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High-relative-accuracy computations with Kac-Murdock-Szegö matrices and their generalizations

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Abstract
A linear-time-complexity method to obtain the bidiagonal decomposition of a generalized Kac-Murdock-Szegö matrix is presented. For convenient values of the parameters, it can be obtained with high relative accuracy and it can be also used to compute all eigenvalues, all singular values, the inverse and the solution of some linear systems of equations with high relative accuracy.

Keywords Accurate computations · Kac-Murdock-Szegö matrix

Mathematics Subject Classification 65F05 · 65F15 · 65G50 · 15A23

1 Introduction

The *Kac-Murdock-Szegö* (KMS) matrix (see Kac et al. 1953, Fikioris 2018, Section 1.3 of Dow (2003) and references in there) is a symmetric matrix with applications to many fields such as statistics, finances or digital signal processing. These matrices have been generalized to the nonsymmetric case (see Section 1.4 of Dow 2003). For them, we present a linear-time-complexity method to derive their bidiagonal decomposition, which can be performed, for adequate parameters, with high relative accuracy. An algorithm computes to high relative accuracy (HRA) when it only uses products, quotients, additions of numbers with the same sign or subtractions of initial data (cf. Demmel et al. 1999, 2008; Demmel and Koev 2005; Koev 2005, 2007). For convenient parameters, the generalized KMS matrices are nonsingular totally positive matrices. A matrix is *totally positive* (TP) if all its minors are nonnegative

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(see Ando 1987; Karlin 1968; Pinkus 2010). TP matrices are also called totally nonnegative (cf. Gantmacher and Krein 2002; Fallat and Johnson 2011).

As presented in Koev (2007), for a nonsingular TP matrix A, if we know its bidiagonal factorization $\mathcal{BD}(A)$, then we can perform many algebraic computations with HRA with the software of Koev (2024). In particular, its eigenvalues, its singular values, its inverse (see also Marco and Martínez 2019) and the solution of linear systems $Ax = b$, where b has alternating signs. The bidiagonal factorization $\mathcal{BD}(A)$ arises naturally in the process of Neville elimination (see Gasca and Peña 1992), which is an elimination procedure alternative to Gaussian elimination and that makes zeros in a column by adding to each row an adequate multiple of the previous one. In Sect. 2 we recall some auxiliary tools and results, including Neville elimination, the bidiagonal factorization and a characterization of nonsingular TP matrices.

Section 3 introduces generalized KMS matrices, that we call nonsymmetric KMS matrices although they include the usual (symmetric) KMS matrices. It provides their bidiagonal decomposition and characterizes when they are nonsingular TP. We also analyze when the bidiagonal decomposition can be performed to HRA and when other linear algebraic computations can be performed to HRA. The fast bidiagonal decomposition algorithm of nonsymmetric KMS matrices is included. Section 4 considers the Hadamard product of nonsymmetric KMS matrices and analyzes parameters domain for which linear algebraic computations with these products can be can be performed to HRA. Section 5 particularizes previous results to symmetric KMS matrices and relates them with other known classes of matrices. Finally, Sect. 6 presents numerical experiments confirming the accuracy of the theoretical results.

2 Notations and auxiliary results

As recalled in the Introduction, *Neville elimination* (NE) is an alternative procedure to Gaussian elimination that produces zeros in a column of a matrix by adding to each row an appropriate multiple of the previous one (see Gasca and Peña 1992, 1994). Given a nonsingular matrix $A = (a_{ij})_{1\le i,j\le n}$, the NE procedure consists of $n-1$ steps and leads to the following sequence of matrices:

$$A =: A^{(1)} \to A^{(2)} \to \cdots \to A^{(n)} = U, \tag{1}$$

where U is an upper triangular matrix and $A^{(k)}$ has zeros below the main diagonal in the first $k-1$ columns ($2 \le k \le n$). For $1 \le k \le n-1$, the matrix $A^{(k+1)} = (a_{ij}^{(k+1)})_{1\le i,j\le n}$ is obtained from the matrix $A^{(k)} = (a_{ij}^{(k)})_{1\le i,j\le n}$ by using the formula:

$$a_{ij}^{(k+1)} = \begin{cases} a_{ij}^{(k)} - \dfrac{a_{ik}^{(k)}}{a_{i-1,k}^{(k)}} a_{i-1,j}^{(k)}, & \text{if } k \le j \le n,\ k+1 \le i \le n \text{ and } a_{i-1,k}^{(k)} \ne 0, \\ a_{ij}^{(k)}, & \text{otherwise}, \end{cases} \tag{2}$$

