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Hammerstein Nonlinear Integral Equations and Iterative Methods for the Computation of Common Fixed Points

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Abstract: In the first part of this article, a special type of Hammerstein nonlinear integral equation is studied. A theorem of the existence of solutions is given in the framework of $\mathcal{L}^2$-spaces. Afterwards, an iterative method for the resolution of this kind of equations is considered, and the convergence of this algorithm towards a solution of the equation is proved. The rest of the paper considers two modifications of the algorithm. The first one is devoted to the sought of common fixed points of a family of nearly asymptotically nonexpansive mappings. The second variant focuses on the search of common fixed points of a finite number of nonexpansive operators. The characteristics of convergence of these methods are studied in the context of uniformly convex Banach spaces. The iterative scheme is applied to approach the common solution of three nonlinear integral equations of Hammerstein type.

Keywords: Hammerstein integral equations; common fixed points; nonexpansive maps; nearly asymptotically nonexpansive mappings; iterative methods

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1. Introduction

A. Hammerstein introduced, in the reference [1], a new type of nonlinear integral equation, given by the expression:

$$f(x) + \int_a^b k(x,y)\psi(y,f(y))dy = 0,$$

for $x \in [a,b]$ and $k(x,y)$ and $\psi(x,y)$ given functions. Equations of this kind appear in a great number of fields of physics, especially in electro-magnetic fluid dynamics. Similar to almost every integral equation, it can be reformulated in terms of a problem of ordinary or partial differential equations.

Hammerstein considered as hypotheses that the operator associated to k is symmetric and positive, and ψ is such that

$$|\psi(x,y)| \leq C_1|y| + C_2,$$

where $C_1, C_2 \geq 0$. He proved that, if $C_1 < \lambda$, where λ is the first eigenvalue of the kernel, and k and ψ are continuous functions, then the equation has a continuous solution. If, in addition, the mapping $\psi(x,\cdot)$ is non-decreasing for any $x \in (a,b)$, the solution is unique. Brézis and Browder gave in the reference [2] sufficient conditions for the existence and uniqueness of solution of Hammerstein equations in $\mathcal{L}^p$-spaces. For instance, they proved that, if $k(x,y)$ and $\psi(x,y)$ are such that

- $\psi(x, y)$ is continuous in y for almost all x, and measurable in x for any fixed y,
- $\psi(x, \cdot)$ is nondecreasing for any fixed x,
- $\psi(\cdot, y) \in \mathcal{L}^1(\Omega)$ for any fixed y, and
- the linear integral operator associated with $k(x, y)$ is bounded from $\mathcal{L}^1(\Omega)$ into $\mathcal{L}^\infty(\Omega)$ and positive,

then the equation

$$f(x) + \int_a^b k(x, y)\psi(y, f(y))dy = h(x)$$

has a unique solution $f \in \mathcal{L}^\infty(\Omega)$ for each $h \in \mathcal{L}^\infty(\Omega)$, and f varies continuously with h. The numerical resolution of integral equations can be approached in very different ways [3]. A standard procedure is the collocation method (see, for instance, [4]). More recent articles about Hammerstein equations are, among many others, the references [5,6].

In the first part of this article, a special type of Hammerstein nonlinear equation is studied. A theorem of existence of solutions is given, in the framework of $\mathcal{L}^2$-spaces. The solution may not be unique. Afterwards, an iterative method for the resolution of this kind of equations is considered. The procedure was introduced in the reference [7]. The convergence of this algorithm towards a solution of the equation is proved in the same theorem. The method is illustrated with an example, giving a table of mean-square successive distances between iterations. A figure displays the first approximate solutions provided by the algorithm.

In Section 3, a modification of the algorithm is studied, for the sought of common fixed points of a family of nearly asymptotically nonexpansive mappings (see, for instance, [8]). The approximations properties of this iterative method are studied, and sufficient conditions on the underlying space and the mappings to ensure its convergence are given.

In a previous paper [9], the problem of approximating a common fixed point of two mappings was treated, and an algorithm for the search of common critical points of a finite number m of nonexpansive maps was suggested. However, the properties of approximation and convergence were only studied for $m = 2$. In Section 4, the case where $m > 2$ is analyzed, generalizing the results given for two maps. The iterative method is applied to find the common solution of three nonlinear integral equations of the Hammerstein type. Since the system has an exact solution, a table of approximation errors with respect to the real solution is given.

2. Solution of a Type of Nonlinear Hammerstein Integral Equation

We consider in this section the approximation of a solution of an integral equation of the Hammerstein type [1]. This kind of nonlinear equations are given, in general, by

$$f(x) = g(x) + \int_\Omega k(x, y)\psi(y, f(y))dy, \tag{1}$$

where Ω is a closed subset of $\mathbb{R}^n$ and g, k, and ψ are given functions. In the particular case where $\psi(y, f(y)) = f(y)$, we obtain an integral equation of the Fredholm type. Thus, Hammerstein equations are nonlinear generalizations of the Fredholm integral equations of the second kind. The following result can be consulted in the reference ([10], Lemma 4.2.8).

Lemma 1. *Let J be a compact real interval, let $k : J \times J \to \mathbb{C}$ and $\psi : J \times \mathbb{C} \to \mathbb{C}$ be continuous functions, and assume that there are real numbers $p \geq 1$, $\alpha, \beta \geq 0$ such that*

$$|\psi(x, z)| \leq \alpha + \beta|z|^p,$$

for $x \in J$ and $z \in \mathbb{C}$. Then the Hammerstein operator

$$Af(x) := \int_J k(x,y)\psi(y, f(y))dy, \tag{2}$$

is a continuous operator mapping $\mathcal{L}^p(J)$ into itself.

Let us consider the space $\mathcal{L}^2(\Omega)$. Equation (1) can be written as $f = g + K \circ \Psi f$, where K is the linear operator given by $Kh(x) = \int_\Omega k(x,y)h(y)dy$, and Ψ is defined by $\Psi(f)(y) = \psi(y, f(y))$. The solution of the Hammerstein equation is a fixed point of the operator of $\mathcal{L}^2(\Omega)$:

$$Hf := g + K \circ \Psi f.$$

Let us assume that Ψ is nonexpansive with respect to the 2-norm, and $||K|| = ||k||_2 \leq 1$. Then H is nonexpansive, since

$$||Hf - Hf'||_2 \leq ||K \circ \Psi f - K \circ \Psi f'||_2 \leq ||f - f'||_2.$$

In the reference [7], we proposed an iterative method, called the N-algorithm, to approach a fixed point of an operator $A : C \to C$, where C is a nonempty, closed, and convex subset of a normed space V. It is given by the following iterative scheme:

$$w_n = (1 - c_n)u_n + c_n A u_n, \tag{3}$$

$$v_n = (1 - b_n)u_n + b_n w_n, \tag{4}$$

$$u_{n+1} = (1 - a_n)v_n + a_n A v_n, \tag{5}$$

for $a_n, b_n, c_n \in [0,1]$, $n \geq 0$, and $u_0 \in C$. For $a_n = 0$ and $b_n = 1$ for all n, one obtains the Krasnoselskii–Mann iteration ([11,12]):

$$u_{n+1} = (1 - c_n)u_n + c_n A u_n.$$

Hence the N-algorithm can be considered a generalization of this classical method.

