{l_{k_0}}]$ of $H^{k_0}(K)$, with $a^{k_0}_i \in C^{k_0}(K)$ (that is, the cohomology elements are represented by simplicial cochains). For each cochain $a^{k_0}_i$, take a lifting $A^{k_0}_i = \Theta(a^{k_0}_i) \in A_{PL}(K)$. For each of these elements, add to M a generator of the same degree, $x^{k_0}_i$, with $d(x^{k_0}_i) = 0$ and $\varphi(x^{k_0}_i) = A^{k_0}_i$.

   At this moment, φ induces an isomorphism in cohomology at degree k_0.

3. Now assume that we have already added generators of degree up to $k-1$ in such a way that φ is a $(k-1)$-quasi-isomorphism. In order to increase the degree and get a k-quasi-isomorphism, we add new generators to get also an isomorphism $\varphi^*_k : H^k(M) \xrightarrow{\simeq} H^k(A_{PL}(K))$, without changing the lower degree cohomologies.

Notice that computing the cohomology of $A_{PL}(K)$ directly cannot be done by simple linear algebra, because the graded parts are infinite dimensional. So we will use the isomorphism induced by $\oint$ and work in $H^*(K)$.

This process has two steps. In the first step, we add generators of degree $k-1$ until the map φ_k^* is injective. In the second step, we add generators of degree k in order to make the map φ_k^* surjective:

3.1 If the map φ_k^* is already injective at degree k, we go to step 3.2. Otherwise, consider the map φ^* composed with the isomorphism $\oint^*$. This is a linear map from $H^k(M)$ to $H^k(K)$. This map can be built following these steps: for each generator $[m_i^k]$ of $H^k(M)$, get a representative $m_i^k \in M^k$ and compute $c_i^k = (\oint^k \circ \varphi^k)(m_i^k)$. The cochain c_i^k is closed so it has associated a cohomology class $[c_i^k] = (\oint^* \circ \varphi^*)([m_i^k])$. Doing this for all generators of $H^k(M)$ we can construct the linear map (see Figure 3.8).

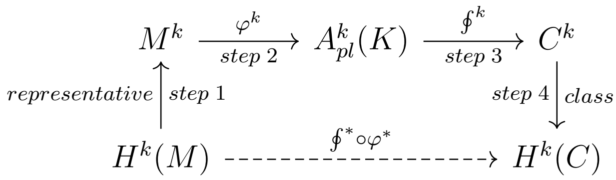


Figure 3.8: The injectivity of $\oint^* \circ \varphi^*$ is checked following the path from steps 1 to 4.

Now, take $[z_0^k], \ldots, [z_{l_k}^k]$ a basis of its kernel and consider representatives $z_0^k, \ldots, z_{l_k}^k \in M$. Compute $C_j^k = \varphi(z_j^k) \in A_{PL}(K)^k$. Since $[z_j^k] \in Ker(\varphi_k^*)$, the element C_j^k must correspond to a trivial cohomology class, so there must be an element $B_j^{k-1} \in A_{PL}(K)^{k-1}$ such that $d(B_j^{k-1}) = C_j^k$. This element B_j^{k-1} can be computed as in section 3.3. So we add to M^{k-1} the generators y_i^{k-1} with $\varphi(y_i^{k-1}) = B_i^{k-1}$ and $d_M(y_i^{k-1}) = z_i^k$.

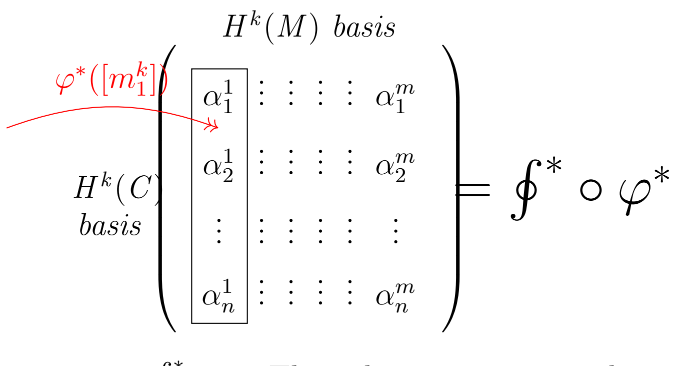


Figure 3.9: The linear map $\oint^* \circ \varphi^*$. The column i corresponds to the coordinates of the generator $[m_i^k] \in H^k(M)$ in the basis of $H^k(C)$. We are interested in the kernel of this map.

Notice that, after adding these generators, new elements of $Ker(\varphi_k^*)$ could have been added, so this step might be needed to be run iteratively until the map φ_k^* is injective.

3.2 Once we have that φ_k^* is injective, we will add new generators of degree k to make it surjective. Consider again the map ϕ_k^* composed with the isomorphism $\oint^*$. Take a basis $[a_0^k], \ldots, [a_{l_k}^k]$ of the complement of the image. As before, each a_i^k is a simplicial cochain that can be lifted to $A_i^k := \Theta(a_i^k) \in A_{PL}(K)$.

Add new generators of degree k to M, $\{x_0^k, \ldots, x_{l_k}^k\}$, with $d(x_i^k) = 0$ and $\phi(x_i^k) = A_i^k$.

In order to obtain an i-minimal model of (A, d_A), repeat the steps 3.1 and 3.2 until $k = i$.

4 Examples of computations

In this section are included some examples in order to illustrate the algorithm and its output.

4.1 The minimal model of $\mathbb{S}^1$

The first example is the computation of the minimal model of the sphere of dimension one, $\mathbb{S}^1$. This space can be modelled as a simplicial complex with the following structure: three vertices or 0-simplices (named 0, 1 and 2), and three edges or 1-simplices (one for each pair of vertices). The orientation of the edges is given just sorting the vertices in an increasing order (see Figure 4.1).

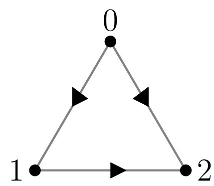


Figure 4.1: The simplicial complex representing $\mathbb{S}^1$ with the minimum number of simplices.

For each 1-simplex, the faces are given by:

$$\begin{aligned}\partial_1([0,1]) = \partial_1([0,2]) = [0]\\ \partial_0([0,1]) = \partial_1([1,2]) = [1]\\ \partial_0([0,2]) = \partial_0([1,2]) = [2]\end{aligned}$$

In this trivial example, $H^1(K)$ is freely generated by the class of the cochain:

$$\begin{array}{rcl} C_1 : \{[0,1],[0,2],[1,2]\} & \longrightarrow & \mathbb{Q} \\ {[0,1]} & \mapsto & 1 \\ {[0,2]} & \mapsto & 0 \\ {[1,2]} & \mapsto & 0 \end{array}$$

So, the lifting of $\Theta(C_1)$ is:

$$\begin{aligned}\Theta(C_1)([0,1]) &= y_1\\ \Theta(C_1)([0,2]) &= 0\\ \Theta(C_1)([1,2]) &= 0\end{aligned}$$

In this case, the minimal model will be generated by only one element of degree 1, x_0^1, with zero differential, and $\varphi(x_0^1) = \Theta(C_1)$. As the generator has odd degree, the minimal model is trivial for degrees greater than 1 and the algorithm finishes. Notice that, although the space is not simply connected, it is possible to compute the complete minimal model of the space. This is because this example is trivial, but in general, if the space is not simply connected, usually we will find that the algorithm gets stuck at some degree adding an infinite number of generators. This is the case of the next example.

4.2 The minimal model of $\mathbb{S}^1 \wedge \mathbb{S}^1$

Consider now the space formed by the wedge of two $\mathbb{S}^1$, i.e., two circles are glued together at a vertex (see Figure 4.2).

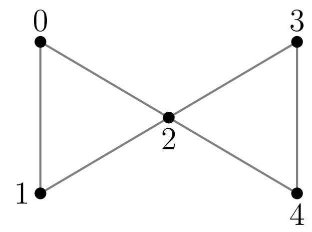


Figure 4.2: The simplicial complex representing the wedge of two $\mathbb{S}^1$.

