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Communication

Mueller Matrix Associated with an Arbitrary 4×4 Real Matrix. The Effective Component of a Mueller Matrix

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Abstract: Due to the limited accuracy of experimental data, Mueller polarimetry can produce real 4×4 matrices that fail to meet required covariance or passivity conditions. A general and simple procedure to convert any real 4×4 matrix into a valid Mueller matrix by adding a portion of polarimetric white noise is presented. This approach provides deeper insight into the structure of Mueller matrices and has a subtle relation to the effective component of the Mueller matrix, which is defined through the subtraction of the fully random component of the characteristic decomposition. Up to a scale coefficient determined by the third index of polarimetric purity of the original Mueller matrix, the effective component retains complete information on the polarimetric anisotropies.

Keywords: Mueller matrices; Mueller polarimetry; Mueller filtering

1. Introduction

The wide scope of applications of Mueller polarimetry in a great variety of areas of science, industry and medical diagnosis supports the need to have the best possible knowledge of the mathematical structure and properties of Mueller matrices. In this work, the background of well-established results on the characteristic properties of Mueller matrices is applied to answer the following question: given a 4×4 real matrix **A**, how can it be transformed into a Mueller matrix through a simple and consistent procedure?

The problem is solved by means of two consecutive steps: (1) the analysis of the eigenvalue structure of a Hermitian matrix associated with the given 4×4 real non-Mueller matrix **A** and (2) the addition of an appropriate portion of polarimetric white noise via the characteristic decomposition of **A**. The transformed matrix coincides with the original one except for the element a_{00} (the matrix elements being denoted as a_{ij} with $i, j = 0, 1, 2, 3$), in such a manner that all polarimetric anisotropies are preserved up to a global scale coefficient.

When dealing with a Mueller matrix, **M**, the subtraction of the polarimetric white noise leads to the introduction of the effective component of **M**, whose polarimetric descriptors are proportional to those of **M** through a coefficient larger than one that is determined by the inverse of the third component of purity of **M**.

To contextualize appropriately the present communication, it is worth summarizing the history of the main works that have dealt with the mathematical representation of linear polarimetric transformations.

A general characterization has been achieved through the so-called ensemble criterion, constituted by the combination of the covariance and passivity criteria [1–14]. The covariance criterion, introduced by Cloude [1] and independently by Arnal [4] consists of the nonnegativity of the covariance (or coherency) Hermitian matrix associated with a given

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Mueller matrix [1,4,8], while the passivity criterion states that the linear transformation represented by a Mueller matrix never increases the intensity of light in either forward or reverse natural interactions [8,13]. To avoid confusion, we will call Mueller–covariance matrices those that satisfy the covariance conditions regardless of whether or not they satisfy the passivity criterion, while the term Mueller matrix will be applied to matrices that satisfy both covariance and passivity criteria.

The first formulation of the linear transformations of Stokes parameters was due to Soleillet [15,16], while related independent approaches were published by Perrin [17] and Parke III [18,19] (a PhD student of H. Mueller at the time).

As demonstrated by Bolshakov, van der Mee, Ran, Reichstein and Rodman [20,21] and independently by Gopala, Mallesh and Sudha [22,23], the covariance conditions are equivalent to the nonnegativity of the four eigenvalues of the matrix $\mathbf{N} = \mathbf{G}\,\mathbf{M}^{\mathrm{T}}\mathbf{G}\,\mathbf{M}$ (where $\mathbf{M}$ is the Mueller matrix, $\mathbf{G} \equiv \mathrm{diag}(1, -1, -1, -1)$, and the superscript T stands for the transpose matrix). This approach is closely related to the so-called normal form of $\mathbf{M}$, expressed as the serial decomposition $\mathbf{M} = \mathbf{M}_{J2}\,\mathbf{M}_{\Delta d}\,\mathbf{M}_{J1}$, ($\mathbf{M}_{J1}$ and $\mathbf{M}_{J2}$ being the pure Mueller matrices of respective nondepolarizing components, while the central depolarizer $\mathbf{M}_{\Delta d}$ takes two alternative forms depending on whether matrix $\mathbf{N}$ is diagonalizable (type-I Mueller matrices) or not (type-II Mueller matrices)). A detailed analysis of the components of the normal form of $\mathbf{M}$, together with the explicit algebraic procedure for their calculation was then presented by Ossikovski [24,25], while a general numerical algorithm to perform such a decomposition, valid for both type-I and type-II measured matrices, can be found in [26].

A number of pioneering works dealing with Jones or Mueller matrices preceded the covariance characterization [27–36], while other later contributions addressed complementary and relevant mathematical aspects [37–70]. Due to the great variety of applications of Mueller polarimetry, other successive complementary approaches of the structure of Jones and Mueller matrices have also been reported from different points of view, namely polarimetric descriptors [71–95], reciprocity [96–99], geometric views [47,100–106], differential formulation [107–120] and statistical analyses [121–125].

2. Theoretical Framework

The transformation of polarized light by the action of a linear medium (under fixed interaction conditions) can always be represented mathematically as $\mathbf{s'} = \mathbf{M}\mathbf{s}$, where $\mathbf{s}$ and $\mathbf{s'}$ are the Stokes vectors that represent the states of polarization of the incident and emerging light beams, respectively, while $\mathbf{M}$ is the Mueller matrix associated with this kind of interaction, which can always be written in the following form [38,126] (throughout this communication, the most common notations are used):

$$\begin{gathered}\mathbf{M} = m_{00}\,\hat{\mathbf{M}}, \quad \hat{\mathbf{M}} = \begin{pmatrix} 1 & \mathbf{D}^{\mathrm{T}} \\ \mathbf{P} & \mathbf{m} \end{pmatrix}, \\ \mathbf{m} = \frac{1}{m_{00}}\begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}, \ \mathbf{D} = \frac{1}{m_{00}}\begin{pmatrix} m_{01} \\ m_{02} \\ m_{03} \end{pmatrix}, \ \mathbf{P} = \frac{1}{m_{00}}\begin{pmatrix} m_{10} \\ m_{20} \\ m_{30} \end{pmatrix}, \end{gathered} \tag{1}$$

