STATIC AND DYNAMIC APPROACHES TO CORPORATE SOCIAL RESPONSIBILITY AS A STRATEGIC TOOL

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Abstract: The aim of this paper is to analyze corporate social responsibility as a strategic tool for firms competing in the market from both static and dynamic perspectives. We analyze competition in a context of strategic delegation in a Cournot duopoly model with homogeneous product, linear demand and constant marginal costs. Firm owners can delegate decisions regarding the quantity produced to their managers using two incentive schemes: one based on a convex linear combination of profits and revenues, and the other on a linear combination of profits and consumer surplus. By resolving the proposed sequential game, we conclude that owners set a social objective for their managers. Furthermore, by introducing the time perspective to the quantity competition stage, we show that the Cournot–Nash equilibrium may become unstable when at least one of the agents adjusts quantities according to an expectations scheme based on marginal utility. An excessively high adjustment speed can lead to a cascade of bifurcations with increasingly complex attractors.

Keywords: Corporative Social Responsibility; Strategic delegation; Cournot–Nash equilibrium; Dynamic stability; Bifurcation

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1. Introduction

In recent years, the concept of corporate social responsibility (CSR) has attracted significant interest in both academia and the broader media. It is generally described as a process in which firms voluntarily integrate social and environmental concerns into their business operations and interactions with stakeholders.

CSR can be defined as a form of self-regulation for firms. This involves adopting socially responsible practices, such as investing in employee well-being, including investment in human capital, health and safety, managing change and natural resources. CSR plays a crucial role in shaping a firm's relationship with its social environment. As pointed out in [4], aspects such as altruism, material incentives, social concerns or self-esteem motivate investors, consumers and employees to engage in prosocial behavior.

CSR extends beyond the limits of the firm, encompassing business partners, suppliers, customers, public authorities and local communities (see [20]). Firms are expected to fulfill their social responsibility by offering products and services that meet consumer needs and desires in an efficient, ethical, and environmentally conscious manner. As a result, CSR should be considered an essential element of a firm's strategic planning and operational performance. Therefore, managers and employees are now required to factor in a wider range of criteria when making business decisions, in addition to those they were traditionally trained to expect, such as profit maximization.

This paper aims to obtain theoretical results on the role of CSR as a strategic variable for firms using both static and dynamic approaches. In the first case, we explore the strategic role of social responsibility in an imperfectly competitive market under strategic delegation. In the second, we examine the potential influence of CSR on the dynamic stability of the market equilibrium.

We analyze competition between two firms using a static approach, based on the Cournot model. In this scenario, we consider ownership and control are separate, allowing owners to delegate production decisions to their managers. When delegation occurs, owners have the option to incentivize their managers by combining either profits and revenues or profits and a social objective represented by consumer surplus. A three-stage sequential game was devised for this project. In the first stage, owners decide whether to delegate and, if so, select the incentive system for their managers. In the second stage, if owners have decided to delegate, they set the remuneration for their managers. In the final stage, managers (or owners if they decided not to delegate) compete based on quantities. By resolving this sequential game, we can endogenously obtain the optimal decisions of both owners and managers, which is a contribution to the literature. This analysis also reveals the effects of adopting a CSR approach from a strategic point of view. Interestingly, our research shows that owners always choose to delegate production decisions to managers and introduce CSR in the compensation contract. Additionally, all owners assign the same weight to social responsibility.

From a dynamic perspective, assuming that both firms are socially responsible, the quantity competition subgame is dynamized on a discrete time scale introduced through the expectations scheme followed by managers for adjusting quantities from one period to the next. Two expectation schemes are considered: one based on best response functions and another based on marginal utility. This defines a dynamic system, qualitatively analyzed using analytical tools and simulations. The results show that if managers adopt the so-called gradient rule as an expectation scheme, the market can become destabilized. In fact, there is a threshold for the speed of quantity adjustment that determines the stability or instability of the market equilibrium. Once the threshold is exceeded, a cascade of flip bifurcations appears with increasingly complex dynamics.

The influence of CSR on this threshold, and thus on equilibrium stability, depends on whether managers are homogeneous or heterogeneous in their expectations. If both managers adjust quantities following the gradient rule, the threshold is independent of the CSR level. Moreover, if the weight assigned to social responsibility is the same in both firms, the map defining the dynamic system becomes symmetric. It can then happen that, starting from different initial conditions, the long-term behavior of both firms is identical.

If there is heterogeneity in the expectation scheme followed by the managers, so that one adopts an adaptive expectation scheme and the other follows the gradient rule, the threshold does depend on the level of CSR set by each firm. In this case, social responsibility can play a stabilising or destabilising role in the market equilibrium.

The remainder of this paper is organized as follows. Section 2 reviews the relevant literature. Section 3 describes the microfoundations of the model and the structure of the proposed three-stage sequential game. Section 4 details the static analysis of the model, solving the third stage of the game (competition in quantities), and the second and first stages (owners' decisions). Section 5 presents the dynamic analysis of the quantity competition subgame assuming both firms are socially responsible. Finally, Section 6 outlines the main conclusions and potential research extensions.

2. Literature review

This section offers a concise literature review to frame our research within both a static and a dynamic context.

From a theoretical point of view, there is a line of research that views CSR as a strategic tool capable of improving competitive advantage and increasing profits, aligning with the original idea put forth by Friedman [15], which confines CSR to increasing firm profits. Lambertini and Tampieri [19] investigated the influence of socially responsible behavior on corporate profit and social welfare in scenarios involving external environmental effects. Similarly, Fanti and Bucella [8] studied the strategic role of incorporating a social objective in a duopoly with differentiated products. These authors demonstrated that under quantity competition, and depending on the degree of differentiation and the level of social responsibility, different equilibria may arise: both firms adopt a CSR criterion, one firm is socially responsible and the other prioritizes profit maximization, or both aim to maximize profit. The authors concluded that, under price competition, firms adopt profit maximization as their sole strategy.

The aforementioned publications assume that the decision to pursue a social objective is exogenous, thus limiting the exploration of using CSR as a competitive variable. To address this gap, the present study examines social responsibility within the context of strategic delegation, in line with the seminal contributions in managerial literature [14, 22, 23]. There are previous papers that analyze the role of CSR in contexts of separating ownership and the control of the firms, being significant examples [7, 9, 10, 17, 20, 21].

Buccella et al. [7], consider a managerial Cournot duopoly with pollution externalities and emission taxes and assume that managers can be compensated by a contract that includes an explicit environmental incentive. The authors show that the environmental contract never emerges as the unique subgame perfect Nash equilibrium of the developed non-cooperative game. In fact, if technological investment to reduce emissions is efficient, the only perfect Nash equilibrium involves remunerating managers through the combination of profits and sales.

Fanti and Bucella [9] analyzed a duopoly in which owners delegate production decisions to managers and compensate them according to two types of contracts: one based on the combination of profit and sales, and another that included consumer surplus alongside profit and sales. Their findings suggest that in equilibrium both firms incorporate a social objective into their compensation scheme (consumer surplus) and this resulted in higher profit but lower social welfare.

The same authors, in [10], analyze a Cournot duopoly in which the owners' objective function is defined as the sum of the profit and a share of the consumer surplus. They remunerate their managers through two contracts: one based on sales volume and the other incorporating relative profit. Under both types of contracts, it is shown that the introduction of CSR in the owners' objective modifies the standard results obtained in the literature devoted to the study of competition in the presence of delegation.

Kopel and Brand [17] developed a mixed Cournot duopoly model to analyze competition between a profit-maximizing firm and a socially concerned firm which maximizes its profit plus a share of consumer surplus. Both firms could hire a manager to delegate production decisions. The authors concluded that, in equilibrium, both firms choose to delegate production decisions and, under the assumption of symmetric costs, the socially responsible firm captures a larger market share.

In [20], Planer-Friedrich and Sahm consider that two firms can choose to maximize profits alongside either consumer surplus (CSR) or the well-being of their own customers specifically (customer orientation). Their findings suggest that firms prioritize overall consumer well-being by opting for positive CSR levels. In [21], these authors extend this concept to a Cournot oligopoly, demonstrating that equilibrium involves positive CSR levels. Furthermore, they conclude that CSR acts as a barrier for new firms to entry the market, increasing market concentration.

In the referred papers, either the decision to delegate, or the weight assigned to social responsibility, is exogenously given from the general economic environment or context in which firms operate. In order to mitigate this limitation, in this paper, through the resolution of a three-stage sequential game, both the decision to delegate market competition to a manager, and the weight assigned to the social responsibility objective, given by the consumer surplus, are obtained endogenously.

The majority of theoretical contributions on competition between socially responsible firms are static, with few authors taking a dynamic approach. An exception is Kopel et al. in [18], who investigate quantity competition in an oligopoly from an evolutionary perspective based on two groups of firms: those that prioritize profit maximization and those that maximize a linear combination of profit and consumer surplus. They endogenously deduce that firms can benefit from social responsibility strategies if consumers pay more for them, and that steady states become unstable if firms' propensity to switch strategies is sufficiently high, leading to complex dynamics.

The present work contemplates the potential influence of social responsibility on market stability using non-linear dynamic oligopoly models (see [5] for an excellent review of this literature). Zhu et al. [25] provide a relevant example, analyzing local and global dynamics in a Cournot duopoly game with differentiated products and strategic delegation. In this model, one firm incorporates market share into its objective function alongside profit, while the other incorporates consumer surplus.

Similarly, Askar et al. [3] also study local and global dynamic stability in a Cournot duopoly with social responsibility. In this model, one firm maximizes a weighted average of total surplus and individual profit, while the other maximizes relative profit.

More recently, in the same context [2] investigates the dynamic behavior in a Cournot oligopoly under the assumption of isoelastic demand, with at least one firm having a social responsibility objective. This study reveals that social responsibility can have a stabilizing effect on equilibrium, with the magnitude of said effect increasing as the demand function becomes more elastic.

As will be shown below, our dynamic analysis will allow us to conclude that, in a scenario with linear demand and constant and identical marginal costs, the influence of CSR on equilibrium stability will depend on whether there is homogeneity or heterogeneity in the expectations scheme followed by firms.

3. The model

We explore a duopoly in which firms produce quantities of a homogeneous good, with a linear inverse demand function:

$$p = a - bq,\ a, b > 0,\ q = q_1 + q_2$$

where p is the market price, q is the total output, and q_i is the quantity supply by firm $i = 1, 2$. Parameter a captures the size of the demand and b the degree of price sensitivity of demand.

The marginal costs of both firms are constant and identical, denoted by c, so $0 < c < a$.

In both firms, ownership and control are separate. This means that owners can hire managers and delegate production decisions. As noted in [22], the owner is a decision-making agent with the objective of profit maximization, while the manager is hired by the owner to make real-time operational decisions.

In each firm, the owner can compensate the manager by offering a "take-it-or-leave-it" incentive contract. Indeed, in the strategic delegation literature (e.g., [9, 17, 22, 23]), it is common to assume that owners have full bargaining power. Alternatively, we could assume that each owner negotiates the compensation contract with his manager, in line with [13].

