Basic Set Theory and some cardinals beyond ZFC

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"Anyone not wanting to sink in the wretchedness of the finite is obliged in the most profound sense to struggle with the infinite."

Søren Kierkegaard

"To infinity and beyond!"

Buzz Lightyear

Resumen

La teoría de conjuntos nació cuando Georg Cantor, con la intención de resolver problemas sobre la convergencia de series de Fourier, trató de entender el infinito y las *colecciones* de puntos del plano real. Él mismo formalizó la idea de cardinalidad en base a biyecciones y estableció teoremas clave como "la cardinalidad del continuo es estrictamente mayor que la cardinalidad de los naturales", al igual que propuso problemas que siguen resultando de profundo interés, como la Hipótesis del Continuo.

Con el tiempo, se vio el potencial que tenía esta teoría, y con ella nació el propósito de fundamentar las matemáticas sobre una base sólida y clara, y de formalizar todas las ideas subyacentes a los razonamientos matemáticos: nociones básicas pero no definidas como conjunto, relación, infinito... A medida que se desarrollaba, la teoría de conjuntos se asentó como una rama más de las matemáticas, con su propio interés, con sus propios problemas.[1]

El objetivo principal de este trabajo es mostrar cómo se pueden definir conjuntos infinitos tan grandes que no se puedan construir con la teoría básica de conjuntos, pero que sean consistentes con ella. A eso nos referimos en el título con *más allá de ZFC*.

Además, nos proponemos hacerlo de forma autocontenida: dado que estamos hablando de fundamentos de las matemáticas, el lector no necesita ningún conocimiento matemático previo (esto es algo ideal, y por supuesto, sí se necesita capacidad de razonamiento lógico y familiaridad con la notación matemática).

Para ello, comenzamos presentando la lista de axiomas más aceptada como base de la teoría de conjuntos: el sistema axiomático de Zermelo-Fraenkel con el Axioma de Elección (abreviado como ZFC). Damos la motivación de cada uno de los axiomas, y mostramos cómo definir en base a ellos algunas construcciones importantes sobre conjuntos, así como los conceptos clave sobre relaciones y funciones.

Después, exponemos las nociones de número ordinal y cardinal, que nos sirven para contar conjuntos bien ordenados, y clasificar conjuntos según su tamaño (ya sean finitos o infinitos). Daremos sus propiedades más importantes, y hablaremos también, gracias a ellos, sobre recursividad e inducción transfinita, métodos que generalizan la recursividad e inducción usual sobre los números naturales. Son resultados de gran utilidad, ya que permiten utilizar argumentos inductivos en conjuntos no numerables, así como definir de forma recursiva conjuntos de este tipo.

Daremos también la noción de cofinalidad, que permite entender cómo de grande es un cardinal infinito viendo "cuantos pasos hacen falta para alcanzarlo" (con pasos más pequeños que ese cardinal). Por ejemplo, para alcanzar $\aleph_0$ con pasos finitos, necesitaremos $\aleph_0$ pasos, así que la cofinalidad de $\aleph_0$ es él mismo; pero para alcanzar $\aleph_{\aleph_0}$ bastan $\aleph_0$ pasos: $\aleph_0, \aleph_1, \aleph_2, \ldots, \aleph_{\aleph_0}$, luego su cofinalidad es estrictamente menor que él. Este concepto da una de las claves para definir los cardinales inaccesibles.

Los ordinales nos proporcionan también una forma de estructurar el universo V de todos los conjuntos como una jerarquía en la que cada nivel consiste en los conjuntos que podemos construir, siguiendo ciertas reglas, a partir de los conjuntos anteriores. Esta construcción, junto con todo lo anterior, permite fundamentar todas las matemáticas habituales y, de hecho, va mucho más allá de ellas.

Hasta ahí llega nuestra presentación de las bases de la teoría de conjuntos. Después, estudiamos qué axiomas de ZFC se satisfacen en cada nivel de la jerarquía que hemos definido. Por ejemplo, en V_ω,

[1] El lector interesado podrá encontrar más información sobre historia de la teoría de conjuntos y sus orígenes en [6] y [7].

que es el conjunto de todos los conjuntos finitos, se satisfacen los axiomas de unión y de elección (entre otros), pero evidentemente no se satisface el Axioma del Infinito: en V_ω no hay un conjunto infinito.

Como hemos señalado antes, nuestro interés radica en definir cardinales de tal magnitud que las operaciones de la teoría de conjuntos no nos permitan alcanzarlos. Daremos una primera definición de cardinal inaccesible, y mostraremos que no podemos probar su existencia en ZFC, que se debe a que están *más allá* de ZFC. Esto quiere decir que si construimos la jerarquía acumulativa de conjuntos hasta llegar a un cardinal inaccesible κ, entonces en V_κ se satisfacen todos los axiomas de ZFC. ¿Por qué esto implica que no podamos probar su existencia? El argumento es aparentemente sencillo pero muy profundo, y apela al Segundo Teorema de Incompletitud de Gödel: si demostramos en ZFC la existencia de tal cardinal κ, y por lo tanto la existencia de V_κ, estaremos demostrando la consistencia de ZFC desde ZFC, contradiciendo dicho teorema.

Veremos también cómo se comportan con respecto a las operaciones básicas de la teoría de conjuntos, lo que dará una buena intuición de por qué llamarlos inaccesibles.

Una vez formalizadas todas estas ideas, definiremos algunos otros tipos de grandes cardinales (hiperinaccesibles y cardinales de Mahlo), y estudiaremos cómo están relacionados y sus propiedades combinatorias.

Para terminar, mostraremos una propiedad interesante que tienen todos los cardinales inaccesibles, que se puede resumir en que "en un universo de conjuntos en el que se llega a un cardinal inaccesible, hay infinitos (y no acotados) sub-universos que modelan ese mismo universo, es decir, todas las fórmulas tienen el mismo valor de verdad en los dos universos". En particular, no solo se satisfacen en V_κ todas las formulas de ZFC, sino que podemos encontrar en él infinitos modelos de la teoría de conjuntos, y tan grandes como queramos.

Para la primera parte hemos seguido dos libros canónicos de la teoría de conjuntos: *Set Theory*, de Thomas Jech [4] y *Set Theory*, de Kenneth Kunen [5].

Para la exposición de los cardinales inaccesibles, el libro de referencia es *Set Theory: an introduction to large cardinals*, de Frank R. Drake [3], y también pero en menor medida los dos anteriores.

Hemos utilizado otros libros de apoyo a lo largo de todo el trabajo que han resultado igualmente fundamentales: *Elements of mathematical logic*, de Alessandro Andretta [1] y *Set Theory: an open introduction*, de Tim Button [2]. Este último no aporta contenido matemático relevante pero es de gran utilidad para entender la filosofía detrás de la teoría de conjuntos.

Contents

Chapter 1

The basics of ZFC Set Theory

In these first sections, we will introduce the basics of Set Theory following the expositions from [4] and [5], and some ideas from [1] and [2].

1 The axioms

We may start by presenting the list of axioms that conform ZFC and, after, discuss them separately. There is no specific order to present them. However, they are usually listed in increasing order of sophistication, starting from the most basic ones.

The axiomatic system of Zermelo-Fraenkel (with Choice) is the following:

Axiom 1. Extensionality.

$$\forall x \forall y (\forall z (z \in x \leftrightarrow z \in y) \to x = y).$$

Axiom 2. Foundation.

$$\forall x [\exists y (y \in x) \to \exists y (y \in x \wedge \neg \exists z (z \in x \wedge z \in y))].$$

Axiom schema[1] 3. Comprehension. For each formula ϕ with free variables among $x, z, w_1, \ldots, w_n$,

$$\forall z \forall w_1, \ldots, w_n \exists y \forall x (x \in y \leftrightarrow x \in z \wedge \phi).$$

Axiom 4. Pairing.

$$\forall x \forall y \exists z (x \in z \wedge y \in z).$$

Axiom 5. Union.

$$\forall \mathcal{F} \exists A \forall Y \forall x (x \in Y \wedge Y \in \mathcal{F} \to x \in A).$$

Axiom schema 6. Replacement. For each formula ϕ with free variables among $x, y, w_1, \ldots, w_n$,

$$\forall A \forall w_1, \ldots, w_n (\forall x \in A \exists ! y \phi \to \exists B \forall x \in A \exists y \in B \phi).$$

With these axioms we can already define, as we will see later, the relation $\subseteq$, the ordinal successor operation S, the intersection $\cap$, and the empty set $\emptyset$, as well as prove the existence of the latter.

Axiom 7. Infinity.

$$\exists x (\emptyset \in x \wedge \forall y \in x (S(y) \in x)).$$

Axiom 8. Power Set.

$$\forall x \exists y \forall z (z \subseteq x \to z \in y).$$

And, after defining functions and notation on them,

[1]Comprehension is not just a single axiom, but rather an axiom *schema*, meaning that there is a distinct axiom for each formula ϕ.

Axiom 9. Choice (AC).

$$\forall \mathcal{F}[\emptyset \notin \mathcal{F} \to \exists f(f : \mathcal{F} \to \bigcup \mathcal{F} \wedge \forall x \in \mathcal{F}(f(x) \in x))].$$

Before discussing them, we introduce some notation:

- Our objects of study are sets. *If there exists* a set A such that $\forall x[x \in A \leftrightarrow \varphi(x)]$, then A is unique by Extensionality and we denote this set by $\{x : \varphi(x)\}$.
- Given a formula ϕ, it is not true in general that $\{x : \phi(x)\}$ is a set. A **class** is any object of that form, and a **proper class** is a class that is not a set. However, proper classes can be useful to shorten notation: instead of writing "let y such that $\phi(y)$", we can write "let $y \in C$".
- We can characterize the property of being a set as $\mathrm{Set}(x) \iff \exists y(x \in y)$.

Note that the sets of the form $\{x : x \in y \wedge \varphi(x)\}$ always exist thanks to the **Axiom of Comprehension** (Axiom 3). The other axiom that we have just mentioned is **Extensionality** (Axiom 1). It asserts that if two sets have exactly the same members, then they are the same set. That is, sets are characterized by their members. We can already define, for example,

$$y \cap x := \{z \in y : z \in x\}, \qquad y \setminus x := \{z \in y : z \notin x\}, \qquad x \subseteq y \iff \forall z(z \in x \to z \in y).$$

We can also define the empty set, $\emptyset$ (but we have to prove that it is a set), and the universe, V:

Definition 1.1. The empty set $\emptyset$ is the only y such that $\forall x(x \notin y)$.

This notion is well defined:

Proposition 1.1. *$\emptyset$ exists as a set and it is unique.*

Proof. Fix any v and let $y = \{x \in v : x \neq x\}$, which exists by the Axiom of Comprehension, and it is empty, since $\forall x(x \notin y)$. So there exists an empty set. Also, y is unique by Extensionality. □

Definition 1.2. The universe of all sets is $V := \{x : x = x\}$.

But now we have that V is not a set, i.e., it is a proper class. Recall Russell's paradox: if $R = \{x : x \notin x\}$ was a set, then $R \in R \iff R \notin R$, a contradiction, so R does not exist. Now, notice that we could define R as $R = \{x \in V : x \notin x\}$ and if V was a set, R should exist by Comprehension.

So far, the only set whose existence we have proved is $\emptyset$. In fact, with these two axioms, it turns out that we cannot prove (nor disprove) the existence of any other set. This leads us to introduce axioms that guarantee the existence of some obvious sets, the most basic ones being **Pairing** (Axiom 4) and **Union** (Axiom 5).

Pairing states that given two sets, x and y, there is a set that contains both. In particular, the set $\{x,y\}$: let z be a set such that $x \in z \wedge y \in z$ (which exists by the Axiom of Pairing), then the set $\{x,y\} = \{w \in z : w = x \vee w = y\}$ exists by Comprehension, and it is unique by Extensionality. Until now we only had the set $\emptyset$, so we can consider for instance the sets $\{\emptyset\}$ (it is the pair $\{\emptyset, \emptyset\}$, which is the same set as $\{\emptyset\}$ by Extensionality) and $\{\emptyset, \{\emptyset\}\}$ (pairing this last set and $\emptyset$). We can also talk about ordered pairs: recall that the main idea of an ordered pair (x,y) is that it satisfies $(x,y) = (z,w) \leftrightarrow x = z \wedge y = w$. There are many ways to define ordered pairs, but the simplest definition, due to Kuratowski, can be given now that we have the Axiom of Pairing: $(x,y) := \{\{x\}, \{x,y\}\}$.