where m_{ij} $(i, j = 0, 1, 2, 3)$ denote the elements of $\mathbf{M}$; superscript T indicates transpose; m_{00} is the mean intensity coefficient (MIC), which quantifies the ratio between the intensities of the output light beam and the input unpolarized light); $\mathbf{m}$ is the normalized 3×3 submatrix associated with $\mathbf{M}$; and vectors $\mathbf{D}$ and $\mathbf{P}$ represent the diattenuation and polarizance vectors, with respective absolute values D (diattenuation) and P (polarizance). The dual role of D and P as diattenuation/polarizance is evidenced by considering the Mueller matrix $\mathbf{M}^{r}$ (reverse Mueller matrix) corresponding to $\mathbf{M}$ when the directions of the incident and emerging light are interchanged, which, in absence of magnetooptical effects, is given by [96,97]

$$\mathbf{M}^r = \mathrm{diag}(1,1,-1,1)\mathbf{M}^{\mathrm{T}}\mathrm{diag}(1,1,-1,1), \quad (2)$$

where diag stands for diagonal matrix, so that D (the diattenuation of $\mathbf{M}$) is the polarizance of $\mathbf{M}^r$, while P (the polarizance of $\mathbf{M}$) is the diattenuation of $\mathbf{M}^r$ [54]. As for the diattenuation and polarizance vectors of $\mathbf{M}^r$, they are given by $\mathbf{D}^r = \mathrm{diag}(1,-1,1)\mathbf{P}$ and $\mathbf{P}^r = \mathrm{diag}(1,-1,1)\mathbf{D}$, respectively.

For certain purposes, it is also useful to represent $\mathbf{M}$ through m_{00} together with the five constitutive vectors: $\mathbf{D}$, $\mathbf{P}$, $\mathbf{k}$, $\mathbf{r}$, and $\mathbf{q}$; where the last three (which determine $\mathbf{m}$) are defined as [14]

$$\mathbf{k} = \frac{1}{\sqrt{3}\,m_{00}}\begin{pmatrix} m_{11} \\ m_{22} \\ m_{33} \end{pmatrix}, \quad \mathbf{r} = \frac{1}{m_{00}}\begin{pmatrix} m_{23} \\ m_{13} \\ m_{12} \end{pmatrix}, \quad \mathbf{q} = \frac{1}{m_{00}}\begin{pmatrix} m_{32} \\ m_{31} \\ m_{21} \end{pmatrix}. \quad (3)$$

The absolute values of all the five constitutive vectors are smaller than one, so that they can be represented in a common Poincaré-type sphere, thus providing a geometric view of the properties of the normalized Mueller matrix $\hat{\mathbf{M}}$.

Linear polarimetric transformations for which the degree of polarization of any states of totally polarized incident light does not decrease can be represented either by Jones matrices [127–133] or pure Mueller matrices (also called nondepolarizing or Mueller–Jones matrices, which hereafter, when appropriate, will be denoted by $\mathbf{M}_J$ to distinguish them from the general, depolarizing Mueller matrices denoted by $\mathbf{M}$). Pure Mueller matrices depend on up to seven independent parameters.

A proper measure of the degree of polarimetric purity of $\mathbf{M}$ is given by the depolarization index [34,71], P_Δ, which can be expressed as

$$P_\Delta = \sqrt{\frac{\sum_{i,j=0}^{3} m_{ij}^2 - m_{00}^2}{3m_{00}^2}} = \sqrt{\frac{D^2 + P^2 + 3P_S^2}{3}} = \sqrt{\frac{2P_P^2}{3} + P_S^2} \quad \left[P_P \equiv \sqrt{\frac{D^2 + P^2}{2}}\right], \quad (4)$$

where P_P is the enpolarizance, also called the degree of polarizance, and P_S is the polarimetric dimension index, also called the degree of spherical purity, defined as [12,82,93]

$$P_S \equiv \frac{\|\mathbf{m}\|_2}{\sqrt{3}} \quad \left[\|\mathbf{m}\|_2 \equiv \frac{1}{m_{00}}\sqrt{\sum_{k,l=1}^{3} m_{kl}^2}\right], \quad (5)$$

$\|\mathbf{m}\|_2$ being the Frobenius norm of $\mathbf{m}$.

Mueller matrices that increase the degree of polarization for some or all incident Stokes vectors, either in forward or reverse interactions, are called enpolarizing; otherwise, they are called nonenpolarizing.

The maximum value, $P_\Delta = 1$, of the degree of polarimetric purity corresponds to nondepolarizing (or pure) media; the minimum value $P_\Delta = 0$ characterizes perfect depolarizers, whose associated Mueller matrix has the form $\mathbf{M}_{\Delta 0} = m_{00}\mathrm{diag}\,(1,0,0,0)$. As for P_S, its maximum value, $P_S = 1$, implies $P_\Delta = 1$ with $P_P = 0$, which correspond to retarders (transparent or exhibiting isotropic attenuation); the minimum, $P_S = 0$, corresponds to media exhibiting $\mathbf{m} = \mathbf{0}$. Maximum enpolarizance, $P_P = 1$, implies $P_\Delta = 1$ and corresponds to perfect polarizers, while the minimum, $P_P = 0$, is exhibited by nonenpolarizing interactions (either depolarizing or pure) [82].

To make the interpretation of Mueller matrices easier, it is sometimes useful to decompose them in terms of combinations of simpler elements. In general, two kinds of decompositions of a Mueller matrix can be performed, namely serial decompositions (products of Mueller matrices) [12,24,30,47,54,56,67,72,98,99,128,134,135] and parallel decompositions (sums of Mueller matrices) [54,136–145]. Both kinds of decompositions can be combined leading to serial–parallel decompositions [146].