In particular, we will assume that the owner $i \in \{1,2\}$ can choose between the following types of contracts:

$$\textit{P-type contract}:\ U_i^P = \Pi_i = (p-c)q_i \quad (1)$$

$$\textit{PR-type contract}:\ U_i^{PR} = \gamma_i \Pi_i + (1-\gamma_i) R_i, \text{ with } 0 < \gamma_i \leq 1 \quad (2)$$

being $R_i = pq_i$ the revenues of firm *i*.

$$\textit{RS-type contract}:\ U_i^{RS} = \Pi_i + \theta_i CS, \text{ with } 0 < \theta_i \leq 1 \quad (3)$$

where $CS = \frac{bq^2}{2}$ is the consumer surplus.

In specification (1), the owner remunerates the manager based on profits, and there is no divergence between the objectives of both parties. This is essentially the same as having no delegation.

In *PR*-type contract, following [17], the manager is remunerated through a convex linear combination of profits and revenues. Thus, the weight assigned to revenues may be greater than, less than or equal to the weight assigned to profits. This compensation scheme can be expressed as $U_i^{PR} = (p - \gamma_i c) q_i$, and represents accounting profits. As noted in [8], the owner can decide the amount of costs for the evaluation of the manager's performance by setting a value of γ_i. A value lower than unity for this parameter incentivizes the manager to be more aggressive in the market, facing a lower marginal cost.

The incentive scheme outlined in (3) incorporates a weighting of consumer surplus, reflecting the strategic role of social responsibility (as seen in [8, 9, 17, 20, 21]). The parameter $\theta_i \in (0,1]$ represents the weight assigned by owner *i* to consumer surplus, thus reflecting the level of social responsibility of firm *i*. In this case, unlike specification (2), this weight cannot exceed the weight assigned to the profits and it does not influence the cost faced by the manager.

We will develop a three-stage sequential game. In the first stage, owners simultaneously decide whether to delegate or not, and if so, choose the incentive scheme for compensating their managers. In the second stage, if they chose to delegate, they simultaneously determine the profit-maximizing remuneration rate (optimal value of γ_i and θ_i). Finally, in the third stage, the managers of each firm (or the owners if they have decided not to delegate) select the quantity produced to maximize their offsetting contract. Solving the game will result in a unique subgame perfect Nash equilibrium that will be symmetric.

Our model differs significantly from those described in [9], [10] and [17]. In [9] delegation is predetermined, assuming that owners invariably opt to hire a manager. In contrast, the decision to delegate in our model is endogenously obtained as an equilibrium outcome. Additionally, a combination of profits and sales volume is introduced in [9] as an incentive scheme, while our model uses a combination of profits and revenues. As highlighted in [22], the output volume should be excluded from the incentive scheme, since owners may be unable to observe it directly, and when a firm produces several goods, it may be difficult to characterize. Both [17] and [9] treat the weight assigned to consumer surplus in managers' objective function as exogenous. In contrast, our model allows this weight to be determined endogenously.

In [10], Fanti and Buccella analyze the equilibrium in a Cournot duopoly in which the owners maximize profit and an exogenous proportion of consumer surplus, and have the option of hiring a manager who receives a fixed salary plus a bonus related to the sales volume.

In [17], Kopel and Brand analyze a mixed duopoly scenario in which, in one firm, owners maximize the linear combination between profit and consumer surplus and in the other firm, they maximize profit. Both firms can hire a manager. In that case, the profit-maximizing firm compensates him through the combination of profit and revenues, and the socially responsible firm compensates its manager through consumer surplus.

It is important to note that in [9], [10] and [17], the owners' objective function incorporates an exogenous fraction of consumer surplus, which may depend on the economic environment in which firms operate. In contrast, in our model, the owners' objective is the maximization of individual profit. Social responsibility, given by the consumer surplus, is interpreted as a part of the managers' remuneration, whose weight in the incentive contract is endogenously determined through the optimizing behavior of the owners. This makes it possible to interpret social responsibility as a competitive instrument and to explore its strategic character.

Finally, in [20], on the basis that there is delegation, firms simultaneously decide their corporate culture, prioritizing either consumer surplus or their own customers exclusively.

4. Static analysis

This section presents the resolution of the three-stage sequential game posed by the backward induction algorithm.

4.1. Quantity competition

The third stage of the game analyzes the decision of managers (or owners if they have decided not to delegate). Following the incentive contract and remuneration set by owners in previous stages, nine different subgames emerge. Three of these subgames correspond to symmetric scenarios in which owners of both firms opt for the same compensation contract.

***Case 1*. In both firms, owners set a *P*-type contract:** $\left(U_1^P, U_2^P\right)$ [1]

[1] U_i^j denotes the utility function of manager *i* assuming that owner *i* chooses a *j*-type contract, with *i* = 1, 2, and *j* = *P*, *PR*, *RS*. Thus, $\left(U_1^j, U_2^k\right)$ denotes the third-stage subgame in which firm 1 has chosen *j*-type contract and firm 2 has chosen *k*-type contract.

In this case, there is no delegation or, equivalently, in both firms the objective of owners and managers is the same.

Each manager determines the quantity according to the maximization problem:

$$\underset{q_i}{Max}\ U_i^P = (p-c)q_i = \left[a - b\left(q_i + q_j\right) - c\right]q_i,\ \ i,j \in \{1,2\}, i \neq j$$

The first-order condition for firm i is given by:

$$\frac{\partial U_i^P}{\partial q_i} = a - c - bq_j - 2bq_i = 0$$

leading to the best response function for the manager of firm i:

$$q_i = R_i\left(q_j\right) = \frac{a - c - bq_j}{2b}$$

The second-order condition is satisfied, given that: $\frac{\partial^2 U_i^P}{\partial q_i^2} = -2b < 0.$

The intersection of the best response functions leads to the quantity combination corresponding to the Cournot–Nash equilibrium of the subgame $\left(U_1^P, U_2^P\right)$:

$$E_{P,P}^c = \left(q_1^*, q_2^*\right) = \left(\frac{a-c}{3b}, \frac{a-c}{3b}\right)$$

the associated profits being[2]:

$$\Pi_1^{P,P} = \Pi_2^{P,P} = \frac{(a-c)^2}{9b} \tag{4}$$

***Case 2*. In both firms, owners set a *PR*-type contract:** $\left(U_1^{PR}, U_2^{PR}\right)$

In this case, we assume that the owners in both firms implement an incentives contract, as described in (2).

Each manager determines the output level that maximizes its objective function:

$$\underset{q_i}{Max}\ U_i^{PR} = (p - \gamma_i c)q_i = \left[a - b\left(q_i + q_j\right) - \gamma_i c\right]q_i,\ \ i,j \in \{1,2\}, i \neq j$$

being the first-order condition for the manager i:

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[2] $\Pi_i^{j,k}$ denotes the owner's profit function of firm i evaluated in the Nash equilibrium of the third stage subgame, assuming that firm 1 chooses a j-type contract and firm 2 a k-type contract, with i = 1, 2, and *j, k = P, PR, RS*.

$$\frac{\partial U_i^{PR}}{\partial q_i} = a - \gamma_i c - bq_j - 2bq_i = 0$$

The best response function for the manager of firm *i* is:

$$q_i = R_i\left(q_j\right) = \frac{a - \gamma_i c - bq_j}{2b}$$

Fulfillment of the second-order condition is ensured, given that $\frac{\partial^2 U_i^{PR}}{\partial q_i^2} = -2b < 0.$

The Cournot–Nash equilibrium of the subgame $\left(U_1^{PR}, U_2^{PR}\right)$ is obtained at the intersection of both reaction functions:

$$E_{PR,PR}^{c} = \left(q_1^*, q_2^*\right) = \left(\frac{a - (2\gamma_1 - \gamma_2)c}{3b}, \frac{a - (2\gamma_2 - \gamma_1)c}{3b}\right)$$

The associated profits are:

$$\left.\begin{aligned} \Pi_1^{PR,PR}(\gamma_1, \gamma_2) &= \frac{\left[a - (3 - \gamma_1 - \gamma_2)c\right]\left[a - (2\gamma_1 - \gamma_2)c\right]}{9b} \\ \Pi_2^{PR,PR}(\gamma_1, \gamma_2) &= \frac{\left[a - (3 - \gamma_1 - \gamma_2)c\right]\left[a - (2\gamma_2 - \gamma_1)c\right]}{9b} \end{aligned}\right\} \quad (5)$$

Case 3*. In both firms, owners set a *RS* –type contract*: $\left(U_1^{RS}, U_2^{RS}\right)$

In this scenario, we asumme that owners remunerate managers according to the incentives contract described in (3), which includes the consumer surplus.

Each manager determines the quantity that maximizes its objective function:

$$\underset{q_i}{Max}\ U_i^{RS} = \left[a - b\left(q_i + q_j\right) - c\right]q_i + \theta_i \frac{b\left(q_i + q_j\right)^2}{2}, \ \ i, j \in \{1, 2\}, i \neq j$$

with the first-order condition given as:

$$\frac{\partial U_i^{RS}}{\partial q_i} = a - c - b(1 - \theta_i)q_j - b(2 - \theta_i)q_i = 0 \quad (6)$$

the sufficient condition being: $\frac{\partial^2 U_i^{RS}}{\partial q_i^2} = -b\left(2 - \theta_i\right) < 0, \ \ \forall \theta_i \in (0, 1].$

From (6), we deduce the reaction functions for both managers:

$$q_i = R_i\left(q_j\right) = \frac{a - c - b(1 - \theta_i)q_j}{(2 - \theta_i)b} \quad (7)$$

The intersection of these functions determines the Cournot–Nash equilibrium of the subgame $\left(U_1^{RS}, U_2^{RS}\right)$:

$$E^{c}_{RS,RS} = \left(q_1^*, q_2^*\right) = \left(\frac{(1+\theta_1-\theta_2)(a-c)}{(3-\theta_1-\theta_2)b}, \frac{(1+\theta_2-\theta_1)(a-c)}{(3-\theta_1-\theta_2)b} \right) \quad (8)$$

And the associated profits are:

$$\left.\begin{aligned} \Pi_1^{RS,RS}(\theta_1,\theta_2) &= \frac{\left[(1-\theta_2)^2-\theta_1^2\right](a-c)^2}{(3-\theta_1-\theta_2)^2 b} \\ \Pi_2^{RS,RS}(\theta_1,\theta_2) &= \frac{\left[(1-\theta_1)^2-\theta_2^2\right](a-c)^2}{(3-\theta_1-\theta_2)^2 b} \end{aligned}\right\} \quad (9)$$

A similar analysis would apply to asymmetric cases. Therefore, the resulting Cournot–Nash equilibria and associated benefits are detailed as follows.

***Case 4*. The owner of one of the firms adopts a *P*-type contract and the other adopts a *PR*-type contract:** $\left(U_1^P, U_2^{PR}\right)$ or $\left(U_1^{PR}, U_2^P\right)$.

