



The Full Electroweak Interaction: An Autonomous Account

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In memory of our dear friend and colleague Florian Scheck.

Abstract. The precise renormalizable interactions in the bosonic sector of electroweak theory are intrinsically determined in the autonomous approach to perturbation theory. This proceeds directly on the Hilbert–Fock space built on the Wigner unirreps of the physical particles, with their given masses: those of three massive vector bosons, a photon, and a massive scalar (the “higgs”). Neither “gauge choices” nor an unobservable “mechanism of spontaneous symmetry breaking” is invoked. Instead, to proceed on Hilbert space requires using string-localized fields to describe the vector bosons. In such a framework, the condition of string independence of the \mathbb{S} -matrix yields consistency constraints on the coupling coefficients, the essentially unique outcome being the experimentally known one. The analysis can be largely carried out for other configurations of massive and massless vector bosons, paving the way towards consideration of consistent mass patterns beyond those of the electroweak theory.

It is a dereliction of duty for philosophers to repeat the physicists’ slogans rather than asking what is the content of the reality that lies behind the veil of gauge.

- John Earman [1]

The concept of symmetry breaking has been borrowed by the elementary particle physicists, but their use of the term is strictly an analogy, whether a deep or a specious one remaining to be understood.

- Philip W. Anderson [2]

1. Introduction

The theory and practice of the autonomous formulation of quantum field theory [3–21], also called “string-localized quantum field theory”, or sQFT for short, were born from dissatisfaction, both with the heuristics permeating the generally used gauge formalism and with the limitations of algebraic field theory [22]. Instead of classical Lagrangians, its building blocks are the free—or asymptotic [23]—quantum fields themselves on Fock–Hilbert space, the underlying one-particle spaces being the irreducible unitary representation spaces of the Poincaré group as classified in terms of mass and spin or helicity by Wigner [24]. Within extant perturbative approaches to the phenomenology of particle theory, this undertaking reasonably claims to be most rigorous, fully enjoying the canonical triad of fundamental quantum requirements: *positivity* (on which every probability interpretation hinges), Poincaré *covariance*, and *locality*, ensuring Einstein causality.

The word “autonomous” warrants an explanation. The free fields of the theory are defined on their physical Hilbert spaces directly, without “canonical quantization” based on classical free Lagrangians, and without the forced detours through indefinite metrics and BRST techniques. A consequence is that vector fields associated with vector bosons necessarily have a weaker localization than usual: they are localized along some auxiliary “string” (whence the name “sQFT”). One must then impose the *principle of string independence* (SI), which posits that the \mathbb{S} -matrix must not depend on the auxiliary string variables. This is necessary and sufficient to keep consistency with the aforementioned triad of quantum requirements. The principle turns out to be extremely restrictive on the allowed interactions, once the field content is specified; essentially it narrows down the set of admissible interactions to precisely those found in Nature.

In its practical implementation, there arise several “obstructions against string independence” at each perturbative order, see Sect. 2.4. The need to cancel all those obstructions enforces a recursive system of conditions on the interaction coefficients. Not least, it shows that couplings to a higgs particle are indispensable in theories with massive vector bosons [29]. It turns out that SI holds great power both as a heuristic device and as a justification tool, dictating *symmetry from interaction*,¹ down to almost every nut and bolt.

The sQFT method to induce higher interactions by imposing the absence of obstructions is in fact an offspring of an analogous program (called “perturbative” or “causal gauge invariance” [26–29], reassembled in the book [30]). That program arrives at very similar results by imposing BRST invariance at all orders. It therefore does not *start* with the fundamental principle of Hilbert space from the outset, but imposes the possibility to *recover* a Hilbert space as its driving mechanism, where sQFT instead imposes string-independence.

¹Thereby reversing Yang’s *dictum* restated in the famous terminological discussion on gauge models between Dirac, Ferrara, Kleinert, Martin, Wigner, Yang himself and Zichichi [25].

In summary, in the autonomous approach the “gauge principle” is replaced by fundamental quantum principles. This reinforces an early objection to regarding gauge invariance as a principle [31].

The heart of the Standard Model (SM), that is, the fermionic sector of the electroweak theory in its coupling to the boson sector, was already investigated in [11] by two of us, together with Jens Mund, on the basis of the sQFT formulation. There we thoroughly showed why and how *chirality*² is an indispensable trait of flavourdynamics. So-called Yukawa couplings arise by way of consistency, and not “in order to give the leptons a mass”.

The purpose of the present work is to show that sQFT leads to an account of the whole electroweak theory from just the knowledge of an (allowed) particle spectrum of specified masses. One recovers precisely the phenomenological couplings of flavourdynamics in the SM, with massive bosons mediating the weak interactions, and the $u(2)$ structure constants—as for instance in [32, 33]. (One cannot say that we recover the usual *formulation* of the SM, since our mathematical description of the boson fields is at variance with gauge theory, and our rule set does not care for Lagrangians. But the coincidence of the couplings will be evident.)

In paper [11], the higgs of the SM was introduced as a partner for the photon,³ similar to the Stückelberg fields in the Proca-like description of the massive intermediate vector bosons, or the “escort fields” of sQFT theory itself—to be introduced below in Eq. (2.6). Such a partner turned out to be extremely convenient, since the proof of chirality required its presence. The main goal of the present work is to complete the tasks in [11] and the results on the Abelian higgs model in [18], by unveiling from first principles: (a) how (at least) one quantum scalar particle is necessarily part of the SM, and (b) what the shape of its self-couplings *must* be, without recourse to alleged, unobservable [35] “spontaneous symmetry breakings” and without pretending that the higgs is “the giver of mass”.⁴ Negative-norm states, ghosts, anti-ghosts, are banished as well.

The fermionic sector and its relation with the bosonic sector having been dealt with in [11], it remains to analyse the purely bosonic sector in the present paper. The main task is the exact determination of all bosonic interactions by consistency at second order, whereas the higgs self-couplings remain undetermined. The third-order consistency argument that fixes those self-couplings will be essentially the same as in the Abelian higgs model [18]. One need only make sure that the nonabelian self-interactions do not interfere with the pertinent conditions.

Our analysis is designed to reach well beyond electroweak theory: we consider here theories with given numbers of massive and massless vector bosons—restricted to only one higgs particle. (The generalization to more than one

²Of the *interaction* as opposed to some “intrinsic nature” of its carriers.

³Following Okun [34], and for obvious grammatical reasons, henceforth we refer to a (physical) Higgs boson as a higgs, with a lower-case h.

⁴SSB is not a physical process. Suffice it to say that the enormous latent heat that would have been released in the early universe is grossly incompatible with observations [36, Sect. 5.C].

higgs is not difficult at second order, compare [29].) Again, the only input is the masses; all coupling coefficients are determined by string independence of the \mathbb{S} -matrix at first, second, and third orders in perturbation theory, compatible with power-counting renormalizability. We are able to derive all conditions as relations between the masses, the Yang–Mills-like structure constants of a reductive Lie algebra, and the higgs couplings and self-couplings. We do not attempt, however, a general solution of these equations, characterizing all possible mass and symmetry patterns. In the special case of the electroweak theory, we give a complete analysis: string independence recovers the empirically known coefficients, or those otherwise predicted by the GWS model.

Plan of the paper. After some technical preparations and a short introduction to string-localized quantum fields describing vector bosons, we solve the constraints imposed by string independence at first order in the interactions (Sect. 2). We then give an outline of “obstruction theory”, which is the utensil to determine higher-order interactions in sQFT (Sect. 3). We continue in Sect. 4 with the concrete case of the *electroweak theory*, whose particle content determines the Lie algebra $\mathfrak{u}(2)$. With the comparison and matching of the sQFT results with the empirical electroweak interactions, otherwise claimed to be consequences of gauge invariance, the main goal of the paper is achieved in Sect. 4.

Some necessary proofs are given in Sect. 5. The main *structural* result at second order is found in Sect. 5.1: the mass-independent part of the self-interactions of massive and massless vector bosons entails the structure constants of a reductive Lie algebra; and an induced interaction—similar to the quartic terms in gauge theory treatments—is predicted. Some more conditions on the coupling coefficients are then identified, and more induced interactions are found.

Bringing from Sect. 5 a few constraints from second-order string independence, we fix all bosonic couplings except for the higgs self-interactions, which require a third-order result. To complete the analysis, a string-independent quartic self-coupling of the higgs must be admitted, necessary to cancel an obstruction present at this order (Sect. 6). Remarkably, the higgs self-couplings turn out to be “universal”: they do not depend on the particulars (masses and Lie algebra) of the models. Also, by fixing parameters, string independence at third order shows other putative second-order induced interactions to be absent.

Technical appendices and a brief discussion of models with a more general particle content are given at the end.

2. Intermediate Vector Boson Theory

2.1. Proca and Maxwell Field Tensors

Let us start by considering the field strengths $G^{\mu\nu}(x)$ for a massive boson of spin 1, and $F^{\mu\nu}(x)$ for a massless boson of helicity one (photon). Both are operator-valued distributions on Hilbert–Fock spaces over the corresponding

Wigner’s unitary irreducible representations of the restricted Poincaré group. Their pertinent time-ordered two-point functions differ only by the mass parameter:

$$\begin{aligned}\langle\langle T G^{\alpha\mu}(x) G_{\beta\rho}(x') \rangle\rangle &= i(\delta_{\beta}^{\alpha} \partial^{\mu} \partial_{\rho} - \delta_{\rho}^{\alpha} \partial^{\mu} \partial_{\beta} - \delta_{\beta}^{\mu} \partial^{\alpha} \partial_{\rho} + \delta_{\rho}^{\mu} \partial^{\alpha} \partial_{\beta}) D_m^F(x - x'), \\ \langle\langle T F^{\alpha\mu}(x) F_{\beta\rho}(x') \rangle\rangle &= i(\delta_{\beta}^{\alpha} \partial^{\mu} \partial_{\rho} - \delta_{\rho}^{\alpha} \partial^{\mu} \partial_{\beta} - \delta_{\beta}^{\mu} \partial^{\alpha} \partial_{\rho} + \delta_{\rho}^{\mu} \partial^{\alpha} \partial_{\beta}) \Delta^F(x - x'),\end{aligned}\tag{2.1}$$

where D_m^F and $\Delta^F \equiv D_0^F$ are, respectively, the standard Feynman propagators for massive and massless scalar particles:

$$D_m^F(x) := \int \frac{d^4 p}{(2\pi)^4} \frac{e^{-i(px)}}{p^2 - m^2 + i0}, \quad \text{so that} \quad (\square + m^2)D_m^F(x) = -\delta(x)\tag{2.2}$$

The corresponding *potential fields* $B(x)$ and $A(x)$ such that $G^{\alpha\mu} = \partial^{\alpha} B^{\mu} - \partial^{\mu} B^{\alpha}$, as well as $F^{\alpha\mu} = \partial^{\alpha} A^{\mu} - \partial^{\mu} A^{\alpha}$, are, however, troublesome—in wildly different ways. On the one hand, it is well known that massive bosons of any spin can be described by potential fields enjoying: (i) positivity, i.e. they are operator-valued distributions on the above indicated Hilbert space; (ii) covariance under the Wigner representation on that space; and (iii) causal localization. A paradigmatic example is provided by the above massive (Proca) particles of spin 1. The trouble is that the high-energy behaviour of such potentials grows steadily worse with spin. Already for B it is rather poor, *no better* than that of G :

$$\langle\langle T B^{\alpha}(x) B_{\beta}(x') \rangle\rangle = -(\delta_{\beta}^{\alpha} + m^{-2} \partial^{\alpha} \partial_{\beta}) D_m^F(x - x'),\tag{2.3}$$

where the derivatives spoil renormalizability in the UV.

Massless bosons of helicity $|h| \geq 1$, on the other hand, have free-field descriptions enjoying those desirable properties, too: think of either the electromagnetic field $F^{\alpha\mu}$ or the linear Riemann–Christoffel field $R^{\alpha\mu\beta\nu}$ for gravitons [19]. Nevertheless, the corresponding *potential fields* do not. The limit as $m \downarrow 0$ of the tensor field $G^{\alpha\mu}(x)$ for the Proca particle *is* of the Faraday–Maxwell field type; both are positive, covariant and local. However, whereas the corresponding field potential $B^{\mu}(x)$ shares these properties, it clearly possesses no limit as $m \downarrow 0$ —and the ordinary electrodynamic potential $A^{\mu}(x)$ is neither positive, nor covariant, nor local. Since violation of positivity conflicts with the probabilistic interpretation of quantum theory, to salvage positivity *for observables* on use of $A^{\mu}(x)$, one is apparently forced to introduce indefinite metrics, ghost fields, and the like.

2.2. Two Birds with One Stone

The sQFT framework addresses those problems of received QFT—and quite a few others. The key point is the definition of *new field potentials* with desirable properties. These potentials depend on spatial directions e (the “strings”) in Minkowski space, both for massive carriers of interaction and for massless ones—like photons, gluons or gravitons. Their definition is identical in the

massive ($s = 1$) and the massless ($|h| = 1$) cases:

$$A^\mu(x, e) := \int_0^\infty ds G^{\mu\nu}(x + se) e_\nu \quad \text{or} \quad A^\mu(x, e) := \int_0^\infty ds F^{\mu\nu}(x + se) e_\nu, \tag{2.4}$$

with $e = (e^0, \mathbf{e})$ denoting a spacelike direction, taken by convention in the hyperboloid $H \subset \mathbb{M}^4$ of unit radius 1, that is, $e_\mu e^\mu = -1$.

The operations (2.4) are invertible, namely, there holds:

$$\begin{aligned} G^{\mu\nu}(x) &= \partial^\mu A^\nu(x, e) - \partial^\nu A^\mu(x, e), \quad \text{resp.} \\ F^{\mu\nu}(x) &= \partial^\mu A^\nu(x, e) - \partial^\nu A^\mu(x, e). \end{aligned} \tag{2.5}$$

The new potential vector for massless vector bosons (“photons”) has positive two-point functions on the same Hilbert space as F ; it is covariant, i.e. the string e is Lorentz-transformed under Poincaré transformations, and localized, i.e. $[A(x, e), A(x', e')] = 0$ whenever the half-lines $\{x + se : s \geq 0\}$ and $\{x' + se' : s \geq 0\}$ are causally disjoint.

In the massive case, where $G^{\mu\nu}(x)$ is the curl of a pointlike Proca field $B^\mu(x)$, one can write

$$A^\mu(x, e) = B^\mu(x) + \partial^\mu \phi(x, e), \quad \text{where} \quad \phi(x, e) := \int_0^\infty ds B^\mu(x + se) e_\mu, \tag{2.6}$$

whereby the scalar “escort” field ϕ carries away the “non-renormalizability”—see [10].⁵ Henceforth we write $A^\mu(x, e)$ for both massive and massless intermediate vector bosons.

It is also true (for any value of the mass) that

$$\partial_\mu A^\mu(x, e) + m^2 \phi(x, e) = 0; \quad e_\mu A^\mu(x, e) = 0. \tag{2.7}$$

These relations follow from the definition (2.4). If $m^2 = 0$, (2.7) constitutes two constraint equations for the massless potential, so it has two degrees of freedom—as it should. Note that for $A^\mu(x, e)$ the limit $m \downarrow 0$ is smooth.

In view of all of the above, and since we do not share the presumption that interactions brought by massless intermediate vector bosons are “more natural” than interactions mediated by massive carriers, we put massive and massless interaction carriers on the same footing—as we have done in our investigation of flavourdynamics [11].

To unify the notation for both massless and massive vector bosons, from now on we shall write $F^{\mu\nu}$ instead of $G^{\mu\nu}$ in the massive case also; thus, $F^{\mu\nu}(x)$ is the curl of $B^\mu(x)$ when $m > 0$, and it is the curl of $A^\mu(x, e)$ in both cases.

⁵One is reminded here of the Stückelberg fields. It is well known that, in the case of a unique Proca field or “massive photon”, perturbative renormalizability of the model can be recovered with their help [37]. However, this fails in the nonabelian cases [38–40].

2.3. Notations and Nomenclature

We use throughout the notation $(VW) \equiv V_\nu W^\nu = \eta_{\mu\nu} V^\mu W^\nu = V^0 W^0 - \mathbf{V} \cdot \mathbf{W}$ for Minkowski products of vectors (including fields or differential operators) on \mathbb{M}^4 . In particular, $(\partial A) \equiv \partial_\mu A^\mu$ denotes a divergence.

String integrations. It is convenient to introduce the notation I_e^μ for integration in the spacelike direction e , which always appears accompanied by multiplication with e^μ ; that is, for any field component (or numerical distribution) $X(x)$ we write

$$I_e^\mu X(x) := e^\mu \int_0^\infty ds X(x + se). \quad (2.8)$$

The formulas (2.4) may thus be rewritten as

$$A_\mu(x, e) := I_e^\nu F_{\mu\nu}(x), \quad (2.9)$$

and in the massive case the escort field (2.6) is given by $\phi(x, e) := I_e^\nu B_\nu(x)$. Assuming that $X(x + se)$ falls off for large s , as is justified (in the sense of correlation functions) for all pertinent fields, the fundamental theorem of calculus reverses this integral transformation:

$$\partial_\mu(I_e^\mu X)(x) = I_e^\mu(\partial_\mu X)(x) = -X(x), \quad (2.10)$$

or more briefly, $(\partial I_e) = (I_e \partial) = -\text{id}$. Indeed,

$$\begin{aligned} \partial_\mu(I_e^\mu X)(x) &= (I_e^\mu \partial_\mu X)(x) = \int_0^\infty ds e^\mu \partial_\mu X(x + se) \\ &= \int_0^\infty ds \frac{\partial}{\partial s} X(x + se) = -X(x). \end{aligned}$$

The fields $A_\mu(x, e)$ and $\phi(x, e)$ are operator-valued distributions in both x and e . As advertised, to get rid of possible singularities in the e -dependence, we smear them in e with an arbitrary sufficiently smooth function $c(e)$, supported in some small region of the hyperboloid H of spacelike unit vectors:

$$A_\mu(x, c) := \int_H d\sigma(e) c(e) A_\mu(x, e), \quad \phi(x, c) := \int_H d\sigma(e) c(e) \phi(x, e). \quad (2.11)$$

Here $d\sigma(e)$ is the Lorentz-invariant measure on H . More generally, we write

$$(I_c^\nu X)(x) := \int_H d\sigma(e) c(e) (I_e^\nu X)(x)$$

for distributions in x .