for $k = 1, \ldots, n-1$. The (i, j) *pivot* of the NE of A is given by

$$p_{ij} = a_{ij}^{(j)}, \quad 1 \le j \le i \le n.$$

If $i = j$ we say that p_{ii} is a *diagonal pivot*. The (i, j) *multiplier* of the NE of A, with $1 \le j \le i \le n$, is defined as

$$m_{ij} = \begin{cases} \dfrac{a_{ij}^{(j)}}{a_{i-1,j}^{(j)}} = \dfrac{p_{ij}}{p_{i-1,j}}, & \text{if } a_{i-1,j}^{(j)} \neq 0, \\ 0, & \text{if } a_{i-1,j}^{(j)} = 0. \end{cases} \tag{3}$$

The multipliers satisfy that

$$m_{ij} = 0 \Rightarrow m_{hj} = 0 \quad \forall h > i.$$

The (i, j) multiplier of the NE of A^T will be denoted by $\widetilde{m}_{ij}$

The following theorem is a consequence of Theorem 4.2 of Gasca and Peña (1996) and the arguments of pp. 116 and 120 of Gasca and Peña (1996) and characterizes nonsingular TP matrices by their *bidiagonal decompositions*.

Theorem 2.1 *A nonsingular matrix* $A = (a_{ij})_{1\le i,j\le n}$ *is TP if and only if it admits the following decomposition:*

$$A = F_{n-1}F_{n-2}\cdots F_1 D G_1 \cdots G_{n-2}G_{n-1}, \tag{4}$$

where D is the diagonal matrix $\operatorname{diag}(p_{11}, \ldots, p_{nn})$ *with positive diagonal entries and F_i, G_i are the nonnegative bidiagonal matrices given by*

$$F_i = \begin{pmatrix} 1 & & & & & & \\ 0 & 1 & & & & & \\ & \ddots & \ddots & & & & \\ & & 0 & 1 & & & \\ & & & m_{i+1,1} & 1 & & \\ & & & & \ddots & \ddots & \\ & & & & & m_{n,n-i} & 1 \end{pmatrix}, \quad G_i = \begin{pmatrix} 1 & 0 & & & & & \\ & 1 & \ddots & & & & \\ & & \ddots & 0 & & & \\ & & & 1 & \widetilde{m}_{i+1,1} & & \\ & & & & 1 & \ddots & \\ & & & & & \ddots & \widetilde{m}_{n,n-i} \\ & & & & & & 1 \end{pmatrix}, \tag{5}$$

for all $i \in \{1, \ldots, n-1\}$. If, in addition, the entries m_{ij} and $\widetilde{m}_{ij}$ satisfy

$$\begin{aligned} m_{ij} = 0 &\Rightarrow m_{hj} = 0 \quad \forall h > i, \\ \widetilde{m}_{ij} = 0 &\Rightarrow \widetilde{m}_{hj} = 0 \quad \forall h > i, \end{aligned} \tag{6}$$

then the decomposition is unique.

In the bidiagonal decomposition given by (4) and (5), the entries m_{ij} and p_{ii} are the multipliers and diagonal pivots, respectively, corresponding to the NE of A (see Theorem 4.2 of Gasca and Peña 1996 and the comment below it) and the entries $\widetilde{m}_{ij}$ are the multipliers of the NE of A^T (see p. 116 of Gasca and Peña 1996). The bidiagonal decomposition can be used to factorize more classes of matrices. If we consider the factorization given by (4)-(6) without any further requirement than the nonsingularity of D, by Proposition 2.2 of Barreras and Peña (2013) the uniqueness of (4) holds. In Koev (2007) the matrix notation $\mathcal{BD}(A)$ was introduced to represent the bidiagonal decomposition of a nonsingular TP matrix,

$$(\mathcal{BD}(A))_{ij} = \begin{cases} m_{ij}, & \text{if } i > j, \\ \widetilde{m}_{ji}, & \text{if } i < j, \\ p_{ii}, & \text{if } i = j. \end{cases} \tag{7}$$

Throughout this paper, $\mathcal{BD}(A)$ will also denote the bidiagonal decomposition of matrices that are not necessarily TP.