We proved that the sequence (u_n) generated by the procedure is such that the sequence $(||u_n - u^*||)$ is convergent for any fixed point u^* of A (see Proposition 4.1 of [7]).

If V is a uniformly convex Banach space and the coefficients of iterative scheme are such that $0 < \inf a_n \leq \sup a_n < 1$ and $0 < \inf c_n \leq \sup c_n < 1$, then the sequence $(||u_n - Au_n||)$ tends to zero (see Theorem 4.3 of [7]).

In the following, we give some sufficient conditions for the existence of solution of Equation (1), as well as the convergence of the N-algorithm to a fixed point of the operator A defined in (2). We recall two previous definitions (see, for instance, the reference [13]).

Definition 1. *Let V be a normed space. A linear mapping $A : V \to V$ is compact if for any bounded sequence $(\bar{v}_n) \subset A$ there is a subsequence $(\bar{v}_{n_j})$ such that $(A\bar{v}_{n_j})$ is convergent.*

Definition 2. *Let V be a normed space. A map $A : C \subseteq V \to V$ is demicompact if any bounded sequence $(\bar{v}_n)$ such that $(A\bar{v}_n - \bar{v}_n)$ is convergent has a convergent subsequence $(\bar{v}_{n_j})$.*

Theorem 1. *Consider $\Omega = [0,1]$ and the operator $A : \mathcal{L}^2(\Omega) \to \mathcal{L}^2(\Omega)$ defined in (2). Let us assume that $k : \Omega \times \Omega \to \mathbb{R}$ and $\psi : \Omega \times \mathbb{R} \to \mathbb{R}$ are continuous functions such that $0 \leq k(x,y) \leq 1$, $0 \leq \psi(y,z) \leq \min\{\alpha + \beta|z|^2, 1\}$, and $\alpha, \beta \geq 0$, and consider that ψ is nonexpansive in the second variable, which is to say,*

$$|\psi(y,z) - \psi(y,z')| \leq |z - z'|, \tag{6}$$

for any $y \in \Omega$ *and* $z, z' \in \mathbb{R}$. *Then the equation*

$$f(x) = \int_\Omega k(x,y)\psi(y, f(y))dy, \tag{7}$$

has at least one solution $f \in \mathcal{L}^2(\Omega)$, *such that* $||f||_2 \leq 1$.

For any $u_0 \in \bar{B}(f_0, 1)$, *where* $\bar{B}(f_0, 1)$ *is the closed ball in* $\mathcal{L}^2(\Omega)$ *with the center in the null function* f_0 *and the radius* 1, *the N-algorithm converges to a solution of the Equation (7) considering coefficients such that* $0 < \inf_n a_n \leq \sup_n a_n < 1$ *and* $0 < \inf_n c_n \leq \sup_n c_n < 1$.

Proof. Let us consider the operator A defined in (2). Then $A = K \circ \Psi$. The inequality (6) implies that Ψ is nonexpansive. As $||k||_2 \leq 1$, we have seen that A is also nonexpansive. Since

$$||Af||_2^2 = \int_\Omega (\int_\Omega k(x,y)\psi(y, f(y))dy)^2 dx \leq 1,$$

A can be restricted to be $A : \bar{B}(f_0, 1) \to \bar{B}(f_0, 1)$. Then A is nonexpansive on a bounded, closed, and convex set, and by Browder's Theorem ([14]) it has a fixed point. Consequently the equation

$$f(x) = \int_\Omega k(x,y)\psi(y, f(y))dy$$

has at least one solution.

Let us see now the convergence of the N-algorithm: It is well known that the operator K defined as $Kh(x) = \int_\Omega k(x,y)h(y)dy$, is nonexpansive and compact if $||k||_2 \leq 1$. Let us check that the restriction of $A = K \circ \Psi$ to $\bar{B}(f_0, 1)$ is demicompact:

Let $(\bar{v}_n) \subset \bar{B}(f_0, 1)$ be such that $(A\bar{v}_n - \bar{v}_n)$ is convergent. Then

$$||\Psi(\bar{v}_n)||_2^2 = \int_\Omega \psi(y, \bar{v}_n(y))^2 dy \leq 1$$

and $(\Psi\bar{v}_n)$ is also bounded. Since K is compact, there is a convergent subsequence $(A\bar{v}_{n_j}) = ((K \circ \Psi)\bar{v}_{n_j})$. Then

$$\bar{v}_{n_j} = (\bar{v}_{n_j} - A\bar{v}_{n_j}) + A\bar{v}_{n_j}$$

is convergent and consequently A is demicompact.

The sequence (u_n) generated by the N-algorithm, with the scalars described, to find a fixed point of A is such that $(Au_n - u_n)$ tends to zero (see Theorem 4.3 of [7]) and (u_n) is bounded. The demicompactness of A implies that there is a convergent subsequence (u_{m_k}). Let $f = \lim_{k \to \infty} u_{m_k}$.

The hypotheses on ψ and Lemma 1 imply that A is continuous. On the other hand, $((Id - A)u_{m_k})$ tends to zero, and f is a fixed point of A, which is to say, a solution of the Equation (7). Since $(||u_n - v^*||)$ is convergent for any fixed point v^* of A (Proposition 4.1 of [7]), the sequence (u_n) tends to f, and the N-method converges strongly to a solution f. □

Example 1. *Let us consider the Hammerstein equation*

$$f(x) = \int_0^1 k(x,y)(\alpha + \beta \sin^2(f(y)))dy, \tag{8}$$

where $k : [0,1] \times [0,1] \to \mathbb{R}$ *is continuous and* $0 \leq k(x,y) \leq 1$. *Here* $\psi(z) = \alpha + \beta \sin^2(z) \leq \alpha + \beta z^2$, *where* $\alpha, \beta \geq 0$ *and* $\alpha + \beta \leq 1$. *It is easy to see that* Ψ *is nonexpansive since* $|\sin^2(z) - \sin^2(z')| \leq |z - z'|$. *According to Theorem 1, Equation (8) has a solution.*

To perform the algorithm, we chose $k(x,y) = (x^2+y^2)/2$, *and* $\alpha = \beta = 1/2$. *The scheme used to approach a solution of this equation according to the N-method has been*

$$w_n(x) = (u_n(x) + \frac{1}{4}\int_0^1 (x^2+y^2)(1+\sin^2(u_n(y))dy)/2, \tag{9}$$

$$v_n(x) = (u_n(x) + w_n(x))/2, \tag{10}$$

$$u_{n+1} = (v_n(x) + \frac{1}{4}\int_0^1 (x^2+y^2)(1+\sin^2(v_n(y))dy)/2. \tag{11}$$

Table 1 collects the distances between 10 successive approximations of the solution: $E_n = ||u_n - u_{n-1}||_2$, $n = 1, 2, \ldots, 10$, *starting from the function* $u_0(x) = x$.