For distributions in two variables like $\delta(x - x')$, we write I_c^ν and $I_c^{\nu'}$ according to the string integration in the variable x or x' . String-integrated delta functions like $(I_c^\nu \delta)(x - x')$ or $(I_c^{\nu'} \delta)(x - x')$ will be called ‘‘string deltas’’. If c has weight one with respect to the integration measure: $\int_H d\sigma(e) c(e) = 1$, then (2.10) becomes

$$(\partial I_c) = (I_c \partial) = -\text{id}, \quad (2.12)$$

and equations (2.5), (2.6), and (2.7) with the second constraint replaced by $I_{c,\mu} A^\mu(x, c) = 0$, hold as well.

The *principle of string independence*—see Sect. 2.4—requires to study string variations, that is, variations of $c(e)$ by arbitrary functions $h(e)$ of weight zero. Let us define

$$\delta_c^h(X(c)) := \left. \frac{d}{dt} X(c + th) \right|_{t=0}.$$

For massive fields, we introduce $u(h) := \delta_c^h(\phi(c))$. The definition implies that

$$\delta_c^h(A_\mu(x, c)) = \delta_c^h(B_\mu(x) + \partial_\mu \phi(x, c)) = \partial_\mu u(x, h). \quad (2.13)$$

Lemma 2.1. *For the massless field with helicity one, although $\phi(x, c)$ is not present, there still exists $u(x, h)$ on the photon’s Hilbert space such that*

$$\delta_c^h(A_\mu(x, c)) = \partial_\mu u(x, h). \quad (2.14)$$

It is given by $u(x, h) := -I_c^\nu(\delta_c^h(A_\nu(x, c)))$.

Proof. Derivatives and string integrations commute. Thus,

$$\begin{aligned} & \partial_\mu I_c^\nu(\delta_c^h(A_\nu(x, c))) + \delta_c^h(A_\mu(x, c)) \\ &= \partial_\mu I_c^\nu(\delta_c^h(A_\nu(x, c))) - \partial_\nu I_c^\nu(\delta_c^h(A_\mu(x, c))) \\ &= I_c^\nu(\delta_c^h(\partial_\mu A_\nu(x, c) - \partial_\nu A_\mu(x, c))) = I_c^\nu(\delta_c^h(F_{\mu\nu}(x))) = 0, \end{aligned}$$

where (2.12) as well as the smeared version of (2.5) have been used. \square

Equations of motion. In the sequel we distinguish different vector bosons, massive or massless, by an index a . For every massive vector boson, there is an associated *family of fields*:

$$[a] := \{F_a^{\mu\nu}, A_{a\mu}(c), B_{a\mu}, \phi_a(c), u_a(h)\}. \quad (2.15)$$

The associated family of fields for each massless vector boson (“photon”) is given by:

$$[a] := \{F_a^{\mu\nu}, A_{a\mu}(c), u_a(h)\}.$$

The family of relevant higgs fields is $[H] := \{H, \partial_\mu H\}$.

All fields within a family satisfy the Klein–Gordon equation with the respective mass m_a (equal to 0 for the photons) or m_H . Within each massive family $[a]$, the following equations of motion hold:

$$\begin{aligned} \partial^{[\mu} B_a^{\nu]} &= \partial^{[\mu} A_a^{\nu]}(c) = F_a^{\mu\nu}, & \partial_\mu F_a^{\mu\nu} &= -m_a^2 B_a^\nu, & \partial_\mu B_a^\mu &= 0, \\ \partial_\mu A_a^\mu(c) &= -m_a^2 \phi_a(c), & \partial_\mu \phi_a(c) &= A_{a\mu}(c) - B_{a\mu}. \end{aligned} \quad (2.16)$$

For the photons the same equations, apart from those involving B_a or $\phi_a(c)$, hold after putting $m_a = 0$. Namely:

$$\partial^{[\mu} A_a^{\nu]}(c) = F_a^{\mu\nu}, \quad \partial_\mu F_a^{\mu\nu} = 0, \quad \text{and} \quad \partial_\mu A_a^\mu(c) = 0.$$

Sectors and types. We shall characterize interaction terms by their “sector” and their “type”. The *sector* specifies the families of the fields making up a Wick product, say $[a][b][H]$ or $[a][b][c]$. The sectors made from such $[a]$ and $[H]$ form the bosonic sector of the electroweak interaction (or generalizations thereof). Those involving $[H]$ are collectively called the *higgs sector*. The electroweak fermionic sector involving leptonic currents was studied in [11].

The *type* specifies the fields in a Wick product, irrespective of their family, such as FAA or ABH . Thus, $\phi_a A_b \partial H$ belongs to the sector $[a][b][H]$ and is of type $\phi A \partial H$.

In the context of the electroweak theory, we use labels $a = W_1, W_2$ (or simply 1, 2) and Z, A or else W_+, W_-, Z, A for the families of the massive vector bosons and the photon, according to the standard terminology. In addition, we shall write

$$W_1 \equiv A_1(c), \quad W_2 \equiv A_2(c), \quad W_{\pm} \equiv \frac{1}{\sqrt{2}}(W_1 \mp iW_2)(c), \quad Z \equiv A_Z(c), \quad A \equiv A_A(c) \quad (2.17)$$

for the string-localized vector fields $A_{a\mu}$. Fields of type A, ϕ , or u are by default string-dependent, while F, B , and H are string-independent.⁶

On notation. We shall drop writing the dependence on c throughout the main body of the paper. To forestall confusion in the presence of many field subindices, the notation $\underline{\delta}$ (rather than δ_c^h) will be used for string variations. The notations $X \stackrel{\partial}{\sim} Y$ and $X \stackrel{\text{mod } I\delta}{\sim} Y$ indicate that X and Y differ by a total derivative: $X = Y + \partial_{\mu} Z^{\mu}$, or by a string delta. In expressions with two or three variables x, x', x'' , the symbols $\mathfrak{S}_{xx'}$ and $\mathfrak{S}_{xx'x''}$ denote symmetrization with respect to them; namely, $\mathfrak{S}_{xx'} f(x, x') := \frac{1}{2}(f(x, x') + f(x', x))$, and analogously for three variables. We omit the notation $:-:$ for Wick products throughout. Operator products and time-ordered products of two Wick products will be expanded into Wick products by Wick’s theorem, of which we need only the tree-level part (one contraction)—see Sect. 3.

2.4. The Principle of String Independence

In the sequel, we adopt the rigorous and flexible Stückelberg–Bogoliubov–Epstein–Glaser formalism of “renormalization without regularization” in perturbation theory [41, 42]. It involves the *construction* of a unitary scattering operator $\mathbb{S}[g, c]$ acting on the Fock spaces of the local free fields, functionally dependent on a multiplet of smooth external fields $g(x)$, with the stock requisites of causality and Lorentz covariance [11, Sect. 3]. One looks for $\mathbb{S}[g, c]$ as a time-ordered exponential series:

$$\mathbb{S}[g, c] = \text{T} \sum_{k=0}^{\infty} \frac{i^k}{k!} \int S_k(x_1, \dots, x_k, c) g(x_1) \cdots g(x_k) dx, \quad (2.18)$$

In the adiabatic limit $g(x) \uparrow g$, this is thought of as the perturbative expansion of the heuristic \mathbb{S} -matrix

$$\text{T} \exp i \int dx L_{\text{int}}(g, c), \quad (2.19)$$

where g is a coupling constant, and in sQFT c denotes a common string smearing function for all fields appearing in L_{int} . With

$$L_{\text{int}} = gL_1 + \frac{1}{2}g^2L_2,$$

⁶There should be no risk of confusion because the gauge potential for the photon, usually called A in textbooks, is not defined on a Hilbert space, and thus, it simply does not appear in sQFT. Recall that, whereas the gauge potential raises “particles” with four states per momentum in an indefinite Fock space, our string-localized photon field $A(c)$ creates precisely the two physical states per momentum on the Hilbert space.

the leading term is $gS_1(x, c) = gL_1(x, c)$ —usually a cubic Wick polynomial in the free fields, chosen in relation to the physics under consideration. It specifies a model. The second-order term is of the form:

$$S_2(x, c) = T[L_1(x, c) L_1(x', c)] - iL_2(x, c) \delta(x - x'). \tag{2.20}$$

The “principle of string independence” posits that, notwithstanding the appearance of the string smearing function c in $\mathbb{S}[g, c]$, the adiabatic limit does not depend on it. In the subsequent sections, it will become clear how this condition serves to determine the interactions, order by order. It already constrains the choice of L_1 (Sect. 2.5); it determines L_2 (Sect. 5); and it must warrant the absence of higher-order interactions as $\propto g^3$, which would violate the power-counting bound (Sect. 6). See also [18] for a systematic coverage of the Abelian case, and [43] for an improved and more general formulation at all orders.

2.5. The Vector Boson Sector at First Order

String independence of the \mathbb{S} -matrix requires that $\delta(S_n)$ be a total derivative for every n . Because $S_1(x, c) = L_1(x, c)$ at first order, this amounts to the condition:

$$\delta(L_1) = \partial_\mu Q_1^\mu \tag{2.21}$$

for an appropriate vector polynomial $Q_1^\mu(x, c)$ in the fields. For the self-interactions, we seek $L_{1,\text{self}}$ as a scalar cubic Wick polynomial in massless and/or massive string-localized vector potentials $A_a^\mu(c)$, their field strengths $F_a^{\mu\nu}$, and their escort fields $\phi_a(c)$, which exist only when $m_a > 0$. Recall that in the latter case $B_{a\mu} := A_{a\mu}(c) - \partial_\mu \phi_a(c)$ is string-independent.

Proposition 2.2. *Apart from the higgs sector, the cubic self-interaction of a string-local theory of interacting bosons with spin $s = 1$ or helicity $|h| = 1$ must be of the form*

$$\begin{aligned} L_{1,\text{self}}(x, c) &\equiv L_{1,\text{self}}^1(x, c) + L_{1,\text{self}}^2(x, c) \\ &= \sum_{abc} f_{abc} F_a^{\mu\nu}(x) A_{b\mu}(x, c) A_{c\nu}(x, c) \\ &\quad + \sum_{abc} f_{abc} m_{abc}^2 B_a^\mu(x) A_{b\mu}(x, c) \phi_c(x, c), \end{aligned} \tag{2.22}$$

where f_{abc} are completely skewsymmetric real coefficients, and

$$m_{abc}^2 := m_a^2 - m_b^2 + m_c^2 = m_{cba}^2.$$

Moreover, if particle b is massless, then particles a and c must have equal mass:

$$m_b^2 = 0 \text{ and } f_{abc} \neq 0 \implies m_a = m_c. \tag{2.23}$$

Consequently, if both particles a and b are massless, then particle c is massless, too.

Proof. (We drop the subscript “self” in $L_{1,\text{self}}$ during this proof.) String independence (2.21) requires that $\underline{\delta}(L_1)$ be a total derivative $\partial_\mu Q_1^\mu$. In the purely massless case, only the fields F and A are available to build L_1 , and (by Lorentz covariance and the power-counting bound) L_1 can only be of type FAA as in L_1^1 . In this case, complete skewsymmetry of the coefficients was proved in [14].

If massive vector bosons are admitted, one may make a most general (hermitian, Lorentz-covariant and power-counting renormalizable) Ansatz:

$$L_1 = \sum_{abc} f_{abc} F_a^{\mu\nu} A_{b\mu} A_{c\nu} + \sum_{abc} g_{abc} B_a^\mu A_{b\mu} \phi_c + \sum_{abc} h_{abc} A_a^\mu A_{b\mu} \phi_c,$$

with real coefficients f_{abc} , g_{abc} , h_{abc} satisfying $f_{abc} = -f_{acb}$ and $h_{abc} = h_{bac}$. Moreover, since B and ϕ fields are necessarily massive, the following supplementary rules apply:

$$\begin{aligned} g_{abc} &= 0 \quad \text{whenever } a \text{ or } c \text{ is massless, and} \\ h_{abc} &= 0 \quad \text{whenever } c \text{ is massless.} \end{aligned} \quad (2.24)$$

We now compute $\underline{\delta}(L_1)$ on use of $\underline{\delta}(\phi) = u$ and $\underline{\delta}(A_\mu) = \partial_\mu u$. To remove terms of type $\partial_\mu u$, we employ

$$XY \partial u = \partial(XY u) - \partial(XY) u \stackrel{\partial}{\sim} -(\partial X Y + X \partial Y) u,$$

as well as the equations of motion (2.16). Since

$$\begin{aligned} \underline{\delta}(F_a^{\mu\nu} A_{b\mu} A_{c\nu}) &= F_a^{\mu\nu} (\partial_\mu u_b A_{c\nu} + A_{b\mu} \partial_\nu u_c) \\ &\stackrel{\partial}{\sim} m_a^2 B_a^\nu u_b A_{c\nu} - m_a^2 B_a^\mu A_{b\mu} u_c - \frac{1}{2} F_a^{\mu\nu} u_b F_{c\mu\nu} - \frac{1}{2} F_a^{\mu\nu} F_{b\nu\mu} u_c, \\ \underline{\delta}(B_a^\mu A_{b\mu} \phi_c) &= B_a^\mu A_{b\mu} u_c + B_a^\mu \partial_\mu u_b \phi_c \stackrel{\partial}{\sim} B_a^\mu A_{b\mu} u_c - B_a^\mu u_b \partial_\mu \phi_c, \quad \text{and} \\ \underline{\delta}(A_a^\mu A_{b\mu} \phi_c) &= \partial^\mu u_a A_{b\mu} \phi_c + A_a^\mu \partial_\mu u_b \phi_c + A_a^\mu A_{b\mu} u_c \\ &\stackrel{\partial}{\sim} m_b^2 u_a \phi_b \phi_c - u_a A_{b\mu} \partial^\mu \phi_c + m_a^2 \phi_a u_b \phi_c - A_a^\mu u_b \partial_\mu \phi_c + A_a^\mu A_{b\mu} u_c, \end{aligned} \quad (2.25)$$

this produces:

$$\begin{aligned} \underline{\delta}(L_1^1) &\stackrel{\partial}{\sim} \sum_{abc} f_{abc} (m_a^2 u_b (B_a A_c) - m_a^2 u_c (B_a A_b) - \frac{1}{2} u_b (F_a F_c) + \frac{1}{2} u_c (F_a F_b)) \\ &+ \sum_{abc} g_{abc} (u_c (B_a A_b) - u_b (B_a (A_c - B_c))) \\ &+ \sum_{abc} h_{abc} (m_b^2 u_a \phi_b \phi_c - u_a (A_b (A_c - B_c)) \\ &+ m_a^2 u_b \phi_a \phi_c - u_b (A_a (A_c - B_c)) + u_c (A_a A_b)). \end{aligned}$$

Next, relabel the indices abc conveniently, using $f_{abc} = -f_{acb}$ and $h_{abc} = h_{bac}$, to write the sum as $\sum_b u_b Z_b$ with

$$\begin{aligned} Z_b &= \sum_{ac} 2m_a^2 f_{abc}(B_a A_c) - f_{abc}(F_a F_c) \\ &\quad + \sum_{ac} g_{acb}(B_a A_c) - g_{abc}(B_a(A_c - B_c)) \\ &\quad + \sum_{ac} 2m_a^2 h_{abc} \phi_a \phi_c - 2h_{abc}(A_a(A_c - B_c)) + h_{acb}(A_a A_c). \end{aligned} \quad (2.26)$$

String independence requires that Z_b must vanish for all b . In the sum over a, c , let us examine the coefficients of $(F_a F_c)$, $(A_a A_c)$, $(B_a B_c)$ and $(B_a A_c)$, in turn.

- Firstly, $\sum_{ac} f_{abc}(F_a F_c) = 0$ requires $f_{abc} + f_{cba} = 0$. Thus, f_{abc} is skewsymmetric under both $b \leftrightarrow c$ and $a \leftrightarrow c$, and hence, it is completely skewsymmetric. This reproduces the result from the massless case.
- For the symmetric coefficients of $(A_a A_c)$, there are two cases to consider. If all fields are massive, then $h_{acb} - 2h_{abc} + [a \leftrightarrow c] = 0$ for all b , that is: $2h_{acb} - 2h_{abc} - 2h_{cba} = 0$, whereby $h_{abc} = h_{acb} - h_{cba} = h_{cab} - h_{cba}$. This is both symmetric and skew under $a \leftrightarrow b$; hence $h_{\bullet\bullet\bullet} = 0$.
- If instead one index, say $b = b'$, refers to a massless field, then a priori $h_{acb'} = h_{cab'} = 0$ by (2.24). In this case, the coefficient of $(A_a A_{b'})$ in Z_c equals $-2h_{acb'} + h_{ab'c} = h_{ab'c}$. Its vanishing, along with that of $a \leftrightarrow c$, again implies $h_{\bullet\bullet\bullet} = 0$. Consequently, the third summation in (2.26) may be dropped altogether.
- The coefficients of $(B_a B_c)$ and of $(B_a A_c)$ in Z_b now serve to determine g_{abc} . Their vanishing gives

$$(I) \quad g_{abc} + g_{cba} = 0, \quad \text{and} \quad (II) \quad g_{abc} - g_{acb} = 2m_a^2 f_{abc}.$$

When a, b, c are all massive, these relations must hold for all permutations, and the relations (I), (II) then also imply:

$$g_{bac} + g_{acb} = -2m_b^2 f_{abc} \quad \text{and} \quad -g_{bac} + g_{abc} = 2m_c^2 f_{abc}.$$

These equations have a unique solution:

$$g_{abc} = (m_a^2 - m_b^2 + m_c^2) f_{abc} =: m_{abc}^2 f_{abc}, \quad (2.27)$$

valid for all permutations.

- On the other hand, if say b' is massless, then a priori only $g_{ab'c}$ and $g_{cb'a}$ can be nonzero by (2.24). Formula (I) implies $g_{ab'c} = -g_{cb'a}$; then (II) together with $a \leftrightarrow c$ yields:

$$2m_a^2 f_{ab'c} = g_{ab'c} = -g_{cb'a} = -2m_c^2 f_{cb'a} = 2m_c^2 f_{ab'c}.$$

This implies $m_a = m_c$ whenever $f_{ab'c} \neq 0$. With this specification, Eq. (2.27) holds again for all permutations. In particular, if any two of the fields a, b, c are massless, then the third one is also massless, and all permutations of g_{abc} vanish. \square

- Remark 2.3.* (i) It follows from (2.23) that $f_{abc} m_{abc}^2 = 0$ whenever a or c is massless. This deletes the non-existent terms “ $B_a A_b \phi_c$ ” in $L_{1,\text{self}}^2$ whenever a or c is massless. Moreover, $L_{1,\text{self}}^2$ contains no terms with more than one massless index, because in that case $f_{abc} m_{abc}^2 = 0$. However, terms with massless b may indeed appear.⁷
- (ii) In Proposition 5.1, we shall show (by SI at second order) that f_{abc} in fact must satisfy the Jacobi identity, and thus, they are the structure constants of a reductive Lie algebra of compact type.
- (iii) If a and b are massless and $f_{abc} \neq 0$, then c is also massless. In view of (ii), this can be reformulated: the structure constants f_{abc} for the massless particles define a Lie subalgebra. The latter may be nonabelian, as for instance in QCD.