3 HRA algorithms for nonsymmetric KMS matrices

First let us introduce the definition of a nonsymmetric Kac-Murdock-Szegö matrix (see Section 1.4 of Dow 2003).

Definition 3.1 The nonsymmetric Kac-Murdock-Szegö (KMS) matrix with parameters ρ and σ is the Toeplitz matrix $A = (a_{ij})_{1\le i,j\le n}$ given by

$$a_{ij} = \begin{cases} \rho^{j-i}, & 1 \le i < j \le n, \\ \sigma^{i-j}, & 1 \le j < i \le n, \\ 1, & \text{otherwise.} \end{cases}$$

For example, the nonsymmertic KMS matrix of order 4 with parameters ρ and σ is given by

$$A = \begin{pmatrix} 1 & \rho & \rho^2 & \rho^3 \\ \sigma & 1 & \rho & \rho^2 \\ \sigma^2 & \sigma & 1 & \rho \\ \sigma^3 & \sigma^2 & \sigma & 1 \end{pmatrix}$$

Remark 1 For the particular case where $\rho = \sigma$, the usual symmetric KMS matrix is obtained (see Section 1.3 of Dow 2003).

In the case that $\rho \ne \sigma$, the multipliers (3) of A^T are analogous to those of A but changing the role of the parameters.

The following result describes the bidiagonal factorization of nonsymmetric KMS matrices, characterizes the parameter values for which they are TP and finds parameter values for which algebraic computations can be performed to HRA.

Theorem 3.2 *If A is the nonsymmetric KMS matrix with parameters ρ and σ then the following properties hold:*

i. *The bidiagonal decomposition of A, $\mathcal{BD}(A)$, is given by*

$$(\mathcal{BD}(A))_{ij} = \begin{cases} \sigma, & \textit{for } j = 1 \textit{ and } i \in \{2,\dots,n\}, \\ \rho, & \textit{for } i = 1 \textit{ and } j \in \{2,\dots,n\}, \\ 1, & \textit{for } i = j = 1, \\ 1 - \sigma\cdot\rho, & \textit{for } i = j \in \{2,\dots,n\}, \\ 0, & \textit{otherwise.} \end{cases} \tag{8}$$

So, A is nonsingular if and only if $\rho\cdot\sigma \ne 1$, and A is nonsingular TP if and only if $\sigma, \rho \ge 0$ and $\sigma\cdot\rho < 1$.

ii. *If $\sigma\cdot\rho \le 0$, or $|\sigma|, |\rho| \le 1$, or $|\sigma|, |\rho| \ge 1$, then the bidiagonal decomposition of A, $\mathcal{BD}(A)$, can be computed to HRA.*

iii. *If ρ, σ are nonnegative parameters satisfying $0 \le \rho\cdot\sigma < 1$, then A is nonsingular TP, and its eigenvalues, its singular values, its inverse A^{-1} and the solution of linear systems $Ax = b$, where b has a pattern of alternating signs, can also be computed to HRA.*

Proof i. Neville elimination of A will provide the bidiagonal decomposition of A. Subtracting σ times the row $i-1$ to the row i for $i = n, \ldots, 2$ and denoting by $E_i(\alpha)$ $(2 \leq i \leq n)$ the $n \times n$ elementary matrix that has unit diagonal, α in place $(i, i-1)$ and 0 elsewhere, the following factorization is obtained

$$E_2(-\sigma) \cdots E_n(-\sigma)A = U,$$

where $U = (u_{ij})$ is the upper triangular matrix whose nonzero entries are $u_{11} = 1$, $u_{ii} = 1-\sigma \cdot \rho$ for $i = 2, \ldots, n$, $u_{1j} = \rho^{j-1}$ for $j = 2, \ldots, n$ and $u_{ij} = \rho^{j-i} \cdot (1-\sigma \cdot \rho)$. Analogously, subtracting ρ times the column $i-1$ to the column i for $i = n, \ldots, 2$ the following factorization is obtained

$$E_2(-\sigma) \cdots E_n(-\sigma)A(E_n(-\rho))^T \cdots (E_2(-\rho))^T = D,$$

where D is the diagonal matrix given by $d_{11} = 1$ and $d_{ii} = 1 - \sigma \cdot \rho$ for $i = 2 \ldots, n$. Then, using that $E_i(\alpha)^{-1} = E_i(-\alpha)$, the bidiagonal factorization of A is deduced

$$A = E_n(\sigma) \cdots E_2(\sigma)D(E_2(\rho))^T \cdots (E_n(\rho))^T, \tag{9}$$