The Cournot–Nash equilibrium of the subgame $\left(U_1^P, U_2^{PR}\right)$ is given as:

$$E^{c}_{P,PR} = \left(q_1^*, q_2^*\right) = \left(\frac{a-(2-\gamma_2)c}{3b}, \frac{a-(2\gamma_2-1)c}{3b} \right)$$

with the associated profits being:

$$\left.\begin{aligned} \Pi_1^{P,PR}(\gamma_2) &= \frac{\left[a-(2-\gamma_2)c\right]^2}{9b} \\ \Pi_2^{P,RP}(\gamma_2) &= \frac{\left[a-(2-\gamma_2)c\right]\left[a-(2\gamma_2-1)c\right]}{9b} \end{aligned}\right\} \quad (10)$$

For the subgame $\left(U_1^{PR}, U_2^P\right)$, the Cournot–Nash equilibrium is given as:

$$E^{c}_{PR,P} = \left(q_1^*, q_2^*\right) = \left(\frac{a-(2\gamma_1-1)c}{3b}, \frac{a-(2-\gamma_1)c}{3b} \right)$$

with the associated profits being:

$$\left.\begin{aligned} \Pi_1^{PR,P}(\gamma_1) &= \frac{\left[a-(2-\gamma_1)c\right]\left[a-(2\gamma_1-1)c\right]}{9b} \\ \Pi_2^{PR,P}(\gamma_1) &= \frac{\left[a-(2-\gamma_1)c\right]^2}{9b} \end{aligned}\right\}$$

Case 5.* The owner of one firm adopts a *P*-type contract and the other adopts an *RS*-type contract*: $\left(U_1^P, U_2^{RS}\right)$ or $\left(U_1^{RS}, U_2^P\right)$.

The Cournot–Nash equilibrium of the subgame $\left(U_1^P, U_2^{RS}\right)$ is defined as:

$$E^{c}_{P,RS}=\left(q_1^*,q_2^*\right)=\left(\frac{(1-\theta_2)(a-c)}{(3-\theta_2)b},\frac{(1+\theta_2)(a-c)}{(3-\theta_2)b}\right)$$

leading to the profit levels:

$$\left.\begin{aligned}\Pi_1^{P,RS}(\theta_2)&=\frac{(1-\theta_2)^2(a-c)^2}{(3-\theta_2)^2 b}\\ \Pi_2^{P,RS}(\theta_2)&=\frac{(1-\theta_2^2)(a-c)^2}{(3-\theta_2)^2 b}\end{aligned}\right\}\tag{11}$$

Similarly, the Cournot–Nash equilibrium and the the profit levels of the subgame $\left(U_1^{RS},U_2^{P}\right)$ are:

$$E^{c}_{RS,P}=\left(q_1^*,q_2^*\right)=\left(\frac{(1+\theta_1)(a-c)}{(3-\theta_1)b},\frac{(1-\theta_1)(a-c)}{(3-\theta_1)b}\right)$$

$$\left.\begin{aligned}\Pi_1^{RS,P}(\theta_1)&=\frac{(1-\theta_1^2)(a-c)^2}{(3-\theta_1)^2 b}\\ \Pi_2^{RS,P}(\theta_1)&=\frac{(1-\theta_1)^2(a-c)^2}{(3-\theta_1)^2 b}\end{aligned}\right\}$$

Case 6*. The owner of one firm adopts a *PR*-type contract and the other adopts an *RS*-type contract*: $\left(U_1^{PR},U_2^{RS}\right)$ or $\left(U_1^{RS},U_2^{PR}\right)$.

The Cournot–Nash equilibrium and the associated profits of the subgame $\left(U_1^{PR},U_2^{RS}\right)$ are, respectively:

$$E^{c}_{PR,RS}=\left(q_1^*,q_2^*\right)=\left(\frac{(1-\theta_2)a+\left[1-\gamma_1(2-\theta_2)\right]c}{(3-\theta_2)b},\frac{(1+\theta_2)a-\left[2-\gamma_1(1-\theta_2)\right]c}{(3-\theta_2)b}\right)$$

$$\left.\begin{aligned}\Pi_1^{PR,RS}(\gamma_1,\theta_2)&=\frac{\left[(1-\theta_2)a-(2-\gamma_1-\theta_2)c\right]\left[(1-\theta_2)a+\left(1-(2-\theta_2)\gamma_1\right)c\right]}{(3-\theta_2)^2 b}\\ \Pi_2^{PR,RS}(\gamma_1,\theta_2)&=\frac{\left[(1-\theta_2)a-(2-\gamma_1-\theta_2)c\right]\left[(1+\theta_2)a-\left(2-(1-\theta_2)\gamma_1\right)c\right]}{(3-\theta_2)^2 b}\end{aligned}\right\}\tag{12}$$

Similarly, for the subgame $\left(U_1^{RS},U_2^{PR}\right)$, the Cournot–Nash equilibrium and the associated profits are, respectively:

$$E^{c}_{RS,PR}=\left(q_1^*,q_2^*\right)=\left(\frac{(1+\theta_1)a-\left[2-\gamma_2(1-\theta_1)\right]c}{(3-\theta_1)b},\frac{(1-\theta_1)a+\left[1-\gamma_2(2-\theta_1)\right]c}{(3-\theta_1)b}\right)$$

$$\left.\begin{aligned}\Pi_1^{RS,PR}(\gamma_2,\theta_1)&=\frac{\left[(1-\theta_1)a-(2-\gamma_2-\theta_1)c\right]\left[(1+\theta_1)a-\left(2-(1-\theta_1)\gamma_2\right)c\right]}{(3-\theta_1)^2 b}\\ \Pi_2^{RS,PR}(\gamma_2,\theta_1)&=\frac{\left[(1-\theta_1)a-(2-\gamma_2-\theta_1)c\right]\left[(1-\theta_1)a+\left(1-(2-\theta_1)\gamma_2\right)c\right]}{(3-\theta_1)^2 b}\end{aligned}\right\}$$

4.2. Owners' decision

If there is delegation, in the second stage of the sequential game, the owners of both firms simultaneously decide on the managers' remuneration. To do so, they determine the values of the profit-maximizing parameters γ_i or $\theta_i, i \in \{1,2\}$, depending on the chosen incentive contract (*PR*-type or *RS*-type). Note that if only one firm delegates, we will be facing an optimization problem with a single decision-maker.

The results derived from the resolution of all possible cases in the second stage are as follows.

a) In both firms, owners set a *PR*–type contract: $\left(\Pi_1^{PR,PR}, \Pi_2^{PR,PR}\right)$[3]

In this case, firm profits are given by (5). The owner *i* chooses the parameter γ_i maximizing its objective function:

$$\underset{\gamma_i}{Max}\ \Pi_i^{PR,PR}(\gamma_i,\gamma_j)=\frac{\left[a-(3-\gamma_i-\gamma_j)c\right]\left[a-(2\gamma_i-\gamma_j)c\right]}{9b},\quad i,j\in\{1,2\}, i\neq j$$

with the first-order condition given as:

$$\frac{\partial \Pi_i^{PR,PR}}{\partial \gamma_i}=\frac{c}{9b}\left[(6-4\gamma_i-\gamma_j)c-a\right]=0$$

the sufficient condition being: $\dfrac{\partial^2 \Pi_i}{\partial \gamma_i^2}=\dfrac{-4c^2}{9b}<0$.

The first-order condition leads to the reaction functions of owner *i*:

$$\gamma_i=R_i(\gamma_j)=\frac{(6-\gamma_j)c-a}{4c}$$

The intersection of $R_1(\gamma_2)$ and $R_2(\gamma_1)$ determines the Nash equilibrium of the subgame, defining the values: $\gamma_1^*=\gamma_2^*=\dfrac{6c-a}{5c}$, which are positive provided that $\dfrac{a}{6}<c$. Recalling

[3] $\left(\Pi_1^{j,k}, \Pi_2^{j,k}\right)$ denotes the second-stage subgame in which firm 1 has chosen *j*-type contract and firm 2 has chosen *k*-type contract, with *j, k = PR, RS*.

that $c < a$, we also have that $\gamma_i^* < 1$, which implies that owners encourage managers to compete more aggressively.

By introducing these values in (5), we obtain the associated profits of the second stage of the sequential game:

$$\Pi_1^{PR,PR} = \Pi_2^{PR,PR} = \frac{2(a-c)^2}{25b} \tag{13}$$

b) In both firms, owners set a *RS* –type contract: $\left(\Pi_1^{RS,RS}, \Pi_2^{RS,RS}\right)$

If the owners of both firms introduce an element of social responsibility in the incentives contracts, profits are given by (9). Resolving the subgame $\left(\Pi_1^{RS,RS}, \Pi_2^{RS,RS}\right)$, the values of the Nash equilibrium are: $\theta_1^* = \theta_2^* = \dfrac{5-\sqrt{17}}{4}$.

The optimum profits are obtained by substituting these values in (9):

$$\Pi_1^{RS,RS} = \Pi_2^{RS,RS} = \frac{3\sqrt{17}-11}{16}\frac{(a-c)^2}{b} \tag{14}$$

c) The owner of one firm adopts a *P*-type contract and the other adopts a *PR*- type contract: $\left(\Pi_1^{P,PR}, \Pi_2^{P,PR}\right)$ or $\left(\Pi_1^{PR,P}, \Pi_2^{PR,P}\right)$

In case $\left(\Pi_1^{P,PR}, \Pi_2^{P,PR}\right)$, the owner of the second firm is the only decision maker, and its profits function, $\Pi_2^{P,RP}(\gamma_2)$, is expressed in (10). By solving the following maximization problem:

$$\underset{\gamma_2}{Max}\ \Pi_2^{P,RP}(\gamma_2) = \frac{\left[a-(2-\gamma_2)c\right]\left[a-(2\gamma_2-1)c\right]}{9b}$$

the optimal parameter value is obtained:

$$\gamma_2^* = \frac{5c-a}{4c} < 1, \text{ for } \frac{a}{5} < c < a$$

and corresponding profits for owners of both firms are:

$$\left.\begin{aligned} \Pi_1^{P,PR} &= \frac{(a-c)^2}{16b} \\ \Pi_2^{P,PR} &= \frac{(a-c)^2}{8b} \end{aligned}\right\} \tag{15}$$

Similarly for case $\left(\Pi_1^{PR,P}, \Pi_2^{PR,P}\right)$ where only the first firm delegates, we obtain:

$$\gamma_1^* = \frac{5c-a}{4c} < 1, \quad \text{for } \frac{a}{5} < c < a$$

$$\left.\begin{aligned}\Pi_1^{PR,P} &= \frac{(a-c)^2}{8b}\\ \Pi_2^{PR,P} &= \frac{(a-c)^2}{16b}\end{aligned}\right\} \qquad (16)$$

d) The owner of one firm adopts a *P*- type contract and the other adopts an *RS*- type contract: $\left(\Pi_1^{P,RS},\Pi_2^{P,RS}\right)$ or $\left(\Pi_1^{RS,P},\Pi_2^{RS,P}\right)$

In case $\left(\Pi_1^{P,RS},\Pi_2^{P,RS}\right)$ we consider that the owner of firm 2 establishes a social incentive for the managers while the owner of firm 1 remunerates according to a *P*-type contract, with the optimal profits of the third stage being expressed in (11).

By solving the following maximization problem:

$$\underset{\theta_2}{Max}\ \Pi_2^{P,RS}(\theta_2) = \frac{\left(1-\theta_2^2\right)(a-c)^2}{(3-\theta_2)^2 b}$$

the optimal value of the parameter that represents the level of social responsibility of the firm 2 is given by $\theta_2^* = \frac{1}{3}$, and the associated profits being:

$$\left.\begin{aligned}\Pi_1^{P,RS} &= \frac{(a-c)^2}{16b}\\ \Pi_2^{P,RS} &= \frac{(a-c)^2}{8b}\end{aligned}\right\} \qquad (17)$$

Similarly, for case $\left(\Pi_1^{RS,P},\Pi_2^{RS,P}\right)$ we obtain $\theta_1^* = \frac{1}{3}$, and the associated profits being:

$$\left.\begin{aligned}\Pi_1^{RS,P} &= \frac{(a-c)^2}{8b}\\ \Pi_2^{RS,P} &= \frac{(a-c)^2}{16b}\end{aligned}\right\} \qquad (18)$$

e) The owner of one firm adopts a *PR*- type contract and the other adopts an *RS*- type contract: $\left(\Pi_1^{PR,RS},\Pi_2^{PR,RS}\right)$ or $\left(\Pi_1^{RS,PR},\Pi_2^{RS,PR}\right)$

Finally, we consider that the owner of one firm uses a *PR*-type contract to remunerate managers, while the other firm uses a *RS*-type contract.