Figure 1 represents the graphs of the approximations of a solution given by the N-iteration, from $u_0(x)$ *to* $u_5(x)$.

Table 1. Distances between 10 successive approximations given by the *N*-algorithm to find a solution of the Hammerstein integral equation.

Iteration (n)	$E_n = ||u_n - u_{n-1}||_2$
1	0.216882
2	0.098364
3	0.043072
4	0.018419
5	0.007786
6	0.003275
7	0.001375
8	0.000577
9	0.000242
10	0.000103

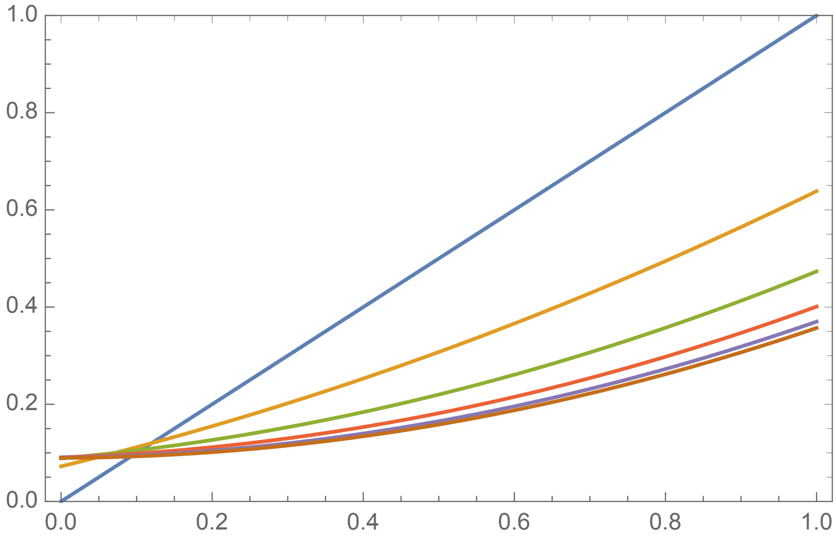

Figure 1. Zero-th (blue), first (yellow), second (green), third (red), fourth (purple), and fifth (brown) approximation of the solution.

3. A Modified Algorithm for the Search of Common Fixed Points of Nearly Asymptotically Nonexpansive Mappings

In this section, we analyze an algorithm to find iteratively common fixed points of nearly asymptotically nonexpansive mappings, in case of their existence.

All throughout this section, we assume a normed space $(V, ||\cdot||)$ and a nonempty subset $C \subseteq V$.

Definition 3. *A mapping $U : C \to V$ is nearly asymptotically nonexpansive if there are positive sequences (k_n), where $k_n \geq 1$, $\lim_n k_n = 1$, and (d_n), where $\lim_n d_n = 0$ fulfills the inequality*

$$||U^n u - U^n v|| \leq k_n ||u - v|| + d_n, \tag{12}$$

for any $u, v \in V$ and $n \in \mathbb{N}$. For $d_n = 0$, for all n, we obtain an asymptotically nonexpansive map. If additionally $k_n = 1$ for all n, then U is nonexpansive.

The structure of the set of fixed points of a nearly asymptotically nonexpansive map was studied in the reference [8].

The objective of the iterative scheme to find a common fixed point of a family $\{T_i\}_{i=1}^m$, $T_i : C \to C$, where C is convex and closed, is to define an approximation sequence (u_n) through the following process: $u_n^1 = u_n$ and

$$u_n^2 = (1 - c_n^1)u_n^1 + c_n^1 T_1^n u_n^1, \tag{13}$$

$$u_n^3 = (1 - c_n^2)u_n^2 + c_n^2 T_2^n u_n^2, \tag{14}$$

$$\ldots\ldots \tag{15}$$

$$u_n^i = (1 - c_n^{i-1})u_n^{i-1} + c_n^{i-1} T_{i-1}^n u_n^{i-1}, \tag{16}$$

$$\ldots\ldots \tag{17}$$

$$u_{n+1} = u_n^{m+1} = (1 - c_n^m)u_n^m + c_n^m T_m^n u_n^m, \tag{18}$$

for $u_0 \in C$ and $0 \leq c_n^i \leq 1$, for $n \geq 0$ and $i = 1, 2, \ldots, m$. The iteration will be called modified common m-steps N-iteration.

All throughout this section, we consider a family of nearly asymptotically nonexpansive mappings $\{T_i\}_{i=1}^m$ defined on a nonempty, closed, and convex set C, $T_i : C \to C$ such that

$$||T_i^n u - T_i^n v|| \leq k_n^i ||u - v|| + d_n^i, \tag{19}$$

where k_n^i and d_n^i are positive and satisfy the properties described in Definition 3. The superindex i is not a power but a counter.

F_{T_i} will denote the set of fixed points of T_i.

We will consider that the sets of fixed points are such that $\cap_{i=1}^m F_{T_i} \neq \emptyset$.

Definition 4. *The sequence $(v_n) \subseteq C$ has the common limit existence property (CLE property) if $\lim_n ||v_n - u^*|| = p \in \mathbb{R}$ for any $u^* \in \cap_{i=1}^m F_{T_i}$.*

The sequence $(v_n) \subseteq C$ has the approximation fixed point property (AF property) with respect to T_i if $\lim_n ||v_n - T_i v_n|| = 0$ (in this case, the existence of a fixed point of T_i is not required).

The next result can be consulted in the reference [15], Lemma 2.5.