In this case, $H^1(K)$ is freely generated by the classes of two cochains:

$$\begin{array}{rcl} C_1 : \{a,b,c,d,e,f\} & \longrightarrow & \mathbb{Q} \\ [0,1] & \mapsto & 1 \\ [0,2] & \mapsto & 0 \\ [1,2] & \mapsto & 0 \\ [2,3] & \mapsto & 0 \\ [2,4] & \mapsto & 0 \\ [3,4] & \mapsto & 0 \end{array} \qquad \begin{array}{rcl} C_2 : \{a,b,c,d,e,f\} & \longrightarrow & \mathbb{Q} \\ [0,1] & \mapsto & 0 \\ [0,2] & \mapsto & 0 \\ [1,2] & \mapsto & 0 \\ [2,3] & \mapsto & 1 \\ [2,4] & \mapsto & 0 \\ [3,4] & \mapsto & 0 \end{array}$$

So, the lifting of $\Theta(C_1)$ is:

$$\begin{array}{rl} \Theta(C_1)([0,1]) = & y_1 \\ \Theta(C_1)([0,2]) = & 0 \\ \Theta(C_1)([1,2]) = & 0 \\ \Theta(C_1)([2,3]) = & 0 \\ \Theta(C_1)([2,4]) = & 0 \\ \Theta(C_1)([3,4]) = & 0 \end{array}$$

and the lifting of $\Theta(C_2)$ is:

$$\begin{array}{rl} \Theta(C_2)([0,1]) = & 0 \\ \Theta(C_2)([0,2]) = & 0 \\ \Theta(C_2)([1,2]) = & 0 \\ \Theta(C_2)([2,3]) = & y_1 \\ \Theta(C_2)([2,4]) = & 0 \\ \Theta(C_2)([3,4]) = & 0 \end{array}$$

The method begins building a GCDA with two generators of degree 1, named x_0^1 and x_1^1, with zero differential, and the morphism associates the generators with the lifted elements:

$$\varphi(x_0^1) = \Theta(C_1)$$
$$\varphi(x_1^1) = \Theta(C_2)$$

The problem in this case arises when the algorithm tries to check injectivity at degree 2. As the current generators have degree 1, the squares are zero, but it is not the case of the cross product $x_0^1 * x_1^1$. This element has degree 2 and it is easy to check that its differential is zero. There is no element of degree 1 whose differential is $x_0^1 * x_1^1$ so it means that this element is closed but not exact (i.e., it is a representative of a non trivial cohomology class).

In order to make the induced morphism $\varphi*$ injective, it is necessary to kill this cohomology class (because the space has trivial cohomology for degrees greater than one). In order to do this, a generator of degree one, y_0^1 is added to the algebra with differential exactly that element:

$$d(y_0^1) = x_0^1 * x_1^1$$

and, due to $\varphi(x_0^1 * x_1^1) = 0$ (because there are no simplices of dimension greater than one), the method just assings zero to the image of the new generator:

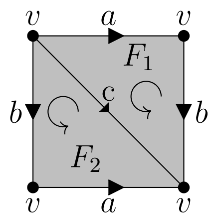
$$\varphi(y_0^1) = 0$$

One could think that the problem has been solved but instead of that, now we have two new elements of degree 2, $y_0^1 * x_0^1$ and $y_0^1 * x_1^1$ corresponding with two new non trivial cohomology classes, so the problem remains and this dynamic just get worst as we add new generators. This example illustrates the problem of computing the minimal model of a non simply connected space.

Just to finish, it is worth noting that the problem is not that the minimal model does not exists, the problem is that the minimal model has infinite generators of degree 1!

4.3 The minimal model of $\mathbb{T}^2$

Consider now the usual triangulation of the torus:



It can be represented with the simplicial set K with the following non degenerate simplices: one in dimension zero (called v), three in dimension one (a, b and c), and two in dimension two F_1, F_2. The face maps are given by

- $\partial_0(a) = \partial_1(a) = \partial_0(b) = \partial_1(b) = \partial_0(c) = \partial_1(c) = v$
- $\partial_0(F_1) = \partial_2(F_2) = b, \qquad \partial_1(F_1) = \partial_1(F_2) = c, \qquad \partial_2(F_1) = \partial_0(F_2) = a$

It is easy to check that $H^1(K)$ is freely generated by the classes of two cochains:

$$\begin{array}{ccc} C_1 : \{a,b,c\} & \longrightarrow & \mathbb{Q} \\ a & \mapsto & 1 \\ b & \mapsto & 0 \\ c & \mapsto & 1 \end{array} \qquad\qquad \begin{array}{ccc} C_2 : \{a,b,c\} & \longrightarrow & \mathbb{Q} \\ a & \mapsto & 0 \\ b & \mapsto & 1 \\ c & \mapsto & 1 \end{array}$$

and $H^2(K)$ is freely generated by

$$\begin{array}{ccc} D : \{F_1,F_2\} & \longrightarrow & \mathbb{Q} \\ F_1 & \mapsto & 1 \\ F_2 & \mapsto & 0 \end{array}$$

With this data, we can now run the algorithm:

1. Since $H^1(K)$ is not trivial, we start with $k_0 = 1$.

2. We take the basis $\langle [C_1], [C_2] \rangle$ of $H^1(K)$. As a cochain, C_1 can be expressed as the sum $C_1 = C_{1a} + C_{1c}$ where:

$$\begin{array}{ccc} C_{1a} : \{a,b,c\} & \longrightarrow & \mathbb{Q} \\ a & \mapsto & 1 \\ b & \mapsto & 0 \\ c & \mapsto & 0 \end{array} \qquad\qquad \begin{array}{ccc} C_{1c} : \{a,b,c\} & \longrightarrow & \mathbb{Q} \\ a & \mapsto & 0 \\ b & \mapsto & 0 \\ c & \mapsto & 1 \end{array}$$

   and by linearity we have that $\Theta(C_1) = \Theta(C_{1a}) + \Theta(C_{1c})$. Now, in order to obtain $\Theta(C_{1a})$ we need to follow the steps mentioned above:

(a) We compute $\theta_1(1) = y_1$

(b) Now, we compute the propagations

$$\Gamma^{1,2}_{\{1\}}(y_1) = \quad -t_2y_1 + t_1y_2 + y_1$$

$$\Gamma^{1,2}_{\{2\}}(y_1) = \quad -t_2y_1 + t_1y_2$$

So, the lifting of $\Theta(C_{1a})$ is:

$$\begin{array}{lcl} \Theta(C_{1a})(O) & = & 0 \\ \Theta(C_{1a})(a) & = & y_1 \\ \Theta(C_{1a})(b) & = & 0 \\ \Theta(C_{1a})(c) & = & 0 \\ \Theta(C_{1a})(F_1) & = & -t_2y_1 + t_1y_2 + y_1 \\ \Theta(C_{1a})(F_2) & = & -t_2y_1 + t_1y_2 \end{array}$$

Doing the analogous computations for C_{1c}, C_{2b} and C_{2c}, we obtain the corresponding liftings of $\Theta(C_1)$ and $\Theta(C_2)$:

$$\begin{array}{lcl lcl} \Theta(C_1)(O) & = & 0 & \Theta(C_2)(O) & = & 0 \\ \Theta(C_1)(a) & = & y_1 & \Theta(C_2)(a) & = & 0 \\ \Theta(C_1)(b) & = & 0 & \Theta(C_2)(b) & = & y_1 \\ \Theta(C_1)(c) & = & y_1 & \Theta(C_2)(c) & = & y_1 \\ \Theta(C_1)(F_1) & = & y_1 + y_2 & \Theta(C_2)(F_1) & = & y_2 \\ \Theta(C_1)(F_2) & = & y_2 & \Theta(C_2)(F_2) & = & y_1 + y_2 \end{array}$$

Now, we add to our model two generators $M = \langle x_0^1, x_1^1 \rangle$ of degree 1 in such a way that $d(x_i^1) = 0$ and $\varphi(x_i^1) = \Theta(C_i)$. At this moment, φ induces an isomorphism at degree 1 given by $\varphi^*([x_i^1]) = [C_i]$. Once we have an isomorphism at degree 1, we go to step 3 in order to check injectivity at degree 2.

3. At this point, the homogeneous part of degree 2 of M is generated by only one element $M^2 = \langle x_0^1 x_1^1 \rangle$, because x_i^1 have odd degree and the square of each one is zero. This generator is closed so it is associated with a cohomology class.

   As we saw before, $H^2(K)$ is a one-dimensional vector space. So, $(\varphi^*)^2$ is injective if and only if the induced image of the generator $[x_0^1 x_1^1]$ is non-zero. In order to check this, we apply the integration map to its image $C_3 : \oint(\Theta(C_1) \cdot \Theta(C_2))$:

$$\begin{array}{ccc} C_3 : K_2 & \longrightarrow & \mathbb{Q} \\ F1 & \mapsto & -\frac{1}{2} \\ F2 & \mapsto & \frac{1}{2} \end{array}$$

   but the image of the coboundary map is generated by the cochain

$$\begin{array}{ccc} K_2 & \longrightarrow & \mathbb{Q} \\ F1 & \mapsto & 1 \\ F2 & \mapsto & 1 \end{array}$$

which is linearly independent with C_3. So we already have an isomorphism and this step of the algorithm is done. Since we have no more generators in cohomology, the algorithm finishes and the minimal model of the torus is $M = \langle x_0^1, x_1^1 \rangle$.