The other basic construction we would like to be able to use is union of sets. This will allow us to widen considerably our universe: in fact, without it we cannot construct sets of more than two elements. The Axiom of Union tells us that if we have a family $\mathcal{F}$ of sets, then there exists a set that contains every member of the members of $\mathcal{F}$. We can define this set as

$$\bigcup \mathcal{F} = \bigcup_{Y \in \mathcal{F}} Y := \{x : \exists Y \in \mathcal{F}(x \in Y)\},$$

and the shorthand $x \cup y := \bigcup\{x,y\}$. In a similar fashion, we define the intersection of a nonempty family of sets:

$$\bigcap \mathcal{F} = \bigcap_{Y \in \mathcal{F}} Y := \{x : \forall Y \in \mathcal{F}(x \in Y)\}.$$

However, for this last definition, the Axiom of Union is not needed: fix $E \in \mathcal{F}$, then $\bigcap \mathcal{F} = \{x \in E : \forall y \in \mathcal{F}(x \in Y)\}$, which exists by Comprehension and is unique by Extensionality.

A very important definition now is the successor operation:

Definition 1.3. The ordinal successor function is $S(x) := x \cup \{x\}$.

With this we can construct every finite ordinal[2]:

$$\begin{aligned} &0 = \emptyset, && 4 = S(3) = \{0,1,2,3\},\\ &1 = S(0) = \{\emptyset\} = \{0\}, && 5 = S(4) = \{0,1,2,3,4\},\\ &2 = S(1) = \{\emptyset,\{\emptyset\}\} = \{0,1\}, && 6 = S(5) = \{0,1,2,3,4,5\},\\ &3 = S(2) = \{\emptyset,\{\emptyset\},\{\emptyset,\{\emptyset\}\}\} = \{0,1,2\}, && \cdots \end{aligned}$$

And in this way we can show the existence of every single finite ordinal, but we cannot construct a set that contains them all. For this, we need the **Axiom of Infinity** (Axiom 7).

Definition 1.4. A set I is **inductive** if $\emptyset \in I$ and $\forall x \in I(S(x) \in I)$.

Thus the Axiom of Infinity says that there exists an inductive set. In particular,

Definition 1.5. $\omega := \bigcap \mathcal{I}$, where $\mathcal{I}$ is the class of all inductive sets.

It is easy to check that ω is inductive and $\forall n \in \omega(n = 0 \vee \exists m \in \omega(n = S(m)))$. We also have immediately:

Theorem 1.1 (Principle of Ordinary Induction)**.** *For any set $X \subseteq \omega$, if X is inductive then $X = \omega$.*

The next axiom, the **Replacement Axiom Schema** (Axiom 6), is strongly related to functional relations, therefore we must start off with some definitions:

Definition 1.6. A **binary relation** is a class of ordered pairs. A binary relation, F, is **functional** if $(x,y),(x,y') \in F \implies y = y'$. If R is a binary relation, we write xRy instead of $(x,y) \in R$, and if R is functional, we write $R(x) =$ the unique y (if it exists) such that $(x,y) \in R$.

Definition 1.7.

- The *domain* of a binary relation R is $\mathrm{dom}(R) = \{x : \exists y((x,y) \in R)\}$.
- The *range* of a binary relation R is $\mathrm{ran}(R) = \{y : \exists x((x,y) \in R)\}$.
- If F is a functional relation, $F[A] = \{F(x) : x \in A \cap \mathrm{dom}(F)\}$.

It is easy to prove that if R is a set, then $\mathrm{dom}(R)$ and $\mathrm{ran}(R)$ are sets, and in particular if a functional relation F is a set then $F[A]$ is also a set. Let us see it for $\mathrm{dom}(R)$ (the other cases are similar): if $x \in \mathrm{dom}(R)$, then $x \in \{x\} \in (x,y) \in R$ for some y, and then $x \in \bigcup(\bigcup R)$, so $\mathrm{dom}(R) \subseteq \bigcup(\bigcup R)$, which is a set, applying the Axiom of Union twice. By Comprehension, $\mathrm{dom}(R)$ is a set, and it is unique by Extensionality. The Axiom of Replacement is a more general statement than this last one: if A is a set and F is a functional relation, which may be a proper class, then $F[A]$ is a set. The original statement of Replacement is

$$\forall x \in A \exists! y \phi(x,y) \rightarrow \exists B \forall x \in A \exists y \in B \phi(x,y).$$

[2]This definition could be understood as a set-theoretic definition of the natural numbers, but with no arithmetical structure. In fact, the numbers defined this way (together with the successor operation) satisfy the Peano Axioms for the natural numbers. The set ω of all finite ordinals would correspond to the set $\mathbb{N}$.

That is indeed the meaning of the axiom: $\forall x \in A \exists! y \phi$ means that ϕ induces a functional relation ($F(x) = y \leftrightarrow \phi(x,y)$), and $F[A]$ satisfies $\forall x \in A \exists y \in F[A] \phi(x,y)$.

We would also like to construct the set of subsets of a given set, and for that we need the **Power Set Axiom** (Axiom 8). Assuming it, we can define the set $\mathcal{P}(x) = \{z : z \subseteq x\}$: let y be a set such that $z \subseteq x \to z \in y$, which exists by the Power Set Axiom, then the set $\mathcal{P}(x) = \{z \in y : z \subseteq x\}$ exists by Comprehension and is unique by Extensionality.

Given two sets A and B, another important set that we can define is the Cartesian product: $A \times B := \{(x,y) : x \in A \wedge y \in B\}$. This is a set since $(x,y) \in \mathcal{P}(\mathcal{P}(A \cup B))$, which is a set (applying the axioms of Union and Power Set), so $A \times B = \{(x,y) \in \mathcal{P}(\mathcal{P}(A \cup B)) : x \in A \wedge y \in B\}$ exists by Comprehension and is unique by Extensionality.

There are still two more axioms that we have not discussed: **Axiom of Foundation** and **Axiom of Choice**. We will do it later for convenience.

2 Relations and functions

We have already defined binary relations and functional relations. We give now more basic definitions of set theory that will be used along the following sections.

Definition 2.1. Let R be a binary relation. We say that:

- R is *transitive* on A if $\forall xyz \in A[xRy \wedge yRz \to xRz]$.
- R is *irreflexive* on A if $\forall x \in A[x \not R x]$.
- R is *reflexive* on A if $\forall x \in A[xRx]$.
- R is *symmetric* on A if $\forall xy \in A[xRy \iff yRx]$.
- R satisfies *trichotomy* on A if $\forall xy \in A[xRy \vee yRx \vee x = y]$.
- R *partially orders* A *strictly* if R is transitive and irreflexive on A.
- R *totally orders* A *strictly* if R is transitive, irreflexive and satisfies trichotomy on A.
- R is an *equivalence relation* on A if R is reflexive, symmetric and transitive on A.

Now we recall some common definitions and notation on functions:

Definition-Notation 2.2.

- $F : A \to B$ means that F is a function, $\mathrm{dom}(F) = A$ and $\mathrm{ran}(F) = B$.
- $F : A \twoheadrightarrow B$ means that $F : A \to B$ and $\mathrm{ran}(F) = B$. We say that F is a *surjection*.
- $F : A \hookrightarrow B$ means $F : A \to B$ and $\forall x, x' \in A[F(x) = F(x') \to x = x']$. We say that F is an *injection*.
- $F : A \leftrightarrows B$ means that F is an injection and a surjection. We say that F is a *bijection*.

Definition 2.3. $B^A = {}^A B$ is the set of all F such that $F : A \to B$.

The existence of that set is justified by the Power Set Axiom, since B^A is a defined subset of $\mathcal{P}(A \times B)$. Also, some more notation:

- $X \preccurlyeq Y$ if there is an injection $F : X \hookrightarrow Y$.
- $X \approx Y$ if there is a bijection $F : X \leftrightarrows Y$.

Note that in sets without any relations or functions defined on them, bijection corresponds with the idea of isomorphism that appears in every mathematical theory. The general intuition is that two sets are in bijection if and only if they *have the same size*, in the sense that they can be put into a 1-1 correspondence. Similarly, $X \preccurlyeq Y$ intuitively means that X is *smaller* than Y.

The next definition is key for the topics that we will cover in the following sections:

Definition 2.4 (Well-ordering)**.**

- Let R be a relation. An element $y \in X$ is R**-minimal** in X if $\neg\exists z(z \in X \wedge zRy)$.
- R is **well-founded** on A if for every non-empty $X \subseteq A$, there is $y \in X$ that is R-minimal in X.
- R is a **well-order** on A if R totally orders A strictly and is well-founded on A.

Examples. $\mathbb{N}$ with the usual $<$ relation is a well-ordering: every subset of $\mathbb{N}$ has a minimum, and $<$ is a total order. On the other hand, the usual relations $<$ in $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ are total but not well-founded: there are subsets of $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ which have no least element. Therefore they are not well-ordered by the usual orderings. However, that does not mean that they cannot be well-ordered: for example, $0 <' -1 <' 1 <' -2 <' 2 <' \cdots$ is a well-ordering on $\mathbb{Z}$.

3 Ordinals

Definition 3.1. A set x is **transitive** if $\forall y \in x[y \subseteq x]$.

Equivalently, $\bigcup x \subseteq x$, and $z \in y \in x \implies z \in x$.

Definition 3.2. A set x is an **ordinal** if it is transitive and is well-ordered by $\in$.

Examples. $\emptyset$ is trivially an ordinal, as well as $S(\emptyset) = \{\emptyset\}$, $S(S(\emptyset)) = \{\emptyset, \{\emptyset\}\}$... and in general, for any ordinal α, its successor is also an ordinal, $S(\alpha) = \alpha \cup \{\alpha\}$. So $0, 1, 2, 3, \dots$ are ordinals, and so are $\omega, \omega + 1 := S(\omega), \dots$

From now on we will use Greek letters to refer to variables that range over the ordinals, and denote by Ord the class of all ordinals.

Proposition 3.1. Ord *is a proper class.*

Proof. First, notice that Ord is a transitive class, that is, $\alpha \in \mathrm{Ord} \wedge \beta \in \alpha \implies \beta \in \mathrm{Ord}$. Indeed, α is a transitive set, so $\beta \subseteq \alpha$, and since $\in$ well-orders α, it well-orders every subset of α, in particular, β is well-ordered by $\in$ (and it is transitive), so $\beta \in \mathrm{Ord}$.

So if Ord was a set, it would be an ordinal, and hence $\mathrm{Ord} \in \mathrm{Ord}$, a contradiction. □

Theorem 3.1. $\in$ *is a well-order on* Ord.

Proof. We have to prove that $\in$ satisfies trichotomy and that it is well-founded on the ordinals.

First, we want to prove that $\forall \alpha, \beta \in \mathrm{Ord}(\alpha \in \beta \vee \alpha = \beta \vee \beta \in \alpha)$. We must prove that $A = \{\alpha \in \mathrm{Ord} \mid \exists \beta \in \mathrm{Ord}(\alpha \notin \beta \wedge \alpha \neq \beta \wedge \beta \notin \alpha)\}$ is empty. If it isn't, then $\exists \bar{\alpha} \in A$ such that $\bar{\alpha} \cap A = \emptyset$. Then $B = \{\beta \in \mathrm{Ord} \mid \beta \notin \bar{\alpha} \wedge \beta \neq \bar{\alpha} \wedge \bar{\alpha} \notin \beta\} \neq \emptyset$ so there is $\bar{\beta} \in B$ such that $\bar{\beta} \cap B = \emptyset$. If $\gamma \in \bar{\alpha}$ then $\gamma \notin A$, so in particular $\bar{\beta} \in \gamma \vee \bar{\beta} = \gamma \vee \gamma \in \bar{\beta}$. The first two possibilities together with transitivity of $\bar{\alpha}$ imply that $\bar{\beta} \in \bar{\alpha}$, against $\bar{\beta} \in B$. Thus $\gamma \in \bar{\beta}$. Being γ arbitrary, we have that $\bar{\alpha} \subseteq \bar{\beta}$. Similarly $\bar{\beta} \subseteq \bar{\alpha}$ and thus $\bar{\alpha} = \bar{\beta}$, a contradiction.

Now, for well-foundedness, let X any non-empty set of ordinals, and fix $\alpha \in X$. If it is the least element, we're done. Otherwise, $\alpha \cap X = \{\beta \in X \mid \beta \in \alpha\} \neq \emptyset$, and $\alpha \cap X$ has a least element, since α is well-ordered by $\in$. Then that element is also the least element of X. □

Notation: we will write $\alpha < \beta$ and $\alpha \leq \beta$ for $\alpha \in \beta$ and $(\alpha \in \beta \vee \alpha = \beta)$.

Definition 3.3. An ordinal α is:

- a **successor** if $\alpha = \mathrm{S}(\beta)$ for some β. We will also write $\beta + 1 := \mathrm{S}(\beta)$.
- a **limit** ordinal if it is neither 0 nor a successor ordinal.