Parallel decompositions, under whose scope the new approach is developed, refer to the expansion of **M** as a convex sum of Mueller matrices and are interpreted as follows: the incident light beam splits into a set of pencils that interact with several material components that cover different and complementary parts of the illuminated area, such that the outgoing beams are incoherently recombined into the whole emerging beam.

As a consequence, and in accordance to the above-mentioned covariance criterion, the coefficients of the Mueller parallel components should be positive and sum to one [54,144].

Given a Mueller matrix **M**, it has an associated positive semidefinite Hermitian coherency matrix $\mathbf{C}(\mathbf{M})$, whose explicit expression in terms of the elements m_{ij} of **M** is [1,14]

$$\mathbf{C}(\mathbf{M}) = \frac{1}{4}\begin{pmatrix} \begin{matrix} m_{00}+m_{11} \\ +m_{22}+m_{33} \end{matrix} & \begin{matrix} m_{01}+m_{10} \\ -i\,(m_{23}-m_{32}) \end{matrix} & \begin{matrix} m_{02}+m_{20} \\ +i\,(m_{13}-m_{31}) \end{matrix} & \begin{matrix} m_{03}+m_{30} \\ -i\,(m_{12}-m_{21}) \end{matrix} \\ \begin{matrix} m_{01}+m_{10} \\ +i\,(m_{23}-m_{32}) \end{matrix} & \begin{matrix} m_{00}+m_{11} \\ -m_{22}-m_{33} \end{matrix} & \begin{matrix} m_{12}+m_{21} \\ +i\,(m_{03}-m_{30}) \end{matrix} & \begin{matrix} m_{13}+m_{31} \\ -i\,(m_{02}-m_{20}) \end{matrix} \\ \begin{matrix} m_{02}+m_{20} \\ -i\,(m_{13}-m_{31}) \end{matrix} & \begin{matrix} m_{12}+m_{21} \\ -i\,(m_{03}-m_{30}) \end{matrix} & \begin{matrix} m_{00}-m_{11} \\ +m_{22}-m_{33} \end{matrix} & \begin{matrix} m_{23}+m_{32} \\ +i\,(m_{01}-m_{10}) \end{matrix} \\ \begin{matrix} m_{03}+m_{30} \\ +i\,(m_{12}-m_{21}) \end{matrix} & \begin{matrix} m_{13}+m_{31} \\ +i\,(m_{02}-m_{20}) \end{matrix} & \begin{matrix} m_{23}+m_{32} \\ -i\,(m_{01}-m_{10}) \end{matrix} & \begin{matrix} m_{00}-m_{11} \\ -m_{22}+m_{33} \end{matrix} \end{pmatrix}. \quad (6)$$

Conversely,

$$\mathbf{M}(\mathbf{C}) = \begin{pmatrix} \begin{matrix} c_{00}+c_{11} \\ +c_{22}+c_{33} \end{matrix} & \begin{matrix} c_{01}+c_{01}^{*} \\ -i\,(c_{23}-c_{23}^{*}) \end{matrix} & \begin{matrix} c_{02}+c_{02}^{*} \\ +i\,(c_{13}-c_{13}^{*}) \end{matrix} & \begin{matrix} c_{03}+c_{03}^{*} \\ -i\,(c_{12}-c_{12}^{*}) \end{matrix} \\ \begin{matrix} c_{01}+c_{01}^{*} \\ +i\,(c_{23}-c_{23}^{*}) \end{matrix} & \begin{matrix} c_{00}+c_{11} \\ -c_{22}-c_{33} \end{matrix} & \begin{matrix} c_{12}+c_{12}^{*} \\ +i\,(c_{03}-c_{03}^{*}) \end{matrix} & \begin{matrix} c_{13}+c_{13}^{*} \\ -i\,(c_{02}-c_{02}^{*}) \end{matrix} \\ \begin{matrix} c_{02}+c_{02}^{*} \\ -i\,(c_{13}-c_{13}^{*}) \end{matrix} & \begin{matrix} c_{12}+c_{12}^{*} \\ -i\,(c_{03}-c_{03}^{*}) \end{matrix} & \begin{matrix} c_{00}-c_{11} \\ +c_{22}-c_{33} \end{matrix} & \begin{matrix} c_{23}+c_{23}^{*} \\ +i\,(c_{01}-c_{01}^{*}) \end{matrix} \\ \begin{matrix} c_{03}+c_{03}^{*} \\ +i\,(c_{12}-c_{12}^{*}) \end{matrix} & \begin{matrix} c_{13}+c_{13}^{*} \\ +i\,(c_{02}-c_{02}^{*}) \end{matrix} & \begin{matrix} c_{23}+c_{23}^{*} \\ -i\,(c_{01}-c_{01}^{*}) \end{matrix} & \begin{matrix} c_{00}-c_{11} \\ -c_{22}+c_{33} \end{matrix} \end{pmatrix}, \quad (7)$$

where the superscript * stands for complex conjugate and c_{ij} $(i,j=0,1,2,3)$ are the complex elements of $\mathbf{C}(\mathbf{M})$.

The above transformation from **M** to **C** can be applied to any 4×4 real matrix **A**, resulting in a matrix $\mathbf{C}(\mathbf{A})$ that is necessarily Hermitian (because of its very construction), but it is not necessarily positive semidefinite (as required when **A** is a Mueller matrix). The passivity constraint (linear polarimetric interactions in nature do not produce increase of the intensity of light) is fully characterized by the inequality $m_{00}(1+Q) \leq 1$ [8,13], with $Q = \max(D,P)$.

As mentioned in the introduction, the ensemble criterion provides a complete characterization of Mueller matrices, which means that a given 4×4 real matrix **M** is a Mueller matrix if and only if it can be represented as a convex sum of pure (and passive) Mueller matrices or, equivalently, the four eigenvalues of $\mathbf{C}(\mathbf{M})$ are nonnegative and, in addition, **M** satisfies the passivity condition.