Solving subgame $\left(\Pi_1^{PR,RS},\Pi_2^{PR,RS}\right)$, whose profits functions are given by (12), allows us to determine the Nash equilibrium:

$$\gamma_1^* = \frac{3a-2\sqrt{3}(a-c)}{3c} < 1$$

$$\theta_2^* = 2-\sqrt{3}$$

leading to profits:

$$\left.\begin{aligned}\Pi_1^{PR,RS} &= \frac{2\sqrt{3}-3}{6}\frac{(a-c)^2}{b}\\ \Pi_2^{PR,RS} &= \frac{2-\sqrt{3}}{3}\frac{(a-c)^2}{b}\end{aligned}\right\} \tag{19}$$

Similarly, for the subgame $\left(\Pi_1^{RS,P}, \Pi_2^{RS,P}\right)$, we obtain:

$$\theta_1^* = 2-\sqrt{3}$$

$$\gamma_2^* = \frac{3a-2\sqrt{3}(a-c)}{3c} < 1$$

and the associated profits being:

$$\left.\begin{aligned}\Pi_1^{RS,PR} &= \frac{2-\sqrt{3}}{3}\frac{(a-c)^2}{b}\\ \Pi_2^{RS,PR} &= \frac{2\sqrt{3}-3}{6}\frac{(a-c)^2}{b}\end{aligned}\right\} \tag{20}$$

Utilizing the results obtained above, we can address the resolution of the first-stage subgame, in which the players (the owners of both firms) decide whether or not to delegate and, if so, the optimal compensation contract. The payoffs given in (13), (14), (15), (16), (17), (18), (19) and (20), can be summarized in the following subgame payoff matrix expressed in normal form (see Table 1):

Table 1. First-stage payoff matrix.

		***Firm* 2**		
		P	***PR***	***RS***
	P	$\frac{(a-c)^2}{9b}, \frac{(a-c)^2}{9b}$	$\frac{(a-c)^2}{16b}, \frac{(a-c)^2}{8b}$	$\frac{(a-c)^2}{16b}, \frac{(a-c)^2}{8b}$
***Firm* 1**	***PR***	$\frac{(a-c)^2}{8b}, \frac{(a-c)^2}{16b}$	$\frac{2(a-c)^2}{25b}, \frac{2(a-c)^2}{25b}$	$\frac{2\sqrt{3}-3}{6}\frac{(a-c)^2}{b}, \frac{2-\sqrt{3}}{3}\frac{(a-c)^2}{b}$
	RS	$\frac{(a-c)^2}{8b}, \frac{(a-c)^2}{16b}$	$\frac{2-\sqrt{3}}{3}\frac{(a-c)^2}{b}, \frac{2\sqrt{3}-3}{6}\frac{(a-c)^2}{b}$	$\frac{3\sqrt{17}-11}{16}\frac{(a-c)^2}{b}, \frac{3\sqrt{17}-11}{16}\frac{(a-c)^2}{b}$

The following proposition is obtained:

Proposition 1. *According to the profit maximization objective, the adoption of the RS strategy by the owners of both firms determines the only subgame perfect Nash equilibrium, which is also symmetric.*

Proof. Iterative elimination of strictly dominated strategies reveals that only profile that survives is (*RS*, *RS*); it is, therefore, the only Nash equilibrium of the first-stage subgame. Owners choose to delegate production quantity decisions to managers and compensate them by combining profits and consumer surplus.

In the associated subgames perfect result, owners compensate managers according to an *RS* contract in the first stage, weight consumer surplus with the value $\theta_1^* = \theta_2^* = \frac{5-\sqrt{17}}{4}$ in the second stage, and managers produce the quantities $q_1^* = q_2^* = \frac{\sqrt{17}-1}{8}\frac{(a-c)}{b}$ in the third stage. □

The intuition behind this result is straightforward. When both firms base incentive contracts on a combination of profit and revenue, the degree of strategic substitutability is higher than with contracts combining profit and consumer surplus. In fact, the slope of the reaction functions in absolute value in the first case is greater than in the second. Therefore, incorporating a social responsibility objective in managers' remuneration encourages managers to be less aggressive in the product market, leading to a more collusive solution.

Our conclusions, therefore, are consistent with those of [9, 20], but we obtained the decisions of all agents endogenously. In [9], Fanti and Bucella define as incentive scheme a linear combination of profits, consumer surplus and sales volume, but assume that the delegation decision and the weight assigned to consumer surplus are exogenous. In [20], Planer-Friedrich and Sahm compare the strategic potential of CSR and customer orientation, revealing that competition with CSR is less severe.

5. Dynamic analysis

In the static analysis developed in the previous section, both owners and managers are assumed to behave rationally, and their optimal decisions are given by the subgame perfect Nash equilibrium. Focusing on the third-stage subgame, if players simultaneously select Nash equilibrium quantities, no agent will have an incentive to unilaterally modify their choice. Therefore, in the absence of coordination and cooperation, quantities will remain at equilibrium levels. However, if the quantities produced deviate from equilibrium, at least one of the players will have incentives to vary production in order to maximize the objective function. Since both agents act similarly, the Nash equilibrium may not be reached, triggering a dynamic adjustment process. The dynamic behavior caused by this process will depend on the time scale considered and the way in which players adjust over time their production quantities. The adjustment mechanism depends, in turn, on the expectations formation rule that is adopted.

This section delves into dynamic adjustment processes in the quantity competition subgame, assuming that firm owners remunerate managers using a combination of profits and consumer surplus. This assumption is justified because one of the objectives of this paper is to determine the potential impact of CSR on the dynamic stability of the Cournot–Nash equilibrium.

In this context, and following the methodology of non-linear dynamic oligopoly models (see [5]), we will analyze the dynamic stability of the equilibria. In addition, we will investigate how the level of social responsibility (parameter θ_i), the size and slope of the demand (*a* and *b*, respectively) and marginal cost (*c*) influence this equilibrium stability.

We consider a discrete time scale and two adjustment mechanisms based on the following expectation rules: adaptive expectations (based on the best response functions) and a gradient rule (based on marginal utility). Each expectations scheme implies a certain degree of bounded rationality. First, we will consider the case of homogeneous expectations. Subsequently, we will analyze the case of heterogeneity in the formation of expectations.

5.1. Homogeneous expectations: The adaptive expectations scheme

We assume manager *i* uses an adjustment mechanism based on the best response function at any period *t* to determine production at period *t*+1. Each player adjusts output quantity proportionally to the difference between the value given by the reaction function and the quantity produced in the last period, as described in [5]. Formally:

$$q_{i,t+1} - q_{i,t} = \beta_i \left(R_i(q_{j,t}) - q_{i,t} \right), i, j \in \{1,2\}, i \neq j, \text{ with } 0 < \beta_i \leq 1$$

From the previous expression we obtain the linear dynamic system:

$$T_A : \begin{cases} q_{1,t+1} = (1-\beta_1)\; q_{1,t} + \beta_1 \, R_1(q_{2,t}) \\ q_{2,t+1} = (1-\beta_2)\; q_{2,t} + \beta_2 \, R_2(q_{1,t}) \end{cases} \tag{21}$$

where $R_1(q_{2,t})$ and $R_2(q_{1,t})$ are given in (7).

Under this expectations scheme, managers possess a comprehensive understanding of the strategic aspects of the game, but operate with a degree of myopia; they adjust their output partially in the current period in response to the observed output of the other player from the previous period. We assume that firms are averse to make changes in the quantity they produce in the current period, influenced by some kind of linkage that prevents them from making excessive changes to their strategic variables from one period to the next.

The parameter β_i measures the extent to which firm *i* follows the best-response signal or, conversely, $1-\beta_i$ represents the strength of the linkage. In the hypothetical extreme case where $\beta_i = 0$, the link would be so strong that firm *i* would never change its initial choice. Conversely, with $\beta_i = 1$, firm *i* adjusts the value of its strategic variable directly to the best response. The latter case would correspond to the concept known as naïve expectations or a Cournot expectations scheme.

By setting the fixed point conditions $q_{i,t+1} = q_{i,t} = q_i$ in the system (21), we obtain a unique steady state, which is the Cournot–Nash equilibrium E^C given in (8).

In a discrete-time dynamic system, local asymptotic stability of an equilibrium demands that the eigenvalues of the Jacobian matrix of a system calculated at the equilibrium point remain within the unit circle. In the two-dimensional case, the condition for local stability of the equilibrium can be given in terms of the trace (*Tr*) and determinant (*Det*) of the associated Jacobian matrix (Schur's conditions, see [16]):

$$\left.\begin{array}{l}(i)\ 1-Tr+Det>0\\(ii)\ 1+Tr+Det>0\\(iii)\ 1-Det>0\end{array}\right\} \tag{22}$$

If any single inequality in (22) becomes an equality, while the other two are simultaneously fulfilled, the equilibrium can lose stability through one of the following bifurcations: a transcritical bifurcation, when $1-Tr+Det=0$; a flip bifurcation, when $1+Tr+Det=0$; or, a Neimark–Sacker bifurcation, when $1-Det=0$.

The Jacobian matrix of system T_A evaluated at E^C is:

$$JT_A\left(E^C\right)=\begin{pmatrix}1-\beta_1 & \beta_1\dfrac{dR_1\left(q_2^*\right)}{dq_1}\\ \beta_2\dfrac{dR_2\left(q_1^*\right)}{dq_2} & 1-\beta_2\end{pmatrix}=\begin{pmatrix}1-\beta_1 & -\beta_1\dfrac{1-\theta_1}{2-\theta_1}\\ -\beta_2\dfrac{1-\theta_2}{2-\theta_2} & 1-\beta_2\end{pmatrix}$$

We can deduce the following result for the local stability of the Nash equilibrium E^C.