We shall also need $Q_{1,\text{self}}^\mu$ so that $\underline{\delta}(L_{1,\text{self}}) = \partial_\mu Q_{1,\text{self}}^\mu$ holds. This can be obtained, for instance, by collecting the total derivatives discarded in the first step of the previous proof.

Proposition 2.4.

$$Q_{1,\text{self}}^\mu \equiv Q_{1,\text{self}}^{1\mu} + Q_{1,\text{self}}^{2\mu} = 2 \sum_{abc} f_{abc} F_a^{\mu\nu} u_b A_{c\nu} + \sum_{abc} f_{abc} m_{abc}^2 B_a^\mu u_b \phi_c. \quad (2.28)$$

Proof. On using $h_{abc} = 0$, we readily retrieve the derivatives discarded in (2.25):

$$\begin{aligned} & \sum_{abc} f_{abc} (\partial_\mu (F_a^{\mu\nu} u_b A_{c\nu}) + \partial_\nu (F_a^{\mu\nu} A_{b\mu} u_c)) + \sum_{abc} g_{abc} \partial_\mu (B_a^\mu u_b \phi_c) \\ &= \partial_\mu \sum_{abc} [2f_{abc} F_a^{\mu\nu} u_b A_{c\nu} + g_{abc} B_a^\mu u_b \phi_c] =: \partial_\mu Q_{1,\text{self}}^\mu. \quad \square \end{aligned}$$

2.6. Let the Higgs be With You

Our $(L_{1,\text{self}}, Q_{1,\text{self}})$ pair above is *not complete*, since bosonic couplings involving massive neutral spinless fields (i.e. higgses) have not been included, and as we shall see, they play an important role in our problem. Such physical point-like scalar fields do not suffer from the renormalizability issues discussed in Sect. 2.1. As already learned in [44], the presence of some higgses, hinted at by the appearance of escort fields, is *required* for consistency of models wherever *massive* A -fields are present. One can study what their couplings ought to be from the standpoint of the SI principle. To simplify matters, we shall suppose that *only one* higgs field $H(x)$ of mass m_H is present—like in the minimal SM.

Proposition 2.5. *The most general (L, Q) pair coupling the vector bosons to the higgs is of the form*

$$L_{1,\text{higgs}} = \sum_{ab} [k_{ab} (A_{a\mu} B_b^\mu H + A_{a\mu} \phi_b \partial^\mu H - \frac{1}{2} m_H^2 \phi_a \phi_b H) + \ell H^3], \quad (2.29)$$

$$Q_{1,\text{higgs}}^\mu = \sum_{ab} k_{ab} (u_b B_a^\mu H + u_b \phi_a \partial^\mu H), \quad (2.30)$$

where the coupling matrix $[k_{ab}]$ is symmetric and links only massive fields.

⁷This qualifies the meaning of the restricted sum \sum' in [11, Eq. (4.1)].

Proof. A general renormalizable cubic Ansatz coupling the higgs to the vector bosons is of the form

$$L_{1,\text{higgs}}^1 := \sum_{ab} k_{ab} A_{a\mu} B_b^\mu H + k'_{ab} A_{a\mu} A_b^\mu H + k''_{ab} A_{a\mu} \phi_b \partial^\mu H + k'''_{ab} \phi_a \phi_b H,$$

with real constants k_{ab}^\bullet such that $k'_{ab} = k'_{ba}$ and $k'''_{ab} = k'''_{ba}$. We require that $\underline{\delta}(L_{1,\text{higgs}})$ equal $\partial_\mu Q_{1,\text{higgs}}^\mu$, with $Q_{1,\text{higgs}}^\mu$ containing no ∂u -terms. On subtracting

$$\sum_{ab} \partial_\mu [k_{ab} u_a B_b^\mu H + 2k'_{ab} u_a A_b^\mu H + k''_{ab} u_a \phi_b \partial^\mu H]$$

from the string variation of $L_{1,\text{higgs}}$, one finds that

$$\begin{aligned} \underline{\delta}(L_{1,\text{higgs}}) &= \sum_{ab} [k_{ab} \partial_\mu u_a B_b^\mu H + 2k'_{ab} \partial_\mu u_a A_b^\mu H \\ &\quad + k''_{ab} (\partial_\mu u_a \phi_b + A_{a\mu} u_b) \partial^\mu H + 2k'''_{ab} u_a \phi_b H] \\ &\stackrel{\partial}{=} \sum_{ab} [k''_{ab} A_{a\mu} u_b \partial^\mu H + 2k'''_{ab} u_a \phi_b H - k_{ab} u_a B_b^\mu \partial_\mu H + 2k'_{ab} m_b^2 u_a \phi_b H \\ &\quad - 2k'_{ab} u_a A_b^\mu \partial_\mu H - k''_{ab} u_a (A_{b\mu} - B_{b\mu}) \partial^\mu H + k''_{ab} m_H^2 u_a \phi_b H]. \end{aligned}$$

The right-hand side (i.e. the last two lines) must vanish, by string independence. Comparing coefficients of $u(A\partial H)$, one gets $k''_{ba} - k''_{ab} = 2k'_{ab}$; this is both symmetric and skew under $(a \leftrightarrow b)$, so $k'_{ab} = 0$ and $k''_{ba} = k''_{ab}$. The coefficient of $u(B\partial H)$ vanishes only if $k''_{ab} = k_{ab}$, and that of $u\phi H$ is zero only if $2k'''_{ab} = -k''_{ab} m_H^2 = -k_{ab} m_H^2$.

As a consequence, $k_{ab} = k_{ba}$, too: the matrix $[k_{ab}]$ is symmetric. To sum up:

$$\begin{aligned} L_{1,\text{higgs}}^1 &= \sum_{ab} k_{ab} (A_{a\mu} B_b^\mu H + A_{a\mu} \phi_b \partial^\mu H - \frac{1}{2} m_H^2 \phi_a \phi_b H), \\ Q_{1,\text{higgs}}^\mu &= \sum_{ab} k_{ab} (u_a B_b^\mu H + u_a \phi_b \partial^\mu H) = \sum_{ab} k_{ab} (u_b B_a^\mu H + u_b \phi_a \partial^\mu H). \end{aligned}$$

We have found it convenient to include in (2.29) the unique (up to a real multiple) renormalizable cubic self-interaction of the higgs, given by $L_{1,\text{higgs}}^2 := \ell H^3$, which trivially satisfies the (L, Q) condition (2.21). \square

The reader may note our assertion that the higgs does not couple to massless fields, notwithstanding that it was first detected at the LHC of CERN by its decay into two photons. The answer is of course that this decay takes place through *loop graphs*. It is well known that in such cases the one-loop contribution is *finite*. The (correct) calculation of this contribution is as old as [45]. This was questioned in a paper by Gastmans, Wu and Wu [46], which actually received some support from other calculations. Consensus around the result in [45] was reestablished in other papers, among them one by Duch, Dütsch and one of us [47].

2.7. Summary: The List of All First-Order Couplings

The complete L_1 and Q_1^μ terms found above are restated as follows:

$$\begin{aligned}
 L_1 &\equiv L_{1,\text{self}}^1 + L_{1,\text{self}}^2 + L_{1,\text{higgs}}^1 + L_{1,\text{higgs}}^2 \\
 &= \sum_{abc} f_{abc} F_a^{\mu\nu} A_{b\mu} A_{c\nu} + \sum_{abc} f_{abc} m_{abc}^2 B_a^\mu A_{b\mu} \phi_c \\
 &\quad + \sum_{ab} k_{ab} (B_a^\mu A_{b\mu} H + \phi_a A_b^\mu \partial_\mu H - \frac{1}{2} m_H^2 \phi_a \phi_b H) + \ell H^3; \quad (2.31a)
 \end{aligned}$$

$$\begin{aligned}
 Q_1^\mu &\equiv Q_{1,\text{self}}^{1\mu} + Q_{1,\text{self}}^{2\mu} + Q_{1,\text{higgs}}^\mu \\
 &= 2 \sum_{abc} f_{abc} F_a^{\mu\nu} u_b A_{c\nu} + \sum_{abc} f_{abc} m_{abc}^2 B_a^\mu u_b \phi_c \\
 &\quad + \sum_{ab} k_{ab} (B_a^\mu u_b H + \phi_a u_b \partial^\mu H). \quad (2.31b)
 \end{aligned}$$

For memory: f_{abc} is completely skewsymmetric, and if b is massless while $f_{abc} \neq 0$, then $m_a^2 = m_c^2$. In particular, $f_{abc} m_{abc}^2$ vanishes whenever a, b, c stand for one massive and two massless particles. Moreover, the matrix $[k_{ab}]$ is symmetric, and it vanishes when either a or b is massless.

The reader should be aware that until now nothing requires the coefficients k_{ab}, ℓ belonging to the higgs sector to be different from zero. Consideration of the scattering matrix *at second order* will allow us to establish that in the presence of massive intermediate bosons these coefficients do not vanish in general—and to extract further relations between them. The quartic interactions L_2 will be determined by the condition of string independence at second order—see Propositions 5.2, 5.5, 5.7 and 5.10. Consideration of the scattering matrix at third order will show that a term proportional to H^4 was indeed needed in L_2 , and will allow for the coefficient ℓ to be computed.

3. Obstruction Theory

At the second order of the \mathbb{S} -matrix, one encounters obstructions against string independence. These arise because time ordering does not commute with derivatives:⁸ in the adiabatic limit,

$$\underline{\delta} \left(\int dx dx' \text{T}[L_1(x)L_1(x')] \right) = \int dx dx' \left(\text{T}[\partial_\mu Q_1^\mu(x)L_1(x')] + [x \leftrightarrow x'] \right) \quad (3.1)$$

does not vanish because $\text{T}[\partial_\mu Q_1^\mu(x)L_1(x')] \neq \partial_\mu \text{T}[Q_1^\mu(x)L_1(x')]$. Quantities of the general form

$$\text{T}[\partial_\mu Q^\mu(x)L(x')] - \partial_\mu \text{T}[Q^\mu(x)L(x')] =: \mathcal{O}_\mu(Q^\mu(x); L(x')), \quad (3.2)$$

⁸One could also question whether $\underline{\delta}$ commutes with time ordering. We assume this to be the case at tree level (which is all we need), thus fixing the propagators involving fields u or ∂u . With a broader view, the commutation between T and $\underline{\delta}$ should be considered as a renormalization condition on loop contributions.

with Wick polynomials Q^μ and L , are the subject of “obstruction theory” [11], see below. Thus, by subtracting $\partial_\mu \mathbb{T}[Q_1^\mu(x)L_1(x')] + [x \leftrightarrow x']$ from the integrand in (3.1), one finds that the total obstruction at second order (that is, the failure of the integrand of (3.1) to be a derivative) is the symmetric sum $\mathcal{O}^{(2)}(x, x')$ displayed in (3.5).

As anticipated in (2.20), it is decisive that the obstruction can be cancelled, and string independence can be restored, by adding “induced” interactions at second order. However, this requires that $\mathcal{O}^{(2)}(x, x')$ be of a form suitable for cancellation: it must be “resolvable”—see Eq. (3.6). For this to be possible, it is instrumental, but of course not sufficient, that expressions like (3.2) contain delta functions, which they do thanks to the subtraction of the derivative term. We aim to show that the string independence condition (that is: resolvability of obstructions) imposes constraints on the (until now) free parameters of the first-order interactions, and allows to compute the induced interactions. Through these systematics, sQFT determines interactions. In this way, in the vein of [11] and [18], one recovers the precise electroweak self-interactions.

In a more ambitious program [43], one can establish that the coupling terms involving the escort field not only make the \mathbb{S} -matrix of sQFT string-independent, but make it coincide with the \mathbb{S} -matrix computed by means of ordinary local fields, like in gauge theory or the Stückelberg field method of [29], once their unphysical degrees of freedom are eliminated.

We now set out to compute and analyse the structure of obstructions. A tree-level analysis is sufficient for the purpose of determining induced interactions. Computing (3.2) at tree level, Wick’s theorem at the tree level entails:

$$\mathcal{O}_\mu(Q^\mu(x); L(x')) = \sum_{\psi, \chi'} \frac{\partial Q^\mu}{\partial \psi} (\langle\langle \mathbb{T} \partial_\mu \psi(x) \chi(x') \rangle\rangle - \partial_\mu \langle\langle \mathbb{T} \psi(x) \chi(x') \rangle\rangle) \frac{\partial L'}{\partial \chi'}.$$
(3.3)

Namely, the uncontracted contributions to the Wick expansion drop out in the difference, and (3.3) are precisely the terms with one contraction. To evaluate (3.5), we therefore only require the *two-point obstructions* of the involved free fields:

$$\mathcal{O}_\mu(\psi, \chi') := \langle\langle \mathbb{T} \partial_\mu \psi(x) \chi(x') \rangle\rangle - \partial_\mu \langle\langle \mathbb{T} \psi(x) \chi(x') \rangle\rangle.$$
(3.4)

The two-point obstructions are computed from time-ordered two-point functions (i.e. the “propagators”) of the free fields and their derivatives. The latter in turn are determined by ordinary two-point functions, up to a certain freedom of renormalization. In local QFT, both terms on the right-hand side of (3.4) coincide except on the singular set $x = x'$; so that the difference is a multiple of (a derivative of) $\delta_{xx'} \equiv \delta(x - x')$. In the string-localized case, these delta functions generalize to string deltas.

The above is discussed in Appendix A, where the two-point obstructions are computed; we present here the results in tabular form. The tables exhibit three real parameters denoted c_H , c_B , c_F , parametrizing the ambiguity of

TABLE 1. Two-point obstructions in the higgs sector

| | H' | $\partial'^{\kappa} H'$ |
|--|-----------------|---|
| $\mathcal{O}_{\mu}(H, \cdot)$ | 0 | $ic_H \delta_{\mu}^{\kappa} \delta_{xx'}$ |
| $\mathcal{O}_{\mu}(\partial^{\mu} H, \cdot)$ | $i\delta_{xx'}$ | $-i(1 + c_H)\partial^{\kappa} \delta_{xx'}$ |

TABLE 2. Two-point obstructions in the Proca sector

| | $F'^{\kappa\lambda}$ | A'^{κ} | B'^{κ} | ϕ' |
|--|--|--|---|---------------------------|
| $\mathcal{O}^{\mu}(F_{\mu\nu}, \cdot)$ | $-i(1 + c_F)\delta_{\nu}^{[\kappa} \partial^{\lambda]} \delta_{xx'}$ | $-i(\delta_{\nu}^{\kappa} - I'_{\nu} \partial^{\kappa})\delta_{xx'}$ | $-i(1 + c_B)\delta_{\nu}^{\kappa} \delta_{xx'}$ | $-iI'_{\nu} \delta_{xx'}$ |
| $\mathcal{O}_{[\mu}(A_{\nu]}, \cdot)$ | $-ic_F \delta_{\mu}^{[\kappa} \delta_{\nu]}^{\lambda]} \delta_{xx'}$ | 0 | 0 | 0 |
| $\mathcal{O}_{\mu}(A^{\mu}, \cdot)$ | $-iI^{[\kappa} \partial^{\lambda]} \delta_{xx'}$ | $-i(I^{\kappa} - (II')\partial^{\kappa})\delta_{xx'}$ | $-iI^{\kappa} \delta_{xx'}$ | $-i(II')\delta_{xx'}$ |
| $\mathcal{O}_{\mu}(B^{\mu}, \cdot)$ | 0 | 0 | $-i(1 + c_B)m^{-2}\partial^{\kappa} \delta_{xx'}$ | $-im^{-2}\delta_{xx'}$ |
| $\mathcal{O}_{\mu}(\phi, \cdot)$ | 0 | 0 | $-ic_B m^{-2} \delta_{\mu}^{\kappa} \delta_{xx'}$ | 0 |
| $\mathcal{O}_{\mu}(u, \cdot)$ | 0 | 0 | 0 | 0 |

some propagators—see Appendix A. The freedom to choose the values of these parameters may be exploited later to secure string independence.

For the massless photon fields (F, A, u) , Table 2 holds without the rows and columns for the non-existent fields $\phi(c)$ and $B = A(c) - \partial\phi(c)$.

3.1. Resolution of Obstructions

In order to establish string independence of the \mathbb{S} -matrix, one must first compute the second-order obstruction

$$\mathcal{O}^{(2)}(x, x') := \mathcal{O}_{\mu}(Q_1^{\mu}(x); L_1(x')) + [x \leftrightarrow x'], \quad (3.5)$$

with the tree-level formula given in (3.3). Then one needs to ensure that this obstruction is “resolvable”, that is, the result must be of the form [20]:

$$\mathcal{O}^{(2)}(x, x') = i\delta(L_2(x))\delta(x - x') - i\mathfrak{S}_{xx'}\partial_{\mu}^x Q_2^{\mu}(x, x'). \quad (3.6)$$

If (3.6) can be fulfilled, on then adding to $(i^2/2)g^2 T[L_1 L_1']$ the “induced” interaction $(i/2)g^2 L_2$, the obstruction of the \mathbb{S} -matrix is cancelled at second order—up to a derivative yielding zero in the adiabatic limit. By (3.3), since $S_1 = L_1$ is cubic in the fields, L_2 will be quartic.

The resolution of obstructions is *the* method to determine higher-order interactions in sQFT. Not least, condition (3.6) will require polynomial relations among the masses, the coefficients f_{abc} in the self-couplings $L_{1,\text{self}}^1$, $L_{1,\text{self}}^2$, and the k_{ab} , ℓ in the higgs couplings $L_{1,\text{higgs}}^1$, $L_{1,\text{higgs}}^2$, see (2.31a). For the electroweak theory, these determine all the cubic couplings except ℓ , as well as most quartic terms. String independence at third order will ultimately determine the coefficients of the cubic and quartic self-couplings of the higgs field.