Theorem 3.2. *ω is the smallest limit ordinal.*

Proof. ω is an ordinal and there are no limit ordinals less than ω. It is enough to check that ω is not a successor. If $\omega = \mathrm{S}(\alpha)$ for some ω, then $\alpha \in \omega$, so $\mathrm{S}(\alpha) \in \omega$, that is, $\omega \in \omega$, a contradiction. □

Definition 3.4. A set A is **finite** if $A \preccurlyeq n$ for some $n \in \omega$. **Infinite** means not finite.

Examples. $1, 2, 3, \ldots$ are finite successor ordinals, $\omega + 1, \omega + 2, \ldots$ are infinite successor ordinals, as well as $3 \cdot \omega^{\omega^2} + 5$, for instance; and $\omega, 2 \cdot \omega, 3 \cdot \omega, \ldots$ and $\omega^2, \omega^3, \ldots, \omega^{\omega^\omega + 1}, \ldots$ are limit ordinals. Every limit ordinal is infinite.

3.1 Classifying well-orders

We have introduced the notion and basic properties of ordinals. The main reason to define ordinals is that they allow us classify well-orderings. That is, two ordinals are isomorphic if and only if they are equal, and every well-ordered set is isomorphic to some ordinal. This is not trivial, and to prove it in a nice way we first need some lemmas:

Lemma 3.3. *Let $\langle A, \leq \rangle$ be a well-ordered class.*

1. *If $f : A \to A$ is increasing then $\forall a \in A (a \leq f(a))$. Moreover, if f is bijective then $f = \mathrm{id}_A$.*
2. *If $\langle A, \leq_A \rangle$ and $\langle B, \leq_B \rangle$ are isomorphic well-ordered classes, then the isomorphism is unique.*
3. *Denote $\mathrm{pred}(a,A;\leq) := \{b \in A \mid b \leq a\}$. If $a \in A$, then $\langle A, \leq \rangle$ and $\langle \mathrm{pred}(a,A;\leq), \leq \rangle$ are not isomorphic.*

Proof.
1. Towards a contradiction, let $\bar{a} \in A$ the least such that $f(\bar{a}) \leq \bar{a}$. By minimality, $f(f(\bar{a})) \geq f(\bar{a})$, and $f(f(\bar{a})) < f(\bar{a})$ since f is increasing, which yields the desired contradiction. Now, if f is bijective, $f(a) \geq a$ and $f^{-1}(a) \geq a$, so $f(a) = a = \mathrm{id}_A(a)$.
2. Let $f, g : A \to B$ isomorphisms. Then $g^{-1} \circ f : A \to A$ is an isomorphism, so $g^{-1}(f(a)) = a$, that is, $f(a) = g(a)$.
3. If $f : A \to \mathrm{pred}(a,A;\leq)$ is an isomorphism, then $f : A \to A$ is increasing, so $\forall x \in A (x \leq f(x))$ and hence $a \leq f(a)$, a contradiction. □

Now, every $\alpha \in \mathrm{Ord}$ yields a well-order $\langle \alpha, \in \rangle$, and if $\beta \in \alpha$, then $\beta = \mathrm{pred}(\beta, \alpha; \in)$, so $\langle \alpha, < \rangle \cong \langle \beta, < \rangle \iff \alpha = \beta$. This, together with the lemma we just proved, implies:

Lemma 3.4.

1. *If $f : \alpha \to \beta$ is increasing, then $\forall \gamma \in \alpha (\gamma \leq f(\gamma))$ and $\alpha \leq \beta$.*
2. *If $f : \alpha \to \beta$ is an isomorphism, then $\alpha = \beta$ and f is the identity.*

Now we can prove the main theorem:

Theorem 3.5. *Every well-ordered set is isomorphic to an ordinal (and every well-ordered class is isomorphic to* Ord*). Moreover, the ordinal and the isomorphism are unique.*

Proof. Let $\langle X,<\rangle$ be a well-ordered set, and let $A = \{\alpha \in \text{Ord} \mid \exists x \in X(\langle \alpha, \in\rangle \cong \langle \text{pred}(x),<\rangle)\}$. Let $\alpha \in A$ and a bijection $f : \langle \alpha, \in\rangle \leftrightarrows \langle \text{pred}(x),<\rangle$ witnessing $\alpha \in A$. If $\beta \in \alpha$, then the restriction $f \restriction \beta : \langle \beta, \in\rangle \to \langle \text{pred}(f(\beta)),<\rangle$ is an isomorphism, so $\beta \in A$. Therefore A is transitive so it is an ordinal. Let $f : A \to X$ be the map assigning to every $\alpha \in A$ the unique $x \in X$ such that $\langle \alpha, \in\rangle \cong \langle \text{pred}(x),<\rangle$. Thus $\text{ran}(f)$ is an initial segment of X. If $\text{ran}(f) \neq X$, then $\text{ran}(f) = \text{pred}(\bar{x})$, for some $\bar{x} \in X$ and the isomorphism $f : \langle A, \in\rangle \to \langle \text{pred}(\bar{x}),<\rangle$ witnesses that the ordinal A belongs to the set A, a contradiction. Uniqueness follows from the preceding lemmas. (The case for proper classes is similar). □

This allows us to define the order type of well-ordered sets:

Definition 3.5. If R well-orders A, then $\text{type}(A;R)$ is the unique $\alpha \in \text{Ord}$ such that $\langle A,R\rangle \cong \langle \alpha, \in\rangle$. We ommit A or R when they are clear from the context and just write $\text{type}(A)$ or $\text{type}(R)$.

Examples. The order type of $\mathbb{N}$ with the usual $<$ is ω. Of course, there is no ordering of $\mathbb{N}$ of type less than ω, since $\mathbb{N}$ is countable and every $\alpha < \omega$ is finite. But we can define many well-orderings on $\mathbb{N}$ of order type larger than ω. For example, if we define the well-order $<_1$ as $\forall x \neq 0[\forall y(x <_1 y \iff x < y) \land x <_1 0]$, that is, the usual ordering but 0 is larger than every number:

$$\underbrace{1 <_1 2 <_1 3 <_1 \cdots}_{\omega} <_1 0.$$

Here the order type of $\langle \mathbb{N}, <_1\rangle$ is $\omega+1$. Analogously, for every $n < \omega$ we can define an order such that $\langle \mathbb{N}, <_n\rangle \cong \langle \omega+n, \in\rangle$, i.e., the order type of $<_n$ is $\omega+n$:

$$\underbrace{n <_n n+1 <_n n+2 <_n \cdots}_{\omega} <_n \underbrace{0 <_n 1 <_n 2 <_n \cdots <_n n-1}_{n}.$$

We can also define an order $<'$ such that $\langle \mathbb{N}, <'\rangle \cong \langle \omega \cdot 2, \in\rangle$:

$$\underbrace{0 <' 2 <' 4 <' 6 <' \cdots}_{\omega} <' \underbrace{1 <' 3 <' 5 <' \cdots}_{\omega}.$$

Analogously for $\omega \cdot 3, \omega \cdot 4, \ldots$, and we could go on and define more complex orders of type $\omega \cdot \omega, \omega^3, \ldots$.

3.2 Transfinite induction and recursion

We introduce now a fundamental tool to work with ordinals: a generalization, for the class of all ordinals, of the ideas of induction and recursion that we already have for ω.

Theorem 3.6 (Transfinite induction)**.** *Let Ω be a class of ordinals. If the following conditions are satisfied:*

1. $0 \in \Omega$,
2. $\alpha \in \Omega \implies \alpha+1 \in \Omega$,
3. *if $\alpha \neq 0$ is a limit ordinal and $\beta \in \Omega$, $\forall \beta < \alpha$, then $\alpha \in \Omega$,*

then $\Omega = \text{Ord}$, the class of all ordinals.

Proof. If those conditions are satisfied but $\Omega \neq \text{Ord}$, then let α be the least ordinal $\notin \Omega$ and applying either (1), (2) or (3) we will have $\alpha \in \Omega$, a contradiction. □

This yields a way of defining "transfinite sequences":

Theorem 3.7 (Transfinite recursion)**.** *Let $G : V \to V$ be a function, then there is a unique function $F : \mathrm{Ord} \to V$ such that*

$$F(\alpha) = G(F \restriction \alpha)$$

for each α. In other words, let $a_\alpha = F(\alpha)$, then for each α,

$$a_\alpha = G(\langle a_\xi : \xi < \alpha \rangle).$$

Proof. Let $F(\alpha) = x \iff$ there is a sequence $\langle a_\xi : \xi < \alpha \rangle$ such that:

1. $\forall \xi < \alpha \left(a_\xi = G(\langle a_\eta : \eta < \xi \rangle) \right)$,
2. $x = G(\langle a_\xi : \xi < \alpha \rangle)$.

Now, for any α, if an α-sequence satisfies (1), then the sequence is unique: if $\langle a_\xi : \xi < \alpha \rangle$ and $\langle b_\xi : \xi < \alpha \rangle$ satisfy (1) then, by induction on ξ, $a_\xi = b_\xi$ for every $\xi < \alpha$. Therefore $F(\alpha)$ is uniquely determined by (2), and hence F is a function. Now, again by induction, for every α there must exist an α-sequence satisfying (1) (at limit steps, we use Replacement to get the α-sequence as the union of all the ξ-sequences for $\xi < \alpha$). Thus F is defined for every $\alpha \in \mathrm{Ord}$. It is clear that F satisfies $F(\alpha) = G(F \restriction \alpha)$.

To finish the proof, notice that if $\hat{F}$ a function on Ord that also satisfies $\hat{F}(\alpha) = G(\hat{F} \restriction \alpha)$ then it follows by induction that $\hat{F}(\alpha) = F(\alpha)$ for all α. □

This theorem tells us that we can define transfinite sequences (sequences indexed in some ordinal, or even indexed in Ord) just by defining its elements in function of the previous ones.

Note that we need Replacement for transfinite recursion, since we need $F \restriction \alpha$ to be a set for each α.

Definition 3.6. Let $\alpha > 0$ a limit ordinal, and let $\langle \gamma_\xi : \xi < \alpha \rangle$ be a non decreasing sequence of ordinals. We define the **limit** of the sequence as:

$$\lim_{\xi \to \alpha} \gamma_\xi = \sup\{\gamma_\xi : \xi < \alpha\}.$$

3.3 Ordinal arithmetic

So far we only have the operation $\mathrm{S}(\alpha)$ in ordinals. We would like to define addition and multiplication in a way that generalizes the usual operations for natural numbers to every ordinal. That way we can see the "list" of ordinals as:

$$0, 1, 2, 3, \ldots, n, n+1, \ldots, \omega, \omega+1, \omega+2, \ldots, \omega \cdot 2, \omega \cdot 2+1, \ldots, \omega \cdot 3, \ldots, \omega \cdot \omega, \ldots, \omega^3, \ldots, \omega^\omega, \ldots$$

Transfinite recursion allows us to define ordinal arithmetic in an intuitive way:

$$\alpha + \beta = \begin{cases} \alpha & \text{if } \beta = 0, \\ \mathrm{S}(\alpha + \gamma) & \text{if } \beta = \mathrm{S}(\gamma), \\ \sup_{\gamma < \beta}(\alpha + \gamma) & \text{if } \beta \text{ is limit.} \end{cases}$$

$$\alpha \cdot \beta = \begin{cases} 0 & \text{if } \beta = 0, \\ (\alpha \cdot \gamma) + \alpha & \text{if } \beta = \mathrm{S}(\gamma), \\ \sup_{\gamma < \beta} \alpha \cdot \gamma & \text{if } \beta \text{ is limit.} \end{cases}$$

$$\alpha^\beta = \begin{cases} 1 & \text{if } \beta = 0, \\ (\alpha^\gamma) \cdot \alpha & \text{if } \beta = \mathrm{S}(\gamma), \\ \sup_{\gamma < \beta} \alpha^\gamma & \text{if } \beta \text{ is limit.} \end{cases}$$

Notice that ordinal addition is noncommutative:

$$1 + \omega = \sup_{n < \omega}(1 + n) = \omega, \quad \text{but} \quad \omega + 1 = \mathrm{S}(\omega) \neq \omega.$$

And also ordinal multiplication is noncommutative:

$$\omega \cdot 2 = (\omega \cdot 1) + \omega = \omega + \omega, \quad \text{but} \quad 2 \cdot \omega = \sup_{n < \omega}(2 \cdot n) = \omega.$$

4 Cardinals

We have seen that with ordinals we are able to classify *ordered* sets. Now, we want to classify all sets by *size*, following the idea that two sets are of the same size if and only if they can be put in bijection. Cardinals will be the representatives of these classes. This can be done by taking representatives in the class of ordinals.

Definition 4.1. A **cardinal** is an ordinal κ which is not in bijection with any ordinal $\alpha < \kappa$.

So, intuitively, the idea is to take as cardinals the first ordinal of each *size*.