which can be expressed in compact form by (8). Observe that, by the unicity of (4), $F_i = E_{i+1}(\sigma)$ and $G_i = (E_{i+1}(\rho))^T$.
From the bidiagonal decomposition of A it is deduced that $\det A = (1-\sigma \cdot \rho)^{n-1}$ (which was given in Section 1.4 of Dow 2003), and so, A is nonsingular if and only if $\rho \cdot \sigma \neq 1$. Applying Theorem 2.1 to the bidiagonal decomposition obtained, it is concluded that A is nonsingular TP if and only if $\sigma, \rho \geq 0$ and $\sigma \cdot \rho < 1$.

ii. If $\sigma \cdot \rho \leq 0$, $1 - \sigma \cdot \rho$ is a sum of nonnegative numbers and it is computed to HRA in an straightforward way. In the case that $\sigma \cdot \rho > 0$, it can be used the following equality

$$1 + a \cdot b = \frac{1}{2}\left[(1+a)(1+b) + (1-a)(1-b)\right]. \tag{10}$$

For example, if $\sigma, \rho > 0$, using (10) with $a = -\rho$ and $b = \sigma$, we obtain a sum of positive numbers for $0 < \sigma, \rho < 1$ and a sum of negative numbers for $\sigma, \rho > 1$. The case when $\sigma, \rho < 0$ can be analyzed analogously.

iii. By i, the matrix A is nonsingular TP. By ii, $\mathcal{BD}(A)$ can be computed to HRA. Then, the methods of Koev (2007); Marco and Martínez (2019) can be used to solve the four algebraic computations mentioned with HRA.

□

Algorithm 1 shows the pseudocode for the computation of $\mathcal{BD}(A)$ to HRA for a nonsymmetric KMS matrix A of parameters ρ, σ satisfying the hypotheses in Theorem 3.2 ii.

The following remark deals with the computational cost of applying Algorithm 1 for the computation of the bidiagonal decomposition of a nonsymmetric KMS matrix, showing that it leads to fast algorithms to compute linear algebra computations with these matrices.

Remark 2 The complexity of performing Algorithm 1 for an $n \times n$ nonsymmetric KMS matrix is $\mathcal{O}(n)$. So, using it and the algorithms given in Koev (2007) and Marco and Martínez (2019) for the corresponding algebraic computations with these matrices, the complexity to compute the singular values and eigenvalues is $\mathcal{O}(n^2)$ elementary operations, whereas the complexity to compute its inverse or solve a linear system of equations is $\mathcal{O}(n)$.

The next section provides a new source of nonsymmetric KMS matrices and HRA computations.

Algorithm 1 Computation of $\mathcal{BD}(A)$ of a nonsymmetric KMS matrix A of order n

Require: n, ρ and σ
Ensure: B bidiagonal decomposition of A

```
B = zeros(n, n)
for i = 2 : n do
    B(1, i) = ρ
    B(i, 1) = σ
end for
B(1, 1) = 1
diag = 1/2 [(1 − ρ)(1 + σ) + (1 + ρ)(1 − σ)]
for i = 2 : n do
    B(i, i) = diag
end for
```

4 Hadamard product of nonsymmetric KMS matrices

Given two $n \times m$ matrices $A = (a_{ij})$ and $B = (b_{ij})$, we can define the Hadamard (or entry-wise) product of A and B is the $n \times m$ matrix $C = A \circ B$ by $c_{ij} = a_{ij}b_{ij}$. Recently, the Hadamard product of Green matrices was considered in Delgado et al. (2023a). The following result shows that the Hadamard product of nonsymmetric KMS matrices is also a nonsymmetric KMS matrix and finds parameter values for which the bidiagonal decomposition of their Hadamard product can be calculated to HRA.