Remark 3.1. The induced interactions L_2 are determined by the parts of (3.5) without string deltas. All obstructions with string deltas must be total derivatives, contributing a part $Q_2|_{I\delta}$ to the fields $Q_2^\mu(x, x')$. Lemma B.2 in Appendix B shows that this is automatically the case. In order to show that there are no induced third-order interactions L_3 (see Sect. 6), we need only third-order obstructions without string deltas, to which $Q_2|_{I\delta}$ does not contribute. We may therefore ignore the string deltas in Table 2 altogether.

3.2. Preliminaries on “Crossings”

We denote pieces of $\mathcal{O}_\mu(Q_1^\mu, L_1)$, corresponding to the pieces of Q_1 and L_1 in (2.31) as *crossings*, and write the latter in a somewhat more readable way as

$$Q \boxtimes L' := \mathcal{O}_\mu(Q^\mu; L'),$$

keeping in mind that according to (3.5) one must always add $Q' \boxtimes L$ to this. The full obstruction that has to be resolved is a sum of all crossings. Each crossing may have contributions belonging to several sectors. Because cancellations are only possible within sectors, we shall organize these crossings by field content, that is, by the sectors and types of the outcome, as in [11].

4. The Electroweak Theory

As regards the chirality of the Standard Model, we proved in [11] that the *physical particle spectrum* with specified masses *forces* the couplings of the massive bosons with the fermions to be parity-violating. The proof involved the crossings of the fermionic interaction set L_1^F with the fermionic Q_1^F and bosonic Q_1 operators. That is, along with the fermionic obstructions, fermionic-bosonic ones were required there. Now, the *entire determination* of the electroweak theory from that spectrum is finally reached in the present paper, by crossing the bosonic Q_1 with the bosonic vertices in L_1 . (The crossing of fermionic Q_1^F with bosonic L_1 is inert.)

In Sects. 5 and 6, we offer a comprehensive computation of the obstructions appearing at second and third orders in a more general situation than is necessary in this section: any number of massive vector bosons and photons, plus a single higgs. String independence is achieved provided several algebraic relations among the masses, the structure constants f_{abc} , the higgs couplings k_{ab} , and the cubic and quartic higgs self-couplings ℓ, ℓ' are satisfied. (The term $\ell' H^4$ must be introduced at second order to cancel an upcoming obstruction at third order.)

In electroweak force theory not all those relations are needed, and some are redundant. We just anticipate a few pertinent ones from the general setup, enough to determine the bosonic interactions of the theory, whose input in sQFT is just *the particle content*, namely the massless photon A , three massive vector bosons W_1, W_2, Z , and the higgs H .

It is assumed that the photon couples to the W -bosons, that is, $f_{12A} \neq 0$. Hence by Eq. (2.23) those have equal masses m_W . It is also assumed that $m_Z \neq m_W$, whereby $f_{AZ1} = f_{AZ2} = 0$, again by Eq. (2.23). (Even so, the

limiting case $m_Z = m_W$, also with $f_{AZ1} = f_{AZ2} = 0$, does yield an acceptable model, but with a completely decoupled photon.) Thus, f_{abc} are the structure constants of the Lie algebra $\mathfrak{g} = \mathfrak{u}(2)$ in a suitable basis. (The Jacobi identity does not constrain f_{12A} and f_{12Z} .) We shall also assume that $f_{12Z} \neq 0$. For the possibility $f_{12Z} = 0$, not realized in Nature, see Remark 4.1.

By a change of basis in the 1-2 mass eigenspace, one may achieve $k_{12} = 0$; however, the relations $k_{1Z} = k_{2Z} = 0$ are not assumed for the higgs couplings: that will follow from string independence. Therefore, the only nontrivial coupling coefficients at first order are f_{12A} and f_{12Z} , the symmetric matrix of higgs couplings k_{ab} (with $a, b = 1, 2, Z$), and the higgs self-coupling ℓ . From the principles of sQFT and that particle content we *predict*, all the characteristic relations of the electroweak interactions usually ascribed to gauge theory and “spontaneous symmetry breaking”—none of these constructs have a place in sQFT.

We first work out the consequences of the string independence condition from Proposition 5.7 further down, to the effect that

$$C_{abc} := \sum_{e=1,2,Z} \frac{k_{ae}}{m_e^2} f_{ebc} m_{ebc}^2 + [a \leftrightarrow c] = C_{bca} = C_{cab} \quad (\text{for } a, b, c = 1, 2, Z, A).$$

Since C_{abc} vanishes identically for massless a or c —see Remark 5.6(i)—it must also vanish for massless b . With $f_{ZA_c} = 0$ for all c , the condition $C_{ZA1} = 0$ yields $k_{Z2} f_{2A1} m_{2A1}^2 = 0$, which implies $k_{Z2} = 0$. Similarly, $k_{Z1} = 0$. Thus, $k_{ab} =: k_a \delta_{ab}$ is diagonal. Then condition $C_{1A2} = 0$ implies $k_1 = k_2$, and:

$$C_{12Z} = C_{2Z1} = C_{Z12} \implies \frac{k_1}{m_W^2} = \frac{k_Z}{m_Z^2} \implies k_a = K m_a^2 \quad (4.1)$$

with some constant K (having dimension of inverse mass). In particular, $k_1 = k_2 =: k_W$.

With this information, we anticipate and interpret the “sum rule” (5.9), a special case of Eq. (5.7). For $a = 1$ and $b = c = 2$, resp. $b = c = Z$, one finds:

$$(f_{12A})^2 (4m_W^2) + (f_{12Z})^2 (4m_W^2 - 3m_Z^2) \stackrel{!}{=} k_W^2 = K^2 m_W^4,$$

and

$$(f_{12Z})^2 \left[2m_Z^2 + \frac{(m_W^2 - m_Z^2)^2 - m_W^4}{m_W^2} \right] \equiv (f_{12Z})^2 \frac{m_Z^4}{m_W^2} \stackrel{!}{=} k_W k_Z = K^2 m_W^2 m_Z^2.$$

These imply the equalities

$$\frac{(f_{12A})^2}{(f_{12Z})^2} = \frac{m_Z^2 - m_W^2}{m_W^2} \quad \text{and} \quad f_{12A}^2 + f_{12Z}^2 = K^2 m_W^2. \quad (4.2)$$

In particular, $m_W < m_Z$ must hold.

The above results imply that the stronger conditions in (5.7) are identically satisfied for all a, b, c, d . Notice that $K \neq 0$ unless all couplings are zero; thus, *string independence imposes the need for the higgs*. The narrative of “spontaneous symmetry breaking” is nowhere required. Together with the higgs self-couplings that will be given by (6.5) in Proposition 6.2, the previous relations secure string independence at all orders.

Let us now compare these results to the Glashow–Weinberg–Salam model, see [48]. In terms of $v := 1/gK$ and $\lambda := g^2 K^2 m_H^2/4$, result (6.5) becomes

$$g\ell H^3 + \frac{1}{2}g^2\ell' H^4 = -\frac{1}{2}\lambda(4vH^3 + H^4), \quad (4.3)$$

which can be written as

$$\frac{1}{2}m_H^2 H^2 - \frac{1}{2}\lambda((v + H)^2 - v^2)^2.$$

In the GWS model, this expression is interpreted as the contribution of the “higgs potential” to the interaction Lagrangian. In sQFT, such a construct is devoid of meaning (only the interaction part makes sense). In our treatment, which starts from the particle content of the theory in order to determine the interactions, it makes sense to define the Weinberg angle Θ via the mass ratio:

$$\cos \Theta := \frac{m_W}{m_Z}. \quad (4.4)$$

This implies

$$\frac{f_{12A}}{f_{12Z}} = \tan \Theta \quad (4.5)$$

up to signs that can be absorbed in redefinitions $A \mapsto -A$ or $Z \mapsto -Z$.

By contrast, in the GWS model the photon field is the gauge field for the unbroken $U(1)$. That determines the Weinberg angle of the corresponding rotation in terms of the two gauge coupling constants for $U(1)$ and $SU(2)$, namely $\tan \Theta := g_1/g_2$. The same relations (4.4) follow, but here the egg and the chicken are interchanged. For the sake of comparison, let us identify the WWZ -coupling $gf_{Z12}F_Z^{\mu\nu}W_{1\mu}W_{2\nu}$ of sQFT with the corresponding term in the $U(1) \times SU(2)$ Yang–Mills self-interaction. This amounts to identifying gf_{12Z} with $-\frac{1}{2}g_2\varepsilon_{123}\cos\Theta$. We may then exploit the freedom of rescaling $g \mapsto sg$, $K \mapsto s^{-1}K$ to identify g with $-g_2$, the $SU(2)$ coupling constant.⁹ Then, the preceding relations imply

$$f_{12Z} = \frac{1}{2}\cos\Theta, \quad f_{12A} = \frac{1}{2}\sin\Theta, \quad gk_a = gKm_a^2 = v^{-1}m_a^2, \\ m_W^2 = (4K^2)^{-1} = \frac{1}{4}g_2^2v^2.$$

This completes the identification between the input parameters of sQFT: g , $m_W = m_Z \cos \Theta$, m_Z , m_H , with those of the gauge theory approach: $g_2, g_1 = g_2 \tan \Theta, \lambda, v$ in the bosonic sector of the electroweak theory.

With the standard notations for the particle–antiparticle pair $W_{\pm} := (W_1 \mp W_2)/\sqrt{2}$, upon using $B = A + \partial\phi$ and dropping all couplings involving the escort field ϕ , there emerges the total electroweak interaction of sQFT, consisting of:

- the cubic self-couplings

$$\frac{1}{2}g \left[\sin \Theta \sum_{abc=1,2,A} \varepsilon_{abc} F_a^{\mu\nu} A_{b\mu} A_{c\nu} + \cos \Theta \sum_{abc=1,2,Z} \varepsilon_{abc} F_a^{\mu\nu} A_{b\mu} A_{c\nu} \right] \\ = \frac{i}{2}g_2 (F_3^{\mu\nu} W_{\nu}^{+} W_{\nu}^{-} + F^{+\mu\nu} W_{\mu}^{-} W_{3\nu} + F^{-\mu\nu} W_{3\mu} W_{\nu}^{+}),$$

⁹This choice is in accord with the conventions in [11], producing [11, Eq. (4.4)] crucially used in that paper.

where we abbreviate $W_3 := A \sin \Theta + Z \cos \Theta$, a combination of fields of different masses [11, 48], so that the unit of electric charge¹⁰ is $e = g_2 \sin \Theta$;

- the quartic self-couplings

$$\begin{aligned} & -\frac{1}{4}g^2 \sum_{abcde=1,2,A,Z} f_{abc}f_{ade}(A_b A_d)(A_c A_e) \\ & = -\frac{1}{2}g_2^2 [(W^+ W^-)^2 - (W^+ W^+)(W^- W^-) \\ & \quad + 2(W^+ W^-)(W_3 W_3) - 2(W^+ W_3)(W^- W_3)]; \end{aligned}$$

- the couplings to the higgs field:

$$g[2k_W W_\mu^+ W^{-\mu} + k_Z Z_\mu Z^\mu] H = v^{-1} [2m_W^2 W_\mu^+ W^{-\mu} + m_Z^2 Z_\mu Z^\mu] H;$$

- and finally, the higgs self-couplings (4.3).

Beyond the displayed terms, there are several couplings involving escort fields of several types, which have no place in gauge theory – much as couplings involving ghost fields have no place in sQFT. Replacing $A(c)$ by the gauge potential A and ignoring the escort couplings (whose role is to ensure that the \mathbb{S} -matrix computed with $A(c)$ is the same as that computed with A), the bosonic interaction Lagrangian of gauge theory would be recovered.¹¹

We recall that the leptonic sector was completed already in [11]. (The only technical difference being the use in [11] of lightlike strings which need no smearing.)

Remark 4.1. There is another solution with the field content A, W_1, W_2, Z , compatible with all constraints imposed by string independence, if one were to admit $f_{12Z} = 0$ while $f_{12A} \neq 0$. In physical terms, the W -bosons would be electrically charged, but they would not couple to the Z -boson. The constraints are solved with $k_{1Z} = k_{2Z} = k_{ZZ} = 0$. Hence, the Z -boson decouples completely, and (4.5) does *not* hold. The resulting admissible theory is a model of a photon and a charged pair of massive W -bosons with a higgs field, tensored with a “massive QED” in which the massive Z -boson replaces the photon.

5. The Boson Sector at Second Order

As mentioned, we reorganize the various bosonic crossings sector by sector (and by type). The analysis is not restricted to the electroweak theory only, but is presented with arbitrary given numbers of massive and massless vector bosons—limited, however, to only one higgs particle. The generalization to more than one higgs is not difficult at second order, cf. [29].

¹⁰Hence $\Theta = 0$ in the limiting case $m_Z = m_W$, so that $e = 0$ and the photon decouples.

¹¹However, a term of type $AAHH$ is missing, due to the use of the non-kinematic propagator with $c_B = -1$. The coupling would be present with $c_B = 0$ via L_2^* , see Appendix B.2.

5.1. The Yang–Mills–Like Sector

We begin with the main *structural* result at second order: the Jacobi identity for the completely skewsymmetric coefficients f_{abc} in (2.22) is a necessary condition for string independence at second order. It follows that f_{abc} are actually the structure constants of a reductive Lie algebra of compact type (i.e. with negative semidefinite Killing form).

This result follows from the resolution of the obstructions of types $uFAA$ and $\partial uAAA$ in the higgs-free sector. Only the crossings of $Q_{1,\text{self}}^1$ with $L_{1,\text{self}}^1$ can produce these types. The case of only massless vector bosons (like QED) was exhaustively examined in [14] and [49], following [15]. One finds, as well, a quartic induced interaction L_2^1 of type $AAAA$, familiar from gauge theory. Interestingly enough, this standard outcome remains valid when massive vector bosons are present; indeed, the proof is essentially the same as in the massless case.

Let us first look, then, at the types $uFAA$ and $\partial uAAA$ in the higgs-free sector.

Proposition 5.1. *String independence requires that the coefficients f_{abc} of the Yang–Mills type cubic self-coupling (2.22) satisfy the Jacobi identity. Thus, they are the structure constants of a reductive Lie algebra of compact type, that is, a direct sum of abelian and simple compact Lie algebras.*

Proposition 5.2. *String independence determines the quartic self-coupling to be of the form*

$$L_2^1 = -2(1 + c_F) \sum_{abcde} f_{abc} f_{ade} (A_b A_d) (A_c A_e). \quad (5.1)$$

Proof. We prove both propositions by the same analysis. (The notation L_2^1 anticipates that the induced interaction L_2 is a sum of several pieces.)

Step 1: the Jacobi identity. Complete skewsymmetry of $\{f_{abc}\}$ is proved in Proposition 2.2. To establish the Jacobi identity, we study the obstruction to string independence arising from the crossing $Q_{1,\text{self}}^1 \boxtimes L_{1,\text{self}}^1 = 2f_{def} F_d^{\mu\nu} A_{f\nu} u_e \boxtimes f_{abc} F_a^{\prime\alpha\beta} A'_{b\alpha} A'_{c\beta}$. It contains terms of type $uFAA$ and $\partial uAAA$ that cannot arise from other crossings and must therefore be separately resolvable. In view of Lemma B.2, we may drop the contribution with a string delta. The crossing at hand involves three of the obstructions in Table 2:

$$\begin{aligned} \mathcal{O}_\mu(F^{\mu\nu}, F^{\prime\alpha\beta}) &= -i(1 + c_F)(\eta^{\alpha\nu} \partial^\beta - \eta^{\beta\nu} \partial^\alpha) \delta_{xx'}, \\ \mathcal{O}_\mu(F^{\mu\nu}, A'_\alpha) &\stackrel{\text{mod } I^\delta}{=} -i\delta_\alpha^\nu \delta_{xx'}, \\ \mathcal{O}_{[\mu}(A_{\nu]}, F^{\prime\alpha\beta}) &= -ic_F(\delta_\mu^\alpha \delta_\nu^\beta - \delta_\mu^\beta \delta_\nu^\alpha) \delta_{xx'}. \end{aligned} \quad (5.2)$$

When pairing $F_d^{\mu\nu} A_{f\nu}$ with $A'_{b\alpha}$, on writing $2F_d^{\mu\nu} A_{f\nu} = F_d^{\mu\nu} A_{f\nu} - F_d^{\lambda\mu} A_{f\lambda}$, the relevant obstruction is $\mathcal{O}_\mu(A_\nu, A'_\alpha) - \mathcal{O}_\nu(A_\mu, A'_\alpha) = \mathcal{O}_{[\mu}(A_{\nu]}, A'_\alpha) = 0$; so

we omit this trivial pairing. Now observe that

$$\begin{aligned}
 Q_{1,\text{self}}^1 \boxtimes L_{1,\text{self}}^1 &= \mathcal{O}_\mu(2f_{def}F_d^{\mu\nu}A_{f\nu}u_e; f_{abc}F_a^{\prime\alpha\beta}A'_{b\alpha}A'_{c\beta}) \\
 &= 2f_{abc}u_e[f_{aef}A_{f\nu}\mathcal{O}_\mu(F_d^{\mu\nu}, F_a^{\prime\alpha\beta})A'_{b\alpha}A'_{c\beta} \\
 &\quad + f_{bef}A_{f\nu}\mathcal{O}_\mu(F_d^{\mu\nu}, A'_{b\alpha})F_a^{\prime\alpha\beta}A'_{c\beta} + f_{cef}A_{f\nu}\mathcal{O}_\mu(F_d^{\mu\nu}, A'_{c\beta})F_a^{\prime\alpha\beta}A'_{b\alpha} \\
 &\quad + \frac{1}{2}f_{dea}F_d^{\mu\nu}\mathcal{O}_{[\mu}(A_{f\nu}), F_a^{\prime\alpha\beta}]A'_{b\alpha}A'_{c\beta}] \\
 &\stackrel{\text{mod } I^\delta}{=} 2if_{abc}u_e[-(1+c_F)f_{aed}A_{d\nu}(A'_b{}^\nu A'_{c\beta}\partial^\beta\delta_{xx'} - A'_{b\alpha}A'_c{}^\nu\partial^\alpha\delta_{xx'}) \\
 &\quad - f_{bed}A_{d\nu}F_a^{\prime\nu\beta}A'_{c\beta}\delta_{xx'} - f_{ced}A_{d\nu}F_a^{\prime\alpha\nu}A'_{b\alpha}\delta_{xx'} - c_F f_{dea}F_d^{\mu\nu}A'_{b\mu}A'_{c\nu}\delta_{xx'}].
 \end{aligned} \tag{5.3}$$

The first line on the right-hand side of (5.3) consists of two identical terms, summing to:

$$\begin{aligned}
 &4i(1+c_F)f_{abc}f_{aed}u_e(A'_c A_d)A'_{b\alpha}\partial^\alpha\delta_{xx'} \\
 &= 4i(1+c_F)f_{abc}f_{aed}\{u_e A'_{b\alpha}(A'_c A_d)\delta_{xx'}\} - \partial^\alpha u_e A'_{b\alpha}(A'_c A_d)\delta_{xx'} \\
 &\quad - 2i(1+c_F)f_{abc}f_{aed}u_e F_d^{\alpha\beta}A'_{b\alpha}A'_{c\beta}\delta_{xx'}.
 \end{aligned} \tag{5.4}$$

To this expression, we add the last line of (5.3).