4.4 The K3 Surface

For the last example, a space with a higher order of complexity has been chosen. This space is known as the K3 surface, it is simply connected and it is available an implementation at the SageMath repository [2]. The simplicial complex has dimension 4 and it is build with 16 vertices and 288 facets (in this case, all facets are simplices of dimension 4). In Table 4.1 are listed the number of simplices for each dimension. It was constructed by Casella and Kühnel in [1]. The implementation in SageMath uses the labeling from Spreer and Kühnel [11].

	Dimension				
	0	**1**	**2**	**3**	**4**
Nº Simplices	16	120	560	720	288

Table 4.1: Number of simplices of the K3 Surface in each dimension.

The Betti numbers of the space are listed in Table 4.2. Omitting degree 0, the cohomology has 22 generators of degree 2 and one generator of degree 4.

	Degree				
	0	**1**	**2**	**3**	**4**
Nº of Cohomology Generators	1	0	22	0	1

Table 4.2: Number of cohomology generators of the K3 surface for each degree.

Due to Terzić [12], it is known that:

Theorem 4.35. *Let M be a closed oriented simply connected four-manifold and b_2 its second Betti number. Then:*

1. *If $b_2 = 0$ then $\operatorname{rk} \pi_4(M) = \operatorname{rk} \pi_7(M) = 1$ and $\pi_p(M)$ is finite for $p \neq 4, 7$,*
2. *If $b_2 = 1$ then $\operatorname{rk} \pi_2(M) = \operatorname{rk} \pi_5(M) = 1$ and $\pi_p(M)$ is finite for $p \neq 2, 5$,*
3. *If $b_2 = 2$ then $\operatorname{rk} \pi_2(M) = \operatorname{rk} \pi_3(M) = 2$ and $\pi_p(M)$ is finite for $p \neq 2, 3$,*
4. *If $b_2 > 2$ then $\dim \pi_*(M) \otimes \mathbb{Q} = \infty$ and*

$$\operatorname{rk} \pi_2(M) = b_2, \quad \operatorname{rk} \pi_3(M) = \frac{b_2(b_2+1)}{2} - 1, \quad \operatorname{rk} \pi_4(M) = \frac{b_2(b_2^2-4)}{3}.$$

In particular, as the $K3$ surface has $b_2 = 22$, and following the theorem 4.35, the rank of the first homotopy groups are (see Table 4.3):

	$\pi_n(K3)$			
	1	**2**	**3**	**4**
rank	0	22	252	3520

Table 4.3: Rank of the first homotopy groups of K3.

The method is able to compute the 4-minimal model of the space. The algorithm begins at degree 2, lifting the 22 cohomology generators to representatives $a_i^2 \in A_{PL}^2(K3)$ and adding the corresponding generators x_i^2 (with $d_M(x_i^2) = 0$ and $\varphi(x_i^2) = a_i^2$) to the model. At this point, the minimal model induces isomorphism in cohomology up to degree 2, but, due to the emptiness of the homogeneous part of degree 3 of $H^3(K3)$ and M^3, it also induces isomorphism at degree 3.

Now the algorithm checks injectivity at degree 4 and finds that:

1. The element $(x_0^2)^2 \in M^4$ is a representative of the single generator of degree 4 of $H^4(K3$.

2. There are 252 elements generating the same number of cohomology classes in $H^4(M)$.

Notice that (2) is coherent with the results of Theorem 4.35 and Table 4.3.

In order to make the induced morphism $\varphi*$ injective at degree 4, the algorithm adds as many generators as cohomology classes needs to kill, pointing the differential of each new generator $y_i^3 \in M^3$ to a representative of a different cohomology class. Due to the extension, the generators and their differentials are included in Appendix I.

Once the algorithm has reached injectivity at degree 4 (and also surjectivity), the method continues killing the rest of the non trivial cohomology classes for degree 5, adding the corresponding generators y_i^4. In that case, as there are no simplices of dimension 5, notice that $d(y_i^4) = 0, \forall i$. So, for our purposes, the interesting degrees of the minimal model are degrees 2, 3 and 4, where the morphism φ carries information about the $A_{PL}(K3)$.

5 Conclusions and Future Work

In this work it has been presented an effective method for the computation of the Sullivan minimal model of a given topological space. In order to carry the work out, it has been necessary to understand the basis of Rational Homotopy Theory and how is defined the Sullivan model of a topological space.

For that purpose, a review of the state of the art about previous works on effective methods for the computation of Sullivan minimal algebras has been made in order to find possible obstacles on these methods when the homogeneous parts of the algebra of interest are infinite dimensional. After that, viewing that these methods can not work with this kind of algebras, a solution has been proposed, adapting the corresponding parts of the algorithm. Last but not least, the method has been implemented in a Computer Algebra System (SageMath). The source code of the implementation can be found at `https://riemann.unizar.es/git/calquezar/AplK`.

As far as we know, it is the first time that such a method has been designed and implemented in a Computer Algebra System. However, the development of this work has pointed out some questions that, although they are beyond the scope of this work, it will be necessary to address in the future. Some of these questions are listed below.

The kernel of the differential of $A_{PL}(K)$.

During the development of this work, some key questions have arisen for which a mathematical answer is needed. In particular, at the injectivity step of the algorithm (see section 3.1), when the algorithm needs to find a preimage by the differential of an element of $A_{PL}(K)$, our method needs to add some kernel elements to the primitive basis of each simplex, but it is not clear, for a generic element, how many elements one needs to add in order to guarantee that the method finds a solution for the system. This is an interesting point that should be reviewed in order to adjust the method properly and to be sure that there are no unuseful terms that can worsen its performance.

Δ-Complexes.

Other idea related to improve the computational performance is to adapt the method to work with other useful constructions similar but different of simplicial complexes. For example, Δ-*Complexes* are essentially a generalization of simplicial complexes where it is not required the condition that each face of a simplex is unique. This relaxation makes this construction more flexible and allows one to build equivalent complexes with less number of simplices.

An example to show the difference between simplicial complexes and Δ-complexes is the following:

Example 5.36 (Cone)**.** Consider the problem of modelling a cone C both as a simplicial complex and Δ-complex. As a simplicial complex, consider the 3-simplex of figure 2.1d and remove one face (i.e. remove one 2-simplex). This gives you a collection of 3 simplices of dimension 2 and their faces (in total 13 simplices: four 0-simplices, six 1-simplices, and three 2-simplices), all glued in the same way they were at the beginning. On the other hand, as a Δ-complex, it is possible to model this space just taking the 2-simplex of the figure 2.1c and identifying two of its faces of dimension 1. This construction gives you a collection of one 2-simplex, two 1-simplices and two 0-simplices (in total 5 simplices).

Taking into account the following theorem:

Theorem 5.37 ([9, p. 107])**.** *Every Δ-complex can be subdivided to be a simplicial complex. In particular, every Δ-complex is homeomorphic to a simplicial complex.*

we can extend the results presented in this work in order to make use of Δ-complexes.

Formality

Finally, the method presented in this work (and in particular the implementation in a Computer Algebra System) allows us to tackle the interesting topic of formality from a computational point of view.

Basically, a topological space is called *formal* if the minimal model of the space and the model of the cohomology algebra are isomorphic (see Figure 5.1). On the other hand, if the two models are not isomorphic, it is said that the space is not formal (see Figure 5.2).

$$\begin{array}{ccc} & M_A \simeq M_H & \\ {}^{\varphi_A}\swarrow & & \searrow^{\varphi_H} \\ A_{PL}(K) & & H(K) \end{array}$$

Figure 5.1: Condition satisfied for a space to be formal.

$$\begin{array}{cc} M_A & M_H \\ {\scriptstyle\varphi_A}\downarrow & {\scriptstyle\varphi_H}\downarrow \\ A_{PL}(K) & H(K) \end{array}$$

Figure 5.2: If the minimal model of the space M_A and the model of the cohomology algebra M_H are not isomorphic, it is said that the space is not formal.

The key point in the algorithm to study if a space if formal is found at injectivity step (see section 3.1). The main difference between computing the minimal model of the cohomology algebra and the minimal model of the space is that in the former, checking injectivity at degree p only involves three objects: the model in construction M, its cohomology $H^p(M)$, and the cohomology algebra $H^p(K)$ (see Figure 5.3).