$\mathbf{C}\,(\mathbf{M})$ can always be diagonalized through a unitary transformation:

$$\mathbf{C} = \mathbf{U}\mathrm{diag}(\lambda_0, \lambda_1, \lambda_2, \lambda_3)\mathbf{U}^{\dagger}. \tag{8}$$

where λ_i are the four nonnegative eigenvalues of $\mathbf{C}$, which satisfy $\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 = m_{00}$ and are taken in decreasing order $(0 \leq \lambda_3 \leq \lambda_2 \leq \lambda_1 \leq \lambda_0)$. The columns $\mathbf{u}_i$ $(i = 0, 1, 2, 3)$ of the unitary matrix $\mathbf{U}$ constitute the four orthonormal eigenvectors of $\mathbf{C}$. Therefore, $\mathbf{C}$ admits the following expansion in terms of four coherency matrices representing respective pure systems:

$$\mathbf{C} = \sum_{i=0}^{3} \frac{\lambda_i}{\mathrm{tr}\mathbf{C}} \mathbf{C}_{Ji}, \;\; \mathbf{C}_{Ji} \equiv (\mathrm{tr}\mathbf{C})\left(\mathbf{u}_i \otimes \mathbf{u}_i^{\dagger}\right). \tag{9}$$

This spectral decomposition (or Cloude's decomposition) can also be expressed as follows in terms of Mueller matrices:

$$\begin{gathered} \mathbf{M} = \sum_{i=0}^{3} \hat{\lambda}_i \left(m_{00}\, \hat{\mathbf{M}}_{Ji}\right) \\ \left[m_{00} = \mathrm{tr}\mathbf{C},\;\; \hat{\lambda}_i = \lambda_i / m_{00},\;\; \mathbf{M}_{Ji} = \mathbf{M}_{Ji}(\mathbf{C}_{Ji})\right], \end{gathered} \tag{10}$$

which shows that any linear polarimetric transformation can be represented as a parallel combination of up to four pure components determined by the respective eigenvectors of $\mathbf{C}$ and whose relative weights are the normalized eigenvalues $\hat{\lambda}_i$ of $\mathbf{C}$. The spectral decomposition constitutes a particular case of the more general arbitrary decomposition [54,144], which provides infinite possible decompositions in terms of up to four independent (not mutually proportional) pure components.

From a general point of view, and regarding the covariance properties, the complete set of 4×4 real matrices can be divided into two complementary subsets, namely non-Mueller-covariance matrices, whose associated Hermitian matrix $\mathbf{C}$ (via Equation (6)) has at least one negative eigenvalue, and Mueller–covariance matrices, for which $\mathbf{C}$ has nonnegative eigenvalues.

The pure components of the spectral decomposition can be regrouped into different sets, some of which are depolarizing. A particularly interesting case is the so-called characteristic decomposition (or trivial decomposition) [54], which is formulated as an extension of the characteristic decomposition of three-dimensional coherency matrices [147], which in turn is an extension of the well-known decomposition of two-dimensional polarization matrices into a sum of those of a totally polarized state and a unpolarized state.

The characteristic decomposition of the coherency matrix $\mathbf{C}$ is defined as follows [54]:

$$\begin{gathered} \mathbf{C} = (\hat{\lambda}_0 - \hat{\lambda}_1)\mathbf{C}_{J0}{+}2(\hat{\lambda}_1 - \hat{\lambda}_2)\mathbf{C}_1{+}3(\hat{\lambda}_2 - \hat{\lambda}_3)\mathbf{C}_2 + 4\hat{\lambda}_3\mathbf{C}_{\Delta 0}, \\ \left[\begin{array}{l} \mathbf{C}_{J0} = (\mathrm{tr}\mathbf{C})\left[\mathbf{U}\mathrm{diag}(1,0,0,0)\mathbf{U}^{\dagger}\right] = \;(\mathrm{tr}\mathbf{C})\left(\mathbf{u}_0 \otimes \mathbf{u}_0^{\dagger}\right), \\ \mathbf{C}_1 = \dfrac{1}{2}(\mathrm{tr}\mathbf{C})\left[\mathbf{U}\mathrm{diag}(1,1,0,0)\mathbf{U}^{\dagger}\right] = \dfrac{1}{2}(\mathrm{tr}\mathbf{C})\sum\limits_{i=0}^{1} \mathbf{u}_i \otimes \mathbf{u}_i^{\dagger}, \\ \mathbf{C}_2 = \dfrac{1}{3}(\mathrm{tr}\mathbf{C})\left[\mathbf{U}\mathrm{diag}(1,1,1,0)\mathbf{U}^{\dagger}\right] = \dfrac{1}{3}(\mathrm{tr}\mathbf{C})\sum\limits_{i=0}^{2} \mathbf{u}_i \otimes \mathbf{u}_i^{\dagger}, \\ \mathbf{C}_{\Delta 0} = \dfrac{1}{4}(\mathrm{tr}\mathbf{C})\left[\mathbf{U}\mathrm{diag}(1,1,1,1)\mathbf{U}^{\dagger}\right] = \dfrac{1}{4}(\mathrm{tr}\mathbf{C})\sum\limits_{i=0}^{3} \mathbf{u}_i \otimes \mathbf{u}_i^{\dagger} = \dfrac{1}{4}(\mathrm{tr}\mathbf{C})\mathbf{I}_4. \end{array}\right] \end{gathered} \tag{11}$$

where $\mathbf{I}_4$ is the 4×4 identity matrix; $\mathbf{C}_{J0}$ is a pure component; $\mathbf{C}_i$ $(i = 1, 2)$ are specific nonpure components with $\mathrm{rank}\mathbf{C}_i = i + 1$ and with $i + 1$ degenerate nonzero eigenvalues; and $\mathbf{C}_{\Delta 0}$ is proportional to the identity matrix and represents a perfect depolarizer. The total symmetry of the structure of $\mathbf{C}_{\Delta 0}$ corresponds to a neutral effect that can be interpreted as polarimetric white noise [141]. As in some related works, matrices associated with the pure component are labeled with the subscript "*J*0" to indicate that they are linked to the first eigenvalue λ_0 (and not to other eigenvalues), while the subscript $\Delta 0$ affecting the fully random component indicates that it corresponds to a perfect depolarizer.