Theorem 4.1 *Let A and B two nonsymmetric KMS matrices of parameters ρ_1, σ_1 and ρ_2, σ_2, respectively, and $C = A \circ B$ the Hadamard product of A and B. Then:*

i. *The Hadamard product $A \circ B$ is also a nonsymmetric KMS matrix with parameters $\rho = \rho_1 \cdot \rho_2$ and $\sigma = \sigma_1 \cdot \sigma_2$.*
ii. *If $\rho_1, \sigma_1, \rho_2, \sigma_2$ are nonnegative parameters satisfying $\rho_1 \cdot \sigma_1 \cdot \rho_2 \cdot \sigma_2 < 1$, then $A \circ B$ is nonsingular TP, the bidiagonal decomposition of $A \circ B$, $\mathcal{BD}(A \circ B)$, can be computed to HRA, and so the eigenvalues, the singular values, the inverse $(A \circ B)^{-1}$ and the solution of linear systems $(A \circ B)x = b$ where b has a pattern of alternating signs can also be computed to HRA.*

Proof i. The entries of the Hadamard product $C = A \circ B$ are given by

$$c_{ij} = \rho_1^{j-i} \cdot \rho_2^{j-i} = (\rho_1 \cdot \rho_2)^{j-i} \quad \text{for } 1 \le i < j \le n,$$
$$c_{ij} = \sigma_1^{j-i} \cdot \sigma_2^{j-i} = (\sigma_1 \cdot \sigma_2)^{j-i} \quad \text{for } 1 \le j < i \le n$$

and $c_{ii} = 1$ for $1 \le i \le n$. So, taking into account Definition 3.1 for $\rho = \rho_1 \cdot \rho_2$ and $\sigma = \sigma_1 \cdot \sigma_2$ the result follows.

ii. Taking into account the result in i, the bidiagonal decomposition of $C = A \circ B$, $\mathcal{BD}(C)$, is given by (8) for $\rho = \rho_1 \cdot \rho_2$ and $\sigma = \sigma_1 \cdot \sigma_2$. By i, $C = A \circ B$ is a nonsymmetric KMS matrix of parameters $\rho = \rho_1 \cdot \rho_2$ and $\sigma = \sigma_1 \cdot \sigma_2$. If $0 < \rho_1, \sigma_1, \rho_2, \sigma_2 < 1$, then $0 < \rho, \sigma < 1$. By iii of Theorem 3.2 we deduce that $C = A \circ B$ is a nonsingular TP matrix. It is clear that the off-diagonal entries of $\mathcal{BD}(C)$ (see (7)) can be computed to HRA. For the diagonal entries $(\mathcal{BD}(C))_{ii} = 1 - (\rho_1 \cdot \rho_2) \cdot (\sigma_1 \cdot \sigma_2)$ formula (10) can be used twice in case that $\rho_1 \cdot \rho_2 \cdot \sigma_1 \cdot \sigma_2 > 0$. The first time can be used taking $a = -\sigma_1 \cdot \sigma_2$ and $b = \rho_1 \cdot \rho_2$ in (10)

$$1 - \sigma_1 \cdot \sigma_2 \cdot \rho_1 \cdot \rho_2 = \frac{1}{2}\left[(1 - \sigma_1 \cdot \sigma_2)(1 + \rho_1 \cdot \rho_2) + (1 + \sigma_1 \cdot \sigma_2)(1 - \rho_1 \cdot \rho_2)\right].$$

Now, formula (10) must be applied to the two factors of the right hand side of the previous expression corresponding to subtractions (only one or none factor if $\sigma_1 \cdot \sigma_2 = 0$ or/and $\rho_1 \cdot \rho_2 = 0$).

Then, from $\mathcal{BD}(C)$ to HRA, the four algebraic problems mentioned can be computed to HRA by using the methods in Koev (2007); Marco and Martínez (2019).

□

5 Symmetric KMS matrices

As mentioned in Remark 1, the symmetric KMS matrices satisfy Definition 3.1 when both parameters coincide. Let us recall the definition of a Green matrix (cf. Pinkus 2010). Given two sequences of nonzero real numbers $(u_i)_{1\leq i\leq n}$, $(v_i)_{1\leq i\leq n}$, a *Green matrix* $A = (a_{ij})_{1\leq i,j\leq n}$ is the symmetric matrix given by $a_{ij} = u_i v_j$ if $i < j$ (or, equivalently, $a_{ij} = u_{\min\{i,j\}} v_{\max\{i,j\}}$ for all i, j). So, taking $u_i = \rho^{-i}$ and $v_i = \rho^i$, a symmetric KMS matrix is a Green matrix (see also Fikioris 2018; Delgado et al. 2023b).

Since symmetric KMS matrices are Green matrices, when they are nonsingular TP matrices, we can deduce from Theorem 3 of Delgado et al. (2016) that they are also Schoenmakers-Coffey matrices, a class of matrices with important finance applications (cf. Schoenmakers and Coffey 2003; Lord and Pelsser 2007; Delgado et al. 2016). The following result collects the application of the results in previous sections to symmetric KMS matrices. Observe that the condition $\rho_1 \in [0, 1)$ of iii to guarantee the nonsingularity and total positivity of a symmetric KMS matrix was already obtained in Fikioris (2018).