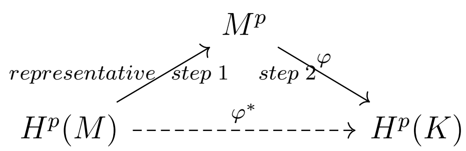


Figure 5.3: Working with a finitely generated GCDA. The injectivity of the morphism $\varphi*$ is checked combining steps 1 and 2.

However, for the computation of the minimal model of a space K, the algorithm needs to use the algebra $A_{PL}(K)$ in order to connect the cohomology of the model $H(M)$ and the cohomology of the space $H(K)$ (see Figure 5.4). It is in this connection where formality can be broken and it is an interesting question what kind of topological structures do not satisfy this property.

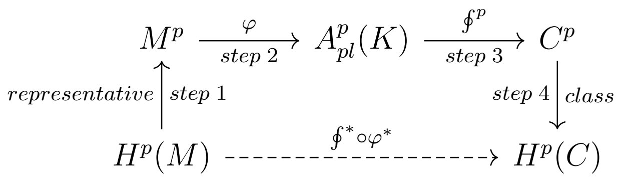


Figure 5.4: Working with $A_{pl}(K)$. The injectivity of the morphism $\oint^* \circ \varphi^*$ is checked following the path from steps 1 to 4.

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Appendix I: The homogeneous part of M^3 for the K3 surface

Following the Example 4.4, the list of generators of degree 3 of the minimal model of the K3 surface are listed within their differentials (i.e, homogeneous elements of degree 4 in M).

Generator	**Differential**
y_0^3	$(x_1^2)^2 - (x_{21}^2)^2$
y_1^3	$(x_0^2)^2 - (x_{21}^2)^2$
y_2^3	$x_0^2x_3^2 - \frac{1}{2}(x_{21}^2)^2$
y_3^3	$x_0^2x_1^2 - \frac{1}{2}(x_{21}^2)^2$
y_4^3	$x_1^2x_2^2 - \frac{1}{2}(x_{21}^2)^2$
y_5^3	$x_0^2x_2^2 - \frac{1}{2}(x_{21}^2)^2$
y_6^3	$(x_2^2)^2 - (x_{21}^2)^2$
y_7^3	$x_1^2x_3^2 - \frac{1}{2}(x_{21}^2)^2$
y_8^3	$x_2^2x_3^2$
y_9^3	$(x_3^2)^2 - (x_{21}^2)^2$
y_{10}^3	$x_0^2x_4^2 + \frac{1}{2}(x_{21}^2)^2$
y_{11}^3	$x_1^2x_4^2 + \frac{1}{2}(x_{21}^2)^2$
y_{12}^3	$x_2^2x_4^2 + \frac{1}{2}(x_{21}^2)^2$
y_{13}^3	$x_3^2x_4^2$
y_{14}^3	$(x_4^2)^2 - (x_{21}^2)^2$
y_{15}^3	$x_0^2x_5^2$
y_{16}^3	$x_1^2x_5^2 - \frac{1}{2}(x_{21}^2)^2$
y_{17}^3	$x_2^2x_5^2$
y_{18}^3	$x_3^2x_5^2$
y_{19}^3	$x_4^2x_5^2 + \frac{1}{2}(x_{21}^2)^2$
y_{20}^3	$(x_5^2)^2 - (x_{21}^2)^2$
y_{21}^3	$x_0^2x_6^2 + (x_{21}^2)^2$
y_{22}^3	$x_1^2x_6^2$
y_{23}^3	$x_2^2x_6^2$
y_{24}^3	$x_3^2x_6^2 + (x_{21}^2)^2$
y_{25}^3	$x_4^2x_6^2 + \frac{1}{2}(x_{21}^2)^2$
y_{26}^3	$x_1^2x_7^2$
y_{27}^3	$x_4^2x_7^2$
y_{28}^3	$x_5^2x_6^2 - \frac{1}{2}(x_{21}^2)^2$
y_{29}^3	$(x_6^2)^2 - 3(x_{21}^2)^2$
y_{30}^3	$x_0^2x_7^2 - (x_{21}^2)^2$
y_{31}^3	$x_2^2x_7^2 - \frac{1}{2}(x_{21}^2)^2$
y_{32}^3	$x_3^2x_7^2 - \frac{1}{2}(x_{21}^2)^2$
y_{33}^3	$x_5^2x_7^2 + \frac{1}{2}(x_{21}^2)^2$
y_{34}^3	$(x_7^2)^2 - 2(x_{21}^2)^2$
y_{35}^3	$x_0^2x_8^2 - \frac{1}{2}(x_{21}^2)^2$
y_{36}^3	$x_6^2x_7^2 + 2(x_{21}^2)^2$

y^3_{37}	$x^2_1 x^2_8 + \frac{1}{2}(x^2_{21})^2$
y^3_{38}	$x^2_2 x^2_8$
y^3_{39}	$x^2_3 x^2_8 - \frac{1}{2}(x^2_{21})^2$
y^3_{40}	$x^2_4 x^2_8$
y^3_{41}	$x^2_5 x^2_8 + \frac{1}{2}(x^2_{21})^2$
y^3_{42}	$x^2_7 x^2_8 - \frac{3}{2}(x^2_{21})^2$
y^3_{43}	$(x^2_8)^2 - 2(x^2_{21})^2$
y^3_{44}	$x^2_0 x^2_9$
y^3_{45}	$x^2_1 x^2_9 - \frac{1}{2}(x^2_{21})^2$
y^3_{46}	$x^2_6 x^2_8 + \frac{3}{2}(x^2_{21})^2$
y^3_{47}	$x^2_2 x^2_9$
y^3_{48}	$x^2_3 x^2_9$
y^3_{49}	$x^2_4 x^2_9$
y^3_{50}	$x^2_5 x^2_9 - \frac{1}{2}(x^2_{21})^2$
y^3_{51}	$x^2_6 x^2_9 - \frac{1}{2}(x^2_{21})^2$
y^3_{52}	$x^2_7 x^2_9 + \frac{1}{2}(x^2_{21})^2$
y^3_{53}	$x^2_8 x^2_9 + (x^2_{21})^2$
y^3_{54}	$(x^2_9)^2 - (x^2_{21})^2$
y^3_{55}	$x^2_1 x^2_{10}$
y^3_{56}	$x^2_0 x^2_{10} - (x^2_{21})^2$
y^3_{57}	$x^2_3 x^2_{10} - \frac{1}{2}(x^2_{21})^2$
y^3_{58}	$x^2_2 x^2_{10} - \frac{1}{2}(x^2_{21})^2$
y^3_{59}	$x^2_4 x^2_{10} + \frac{1}{2}(x^2_{21})^2$
y^3_{60}	$x^2_5 x^2_{10}$
y^3_{61}	$x^2_9 x^2_{10} + \frac{1}{2}(x^2_{21})^2$
y^3_{62}	$x^2_7 x^2_{10} - \frac{3}{2}(x^2_{21})^2$
y^3_{63}	$x^2_2 x^2_{11}$
y^3_{64}	$x^2_6 x^2_{10} + \frac{3}{2}(x^2_{21})^2$
y^3_{65}	$x^2_8 x^2_{10} - \frac{3}{2}(x^2_{21})^2$
y^3_{66}	$(x^2_{10})^2 - 2(x^2_{21})^2$
y^3_{67}	$x^2_0 x^2_{11} + \frac{1}{2}(x^2_{21})^2$
y^3_{68}	$x^2_1 x^2_{11} + \frac{1}{2}(x^2_{21})^2$
y^3_{69}	$x^2_3 x^2_{11} + \frac{1}{2}(x^2_{21})^2$
y^3_{70}	$x^2_4 x^2_{11} - \frac{1}{2}(x^2_{21})^2$
y^3_{71}	$x^2_5 x^2_{11} + \frac{1}{2}(x^2_{21})^2$
y^3_{72}	$x^2_7 x^2_{11} + \frac{1}{2}(x^2_{21})^2$
y^3_{73}	$x^2_6 x^2_{11} - \frac{1}{2}(x^2_{21})^2$
y^3_{74}	$x^2_8 x^2_{11}$
y^3_{75}	$x^2_9 x^2_{11} + \frac{1}{2}(x^2_{21})^2$
y^3_{76}	$(x^2_{11})^2 - (x^2_{21})^2$
y^3_{77}	$x^2_0 x^2_{12} + \frac{1}{2}(x^2_{21})^2$
y^3_{78}	$x^2_1 x^2_{12} + \frac{1}{2}(x^2_{21})^2$
y^3_{79}	$x^2_{10} x^2_{11} + \frac{1}{2}(x^2_{21})^2$
y^3_{80}	$x^2_2 x^2_{12}$
y^3_{81}	$x^2_3 x^2_{12} + \frac{1}{2}(x^2_{21})^2$
y^3_{82}	$x^2_4 x^2_{12} - \frac{1}{2}(x^2_{21})^2$