Definition 4.2. Let X be a well-orderable set. Its **cardinality** is the smallest ordinal κ in bijection with X. We denote it by $|X|$.

In particular $|\alpha|$ is the smallest ordinal $\beta \approx \alpha$ and hence $|\alpha| \leq \alpha$.

Notice that this requires for X to be well-orderable. This comes from how we have defined cardinals. To talk about the cardinality of any set, we will need the Axiom of Choice. We will discuss this in detail in a few sections.

Before giving results on cardinals, we may recall a well-known and very useful result (a proof can be found in [1, Theorem 11.11] and [2, Appendix 7.5]):

Lemma 4.1 (Cantor-Schröder-Bernstein Theorem)**.** *If X, Y are sets, then*

$$X \approx Y \iff X \preccurlyeq Y \wedge Y \preccurlyeq X.$$

Proposition 4.1. *Let $\kappa, \lambda \in \mathrm{Card}$,*

1. $\kappa = \lambda \iff \kappa \approx \lambda$,
2. $\kappa \leq \lambda \iff \kappa \preccurlyeq \lambda$.

Proof. 1. Assume that $\kappa \approx \lambda$ but $\kappa \neq \lambda$, for example, $\kappa < \lambda$. Then λ would be in injection with a strictly smaller ordinal; a contradiction.

2. Assume that $\kappa \preccurlyeq \lambda$ but $\lambda < \kappa$, then let $\mathrm{id} : \lambda \to \kappa$ the identity. By the Cantor-Schröder-Bernstein Theorem, $\kappa \approx \lambda$ so $\kappa = \lambda$ by the preceding result; a contradiction. □

Therefore we have that for any well-ordered set A, $|A|$ is a cardinal, $|A| = |B| \iff A \approx B$ and $|A| \leq |B| \iff A \preccurlyeq B$.

Theorem 4.2 (Hartogs)**.** *For every set A, there is a cardinal κ such that $\kappa \not\preccurlyeq A$.*

Proof. Let W be the set of pairs (X, R) such that $R \subseteq X \times X$ well-orders $X \in \mathcal{P}(A)$. That is, W is the set of well-orderings on all the subsets of A. Note that $\alpha \preccurlyeq X \iff \alpha = \mathrm{type}(X; R)$ for some $(X; R) \in W$. Let $\kappa = \sup\{\mathrm{type}(X; R) + 1 \mid (X; R) \in W\}$. Then κ is a cardinal and $\kappa > \alpha$ for any $\alpha \preccurlyeq A$, so $\kappa \not\preccurlyeq A$. □

In particular, for any cardinal λ, there is always a bigger cardinal $\kappa > \lambda$. A similar result is Cantor's theorem, which explicitly gives us a bigger cardinal. We will discuss it in a few sections, since we need to introduce the Axiom of Choice for that.

Therefore, the class of cardinals is unbounded, and since it is well-ordered, we can make the following definitions:

Definition 4.3. For each cardinal, κ, we write κ^+ for the least cardinal greater than κ, and we call it its *successor*. We will also write $\omega_1 := \omega^+$.

Definition 4.4. $\aleph : \mathrm{Ord} \to \mathrm{Card}$ is the class-function enumerating the class of infinite cardinals:

$$\aleph_0 = \omega,$$

$$\aleph_{\alpha+1} = (\aleph_\alpha)^+,$$

$$\aleph_\lambda = \sup_{\alpha<\lambda} \aleph_\alpha, \text{ for } \lambda \text{ limit.}$$

4.1 Cardinal addition and multiplication

Definition 4.5 (Cardinal addition and multiplication)**.**

$$\kappa+\lambda = |\{0\}\times\kappa\cup\{1\}\times\lambda|, \qquad\qquad \kappa\cdot\lambda = |\kappa\times\lambda|.$$

Given any ordinal ξ, the canonical well-order on $\xi\times\xi$ is defined by:

$$(\alpha,\beta) <_C (\gamma,\delta) \iff [\max(\alpha,\beta) < \max(\gamma,\delta) \vee (\max(\alpha,\beta) = \max(\gamma,\delta) \wedge (\alpha,\beta) <_{\text{lex}} (\gamma,\delta))].$$

That is, the pair that has the biggest element is the biggest pair, and if both have the same biggest element, then apply lexicographic order.

Theorem 4.3. *Let κ be an infinite cardinal. Then* $\text{type}(\kappa\times\kappa,<_C) = \kappa$ *and* $|\kappa\times\kappa| = \kappa$.

Proof. The function $\langle\kappa,<\rangle \to \langle\kappa\times\kappa,<_C\rangle$, $\alpha\mapsto(\alpha,0)$ is increasing so that $\kappa\le\text{type}(\kappa\times\kappa,<_C)$. Therefore it is enough to show that $\text{type}(\kappa\times\kappa,<_C)\le\kappa$, so that $|\kappa\times\kappa| = \kappa$. We can prove this by induction on $\kappa\ge\omega$:

Let $\alpha<\kappa$. If $\alpha<\omega$ then $|\alpha\times\alpha|<\omega$. Otherwise, $\omega\le|\alpha|<\kappa$ so by inductive assumption $|\alpha|\times|\alpha|$ is of cardinality $|\alpha|$. As $|\alpha|\times|\alpha|\approx\alpha\times\alpha$, then $|\alpha\times\alpha|<\kappa$. Therefore we have shown that $\forall\alpha<\kappa(|\alpha\times\alpha|<\kappa)$. Fix $\alpha,\beta<\kappa$. The set $\text{pred}(\alpha,\beta)$ of $<_C$-predecessors of (α,β) is included in $\nu\times\nu$, where $\nu = \max\alpha,\beta+1$, so $|\text{pred}(\alpha,\beta)|\le|\nu\times\nu|<\kappa$. Therefore we have shown that $\forall\alpha,\beta<\kappa(\text{type}(\text{pred}(\alpha,\beta)<\kappa)$ and hence $\text{type}(\kappa\times\kappa,<_C)\le\kappa$. □

From this theorem and noticing that for κ,λ cardinals such that either $2\le\min(\kappa,\lambda)$ or else $1 = \min(\kappa,\lambda)\wedge\omega\le\max(\kappa,\lambda)$, then

$$\max(\kappa,\lambda)\le\kappa+\lambda\le\kappa\cdot\lambda\le\max(\kappa,\lambda)\cdot\max(\kappa,\lambda),$$

we have the following corollary:

Corollary 4.3.1. *If $\kappa,\lambda\ne 0$ are cardinals and at least one of them is infinite, then*

$$\max(\kappa,\lambda) = \kappa+\lambda = \kappa\times\lambda.$$

Also,

Corollary 4.3.2. *If $2\le\kappa\le\lambda$ and λ is an infinite cardinal, then ${}^{\lambda}2\approx{}^{\lambda}\kappa\approx{}^{\lambda}\lambda$.*

Proof.

$${}^{\lambda}2\preccurlyeq{}^{\lambda}\kappa\preccurlyeq{}^{\lambda}\lambda\preccurlyeq\mathcal{P}(\lambda\times\lambda)\approx\mathcal{P}(\lambda)\approx{}^{\lambda}2.$$

The first two injections are clear: they follow from the fact that ${}^{\lambda}2\subseteq{}^{\lambda}\kappa$ and ${}^{\lambda}\kappa\subseteq{}^{\lambda}\lambda$, since $2\subseteq\kappa\subseteq\lambda$.

For the last part, notice that ${}^{\lambda}\lambda\subseteq\mathcal{P}(\lambda\times\lambda)\approx\mathcal{P}(\lambda)$, where the last assertion holds since $\lambda\times\lambda\approx\lambda$, by Theorem 4.3; and finally $\mathcal{P}(\lambda)\approx{}^{\lambda}2$. □

4.2 Cardinal exponentiation

So far we have only defined sum and product of ordinals and cardinals. These operations are very straightforward and do not raise significant questions. We are now going to introduce the most important operation on cardinals: exponentiation. It is of deep interest since it brings up many questions in set theory. First we need to introduce the **Axiom of Choice (AC)**, that will play a crucial role from now on.

The Axiom of Choice (Axiom 9) states that every family $\mathcal{F}$ of nonempty sets has a choice function, that is, a function f on $\mathcal{F}$ such that $f(X)\in X$ for every $X\in\mathcal{F}$.

The formalization of the Axiom of Choice was introduced after Zermelo proved that $\mathbb{R}$ is well-orderable, but without giving an explicit well-order for it. He was implicitly using AC, and the skepticism on the validity of his proof led to a closer investigation of the underlying reasoning, which led to the formalization and introduction of AC as an axiom. AC is in fact equivalent to "every set is well-orderable". Notice that every well-ordered set has a choice function: it suffices to take the least element. The reverse implication is less trivial:

Theorem 4.4. *Assuming AC, every set is well-orderable.*

Proof. Let $X \neq \emptyset$ be a set. To well-order X, it is enough to construct a transfinite sequence $\langle x_\beta : \beta < \alpha \rangle$ that enumerates X (for some unknown ordinal α). We may achieve this inductively: let f a choice function in $\mathcal{P}(X)$, and for every β, let

$$x_\beta = f(X \setminus \{x_\xi : \xi < \beta\}), \quad \text{if } X \setminus \{x_\xi : \xi < \beta\} \neq \emptyset.$$

Let α be the least ordinal such that $X = \{x_\xi : \xi < \alpha\}$. Clearly, $\langle x_\beta : \beta < \alpha \rangle$ enumerates X. □

Recall that cardinality is only defined for well-orderable sets, and we would like to talk about the cardinality of any set. Now, assuming the Axiom of Choice, we can well-order any set and therefore talk about its cardinality. So now we can define cardinal exponentiation in the following way:

Definition 4.6. Cardinal exponentiation is defined by

$$\lambda^\kappa = |{}^\kappa\lambda|.$$

Recall that ${}^\kappa\lambda$ is the set of functions $\kappa \to \lambda$. This definition requires that the set ${}^\kappa\lambda$ is well-ordered, so **cardinal exponentiation is defined assuming the Axiom of Choice**. Thus, we will work under AC from now on, unless we explicitly state otherwise.

Cardinal exponentiation satisfies the usual properties of exponentiation:

$$(\kappa^\lambda)^\mu = \kappa^{\lambda\cdot\mu},$$

$$\kappa^{\lambda+\mu} = \kappa^\lambda \cdot \kappa^\mu,$$

$$(\kappa\cdot\lambda)^\mu = \kappa^\mu \cdot \lambda^\mu,$$

$$\kappa^\lambda \leq \nu^\mu \quad \text{if } \kappa \leq \nu \text{ and } \lambda \leq \mu.$$

And we can interpret Corollary 4.3.2 as $2^\lambda = \kappa^\lambda = \lambda^\lambda$, for $2 \leq \kappa \leq \lambda$.

Theorem 4.5 (Cantor)**.** *For every set X, there is no surjection from X onto $\mathcal{P}(X)$. Consequently, for every cardinal, $\lambda < 2^\lambda$.*

Proof. Let $f : X \to \mathcal{P}(X)$ and consider the set $Y = \{x \in X : x \notin f(x)\}$. This set is not in $\operatorname{ran}(f)$: if there was some $z \in X$ such that $f(z) = Y$, then $z \in Y \iff z \notin Y$, a contradiction. Hence f is not a surjection.

On the other hand, it is clear that there exists an injection $g : X \to \mathcal{P}(X) \ni f(x) = \{x\}$. Therefore $|X| < |\mathcal{P}(X)|$, which can be written as $\lambda < 2^\lambda$ for every cardinal λ since for every set X, $\mathcal{P}(X) \approx 2^X$: for every subset S of X we can consider the characteristic map $\chi_S : X \to 2 = \{0,1\}$ such that $\chi_S(x) = 1 \iff x \in S$, and viceversa, every such map defines a different element of $\mathcal{P}(X)$. □

Now we know that for every cardinal λ, the cardinal 2^λ is bigger. Is it the case that $\lambda^+ = 2^\lambda$? That is, is 2^λ the cardinal immediately after λ? The affirmative answer is the Generalized Continuum Hypothesis. More formally, the **Continuum Hypothesis** is that $\aleph_1 = 2^{\aleph_0}$, and the **Generalized Continuum Hypothesis** is that $\forall\alpha \in \mathrm{Ord}(2^{\aleph_\alpha} = \aleph_{\alpha+1})$. Both questions are known to be undecidable in ZFC after the works of Gödel (1940) and Cohen (1963).

Definition 4.7. The beth function is the function $\beth : \mathrm{Ord} \to \mathrm{Card}$ defined recursively by

$$\beth_0 = \omega, \qquad \beth_{\alpha+1} = 2^{\beth_{\alpha+1}},$$

$$\beth_\lambda = \sup_{\alpha<\lambda} 2^{\beth_\alpha}, \text{ for } \lambda \text{ limit}.$$

And this way, CH is equivalent to $\beth_1 = \aleph_1$, and GCH is equivalent to $\beth_\alpha = \aleph_\alpha$ for every ordinal α.