Proposition 2. *Under an adaptive expectations scheme, for all* $0<\beta_i\le 1,\ 0<\theta_i\le 1, i\in\{1,2\}$, *a > c > 0 and b > 0, the Cournot–Nash equilibrium is locally asymptotically stable.*

Proof. The trace and determinant of the Jacobian matrix $JT_A\left(E^c\right)$ are:

$$Tr=2-\left(\beta_1+\beta_2\right)\ge 0$$

$$Det=Tr-1+\beta_1\beta_2\frac{3-\theta_1-\theta_2}{\left(2-\theta_1\right)\left(2-\theta_2\right)}$$

Substituting these expressions into (22), we deduce that Schur's conditions are satisfied:

$$(i)\ 1-Tr+Det=\beta_1\beta_2\frac{3-\theta_1-\theta_2}{\left(2-\theta_1\right)\left(2-\theta_2\right)}>0$$

$$(ii)\ 1+Tr+Det=2Tr+\beta_1\beta_2\frac{3-\theta_1-\theta_2}{\left(2-\theta_1\right)\left(2-\theta_2\right)}>0$$

$$(iii)\ 1-Det=\beta_1+\beta_2-\beta_1\beta_2\frac{3-\theta_1-\theta_2}{\left(2-\theta_1\right)\left(2-\theta_2\right)}=\beta_1+\beta_2\left[1-\beta_1\frac{3-\theta_1-\theta_2}{\left(2-\theta_1\right)\left(2-\theta_2\right)}\right]>0,$$

given that: $1-\beta_1\dfrac{3-\theta_1-\theta_2}{\left(2-\theta_1\right)\left(2-\theta_2\right)}\ge 1-\dfrac{3-\theta_1-\theta_2}{\left(2-\theta_1\right)\left(2-\theta_2\right)}=\dfrac{\left(1-\theta_1\right)\left(1-\theta_2\right)}{\left(2-\theta_1\right)\left(2-\theta_2\right)}>0$ □

This result is consistent with findings in the literature on dynamic oligopoly models, ensuring that deviations from equilibrium in players' decisions will be corrected in the long run. In fact, a more detailed analysis reveals that $\left|\frac{dR_1\left(q_2^*\right)}{dq_1}\frac{dR_2\left(q_1^*\right)}{dq_2}\right|<1$ is a necessary and sufficient condition for the local asymptotic stability of the Nash equilibrium (see [1]), which holds under our assumptions of constant marginal costs and linear demand function. Furthermore, it can be deduced that this result is robust even when considering the incentive contract defined in (2).

5.2. Homogeneous expectations: The gradient rule scheme

We now assume firms use a gradient rule for adjusting production levels. In this scenario, managers lack complete market knowledge and rely on local information to improve their utility. Thus, they increase (decrease) their output for the period *t*+1 based on whether their marginal utility at time *t* is positive (negative), as described in [5]. Interestingly, the gradient rule adjustment does not need the computation of best responses; only local information about utility functions is required.

Formally:

$$q_{i,t+1}-q_{i,t}=\alpha_i\left(q_{i,t}\right)\frac{\partial U_i\left(q_{i,t},q_{j,t}\right)}{\partial q_{i,t}},i,j\in\{1,2\},i\neq j$$

where $\alpha_i\left(q_{i,t}\right)$ is a positive function that determines the extent to which firm *i* adjusts its quantity based on a given marginal utility signal. A linear function is usually assumed, $\alpha_i\left(q_{i,t}\right)=\alpha_i q_{i,t}$, where α_i is a positive parameter representing in this literature the speed of adjustment for player *i*. $U_i\left(q_{i,t},q_{j,t}\right)$ denotes the objective function for each manager as given in (3).

This yields the following dynamic system:

$$T_G:\begin{cases} q_{1,t+1}=q_{1,t}+\alpha_1 q_{1,t}\dfrac{\partial U_{1,t}^{RS}}{\partial q_{1,t}} \\ q_{2,t+1}=q_{2,t}+\alpha_2 q_{2,t}\dfrac{\partial U_{2,t}^{RS}}{\partial q_{2,t}} \end{cases} \tag{23}$$

with $\frac{\partial U_{1,t}^{RS}}{\partial q_{1,t}}$ and $\frac{\partial U_{2,t}^{RS}}{\partial q_{2,t}}$ given in (6).

Substituting $\frac{\partial U_{1,t}^{RS}}{\partial q_{1,t}}$ and $\frac{\partial U_{2,t}^{RS}}{\partial q_{2,t}}$ in (23), we obtain the two-dimensional system that describes the game's dynamics as the following nonlinear map:

$$T_G:\begin{cases} q_{1,t+1}=q_{1,t}+\alpha_1 q_{1,t}\left[a-c-b(1-\theta_1)q_{2,t}-b(2-\theta_1)q_{1,t}\right] \\ q_{2,t+1}=q_{2,t}+\alpha_2 q_{2,t}\left[a-c-b(1-\theta_2)q_{1,t}-b(2-\theta_2)q_{2,t}\right] \end{cases} \tag{24}$$

This system has a unique interior steady state, the Cournot–Nash equilibrium E^C, given in (8), and three boundary steady states: $E_0 = (0,0)$, $E_1 = \left(\dfrac{a-c}{b(2-\theta_1)}, 0 \right)$ and $E_2 = \left(0, \dfrac{a-c}{b(2-\theta_2)} \right)$. E_1 and E_2 represent monopoly situations. In Appendix A, we deduce that the boundary equilibria are unstable. Now, we focus solely on analyzing the local stability of the Cournot-Nash equilibrium.

The Jacobian matrix of T_G is as follows:

$$JT_G(q_1, q_2) = \begin{pmatrix} 1 + \alpha_1 \dfrac{\partial U_1^{RS}}{\partial q_1} + \alpha_1 q_1 \dfrac{\partial^2 U_1^{RS}}{\partial q_1^2} & \alpha_1 q_1 \dfrac{\partial^2 U_1^{RS}}{\partial q_1 \partial q_2} \\ \alpha_2 q_2 \dfrac{\partial^2 U_2^{RS}}{\partial q_2 \partial q_1} & 1 + \alpha_2 \dfrac{\partial U_2^{RS}}{\partial q_2} + \alpha_2 q_2 \dfrac{\partial^2 U_2^{RS}}{\partial q_2^2} \end{pmatrix}$$

This matrix evaluated at the Nash equilibrium E^C leads to:

$$JT_G\left(E^C\right) = \begin{pmatrix} 1 - \alpha_1 b(2-\theta_1) q_1^* & -\alpha_1 b(1-\theta_1) q_1^* \\ -\alpha_2 b(1-\theta_2) q_2^* & 1 - \alpha_2 b(2-\theta_2) q_2^* \end{pmatrix} \tag{25}$$

For the sake of simplicity, in what follows, it is assumed that $\alpha_1 = \alpha_2 = \alpha > 0$.[4]

The trace and the determinant of matrix given in (25) are, respectively:

$$\left. \begin{aligned} Tr &= 2 - \alpha b \left[(2-\theta_1) q_1^* + (2-\theta_2) q_2^* \right] \\ Det &= Tr - 1 + \alpha^2 b^2 (3 - \theta_1 - \theta_2) q_1^* q_2^* \end{aligned} \right\} \tag{26}$$

Proposition 3. *Assuming that both firms follow a gradient rule expectations scheme, the Cournot–Nash equilibrium is locally asymptotically stable for all* $0 < \theta_i \le 1, i \in \{1,2\}$, *a > c > 0 and b > 0, provided that* $\alpha < \alpha_G(a,c)$, *being* $\alpha_G(a,c) = \dfrac{2}{a-c}$.

Proof. Introducing (26) into the stability conditions given in (22), we deduce that condition (*i*) is always satisfied:

$$(i)\ 1 - Tr + Det = \alpha^2 b^2 (3 - \theta_1 - \theta_2) q_1^* q_2^* > 0$$

Condition (*iii*) is satisfied, provided that:

$$1 - Det = \alpha \left[b \left[(2-\theta_1) q_1^* + (2-\theta_2) q_2^* \right] - \alpha b^2 (3 - \theta_1 - \theta_2) q_1^* q_2^* \right] > 0 \Leftrightarrow$$

$$\alpha < \hat{\alpha} = \frac{(2-\theta_1) q_1^* + (2-\theta_2) q_2^*}{b\,(3 - \theta_1 - \theta_2) q_1^* q_2^*}$$

[4] This assumption is usually made in the literature in order to simplify the dynamic analysis and to obtain formal results. See, among others, [2, 11].

Condition (*ii*) is satisfied when:

$$1+Tr+Det=4-2b\left[(2-\theta_1)q_1^*+(2-\theta_2)q_2^*\right]\alpha+b^2(3-\theta_1-\theta_2)q_1^*q_2^*\alpha^2>0$$

To determine the values of α that satisfy condition (*ii*), we define the following parabola:

$y=A\alpha^2+B\alpha+C$, being $A=b^2(3-\theta_1-\theta_2)q_1^*q_2^*$, $B=-2b\left[(2-\theta_1)q_1^*+(2-\theta_2)q_2^*\right]$, $C=4$

The coordinates of the vertex of the parabola are:

$$\alpha_v=\frac{(2-\theta_1)q_1^*+(2-\theta_2)q_2^*}{b\,(3-\theta_1-\theta_2)q_1^*q_2^*}=\hat{\alpha}$$

$$y_v=y(\alpha_v)=4-\frac{\left[(2-\theta_1)q_1^*+(2-\theta_2)q_2^*\right]^2}{(3-\theta_1-\theta_2)q_1^*q_2^*}<0$$

Since $A>0$, $y_v<0$ and $C>0$, the parabola intersects the abscissa axis at two positive values, denoted by α_1 and α_2. Consequently, the values of the speed of adjustment that satisfy condition (*ii*) belong to the set $(0,\alpha_1)\cup(\alpha_2,+\infty)$:

$$\alpha_1=\frac{(2-\theta_1)q_1^*+(2-\theta_2)q_2^*-\sqrt{\left[(2-\theta_1)q_1^*\right]^2+\left[(2-\theta_2)q_2^*\right]^2-2(2-\theta_1\theta_2)q_1^*q_2^*}}{b(3-\theta_1-\theta_2)q_1^*q_2^*}$$

$$\alpha_2=\frac{(2-\theta_1)q_1^*+(2-\theta_2)q_2^*+\sqrt{\left[(2-\theta_1)q_1^*\right]^2+\left[(2-\theta_2)q_2^*\right]^2-2(2-\theta_1\theta_2)q_1^*q_2^*}}{b(3-\theta_1-\theta_2)q_1^*q_2^*}$$

From the comparison of α_1,α_2 and α_v, we deduce that $0<\alpha_1<\alpha_v<\alpha_2$. In consequence, the three stability conditions given in (22) are fulfilled and the Nash equilibrium is therefore asymptotically stable for all $0<\alpha<\alpha_1$. Substituting the equilibrium quantities given in (8), the stability threshold α_1 is $\alpha_G(a,c)=\dfrac{2}{a-c}$ □

Based on the above proposition, we deduce that the Cournot–Nash equilibrium undergoes a flip bifurcation, losing its dynamic stability when $\alpha=\alpha_G(a,c)$. Slower firm adjustments lead to improved market stability; more precisely, if the speed of adjustment is below the threshold $\alpha_G(a,c)=\dfrac{2}{a-c}$, any disturbances that move the market away from the Nash equilibrium disappear in the long run.

To illustrate the formal results obtained and the long-term behavior of the solution trajectories once the threshold is exceeded, we include numerical simulations of the dynamics generated by the map (24). For this purpose, we assume the CSR parameters for both firms take the same value, $\theta_1=\theta_2=\dfrac{5-\sqrt{17}}{4}$, in agreement with the Nash equilibrium obtained in the second stage of the sequential game.

The remaining parameters in all simulations are set as: $a = 2,\ c = 1,\ b = 1$, with initial conditions $q_{1,0} = 0.6,\ q_{2,0} = 0.2$. As shown in Figure 1, the Nash equilibrium ceases to be an attractor when the adjustment speed of the firms exceeds the stability threshold (in this case $\alpha_G(2,1) = 2$), and more complex attractors (from a steady state to a strange attractor) appear as the adjustment speed increases.