y^3_{83}	$x^2_5x^2_{12} + \frac{1}{2}(x^2_{21})^2$
y^3_{84}	$x^2_6x^2_{12}$
y^3_{85}	$x^2_7x^2_{12}$
y^3_{86}	$x^2_8x^2_{12}$
y^3_{87}	$x^2_9x^2_{12}$
y^3_{88}	$x^2_{10}x^2_{12} + \frac{1}{2}(x^2_{21})^2$
y^3_{89}	$x^2_{11}x^2_{12} - \frac{1}{2}(x^2_{21})^2$
y^3_{90}	$(x^2_{12})^2 - (x^2_{21})^2$
y^3_{91}	$x^2_0x^2_{13} + \frac{1}{2}(x^2_{21})^2$
y^3_{92}	$x^2_1x^2_{13} + \frac{1}{2}(x^2_{21})^2$
y^3_{93}	$x^2_2x^2_{13} + \frac{1}{2}(x^2_{21})^2$
y^3_{94}	$x^2_3x^2_{13}$
y^3_{95}	$x^2_4x^2_{13} - \frac{1}{2}(x^2_{21})^2$
y^3_{96}	$x^2_5x^2_{13}$
y^3_{97}	$x^2_9x^2_{13}$
y^3_{98}	$x^2_7x^2_{13}$
y^3_{99}	$x^2_8x^2_{13} - \frac{1}{2}(x^2_{21})^2$
y^3_{100}	$x^2_6x^2_{13} + \frac{1}{2}(x^2_{21})^2$
y^3_{101}	$x^2_{11}x^2_{13}$
y^3_{102}	$x^2_{10}x^2_{13}$
y^3_{103}	$x^2_{12}x^2_{13}$
y^3_{104}	$(x^2_{13})^2 - (x^2_{21})^2$
y^3_{105}	$x^2_1x^2_{14} - \frac{1}{2}(x^2_{21})^2$
y^3_{106}	$x^2_2x^2_{14}$
y^3_{107}	$x^2_0x^2_{14} - \frac{1}{2}(x^2_{21})^2$
y^3_{108}	$x^2_3x^2_{14} - \frac{1}{2}(x^2_{21})^2$
y^3_{109}	$x^2_5x^2_{14} - \frac{1}{2}(x^2_{21})^2$
y^3_{110}	$x^2_4x^2_{14} + \frac{1}{2}(x^2_{21})^2$
y^3_{111}	$x^2_6x^2_{14}$
y^3_{112}	$x^2_8x^2_{14} + \frac{1}{2}(x^2_{21})^2$
y^3_{113}	$x^2_9x^2_{14} - \frac{1}{2}(x^2_{21})^2$
y^3_{114}	$x^2_7x^2_{14}$
y^3_{115}	$x^2_{10}x^2_{14}$
y^3_{116}	$x^2_{11}x^2_{14} + \frac{1}{2}(x^2_{21})^2$
y^3_{117}	$x^2_{12}x^2_{14} + \frac{1}{2}(x^2_{21})^2$
y^3_{118}	$x^2_{13}x^2_{14} + \frac{1}{2}(x^2_{21})^2$
y^3_{119}	$(x^2_{14})^2 - (x^2_{21})^2$
y^3_{120}	$x^2_0x^2_{15} + 2(x^2_{21})^2$
y^3_{121}	$x^2_2x^2_{15} + (x^2_{21})^2$
y^3_{122}	$x^2_1x^2_{15} + \frac{1}{2}(x^2_{21})^2$
y^3_{123}	$x^2_3x^2_{15} + \frac{3}{2}(x^2_{21})^2$
y^3_{124}	$x^2_4x^2_{15} - \frac{1}{2}(x^2_{21})^2$
y^3_{125}	$x^2_5x^2_{15} - \frac{1}{2}(x^2_{21})^2$
y^3_{126}	$x^2_6x^2_{15} - \frac{7}{2}(x^2_{21})^2$
y^3_{127}	$x^2_7x^2_{15} + 3(x^2_{21})^2$
y^3_{128}	$x^2_8x^2_{15} + 2(x^2_{21})^2$

y^3_{129}	$x^2_9 x^2_{15} - \frac{1}{2}(x^2_{21})^2$
y^3_{130}	$x^2_{10} x^2_{15} + \frac{5}{2}(x^2_{21})^2$
y^3_{131}	$x^2_{11} x^2_{15} - (x^2_{21})^2$
y^3_{132}	$x^2_{12} x^2_{15} - \frac{1}{2}(x^2_{21})^2$
y^3_{133}	$x^2_{13} x^2_{15} - \frac{1}{2}(x^2_{21})^2$
y^3_{134}	$x^2_{14} x^2_{15} + \frac{1}{2}(x^2_{21})^2$
y^3_{135}	$x^2_0 x^2_{16} - \frac{5}{2}(x^2_{21})^2$
y^3_{136}	$x^2_1 x^2_{16} - \frac{1}{2}(x^2_{21})^2$
y^3_{137}	$(x^2_{15})^2 - 6(x^2_{21})^2$
y^3_{138}	$x^2_2 x^2_{16} - (x^2_{21})^2$
y^3_{139}	$x^2_3 x^2_{16} - \frac{3}{2}(x^2_{21})^2$
y^3_{140}	$x^2_4 x^2_{16} + \frac{1}{2}(x^2_{21})^2$
y^3_{141}	$x^2_5 x^2_{16} + \frac{1}{2}(x^2_{21})^2$
y^3_{142}	$x^2_6 x^2_{16} + 4(x^2_{21})^2$
y^3_{143}	$x^2_7 x^2_{16} - \frac{7}{2}(x^2_{21})^2$
y^3_{144}	$x^2_9 x^2_{16} + (x^2_{21})^2$
y^3_{145}	$x^2_8 x^2_{16} - \frac{5}{2}(x^2_{21})^2$
y^3_{146}	$x^2_{12} x^2_{16} + \frac{1}{2}(x^2_{21})^2$
y^3_{147}	$x^2_{10} x^2_{16} - 3(x^2_{21})^2$
y^3_{148}	$x^2_{13} x^2_{16} + \frac{1}{2}(x^2_{21})^2$
y^3_{149}	$x^2_{11} x^2_{16} + (x^2_{21})^2$
y^3_{150}	$x^2_0 x^2_{17}$
y^3_{151}	$x^2_{14} x^2_{16} - \frac{1}{2}(x^2_{21})^2$
y^3_{152}	$x^2_2 x^2_{17}$
y^3_{153}	$x^2_1 x^2_{17} + \frac{1}{2}(x^2_{21})^2$
y^3_{154}	$x^2_3 x^2_{17}$
y^3_{155}	$(x^2_{16})^2 - 8(x^2_{21})^2$
y^3_{156}	$x^2_{15} x^2_{16} + \frac{13}{2}(x^2_{21})^2$
y^3_{157}	$x^2_4 x^2_{17} - \frac{1}{2}(x^2_{21})^2$
y^3_{158}	$x^2_5 x^2_{17} + \frac{1}{2}(x^2_{21})^2$
y^3_{159}	$x^2_9 x^2_{17}$
y^3_{160}	$x^2_7 x^2_{17} - \frac{1}{2}(x^2_{21})^2$
y^3_{161}	$x^2_6 x^2_{17} + (x^2_{21})^2$
y^3_{162}	$x^2_8 x^2_{17} - \frac{1}{2}(x^2_{21})^2$
y^3_{163}	$x^2_{10} x^2_{17}$
y^3_{164}	$x^2_{11} x^2_{17}$
y^3_{165}	$x^2_{12} x^2_{17} - \frac{1}{2}(x^2_{21})^2$
y^3_{166}	$x^2_{13} x^2_{17}$
y^3_{167}	$x^2_{14} x^2_{17}$
y^3_{168}	$(x^2_{17})^2 - (x^2_{21})^2$
y^3_{169}	$x^2_{15} x^2_{17} + (x^2_{21})^2$
y^3_{170}	$x^2_{16} x^2_{17} - (x^2_{21})^2$
y^3_{171}	$x^2_0 x^2_{18} + \frac{1}{2}(x^2_{21})^2$
y^3_{172}	$x^2_2 x^2_{18} + \frac{1}{2}(x^2_{21})^2$
y^3_{173}	$x^2_1 x^2_{18} + \frac{1}{2}(x^2_{21})^2$
y^3_{174}	$x^2_3 x^2_{18} + \frac{1}{2}(x^2_{21})^2$