To achieve string independence, the final result of (5.3) must be the sum of a total derivative and the string variation of an induced quartic interaction [18, 20]. The first term on the right-hand side of (5.4) is a total derivative, and the second one is a string variation. Indeed:

$$\begin{aligned}
 &f_{abc}f_{aed}\underline{\delta}[(A_e A_b)(A_c A_d)] \\
 &= f_{abc}f_{aed}\underline{\delta}[A_e^\alpha]A_{b\alpha}(A_c A_d) + f_{aed}f_{abc}\underline{\delta}[A_b^\alpha]A_{e\alpha}(A_d A_c) \\
 &\quad + f_{ade}f_{acb}\underline{\delta}[A_c^\alpha]A_{d\alpha}(A_e A_b) + f_{acb}f_{ade}\underline{\delta}[A_d^\alpha]A_{c\alpha}(A_b A_e) \\
 &= 4f_{abc}f_{aed}\underline{\delta}[A_e^\alpha]A_{b\alpha}(A_c A_d),
 \end{aligned} \tag{5.5}$$

using symmetry $(b, c) \leftrightarrow (e, d)$, and skewsymmetry $(b \leftrightarrow c)$ and $(e \leftrightarrow d)$ to verify that the four summands are all equal. With $\underline{\delta}[A_e^\alpha] = \partial^\alpha u_e$, this is indeed the second term in (5.4) without the factor $-i(1+c_F)\delta_{xx'}$.

It remains to consider the sum of the last term in (5.4) and the last line of (5.3). Notice the cancellation of the c_F -parts. Dropping the primes in the presence of $\delta_{xx'}$, one gets:

$$\begin{aligned}
 &-2if_{abc}u_e(f_{bed}F_a^{\nu\beta}A_{d\nu}A_{c\beta} + f_{ced}F_a^{\alpha\nu}A_{b\alpha}A_{d\nu} + f_{aed}F_d^{\alpha\beta}A_{b\alpha}A_{c\beta})\delta_{xx'} \\
 &= -2iu_e(f_{bca}f_{bed}F_a^{\alpha\beta}A_{d\alpha}A_{c\beta} + f_{cab}f_{ced}F_a^{\alpha\beta}A_{b\alpha}A_{d\beta} \\
 &\quad + f_{abc}f_{aed}F_d^{\alpha\beta}A_{b\alpha}A_{c\beta})\delta_{xx'} \\
 &= -2iu_e(f_{acd}f_{aeb}F_d^{\alpha\beta}A_{b\alpha}A_{c\beta} + f_{adb}f_{aec}F_d^{\alpha\beta}A_{b\alpha}A_{c\beta} \\
 &\quad + f_{abc}f_{aed}F_d^{\alpha\beta}A_{b\alpha}A_{c\beta})\delta_{xx'} \\
 &= 2i(f_{abe}f_{acd} + f_{abd}f_{aec} + f_{abc}f_{ade})u_e F_d^{\alpha\beta}A_{b\alpha}A_{c\beta}\delta_{xx'}.
 \end{aligned} \tag{5.6}$$

Since this expression is neither a derivative nor a string variation, it must vanish in order to achieve string independence. Therefore, the coefficient in

parentheses must vanish—yielding precisely the Jacobi identity for the coefficients f_{abc} .

Step 2: the Lie algebra. The formula $[\xi_a, \xi_b] := \sum_c f_{abc} \xi_c$ defines a Lie algebra \mathfrak{g} . Because the f_{abc} are completely skewsymmetric, the scalar product $(\xi_a, \xi_b) := \delta_{ab}$ is invariant under the adjoint representation: $([\xi, \xi'], \xi'') + (\xi', [\xi, \xi'']) = 0$ for $\xi, \xi', \xi'' \in \mathfrak{g}$. Because this scalar product is nondegenerate, the adjoint representation is completely reducible; i.e. \mathfrak{g} is reductive. And since the f_{abc} are real, the adjoint representers $\text{ad}(\xi_b)_{ac} = f_{abc}$ are skewsymmetric, the adjoint Casimir operator $\sum_a \text{ad}(\xi_a)^2$ is negative semidefinite; thus, \mathfrak{g} is of compact type. Proposition 5.1 is proved.

Step 3: the induced interaction. In order to determine the induced interaction L_2 such that $L_{\text{int}} = gL_1 + \frac{1}{2}g^2L_2$, one must now add to the string variation term in the obstruction (5.3) the (identical) one from the symmetrized crossing $x \leftrightarrow x'$ and equate the result with $i\delta(L_2)\delta_{xx'}$, see formula (3.6). Thus, $-2(1 + c_F)$ multiplied by the result of (5.5) produces $\delta(L_2^1)$. Adjusting the labelling of indices, this yields (5.1), concluding the proof of Proposition 5.2. \square

Note that the derivative terms discarded along the way are still needed, since they contribute to $Q_2|_\delta$ —see formula (5.22a).

Remark 5.3. By the key equation (2.23), the “massless generators” ξ^a (with $m_a = 0$) generate a Lie subalgebra \mathfrak{h} , which may be nonabelian, as for instance in QCD. Also by Eq. (2.23), their adjoint action on the massive generators assembles the massive fields into representations of \mathfrak{h} with constant mass.

Remark 5.4. Expression L_2^1 in Proposition 5.2 coincides with the quartic self-interaction typical in gauge theory, with the Feynman gauge potentials replaced by the string-localized potentials, but with an extra factor $(1 + c_F)$. After the cancellations in the above proof, this is the only place where c_F appears at second order, and—see the proof of Proposition 6.4—it remains undetermined by string independence also at third order. This situation is reminiscent of scalar QED, where the renormalization of the propagator of two derivatives of the scalar field admits a free parameter, which then changes the coefficient of the quartic interaction, familiar from the gauge theory treatment. We are therefore free to set $c_F = 0$. See also footnote 11 and Appendix B.2 for another instance of the freedom to absorb couplings in renormalizations of propagators.

5.2. Resolution of Obstructions in the Sectors $[a][b][c][d]$

From here on, we choose the renormalization constants $c_H = -1$, $c_B = -1$; see Appendix B.2. We proceed sector by sector.

The following consistency condition (5.7) provides a link between the self-coupling constants f_{abc} and the higgs coupling coefficients k_{ab} and puts constraints on admissible mass patterns.

Proposition 5.5. *String independence in the higgs-free sectors with field content $[a][b][c][d]$ requires the self-couplings and higgs couplings to satisfy the relation*

$$\begin{aligned} \sum_e \left[\theta(m_e) \left(f_{cae} f_{ebd} \frac{m_{cae}^2 m_{ebd}^2}{m_e^2} - [a \leftrightarrow b] \right) - 2f_{eab} f_{ced} m_{ced}^2 \right] \\ = k_{ac} k_{bd} - k_{ad} k_{bc} \end{aligned} \quad (5.7)$$

for all a, b and all massive c, d .

Then, all obstructions in the sectors with field content $[a][b][c][d]$ can be resolved. As well as L_2^1 in Proposition 5.2, there is a second higgs-free induced interaction:

$$L_2^2 = -\frac{1}{4} \mu_H^2 \left(\sum_{ab} k_{ab} \phi_a \phi_b \right)^2. \quad (5.8)$$

Remark 5.6. (i) The left-hand side of (5.7) is skewsymmetric under both $a \leftrightarrow b$ and $c \leftrightarrow d$. Moreover, if c or d is massless, then $f_{cae} m_{cae}^2 = f_{ebd} m_{ebd}^2 = f_{ced} m_{ced}^2 = 0$ from (2.23)—consult Proposition 2.2—and in this case the expression identically vanishes.

(ii) Equation (5.7) is equivalent *mutatis mutandis* to [29, Eq. (20)], which was derived in a different setting in terms of Stückelberg fields and BRST invariance, rather than string independence.

(iii) A useful special case of (5.7) is $d = a$, which gives the “sum rule”:

$$\begin{aligned} \sum_e f_{aeb} f_{aec} \left[m_{aeb}^2 + m_{aec}^2 + \theta(m_e) \frac{(m_a^2 - m_b^2)(m_a^2 - m_c^2) - m_e^4}{m_e^2} \right] \\ = k_{aa} k_{bc} - k_{ab} k_{ac}. \end{aligned} \quad (5.9)$$

This sum rule is trivially satisfied for massless a, b or c .

(iv) The sum rule shows that the higgs couplings are *indispensable* for string independence in theories with nonabelian massive vector bosons. To wit, one can always find labels a, b, c for which the left-hand side is nonzero. For instance, when all masses are equal, the sum over all b on the left-hand side gives m^2 times the quadratic Casimir operator in the adjoint representation, which is not zero. Hence, the matrix $[k_{ab}]$ cannot vanish.

Proof of Proposition 5.5. By Lemma B.2, we may discard all obstructions with string deltas. Tables 1 and 2 show that all obstructions without string deltas in the sectors $[a][b][c][d]$ arise from

$$Q_{1,\text{higgs}} \boxtimes L_{1,\text{higgs}}' + Q_{1,\text{self}} \boxtimes L_{1,\text{self}}' + [x \leftrightarrow x']. \quad (5.10)$$

Step 1. Higgs-free obstruction arising from higgs–higgs crossings: of the first summand in Eq. (5.10) obviously we need only the terms arising through pairings of the higgs fields. We use Table 1. With $c_H = -1$, this gives

$$\begin{aligned} \left[\frac{\partial Q_{1,\text{higgs}}}{\partial(\partial H)} \frac{\partial L_{1,\text{higgs}}'}{\partial H} - \frac{\partial Q_{1,\text{higgs}}}{\partial H} \frac{\partial L_{1,\text{higgs}}'}{\partial(\partial H)} \right] i\delta_{xx'} + [x \leftrightarrow x'] \\ = 2 \sum_{abcd} [k_{ad} k_{bc} u_a \phi_d ((A_b B_c) - \frac{1}{2} m_H^2 \phi_b \phi_c) - k_{ac} k_{bd} u_a (B_c A_b) \phi_d] i\delta_{xx'}, \end{aligned}$$

so that one obtains

$$Q_{1,\text{higgs}} \boxtimes L'_{1,\text{higgs}} + [x \leftrightarrow x'] = \underline{\delta}(L_2^2) i\delta_{xx'} + \mathcal{O}_{\text{higgs}} i\delta_{xx'}, \quad (5.11)$$

with $L_2^2 = -\frac{1}{4}m_H^2 \sum_{abcd} k_{ab}k_{cd} \phi_a \phi_b \phi_c \phi_d$, and the non-resolvable obstruction is of the form

$$\mathcal{O}_{\text{higgs}} := 2 \sum_{abcd} (k_{ad}k_{bc} - k_{ac}k_{bd}) u_a (A_b B_c) \phi_d. \quad (5.12)$$

Notice the manifest skewsymmetry of the coefficients in $a \leftrightarrow b$ and in $c \leftrightarrow d$.

Step 2. Higgs-free obstruction arising from self–self-crossings: we compute the second summand in Eq. (5.10), using Table 2. In computations of this type, it is important to keep Eq. (2.23) in mind. The vanishing $m_{abe}^2 = m_{eba}^2 = 0$ when e is massless eliminates the corresponding coupling term, hence it always overrules the occurrence of factors m_e^{-2} coming from pairings of such terms, as in Eq. (5.13). We multiply such terms by $\theta_e \equiv \theta(m_e)$.

One may drop all two-point obstructions involving string deltas, again by Lemma B.2. The crossing $FuA \boxtimes F'A'A'$ has been dealt with in Proposition 5.2, and there is a null crossing: $Bu\phi \boxtimes F'A'A' = 0$. We compute the remaining crossings in $Q_{1,\text{self}}^\mu \boxtimes L_{1,\text{self}}^{2'}$. With $c_B = -1$, one gets:

$$\begin{aligned} & \sum_e \left[\frac{\partial Q_{1,\text{self}}}{\partial \phi_e} m_e^{-2} \frac{\partial L_{1,\text{self}}^{2'}}{\partial B_e} - \frac{\partial Q_{1,\text{self}}}{\partial B_e} m_e^{-2} \frac{\partial L_{1,\text{self}}^{2'}}{\partial \phi_e} - \frac{\partial Q_{1,\text{self}}}{\partial F_e} \frac{\partial L_{1,\text{self}}^{2'}}{\partial A_e} \right] i\delta_{xx'} \\ & + [x \leftrightarrow x'] \\ & = 2 \left[\sum_{abcde} \theta_c \theta_d \theta_e [f_{cae} m_{cae}^2 f_{ebd} m_{ebd}^2 m_e^{-2} u_a (B_c A_b) \phi_d \right. \\ & \quad \left. - f_{ead} m_{ead}^2 f_{cbe} m_{cbe}^2 m_e^{-2} u_a \phi_d (B_c A_b) \right] \\ & - 2 \sum_{abcde} \theta_c \theta_d f_{eab} f_{ced} m_{ced}^2 u_a (A_b B_c) \phi_d \Big] i\delta_{xx'} \\ & = 2 \sum_{abcd} \sum_e \left[\theta_e \left[f_{cae} f_{ebd} \frac{m_{cae}^2 m_{ebd}^2}{m_e^2} - [a \leftrightarrow b] \right] - 2 f_{eab} f_{ced} m_{ced}^2 \right] \\ & \quad u_a (A_b B_c) \phi_d i\delta_{xx'}. \end{aligned} \quad (5.13)$$

We notice that this term is of the same type $uAB\phi$ as (5.12), with the same skewsymmetry of the coefficients in $a \leftrightarrow b$ and $c \leftrightarrow d$. In view of (5.7), it cancels (5.12). \square

5.3. Resolution of Obstructions in Sectors $[a][b][c][H]$

String independence at second order admits an induced interaction of type $\phi\phi\phi H$, that will be eliminated later at third order by means of Proposition 6.2.

Proposition 5.7. *The obstruction in the sectors with field content $[a][b][c][H]$ equals:*

$$\begin{aligned} & Q_{1,\text{self}} \boxtimes L'_{1,\text{higgs}} + Q_{1,\text{higgs}} \boxtimes L'_{1,\text{self}} + [x \leftrightarrow x'] \\ &= \sum_{abc} [C_{abc} m_H^2 \phi_a u_b \phi_c H - 2(C_{abc} - C_{acb})(B_a u_b A_c H + \phi_a u_b A_c \partial H)] i\delta_{xx'}, \end{aligned} \quad (5.14)$$

where

$$C_{abc} := \sum_e \left[\frac{k_{ae}}{m_e^2} f_{ebc} m_{ebc}^2 + \frac{k_{ce}}{m_e^2} f_{eba} m_{eba}^2 \right] = C_{cba}. \quad (5.15)$$

String independence requires that C_{abc} be completely symmetric in a, b, c :

$$C_{abc} = C_{acb} = C_{bac}. \quad (5.16)$$

Then the obstruction is resolved by the induced interaction:

$$L_2^5 := \frac{1}{3} \sum_{abc} C_{abc} m_H^2 \phi_a \phi_b \phi_c H. \quad (5.17)$$

Proof. Refer again to the (L_1, Q_1) pair listed in (2.31). The relevant obstructions in sectors $[a][b][c][H]$ arise only in the crossings:

$$\begin{aligned} & \sum_{abc} f_{abc} [2F_a^{\mu\nu} u_b A_{c\nu} + m_{abc}^2 B_a^\mu u_b \phi_c] \boxtimes \\ & \quad \sum_{de} k_{de} [A'_d B'_e H' + A'_d \phi'_e \partial' H' - \frac{1}{2} m_H^2 \phi'_d \phi'_e H'] \\ & + \sum_{de} k_{de} [u_d B_e^\mu H + u_d \phi_e \partial^\mu H] \boxtimes \\ & \quad \sum_{abc} f_{abc} [F_a^{\mu\nu} A'_{b\mu} A'_{c\nu} + m_{abc}^2 B'_{a\nu} A'_b{}^\nu \phi'_c] + [x \leftrightarrow x']. \end{aligned} \quad (5.18)$$

By inspection of Table 2, again with $c_B = -1$, one sees that the only pairings that contribute are $\mathcal{O}_\mu(F, A')$, $\mathcal{O}_\mu(B, \phi')$ and $\mathcal{O}_\mu(\phi, B')$. The resulting structures have only the types $u\phi\phi H$, $uBAH$ and $u\phi A\partial H$.

We compute the coefficients of monomials $\phi_a u_b \phi_c H$, emerging from (5.18) (relabelling the respective contracted index as e , and adjusting the remaining indices):

$$\begin{aligned} & \sum_{abc} f_{abc} m_{abc}^2 B_a u_b \phi_c \boxtimes - \frac{1}{2} m_H^2 \sum_{de} k_{de} \phi'_d \phi'_e H' \\ &= m_H^2 \sum_{ebcd} f_{ebc} m_{ebc}^2 u_b \phi_c m_e^{-2} k_{de} \phi_d H i\delta_{xx'} \equiv \frac{1}{2} m_H^2 \sum_{abc} C_{abc} \phi_a u_b \phi_c H i\delta_{xx'}. \end{aligned}$$

The monomials $B_a u_b A_c H$ and $\phi_a u_b A_c \partial H$ each arise from three kinds of crossings: the first from $FuA \boxtimes A'B'H'$, $uBH \boxtimes B'A'\phi'$ and $Bu\phi \boxtimes A'B'H'$ and

the second from $FuA \boxtimes A'\phi'\partial'H'$, $Bu\phi \boxtimes A'\phi'\partial'H'$ and $u\phi\partial H \boxtimes B'A'\phi'$. These yield (with string deltas dropped):

$$\begin{aligned}
 & - \sum_{abc} \left[\sum_e (2f_{ebc}k_{ae} + f_{ace}m_{ace}^2 k_{be}m_e^{-2} - k_{ce}m_e^{-2} f_{abe}m_{abe}^2) B_a u_b A_c H \right. \\
 & \quad \left. + \sum_e (2f_{ebc}k_{ae} + f_{eba}m_{eba}^2 k_{ce}m_e^{-2} - k_{be}m_e^{-2} f_{eca}m_{eca}^2) \phi_a u_b A_c \partial H \right] i\delta_{xx'}.
 \end{aligned}$$

Because $f_{a\bullet e}m_{a\bullet e}^2 = -f_{e\bullet a}m_{e\bullet a}^2$, the two sums over e are actually identical. Moreover, replacing the 2 in the first summands by $2 = (m_{ebc}^2 + m_{ecb}^2)m_e^{-2}$, one easily sees that the sums over e equal $C_{abc} - C_{acb}$. This proves (5.14).