4.3 Cofinality

Definition 4.8. Let $\alpha > 0$ be a limit ordinal. We say that an increasing sequence $\langle \alpha_\xi : \xi < \beta \rangle$ is **cofinal** in α if $\sup\{\alpha_\xi : \xi < \beta\} = \alpha$. The **cofinality** of α is the least limit ordinal β such that there exist an increasing β-sequence that is cofinal in α. It is denoted as $\mathrm{cf}(\alpha) = \beta$.

Similarly, we can say that a function $f : \beta \to \alpha$ is cofinal (in α) if $\mathrm{ran}(f)$ is unbounded in α, that is, $\forall \alpha' < \alpha \exists \beta' < \beta (\alpha' \leq f(\beta'))$, and this way $\mathrm{cf}(\alpha) =$ the least β sucht that there is a cofinal $f : \beta \to \alpha$.

In some sense, the cofinality of a limit ordinal α tells how many *steps* does it take to *scale* it, with steps smaller than α.

Note that always $\mathrm{cf}(\alpha) \leq \alpha$. Some examples of cofinalities: $\mathrm{cf}(\omega) = \omega$, and under AC, $\mathrm{cf}(\omega_1) = \omega_1$. On the other hand, $\mathrm{cf}(\aleph_\omega) = \omega$, since $\langle \aleph_n : n \in \omega \rangle$ is cofinal.

Definition 4.9. A limit ordinal λ is **regular** if $\mathrm{cf}(\lambda) = \lambda$. Otherwise, it is **singular**. If λ is an infinite cardinal, we will speak about regular of singular cardinals.

Directly from these definitions, the following properties hold:

Proposition 4.2. *For any limit ordinal α,*

1. *If $A \subseteq \alpha$ and $\sup(A) = \alpha$ then $\mathrm{cf}(\alpha) = \mathrm{cf}(\mathrm{type}(A))$.*
2. $\mathrm{cf}(\mathrm{cf}(\alpha)) = \mathrm{cf}(\alpha)$*, so* $\mathrm{cf}(\alpha)$ *is regular.*
3. $\omega \leq \mathrm{cf}(\alpha) \leq |\alpha| \leq \alpha$.
4. *If α is regular, then it is a cardinal. In particular, cofinalities are regular cardinals.*

The following theorem gives us a good intuition on cofinality and regularity (a proof can be found in [5, Theorem I.13.11]):

Theorem 4.6. *Let θ be any infinite cardinal.*

1. *If θ is regular, and $\mathcal{F}$ is a family of sets with $|\mathcal{F}| < \theta$, and $|S| < \theta$ for all $S \in \mathcal{F}$, then $|\bigcup \mathcal{F}| < \theta$.*
2. *If $\mathrm{cf}(\theta) = \lambda < \theta$, then there is a family $\mathcal{F}$ of subsets of θ with $|\mathcal{F}| = \lambda$ and $\bigcup \mathcal{F} = \theta$, such that $|S| < \theta$ for all $S \in \mathcal{F}$.*

5 The universe V and the cumulative hierarchy of sets

Now, we want to describe more explicitly the universe of all sets, V. Here, the **Axiom of Foundation** (Axiom 2) plays a crucial role. It says that the $\in$ relation is well-founded. We stated it as

$$\forall x[\exists y(y \in x) \to \exists y(y \in x \wedge \neg \exists z(z \in x \wedge z \in y))],$$

which can be rewritten, now that we have more notation, as

$$\forall x[x \neq \emptyset \to \exists y \in x(y \cap x = \emptyset)].$$

In other words, every non empty set has an $\in$-minimal element, that is, $\in$ is well founded. As a consequence, there is no infinite sequence $x_0 \ni x_1 \ni x_2 \ni \cdots$. In particular, there is no set such that $x \in x$, and there are no *cycles* ($x_0 \in x_1 \in \cdots \in x_0$).

Our purpose is to describe V as a hierarchy, explicitly showing how every set can be constructed from the sets that we had *before*. The majority of our axioms serve this purpose: Union, Pairing and Power Set allow us to construct sets from other ones; also Comprehension, Replacement and Infinity are statements about the existence of sets. So to construct V we just have to apply these axioms to the sets we know. The problem is that we need to have control over where we are starting: without Foundation we could have pathological sets with infinite descending chains or cycles, Foundation excludes all these pathological objects, and thanks to this we can start constructing V from $\emptyset$.

Let us give some intuition on what we will do: taking into account that the relation $\in$ is well founded in V (Axiom of Foundation), and that $\emptyset$ has no elements (and it is the only set with no elements, by Extensionality), we know that it should be the first (*lowest*) element in the hierarchy. And in general, how do we build the next *stage* from the previous ones? That is, how do we get bigger sets? We have two fundamental operations for doing that: union and power set.

So we define, by transfinite induction,

$$V_0 = \emptyset,$$

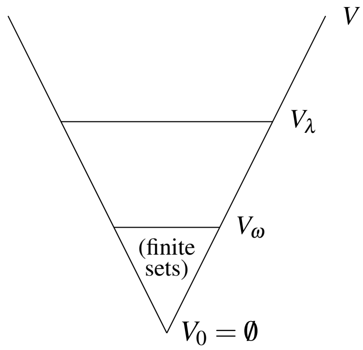
$$V_{\alpha+1} = \mathcal{P}(V_\alpha),$$

$$V_\alpha = \bigcup_{\beta<\alpha} V_\beta, \text{ for } \alpha \text{ limit.}$$

The following properties are deduced immediately (by induction):

1. Every V_α is transitive,
2. $V_\alpha \subset V_\beta$ for $\alpha < \beta$,
3. $\alpha \subset V_\alpha$.

This hierarchy is usually depicted in a diagram as:



And now, let us prove that we indeed achieved our goal: to describe in this hierarchy the whole universe of sets.

Theorem 5.1. *For every set x there is an ordinal α such that $x \in V_\alpha$, that is,*

$$V = \bigcup_{\alpha \in \mathrm{Ord}} V_\alpha.$$

Proof. Let C be the class of all sets that are not in any V_α. If $C \neq \emptyset$, then C has an $\in$-minimal element, x. That is, $x \in C$ and $z \in \bigcup_\alpha V_\alpha$ for every $z \in x$. Hence $x \subset \bigcup_\alpha V_\alpha$, and then there exists an ordinal γ such that $x \subset \bigcup_{\alpha<\gamma} V_\alpha$ (by Replacement). Hence $x \subset V_\gamma$, and so $x \in V_{\gamma+1}$: a contradiction. □

Chapter 2

Cardinals beyond ZFC

The essential bibliography for this chapter is [3], for its introduction to large cardinals, and especially Mahlo cardinals; and [1], for the more model-theoretic sections, as well as the notion of universe in section 7.1 and the related results. We have also relied, but more lightly, on [4].

6 Models of Set Theory

The goal of this thesis is to give some examples of cardinals whose existence goes beyond ZFC. The general idea that we will follow in order to prove that a cardinal κ is beyond ZFC (that is, its existence is not provable from the axioms of ZFC) is to see that V_κ is a model of ZFC, that is, every axiom of ZFC is satisfied in V_κ. But then, if κ existed, ZFC would prove its own consistency, contradicting Gödel's second theorem. Of course, we will follow each step in detail.

We may start by exploring the following question:

What axioms of ZFC are true in V_α?

Before trying to answer this question, we need some definitions and lemmas.

A language $\mathcal{L}$ is a set of symbols, containing relation symbols, function symbols and constant symbols. Denote by $\mathcal{L}_\in$ the language of set theory[1].

A structure for $\mathcal{L}_\in$ is a pair $\langle M, E\rangle$ where M is a nonempty set and $E \subseteq M \times M$ a relation on M. In particular, for our question we will consider the structure $\langle V_\alpha, \in\rangle$, with $\alpha > 0$.

Definition 6.1. An $\mathcal{L}_\in$-formula is Δ_0 if it belongs to the smallest class containing all atomic formulae and it is closed under connectives and **bounded** quantifications, that is:

- atomic formulae are Δ_0,
- if ψ, ϕ are Δ_0, then so are $\neg\psi$ and $\psi \star \phi$, where $\star$ is any binary connective,
- if ϕ is Δ_0, then so is $\forall y(y \in x \implies \phi)$ and $\exists y(y \in x \wedge \phi)$,

and nothing else is a Δ_0-formula.

[1] $\mathcal{L}_\in$ is the set of symbols with just one element: the relation symbol $\in$, which we use together with the symbols of first order logic (the parenthesis, the connectives $\wedge, \vee, \rightarrow, \leftrightarrow, \neg$. the quantifiers $\forall, \exists$, the equality symbol $=$, and an infinite number of variables $v_0, v_1, v_2, \ldots$).

Given a language $\mathcal{L}$, the set of $\mathcal{L}$-terms is defined recursively as: (1) every variable and every constant symbol is a term, and (2) for every function symbol f with arity k and terms $t_1, \ldots, t_k$, then $f(t_1, \ldots, t_k)$ is a term. In $\mathcal{L}_\in$ there are no constant or function symbols, so $\mathcal{L}_\in$-terms are just variables.

Given a language $\mathcal{L}$, we say that ϕ is an **atomic formula** if it is of the form $R(t_1, \ldots, t_k)$ or $t_1 = t_2$, where R is k-ary a relation symbol and $t_1, \ldots, t_k$ are $\mathcal{L}$-terms.

The set of $\mathcal{L}$**-formulae** is defined recursively as: (1) atomic formulae are $\mathcal{L}$-formulae; (2) if ϕ, ψ are $\mathcal{L}$-formulae and $\star$ is any binary connective of first order logic, then $(\phi \star \psi)$ and $\neg\phi$ are $\mathcal{L}$-formulae; and (3) if ϕ is an $\mathcal{L}$-formula, x is a variable and Q a quantifier, then $(Qx\phi)$ is an $\mathcal{L}$-formula.

Definition 6.2. An $\mathcal{L}_\in$-formula is Σ_1 if it is of the form $\exists x\phi$, with ϕ a Δ_0-formula; and is Π_1 if it is of the form $\forall x\phi$, with ϕ a Δ_0-formula.

Definition 6.3. Let M be a non-empty set. We say that $\phi(x_1,\dots,x_n)$ is:

- **upward absolute between** M **and** V if

$$\forall a_1,\dots,a_n \in M\,[(\langle M,\in\rangle \models \phi(a_1,\dots,a_n)) \implies \phi(a_1,\dots,a_n)]\,;$$

- **downward absolute between** M **and** V if

$$\forall a_1,\dots,a_n \in M\,[\phi(a_1,\dots,a_n) \implies (\langle M,\in\rangle \models \phi(a_1,\dots,a_n))]\,;$$

- **absolute between** M **and** V if it is both upward and downward absolute, that is,

$$\forall a_1,\dots,a_n \in M\,[(\langle M,\in\rangle \models \phi(a_1,\dots,a_n)) \iff \phi(a_1,\dots,a_n)]\,.$$

From the definition, it follows that ϕ is upward absolute if and only if $\neg\phi$ is downward absolute, and that if ϕ,ψ are upward or downward absolute, then so are $\phi\wedge\psi$ and $\phi\vee\psi$. Therefore the class of absolute formulae between M and V is closed under all connectives.