y^3_{175}	$x^2_5 x^2_{18}$
y^3_{176}	$x^2_4 x^2_{18} - \frac{1}{2}(x^2_{21})^2$
y^3_{177}	$x^2_9 x^2_{18}$
y^3_{178}	$x^2_6 x^2_{18} - \frac{1}{2}(x^2_{21})^2$
y^3_{179}	$x^2_7 x^2_{18} + \frac{1}{2}(x^2_{21})^2$
y^3_{180}	$x^2_8 x^2_{18} + \frac{1}{2}(x^2_{21})^2$
y^3_{181}	$x^2_{11} x^2_{18} - \frac{1}{2}(x^2_{21})^2$
y^3_{182}	$x^2_{10} x^2_{18} + \frac{1}{2}(x^2_{21})^2$
y^3_{183}	$x^2_{12} x^2_{18} - \frac{1}{2}(x^2_{21})^2$
y^3_{184}	$x^2_{13} x^2_{18} - \frac{1}{2}(x^2_{21})^2$
y^3_{185}	$x^2_{14} x^2_{18} + \frac{1}{2}(x^2_{21})^2$
y^3_{186}	$x^2_{15} x^2_{18} - \frac{3}{2}(x^2_{21})^2$
y^3_{187}	$x^2_{17} x^2_{18}$
y^3_{188}	$x^2_{16} x^2_{18} + \frac{3}{2}(x^2_{21})^2$
y^3_{189}	$(x^2_{18})^2 - (x^2_{21})^2$
y^3_{190}	$x^2_0 x^2_{19} - \frac{3}{2}(x^2_{21})^2$
y^3_{191}	$x^2_1 x^2_{19} - \frac{1}{2}(x^2_{21})^2$
y^3_{192}	$x^2_2 x^2_{19} - \frac{1}{2}(x^2_{21})^2$
y^3_{193}	$x^2_5 x^2_{19}$
y^3_{194}	$x^2_3 x^2_{19} - (x^2_{21})^2$
y^3_{195}	$x^2_4 x^2_{19} + \frac{1}{2}(x^2_{21})^2$
y^3_{196}	$x^2_9 x^2_{19} + \frac{1}{2}(x^2_{21})^2$
y^3_{197}	$x^2_7 x^2_{19} - 2(x^2_{21})^2$
y^3_{198}	$x^2_8 x^2_{19} - \frac{3}{2}(x^2_{21})^2$
y^3_{199}	$x^2_6 x^2_{19} + 2(x^2_{21})^2$
y^3_{200}	$x^2_{10} x^2_{19} - 2(x^2_{21})^2$
y^3_{201}	$x^2_{11} x^2_{19} + (x^2_{21})^2$
y^3_{202}	$x^2_{12} x^2_{19} + \frac{1}{2}(x^2_{21})^2$
y^3_{203}	$x^2_{13} x^2_{19}$
y^3_{204}	$x^2_{14} x^2_{19}$
y^3_{205}	$x^2_{17} x^2_{19}$
y^3_{206}	$x^2_2 x^2_{20}$
y^3_{207}	$x^2_{15} x^2_{19} + \frac{7}{2}(x^2_{21})^2$
y^3_{208}	$x^2_{16} x^2_{19} - 4(x^2_{21})^2$
y^3_{209}	$x^2_{18} x^2_{19} + (x^2_{21})^2$
y^3_{210}	$(x^2_{19})^2 - 3(x^2_{21})^2$
y^3_{211}	$x^2_0 x^2_{20} + \frac{1}{2}(x^2_{21})^2$
y^3_{212}	$x^2_1 x^2_{20} + \frac{1}{2}(x^2_{21})^2$
y^3_{213}	$x^2_3 x^2_{20} + \frac{1}{2}(x^2_{21})^2$
y^3_{214}	$x^2_4 x^2_{20}$
y^3_{215}	$x^2_5 x^2_{20} + \frac{1}{2}(x^2_{21})^2$
y^3_{216}	$x^2_7 x^2_{20}$
y^3_{217}	$x^2_8 x^2_{20}$
y^3_{218}	$x^2_9 x^2_{20}$
y^3_{219}	$x^2_6 x^2_{20} - \frac{1}{2}(x^2_{21})^2$
y^3_{220}	$x^2_{10} x^2_{20}$

y^3_{221}	$x^2_{11}x^2_{20} - \frac{1}{2}(x^2_{21})^2$
y^3_{222}	$x^2_{13}x^2_{20}$
y^3_{223}	$x^2_{14}x^2_{20} + \frac{1}{2}(x^2_{21})^2$
y^3_{224}	$x^2_{12}x^2_{20} - \frac{1}{2}(x^2_{21})^2$
y^3_{225}	$x^2_{18}x^2_{20}$
y^3_{226}	$x^2_{17}x^2_{20} - \frac{1}{2}(x^2_{21})^2$
y^3_{227}	$x^2_{15}x^2_{20} - \frac{1}{2}(x^2_{21})^2$
y^3_{228}	$x^2_{16}x^2_{20} + (x^2_{21})^2$
y^3_{229}	$(x^2_{20})^2 - (x^2_{21})^2$
y^3_{230}	$x^2_{1}x^2_{21} + \frac{1}{2}(x^2_{21})^2$
y^3_{231}	$x^2_{0}x^2_{21}$
y^3_{232}	$x^2_{19}x^2_{20} + \frac{1}{2}(x^2_{21})^2$
y^3_{233}	$x^2_{2}x^2_{21}$
y^3_{234}	$x^2_{3}x^2_{21}$
y^3_{235}	$x^2_{4}x^2_{21} - \frac{1}{2}(x^2_{21})^2$
y^3_{236}	$x^2_{5}x^2_{21} + \frac{1}{2}(x^2_{21})^2$
y^3_{237}	$x^2_{7}x^2_{21} - \frac{1}{2}(x^2_{21})^2$
y^3_{238}	$x^2_{8}x^2_{21} - (x^2_{21})^2$
y^3_{239}	$x^2_{6}x^2_{21} + (x^2_{21})^2$
y^3_{240}	$x^2_{9}x^2_{21} + \frac{1}{2}(x^2_{21})^2$
y^3_{241}	$x^2_{11}x^2_{21}$
y^3_{242}	$x^2_{10}x^2_{21} - \frac{1}{2}(x^2_{21})^2$
y^3_{243}	$x^2_{12}x^2_{21} - \frac{1}{2}(x^2_{21})^2$
y^3_{244}	$x^2_{13}x^2_{21} - \frac{1}{2}(x^2_{21})^2$
y^3_{245}	$x^2_{14}x^2_{21} + \frac{1}{2}(x^2_{21})^2$
y^3_{246}	$x^2_{18}x^2_{21}$
y^3_{247}	$x^2_{17}x^2_{21} - \frac{1}{2}(x^2_{21})^2$
y^3_{248}	$x^2_{15}x^2_{21} + (x^2_{21})^2$
y^3_{249}	$x^2_{16}x^2_{21} - (x^2_{21})^2$
y^3_{250}	$x^2_{20}x^2_{21} - \frac{1}{2}(x^2_{21})^2$
y^3_{251}	$x^2_{19}x^2_{21} - \frac{1}{2}(x^2_{21})^2$

Table I.1: The generators of degree 3 of the model and their differentials.

Appendix II: Example of an element $p \in A^3_{PL}(K3)$

To illustrate how complicated the output is, we show an example of two elements of $A^3_{PL}(K3)$ and $A^4_{PL}(K3)$.