Theorem 5.1 *Let A and B two symmetric KMS matrices of parameter ρ_1 and ρ_2, respectively. Then:*

i. *The bidiagonal decomposition of A, $\mathcal{BD}(A)$, is given by*

$$(\mathcal{BD}(A))_{ij} = \begin{cases} 1, & \text{for } i = j = 1, \\ 1-\rho_1^2, & \text{for } i = j \in \{2,\ldots,n\}, \\ \rho_1, & \text{for } i = 1 \text{ and } j \in \{2,\ldots,n\}, \text{ or } j = 1 \text{ and } i \in \{2,\ldots,n\}, \\ 0, & \text{otherwise.} \end{cases}$$

ii. *The bidiagonal decomposition of A, $\mathcal{BD}(A)$, can be computed to HRA.*

iii. *If $0 \leq \rho_1 < 1$, A is a nonsingular TP matrix, and the eigenvalues, the singular values, the inverse A^{-1} and the solution of linear systems $Ax = b$ where b has a pattern of alternating signs can also be computed to HRA.*

iv. *The Hadamard product $A \circ B$ is also a symmetric KMS matrix of parameter $\rho = \rho_1 \cdot \rho_2$.*

v. *If $0 \leq \rho_1 \cdot \rho_2 < 1$, $A \circ B$ is a nonsingular TP matrix. In addition, if $0 < \rho_1, \rho_2 < 1$, its eigenvalues, its singular values, its inverse $(A \circ B)^{-1}$ and the solution of linear systems $(A \circ B)x = b$, where b has a pattern of alternating signs, can also be solved to HRA.*

Proof i. It follows from Theorem 3.2 i taking $\sigma = \rho = \rho_1$.

ii. Taking into account that $1-\rho_1^2 = (1-\rho_1)(1+\rho_1)$ the bidiagonal decomposition $\mathcal{BD}(A)$ in i can be computed to HRA for all ρ_1.

iii. For $\sigma = \rho = \rho_1 \in [0, 1)$, the matrix A is nonsingular. In addition, in this case each factor in (9) is a bidiagonal or diagonal nonnegative matrix, and so, TP. Then the matrix A is TP since it is a product of TP matrices (cf. Theorem 3.1 of Ando 1987). By ii, $\mathcal{BD}(A)$ can be computed to HRA. Then, the methods of Koev (2007); Marco and Martínez (2019) can be used to solve the four mentioned algebraic computations with HRA.

iv. It follows from i of Theorem 4.1 for $\sigma_1 = \rho_1$ and $\sigma_2 = \rho_2$.
v. If $0 \leq \rho_1 \cdot \rho_2 < 1$, by i of Theorem 4.1 and iii, $A \circ B$ is a nonsingular TP matrix. In addition, if $0 \leq \rho_1, \rho_2 < 1$, then by using (10) to compute $1-(\rho_1 \cdot \rho_2)^2 = (1-\rho_1 \cdot \rho_2)(1+\rho_1 \cdot \rho_2)$, $\mathcal{BD}(A \circ B)$ can be computed to HRA. Then, the four algebraic problems mentioned can be computed to HRA by using the methods in Koev (2007); Marco and Martínez (2019). □

Algorithm 2 shows the pseudocode for the computation of $\mathcal{BD}(A)$ to HRA for a symmetric KMS matrix A of parameter ρ.

Algorithm 2 Computation of $\mathcal{BD}(A)$ of a symmetric KMS matrix A of order n

Require: n and ρ
Ensure: B bidiagonal decomposition of A

```
B = zeros(n, n)
for i = 2 : n do
    B(1, i) = ρ
    B(i, 1) = ρ
end for
B(1, 1) = 1
diag = (1 − ρ)(1 + ρ)
for i = 2 : n do
    B(i, i) = diag
end for
```

Let us recall that we can apply Remark 2 to compute the algebraic problems mentioned in the previous result with fast algorithms.