Lemma 6.1. *Let M be a non-empty transitive set.*

1. *Every Δ_0-formula is absolute between M and V.*
2. *Every Σ_1 (resp. Π_1) is upward (resp. downward) absolute between M and V.*

Proof. 1. Note that every quantifier-free formula is absolute between $M\neq\emptyset$ and V, and therefore it is enough to consider formulae of the form $\forall y\in x_i\phi(y,x_1,\dots,x_n)$. Fix $a_1,\dots,a_n\in M$. Since M is transitive,

$$\begin{aligned}\langle M,\in\rangle \models \forall y\in x_i\phi[\vec{a}] &\iff \forall b\in M(b\in a_i \implies \langle M,\in\rangle\models\phi[b,\vec{a}])\\ &\iff \forall b\in a_i\langle M,\in\rangle\models\phi[b,\vec{a}]\\ &\iff \forall y\in a_i\phi(\vec{a}).\end{aligned}$$

2. It is enough to prove that Σ_1 formulae are upward absolute. Let $\phi(y_1,\dots,y_k,x_1,\dots,x_n)$ a Δ_0-formula, and let $a_1,\dots,a_n\in M$. Suppose that $\langle M,\in\rangle\models\exists y_1,\dots,y_k\phi[a_1,\dots,a_n]$. Fix $b_1,\dots,b_k\in M$ such that $\langle M,\in\rangle\models\phi[b_1,\dots,b_k,a_1,\dots,a_n]$. By the preceding point, $\phi(b_1,\dots,b_k,a_1,\dots,a_n)$ holds, and hence $\exists y_1,\dots,y_k\phi(a_1,\dots,a_n)$. □

Now it is straightforward to prove the following theorem, which will be of great help for our purpose:

Theorem 6.2. *Let M a non-empty transitive set. Then,*

1. *$\langle M,\in\rangle$ satisfies the Axioms of Extensionality and Foundation.*
2. *If $\{a,b\}\in M$ for all $a,b\in M$, then $\langle M,\in\rangle$ satisfies the Axiom of Pairing.*
3. *If $\bigcup a\in M$ for all $a\in M$, then $\langle M,\in\rangle$ satisfies the Axiom of Union.*
4. *If $\forall a\in M(\mathcal{P}(a)\cap M\in M)$, then $\langle M,\in\rangle$ satisfies the Power Set Axiom.*
5. *If $\omega\in M$, then $\langle M,\in\rangle$ satisfies the Axiom of Infinity.*
6. *If $\forall a\in M\forall b\subseteq a(b\in M)$ then $\langle M,\in\rangle$ satisfies the Axiom (schema) of Comprehension.*
7. *If $\forall a\in M$ and $\forall f: a\to M$ there is $b\in M$ such that $\operatorname{ran}(f)\subseteq b$, then $\langle M,\in\rangle$ satisfies the Axiom (schema) of Replacement.*

8. *$\langle M, \in \rangle$ satisfies the Axiom of Choice if and only if $\forall \mathcal{F} \in M(\forall X \in \mathcal{F}(X \neq \emptyset) \implies \exists f \in M(f$ is a choice function for $\mathcal{F}))$.*

Proof.

1. (Extensionality and Foundation) The Axioms of Extensionality and Foundation are the universal closure of the Δ_0-formulae
$$\forall z \in x(z \in y) \wedge \forall z \in y(z \in x) \implies x = y$$
and
$$\exists y \in x(y = y) \implies \exists y \in x \forall z \in y(z \notin x),$$
so they are downward absolute. Since both axioms hold in V, they hold in $\langle M, \in \rangle$.

2. (Pairing) Same reasoning, since $z = \{x, y\}$ is Δ_0.

3. (Union) $v = \bigcup u$ is Δ_0.

4. (Power Set) Fix $a \in M$ and let $b = \mathcal{P}(a) \cap M$. As $z \subseteq x$ is Δ_0, then $\langle M, \in \rangle$ satisfies $\forall z(z \subseteq x \iff z \in y)$ (the Power Set Axiom), giving x, y the values of a, b.

5. (Infinity) The Axiom of Infinity is $\exists x \phi(x)$, where $\phi(x)$ is the Δ_0-formula $\emptyset \in x \wedge \forall y \in x(\mathrm{S}(y) \in x)$, so by absoluteness $\langle M, \in \rangle$ satisfies the Axiom of Infinity if and only if $\exists x \in M \phi(x)$. As ω satisfies ϕ, if $\omega \in M$ then $\langle M, \in \rangle$ satisfies the Axiom of Infinity.

6. (Comprehension) We must show that for any formula $\phi(x, y, \vec{w})$ and given $a, \vec{c} \in M$, the set $b = \{d \in a \mid \langle M, \in \rangle \vDash \phi[d, a, \vec{c}]\}$ belongs to M. This follows directly from the assumption of the statement and by $b \subseteq a$.

7. (Replacement) We must show that, given $\phi(x, y, z, \vec{w})$ and $a, \vec{c} \in M$, if $\langle M, \in \rangle \vDash \forall x \in z \exists! y \phi[a, \vec{c}]$ then there is $b \in M$ such that $\langle M, \in \rangle \vDash \forall x \in z \exists y \in v \phi[a, \vec{c}, b]$, with b assigned to the variable v. Then $\phi, a, \vec{c}$ yield a function $f : a \to M$ and by hypothesis there is $b \in M$ such that $\mathrm{ran}(f) \subseteq b$, which is the b we are looking for.

8. (AC) The result follows from the fact that the formula $\phi(f, x)$ that says "$x \neq \emptyset$, every element of x is non empty and $f : x \to \bigcup x$ is a choice function" is Δ_0. □

And now we have a clear answer to our question:

Theorem 6.3.

1. *All axioms of ZFC except the Axiom of Infinity hold in V_ω.*

2. *All axioms of ZF except possibly Replacement hold in V_λ, for $\lambda > \omega$ limit.*

3. *Assuming choice, AC holds in V_λ, for λ limit.*

Proof.

1. It is enough to check it for Replacement and Choice. Since every V_n is finite, every element of V_ω is finite (every element of V_ω belongs to some V_n). It follows that every $x \in V_\omega$ is well-orderable and therefore AC holds. On the other hand, if $A \in V_\omega$ and $F : A \to V_\omega$, then $F[A]$ is finite, $F[A] = \{a_0, \dots, a_n\}$. For every $i \leq n$, let $m_i < \omega$ be such that $a_i \in V_{m_i}$. Then $F[A] \subseteq V_m$, where $m = \max\{m_0, \dots, m_n\}$, hence $F[A] \in V_{m+1}$.

2. If $\lambda > \omega$, then $\omega \in V_\lambda$. It suffices to apply the previous theorem.

3. Let $\mathcal{F} \in V_\lambda$ a non empty family of non empty sets, by AC there is a choice function $f : \mathcal{F} \to \bigcup \mathcal{F}$. If $\alpha < \lambda$ is such that $\mathcal{F} \in V_{\alpha+1}$, then $f \in \mathcal{P}(\mathcal{F} \times \bigcup \mathcal{F}) \subseteq V_{\alpha+3} \implies f \in V_{\alpha+3}$, and the result comes from the last theorem. □

Notice that the second statement of the theorem says "except *possibly* Replacement". In general, $V_\lambda \not\models$ Replacement, for $\lambda > \omega$ limit. The first limit ordinal for which Replacement is not satisfied is $\omega+\omega$. Indeed, if $V_{\omega+\omega} \models$ Replacement, then consider the function $f : \omega \to V_{\omega+\omega}$ given by $f(\alpha) = \omega + \alpha$. Thus $\mathrm{ran}(f) = \omega + \omega$ must be a set in $V_{\omega+\omega}$, that is, $\omega+\omega \in V_{\omega+\omega}$, a contradiction.

An interesting observation is that $V_{\omega+\omega}$ contains everything needed for elementary mathematical analysis, so for most of basic mathematics it is not necessary to assume Replacement. Indeed, we could identify $\mathbb{N} = \omega$, and $\mathbb{R} = 2^\omega$, and construct $\mathbb{R}^n$ and $\mathbb{C}^n$ for any n, which will be in $V_{\omega+\omega}$, as well as integration, derivation or the Lebesgue measure can be seen as functions that are contained in $V_{\omega+\omega}$.

This does not diminish the importance of Replacement in set theory. Quite the opposite, we have just seen that it is necessary to show the existence of $\omega+\omega$, and everything above.

7 Inaccessible cardinals

Definition 7.1.

- A cardinal κ is a **limit** if $\lambda^+ < \kappa$ for every $\lambda < \kappa$.
- A cardinal κ is a **strong limit** if $2^\lambda < \kappa$ for every $\lambda < \kappa$.

An equivalent definition for limit (resp., strong limit) cardinals is that their index in the aleph (resp., beth) notation is a limit ordinal. The first limit cardinal is $\aleph_\omega$, which is not regular since $\mathrm{cf}(\aleph_\omega) = \omega$, as we pointed out earlier. Clearly, every strong limit cardinal is a limit cardinal. If the Generalized Continuum Hypothesis holds, then the converse is true.

Definition 7.2. A cardinal $\kappa > \omega$ is **weakly inaccessible** if it is regular and a limit, and it is **strongly inaccessible** if it is regular and a strong limit.

If κ is weakly inaccessible then $\kappa = \aleph_\kappa$: let $\aleph_\kappa$ be a weakly inaccessible cardinal. It is a limit, so its index κ is a limit ordinal. Now, the κ-sequence $\langle \aleph_\alpha : \alpha < \kappa \rangle$ is cofinal in $\aleph_\kappa$, giving us $\mathrm{cf}(\aleph_\kappa) \le \kappa$. By regularity, $\mathrm{cf}(\aleph_\kappa) = \aleph_\kappa$, so the last inequality becomes $\aleph_\kappa \le \kappa$, which together with the general inequality $\kappa \le \aleph_\kappa$ yields the desired result. Analogously, if κ is strongly inaccessible then $\beth_\kappa = \kappa$.

The converse does not hold: the least fixed point of the $\aleph$ function is of cofinality ω, hence not regular. Indeed, if κ is the least fixed point of the $\aleph$ function, then the ω-sequence $\aleph_0, \aleph_{\aleph_0}, \aleph_{\aleph_{\aleph_0}}, \dots$ is cofinal in κ.

There are two ways of understanding inaccessible cardinals, and why they are called *inaccessible*. The first one is noticing their combinatorial properties, and the second one is that they are big enough to satisfy that $V_\kappa \models \mathrm{ZFC}$.

7.1 Combinatorics of inaccessible cardinals

We have implicitly showed an equivalent way of defining inaccessible ordinals, which gives more of a combinatorial insight:

Proposition 7.1. *An ordinal κ is weakly inaccessible if and only if it is regular and it is a fixed point of the aleph function, i.e., $\aleph_\kappa = \kappa$, and is strongly inaccessible if and only if it is regular and $\beth_\kappa = \kappa$.*

We know that for every $\alpha < \kappa$, $2^\alpha < \kappa$. Notice that 2^α is the power set of α, meaning that κ cannot be reached by the power set operation. Also, the regularity of κ will imply that κ is also closed under unions of collections of size $< \kappa$. Therefore strongly inaccessible cardinals cannot be reached by taking unions or power sets of sets below in any way, hence the name of inaccessibles.

Let us present these ideas rigorously. We shall start by defining the following notion:

Definition 7.3. A **universe** is a transitive set U such that:

- $\omega \in U$,

- U is closed under the operation $x \mapsto \mathcal{P}(x)$,

- $\forall I \in U \forall f : I \to U (\bigcup_{i \in I} f(i) \in U)$.

Lemma 7.1. *If U is a universe, then*

1. $x \subseteq y \in U \implies x \in U$,

2. $x, y \in U \implies x \cup y \in U$,

3. $x, y \in U \implies \{x, y\} \in U$ *(and hence $(x, y) \in U$),*

4. $x, y \in U \implies x \times y \in U$ *and* ${}^x y \in U$,

5. *if $f : I \to U$ and $I \in U$ then* $\mathrm{ran} f \in U$ *and* $f \in U$.

Proof. 1. $x \in \mathcal{P}(y) \in U$, and transitivity implies $x \in U$.

2. $2 \in \omega \in U$, so $2 \in U$ again by transitivity, and $x \cup y = \bigcup_{i \in 2} f(i) \in U$, where $f : 2 \to U$ is defined by $f(0) = x$ and $f(1) = y$.

3. $x \in U \implies \{x\} \in \mathcal{P}\mathcal{P}(x) \in U$, so $\{x\} \in U$. Thus $x, y \in U \implies \{x\}, \{y\} \in U$, so $\{x, y\} = \{x\} \cup \{y\} \in U$, and therefore $(x, y) \in U$.

4. $x, y \in U \implies x \times y \subseteq \mathcal{P}\mathcal{P}(x \cup y) \in U \implies x \times y \in U$. The second assertion follows from ${}^x y \subseteq \mathcal{P}(x \times y) \in U$.

5. Let $g : I \to U \ni i \mapsto \{f(i)\}$, then $\mathrm{ran}(f) = \bigcup_{i \in I} g(i) \in U$. Moreover, $f \subseteq I \times \mathrm{ran}(f) \in U$, hence $f \in U$. □

Theorem 7.2. *U is a universe if and only if $U = V_\kappa$ for some strongly inaccessible cardinal κ.*

Proof. ($\implies$) Suppose that U is a universe, and let $\kappa = U \cap \mathrm{Ord}$. U is closed under the successor ordinal operation, so κ is limit, and $\kappa \notin U$. If $\gamma < \kappa$ and $f : \gamma \to \kappa$, then $\sup(\mathrm{ran}(f)) = \bigcup_{\alpha < \kappa} f(\alpha) \in U$, hence f cannot be cofinal in κ. Thus κ is regular. If $2^\lambda \geq \kappa$ for some infinite cardinal $\lambda < \kappa$, there would exist a surjection $f : \mathcal{P}(\lambda) \twoheadrightarrow \kappa \subseteq U$. But $\mathcal{P}(\lambda) \in U$, a contradiction. It follows that κ is a strongly inaccessible cardinal.