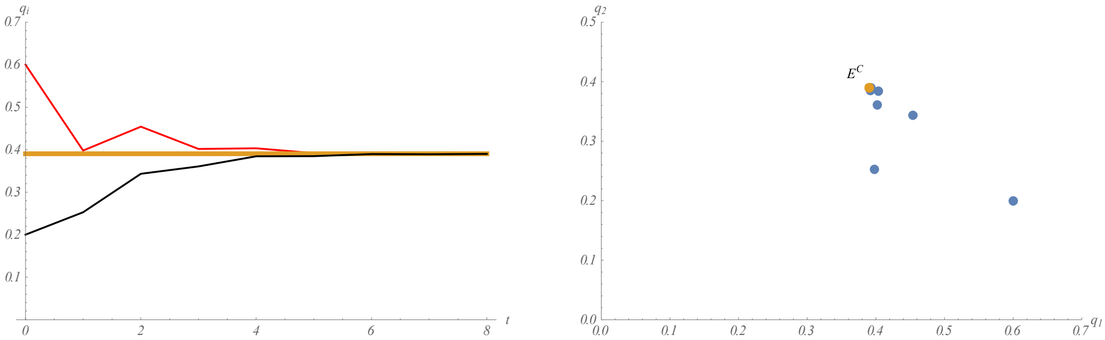

Figure 1a. Attractor (orange): the Nash Equilibrium E^C for $\alpha = 1.5$.

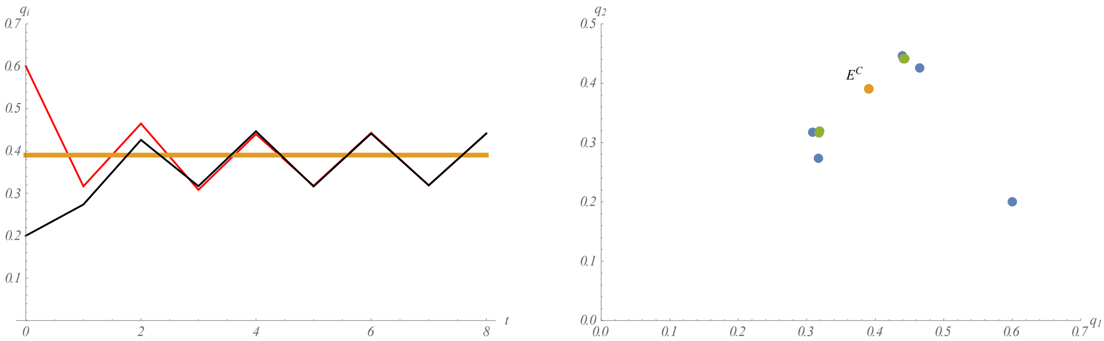

Figure 1b. Nash Equilibrium (orange); Attractor (green): a 2-cycle for $\alpha = 2.1$.

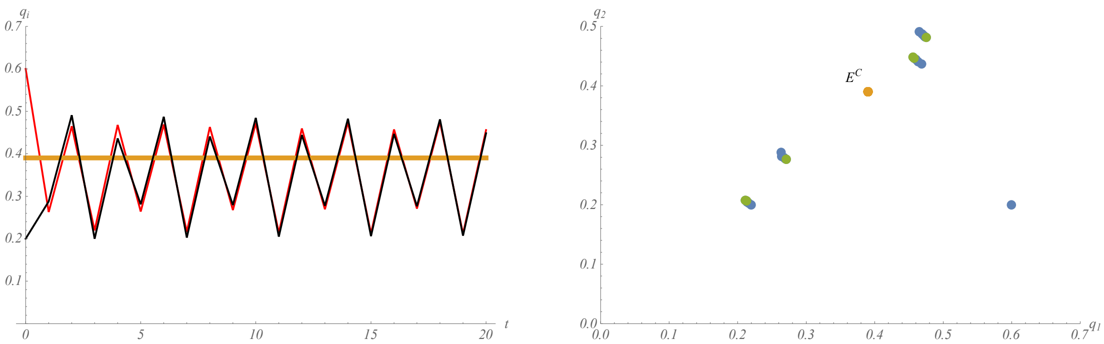

Figure 1c. Nash Equilibrium (orange); Attractor (green): a 2^2-cycle for $\alpha = 2.5$.

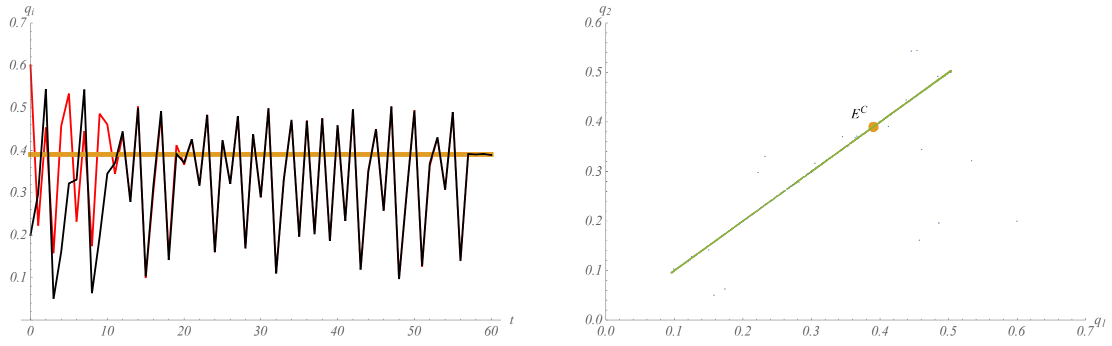

Figure 1d. Nash Equilibrium (orange); Attractor (green): a strange attractor for $\alpha = 2.8$.

The figures above illustrate the dynamic behavior changes in (24), which are corroborated by the bifurcation diagram of q_i , $i \in \{1,2\}$, and the maximum Lyapunov exponent with respect to the adjustment speed (see Figure 2).

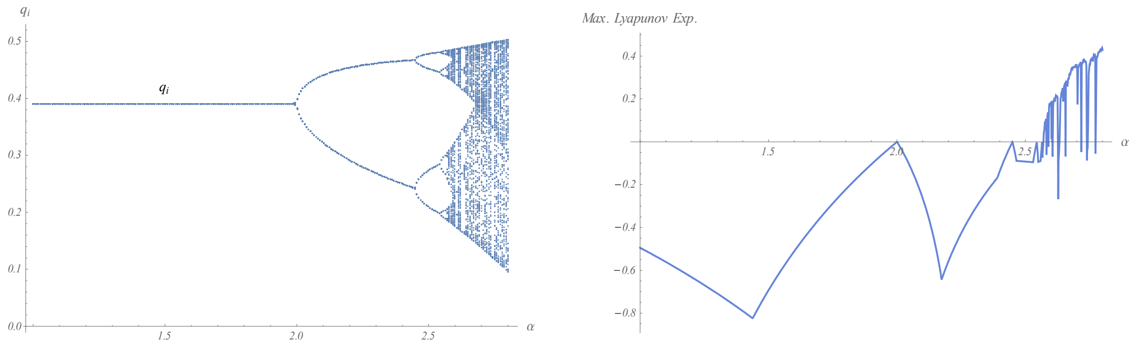

Figure 2. Bifurcation diagrams q_i (i = 1, 2) and the maximum Lyapunov exponent with respect to parameter α .

Proposition 3 and numerical simulations reveal that the Cournot–Nash equilibrium ceases to be an attractor through a period-doubling bifurcation when the adjustment speed reaches the value $\alpha_G(a,c) = \dfrac{2}{a-c}$. A cascade of flip bifurcations appears as the speed of adjustment increases, leading to increasingly complex attractors (from 2^n-cycles in Figure 1b and Figure 1c, to a strange attractor in Figure 1d). The long-term behavior of quantities around the Nash equilibrium becomes more complex the faster the adjustment process is, giving the market greater instability.

Under the gradient rule expectation scheme, the adjustment speed plays a pivotal role in the local asymptotic stability of the Cournot–Nash equilibrium. The destabilizing effect of sufficiently high values of parameter α has been reported by other authors (see, among others, [1, 2, 11, 12]).

Interestingly, if $\theta_1 = \theta_2 = \theta$ (identical firms), the map T_G exhibits a symmetry property, implying the invariance of the diagonal $q_1 = q_2$, enabling synchronized dynamics. Figures 1 and 2 support this observation.[5]

From the expression $\alpha_G(a,c) = \dfrac{2}{a-c}$, it is clear that the local stability of the equilibrium is higher the smaller the demand size (given by the parameter *a*) and the higher the marginal cost (value of *c*). This result differs from that obtained in [2], which highlights the sensitivity of these results to the specification of the demand function.

Finally, the adjustment speed threshold, $\alpha_G(a,c) = \dfrac{2}{a-c}$, in line with the conclusions obtained in [2] under the assumption of isoelastic demand, does not depend on the level of social responsibility. Moreover, its expression coincides with that obtained in [12] considering a Cournot duopoly with relative profit delegation.

It should be noted that this last result is not robust either to the specific contract of the manager or to the behavior followed by the managers when adjusting the quantities. Thus, it can be deduced that when both managers maximize a *PR*-type contract, the parameter γ_i influences the adjustment speed threshold through the marginal cost. Intuitively, in a *RS*-type contract, the change in utility with respect to a change in the level of social responsibility is the same for both managers, i.e., consumer surplus. In contrast, in a *RP*-type contract, the marginal effect on utility of a change in parameter γ_i is the individual production cost of each manager.

Moreover, in line with the conclusions offered in [2], under heterogeneous expectations schemes, the weight given to consumer surplus can affect the stability of the Cournot-Nash equilibrium. This aspect will be further developed below.

5.3. Heterogeneous expectations

In this subsection, we assume that managers are heterogeneous with respect to the expectations rules. The manager of firm 1 uses an adaptive expectations rule for adjusting production levels and the manager of firm 2 follows the gradient rule. Then, the dynamic system is given by the nonlinear map:

$$T_H : \begin{cases} q_{1,t+1} = (1-\beta)\, q_{1,t} + \beta R_1(q_{2,t}) \\ q_{2,t+1} = q_{2,t} + \alpha\, q_{2,t} \dfrac{\partial U_{2,t}^{RS}}{\partial q_{2,t}} \end{cases} \tag{27}$$

[5] Under the assumption of symmetry, the dynamical behavior of the restriction of T_G on the diagonal, characterized by synchronized dynamics of identical players, can be obtained from the well-known behavior of the standard logistic map. Furthermore, we can establish that the direction transverse to the diagonal can be attracting depending on the parametric values (see, [6]).

where $R_1(q_{2,t})$ and $\dfrac{\partial U_{2,t}^{RS}}{\partial q_{2,t}}$ are given in (7) and (6), respectively, and $0<\beta\le 1,\ \alpha>0$.

Substituting $R_1(q_{2,t})$ and $\dfrac{\partial U_{2,t}^{RS}}{\partial q_{2,t}}$ in (27), we obtain:

$$T_H:\begin{cases} q_{1,t+1}=(1-\beta)\ q_{1,t}+\beta\dfrac{a-c-b(1-\theta_1)q_{2,t}}{(2-\theta_1)b} \\ q_{2,t+1}=q_{2,t}+\alpha\, q_{2,t}\left[a-c-b(1-\theta_2)q_{1,t}-b(2-\theta_2)q_{2,t}\right] \end{cases}$$

This system has a unique interior steady state, the Cournot–Nash equilibrium E^C given in (8), and a boundary steady state $E_1=\left(\dfrac{a-c}{b(2-\theta_1)},0\right)$ representing a monopoly situation of firm 1. The instability of boundary steady state is deduced in Appendix B. Next, we analize the local stability of E^C.