Lemma 2. *Let (α_n), (β_n) and (γ_n) be sequences such that $\beta_n \geq 1$ and*

$$\alpha_{n+1} \leq \beta_n \alpha_n + \gamma_n \tag{20}$$

for $n \in \mathbb{N}$*. If* $\sum_{n=1}^{\infty}(\beta_n - 1) < \infty$ *and* $\sum_{n=1}^{\infty}\gamma_n < \infty$*, then the sequence* (α_n) *is convergent.*

Theorem 2. *Let* V *be a Banach space, and let* $C \subseteq V$ *be nonempty, closed, and convex. Let us consider a family of nearly asymptotically nonexpansive maps* $\{T_i\}_{i=1}^{m}$ *such that* $\cap_{i=1}^{m} F_{T_i} \neq \emptyset$ *with constants* k_n^i *and* d_n^i *satisfying Definition 3 for* $i = 1, \ldots, m$*. Let us consider*

$$\beta_n = \prod_{j=1}^{m} k_n^j$$

and

$$\gamma_n = (\prod_{j=2}^{m} k_n^j)d_n^1 + (\prod_{j=3}^{m} k_n^j)d_n^2 + \ldots + (\prod_{j=m-1}^{m} k_n^j)d_n^{m-2} + k_n^m d_n^{m-1} + d_n^m,$$

and assume that $\sum_{n=1}^{\infty}(\beta_n - 1) < \infty$ *and* $\sum_{n=1}^{\infty}\gamma_n < \infty$*, then* (u_n) *has the CLE property.*

Proof. Let $u^* = \cap_{i=1}^{m} F_{T_i}$. Then, applying the first step of the algorithm and the bounds of c_n^1, we have

$$||u_n^2 - u^*|| \leq (1 + c_n^1(k_n^1 - 1))||u_n^1 - u^*|| + c_n^1 d_n^1 \leq k_n^1||u_n^1 - u^*|| + d_n^1.$$

For the second step of the algorithm, we have

$$||u_n^3 - u^*|| \leq k_n^2||u_n^2 - u^*|| + d_n^2 \leq k_n^2 k_n^1||u_n^1 - u^*|| + k_n^2 d_n^1 + d_n^2.$$

Similarly,

$$||u_n^4 - u^*|| \leq (\prod_{j=1}^{3} k_n^j)||u_n^1 - u^*|| + (\prod_{j=2}^{3} k_n^j)d_n^1 + k_n^3 d_n^2 + d_n^3.$$

In general,

$$||u_n^i - u^*|| \leq \beta_n^i||u_n^1 - u^*|| + \gamma_n^i, \tag{21}$$

for $i = 3, \ldots m$, where

$$\beta_n^i := \prod_{j=1}^{i-1} k_n^j, \tag{22}$$

$$\gamma_n^i := (\prod_{j=2}^{i-1} k_n^j)d_n^1 + (\prod_{j=3}^{i-1} k_n^j)d_n^2 + \ldots + (\prod_{j=i-2}^{i-1} k_n^j)d_n^{i-3} + k_n^{i-1}d_n^{i-2} + d_n^{i-1}. \tag{23}$$

The last step of the procedure provides

$$||u_{n+1} - u^*|| = ||u_{n+1}^1 - u^*|| = ||u_n^{m+1} - u^*|| \leq \beta_n||u_n - u^*|| + \gamma_n,$$

where

$$\beta_n := \prod_{j=1}^{m} k_n^j \tag{24}$$

and

$$\gamma_n := (\prod_{j=2}^{m} k_n^j)d_n^1 + (\prod_{j=3}^{m} k_n^j)d_n^2 + \ldots + (\prod_{j=m-1}^{m} k_n^j)d_n^{m-2} + k_n^m d_n^{m-1} + d_n^m. \tag{25}$$

If $\sum_{n=1}^{\infty}(\beta_n - 1) < \infty$ and $\sum_{n=1}^{\infty}\gamma_n < \infty$, according to Lemma 2, the sequence $(||u_n - u^*||)$ is convergent, and (u_n) has the CLE property. □

Definition 5. *Let V be a normed space and $C \subseteq X$. A mapping $N : C \to X$ is asymptotically regular if*

$$\lim_{n\to\infty} ||N^{n+1}u - N^n u|| = 0,$$

for any $u \in C$.

The next result can be consulted in the reference [16].

Lemma 3. *Let V be a uniformly convex Banach space, and a sequence $(\gamma_n) \subseteq V$ be such that there is an $a, b \in \mathbb{R}$ satisfying the condition $0 < a \leq \inf\{\gamma_n\} \leq \sup\{\gamma_n\} \leq b < 1$. Let (u_n), (v_n) be sequences in V such that $\limsup_{n\to\infty} ||u_n|| \leq p$, $\limsup_{n\to\infty} ||v_n|| \leq p$, and $\limsup_{n\to\infty} ||\gamma_n u_n + (1-\gamma_n)v_n|| = p$ for some $p \geq 0$. Then $\lim_{n\to\infty} ||u_n - v_n|| = 0$.*

Theorem 3. *Let V be a uniformly convex Banach space, and let C be a subset of V that is convex and closed. Let $\{T_i\}_{i=1}^m$ be a family of nearly asymptotically nonexpansive operators, and let $T_i : C \to C$ be such that $\cap_{i=1}^m F_{T_i} \neq \emptyset$. Let us assume that β_n and γ_n defined in (24) and (25), respectively, are such that $\sum_{n=1}^\infty (\beta_n - 1) < \infty$ and $\sum_{n=1}^\infty \gamma_n < \infty$. Let (u_n) be the modified common m-steps N-iteration for $u_0 \in C$, where the scalars are chosen such that*

$$0 < \inf_n c_n^i \leq \sup_n c_n^i < 1, \tag{26}$$

for $i = 1, 2, \ldots, m$. Then

1. *the map N defined as $Nu_n = u_{n+1}$ is asymptotically regular, and,*
2. *if additionally T_1 is uniformly continuous, the sequence (u_n) has the AF property with respect to T_1.*

Proof. We first note that β_n^i and γ_n^i defined in (22) and (23), respectively, are such that $\lim_{n\to\infty} \beta_n^i = 1$ and $\lim_{n\to\infty} \gamma_n^i = 0$, due to the conditions on k_n^i and d_n^i. These limits hold for $i = 1, 2, \ldots, m$.

Theorem 2 implies that, for $u^* \in \cap_{i=1}^m F_{T_i}$,

$$p = \lim_{n\to\infty} ||u_{n+1} - u^*|| = \lim_{n\to\infty} ||(1-c_n^m)(u_n^m - u^*) + c_n^m(T_m^n u_n^m - u^*)||.$$

Since, by (21),

$$\limsup_{n\to\infty} ||u_n^m - u^*|| \leq \limsup_{n\to\infty} \beta_n^m ||u_n - u^*|| + \gamma_n^m = p$$

and, due to the nearly asymptotic nonexpansiveness of T_m,

$$\limsup_{n\to\infty} ||T_m^n u_n^m - u^*|| \leq \limsup_{n\to\infty} k_n^m ||u_n^m - u^*|| + d_n^m \leq p,$$

applying Lemma 3, we obtain the following:

$$\lim_{n\to\infty} ||u_n^m - T_m^n u_n^m|| = 0. \tag{27}$$

Moreover,

$$||u_{n+1} - u^*|| \leq ||u_n^m - u^*|| + c_n^m ||T_m^n u_n^m - u_n^m||$$

Therefore,

$$p \leq \liminf_{n\to\infty} ||u_n^m - u^*||,$$

and $\lim_{n\to\infty}||u_n^m - u^*|| = p$. Now

$$p = \lim_{n\to\infty}||u_n^m - u^*|| = \lim_{n\to\infty}||(1-c_n^{m-1})(u_n^{m-1} - u^*) + c_n^{m-1}(T_{m-1}^n u_n^{m-1} - u^*)||.$$

$$\limsup_{n\to\infty}||u_n^{m-1} - u^*|| \leq \limsup_{n\to\infty}(\beta_n^{m-1}||u_n - u^*|| + \gamma_n^{m-1}) = p$$

and

$$\limsup_{n\to\infty}||T_{m-1}^n u_n^{m-1} - u^*|| \leq \limsup_{n\to\infty}(k_n^{m-1}||u_n^{m-1} - u^*|| + d_n^{m-1}) \leq p.$$

Applying Lemma 3, we obtain the following:

$$\lim_{n\to\infty}||u_n^{m-1} - T_{m-1}^n u_n^{m-1}|| = 0. \tag{28}$$

Iteratively, we obtain that

$$\lim_{n\to\infty}||u_n^i - T_i^n u_n^i|| = 0, \tag{29}$$

for $i = 1, 2, \ldots, m$.