Inspection of the three field structures in (5.14) reveals that only the symmetrized sum over the first of them is resolvable, namely it is a string variation proportional to $\delta[\phi_a\phi_b\phi_c H]$. This proves condition (5.16) for string independence. Formula (5.17) follows at once. \square

We shall soon find, in Proposition 6.2, that $[k_{ab}]$ if nondegenerate¹² must be diagonal, and $C_{abc} = 0$. That has the following important consequence.

Proposition 5.8. *Assume that $k_{ab} = k_a \delta_{ab}$ is diagonal. Then $C_{abc} = 0$ implies*

$$\frac{k_a}{m_a^2} = \frac{k_c}{m_c^2} =: K \tag{5.19}$$

whenever there is a field c such that $f_{abc} \neq 0$.

Proof. With k_{ab} diagonal, (5.15) reduces to

$$C_{abc} = \frac{k_a}{m_a^2} m_{abc}^2 f_{abc} + [a \leftrightarrow c] = \left(\frac{k_a}{m_a^2} - \frac{k_c}{m_c^2} \right) m_{abc}^2 f_{abc}.$$

Thus, Eq. (5.19) follows unless $m_a^2 + m_c^2 = m_b^2$ by chance. But if so, the relations $m_a^2 + m_b^2 = m_c^2$ and $m_b^2 + m_c^2 = m_a^2$ cannot both hold, whereby $k_a m_a^{-2} = k_b m_b^{-2}$ or $k_b m_b^{-2} = k_c m_c^{-2}$, which at any rate implies that $k_i = K/m_i^2$. \square

Remark 5.9. (i) The quotient K defined in (5.19) is only constant over fields linked by the structure constants of the Lie algebra \mathfrak{g} . If that Lie algebra is the direct sum of two or more simple nonabelian Lie algebras, these may have independent constants K . This possibility will be excluded by Proposition 6.2(ii).

(ii) It is desirable to show that (5.16), in combination with other conditions from string independence such as (5.9), has only diagonal solutions $k_{ab} = k_a \delta_{ab}$, so that Proposition 5.8 applies. Unfortunately, we are at present unable to do so at second order, although Propositions 5.7 and 5.8, possibly combined with other constraints like (5.9), point to such a result. It does hold for the electroweak theory—consult Sect. 4.

¹²For an example with degenerate $[k_{ab}]$, recall Remark 4.1. Still, $[k_{ab}]$ is diagonal in that example, too. On the other hand, the analysis in Prop. 6.2 shows that in general $[m_a m_b k_{ab}]$ can only be expected to be a multiple of a projection matrix. The analogous argument in the local approach [29,30] in favour of diagonality is flawed.

5.4. Resolution of Obstructions in Sectors $[a][b][H][H]$

We continue with the determination of the induced interaction L_2 at second order. Beyond the pieces L_2^1 in (5.1) and L_2^2 in (5.8), as well as L_2^5 in (5.17) that is to be discarded later, there is another piece L_2^3 that we identify now. This piece will be used at third order to determine the higgs self-couplings.

Proposition 5.10. *String independence in the sectors of the form $[a][b][H][H]$ requires the induced interaction:*

$$L_2^3 = \sum_{ab} (3\ell k_{ab} + m_H^2 k_{ab}^*) \phi_a \phi_b H^2, \quad (5.20)$$

where $k_{ab}^* := \sum_c k_{ac} m_c^{-2} k_{cb}$.

Proof. The obstruction in the sectors of the form $[a][b][H][H]$ arises from the terms $Q_{1,\text{higgs}} \boxtimes L'_{1,\text{higgs}}$ in (2.31). We compute:

$$\begin{aligned} \sum_{ab} k_{ab} u_a \phi_b \mathcal{O}_\mu(\partial^\mu H, H') 3\ell H'^2 &= 3\ell \sum_{ab} k_{ab} u_a \phi_b H^2 i\delta_{xx'}, \\ -m_H^2 \sum_{abc} k_{ac} u_a H \mathcal{O}_\mu(B_c^\mu, \phi'_c) k_{cb} \phi'_b H' &= m_H^2 \sum_{abc} k_{ac} m_c^{-2} k_{cb} u_a \phi_b H^2 i\delta_{xx'}. \end{aligned}$$

These are the only nonzero crossings (always assuming that $c_B = -1$) without string deltas that have not yet been accounted for. Adding $[x \leftrightarrow x']$, one gets a string variation:

$$\mathcal{O}^{(2)}|_{[a][b][H][H]} = \underline{\delta}[\sum_{ab} (3\ell k_{ab} + m_H^2 k_{ab}^*) \phi_a \phi_b H^2] i\delta_{xx'},$$

giving rise to the induced interaction term (5.20). \square

5.5. Wrapping Up: The Induced Interactions at Second Order

We are still free to add to L_2 a quartic higgs self-coupling $L_2^4 := \ell' H^4$. This piece will be needed at third order, where the coefficient ℓ of H^3 in (2.31) and the new ℓ' of H^4 are determined in Proposition 6.4.

We now collect the complete second-order interaction:

$$\begin{aligned} L_2 &\equiv L_2^1 + L_2^2 + L_2^3 + L_2^4 + L_2^5 \\ &= -2(1 + c_F) \sum_{abcde} f_{abe} f_{cde} (A_a A_c)(A_b A_d) - \frac{1}{4} m_H^2 \left(\sum_{ab} k_{ab} \phi_a \phi_b \right)^2 \\ &\quad + \sum_{ab} (3\ell k_{ab} + m_H^2 k_{ab}^*) \phi_a \phi_b H^2 + \ell' H^4 + \frac{1}{3} \sum_{abc} C_{abc} m_H^2 \phi_a \phi_b \phi_c H. \end{aligned} \quad (5.21)$$

Remember that these terms were computed under the assumption $c_H = c_B = -1$. Additional terms appear for general values of c_H and c_B ; those are outlined in Appendix B.2.

Remark 5.11. We shall see in Proposition 6.2 that L_2^5 leads to non-resolvable obstructions at third order. Therefore $C_{abc} = 0$ is forced, and then L_2^5 is discarded. (For good measure, we already saw in Sect. 4 that $C_{abc} = 0$ in the electroweak theory.)

Next, $Q_2|_{I\delta} := 2 \sum_a u_{2a} \partial L_1 / \partial A_a$ is given by Lemma B.2, while $Q_2|_\delta$ is retrieved from the total derivative in (5.4):

$$Q_2^\mu|_\delta(x, x') = (1 + c_F) \sum_{abcde} 8f_{abe} f_{cde} u_a A_c^\mu(A_b A_d) i\delta_{xx'}, \quad (5.22a)$$

which equals

$$Q_2^\mu|_\delta = \sum_a u_a \frac{\partial L_2}{\partial A_{a\mu}} i\delta_{xx'}. \quad (5.22b)$$

Notice that no other total derivatives appear in the computations of L_2^2 , L_2^3 or L_2^5 .

6. The Boson Sector at Third Order

At third order, we see that the process of inducing higher interactions terminates, and the key parameters of the previous induced interactions are fixed. We retain the values $c_H = c_B = -1$, see Remark B.4.

6.1. Cancellation of Obstructions at Third Order

We are now led to compute and resolve the obstructions [18, 43]:

$$\begin{aligned} \mathcal{O}^{(3)}(x, x', x'') &:= -3i\mathfrak{S}_{xx'x''}(\mathcal{O}(Q_2; L_1'') + \mathcal{O}(Q_1; L_2'') \delta_{x'x''}) \\ &\stackrel{!}{=} \underline{\delta}(L_3) \delta_{xx'x''} - \mathfrak{S}_{xx'x''} \partial_\mu Q_3^\mu(x, x', x''). \end{aligned} \quad (6.1)$$

We give a quick schematic derivation of (6.1). At third order, $\underline{\delta}(S)$ is given by $(ig)^3/6$ times

$$\iiint (3T \underline{\delta}(L_1) L_1' L_1'' - 3i\delta_{xx'}(T \underline{\delta}(L_1) L_2'' + T \underline{\delta}(L_2) L_1'') - \delta_{xx'x''} \underline{\delta}(L_3)),$$

with some dummy delta functions inserted to represent it as a triple integral. In order to express it in terms of obstruction maps, we subtract

$$\iiint (3\partial T Q_1 L_1' L_1'' - 3i(\delta_{xx'} \partial T Q_1 L_2'' + \partial T Q_2 L_1'')),$$

and use $\underline{\delta}(L_1) = \partial Q_1$ and $\delta_{xx'} \underline{\delta}(L_2) = \partial Q_2 - i\mathcal{O}^{(2)}$. This yields

$$\iiint (3[T, \partial]Q_1 L_1' L_1'' - 3i\delta_{xx'}[T, \partial]Q_1 L_2'' - 3i[T, \partial]Q_2 L_1'' - 3T \mathcal{O}^{(2)} L_1'' - \delta_{xx'x''} \underline{\delta}(L_3)). \quad (6.2)$$

Here, the fourth term cancels the first by virtue of the Master Ward Identity [51] adapted to sQFT [18, 43, 50],

$$[T, \partial]Q_1 L_1' L_1'' = T \mathcal{O}(Q_1; L_1') L_1'' + T \mathcal{O}(Q_1; L_1'') L_1' = T \mathcal{O}^{(2)}(x, x') L_1''$$

after symmetrization. The formula (6.1) is the statement that the symmetrized integrand vanishes up to another total derivative.

It suffices to resolve only the parts of (6.1) without string deltas, as already discussed.¹³

Remark 6.1. By a power-counting argument (obstruction maps preserve the total scaling dimension), we know that L_3 must have scaling dimension 4, which is impossible for Wick polynomials of degree 5.¹⁴ However, because u has scaling dimension 0, $\mathcal{O}^{(3)}|_\delta$ may contain terms of types $uAAAA$ or $u\phi^k H^{4-k}$. These would not be resolvable, and we must show that such terms do not arise.

Thus, we must resolve

$$\mathfrak{S}_{xx'x''}(Q_2|_\delta \boxtimes L_1'' + Q_1 \boxtimes (L_2^{1'} + L_2^{2'} + L_2^{3'} + L_2^{4'} + L_2^{5'}) \delta_{x'x''})|_\delta. \quad (6.3)$$

We begin with the higgs-odd sectors, which, among other things, determine ℓ and ℓ' .

Proposition 6.2. *In the higgs-odd sectors $[a][b][H][H][H]$ and $[a][b][c][d][H]$, string independence demands that the following conditions be met.*

- (i) *The symmetric tensor C_{abc} in Proposition 5.7 vanishes, which entails $L_2^5 = 0$.*
- (ii) *The symmetric matrix of higgs couplings $[k_{ab}]$ is of the form*

$$k_{ab} = Km_a m_b P_{ab}, \quad (6.4)$$

where the matrix P is a projector: $P^2 = P = P^t$, and K is real. If $[k_{ab}]$ is nondegenerate, then $P = \mathbf{1}$, hence $[k_{ab}]$ is diagonal and satisfies (5.19), namely, $k_{ab} = Km_a^2 \delta_{ab}$.

- (iii) *The values of the cubic and quartic higgs self-couplings are determined to be:*

$$\ell = -\frac{1}{2}Km_H^2, \quad \ell' = -\frac{1}{4}K^2m_H^2. \quad (6.5)$$

With these conditions satisfied, all obstructions in these sectors cancel each other. In particular, there are no higgs-odd induced interactions.

Remark 6.3. If (5.19) holds by virtue of condition (ii), then C_{abc} is identically zero: see the proof of Proposition 5.8. Its vanishing by condition (i) imposes no further constraints.

Proof of Proposition 6.2. Notice that L_2 and $Q_2|_\delta$ contain no terms in the sectors $[a][H][H][H]$, since there are no crossings that could produce them at second order. Indeed, $Q_2|_\delta$ is of type $uAAA$ by (5.22a).

Step 1: the constraint in sectors $[a][b][H][H][H]$. The crossings that produce obstructions in these sectors are $Q_{1,\text{higgs}} \boxtimes L_2^{4'}$ with the pairing $\mathcal{O}_\mu(\partial^\mu H, H')$, and $Q_{1,\text{higgs}} \boxtimes L_2^{3'}$ with the pairing $\mathcal{O}_\mu(B^\mu, \phi')$; plus the $[x \leftrightarrow x']$ terms.

¹³ For the string delta parts, there are substantial a priori cancellations. Consult [50, Lemma 4.3].

¹⁴We thank the anonymous referee for this argument.

We compute:

$$\begin{aligned} \sum_{ab} k_{ab} u_a \phi_b \mathcal{O}_\mu(\partial^\mu H, H') 4\ell' H'^3 &= 4\ell' \sum_{ab} k_{ab} u_a \phi_b H^3 i\delta_{xx'}, \\ \sum_{abc} k_{ac} u_a H \mathcal{O}_\mu(B_c^\mu, \phi'_c) (6\ell k_{cb} + 2m_H^2 k_{cb}^*) \phi'_b H'^2 \\ &= -i \sum_{abc} k_{ab} m_c^{-2} (6\ell k_{cb} + 2m_H^2 k_{cb}^*) u_a \phi_b H^3 \delta_{xx'}. \end{aligned}$$

Adding the $[x \leftrightarrow x']$ part, the total obstruction in this sector is:

$$\mathcal{O}^{(2)} \Big|_{[a][b][H][H][H]} = 4 \sum_{ab} (2\ell' k_{ab} - 3\ell k_{ab}^* - m_H^2 k_{ab}^{**}) u_a \phi_b H^3 i\delta_{xx'}, \quad (6.6)$$

where $k_{ab}^* := \sum_c k_{ac} m_c^{-2} k_{cb}$ as before, and

$$k_{ab}^{**} := \sum_c k_{ac} m_c^{-2} k_{cb}^* = \sum_{cd} k_{ac} m_c^{-2} k_{cd} m_d^{-2} k_{db}.$$

Note in passing that the matrices $[k_{ab}^*]$ and $[k_{ab}^{**}]$ are symmetric. The quantity (6.6) is the string variation of terms of the type $\phi\phi HHH$. Because these have dimension 5, they are not power-counting renormalizable. Thus, for the obstruction to be resolvable, its total coefficient must vanish, i.e. $2\ell' k_{ab} - 3\ell k_{ab}^* - m_H^2 k_{ab}^{**} = 0$.

Step 2: constraints in the sectors $[a][b][c][d][H]$. These sectors contain contributions from the crossings $Q_2|_\delta \boxtimes L_{1,\text{higgs}}$, $Q_{1,\text{higgs}} \boxtimes L_2'$, $Q_{1,\text{higgs}} \boxtimes L_2^{3'}$ and $Q_{1,\text{self}} \boxtimes L_2^{5'}$. In the evaluation of $Q_2|_\delta \boxtimes L_{1,\text{higgs}}$, one must use Lemma B.5: the factor $\delta_{xx'}$ included in $Q_2|_\delta(x, x')$ can be pulled out of the \boxtimes operation at the price of a factor $\frac{1}{2}$. Now, this crossing only produces string deltas, see Remark A.1, so we ignore it here.

Once more from Table 2 we get, in turn, with sums over all indices understood:

$$\begin{aligned} k_{ae} u_a B_e^\mu H \boxtimes -\frac{1}{4} m_H^2 k_{rb} k_{cd} \phi'_r \phi'_b \phi'_c \phi'_d &= m_H^2 k_{ab}^* k_{cd} u_a \phi_b \phi_c \phi_d H i\delta_{xx'}, \\ k_{ab} u_a \phi_b \partial^\mu H \boxtimes (3\ell k_{cd} + m_H^2 k_{cd}^*) \phi'_c \phi'_d H'^2 &= (6\ell k_{ab} k_{cd} + 2m_H^2 k_{ab} k_{cd}^*) \\ &\quad u_a \phi_b \phi_c \phi_d H i\delta_{xx'}, \end{aligned} \quad (6.7)$$

as well as

$$f_{cde} m_{cde}^2 B_c u_d \phi_e \boxtimes \frac{1}{3} C_{abr} \phi'_a \phi'_b \phi'_r H' = 2f_{cde} m_{cde}^2 m_c^{-2} C_{abc} u_d \phi_a \phi_b \phi_e H i\delta_{xx'}. \quad (6.8)$$

On the right-hand side of (6.8), we write $f_{cde} m_{cde}^2 m_c^{-2} = f_{cde} + f_{cde} (m_e^2 - m_d^2) m_c^{-2}$. The first summand is skewsymmetric and the second is symmetric under $d \leftrightarrow e$. Since all other contributions of type $u\phi\phi\phi H$ in the above list have coefficients that are symmetric in two pairs of indices, the part $\sum_c f_{cde} C_{abc} u_d \phi_a \phi_b \phi_e H$ cannot be cancelled by any other term. And because it is not a total derivative, every $\sum_c f_{cde} C_{abc}$ must vanish. This forces $C_{abc} = 0$.

Adding $[x \leftrightarrow x']$ to the remaining terms (6.7), one gets

$$\mathcal{O}^{(2)} \Big|_{[a][b][c][d][H]} = 2 \sum_{abcd} (6\ell k_{ab} k_{cd} + m_H^2 k_{ab}^* k_{cd} + 2m_H^2 k_{ab} k_{cd}^*) u_a \phi_b \phi_c \phi_d H i\delta_{xx'}. \quad (6.9)$$

These coefficients must be zero, along with those of (6.6). With the vanishing of (6.6) and (6.9), no obstructions in the higgs-odd sectors remain.