In terms of Mueller matrices, the characteristic decomposition takes the form

$$\begin{array}{l} \mathbf{M} = P_1\, m_{00}\, \hat{\mathbf{M}}_{J0} + (P_2 - P_1)\, m_{00}\, \hat{\mathbf{M}}_1 + (P_3 - P_2)\, m_{00}\, \hat{\mathbf{M}}_2 + (1 - P_3)\, m_{00}\, \hat{\mathbf{M}}_{\Delta 0}, \\ P_1 = \hat{\lambda}_0 - \hat{\lambda}_1\,,\ \ P_2 = \hat{\lambda}_0 + \hat{\lambda}_1 - 2\hat{\lambda}_2\,,\ \ P_3 = \hat{\lambda}_0 + \hat{\lambda}_1 + \hat{\lambda}_2 - 3\hat{\lambda}_3 = 1 - 4\hat{\lambda}_3, \\ \left[m_{00}\, \hat{\mathbf{M}}_{J0} = \mathbf{M}(\mathbf{C}_{J0}),\ \ m_{00}\, \hat{\mathbf{M}}_i = \mathbf{M}(\mathbf{C}_i)\ \ (i = 1,2),\ \ m_{00}\, \hat{\mathbf{M}}_{\Delta 0} = \mathbf{M}(\mathbf{C}_{\Delta 0}) \right], \end{array} \tag{12}$$

where P_1, P_2, P_3 are the indices of polarimetric purity (IPP) [54,83,91], which provide complete information on the polarimetric purity–randomness of the linear interaction represented by $\mathbf{M}$.

Note that the normalized pure component $\hat{\mathbf{M}}_{J0}$ of the above decomposition coincides with the normalized component of the spectral decomposition with largest associated eigenvalue. However, a portion $(1 - P_1)\, m_{00}\, \hat{\mathbf{M}}_{J0}$ of the pure component is embedded within the perfect depolarizer $(1 - P_3)\, m_{00}\, \hat{\mathbf{M}}_{\Delta 0}$ of the characteristic decomposition.

3. Mueller Matrix Associated with a 4×4 Real Non-Mueller Matrix

Let us consider a 4×4 real matrix $\mathbf{A}$ that does not satisfy the covariance conditions, i.e., at least one of the eigenvalues of $\mathbf{C}(\mathbf{A})$ is negative. In that case, we maintain the criterion of labeling the eigenvalues of $\mathbf{C}(\mathbf{A})$ in decreasing order, $\lambda_0 \geq \lambda_1 \geq \lambda_2 \geq \lambda_3$, so λ_3 is the negative eigenvalue with largest absolute value among the negative eigenvalues. Let us now build the matrix sum

$$\begin{array}{l} \overline{\mathbf{C}}(\mathbf{A}) = \mathbf{C}(\mathbf{A}) + 4|\lambda_3|(\mathbf{I}_4/4) \\ \quad = \mathbf{U}\mathrm{diag}(\lambda_0, \lambda_1, \lambda_2, \lambda_3)\mathbf{U}^{\dagger} + \mathbf{U}(4|\lambda_3|)\mathrm{diag}(1/4, 1/4, 1/4, 1/4)\mathbf{U}^{\dagger}\ . \\ \quad = \mathbf{U}\mathrm{diag}(\lambda_0 + |\lambda_3|, \lambda_1 + |\lambda_3|, \lambda_2 + |\lambda_3|, 0)\mathbf{U}^{\dagger}, \end{array} \tag{13}$$

where $\mathbf{I}_4/4 = \mathbf{C}(\hat{\mathbf{M}}_{\Delta 0})$ and $\lambda_0 + |\lambda_3| \geq \lambda_1 + |\lambda_3| \geq \lambda_2 + |\lambda_3| \geq 0$, so that the resulting coherency matrix lacks negative eigenvalues and has a number of zero eigenvalues equal to the degeneracy of the eigenvalue λ_3 of $\mathbf{C}(\mathbf{A})$. In the Mueller representation,

$$\overline{\mathbf{M}}(\mathbf{A}) = \mathbf{A} + 4|\lambda_3|\hat{\mathbf{M}}_{\Delta 0}. \tag{14}$$

Consequently, through the above procedure, which only involves the addition of a portion $4|\lambda_3|$ of polarimetric white noise, a nonnegative definite Hermitian coherency matrix $\overline{\mathbf{C}}(\mathbf{A})$ is associated with $\mathbf{A}$ in a straightforward manner. Therefore, by applying Equation (7) to $\overline{\mathbf{C}}(\mathbf{A})$, a Mueller–covariance matrix $\overline{\mathbf{M}}(\mathbf{A})$ associated with the original non-Mueller matrix $\mathbf{A}$ is calculated. Obviously, any transformation $\mathbf{A} + c\,\hat{\mathbf{M}}_{\Delta 0}$, with $c > 4|\lambda_3|$, leads to a covariance Mueller matrix whose fully random component does not vanish; thus, a natural choice for the Mueller representative of $\mathbf{A}$ is $\overline{\mathbf{M}}(\mathbf{A})$, for which the amount of added $\hat{\mathbf{M}}_{\Delta 0}$ is minimal.