We have to check that $V_\kappa = U$. First, let us show that $V_\alpha \in U$ for every $\alpha < \kappa$, and so $V_\kappa \subseteq U$. As U is closed under the $\mathcal{P}$ operation, then $\bar{\kappa} = \{\alpha < \kappa : V_\alpha \in U\}$ is a limit ordinal. All we have to see is that $\bar{\kappa} = \kappa$. If $\bar{\kappa} < \kappa$, then using the function $\bar{\kappa} \to \kappa \ni \alpha \mapsto V_\alpha$, we would have that $V_{\bar{\kappa}} = \bigcup_{\alpha < \bar{\kappa}} V_\alpha \in U$, so that $\bar{\kappa} \in \bar{\kappa}$, a contradiction. Therefore $V_\kappa \subseteq U$.

Now let us prove that $U \subseteq V_\kappa$, and we will have proved the equality. If $U \nsubseteq V_\kappa$, for each $x \in U \setminus V_\kappa$, let $\mathrm{rank}(x) := \min\{\alpha : x \in V_{\alpha+1}\}$, and let $y \in U \setminus V_\kappa$ be of least rank. Then $\mathrm{rank}(x) = \kappa$, so the map $x \to \kappa \ni y \mapsto \mathrm{rank}(y)$ is cofinal in κ, thus $\kappa = \sup_{y \in x} \mathrm{rank}(y) \in U$, a contradiction. Therefore $V_\kappa = U$, where κ is a strongly inaccessible cardinal.

($\impliedby$) Assume now that κ is a strongly inaccessible cardinal, and let us check that V_κ is a universe. Suppose $f : I \to V_\kappa$, with $I \in V_\kappa$. Then the function $I \to \kappa$ given by $i \mapsto \mathrm{rank}(f(i))$ is bounded in κ, since $|I| < \kappa$ (otherwise f would be cofinal, contradicting the regularity of κ). So $\mathrm{ran}(f) \subseteq V_\alpha$ for some $\alpha < \kappa$. Therefore $\bigcup_{i \in I} f(i) \subseteq V_\alpha$, and hence $\bigcup_{i \in I} f(i) \in V_{\alpha+1} \subseteq V_\kappa$. The other clauses in the definition of universe are immediate by the inaccessibility of κ. □

7.2 Model-theoretic properties of inaccessible cardinals

Now, we want to show that inaccessible cardinals are beyond ZFC in the sense that $V_\kappa \vDash$ ZFC for any strongly inaccessible cardinal κ, which will imply that we cannot prove the existence of κ from ZFC. This point of view shares fundamental ideas with the combinatorial one: in our process of "recreating" the universe with the V_αs, we cannot get to V_κ from the stages below, since we construct every V_α with unions and power sets.

Lemma 7.3. *Let κ be strongly inaccessible. Then $|V_\alpha| < \kappa$ for all $\alpha < \kappa$. In particular, $|X| < \kappa$ for all $X \in V_\kappa$.*

Proof. By induction on α. If $|V_\alpha| < \kappa$ then $|V_{\alpha+1}| = 2^{|V_\alpha|} < \kappa$, as κ is strong limit. For α limit, $|V_\alpha| = |\bigcup_{\beta<\alpha} V_\beta| \leq |\alpha| \cdot \sup_{\beta<\alpha} |V_\beta| < \kappa$, as κ is regular. □

Theorem 7.4. *If κ is strongly inaccessible, then $V_\kappa \vDash$* ZFC.

Proof. By Theorem 6.3, given that $\kappa > \omega$ is a limit ordinal, we already have that V_κ satisfies the axioms of Extensionality, Foundation, Comprehension, Pairing, Union, Infinity, Power Set and Choice. Therefore, it is enough to show that V_κ satisfies Replacement, that is, for every function F from some $X \in V_\kappa$ into V_κ, then $F \in V_\kappa$. Since κ is inaccessible, $|V_\kappa| = \kappa$ and $|X| < \kappa$ for all $X \in V_\kappa$. If F is a function from $X \in V_\kappa$ into V_κ, then $|F(X)| \leq |X| < \kappa$ and, since κ is regular, $F(X) \subset V_\alpha$ for some $\alpha < \kappa$ (otherwise, F would be unbounded in κ such that $|F(X)| < \kappa$, contradicting regularity of κ). It follows that $F \in V_\kappa$. □

This theorem has a critical consequence: since $V_\kappa \vDash$ ZFC, proving the existence of κ (and therefore of V_κ) would mean proving the consistency of ZFC from ZFC, which contradicts Gödel's Second Incompleteness Theorem: *in any consistent formal system that is sufficiently strong to express arithmetic, it is impossible to prove the consistency of the system within itself.* In other words, if a formal system is consistent (meaning it does not lead to contradictions), then the system cannot prove its own consistency. Therefore,

Metatheorem 7.1. *The existence of inaccessible cardinals cannot be proved in ZFC (if ZFC is consistent).*

However, the existence of an inaccessible cardinal is consistent with ZFC, meaning that we can add it as an axiom (similarly as it is done for the existence of ω):

Axiom of Inaccessible Cardinals: $\forall \alpha \in \mathrm{Ord}\, \exists \kappa (\kappa \text{ is a strongly inaccessible cardinal} \wedge \alpha < \kappa)$.

Now, this axiom yields a transfinite sequence $\theta_0 < \theta_1 < \cdots < \theta_\alpha < \cdots$ of inaccessible cardinals, one for each ordinal α: let θ_0 be the first inaccessible cardinal. Let $\theta_{\alpha+1}$ be the first inaccessible cardinal that satisfies $\exists \kappa (\kappa \text{ is strongly inaccessible} \wedge \theta_\alpha < \kappa)$. Lastly, for λ limit, let θ_λ be the first cardinal that satisfies $\exists \kappa (\kappa \text{ is strongly inaccessible} \wedge \forall \alpha (\alpha < \lambda \rightarrow \theta_\alpha < \kappa))$.

Then, we can define *hyperinaccessible* cardinals to be the regular cardinals κ such that $\theta_\kappa = \kappa$ (κ is strongly inaccessible and there are κ strongly inaccessible cardinals below κ). It can be shown that if κ is the first hyperinaccessible cardinal, then V_κ is a model of ZFC+Axiom of Inaccessible Cardinals+"there are no hyperinaccessible cardinals". Therefore, we have found bigger cardinals whose existence cannot be proved (nor disproved) in ZFC+Axiom of Inaccessible Cardinals.

Now, if we add an axiom to ensure the existence of hyperinaccessible cardinals, in a similar way as before, we could define: κ is *hyper-hyperinaccessible* if κ is hyperinaccessible and there are κ hyperinaccessible cardinals below κ. Again, let κ the first such cardinal, and $V_\kappa \vDash$ZFC+Axiom of Hyperinaccessibles+"there are no hyper-hyperinaccessible cardinals".

And we could go on with this process indefinitely, which prompts us to search for more general way of postulating the existence of inaccessible cardinals.

8 Hyperinaccessible cardinals

We shall start by making some definitions that will allow us to generalize that process of getting bigger and bigger inaccessible cardinals.

Definition 8.1. A function $f : \mathrm{Ord} \to \mathrm{Ord}$ is a **normal function** if

1. f is *increasing*: $\alpha < \beta \implies f(\alpha) < f(\beta)$; and
2. f is *continuous*: if α is limit, then $f(\alpha) = \bigcup_{\xi<\alpha} f(\xi)$.

Notice that both $\aleph$ and $\beth$ are normal functions.

Definition 8.2. A subset $X \subset \mathrm{Ord}$ is **closed** (with respect to the order topology on the ordinals) if $\emptyset \neq Y \subset X \implies \bigcup Y \in X$ (equivalently, $\sup Y \in X$).

Lemma 8.1. *The range of a normal function is closed and unbounded in* Ord*, and every closed unbounded class of ordinals is the range of a unique normal function, which is its enumerating function.*

Proof. The range is unbounded because $\alpha \leq f(\alpha)$, and is closed since f is continuous and Ord is closed.

Now, given a closed and unbounded class X, transfinite induction lets us consider its enumeration f, which will be a normal function, and will be defined for all ordinals since X must be a proper class. □

Definition 8.3. An ordinal α is a **fixed point** of f if $f(\alpha) = \alpha$.

Theorem 8.2. *The fixed points of a normal function form a closed unbounded class.*

Proof. Let $f : \mathrm{Ord} \to \mathrm{Ord}$ a normal function.

1. The class of fixed points is unbounded: given any $\alpha \in \mathrm{Ord}$, define:

$$f^0(\alpha) = \alpha, \quad f^{n+1}(\alpha) = f(f^n(\alpha)), \quad f^\omega(\alpha) = \bigcup_{n<\omega} f^n(\alpha).$$

If α is a fixed point, then all of these will be α. If not, an easy induction shows that $\alpha < f(\alpha)$, and since f is increasing, the sequence $f^n(\alpha)$ will be strictly increasing for $n < \omega$. Then $\bar{\alpha} := f^\omega(\alpha)$ is a limit ordinal, and since f is continuous,

$$f(\bar{\alpha}) = \bigcup_{\xi<\bar{\alpha}} f(\xi).$$

But if $\xi < \bar{\alpha}$ then $\xi < f^n(\alpha)$ for some $n < \omega$, so $f(\xi) < f^{n+1}(\alpha)$ and thus

$$f(\bar{\alpha}) \leq \bigcup_{n<\omega} f^{n+1}(\alpha) = \bar{\alpha}.$$

Hence, for every ordinal α, $\bar{\alpha}$ is a fixed point, and $\alpha \leq \bar{\alpha}$, so the fixed points are unbounded in Ord.

2. The class of fixed points is closed: let Y be a set of fixed points such that $\bigcup Y \notin Y$ (otherwise it is trivial). Then $\bigcup Y$ must be a limit, and $\alpha < \bigcup Y \iff \exists \xi \in Y(\alpha < \xi)$. So $f(\bigcup Y) = \bigcup_{\xi \in Y} f(\xi) = \bigcup_{\xi \in Y} \xi = \bigcup Y$, and therefore $\bigcup Y$ is a fixed point. □

Corollary 8.2.1. *The enumeration of the fixed points of any normal function is also a normal function.*

Notice that therefore the enumeration of fixed points will have its own class of fixed points, which we can enumerate, yielding another normal function with its own fixed points, and so on. This is exactly what we were doing when defining hyperinaccessible, hyper-hyperinaccessible, etc. cardinals: the functions $\aleph$ and $\beth$ are normal functions. Fixed points of these functions will be ordinals α such that $\aleph_\alpha = \alpha$ and $\beth_\alpha = \alpha$, and the regular fixed points of these two functions will be the inaccessible cardinals (weak and strongly, respectively): if α is a fixed point of $\aleph$ (resp., $\beth$), then α is a limit ordinal, and in particular it is a limit cardinal since it is equal to $\aleph_\alpha$, whose index is a limit ordinal. Adding the condition of regularity we get that α is a weakly (resp., strongly) inaccessible cardinal. So we can generalize the Axiom of Inaccessible Cardinals:

Generalized Axiom of Inaccessible Cardinals: Every normal function has a regular fixed point.

It is indeed a generalization: we can enumerate with a normal function the beth numbers greater than a given ordinal α, and it will have a regular fixed point greater than α, which will be a strongly inaccessible cardinal. Therefore the original Axiom of Inaccessible Cardinals follows. Now, we can enumerate the class of strongly inaccessibles by a normal function, and its fixed points will be the hyperinaccessible cardinals. And of course we can repeat this process to get classes of bigger and bigger cardinals.

Definition 8.4 (Normal function on an ordinal)**.** For a given ordinal α, a function $f : \beta \to \alpha$ is a normal function on α if it is increasing, continuous, and its range is unbounded in α.

We are only interested in this notion when α is a regular cardinal, and we have that:

Theorem 8.3. *The following are equivalent:*

1. *Every normal function on α has a fixed point.*

2. *α is regular and $\alpha > \omega$.*

Proof. (2) $\Longrightarrow$ (1): Let $f : \beta \to \alpha$ a normal function. Since α is regular and f is unbounded in α, we must have $\beta \leq \alpha$; and since f is increasing, we must have $\beta = \alpha$. Then, as $\alpha > \omega$, we can construct a fixed point of f as in the proof of Theorem 8.2.

(1) $\Longrightarrow$ (2): First, $\alpha > \omega$ because ω has normal functions without fixed points (for example, $g : \omega \to \omega$ given by $g(n) = n+1$). Now, α is limit since, otherwise, no function could be unbounded in α; and if $\mathrm{cf}(\alpha) = \beta < \alpha$, then we can find $f : \beta \to \alpha$ unbounded, increasing and continuous, and such that $f(\xi) > \beta$ for all $\xi < \beta$, i.e., if α is not regular then there is a normal function on α without fixed points. □

The following definition covers all possible kinds of hyperinaccessibles. In the next section, we will use the concepts which have been just introduced in order to find cardinals even larger than any hyperinaccessible.