Step 3: fixing the higgs couplings. Consider the symmetric matrix μ with entries $\mu_{ab} := m_a^{-1} k_{ab} m_b^{-1}$ and notice that $(\mu^2)_{ab} = m_a^{-1} k_{ab}^* m_b^{-1}$ and $(\mu^3)_{ab} = m_a^{-1} k_{ab}^{**} m_b^{-1}$. One can then rewrite the vanishing of the respective coefficient matrices in (6.6) and (6.9) as:

$$2\ell'\mu - 3\ell\mu^2 - m_H^2\mu^3 = 0, \quad 6\ell\mu \otimes \mu + m_H^2\mu^2 \otimes \mu + 2m_H^2\mu \otimes \mu^2 = 0. \quad (6.10)$$

Tensoring the first with 2μ from the left, multiplying the second by $1 \otimes \mu$, and adding the results, one is left with:

$$4\ell'\mu \otimes \mu = -m_H^2\mu^2 \otimes \mu^2. \quad (6.11)$$

This is only possible if μ^2 is a multiple of μ , hence μ is a multiple of a projector P , that is to say, $\mu = KP$ for some real number K . With that, (6.11) yields $\ell' = -\frac{1}{4}K^2m_H^2$. Then, with $\mu^2 = K\mu$ because $P^2 = P$, the second equality in (6.10) also gives $\ell = -\frac{1}{2}Km_H^2$.

Now $P^2 = P$ also implies $k_{ab}^* = Kk_{ab}$ and $k_{ab}^{**} = K^2k_{ab}$. And if $[k_{ab}]$ is nondegenerate, then necessarily $P = \mathbf{1}$ and $k_{ab} = m_a\mu_{ab}m_b = Km_a^2\delta_{ab}$ makes $[k_{ab}]$ diagonal; and the relation (5.19) is satisfied (with the same constant K). \square

We arrive at a happy conclusion.

Proposition 6.4. *There are no obstructions at third order in the higgs-even sectors either, and hence no induced interaction: $L_3 = 0$. This is true irrespective of the value of c_F .*

Proof. With the results of Proposition 6.2, inspection of Tables 1 and 2 shows that only a few other crossings may contribute terms without string deltas:

- (i) $Q_{1,\text{self}} \boxtimes L_2' = Q_{1,\text{self}} \boxtimes -\frac{1}{4}m_H^2 (\sum_{ab} k_{ab} \phi'_a \phi'_b)^2$ in sectors $[a][b][c][d][e]$;
- (ii) $Q_{1,\text{self}} \boxtimes L_2^3 = Q_{1,\text{self}} \boxtimes (3\ell + m_H^2 K) (\sum_{ab} k_{ab} \phi'_a \phi'_b) H'^2$ in sectors $[a][b][c][H][H]$;
- (iii) $Q_{1,\text{self}} \boxtimes L_2^1$ in sectors $[a][b][c][d][e]$;
- (iv) $Q_{2|\delta} \boxtimes L_1^1$ in sectors $[a][b][c][d][e]$.

For items (i) and (ii), it is enough to consider the crossing:

$$\begin{aligned} Q_{1,\text{self}} \boxtimes \sum_{ad} k_{ad} \phi'_a \phi'_d &= \sum_{abcde} f_{ebc} m_{ebc}^2 B_e u_b \phi_c \boxtimes k_{ad} \phi'_a \phi'_d \\ &= - \sum_{abce} 2f_{ebc} m_{ebc}^2 m_e^{-2} k_{ae} u_b \phi_c \phi_a i\delta_{xx'} = - \sum_{abc} C_{abc} u_b \phi_a \phi_c i\delta_{xx'} = 0, \end{aligned}$$

by Proposition 6.2(i). Thus, the crossings in (i) and (ii) vanish.

Now we examine item (iii), of the type $FuA + Bu\phi \boxtimes (AAAA)''$, which comes with a factor $(1 + c_F)$. Here we need only the delta part of $\mathcal{O}_\mu(F^{\mu\nu}, A')$ in Table 2. To shorten the expressions, one may write contractions with structure constants symbolically as commutators:

$$\sum_{abc} if_{abc} X_a Y_b Z_c =: \sum_c [X, Y]_c Z_c = \sum_a X_a [Y, Z]_a. \quad (6.12)$$

With this notation, omitting the factor $(1 + c_F)$ in $L_2^{1'}$, we reach:

$$\begin{aligned} Q_{1,\text{self}}^1 &= -2i \sum_a F_a^{\mu\nu} [u, A_\nu]_a, \\ L_2^{1'} &= 2 \sum_a [A'^\kappa, A'^\lambda]_a [A'_\kappa, A'_\lambda]_a = 2 \sum_a A'_a{}^\kappa [A'^\lambda, [A'_\kappa, A'_\lambda]]_a. \end{aligned}$$

This yields

$$\begin{aligned} Q_{1,\text{self}}^1 \boxtimes L_2^{1'} &\stackrel{\text{mod } I\delta}{=} 4 \sum_a [u, A_\nu]_a [A'^\lambda, [A'_\kappa, A'_\lambda]]_a \eta^{\nu\kappa} \delta_{xx'} \\ &= 4 \sum_a [[u, A^\kappa], A^\lambda]_a [A_\kappa, A_\lambda]_a \delta_{xx'}. \end{aligned}$$

The Jacobi identity $[[u, A^\kappa], A^\lambda] - [[u, A^\lambda], A^\kappa] = [u, [A^\kappa, A^\lambda]]$ reduces this to

$$2 \sum_a [u, [A^\kappa, A^\lambda]]_a [A_\kappa, A_\lambda]_a \delta_{xx'} = 2 \sum_a u_a [[A^\kappa, A^\lambda], [A_\kappa, A_\lambda]]_a \delta_{xx'} = 0.$$

For item (iv), of type $uAAA \boxtimes (FAA + BA\phi)''$, which also comes with a factor $(1 + c_F)$, we require two-point obstructions $\mathcal{O}_\mu(A_\nu, X')$ without Lorentz contraction or skewsymmetrization of μ, ν . These are not listed in Table 2, but since string deltas may be ignored, we may replace such $\mathcal{O}_\mu(A_\nu, X')$ by their delta parts $\mathcal{O}_{[\mu}(A_{\nu]}, X')$ —see Remark A.1. Then, in

$$\mathcal{O}_\mu([u, A_\nu]_a [A^\mu, A^\nu]_a, F_b^{\nu\kappa\lambda} [A'_\kappa, A'_\lambda]_b),$$

we must pair with F' each of the three A -fields on the left. Pairing of A_ν with F' results in:

$$\begin{aligned} c_F \sum_a [u, \delta_\mu^{[\kappa} \delta_\nu^{\lambda]}] [A'_\kappa, A'_\lambda] i\delta_{xx'}]_a [A^\mu, A^\nu]_a \\ = 2c_F \sum_a u_a [[A^\kappa, A^\lambda], [A_\kappa, A_\lambda]]_a i\delta_{xx'} = 0. \end{aligned} \quad (6.13)$$

The pairing of A^μ with F' requires $\mathcal{O}_\mu(A^\mu, F')$, which has no delta part. The skewsymmetrized pairing of A^ν with F' yields

$$\begin{aligned} \frac{1}{2} c_F \sum_a [u, A_\nu]_a [A^\mu, \eta^{\nu\rho} \delta_{[\mu}^{[\kappa} \delta_{\rho]}^{\lambda]}] [A'_\kappa, A'_\lambda] i\delta_{xx'}] \\ = 2c_F \sum_a [[u, A^{[\kappa}], A^{\lambda]}]_a [A'_\kappa, A'_\lambda]_a i\delta_{xx'}. \end{aligned} \quad (6.14)$$

Thanks to the Jacobi identity, this again equals:

$$\begin{aligned} & 2c_F \sum_a [u, [A^\kappa, A^\lambda]]_a [A'_\kappa, A'_\lambda]_a i\delta_{xx'} \\ &= 2c_F \sum_a u_a [[A'^\kappa, A'^\lambda], [A'_\kappa, A'_\lambda]]_a i\delta_{xx'} = 0. \end{aligned} \quad (6.15)$$

Thus all obstructions vanish identically, irrespective of the value of c_F in (iii) and (iv). \square

Remark 6.5. The termination of induced couplings after the second order is a *necessary* feature, because higher-order induced interactions would not be renormalizable.

6.2. Higher Orders $n > 3$

Higher-order interactions L_n are determined by the parts of $\mathcal{O}^{(n)}$ without string deltas. At fourth order, $\mathcal{O}(Q_1; L_3)|_\delta = 0$ because $L_3 = 0$; $\mathcal{O}(Q_3; L_1''')|_\delta = 0$ because $Q_3|_\delta = 0$; and $\mathcal{O}(Q_2|_\delta; L_2'') = 0$ since $Q_2|_\delta$ and L_2 are Wick polynomials in u and A , with $\mathcal{O}_\mu(A^\mu, A')|_\delta = 0$.

This argument generalizes to all orders by induction, see also Remark 6.1. The only open question is whether all obstructions with string deltas are total derivatives, i.e. the existence of $Q_n|_{I\delta}$.

7. Outlook

We have studied interactions between particles of spin and helicity 1 and scalar particles on the string-localized Hilbert-space fields provided by sQFT. Given the particle content of the electroweak theory, the condition of string independence (SI) of the \mathbb{S} -matrix fixes all coupling coefficients, up to a freedom of renormalization, see Remark 5.4, and predicts precisely the known interactions of the Standard Model.

We have also laid the grounds for an SI analysis of more general models of massive and massless vector bosons. Resolution of obstructions to SI in the general case consists of a plethora of polynomial constraints on coupling coefficients and masses. Such a general solution may be quite difficult to characterize. It might be interesting to know whether GUT models with SSB satisfy all consistency constraints of sQFT.

For the models with one scalar particle (one higgs) studied in this paper, we may define skewsymmetric matrices

$$(\gamma^a)_{cd} := \frac{m_{cad}^2}{2m_c m_d} f_{cad} \equiv \frac{m_{cad}^2}{2m_c m_d} \text{ad}(\xi^a)_{cd}, \quad (7.1)$$

whose indices run over the “massive” particles c, d only. For massless a , $m_{cad}^2 = 2m_c m_d$ holds, so $(\gamma^a)_{cd}$ equals $f_{cad} = \text{ad}(\xi^a)_{cd}$, the adjoint representers of the “massless” Lie subalgebra \mathfrak{h} that organizes the massless particles into multiplets (representations of \mathfrak{h}), according to Proposition 2.2.

When the Lie algebra structure constants and the higgs coupling coefficients k_{ab} are expressed in terms of the matrices γ^a and the matrix projector

P of Eq. (6.4), all conditions for string independence, namely conditions (i) and (ii) in Proposition 6.2 together with Eq. (5.7), can be displayed as a system of matrix equations:

$$P^2 = P = P^t, \quad [P, \gamma^a] = 0 \quad (\text{all } a),$$

$$[\gamma^a, \gamma^b] = \sum_{e:m_e=m_b} f_{abe} \gamma^e \quad (a \text{ massless, } b \text{ massive}), \quad (7.2)$$

$$[\gamma^a, \gamma^b] - \sum_e f_{abe} \gamma^e = \frac{1}{4} m_a m_b K^2 P^a \wedge P^b \quad (a, b \text{ massive}), \quad (7.3)$$

where P^a are the column vectors of P and the sums in (7.3) run over all indices e , massive or massless. This rewriting teases out an algebraic structure underlying the SI conditions that could be of use to analyse more general admissible mass patterns. In particular, (7.2) says that the adjoint action of \mathfrak{h} on the space of massive γ^b coincides with its action on the space of massive ξ^b , which splits it into representations of the Lie algebra \mathfrak{h} . By (7.3), the higgs couplings compensate for the failure of the “mass-deformed” massive generators γ^a to satisfy the Lie algebra of the ξ^a .

Remark 7.1. On dividing the sum rule (5.9) by $m_a^2 m_b m_c$, the right-hand side becomes $K^2(P_{aa}P_{bc} - P_{ab}P_{ac})$. Summed over a , this is $K^2(r - 1)P_{bc}$, where r is the rank of the projector P . Thereby, (5.9) gives an explicit formula for P_{ab} in terms of the masses and the structure constants of the Lie algebra. The idempotent property $P^2 = P$ is then a direct constraint relation (not involving k_{ab}) among the latter.

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A. Two-Point Obstructions

This appendix outlines the construction of Tables 1 and 2 in Sect. 3.

A.1. Two-Point Functions

Let $W_m(x - x') = \langle\langle \varphi(x) \varphi(x') \rangle\rangle$ be the two-point function of a free scalar field of mass m , so that $(\square + m^2)W_m(x - x') = 0$.

For two-point functions involving derivatives of fields, we apply the rules

$$\begin{aligned} \langle\langle \partial^\mu X(x) Y(x') \rangle\rangle &= \partial^\mu (\langle\langle X Y \rangle\rangle)(x - x') \quad \text{and} \\ \langle\langle X(x) \partial'^\nu Y(x') \rangle\rangle &= -\partial^\nu (\langle\langle X Y \rangle\rangle)(x - x'). \end{aligned}$$

This settles all two-point functions of the higgs field and its derivatives, in particular:

$$\langle\langle \partial_\mu H(x) \partial'_\nu H(x') \rangle\rangle = -\partial_\mu \partial_\nu W_{m_H}(x - x'). \quad (\text{A.1})$$

Turning to the fields in the Proca sector, recall from Sect. 2.3 that $A_\mu(x, c) = I_c^\nu F_{\mu\nu}(x)$, $B_\nu(x) = -m^{-2} \partial_\mu F^{\mu\nu}(x)$, $\phi(x, c) = I_c^\mu B_\mu(x)$, and $u(x, h) = \underline{\delta}(\phi(x, c))$. One therefore obtains all two-point functions of string-localized fields from

$$\langle\langle F_{\mu\nu}(x) F_{\kappa\lambda}(x') \rangle\rangle = (\eta_{\mu\kappa} \partial_\nu \partial_\lambda - \eta_{\nu\kappa} \partial_\mu \partial_\lambda + \eta_{\nu\lambda} \partial_\mu \partial_\kappa - \eta_{\mu\lambda} \partial_\nu \partial_\kappa) W_m(x - x'), \quad (\text{A.2})$$

by applying the rules:

$$\begin{aligned} \langle\langle I_c^\mu X(x) Y(x') \rangle\rangle &= I_c^\mu (\langle\langle X Y \rangle\rangle)(x - x') \quad \text{and} \\ \langle\langle X(x) I_c^\nu Y(x') \rangle\rangle &= I_c^\nu (\langle\langle X Y \rangle\rangle)(x - x'), \end{aligned}$$

where I_c^ν acts on the argument x' . Let us abbreviate $I^\mu \equiv I_c^\mu$ and $I'^\nu \equiv I_c^\nu$, as well as $X \equiv X(x)$ and $X' \equiv X(x')$ for fields. The argument of every two-point function is $(x - x')$. Formula (2.10) now reads $(\partial I) = (I \partial) = -\text{id}$, and also $(\partial I') = (I' \partial) = +\text{id}$. On using the Klein–Gordon equation, this yields in particular:

$$\langle\langle B_\mu B'_\nu \rangle\rangle \equiv \langle\langle B_\mu(x) B_\nu(x') \rangle\rangle = -(\eta_{\mu\nu} + m^{-2} \partial_\mu \partial_\nu) W_m(x - x'), \quad (\text{A.3})$$

The same rules apply in the photon sector, using the two-point function (A.2) with $m = 0$ for the Faraday tensor, and the definitions $A_\mu(x, c) := I_c^\nu F_{\mu\nu}(x)$ and $u := -I_c^\mu \underline{\delta}(A_\mu)$.

A.2. Propagators

Defining time-ordered vacuum expectation values naïvely with the help of the Heaviside function $\theta(x^0 - x'^0)$ is in general illegitimate, since one is multiplying distributions. For point-localized fields, it is well known that locality and covariance ensure that the naïve definition is well defined outside the “diagonal” set $x = x'$. Therefore, one needs to extend that naïve definition to the points $x = x'$.

One extension is given by the so-called “kinematic” propagator, which amounts to replacing W_m by iD_m^F , where D_m^F denotes the Feynman propagator (2.2) of a scalar field of mass m . However, the extension is in general not unique: one may add (derivatives of) $\delta(x - x')$ with the correct symmetry and Lorentz transformation behaviour. This “renormalization” of propagators is constrained by the condition that it must not exceed the scaling dimension of the kinematic propagator.

For string-localized fields, regarded as distributions in x and e , the situation looks far more delicate because the “string diagonal” consists of all points $x + se = x' + s'e'$ ($s, s' \geq 0$). However, Gaß showed in [16, Thm. 4.5] that when the string-localized fields are smeared with $c(e)$ and regarded as distributions in x only, the relevant diagonal is again $x = x'$. In particular, this rules out nontrivial commutation between the operations of time-ordering T and string variation $\hat{\delta}$, which in principle should be taken into account, since obstructions of this sort vanish after smearing with $c(e)$. Therefore the allowed renormalizations are still just of the type $\delta(x - x')$ and its derivatives, occurring only when the scaling dimension is ≥ 4 .

In the current context, since string integrations lower the scaling dimension, only the propagators of local fields with scaling dimension 2 admit in principle such renormalizations. This pertains to the time-ordering of (A.1), (A.2), and (A.3):

$$\begin{aligned} \langle\langle T \partial_\mu H(x) \partial'_\nu H(x') \rangle\rangle &= -i \partial_\mu \partial_\nu D_{m_H}^F(x - x') + i c_H \eta_{\mu\nu} \delta(x - x'), \\ \langle\langle T F_{\mu\nu}(x) F_{\kappa\lambda}(x') \rangle\rangle &= -i \partial_{[\mu} \eta_{\nu][\kappa} \partial_{\lambda]} D_m^F(x - x') - i c_F \eta_{\mu[\kappa} \eta_{\lambda]\nu} \delta(x - x'), \\ \langle\langle T B_\mu(x) B_\nu(x') \rangle\rangle &= -i (\eta_{\mu\nu} + m^{-2} \partial_\mu \partial_\nu) D_m^F(x - x') + i c_B m^{-2} \eta_{\mu\nu} \delta(x - x'). \end{aligned} \tag{A.4}$$

The real coefficients c_H, c_F, c_B are free parameters at this point. All other propagators are “kinematic”, that is, they are given by replacing W_m by iD_m^F .

A.3. Computing Two-Point Obstructions

We determine here the two-point obstructions (3.4) for all relevant fields. In the first term of (3.4), with the derivative inside the T -product, one uses the equations of motion (2.16) for the fields and computes the resulting propagators as in the previous subsection. One then subtracts the derivatives of the propagators in the second term. The Green function property (2.2) of the Feynman propagators produces delta functions, added to the deltas appearing in the renormalized propagators (when applicable).