The IPP of $\overline{\mathbf{M}}(\mathbf{A})$ are given by

$$\begin{array}{c} P_1 = \frac{\lambda_0 - \lambda_1}{a_0 + 4|\lambda_3|}\,,\ \ P_2 = \frac{\lambda_0 + \lambda_1 - 2\lambda_2}{a_0 + 4|\lambda_3|}\,,\ \ P_3 = \frac{\lambda_0 + \lambda_1 + \lambda_2 - 3\lambda_3}{a_0 + 4|\lambda_3|} = 1\,, \\ \left[a_0 = \mathrm{tr}\mathbf{C}(\mathbf{A}) = \lambda_0 + \lambda_1 + \lambda_2 + \lambda_3,\ \ (\lambda_3 < 0),\ \ 0 \leq P_1 \leq P_2 \leq 1 \right]. \end{array} \tag{15}$$

It should be emphasized that, as shown in Equation (14), the transformation in Equation (13) is entirely equivalent to adding to $\mathbf{A}$ a portion $4|\lambda_3|$ of a normalized perfect depolarizer. Thus, the effect of such a transformation is no other than replacing a_{00} by

$$\overline{m}_{00} = a_{00} + 4|\lambda_3|\,, \tag{16}$$

without affecting the fifteen remaining elements of **M**, namely $\overline{m}_{ij} = a_{ij}$ ($i, j = 0, 1, 2, 3$; $i = j = 0$ excluded), i.e.,

$$\overline{\mathbf{M}}(\mathbf{A}) = \begin{pmatrix} a_{00} + 4|\lambda_3| & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{pmatrix}. \tag{17}$$

This means that, up to a scale coefficient, $\overline{\mathbf{M}}(\mathbf{A})$ preserves all polarimetric anisotropies involved in the non-Mueller matrix **A**. In fact, the five constitutive vectors of $\overline{\mathbf{M}}(\mathbf{A})$ are reduced with respect to those of **A** through the common coefficient $k = a_{00}/\overline{m}_{00}$,

$$\begin{gathered} \mathbf{D}(\overline{\mathbf{M}}) = k\,\mathbf{D}(\mathbf{A}),\ \mathbf{P}(\overline{\mathbf{M}}) = k\,\mathbf{P}(\mathbf{A}), \\ \mathbf{k}(\overline{\mathbf{M}}) = k\,\mathbf{k}(\mathbf{A}),\ \mathbf{r}(\overline{\mathbf{M}}) = k\,\mathbf{r}(\mathbf{A}),\ \mathbf{q}(\overline{\mathbf{M}}) = k\,\mathbf{q}(\mathbf{A}), \\ [k = a_{00}/\overline{m}_{00} = a_{00}/(a_{00} + 4|\lambda_3|) < 1]. \end{gathered} \tag{18}$$

Since $\overline{\mathbf{M}}(\mathbf{A})$ is a Mueller–covariance matrix, each of its five constitutive vectors have absolute values that do not exceed unity and therefore can be represented as Poincaré vectors in the unit sphere.

From the very definition of $\overline{\mathbf{M}}(\mathbf{A})$ its characteristic decomposition takes the form

$$\overline{\mathbf{M}} = P_1\,\mathbf{M}_{J0} + (P_2 - P_1)\ \mathbf{M}_1 + (1 - P_2)\ \hat{\mathbf{M}}_2, \tag{19}$$

where the coherency matrices of the Mueller parallel components are given by

$$\begin{aligned} \mathbf{C}(\mathbf{M}_{J0}) &= \mathbf{U}[(\lambda_0 + |\lambda_3|)\mathrm{diag}(1,0,0,0)]\mathbf{U}^{\dagger}, \\ \mathbf{C}(\mathbf{M}_1) &= \mathbf{U}[(\lambda_1 + |\lambda_3|)\mathrm{diag}(1,1,0,0)]\mathbf{U}^{\dagger}, \\ \mathbf{C}(\mathbf{M}_2) &= \mathbf{U}[(\lambda_2 + |\lambda_3|)\mathrm{diag}(1,1,1,0)]\mathbf{U}^{\dagger}. \end{aligned} \tag{20}$$

This result evidences that the characteristic decomposition of $\overline{\mathbf{M}}(\mathbf{A})$ lacks the fully random component $\hat{\mathbf{M}}_{\Delta 0}$, which agrees with the fact that the third index of polarimetric purity $P_3(\overline{\mathbf{M}})$ equals 1. Consequently, $\mathrm{rank}\overline{\mathbf{C}}(\mathbf{A}) \leq 3$, which means that no more than three independent components are involved in the structure of $\overline{\mathbf{M}}(\mathbf{A})$ (assumed that the starting matrix **A** does not satisfy the covariance conditions). The integer value of $\mathrm{rank}\overline{\mathbf{C}}(\mathbf{A})$ can be interpreted as the minimum number of independent incoherent pure components of $\overline{\mathbf{M}}(\mathbf{A})$ [54,145] and is given by $\mathrm{rank}\overline{\mathbf{C}}(\mathbf{A}) \leq 4 - t$, t being the degeneracy of eigenvalue λ_3 $(1 \leq t \leq 3)$ (the case $t = 4$ is excluded because it corresponds to $\mathbf{A} = \mathbf{M}_{\Delta 0}$, i.e., $\overline{\mathbf{M}}(\mathbf{A}) = \mathbf{0}$).

The above procedure differs from the analysis of the structure of polarimetric randomness performed in [141] and also from the extended representation of Mueller matrices [148], which apply to Mueller–covariance matrices instead of non-Mueller matrices.

As for the filtering of measured Mueller matrices, the method presented does not detract the well-established and well-founded standard approaches [37,149–162], but provides an additional complementary point of view, besides its contribution to clarify certain aspects of the structure of Mueller–covariance matrices like the role played by the relation between the MIC and the remaining fifteen elements of the matrix.