Let us check now the asymptotical regularity of N defined as $Nu_n = u_{n+1}$.

$$||N^{n+1}u_0 - N^n u_0|| = ||u_{n+1} - u_n|| \leq ||u_{n+1} - u_n^m|| + ||u_n^m - u_n^{m-1}|| + \ldots + ||u_n^2 - u_n^1||.$$

According to the algorithm and (29),

$$||u_{n+1} - u_n^m|| = c_n^m||u_n^m - T_m^n u_n^m|| \to 0,$$

$$||u_n^m - u_n^{m-1}|| = c_n^{m-1}||u_n^{m-1} - T_{m-1}^n u_n^{m-1}|| \to 0,$$

etc. Consequently,

$$||N^{n+1}u_0 - N^n u_0|| \to 0, \tag{30}$$

and N is asymptotically regular. We see now that $(u_n^1) = (u_n)$ has the AF property with respect to T_1.

$$||u_n^1 - T_1 u_n^1|| \leq$$
$$||u_n^1 - u_{n+1}^1|| + ||u_{n+1}^1 - T_1^{n+1}u_{n+1}^1|| + ||T_1^{n+1}u_{n+1}^1 - T_1^{n+1}u_n^1|| + ||T_1^{n+1}u_n^1 - T_1 u_n^1||.$$

The first term tends to zero due to the asymptotic regularity of N. The limit of the second term is also null due to (29). The definition of nearly asymptotic nonexpansiveness of T_1 implies that

$$||T_1^{n+1}u_{n+1}^1 - T_1^{n+1}u_n^1|| \leq k_{n+1}^1||u_n^1 - u_{n+1}^1|| + d_{n+1}^1 \to 0.$$

The last term converges to zero due to the uniform continuity of T_1 and (29) for $i = 1$. Consequently,

$$\lim_{n\to\infty}||u_n - T_1 u_n|| = \lim_{n\to\infty}||u_n^1 - T_1 u_n^1|| = 0,$$

and (u_n) has the AF property with respect to T_1. □

In the following, we give some results of convergence of the modified common algorithm for nearly asymptotically nonexpansive mappings. They are based on the approximation properties given by Theorems 2 and 3. We will assume a family of nearly asymptotically nonexpansive mappings $(T_i)_{i=1}^m$ such that $T_i : C \to C$ and $\cap_{i=1}^m F_{T_i} \neq \emptyset$.

Theorem 4. *Let V be a uniformly convex Banach space, and $C \subseteq V$, $C \neq \emptyset$, be compact and convex. Let $\{T_i\}_{i=1}^m$ be a set of uniformly continuous nearly asymptotically nonexpansive mappings with constants k_n^i and d_n^i such that*

$$\beta_n = \prod_{j=1}^m k_n^j \tag{31}$$

and

$$\gamma_n = (\prod_{j=2}^m k_n^j)d_n^1 + (\prod_{j=3}^m k_n^j)d_n^2 + \ldots + (\prod_{j=m-1}^m k_n^j)d_n^{m-2} + k_n^m d_n^{m-1} + d_n^m. \tag{32}$$

are such that $\sum_{n=1}^\infty (\beta_n - 1) < \infty$ and $\sum_{n=1}^\infty \gamma_n < \infty$. Then the modified common algorithm with the scalars chosen such that $0 < \inf_n c_n^i \leq \sup_n c_n^i < 1$ converges strongly to a common fixed point of $\{T_i\}_{i=1}^m$.

Proof. The sequence (u_n) is obviously bounded. Since C is compact, there is a convergent subsequence (u_{m_i}). Let $\lim_{i\to\infty} u_{m_i} = u^*$. The AF property of (u_n) with respect to T_1 (Theorem 3) implies that $\lim_{i\to\infty} ||u_{m_i} - T_1 u_{m_i}|| = 0$. Since T_1 is continuous, $(Id - T_1)u_{m_i} \to (u^* - T_1 u^*) = 0$, and $u^* \in F_{T_1}$. The CLE property of (u_n) implies that $\lim_{n\to\infty} u_n = u^*$. Let us see now that u^* is a common fixed point.

Using the first step of the algorithm and the properties of T_1,

$$||u_n^2 - u^*|| = ||(1 - c_n^1)(u_n^1 - u^*) + c_n^1(T_1^n u_n^1 - u^*)||,$$

so

$$||u_n^2 - u^*|| \leq (1 - c_n^1)||u_n - u^*|| + c_n^1(k_n^1||u_n - u^*|| + d_n^1),$$

and $||u_n^2 - u^*|| \to 0$. Iteratively we find that

$$||u_n^i - u^*|| \to 0,$$

for all $i = 1, 2, \ldots, m$. Using the arguments given at the end of the last theorem, the uniform continuity of T_i implies that $||u_n^i - T_i u_n^i||$ tends to zero and consequently (u_n^i) has the AF property with respect to T_i. Since $\lim_{n\to\infty} u_n^i = u^*$ and T_i is continuous, $(Id - T_i)u^* = 0$ for all i and thus $u^* \in \cap_{i=1}^m F_{T_i}$. Therefore, the iteration (u_n) converges strongly to a common fixed point. □

Corollary 1. *Let V be a uniformly convex Banach space, and let $C \subseteq V$, $C \neq \emptyset$, be compact and convex. Let $\{T_i\}_{i=1}^m$ be a set of asymptotically nonexpansive mappings where $\beta_n = \prod_{i=1}^m k_n^i$ is such that $\sum_{n=1}^\infty (\beta_n - 1) < \infty$. Then the modified common algorithm with the scalars chosen such that $0 < \inf_n c_n^i \leq \sup_n c_n^i < 1$ converges to a common fixed point of $\{T_i\}_{i=1}^m$.*

Proof. Asymptotically nonexpansive mappings are nearly asymptotically nonexpansive mappings with $d_n^i = 0$, and they are uniformly continuous. Therefore, the result is a consequence of the last theorem. □

4. An Iterative Method for the Search of Common Fixed Points of a Finite Number of Quasi-Nonexpansive Operators

For nonexpansive maps, the algorithm studied in the last section may be applicable, since these mappings are nearly asymptotically nonexpansive. However, its implementation is expensive due to the great number of evaluations of the maps required. In the reference [9], we analyzed the properties of an algorithm for the search of common fixed points of two nonexpansive mappings and its application to the approximation of fixed points of fractal convolutions of operators (for the latter concept, see the reference [17]). We proposed also an iterative scheme for finding a common fixed point of a finite collection of mappings $T_1, T_2, \cdots, T_m$. However, the approximation and convergence properties of the algorithm were only studied for $m = 2$. In this section, we consider the general case $m \geq 2$.