6 Numerical tests

Given a nonsingular TP matrix A, in Koev (2007) Koev developed algorithms to compute the eigenvalues, the singular values of A and the solution of linear systems of equations $Ax = b$ from $\mathcal{BD}(A)$. In addition, if this bidiagonal decomposition is known to HRA, the algebraic problems are solved to HRA (for the case of the linear system of equations when b has an alternating sign pattern). Under the same assumptions, in Marco and Martínez (2019) Marco and Martínez provided an algorithm to compute the inverse A^{-1} with HRA. An implementation of these algorithms for Matlab is available in the software library TNTool, which can be downloaded in Koev (2024).

The bidiagonal decomposition of a nonsingular TP nonsymmetric KMS matrix to HRA, deduced in Theorem 3.2 and expressed in pseudocode in Algortihm 1, has been implemented in Matlab function `TNBDNKMS`.

Now let us present a numerical example illustrating the theoretical results given in Theorem 3.2. So, we have considered the nonsymmetric KMS matrix A of order 20 with parameters $\rho = 1 - 1/2^{30}$ and $\sigma = 1 - 1/2^{50}$. By Theorem 3.2 the matrix A is nonsingular TP, $\mathcal{BD}(A)$ can be computed to HRA and the four algebraic problems mentioned before can be solved to HRA. The condition number of A is $\kappa(A) = 8.59e+10$, so it is very ill-conditioned as many TP matrices. In these cases we cannot expect high accurate results from the usual algebraic algorithms and it is desirable to develop algorithms adapted to the structure of the matrix.

First, the eigenvalues of A have been computed with Mathematica using a 200-digits precision. These approximations to the eigenvalues have also been obtained in Matlab with both

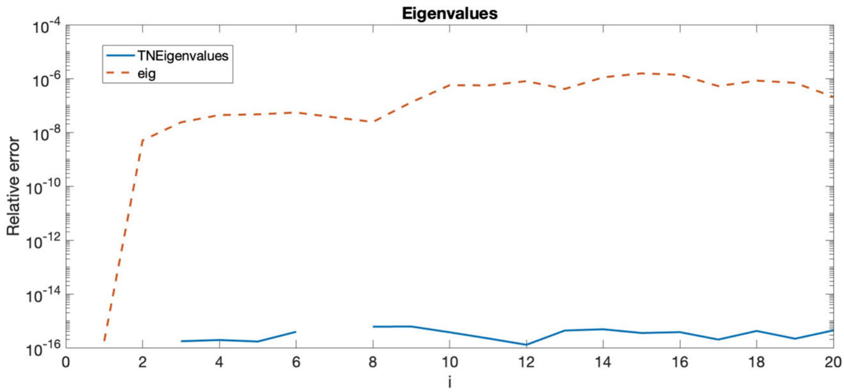

Fig. 1 Relative errors when computing the eigenvalues λ_i, $i = 1, \ldots, 20$, with Matlab

the usual function `eig` and the function `TNEigenValues` using the bidiagonal decomposition $\mathcal{BD}(A)$ provided by `TNBDNKMS` to HRA. Then, the relative errors of the approximation obtained by both algorithms with Matlab have been calculated considering the eigenvalues obtained with Mathematica as exact. In order to show the relative errors, we have considered the eigenvalues in decreasing order: $\lambda_1 > \lambda_2 > \cdots > \lambda_{20} > 0$. In Fig. 1, we can see the relative errors for the approximation to the eigenvalues λ_i, $i = 1, \ldots, 20$, of matrix A by both methods. The breaks in the solid line in the figure correspond to eigenvalues calculated exactly with `TNEigenValues`. From the figure we conclude that `TNBDNKMS` joint with `TNEigenValues` provide more accurate results than these obtained by using the Matlab command `eig`.

Then, the singular values of A have also been computed with Mathematica using a 200-digits precision. These singular values have also been approximated in Matlab with both the usual function `svd` and the function `TNSingularValues` using the bidiagonal decomposition $\mathcal{BD}(A)$ provided by `TNBDNKMS` to HRA. Then, the relative errors for the approximation obtained by both algorithms with Matlab have been calculated considering the singular values obtained with Mathematica as exact. In order to show the relative errors we have considered the singular values in decreasing order: $\sigma_1 > \sigma_2 > \cdots > \sigma_{20} > 0$. In Fig. 2, the relative errors for the approximation of the singular values σ_i, $i = 1, \ldots, 20$, of matrix A by both methods are shown. The breaks in the solid line in the figure correspond to singular values calculated exactly with `TNSingularValues`. From the figure we conclude that `TNBDNKMS` joint with `TNSingularValues` provide more accurate results than these obtained by using the Matlab command `svd`.