As a measure of how far is a non-Mueller matrix **A** from its associated Mueller–covariance matrix $\overline{\mathbf{M}}(\mathbf{A})$ we define the isotropic distance

$$d = \|\overline{\mathbf{M}}(\mathbf{A}) - \mathbf{A}\|_2^2 = 4|\lambda_3|, \tag{21}$$

where $\| \|_2$ represents the Frobenius norm. The quantity d is termed the isotropic distance in the sense that it preserves the proportionalities among the five constitutive vectors and to distinguish it from other possible distances defined through other alternative criteria based on the eigenvalues of $\mathbf{C}(\mathbf{A})$.

From an experimental point of view, the operator of the Mueller polarimeter can establish tolerance criteria (a maximum limit for d, with $\lambda_3 < 0$) for deciding whether a measurement can be considered acceptable or should be discarded due to excessive error.

Regarding the passivity condition, if it is not directly satisfied, it can be achieved by multiplying the normalized matrix $\hat{\overline{\mathbf{M}}}(\mathbf{A}) = \overline{\mathbf{M}}(\mathbf{A})/m_{00}$ by the coefficient $1/(1+Q(\overline{\mathbf{M}}))$ (with $Q(\overline{\mathbf{M}}) \equiv \max[D(\overline{\mathbf{M}}), P(\overline{\mathbf{M}})]$, leading to the passive–representative Mueller matrix [13]

$$\widetilde{\mathbf{M}}(\mathbf{A}) = \frac{1}{1+Q(\overline{\mathbf{M}}(\mathbf{A}))}\hat{\overline{\mathbf{M}}}(\mathbf{A}), \tag{22}$$

which constitutes the Mueller matrix proportional to $\overline{\mathbf{M}}(\mathbf{A})$ with maximal MIC. Obviously, all matrices $k\widetilde{\mathbf{M}}(\mathbf{A})$, with $k \leq 1$, satisfy the passivity condition; however $\widetilde{\mathbf{M}}(\mathbf{A})$ constitutes an appropriate choice as passive–representative when the MIC is unknown, as occurs for instance with certain polarimeters that measure the Mueller matrices up to a global scale coefficient.

4. The Effective Component of a Mueller Matrix

Given a Mueller matrix $\mathbf{M}$, let us consider its characteristic decomposition and perform the polarimetric subtraction [139,141] of its fully random component $\hat{\mathbf{M}}_{\Delta 0}$:

$$\mathbf{M}_e = \frac{\mathbf{M} - (1-P_3)m_{00}\,\hat{\mathbf{M}}_{\Delta 0}}{P_3} = \frac{P_1}{P_3}\,m_{00}\,\hat{\mathbf{M}}_{J0} + \frac{P_2 - P_1}{P_3}m_{00}\,\hat{\mathbf{M}}_1 + \frac{1-P_2}{P_3}m_{00}\,\hat{\mathbf{M}}_2. \tag{23}$$

In accordance to the general procedure described in [139], this operation consists of removing from $\mathbf{M}$ the polarimetric white noise [141], represented by $\hat{\mathbf{M}}_{\Delta 0}$, and keep the part affected by polarimetric anisotropies. As a consequence, the resulting effective component $\mathbf{M}_e(\mathbf{M})$ exhibits polarimetric descriptors that are increased by a scale coefficient $1/P_3(0 < P_3 \leq 1)$, showing higher sensitivity to enpolarizing, retarding and depolarizing properties. In particular,

$$\begin{aligned}
&P(\mathbf{M}_e) = P(\mathbf{M})/P_3(\mathbf{M}),\ D(\mathbf{M}_e) = D(\mathbf{M})/P_3(\mathbf{M}),\\
&P_S(\mathbf{M}_e) = P_S(\mathbf{M})/P_3(\mathbf{M}),\ P_P(\mathbf{M}_e) = P_P(\mathbf{M})/P_3(\mathbf{M}),\\
&P_\Delta(\mathbf{M}_e) = P_\Delta(\mathbf{M})/P_3(\mathbf{M}),\\
&P_1(\mathbf{M}_e) = P_1(\mathbf{M})/P_3(\mathbf{M}),\ P_2(\mathbf{M}_e) = P_2(\mathbf{M})/P_3(\mathbf{M}),\ P_3(\mathbf{M}_e) = 1\,.
\end{aligned} \tag{24}$$

Thus, any Mueller matrix $\mathbf{M}$ has as an associated effective Mueller matrix $\mathbf{M}_e(\mathbf{M})$, which encompass all polarimetric anisotropies and is free of the fully random component $\hat{\mathbf{M}}_{\Delta 0}$. Obviously, when the starting Mueller matrix $\mathbf{M}$ lacks such a component $\hat{\mathbf{M}}_{\Delta 0}$, it coincides with its effective component.

Consequently, when the interest is focused on the polarimetric anisotropies of the measured Mueller matrix, the effective component results to be more sensitive to such anisotropies. In particular, when spatially or temporally distributed Mueller matrices are taken to generate polarimetric images, the use of the corresponding active components leads to an improved contrast [94].

5. Discussion

The structure of a pure Mueller matrix is entirely derived from its expression in terms of the associated Jones matrix. Consequently, the nature of 4×4 real matrices satisfying the covariance conditions lies not only in the property that they are Stokes matrices (i.e., matrices transforming Stokes vectors into Stokes vectors) but their algebraic structure comes from the fact that they can always be expressed as convex sums of pure Mueller matrices. This genuine property is equivalent to the nonnegativity of the associated coherency matrix. Thus, any Mueller–covariance matrix is a Stokes matrix, while the converse is not true. For instance, the matrix $\text{diag}\,(1,1,1,-1)$ is a Stokes matrix that does not satisfy the covariance conditions.