The algorithm requires the structure of a normed space V. We will assume that $\cap_{i=1}^{m} F_{T_i} \neq \emptyset$ and the maps T_i are quasi-nonexpansive for $i = 1, 2, \cdots m$, which is to say, satisfying the inequality

$$||T_i u - u^*|| \leq ||u - u^*||,$$

for any u in the domain of T_i and $u^* \in F_{T_i}$, where F_{T_i} denotes the set of fixed points of T_i as usual. We will assume $T_i : C \subseteq V \to C$, where C is a convex and closed subset.

The algorithm defines a sequence of approximations $u_n := u_n^1$, for $n = 1, 2, \ldots$ beginning from $u_0 := u_0^1 \in C$. The scheme proposed is the following:

$$u_n^2 = (1 - c_n^1) u_n^1 + c_n^1 T_1 u_n^1, \tag{33}$$

$$u_n^3 = (1 - c_n^2) u_n^2 + c_n^2 T_2 u_n^2, \tag{34}$$

$$\ldots \tag{35}$$

$$u_n^i = (1 - c_n^{i-1}) u_n^{i-1} + c_n^{i-1} T_{i-1} u_n^{i-1}, \tag{36}$$

$$\ldots \tag{37}$$

$$u_n^m = (1 - c_n^{m-1}) u_n^{m-1} + c_n^{m-1} T_{m-1} u_n^{m-1}, \tag{38}$$

$$u_{n+1}^1 = u_n^{m+1} = (1 - c_n^m) u_n^m + c_n^m T_m u_n^m, \tag{39}$$

where $0 \leq c_n^i \leq 1$ for $i = 1, 2, \ldots, m$ and $n \geq 0$.

Proposition 1. *The sequence (u_n) generated by the m-steps common fixed point algorithm has the CLE property and, for any $u^* \in \cap_{i=1}^{m} F_{T_i}$,*

$$||u_n^m - u^*|| \leq ||u_n^{m-1} - u^*|| \leq \ldots \leq ||u_n^1 - u^*|| = ||u_n - u^*||, \tag{40}$$

for $n \in \mathbb{N}$.

Proof. Let $u^* \in \cap_{i=1}^{m} F_{T_i}$. The quasi-nonexpansiveness of T_1 implies that

$$||u_n^2 - u^*|| \leq (1 - c_n^1) ||u_n^1 - u^*|| + c_n^1 ||u_n^1 - u^*|| \leq ||u_n^1 - u^*||.$$

In the same way, we obtain

$$||u_n^3 - u^*|| \leq ||u_n^2 - u^*|| \leq ||u_n^1 - u^*||$$

and, in general,

$$||u_n^m - u^*|| \leq ||u_n^{m-1} - u^*|| \leq \ldots \leq ||u_n^1 - u^*|| = ||u_n - u^*||. \tag{41}$$

Using the last step of the algorithm,

$$||u_{n+1} - u^*|| = ||u_{n+1}^1 - u^*|| \leq ||u_n^m - u^*|| \leq ||u_n - u^*||. \tag{42}$$

Consequently, $(||u_n - u^*||)$ is bounded and decreasing, and it has a limit. □

Theorem 5. *If V is a uniformly convex Banach space and $0 < \inf_n c_n^i \leq \sup_n c_n^i < 1$ for all $i = 1, 2, \ldots m$, then the following holds:*

1. *The sequence (u_n^i) has the CLE property for any $i = 1, 2, \ldots m$.*

2. *The sequence* (u_n^i) *has the AF property with respect to* T_i *for any* $i = 1, 2, \dots m$.

Proof. In Proposition 1, we have proved that (u_n) has the CLE property, which is to say, there is a $p = \lim_{n\to\infty} ||u_n - u^*||$. Then

$$p = \lim_{n\to\infty} ||u_{n+1} - u^*|| = \lim_{n\to\infty} ||(1 - c_n^m)(u_n^m - u^*) + c_n^m(T_m u_n^m - u^*)||.$$

Bearing in mind (40),

$$\limsup_{n\to\infty} ||u_n^m - u^*|| \leq \limsup_{n\to\infty} ||u_n - u^*|| = p. \tag{43}$$

The quasi-nonexpansiveness of T_m implies that

$$\limsup_{n\to\infty} ||T_m u_n^m - u^*|| \leq \limsup_{n\to\infty} ||u_n^m - u^*|| \leq p. \tag{44}$$

As a consequence, Lemma 3 implies that

$$\lim_{n\to\infty} ||u_n^m - T_m u_n^m|| = 0, \tag{45}$$

and (u_n^m) has the AF property with respect to T_m. Moreover,

$$||u_{n+1} - u^*|| \leq ||u_n^m - u^*|| + c_n^m ||T_m u_n^m - u_n^m||.$$

Then, by (45),

$$p \leq \liminf_{n\to\infty} ||u_n^m - u^*||. \tag{46}$$

The inequalities (43) and (46) imply that

$$p = \lim_{n\to\infty} ||u_n^m - u^*||, \tag{47}$$

and (u_n^m) has the CLE property. Then

$$p = \lim_{n\to\infty} ||u_n^m - u^*|| = \lim_{n\to\infty} ||(1 - c_n^{m-1})(u_n^{m-1} - u^*) + c_n^{m-1}(T_{m-1} u_n^{m-1} - u^*)||,$$

and thus, by (40),

$$\limsup_{n\to\infty} ||T_{m-1} u_n^{m-1} - u^*|| \leq \limsup_{n\to\infty} ||u_n^{m-1} - u^*|| \leq \lim_{n\to\infty} ||u_n - u^*|| = p. \tag{48}$$

Using the Lemma 3,

$$\lim_{n\to\infty} ||u_n^{m-1} - T_{m-1} u_n^{m-1}|| = 0, \tag{49}$$

and (u_n^{m-1}) has the AF property with respect to T_{m-1}. Considering that

$$||u_n^m - u^*|| \leq ||u_n^{m-1} - u^*|| + c_n^{m-1} ||T_{m-1} u_n^{m-1} - u_n^{m-1}||,$$

by (49),

$$p = \liminf_{n\to\infty} ||u_n^m - u^*|| \leq \liminf_{n\to\infty} ||u_n^{m-1} - u^*||, \tag{50}$$

and thus $p = \lim_{n\to\infty} ||u_n^{m-1} - u^*||$. Repeating the same argument m-times, it follows that

$$\lim_{n\to\infty} ||u_n - T_1 u_n|| = \lim_{n\to\infty} ||u_n^1 - T_1 u_n^1|| = 0.$$

Then (u^i_n) has the AF property with respect to T_i and it has the CLE property for any $i = 1, 2, \ldots, m$. □

Strong Convergence of the m-Steps Common N-Iteration

In this section, we establish some results on the convergence of the m-steps common N-iteration to a common fixed point of a collection of mappings $T_1, T_2, \ldots, T_m$. We will assume that V is a uniformly convex Banach space, the maps $T_i : C \to C$, where $C \subseteq V$ is nonempty, closed, and convex, are quasi-nonexpansive, and $\cap_{i=1}^m F_{T_i} \neq \emptyset$. The symbol Id will denote the identity mapping as before.