The Jacobian matrix of T_H is as follows:

$$JT_H(q_1,q_2)=\begin{pmatrix} 1-\beta & \beta\dfrac{dR_1(q_2)}{dq_1} \\ \alpha\, q_2\dfrac{\partial^2 U_2^{RS}}{\partial q_2\partial q_1} & 1+\alpha\dfrac{\partial U_2^{RS}}{\partial q_2}+\alpha\, q_2\dfrac{\partial^2 U_2^{RS}}{\partial q_2^2} \end{pmatrix}$$

This matrix evaluated at the Nash equilibrium E^C leads to:

$$JT_G\left(E^C\right)=\begin{pmatrix} 1-\beta & -\beta\dfrac{1-\theta_1}{2-\theta_1} \\ -\alpha b\,(1-\theta_2)q_2^* & 1-\alpha b\,(2-\theta_2)q_2^* \end{pmatrix}$$

The trace and the determinant of this matrix are, respectively:

$$\left.\begin{aligned} &Tr=2-\beta-\alpha b\,(2-\theta_2)q_2^* \\ &Det=Tr-1+\alpha\beta b\,\frac{3-\theta_1-\theta_2}{2-\theta_1}q_2^* \end{aligned}\right\} \qquad (28)$$

Proposition 4. *Assuming manager of firm* 1 *follows an adpative expectations scheme and manager of firm* 2 *a gradient rule expectations scheme, the Cournot–Nash equilibrium is locally asymptotically stable for all* $0<\theta_i\le 1, i\in\{1,2\}$, $a>c>0$, $b>0$ *and* $0<\beta\le 1$, *provided that*

$$\alpha<\alpha_H(a,c,\beta,\theta_1,\theta_2)=\frac{2(2-\beta)(2-\theta_1)(3-\theta_1-\theta_2)}{(a-c)(1-\theta_1+\theta_2)\left[(3-\theta_1-\theta_2)(2-\beta)+2(1-\theta_1)(1-\theta_2)\right]}$$

Proof. Introducing (28) into the stability conditions given in (22), we deduce that condition (*i*) and (iii) are always satisfied:

$$(i)\ 1-Tr+Det=\alpha\beta b\,\frac{3-\theta_1-\theta_2}{2-\theta_1}q_2^*>0$$

$$(iii)\ 1-Det=\beta+\alpha b\,q_2^*\frac{(3-\theta_1-\theta_2)(1-\beta)+(1-\theta_1)(1-\theta_2)}{2-\theta_1}>0$$

Condition (ii) is satisfied when:

$$1+Tr+Det=2(2-\beta)-b\,\frac{(3-\theta_1-\theta_2)(2-\beta)+2(1-\theta_1)(1-\theta_2)}{2-\theta_1}q_2^*\alpha>0$$

$$\Leftrightarrow \alpha<\frac{2(2-\beta)(2-\theta_1)}{b\,q_2^*\left[(3-\theta_1-\theta_2)(2-\beta)+2(1-\theta_1)(1-\theta_2)\right]}$$

Substituting the equilibrium quantity q_2^* given in (8), the stability threshold is:

$$\alpha_H(a,c,\beta,\theta_1,\theta_2)=\frac{2(2-\beta)(2-\theta_1)(3-\theta_1-\theta_2)}{(a-c)(1-\theta_1+\theta_2)\left[(3-\theta_1-\theta_2)(2-\beta)+2(1-\theta_1)(1-\theta_2)\right]}\qquad\square$$

The results regarding long-term market dynamics are similar to the case where both managers adjust their quantities according to the gradient rule. This is illustrated in the figures 3 and 4, which correspond to numerical simulations of the dynamic system (27) for the same parameter values and initial conditions as in the previous subsection, with the value of the stability threshold being $\alpha_H=2.7036...$

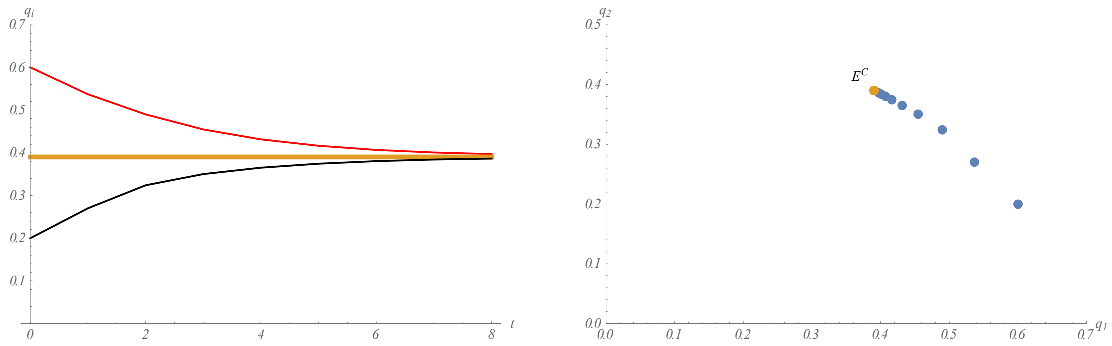

Figure 3a. Attractor (yellow): the Nash Equilibrium E^C for $\alpha=2$.

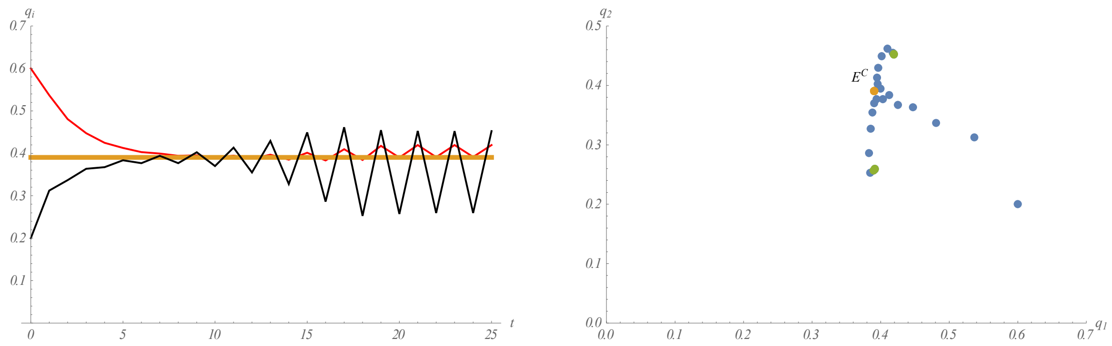

Figure 3b. Nash Equilibrium (yellow); Attractor (green): a 2-cycle for $\alpha=3.2$.

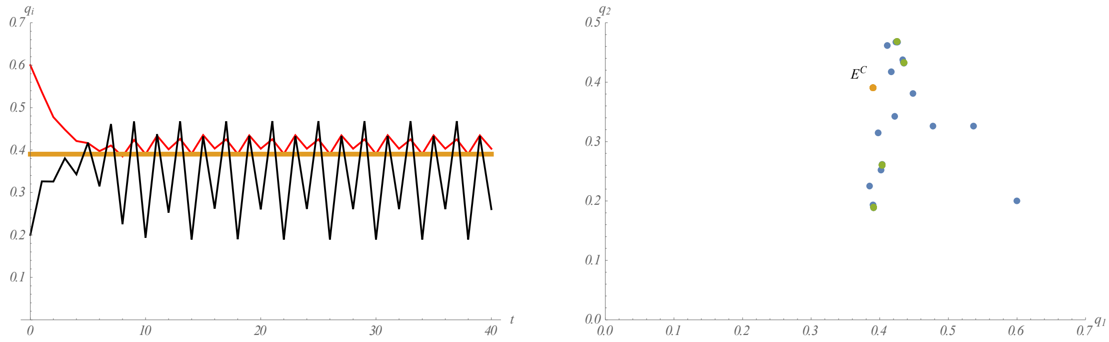

Figure 3c. Nash Equilibrium (yellow); Attractor (green): a 2^2-cycle for $\alpha = 3.6$.

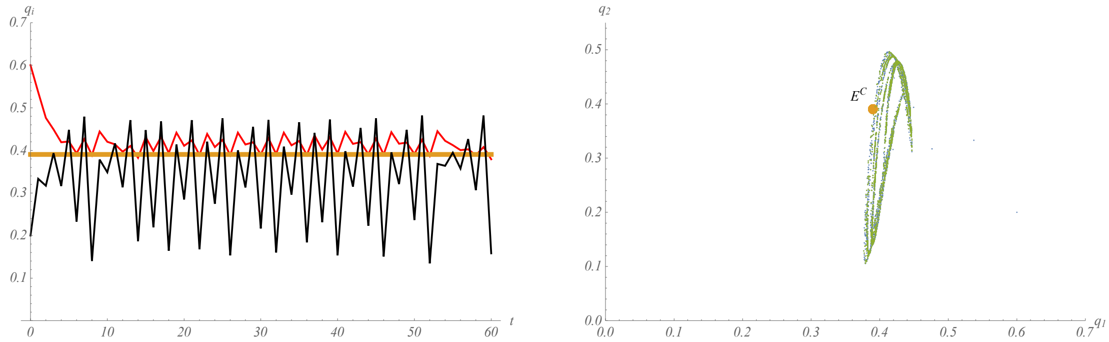

Figure 3d. Nash Equilibrium (yellow); Attractor (green): a strange attractor for $\alpha = 3.8$.

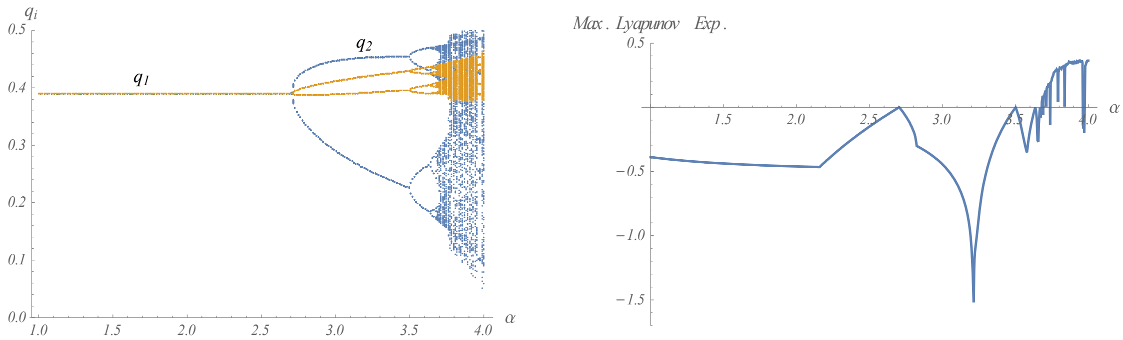

Figure 4. Bifurcation diagrams q_i (i = 1, 2) and the maximum Lyapunov exponent with respect to parameter α .

In this case the synchronization phenomenon does not appear, since the dynamic system (27) is not symmetrical as each firm follows a different expectation scheme.