During the computation of the minimal model of the $K3$ surface, at the injectivity step (see section 3.1), the method finds an element of degree 4 (in particular, the element $x_{19}^2 x_{21}^2 - \frac{1}{2}(x_{21}^2)^2 \in M^4$) that should kill in order to make the quasi-isomorphism φ^* injective at degree 4. The image of this element by the morphism φ is represented in the following table:

Simplex	$\boldsymbol{p \in (A_{PL})^4_n}$
(2, 3, 4, 5, 9)	$4y_1y_2y_3y_4$
(2, 3, 4, 9, 13)	$-4y_1y_2y_3y_4$
(2, 3, 5, 6, 13)	$-4y_1y_2y_3y_4$
(2, 3, 9, 13, 14)	$4y_1y_2y_3y_4$
(2, 3, 10, 13, 14)	$-4y_1y_2y_3y_4$
(2, 4, 5, 9, 12)	$-8y_1y_2y_3y_4$
(2, 4, 7, 12, 14)	$4y_1y_2y_3y_4$
(2, 4, 8, 9, 12)	$-4y_1y_2y_3y_4$
(2, 4, 11, 12, 14)	$4y_1y_2y_3y_4$
(2, 6, 9, 11, 12)	$-4y_1y_2y_3y_4$
(3, 4, 5, 7, 13)	$4y_1y_2y_3y_4$
(3, 4, 5, 7, 15)	$-4y_1y_2y_3y_4$
(3, 4, 5, 8, 15)	$-8y_1y_2y_3y_4$
(3, 4, 6, 7, 12)	$-4y_1y_2y_3y_4$
(3, 4, 6, 7, 16)	$8y_1y_2y_3y_4$
(3, 4, 7, 12, 14)	$4y_1y_2y_3y_4$
(3, 4, 7, 14, 15)	$-4y_1y_2y_3y_4$
(3, 4, 8, 14, 15)	$-4y_1y_2y_3y_4$
(3, 5, 6, 8, 15)	$-4y_1y_2y_3y_4$
(3, 5, 6, 13, 15)	$-4y_1y_2y_3y_4$
(3, 5, 10, 13, 15)	$4y_1y_2y_3y_4$
(4, 5, 7, 9, 12)	$4y_1y_2y_3y_4$
(4, 5, 7, 9, 15)	$4y_1y_2y_3y_4$
(4, 5, 8, 9, 11)	$-4y_1y_2y_3y_4$
(4, 5, 8, 9, 15)	$8y_1y_2y_3y_4$
(4, 7, 8, 9, 12)	$-4y_1y_2y_3y_4$
(5, 6, 11, 13, 15)	$4y_1y_2y_3y_4$
(5, 8, 9, 13, 15)	$-4y_1y_2y_3y_4$
(6, 7, 9, 10, 13)	$-4y_1y_2y_3y_4$
(6, 7, 10, 13, 15)	$-4y_1y_2y_3y_4$

Table II.1: The element of $A^4_{PL}(K3)$ corresponding to $x_{19}^2 x_{21}^2 - \frac{1}{2}(x_{21}^2)^2$.

In order to make the induced morphims in cohomology injective, the algorithm adds the generator $y^3_{251} \in M^3$ whose differential is precisely that element:

$$\begin{array}{rccc} d^3: & M^3 & \longrightarrow & M^4 \\ & y^3_{251} & \to & x^2_{19}x^2_{21} - \frac{1}{2}(x^2_{21})^2. \end{array}$$

The last step is to find an element $p \in A^3_{PL}(K3)$ whose differential is the element $\varphi(x^2_{19}x^2_{21} - \frac{1}{2}(x^2_{21})^2) \in A^4_{PL}(K3)$. This element is presented in the following table:

Simplex	$\boldsymbol{p \in (A_{PL})^3_n}$
(1, 2, 3, 8, 12)	$2y_1y_3y_4 - 5y_2y_3y_4$
(1, 2, 4, 7, 11)	$y_1y_3y_4$
(1, 2, 5, 7, 13)	$-2y_1y_2y_3 + 2y_1y_3y_4$
(1, 2, 5, 7, 15)	$-2y_1y_2y_3 - 2y_1y_2y_4$
(1, 2, 5, 8, 10)	$-2y_1y_2y_3 + 2y_1y_3y_4$
(1, 2, 5, 8, 14)	$-2y_1y_2y_3 - 2y_1y_2y_4$
(1, 2, 5, 14, 15)	$-2y_1y_2y_3 - 2y_1y_2y_4$
(1, 2, 6, 7, 9)	$-2y_1y_2y_3 + y_1y_3y_4$
(1, 2, 6, 7, 13)	$-2y_1y_2y_3 + 2y_1y_3y_4$
(1, 2, 7, 9, 11)	$y_1y_2y_3 + y_1y_2y_4$
(1, 2, 8, 10, 12)	$2y_1y_2y_3 + 2y_1y_2y_4$
(1, 3, 4, 6, 10)	$y_1y_2y_3 - 4y_1y_3y_4$
(1, 3, 4, 6, 14)	$y_1y_2y_3$
(1, 3, 5, 6, 9)	$5y_1y_2y_3 + 5y_1y_2y_4$
(1, 3, 5, 6, 11)	$5y_1y_2y_3 - 4y_1y_3y_4$
(1, 3, 5, 9, 12)	$5y_1y_2y_3 - 5y_1y_3y_4$
(1, 3, 6, 10, 11)	$-4y_1y_2y_3 - 4y_1y_2y_4$
(1, 3, 8, 9, 12)	$-5y_1y_2y_4 - 5y_1y_3y_4$
(2, 3, 4, 5, 9)	$-t_4y_1y_2y_3 + t_3y_1y_2y_4 - t_2y_1y_3y_4 + t_1y_2y_3y_4 + 3y_1y_2y_3 + 4y_1y_2y_4 - 2y_2y_3y_4$
(2, 3, 4, 5, 13)	$3y_1y_2y_3$
(2, 3, 4, 9, 13)	$t_4y_1y_2y_3 - t_3y_1y_2y_4 + t_2y_1y_3y_4 - t_1y_2y_3y_4 + 4y_1y_2y_3 - 5y_1y_3y_4$
(2, 3, 5, 6, 13)	$t_4y_1y_2y_3 - t_3y_1y_2y_4 + t_2y_1y_3y_4 - t_1y_2y_3y_4$
(2, 3, 7, 8, 12)	$-7y_1y_2y_4 - 7y_1y_3y_4$
(2, 3, 7, 12, 14)	$-7y_1y_2y_3 + 7y_1y_3y_4$
(2, 3, 9, 13, 14)	$-t_4y_1y_2y_3 + t_3y_1y_2y_4 - t_2y_1y_3y_4 + t_1y_2y_3y_4 - 5y_1y_2y_3 + 6y_1y_3y_4$
(2, 3, 10, 12, 14)	$7y_1y_2y_4 + 7y_1y_3y_4$
(2, 3, 10, 13, 14)	$t_4y_1y_2y_3 - t_3y_1y_2y_4 + t_2y_1y_3y_4 - t_1y_2y_3y_4 + 7y_1y_2y_4 + 6y_1y_3y_4$
(2, 4, 5, 6, 10)	$-y_1y_3y_4 - y_2y_3y_4$
(2, 4, 5, 9, 12)	$2t_4y_1y_2y_3 - 2t_3y_1y_2y_4 + 2t_2y_1y_3y_4 - 2t_1y_2y_3y_4 - 2y_1y_2y_3$
(2, 4, 6, 10, 11)	$-y_1y_2y_3 - y_1y_2y_4$
(2, 4, 6, 11, 12)	$-y_1y_2y_3 + y_1y_3y_4$
(2, 4, 7, 8, 10)	$-y_1y_2y_4 - y_1y_3y_4$
(2, 4, 7, 8, 12)	$y_1y_2y_4$
(2, 4, 7, 10, 11)	$-y_1y_2y_3 - y_1y_2y_4$
(2, 4, 7, 12, 14)	$-t_4y_1y_2y_3 + t_3y_1y_2y_4 - t_2y_1y_3y_4 + t_1y_2y_3y_4 + y_1y_2y_3$
(2, 4, 8, 9, 10)	$-y_1y_2y_3 - y_1y_2y_4$
(2, 4, 8, 9, 12)	$t_4y_1y_2y_3 - t_3y_1y_2y_4 + t_2y_1y_3y_4 - t_1y_2y_3y_4 - y_1y_2y_3$
(2, 4, 11, 12, 14)	$-t_4y_1y_2y_3 + t_3y_1y_2y_4 - t_2y_1y_3y_4 + t_1y_2y_3y_4 + y_1y_2y_3$