Theorem 6. *If C is compact and T_i is continuous for any i, then the m-steps common fixed point algorithm converges strongly to $u^* \in \cap_{i=1}^m F_{T_i}$.*

Proof. The fact that C is compact implies that there is a convergent subsequence $(u^2_{n_j})$ of (u^2_n). Let $u^* = \lim_{j\to\infty} u^2_{n_j}$. Since $Id - T_2$ is continuous and $((Id - T_2)u^2_{n_j})$ tends to zero, due to the AF property of (u^2_n) with respect to T_2, $(Id - T_2)u^* = 0$ and consequently $u^* \in F_{T_2}$. Bearing in mind that

$$||u^3_{n_j} - u^*|| = ||(1 - c^2_{n_j})(u^2_{n_j} - u^*) + c^2_{n_j}(T_2 u^2_{n_j} - u^*)|| \leq ||u^2_{n_j} - u^*|| \to 0, \tag{51}$$

it follows that $\lim_{j\to\infty} u^3_{n_j} = u^*$. Repeating the same argument for $i = 4, \ldots m$, we find that $\lim_{j\to\infty} u^m_{n_j} = u^*$ and $u^* \in F_{T_m}$. Now, by the last step of the algorithm,

$$||u^1_{n_j+1} - u^*|| = ||(1 - c^m_{n_j})(u^m_{n_j} - u^*) + c^m_{n_j}(T_m u^m_{n_j} - u^*)|| \leq ||u^m_{n_j} - u^*|| \to 0, \tag{52}$$

and $\lim_{j\to\infty} u^1_{n_j+1} = u^*$. Since $Id - T_1$ is continuous and $((Id - T_1)u^1_{n_j+1})$ tends to zero, then $u^* \in F_{T_1}$. Consequently, u^* is a common fixed point of T_i for $i = 1, 2, \ldots, m$.

The CLE property of (u_n) implies that $\lim_{n\to\infty} u_n = u^*$, and the common N-iteration converges strongly to u^*. □

The next type of mapping was introduced in the reference [7], Definition 3.1.

Definition 6. *Let V be a normed space. $T : C \subseteq V \to V$ is a nonexpansive partial contractivity if there is a $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\psi(0) = 0$, such that, for any $u, v \in C$,*

$$||Tu - Tv|| \leq ||u - v|| + \psi(\min\{||u - Tu||, ||v - Tv||\}). \tag{53}$$

Remark 1. *A nonexpansive map is a nonexpansive partial contractivity (taking ψ as the null function).*

Corollary 2. *If C is compact and T_i are continuous nonexpansive partial contractivities for $i = 1, 2, \ldots m$, the m-steps common N-iteration converges to a common fixed point.*

Proof. If a nonexpansive partial contractivity has fixed points, then it is quasi-nonexpansive, and the result is a straightforward consequence of the previous theorem. □

Corollary 3. *If C is compact and T_i are nonexpansive for $i = 1, 2, \ldots m$, the m-steps common N-iteration converges to a common fixed point.*

Proof. A nonexpansive map with some fixed point is continuous and quasi-nonexpansive, and the result is a consequence of Theorem 6. □

The following concept may be consulted, for instance, in the reference [18].

Definition 7. *A map* $T : C \subseteq V \to V$ *is demicompact at zero if a bounded sequence* $(u_n) \subseteq C$ *satisfying* $\lim_{n\to\infty} Tu_n - u_n = 0$ *has a convergent subsequence.*

Theorem 7. *If* $T_i : C \to C$*, where* C *is nonempty, closed, and convex, is quasi-nonexpansive and continuous for* $i = 1, 2, \ldots, m$*, and there is a* $j \in \{1, 2, \ldots, m\}$ *such that* T_j *is demicompact at zero, then the common m-steps algorithm converges strongly to a common fixed point of* $\{T_i\}_{i=1}^m$*.*

Proof. Let us rearrange the steps of the algorithm so that the demicompact at zero map T_j is the new T_2. By Theorem 5, the sequence $(||u_n^2 - v^*||)$ is convergent for any $v^* \in \cap_{i=1}^m F_{T_i}$, so (u_n^2) is bounded and, by the same Theorem, $\lim_{n\to\infty} T_2 u_n^2 - u_n^2 = 0$. Since T_2 is demicompact, there is a convergent subsequence $(u_{n_j}^2)$. Let $\lim_{j\to\infty} u_{n_j}^2 = u^*$.

The continuity of T_2, along with the AF property of (u_n^2), implies that $(Id - T_2)u^* = 0$, and $u^* \in F_{T_2}$. Now, by Proposition 1,

$$||u_n^3 - u^*|| \leq ||u_n^2 - u^*|| \to 0,$$

and consequently $\lim_{j\to\infty} u_{n_j}^3 = u^*$. The continuity of T_3 implies that $u^* \in F_{T_3}$. Repeating the same argument, we have that $\lim_{j\to\infty} u_{n_j}^{m+1} = \lim_{j\to\infty} u_{n_j+1}^1 = u^*$, and $u^* \in \cap_{i=1}^m F_{T_i}$. The CLE property of (u_n) implies that $\lim_{n\to\infty} u_n = \lim_{n\to\infty} u_n^1 = u^*$. ☐

Corollary 4. *If* $T_i : C \to C$*, where* C *is nonempty, closed, and convex, is a continuous nonexpansive partial contractivity for any* $i = 1, 2, \ldots, m$ *and there is a* j *such that* T_j *is demicompact at zero, then the common m-steps algorithm converges strongly to a common fixed point of* $\{T_i\}_{i=1}^m$*.*

Proof. A continuous nonexpansive partial contractivity with fixed points is quasi-nonexpansive, and the hypotheses of the previous theorem are satisfied. ☐

Corollary 5. *If* $T_i : C \to C$*, where* C *is nonempty, closed, and convex, is a nonexpansive for any* $i = 1, 2, \ldots, m$ *and there is a* j *such that* T_j *is demicompact at zero, then the common m-steps algorithm converges strongly to a common fixed point of* $\{T_i\}_{i=1}^m$*.*

Proof. A nonexpansive map is a continuous nonexpansive partial contractivity, and this is a particular case of the previous corollary. ☐

Definition 8. *A map* $U : C \subseteq V \to V$ *is demiclosed if* $u_n \rightharpoonup u$ *(the sequence* (u_n) *converges weakly to* u*) and* $Uu_n \to v$ *imply that* $v = Uu$*.*

The following is a statement of demiclosedness for nonexpansive mappings. The reference [19], Theorem 10.4, may be consulted.