The procedure described in Section 3 allows for transforming any 4×4 real matrix $\mathbf{A}$ to its associated Mueller–covariance matrix $\overline{\mathbf{M}}\,(\mathbf{A})$ through the appropriate increase of the element a_{00}. This approach resembles the case of four-dimensional vectors with real components that are not Stokes vectors, which, through the addition of a unpolarized Stokes vector, can always be converted to Stokes vectors, $(s_0, s_1, s_2, s_3)^T$, satisfying the characteristic conditions $s_1^2 + s_2^2 + s_3^2 \leq s_0^2$ and $s_0 \geq 0$. Such conditions determine the positive branch of a four-dimensional cone whose height is s_0; the cut of such a cone with the plane $s_0 = 1$ determining the well-known Poincaré sphere.

Obviously, the Mueller case (sixteen elements) is rather more involved than that of the Stokes vectors (four elements). For instance, a non-Stokes vector $(v_0, v_1, v_2, v_3)^T$ can always be converted to a pure (totally polarized) Stokes vector by replacing its first component v_0 by $\sqrt{v_1^2 + v_2^2 + v_3^2}$; nevertheless, $\overline{\mathbf{M}}\,(\mathbf{A})$ is not a pure Mueller–covariance matrix in general. Another example of the greater complexity of Mueller matrices with respect to Stokes vectors is the fact that a complete quantitative characterization of the purity–randomness exhibited by $\mathbf{M}$ requires the consideration of the three IPP instead of the single parameter $P \equiv \sqrt{s_1^2 + s_2^2 + s_3^2}/s_0$ characterizing the degree of polarization. Additionally, it is remarkable that the single and simple transformation in Equation (17) allows the conversion of an arbitrary 4×4 real matrix $\mathbf{A}$ to a matrix $\mathbf{M}$ satisfying the four covariance conditions.

Among the great variety of mathematical configurations of matrices $\mathbf{A}$ that do not satisfy the covariance conditions, it is particularly interesting the case of Stokes matrices that involve certain inversions of the Stokes parameters of the interacting light, which are not compatible with Mueller (hence physical) transformations. A simple example is the Stokes matrix $\text{diag}\,(1,-1,-1,-1)$ for which the eigenvalues of the associated coherency matrix are $1/2,\ 1/2,\ 1/2,\ -1/2$. After transformation (17), the associated (depolarizing) Mueller matrix is given by $\text{diag}\,(3,-1,-1,-1)$, whose corresponding coherency matrix has the eigenvalues $1,\ 1,\ 1,\ 0$.

Another simple and illustrative case is $\mathbf{A} = \text{diag}\,(-c,1,1,1)$, with $c \geq 0$, for which the element $a_{00} = -c$ should be replaced by $m_{00} = a_{00} + c + 1 = 1$, leading to the Mueller matrix $\mathbf{M} = \text{diag}\,(1,1,1,1)$. Obviously, infinite different situations can be considered, but it should be stressed that the procedure for the calculation of the Mueller matrix associated with a non-Mueller matrix is applicable without exception.

From the point of view of filtering polarimetric measurements, the indicated approach is based on removing completely the spurious polarimetric white noise affecting the measured non-Mueller matrix $\mathbf{A}$. The resulting Mueller matrix associated with $\mathbf{A}$ can be further represented by a parallel composition of up to three independent (i.e., not mutually proportional) pure Mueller matrices.

Regarding the effective component of a Mueller matrix $\overline{\mathbf{M}}$ associated with $\mathbf{A}$ (assumed that $\mathbf{A}$ does not satisfy the covariance conditions), it coincides with $\overline{\mathbf{M}}\,(\mathbf{A})$ itself because of the lack of the fully random component $\hat{\mathbf{M}}_{\Delta 0}$ of the characteristic decomposition. In the most general case where the Mueller matrix $\mathbf{M}$ includes a nonvanishing $\hat{\mathbf{M}}_{\Delta 0}$ component in its characteristic decomposition, the effective component replicates, up to a scale coefficient $1/P_3 \geq 1$, the polarimetric properties of the original matrix $\mathbf{M}$. As a consequence, the use of the effective component has been proven to be useful to increase the contrast of polarimetric images [94] (note that the name "effective component", coined here for the first time, was not used in the referenced paper).

As for the geometric representation of $\mathbf{M}$ through its canonical or characteristic ellipsoids [14,103], their shapes are preserved while their sizes are enlarged when it is transformed to its associated effective representative $\mathbf{M}_e$. Accordingly, the five constitutive vectors [14] of $\mathbf{M}_e$ coincide with those of $\mathbf{M}$ except for the increase of their absolute values by a coefficient $1/P_3$.

6. Conclusions

Despite the notable complexity of the mathematical structure of Mueller matrices, there exists a simple procedure to transform any arbitrary 4×4 real matrix $\mathbf{A}$ to a Mueller matrix. To do so, it is enough to appropriately increase the value of the element a_{00} by just the quantity that produces a zero coefficient for the fully random component of the characteristic decomposition of the transformed matrix $\overline{\mathbf{M}}(\mathbf{A})$. Then, $\overline{\mathbf{M}}(\mathbf{A})$ satisfies the covariance conditions required to be a Mueller matrix, while, as indicated at the end of Section 3, the fulfillment of the passivity conditions only requires, if necessary, a simple change of the global coefficient affecting the matrix.

When dealing with Mueller matrices, the removal of the fully random component of the characteristic decomposition of a given Mueller matrix $\mathbf{M}$ leads to the introduction of the associated effective Mueller matrix $\mathbf{M}_e(\mathbf{M})$, which, up to a global scale coefficient given by $1/P_3$ (P_3 being the third degree of polarimetric purity of $\mathbf{M}$, which represents the parallel portion of $\mathbf{M}$ that is not fully random), preserves the polarimetric anisotropies. The fact that $1/P_3 \geq 1$ implies that the values of the main polarimetric descriptors of $\mathbf{M}_e$ are increased with respect to those of $\mathbf{M}$. Consequently, all information contained in $\mathbf{M}$ is completely encoded by the effective matrix $\mathbf{M}_e$ together with the scalar parameter P_3, so that $\mathbf{M}_e$ provides an augmented view of the polarimetric features of the interaction represented by $\mathbf{M}$.

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