(2, 5, 6, 10, 11)	$-y_1y_2y_3 - y_1y_2y_4$
(2, 5, 6, 11, 13)	$-y_1y_2y_3$
(2, 6, 7, 9, 11)	$y_1y_2y_3 - y_1y_3y_4$
(2, 6, 9, 11, 12)	$t_4y_1y_2y_3 - t_3y_1y_2y_4 + t_2y_1y_3y_4 - t_1y_2y_3y_4 - y_1y_2y_3$
(3, 4, 5, 7, 13)	$-t_4y_1y_2y_3 + t_3y_1y_2y_4 - t_2y_1y_3y_4 + t_1y_2y_3y_4 - y_1y_2y_3 - 3y_1y_2y_4 + y_2y_3y_4$
(3, 4, 5, 7, 15)	$t_4y_1y_2y_3 - t_3y_1y_2y_4 + t_2y_1y_3y_4 - t_1y_2y_3y_4 - y_1y_2y_3$
(3, 4, 5, 8, 11)	$-2y_1y_2y_3$
(3, 4, 5, 8, 15)	$2t_4y_1y_2y_3 - 2t_3y_1y_2y_4 + 2t_2y_1y_3y_4 - 2t_1y_2y_3y_4 - 2y_1y_2y_3$
(3, 4, 6, 7, 12)	$t_4y_1y_2y_3 - t_3y_1y_2y_4 + t_2y_1y_3y_4 - t_1y_2y_3y_4 + 2y_1y_2y_3 + 3y_1y_2y_4$
(3, 4, 6, 7, 16)	$-2t_4y_1y_2y_3 + 2t_3y_1y_2y_4 - 2t_2y_1y_3y_4 + 2t_1y_2y_3y_4 + 2y_1y_2y_3$
(3, 4, 6, 8, 14)	$-y_1y_2y_4 - y_1y_3y_4$
(3, 4, 6, 10, 12)	$3y_1y_2y_3 + 3y_1y_2y_4$
(3, 4, 7, 12, 14)	$-t_4y_1y_2y_3 + t_3y_1y_2y_4 - t_2y_1y_3y_4 + t_1y_2y_3y_4 - y_1y_2y_4$
(3, 4, 7, 14, 15)	$t_4y_1y_2y_3 - t_3y_1y_2y_4 + t_2y_1y_3y_4 - t_1y_2y_3y_4 - y_1y_2y_3$
(3, 4, 8, 14, 15)	$t_4y_1y_2y_3 - t_3y_1y_2y_4 + t_2y_1y_3y_4 - t_1y_2y_3y_4 - y_1y_2y_3$
(3, 5, 6, 8, 11)	$-y_1y_2y_3 - y_1y_2y_4$
(3, 5, 6, 8, 15)	$t_4y_1y_2y_3 - t_3y_1y_2y_4 + t_2y_1y_3y_4 - t_1y_2y_3y_4 - y_1y_2y_3$
(3, 5, 6, 13, 15)	$t_4y_1y_2y_3 - t_3y_1y_2y_4 + t_2y_1y_3y_4 - t_1y_2y_3y_4 - y_1y_2y_3$
(3, 5, 7, 13, 14)	$y_1y_2y_3 - y_1y_3y_4$
(3, 5, 10, 13, 14)	$y_1y_2y_3 - y_1y_3y_4$
(3, 5, 10, 13, 15)	$-t_4y_1y_2y_3 + t_3y_1y_2y_4 - t_2y_1y_3y_4 + t_1y_2y_3y_4 + y_1y_2y_3$
(4, 5, 7, 9, 12)	$-t_4y_1y_2y_3 + t_3y_1y_2y_4 - t_2y_1y_3y_4 + t_1y_2y_3y_4 + y_1y_2y_3$
(4, 5, 7, 9, 15)	$-t_4y_1y_2y_3 + t_3y_1y_2y_4 - t_2y_1y_3y_4 + t_1y_2y_3y_4 + y_1y_2y_3$
(4, 5, 8, 9, 11)	$t_4y_1y_2y_3 - t_3y_1y_2y_4 + t_2y_1y_3y_4 - t_1y_2y_3y_4 + y_1y_2y_3 + 2y_1y_2y_4$
(4, 5, 8, 9, 15)	$-2t_4y_1y_2y_3 + 2t_3y_1y_2y_4 - 2t_2y_1y_3y_4 + 2t_1y_2y_3y_4 + y_1y_2y_3$
(4, 7, 8, 9, 12)	$t_4y_1y_2y_3 - t_3y_1y_2y_4 + t_2y_1y_3y_4 - t_1y_2y_3y_4 + y_1y_2y_4$
(5, 6, 11, 13, 15)	$-t_4y_1y_2y_3 + t_3y_1y_2y_4 - t_2y_1y_3y_4 + t_1y_2y_3y_4 + y_1y_2y_3$
(5, 8, 9, 13, 15)	$t_4y_1y_2y_3 - t_3y_1y_2y_4 + t_2y_1y_3y_4 - t_1y_2y_3y_4 + y_1y_2y_4$
(6, 7, 9, 10, 13)	$t_4y_1y_2y_3 - t_3y_1y_2y_4 + t_2y_1y_3y_4 - t_1y_2y_3y_4 - y_1y_3y_4$
(6, 7, 10, 13, 15)	$t_4y_1y_2y_3 - t_3y_1y_2y_4 + t_2y_1y_3y_4 - t_1y_2y_3y_4 - y_1y_2y_3$
(1, 2, 5, 7)	$-2y_1y_2y_3$
(1, 2, 5, 8)	$-2y_1y_2y_3$
(1, 2, 5, 14)	$-2y_1y_2y_3$
(1, 2, 5, 15)	$-2y_1y_2y_3$
(1, 2, 6, 7)	$-2y_1y_2y_3$
(1, 2, 7, 9)	$y_1y_2y_3$
(1, 2, 7, 11)	$y_1y_2y_3$
(1, 2, 7, 13)	$2y_1y_2y_3$
(1, 2, 8, 10)	$2y_1y_2y_3$
(1, 2, 8, 12)	$2y_1y_2y_3$
(1, 3, 4, 6)	$y_1y_2y_3$
(1, 3, 5, 6)	$5y_1y_2y_3$
(1, 3, 5, 9)	$5y_1y_2y_3$
(1, 3, 6, 10)	$-4y_1y_2y_3$
(1, 3, 6, 11)	$-4y_1y_2y_3$
(1, 3, 8, 12)	$-5y_1y_2y_3$

(1, 3, 9, 12)	$-5y_1y_2y_3$
(2, 3, 4, 5)	$3y_1y_2y_3$
(2, 3, 4, 9)	$4y_1y_2y_3$
(2, 3, 7, 12)	$-7y_1y_2y_3$
(2, 3, 8, 12)	$-7y_1y_2y_3$
(2, 3, 9, 13)	$-5y_1y_2y_3$
(2, 3, 10, 14)	$7y_1y_2y_3$
(2, 3, 12, 14)	$7y_1y_2y_3$
(2, 3, 13, 14)	$6y_1y_2y_3$
(2, 4, 5, 9)	$-2y_1y_2y_3$
(2, 4, 6, 10)	$-y_1y_2y_3$
(2, 4, 6, 11)	$-y_1y_2y_3$
(2, 4, 7, 10)	$-y_1y_2y_3$
(2, 4, 7, 11)	$-y_1y_2y_3$
(2, 4, 7, 12)	$y_1y_2y_3$
(2, 4, 8, 9)	$-y_1y_2y_3$
(2, 4, 8, 10)	$-y_1y_2y_3$
(2, 4, 11, 12)	$y_1y_2y_3$
(2, 5, 6, 10)	$-y_1y_2y_3$
(2, 5, 6, 11)	$-y_1y_2y_3$
(2, 6, 7, 9)	$y_1y_2y_3$
(2, 6, 9, 11)	$-y_1y_2y_3$
(3, 4, 5, 7)	$-y_1y_2y_3$
(3, 4, 5, 8)	$-2y_1y_2y_3$
(3, 4, 5, 13)	$-3y_1y_2y_3$
(3, 4, 6, 7)	$2y_1y_2y_3$
(3, 4, 6, 10)	$3y_1y_2y_3$
(3, 4, 6, 12)	$3y_1y_2y_3$
(3, 4, 6, 14)	$-y_1y_2y_3$
(3, 4, 7, 14)	$-y_1y_2y_3$
(3, 4, 8, 14)	$-y_1y_2y_3$
(3, 5, 6, 8)	$-y_1y_2y_3$
(3, 5, 6, 11)	$-y_1y_2y_3$
(3, 5, 6, 13)	$-y_1y_2y_3$
(3, 5, 7, 13)	$y_1y_2y_3$
(3, 5, 10, 13)	$y_1y_2y_3$
(3, 5, 13, 14)	$-y_1y_2y_3$
(4, 5, 7, 9)	$y_1y_2y_3$
(4, 5, 8, 9)	$y_1y_2y_3$
(4, 5, 8, 11)	$2y_1y_2y_3$
(4, 7, 8, 12)	$y_1y_2y_3$
(5, 6, 11, 13)	$y_1y_2y_3$
(5, 8, 9, 15)	$y_1y_2y_3$
(6, 7, 10, 13)	$-y_1y_2y_3$

Table II.2: The element of $A^3_{PL}(K3)$ corresponding to generator y^3_{251}.