Theorem 8. *If* V *is a uniformly convex Banach space and* $T : C \to V$*, where* $C \subseteq V$ *is nonempty, closed, and convex, is nonexpansive, then* $Id - T$ *is demiclosed.*

Definition 9. *A map* $U : C \subseteq V \to V$ *is completely continuous if* $u_n \rightharpoonup u$ *implies that* $Uu_n \to Uu$*.*

Recall that, all throughout this subsection, V is a uniformly convex Banach space, $T_i : C \to C$, and $\cap_{i=1}^m F_{T_i} \neq \emptyset$.

Proposition 2. *Let* $C \subseteq V$ *be nonempty, bounded, closed, and convex. If* T_i *is nonexpansive for any* $i = 1, 2, \ldots m$ *and there is a* $j \in \{1, 2, \ldots, m\}$ *such that* T_j *is completely continuous, then the common m-steps algorithm converges strongly to a common fixed point of* $\{T_i\}_{i=1}^m$*.*

Proof. Let us rearrange the collection of maps so that T_j is the new T_2. The properties of boundedness, closedness, and convexity imply that (u_n^2) has a weakly convergent subsequence $(u_{n_j}^2)$. Let $u_{n_j}^2 \rightharpoonup u^*$. The AF property of (u_n^2) with respect to T_2 implies that $(||u_{n_j}^2 - T_2 u_{n_j}^2||)$ tends to zero. Since $Id - T_2$ is demiclosed (Theorem 8), then $(Id - T_2)u^* = 0$ and $u^* \in F_{T_2}$.

The complete continuity of T_2 implies that $\lim_{j\to\infty} T_2 u_{n_j}^2 = T_2 u^* = u^*$. Then

$$u_{n_j}^2 = u_{n_j}^2 - T_2 u_{n_j}^2 + T_2 u_{n_j}^2 \to u^*. \tag{54}$$

The fact that

$$||u_{n_j}^3 - u^*|| \leq ||u_{n_j}^2 - u^*|| \to 0,$$

along with the AF property of $(u_{n_j}^3)$ and the continuity of T_3 imply that $u^* \in F_{T_3}$.

Iteratively, we find that $u_{n_j}^m \to u^*$ and $u^* \in F_{T_m}$. Applying the last step of the algorithm, we find that

$$||u_{n_j+1}^1 - u^*|| \leq ||u_{n_j}^m - u^*|| \to 0.$$

We know also that $(Id - T_1)u_{n_j+1}^1 \to 0$ and $Id - T_1$ is demiclosed, so $u^* \in F_{T_1}$. Consequently, $u^* \in \cap_{i=1}^m F_{T_i}$. The CLE property of $u_n = u_n^1$ implies that $\lim_{n\to\infty} u_n = u^*$. □

In the following, we give an example of an application of the common m-steps N-algorithm.

Example 2. *The next three integral equations of the Hammerstein type have a common solution at* $f(x) = x$ *(see the references [20,21]).*

$$f(x) = x - \frac{1}{40}(e-1)e^{x^4} + \frac{1}{10}\int_0^1 e^{x^4+y^4} f^3(y)dy. \tag{55}$$

$$f(x) = x + \frac{1}{20}\cos(e+t) - \frac{1}{20}\cos(1+t) + \frac{1}{20}\int_0^1 \sin(x+e^y)e^{f(y)}dy. \tag{56}$$

$$f(x) = x + 0.2 - 0.4\int_0^1 f(y)dy. \tag{57}$$

The common solution is a common fixed point of the Hammerstein operators:

$$T_1 u(t) = g_1(t) + \int_0^1 k_1(s,t)\psi_1(s,u(s))ds, \tag{58}$$

$$T_2 u(t) = g_2(t) + \int_0^1 k_2(s,t)\psi_2(s,u(s))ds, \tag{59}$$

$$T_3 u(t) = g_3(t) + \int_0^1 k_3(s,t)\psi_3(s,u(s))ds, \tag{60}$$

where

$$g_1(t) = t - (e-1)e^{t^4}/40, \qquad k_1(s,t) = e^{s^4+t^4}/10, \qquad \psi_1(s,u(s)) = u^3(s),$$

$$g_2(t) = t + (\cos(e+t) - \cos(1+t))/20, \qquad k_2(s,t) = \sin(t+e^s)/20, \qquad \psi_2(s,u(s)) = e^{u(s)},$$

$$g_3(t) = t + 0.2, \qquad k_3(s,t) = 0.4, \qquad \psi_3(s,u(s)) = -u(s).$$

The common three steps N-iteration has been implemented, using the coefficients $c_n^1 = c_n^2 = c_n^3 = 1/2$. *Thus, the scheme for* $u_n = u_n^1$ *is the following:*

$$u_n^2 = (u_n^1 + T_1 u_n^1)/2,$$

$$u_n^3 = (u_n^2 + T_2 u_n^2)/2,$$

$$u_n^4 = (u_n^3 + T_3 u_n^3)/2.$$

Table 2 gathers the distances between six successive approximations of the algorithm and the solution $f(x) = x$, *(*$E_n = ||u_n - f||_2$, $n = 0, 1, 2, \ldots, 5$*) starting from* $u_0(x) = 0$ *for any* $x \in [0, 1]$.

Table 2. Distances between the successive approximations (from zeroth to fifth) and the exact solution given by the common three-steps N- algorithm.

Iteration (n)	$E_n = \|\|u_n - f\|\|_2$
0	0.57735
1	0.056664
2	0.006714
3	0.000841
4	0.000108
5	0.000014

5. Conclusions

A particular case of Hammerstein nonlinear integral equation was studied first. Sufficient conditions for the existence of solutions in the space of square-integrable functions on an interval were given. In the reference [7], an iterative method to solve equations of any kind was proposed. This procedure was proved to be convergent, and hence suitable for the solution of the integral equation considered.

Afterwards, two different variants of the algorithm were proposed. The first one was an iterative method designed to find common fixed points of a set of nearly asymptotically nonexpansive mappings, and the second one had the same purpose, but the target was a system of nonexpansive mappings. In both cases, some theorems of strong convergence of the successive approximations provided by the algorithm to a common fixed point were proved.

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