A relevant result under the assumption of heterogeneous expectations is that the stability threshold depends on the CSR level set by each firm. To analyze this dependence, we will assume that both firms set the same level of CSR, $\theta_1 = \theta_2 = \theta$, in line with the result obtained in the static analysis. Thus, the stability threshold is as follows:

$$\alpha_H(a,c,\beta,\theta) = \frac{2(2-\beta)(2-\theta)(3-2\theta)}{(a-c)\left[(3-2\theta)(2-\beta)+2(1-\theta)^2\right]}$$

Deriving whit respect to θ:

$$\frac{\partial \alpha_H}{\partial \theta}(a,c,\beta,\theta)=\frac{2(2-\beta)\left[\beta(3-2\theta)^2-2(2-\theta)^2\right]}{(a-c)\left[(3-2\theta)(2-\beta)+2(1-\theta)^2\right]^2}<0 \Leftrightarrow \beta<2\left(\frac{2-\theta}{3-2\theta}\right)^2$$

Given that:

$$2\left(\frac{2-\theta}{3-2\theta}\right)^2 \leq 1 \Leftrightarrow \theta \leq 1-\frac{\sqrt{2}}{2}=0.29283...$$

it is deduced $\forall 0<c<a$:

$$0<\theta<1-\frac{\sqrt{2}}{2}=0.29283... \Rightarrow \begin{cases} \frac{\partial \alpha_H}{\partial \theta}(a,c,\beta,\theta)<0, & \text{if } 0<\beta<2\left(\frac{2-\theta}{3-2\theta}\right)^2 \\ \frac{\partial \alpha_H}{\partial \theta}(a,c,\beta,\theta)>0, & \text{if } 2\left(\frac{2-\theta}{3-2\theta}\right)^2<\beta\leq 1 \end{cases}$$

$$\theta>1-\frac{\sqrt{2}}{2}=0.29283... \Rightarrow \frac{\partial \alpha_H}{\partial \theta}(a,c,\beta,\theta)<0, \quad \forall 0<\beta\leq 1$$

In short, if the level of social responsibility set by both owners is sufficiently high, $\left(\theta>1-\frac{\sqrt{2}}{2}\right)$, the stability threshold decreases the greater the weight assigned to consumer surplus. Consequently, the CSR is a market destabilizing factor.

However, if the weight assigned to consumer surplus is below a certain level, $\left(\theta<1-\frac{\sqrt{2}}{2}\right)$, and the manager following an adaptive expectations scheme is more inclined to adjust its quantity according to its best response function, $\left(\beta>2\left(\frac{2-\theta}{3-2\theta}\right)^2\right)$, the CSR can be a stabilizing factor in the market. As the manager adopting an adaptive expectations scheme is more reluctant to change its quantity from one period to the subsequent one, $\left(\beta<2\left(\frac{2-\theta}{3-2\theta}\right)^2\right)$, the CSR regains its destabilizing role.

In addition, the behavior of the manager who adjusts his quantities according to an adaptive expectations scheme also influences the stability of the equilibrium. Specifically, it follows:

$$\frac{\partial \alpha_H}{\partial \beta}(a,c,\beta,\theta_1,\theta_2)=-\frac{4(2-\theta_1)(3-\theta_1+\theta_2)(1-\theta_1)(1-\theta_2)}{(a-c)(1-\theta_1+\theta_2)\left[(3-\theta_1+\theta_2)(2-\beta)+2(1-\theta_1)(1-\theta_2)\right]^2}<0$$

Therefore, a higher weight on the best-response function when adjusting quantity would destabilize the market.

6. Conclusions

Our analysis explores CSR as a strategic tool in a duopoly with strategic delegation from a static and dynamic approach. To this end, we develop a sequential game where profit-maximizing owners have the option of hiring the services of a manager, delegate production decisions and choose the optimal incentive contract (linear combination of profits and revenues or a linear combination of profits and consumer surplus). If delegation is chosen, managers compete in quantities in the product market, according to the compensation contract set by the owners. Our analysis of the game, specifically by finding the subgame perfect Nash equilibrium, reveals some key insights. From the static approach, it is endogenously deduced that owners always delegate production decisions to their managers, with the incentive contract defined by a linear combination of profits and consumer surplus. Additionally, both firms end up assigning the same weight to CSR, which means that the solution obtained is symmetrical.

With this compensation scheme, firms earn higher profits compared to a plan based solely on profits and revenue. This finding highlights the strategic value of CSR and its potential for increasing firms' market power. The higher the weight assigned to consumer surplus in managers' compensation contracts, the lower the level of strategic quantity substitutability. As a result, the equilibrium price increases and the total quantity offered falls.

The static analysis corroborates existing research and contributes further by endogenously obtaining the owners' decision to delegate and, if they do so, the optimal incentive contract—determined as the outcome of a subgame perfect Nash equilibrium.

The dynamization of the quantity competition stage was performed in a discrete context. We assumed managers are compensated based on a combination of profit and consumer surplus, and they adjust quantities according to an expectations scheme based on either their best response functions, or the gradient rule, based on marginal utility.

Our dynamic study revealed that under the adaptive expectations approach, the Cournot–Nash equilibrium is locally asymptotically stable over the entire parameter space. However, it is sufficient for a manager to make its decisions under the gradient rule for instability to occur. In this case, there is a threshold of the adjustment speed that induces instability of the Cournot–Nash equilibrium. Exceeding this limit leads to a cascade of flip bifurcations with increasingly complex attractors. This result is in line with the literature on nonlinear oligopoly dynamics.

We also found that social responsibility does not affect the stability threshold if both managers follow the gradient rule as an expectation scheme. Moreover, in this case, the map defining the dynamics becomes symmetrical and a phenomenon called synchronization may appear. Despite starting from different initial conditions, in the long run, firms might evolve over time to behave identically, reaching either the Cournot–Nash equilibrium or a more complex attractor.

However, if there is heterogeneity in the expectations followed by the managers, the stability threshold does depend on the level of social responsibility. Specifically, if the weight assigned to the consumer surplus is sufficiently low and the manager following an adaptive expectation scheme is reluctant to modify its quantity, a higher CSR provides greater stability to the market.

Finally, both in a context of homogeneous and heterogeneous expectations, a higher marginal cost of the firms or a smaller size of demand leads to a more stable equilibrium.

It is important to note that the results obtained rely on the assumptions considered, for instance, linearity in demand, as well as constant and symmetric costs. Therefore, future research could explore scenarios with product differentiation, non-linear cost/demand functions, asymmetric costs, alternative incentive contracts, such as a linear combination of profit and market share or relative performance, or different expectation schemes like the local monopolistic approximation (LMA).

Appendix A

Considering the dynamic system given in (24), the Jacobian matrix of T_G is as follows:

$$JT_G(q_1,q_2)=\begin{pmatrix} 1+\alpha_1\dfrac{\partial U_1^{RS}}{\partial q_1}+\alpha_1 q_1\dfrac{\partial^2 U_1^{RS}}{\partial q_1^2} & \alpha_1 q_1\dfrac{\partial^2 U_1^{RS}}{\partial q_1 \partial q_2} \\ \alpha_2 q_2\dfrac{\partial^2 U_2^{RS}}{\partial q_2 \partial q_1} & 1+\alpha_2\dfrac{\partial U_2^{RS}}{\partial q_2}+\alpha_2 q_2\dfrac{\partial^2 U_2^{RS}}{\partial q_2^2} \end{pmatrix}$$

This matrix evaluated at the steady state E_0 leads to:

$$JT_G(E_0)=\begin{pmatrix} 1+\alpha_1(a-c) & 0 \\ 0 & 1+\alpha_2(a-c) \end{pmatrix}$$

whose eigenvalues are:

$$\lambda_1=1+\alpha_1(a-c)>1$$
$$\lambda_2=1+\alpha_2(a-c)>1$$

Then, the boundary steady state is an unstable equilibrium of (24) (specifically, it is a source).

The matrix $JT_G(q_1,q_2)$ evaluated at the steady state E_1 leads to:

$$JT_G(E_1)=\begin{pmatrix} 1-\alpha_1(a-c) & -\alpha_1\dfrac{(a-c)(1-\theta_1)}{2-\theta_1} \\ 0 & 1+\alpha_2\dfrac{(a-c)(1-\theta_1+\theta_2)}{2-\theta_1} \end{pmatrix}$$

whose eigenvalues are:

$$\lambda_1=1-\alpha_1(a-c)\begin{cases} <-1 \Leftrightarrow \alpha_1<\dfrac{2}{a-c} \\ \in(-1,1)\Leftrightarrow \alpha_1>\dfrac{2}{a-c} \end{cases}$$

$$\lambda_2=1+\alpha_2\frac{(a-c)(1-\theta_1+\theta_2)}{2-\theta_1}>1$$

and we deduce E_1 is an unstable boundary equilibrium of (24). In addition, it is also verified that:

$$\alpha_1 < \frac{2}{a-c} \Rightarrow E_1 \text{ is a saddle point of (24)}$$
$$\alpha_1 > \frac{2}{a-c} \Rightarrow E_1 \text{ is a source of (24)}$$

Note $\alpha_1 = \frac{2}{a-c}$ is the stability threshold of the steady state E_1 in the standard logistic map corresponding to the projection of the map T_G on the coordinate axes for $q_2 = 0$.

The matrix $JT_G(q_1,q_2)$ evaluated at the steady state E_2 leads to:

$$JT_G(E_2) = \begin{pmatrix} 1+\alpha_1 \frac{(a-c)(1-\theta_2+\theta_1)}{2-\theta_2} & 0 \\ -\alpha_2 \frac{(a-c)(1-\theta_2)}{2-\theta_2} & 1-\alpha_2(a-c) \end{pmatrix}$$

whose eigenvalues are:

$$\lambda_1 = 1+\alpha_1 \frac{(a-c)(1-\theta_2+\theta_1)}{2-\theta_2} > 1$$
$$\lambda_2 = 1-\alpha_2(a-c) \begin{cases} < -1 \Leftrightarrow \alpha_2 < \frac{2}{a-c} \\ \in (-1,1) \Leftrightarrow \alpha_2 > \frac{2}{a-c} \end{cases}$$

and we deduce that E_2 is an unstable boundary equilibrium of (24). Moreover, it is further verified that:

$$\alpha_2 < \frac{2}{a-c} \Rightarrow E_2 \text{ is a saddle point of (24)}$$
$$\alpha_2 > \frac{2}{a-c} \Rightarrow E_2 \text{ is a source of (24)}$$

Note $\alpha_2 = \frac{2}{a-c}$ is the stability threshold of the steady state E_2 in the standard logistic map corresponding to the projection of the map T_G on the coordinate axes for $q_1 = 0$.

Appendix B

Considering the dynamic system given in (27), the Jacobian matrix of T_H is as follows:

$$JT_H(q_1,q_2) = \begin{pmatrix} 1-\beta & \beta \frac{dR_1(q_2)}{dq_1} \\ \alpha q_2 \frac{\partial^2 U_2^{RS}}{\partial q_2 \partial q_1} & 1+\alpha \frac{\partial U_2^{RS}}{\partial q_2} + \alpha q_2 \frac{\partial^2 U_2^{RS}}{\partial q_2^2} \end{pmatrix}$$

This matrix evaluated at the steady state E_1 leads to:

$$JT_H\left(E_1\right) = \begin{pmatrix} 1-\beta & -\beta\dfrac{1-\theta_1}{2-\theta_1} \\ 0 & 1+\alpha(a-c)\dfrac{1-\theta_1+\theta_2}{2-\theta_1} \end{pmatrix}$$

whose eigenvalues are:

$$\lambda_1 = 1-\beta \in \left[0,1\right)$$
$$\lambda_2 = 1+\alpha(a-c)\frac{1-\theta_1+\theta_2}{2-\theta_1} > 1$$

Then, the boundary steady state is an unstable equilibrium of (27) (specifically, it is a saddle point).

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