An example may suffice to illustrate the general procedure. From $\langle\langle B^\mu(x) \phi(x') \rangle\rangle = I'_\kappa(-\eta^{\mu\kappa} - m^{-2} \partial^\mu \partial^\kappa) W_m(x-x') = -(I'^\mu + m^{-2} \partial^\mu) W_m(x-x')$ using $(I' \partial) = \text{id}$, we conclude $\langle\langle \text{T} B^\mu(x) \phi(x') \rangle\rangle = -(I'^\mu + m^{-2} \partial^\mu) iD_m^F(x-x')$. Thus,

$$\begin{aligned} \mathcal{O}_\mu(B^\mu, \phi') &:= \langle\langle \text{T} \partial_\mu B^\mu(x) \phi(x') \rangle\rangle - \partial_\mu \langle\langle \text{T} B^\mu(x) \phi(x') \rangle\rangle \\ &= \partial_\mu (I'^\mu + m^{-2} \partial^\mu) iD_m^F(x-x') = m^{-2} (\square + m^2) iD_m^F(x-x') \\ &= -m^{-2} i\delta(x-x'). \end{aligned}$$

This results in Tables 1 and 2 of two-point obstructions. The last line of Table 2 is obtained by string variation $\underline{\delta}$ of the line before it. All entries also pertain to $\mathcal{O}_\mu(X(x, c), Y(x', c'))$ with c' independent of c .

Remark A.1. Table 2 displays only the Lorentz-contracted and skewsymmetrized parts of the two-point obstructions $\mathcal{O}_\mu(A_\nu, X')$ that are needed at second order. The traceless symmetric part is not obtained with this approach, because one cannot use the equations of motion for propagators $\langle\langle \text{T} \partial_\mu A_\nu X' \rangle\rangle$. Fortunately, those are not needed at second order, and at third order only the part of $\mathcal{O}_\mu(A_\nu, X')$ without string deltas is required. This is easily found: because $\langle\langle \text{T} \partial_\mu A_\nu X' \rangle\rangle$ for $X' = A', B', \phi'$ have respective scaling dimensions 3, 3, 2, the corresponding $\mathcal{O}_\mu(A_\nu, X')$ cannot include delta parts (having dimension at least 4). For $X' = F'$, the propagator $\langle\langle \text{T} \partial_\mu A_\nu F' \rangle\rangle$ of dimension 4 does admit a delta part of $\mathcal{O}_\mu(A_\nu, F'^{\kappa\lambda})$. Now, for Lorentz-symmetry reasons, it must be skewsymmetric in $\mu \leftrightarrow \nu$, and therefore it equals $\frac{1}{2} \mathcal{O}_{[\mu}(A_{\nu]}, X')$ in Table 2. The suppressed traceless symmetric parts of $\mathcal{O}_\mu(A_\nu, X')$ are purely string deltas.

B. Some Details of the Second-Order Resolutions

Here we give a few lemmas that complete the determination of L_2 and Q_2 in the resolution (3.6) at second order.

B.1. Disposing of String Deltas

We show that all second-order obstructions involving string deltas are automatically derivatives, contributing to $Q_2|_{I\delta}$. See Remark 3.1.

First comes a preparatory observation.

Lemma B.1. *If $\underline{\delta} L_1(c) = \partial Q_1$ where L_1 is a Wick polynomial in the fields A_a , ϕ_a and string-independent fields, and Q_1 is linear in u_a , then:*

$$\begin{aligned} \text{(i)} \quad \frac{\partial L_1}{\partial A_{a\mu}} &= \frac{\partial Q_1^\mu}{\partial u_a}, & \text{(ii)} \quad \frac{\partial L_1}{\partial \phi_a} &= \partial_\mu \left(\frac{\partial Q_1^\mu}{\partial u_a} \right), \\ \text{(iii)} \quad \frac{\partial L_1}{\partial \phi_a} &= \partial_\mu \left(\frac{\partial L_1}{\partial A_{a\mu}} \right). \end{aligned} \tag{B.1}$$

In particular, when L_1 does not contain ϕ_a , the quantity $\partial L_1 / \partial A_{a\mu}$ is conserved.

The latter case applies for the photon field, where ϕ_a does not exist.

Proof. The comparison of

$$\underline{\delta}L_1 = \sum_a \frac{\partial L_1}{\partial \phi_a} u_a + \frac{\partial L_1}{\partial A_{a\mu}} \partial_\mu u_a$$

with

$$\partial_\mu Q_1^\mu = \sum_a \partial_\mu \left(\frac{\partial Q_1^\mu}{\partial u_a} u_a \right) = \sum_a \partial_\mu \left(\frac{\partial Q_1^\mu}{\partial u_a} \right) u_a + \frac{\partial Q_1^\mu}{\partial u_a} \partial_\mu u_a$$

immediately yields (i) and (ii). Formula (iii) and the last statement are obvious consequences of (i) and (ii). \square

Lemma B.2. *For the interactions $S_1 = L_1$ and Q_1^μ as specified in Eq. (2.31), all second-order obstructions involving string deltas are total derivatives. They determine the part $Q_2^\mu|_{I\delta}(x, x')$ of Q_2^μ to arise by a simple replacement of $u(x)$ by $2u_2(x, x')$ in Q_1^μ :*

$$Q_2^\mu|_{I\delta}(x, x') = 2 \sum_a \frac{\partial Q_1^\mu}{\partial u_a}(x) u_{2a}(x, x'), \quad (\text{B.2})$$

where

$$u_{2a}(x, x') := - \sum_{bc} f_{abc} u_b(x') A_{c\nu}(x') I^\nu \delta_{xx'}. \quad (\text{B.3})$$

Proof. Because Q_1^μ contains the fields A_ν only in the skewsymmetric combination $F^{\mu\nu} A_\nu$, the third line of Table 2 does not contribute to $\mathcal{O}(S_1; S_1')$. Therefore, the string deltas may only arise through $\mathcal{O}_\mu(F^{\mu\nu}, A')$ and $\mathcal{O}_\mu(F^{\mu\nu}, \phi')$. They contribute

$$\begin{aligned} & \frac{\partial Q_1^\mu}{\partial F_a^{\mu\nu}} \left[-i I'_\nu \delta_{xx'} \frac{\partial L'_1}{\partial \phi'_a} - i (\partial'^\kappa I'_\nu \delta_{xx'}) \frac{\partial L'_1}{\partial A'^\kappa_a} \right] \\ &= \partial'^\kappa \left[-i \frac{\partial Q_1^\mu}{\partial F_a^{\mu\nu}} I'_\nu \delta_{xx'} \frac{\partial L'_1}{\partial A'^\kappa_a} \right] = \partial'^\kappa \left[-i \frac{\partial Q_1^\mu}{\partial F_a^{\mu\nu}} I'_\nu \delta_{xx'} \frac{\partial Q_1^{\kappa'}}{\partial u'_a} \right] \end{aligned}$$

by Lemma B.1(iii) and (i).

Now (B.3) follows from the formula (2.31b) and the condition (3.6) for resolving second-order obstructions that determines $Q_2^\mu(x, x')$. \square

B.2. The Case of General c_H and c_B

In the main body of the paper, we have computed second-order obstructions with the choice of renormalization constants $c_H = c_B = -1$. The next Lemma shows that the additional contributions for general c_H, c_B are always resolvable and have a rather simple form. The result reflects the circumstance that a delta function in a propagator amounts to the contraction to a new quartic vertex of two cubic vertices connected by that propagator.

Lemma B.3. *The additional second-order obstruction when c_H and c_B differ from the distinguished choice $c_H = c_B = -1$ is*

$$\mathcal{O}^{(2)*}(x, x') = \underline{\delta}[L_2^*] - \mathfrak{S}_{xx'} \partial_\mu Q_2^{*\mu}$$

with

$$\begin{aligned}
 L_2^* &= (1 + c_H) \frac{\partial L_1}{\partial(\partial_\kappa H)} \frac{\partial L_1}{\partial(\partial^\kappa H)} + (1 + c_B) \sum_e m_e^{-2} \frac{\partial L_1}{\partial B_{e\kappa}} \frac{\partial L_1}{\partial B_e^\kappa}, \\
 Q_2^{*\mu} &= 2(1 + c_H) \frac{\partial Q_1^\mu}{\partial(\partial_\kappa H)} \frac{\partial L_1}{\partial(\partial^\kappa H)} i\delta_{xx'} + 2(1 + c_B) \sum_e \frac{\partial Q_1^\mu}{\partial B_{e\kappa}} \frac{\partial L_1}{\partial B_e^\kappa} i\delta_{xx'}.
 \end{aligned} \tag{B.4}$$

The relation (5.22b) also holds for Q_2^* and L_2^* , namely $Q_2^*|_\delta = \sum_a u_a \frac{\partial L_2^*}{\partial A_{a\mu}} i\delta_{xx'}$.

Remark B.4. L_2^* has terms of type $AA\phi\phi$, $AAHH$ and $AA\phi H$ (where the last of these also comes with coefficients C_{abc}). We expect, from experience with a simpler model with all masses equal [50], that the string delta parts of the third-order obstructions (which we are not considering here, but must be separately derivatives because they cannot be part of $\underline{\delta}(L_3)$) put further constraints on the renormalization constants, leaving only the choice $c_H = c_B = -1$. The rationale is similar to that of [18], where that choice was motivated by the complete absence of string deltas. This shortcut is not possible in nonabelian models, because of Lemma B.2.

Proof of Lemma B.3. Notice that $\partial Q_1^\mu/\partial(\partial_\alpha H)$ and $\partial Q_1^\mu/\partial B_\alpha$ both contain a factor $\eta^{\mu\alpha}$. It is convenient to write:

$$\frac{\partial Q_1^\mu}{\partial(\partial_\alpha H)} =: \eta^{\mu\alpha} \frac{\partial Q_1}{\partial(\partial H)} \quad \text{and} \quad \frac{\partial Q_1^\mu}{\partial(B_{e\alpha})} =: \eta^{\mu\alpha} \frac{\partial Q_1}{\partial(\partial B_e)}.$$

Similarly, one can abbreviate

$$\frac{\partial Q_1^\mu}{\partial F_{e\alpha\beta}} = \frac{1}{2} \sum_{ab} \eta^{\mu\alpha} f_{eab} (2u_a A_b^\beta) - [\alpha \leftrightarrow \beta] =: \frac{1}{2} \eta^{\mu\alpha} \left(\frac{\partial Q_1}{\partial(\partial F)} \right)^\beta - [\alpha \leftrightarrow \beta].$$

After inspection of the c_H - and c_B -dependent entries in Tables 1 and 2, one must compute:

$$\begin{aligned}
 \frac{\partial}{\partial c_H} Q_1 \boxtimes L_1' &= \left[\frac{\partial Q_1^\mu}{\partial H} i\delta_{xx'} - \frac{\partial Q_1}{\partial(\partial H)} i\partial^\mu \delta_{xx'} \right] \frac{\partial L_1'}{\partial(\partial'^\mu H')} \\
 &= \left[\frac{\partial Q_1^\mu}{\partial H} + \partial^\mu \left(\frac{\partial Q_1}{\partial(\partial H)} \right) \right] \frac{\partial L_1'}{\partial(\partial'^\mu H')} i\delta_{xx'} \\
 &\quad - \partial^\mu \left(\frac{\partial Q_1}{\partial(\partial H)} \frac{\partial L_1'}{\partial(\partial'^\mu H')} i\delta_{xx'} \right)
 \end{aligned} \tag{B.5}$$

and

$$\begin{aligned}
 \frac{\partial}{\partial c_B} Q_1 \boxtimes L_1' &= \sum_e \left[- \left(\frac{\partial Q_1}{\partial F_e} \right)^\mu i\delta_{xx'} - m_e^{-2} \frac{\partial Q_1^\mu}{\partial \phi_e} i\delta_{xx'} - m_e^{-2} \frac{\partial Q_1}{\partial B_e} i\partial^\mu \delta_{xx'} \right] \frac{\partial L_1'}{\partial B_e'^\mu} \\
 &= \sum_e m_e^{-2} \left[-m_e^2 \left(\frac{\partial Q_1}{\partial F_e} \right)^\mu - \frac{\partial Q_1^\mu}{\partial \phi_e} + \partial^\mu \left(\frac{\partial Q_1}{\partial B_e} \right) \right] \frac{\partial L_1'}{\partial B_e'^\mu} i\delta_{xx'} \\
 &\quad - \partial^\mu \left(\sum_e m_e^{-2} \frac{\partial Q_1}{\partial B_e} \frac{\partial L_1'}{\partial B_e'^\mu} i\delta_{xx'} \right).
 \end{aligned} \tag{B.6}$$

Next, there are the remarkable relations:

$$\begin{aligned} \frac{\partial Q_1^\mu}{\partial H} + \partial^\mu \left(\frac{\partial Q_1}{\partial(\partial H)} \right) &= \underline{\delta} \left[\frac{\partial L_1}{\partial(\partial_\mu H)} \right], \\ \left(-m_e^2 \frac{\partial Q_1}{\partial F_e} \right)^\mu - \frac{\partial Q_1^\mu}{\partial \phi_e} + \partial^\mu \left(\frac{\partial Q_1}{\partial B_e} \right) &= \underline{\delta} \left[\frac{\partial L_1}{\partial B_{e\mu}} \right]. \end{aligned} \quad (\text{B.7})$$

These are verified by direct computation. For the first:

$$\begin{aligned} \sum_{ab} k_{ab} (B_a^\mu u_b + \partial^\mu (\phi_a u_b)) &= \sum_{ab} k_{ab} (A_a^\mu u_b + \partial^\mu u_a \phi_b) \\ &= \underline{\delta} \left[\sum_{ab} k_{ab} A_a^\mu \phi_b \right] = \underline{\delta} \left[\frac{\partial L_1}{\partial(\partial_\mu H)} \right], \end{aligned}$$

and the second for $Q_{1,\text{higgs}}$:

$$\sum_b k_{eb} (-u_b \partial^\mu H + \partial^\mu (u_b H)) = \sum_b k_{eb} \partial^\mu u_b H = \underline{\delta} \left[\sum_b k_{eb} A_b^\mu H \right] = \underline{\delta} \left[\frac{\partial L_{1,\text{higgs}}}{\partial B_{e\mu}} \right],$$

and for $Q_{1,\text{self}}$:

$$\begin{aligned} &\sum_{bc} -2m_e^2 f_{ebc} u_b A_c^\mu - f_{bce} m_{bce}^2 B_b^\mu u_c + f_{ebc} m_{ebc}^2 \partial^\mu (u_b \phi_c) \\ &= \sum_{bc} -(m_{ebc}^2 + m_{ecb}^2) f_{ebc} u_b A_c^\mu + f_{ebc} m_{ebc}^2 B_c^\mu u_b + f_{ebc} m_{ebc}^2 (\partial^\mu u_b \phi_c + u_b \partial^\mu \phi_c) \\ &= \sum_{bc} f_{ebc} m_{ebc}^2 (A_b^\mu u_c + \partial^\mu u_b \phi_c) = \sum_{bc} f_{ebc} m_{ebc}^2 \underline{\delta} [A_b^\mu \phi_c] = \underline{\delta} \left[\frac{\partial L_{1,\text{self}}}{\partial B_{e\mu}} \right]. \end{aligned}$$

When the relations (B.7) are inserted into (B.6), and $[x \leftrightarrow x']$ is added, the formulas (B.4) follow from Eq. (3.6). The final statement is a consequence of the relations:

$$\frac{\partial Q_1^\mu}{\partial(\partial^\kappa H)} = \sum_b u_b \frac{\partial}{\partial A_{b\mu}} \left(\frac{\partial L_1}{\partial(\partial^\kappa H)} \right) \quad \text{and} \quad \frac{\partial Q_1^\mu}{\partial B_e^\kappa} = \sum_b u_b \frac{\partial}{\partial A_{b\mu}} \left(\frac{\partial L_1}{\partial B_e^\kappa} \right)$$

which hold by inspection. \square

B.3. Delta Functions Within Q_2^μ

The resolution of second-order obstructions of the form $\partial_\mu Y^\mu(x) \delta_{xx'}$ and the computation of third-order obstructions of the form $Q_2(x, x') \boxtimes L(x'')$, where $Q_2(x, x') = Y(x) \delta_{xx'}$, bring in some unexpected factors of 2.

Lemma B.5. *For $Q^\mu(x, x')$ of the form $Q^\mu(x, x') = Y^\mu(x) \delta_{xx'}$, the following relations hold:*

- (i) $2\mathfrak{S}_{xx'}(\partial_\mu Q^\mu(x, x')) = \partial_\mu Y^\mu(x) \delta_{xx'}$,
- (ii) $2\mathfrak{S}_{xx'}(Y(x) \delta_{xx'}) \boxtimes X(x'') = (Y(x) \boxtimes X(x'')) \delta_{xx'}$.

Proof. From $(\partial + \partial')\delta_{xx'} = 0$, it follows that

$$(\partial_x + \partial'_x)(f(x) \delta_{xx'}) = \delta_{xx'} (\partial_x + \partial'_x)f(x) = \partial f(x) \delta_{xx'}. \quad (\text{B.8})$$

The left-hand side of (i) is

$$2\mathfrak{S}_{xx'}(\partial_x Q)(x, x') = (\partial_x Q)(x, x') + [x \leftrightarrow x'] = (\partial_x + \partial_{x'})(Y(x) \delta_{xx'}) = (\partial Y(x)) \delta_{xx'}.$$

The left-hand side of (ii) is

$$\begin{aligned} & Y(x) \delta_{xx'} \boxtimes X(x'') + [x \leftrightarrow x'] \\ &= \mathbb{T}[(\partial_x + \partial_{x'})(Y(x) \delta_{xx'})X(x'')] - (\partial_x + \partial_{x'}) \mathbb{T}[Y(x) \delta_{xx'} X(x'')]. \end{aligned}$$

Using (B.8) again, one gets

$$\begin{aligned} & \mathbb{T}[(\partial Y(x)) \delta_{xx'} X(x'')] - (\partial_x + \partial_{x'}) (\delta_{xx'} \mathbb{T}[Y(x)X(x'')]) \\ &= (\mathbb{T}[\partial Y(x)X(x'')] - \partial \mathbb{T}[Y(x)X(x'')]) \delta_{xx'}. \end{aligned} \quad \square$$

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