Now let us consider the linear system of equations $Ax = b$, where b has an alternating pattern of signs and the absolute values of its entries have been randomly generated as integers in the interval $[1, 1000]$. The exact solution x has been obtained with Mathematica. Then, approximations to the exact solution of the system have been calculated with the usual Matlab command \ and with `TNSolve` using $\mathcal{BD}(A)$ to HRA. Finally, relative errors have been computed componentwise. Figure 3 shows these componentwise relative errors. The breaks in the solid line in the figure correspond to components of the solution calculated exactly with `TNSolve`.

Finally, we have computed the inverse A^{-1} using Mathematica with exact arithmetic. Then we have obtained with Mathematica numerical approximations of the exact entries of this inverse. In addition, we have also computed approximations to the inverse A^{-1} with the usual

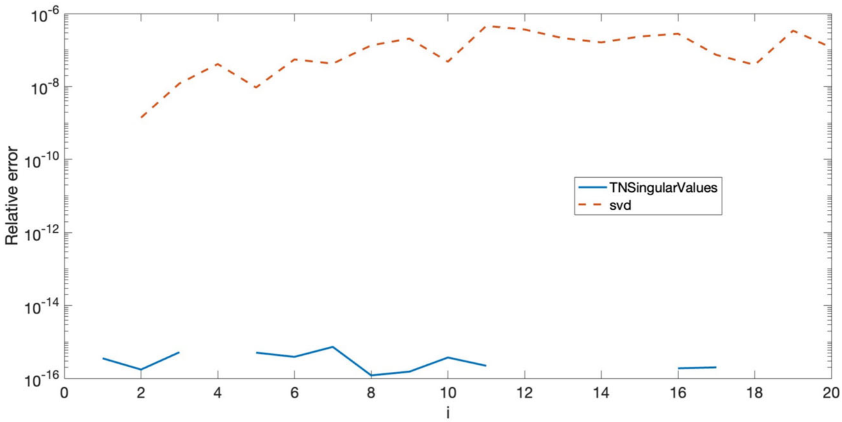

Fig. 2 Relative errors when computing the singular values σ_i, $i = 1, \ldots, 20$, with Matlab

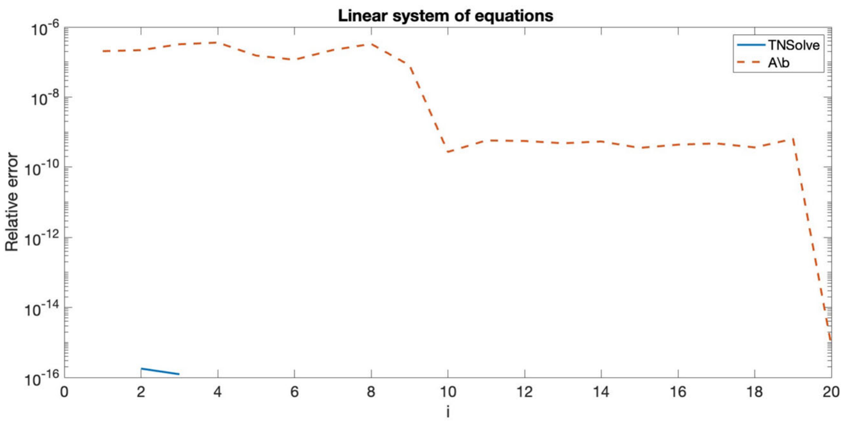

Fig. 3 Componentwise relative errors when solving $Ax = b$ with Matlab

Table 1 Relative errors when computing the inverse of the nonsymmetric KMS matrix A

	`TNInverseExpand`	`inv`
Average rel. error	0	4.496036037402022e−10
Maximum rel. error	0	9.313216873027248e−10

Matlab command `inv` and with `TNInverseExpand` using the bidiagonal decomposition given to HRA by `TNBDNKMS`. Then, taking into account that it is known that the inverse A^{-1} is a tridiagonal matrix, the componentwise relative errors of both approximations for the nonzero entries have been obtained considering the inverse obtained with Mathematica as exact. These data can be seen in Table 1.

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Data availability The authors confirm that the data supporting the findings of this study are available within the manuscript. The Matlab and Mathematica codes to run the numerical experiments are available upon request.

Declarations

Conflict of interest The authors declare no conflict of interest.

Ethical approval Not Applicable.

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