Definition 8.5 (Hyperinaccessible cardinals)**.**

- κ is *0-weakly hyperinaccessible* if κ is weakly inaccessible.

- κ is $\alpha+1$-*weakly hyperinaccessible* if κ is regular and there are κ α-weakly hyperinaccessible cardinals below κ.

- For λ limit, κ is λ-*weakly hyperinaccessible* if κ is α-weakly hyperinaccessible for all $\alpha < \lambda$.

We can define analogously α-*strongly hyperinaccessible* cardinals.

9 Mahlo cardinals

Definition 9.1. An ordinal κ is **weakly Mahlo** if every normal function on κ has a regular fixed point, and κ is **strongly Mahlo** if every normal function on κ has a strongly inaccessible fixed point.

By Theorem 8.3, weakly and strongly Mahlo ordinals are regular, so they are in fact cardinals. We shall use just the term *Mahlo cardinals* for strongly Mahlo cardinals.

These cardinals are indeed inaccessible and hence their existence goes beyond ZFC as well:

Lemma 9.1. *If κ is weakly Mahlo, then κ is weakly inaccessible; if κ is strongly Mahlo, then κ is strongly inaccessible.*

Proof. We have just showed that κ is regular. Suppose that it is not a limit cardinal, then κ is $\aleph_{\alpha+1}$ for some α, and $f:\kappa\to\kappa$ given by $f(\xi)=\alpha+\xi$ is a normal function on κ whose fixed points cannot be regular: if ξ is a fixed point of f, then $\alpha<\xi$. Now for any cofinal sequence in α, $\langle\alpha_\beta:\beta<\mathrm{cf}(\alpha)\rangle$, the sequence $\langle\alpha_\beta+\xi:\beta<\mathrm{cf}(\alpha)\rangle$ is cofinal in $\alpha+\xi=\xi$, but $\mathrm{cf}(\alpha)\le\alpha<\xi$, so $\mathrm{cf}(\xi)<\xi$, thus ξ is not regular. So κ cannot be weakly Mahlo.

If κ is not a strong limit cardinal, then $2^\alpha\ge\kappa$ for some $\alpha<\kappa$, and again $f:\kappa\to\kappa\ni f(\xi)=\alpha+\xi$ is a normal function. In this case its fixed points cannot be strong limit cardinals since if β is a fixed point of f, then $\alpha<\beta<2^\alpha$ ($\alpha<\beta$ because otherwise it would not be a fixed point, and $\beta<2^\alpha$ since $\beta=f(\beta)<\kappa<2^\alpha$), so the fixed points of f are not strongly inaccessible and therefore κ is not strongly Mahlo. □

When discussing hyperinaccessible cardinals, we apparently had a method of defining always larger inaccessible cardinals (recall Definition 8.5). The definition of Mahlo cardinals, however, trascends this method, and therefore Mahlo cardinals are larger than any cardinal that can be defined with the methods presented so far. The following theorem formalizes this last statement.

Theorem 9.2. *If κ is weakly Mahlo, then κ is α-weakly hyperinaccessible for all $\alpha\le\kappa$.*
If κ is strongly Mahlo, then κ is α-strongly hyperinaccessible for all $\alpha\le\kappa$.

Proof. The proof is the same for strong or weakly hyperinaccessibles. We shall write α-hyp to shorten notation, corresponding to either the strong or weak case, depending on which we are considering.

First, note that if κ is weakly Mahlo, any normal function on κ must have κ regular fixed points. For this, suppose $f:\kappa\to\kappa$ is a normal function on κ. Then for any fixed point $\beta<\kappa$, we can define $g:\kappa\to\kappa$ such that $g(\alpha)=f(\beta+\alpha)$, and then g is also normal on κ, so g has a regular fixed point δ. But then $\delta=f(\beta+\delta)$, and since f is increasing, $f(\beta+\delta)\ge\beta+\delta$, so we have $\beta+\delta=\delta=f(\beta+\delta)$, i.e., δ is a regular fixed point of f greater than β. Since κ is regular, this implies that there must be κ regular fixed points. This gives the induction step from $\alpha+1$ to $\alpha+2$ immediately: Suppose that κ is $\alpha+1$-hyp, then κ is a regular fixed point of f_α, where f_α is the normal function enumerating the closure of the class of α-hyp cardinals. Hence $f_\alpha\restriction\kappa$ is a normal function on κ, and so has κ regular fixed points. Each of these is then $\alpha+1$-hyp and less than κ, i.e., κ is an $\alpha+2$-hyp.

The induction step for λ limit is trivial, since we have defined λ-hyp as: α-hyp for all $\alpha<\lambda$. It remains to show that if κ is λ-hyp, where $\lambda<\kappa$ is a limit, then κ is $\lambda+1$-hyp.

So suppose that κ is λ-hyp, and $\lambda<\kappa$. Then for $\delta<\lambda$, let X_δ be the closure of $\{\alpha<\kappa:\alpha\text{ is }\delta\text{-hyp}\}$. Then since κ is $\delta+1$-hyp, $|X_\delta|=\kappa$ for every $\delta<\lambda$, and all regular members of X_δ are δ-hyp. We need to show that $X_\lambda=\bigcap_{\delta<\lambda}X_\delta$ has cardinality κ also. Since X_λ is also closed, its enumerating function will be normal on κ and so will have κ regular fixed points: these will all be λ-hyp, and therefore κ will be $\lambda+1$-hyp.

So suppose that $\beta<\kappa$, and set

$$\alpha_\delta=\min(X_\delta\setminus\beta)\text{ for }\delta<\lambda,$$

and let $\gamma=\bigcup_{\delta<\lambda}\alpha_\lambda$. Since κ is regular and $\lambda<\kappa$, we have $\gamma<\kappa$. But if $\delta<\delta'<\lambda$, then $\alpha_{\delta'}\in X_\delta$, and then γ is a limit of members of X_δ, which is closed, so $\gamma\in X_\delta$ for every $\delta<\lambda$, i.e., $\gamma\in X_\lambda$. Hence X_λ is cofinal in κ and, since κ is regular, it must have cardinality κ, as required. □

10 The Reflection Principle

In this section, we explore an important property of ZF that in simple words can be expressed as *for any formula, there are sets in V that behave just like V with respect to that formula*. Also, this property is in fact equivalent to (Axiom of Infinity ∧ Axiom of Replacement), so it can be seen as an insight into what these axioms say about the cumulative hierarchy. After that, we will prove that inaccessible cardinals satisfy a much stronger reflection property. The original Reflection Principle was stated by Montague and Lévy, and we will express it in easier terms following the exposition of [3, chapters 3, §6 and 4, §4].We shall start with a couple of definitions:

Definition 10.1. Let ϕ be a formula and let X any term not occurring in ϕ, then the formula ϕ *relativized to* X, which we denote by ϕ^X, is the formula that results from ϕ by restricting all the quantifiers to X, i.e., replacing $\forall x$ by $\forall x \in X$, and $\exists x$ by $\exists x \in X$.

It is clear that ϕ^X says about the sets in X what the formula says about the sets in V. Now we can express the notion of reflection:

Definition 10.2. We say that X *reflects the true situation for* ϕ, or just X *reflects* ϕ if the following holds:

$$\forall x_1,\dots,x_n \in X(\phi(x_1,\dots,x_n) \leftrightarrow \phi^X(x_1,\dots,x_n)),$$

where $x_1,\dots,x_n$ are all the free variables of ϕ.

In other words, X reflects ϕ if, with parameters from X, the truth value of ϕ is the same whether we look at it from X or V. We can already state the *Reflection Principle*, which says that for any formula ϕ, there are arbitrary high levels of the cumulative hierarchy which reflect ϕ:

Theorem 10.1 (Montague, Lévy)**.** *If $\phi(x_1,\dots,x_n)$ is a formula without abstraction terms, with no free variables*[2] *other than $x_1,\dots,x_n$, then*

$$(\mathrm{R}_0) \qquad \mathrm{ZF} \vdash \forall\alpha\exists\beta > \alpha\,\forall x_1,\dots,x_n \in V_\beta[\phi(x_1,\dots,x_n) \leftrightarrow \phi^{V_\beta}(x_1,\dots,x_n)]$$

The proof may seem complicated, so let us first give a sketch of the proof:

1. Let ϕ be a formula, and write ϕ as a string of m quantifiers followed by a formula with no quantifiers. Now, for each $1 \le r \le m$ consider the formula ψ_r resulting from removing the first r quantifiers.
2. For each V_ζ, given a tuple of elements of V_ζ, there is a least V_δ where $Q_r y_r \psi_r \leftrightarrow Q_r y_r \in V_\delta \psi_r$, so we can consider the set of every such δ for every tuple of elements in V_ζ, and write $f_r(\zeta)$ for the least ordinal greater than ζ and than all those δs.
3. Now, we construct a sequence $f^n, n < \omega$ such that every f^{n+1} is the maximum among $f_1^n,\dots,f_m^n$, and starting from $f^0 = \zeta$. The supremum of this sequence will be the ordinal we are looking for.

Proof. We may assume that ϕ is written in prenex normal form[3], and we shall write it as

$$Q_1 y_1 \dots Q_m y_m \psi(x_1,\dots,x_n,y_1,\dots,y_m),$$

where each Q_i is either $\exists$ or $\forall$. We write ψ_r for

$$Q_{r+1} y_{r+1} \dots Q_m y_m \psi(x_1,\dots,x_n,y_1,\dots,y_m),$$

for $r = 0,1,\dots,m$ (so ψ_0 is ϕ and ψ_m is ψ).

Now, given r, $1 \le r \le m$, and sets $x_1,\dots,x_n,y_1,\dots,y_{r-1}$, we set:

[2] A variable x is *free* in a formula ϕ if it does not occur under the scope of any quantifier ($\forall$ or $\exists$). Otherwise it is *bound*.

[3] That is, it is written as a string of quantifiers and bound variables, followed by a quantifier-free part. Every formula is logically equivalent to a formula in prenex normal form, and an algorithm to find such a formula is given in [1, page 33].

- if Q_r is $\exists$, then $g_r(x_1,\dots,x_n,y_1,\dots,y_{r-1}) :=$ *the least ordinal* δ *such that*

$$\exists y_r \psi_r \to \exists y_r \in V_\delta \psi_r$$

(and δ is 0 if $\neg\exists y_r \phi_r$).

- If Q_r is $\forall$, then $g_r(x_1,\dots,x_n,y_1,\dots,y_{r-1}) :=$ *the least* δ *such that*

$$\exists y_r(\neg\psi_r) \to \exists y_r \in V_\delta(\neg\psi_r).$$

Now using the Axiom of Replacement, given any ordinal ζ, the collection

$$Y = \{g_r(x_1,\dots,x_n,y_1,\dots,y_{r-1}) \mid x_1,\dots,x_n,y_1,\dots,y_{r-1} \in V_\zeta\}$$

is a set of ordinals, so it has a supremum. We set $f_r(\zeta)$ to be the least ordinal greater than ζ and than all ordinals in Y, that is, $f_r(\zeta) := \max(\zeta+1, \sup(Y)+1)$. Then put

$$\begin{aligned} f^0(\zeta) &= \zeta, \\ f^1(\zeta) &= \max_{1\le r\le m}(f_r(\zeta)), \\ f^{n+1}(\zeta) &= f^1(f^n(\zeta)), \quad \text{for } n \ge 1. \end{aligned}$$

Then again by Replacement and also the Axiom of Infinity, since we need that ω is a set, we put

$$f(\zeta) = \sup_{n<\omega}(f^n(\zeta)).$$

Now $f(\zeta)$ will have the following properties:

- $f(\zeta) > \zeta$,
- for all $x_1,\dots,x_n,y_1,\dots,y_m \in V_{f(\zeta)}$, and for all $1 \le r \le m$,

$$g_r(x_1,\dots,x_n,y_1,\dots,y_{r-1}) < f(\zeta).$$

This will hold since $f(\zeta)$ must be a limit ordinal, and so in fact we must have $x_1,\dots,x_n,y_1,\dots,y_m \in V_{f^p(\zeta)}$, for some $p < \omega$, so that

$$g_r(x_1,\dots,x_n,y_1,\dots,y_{r-1}) < f^{p+1}(\zeta) < f(\zeta).$$

But this means that

$$Q_r y_r \psi_r \leftrightarrow Q_r y_r \in V_{f(\zeta)} \psi_r,$$

by the definition of g_r. Write X for $V_{f(\zeta)}$; then, ψ_m has no quantifiers (so that $\psi_m^X = \psi_m$ and therefore X reflects ψ_m), so we have successively for $r = m, m-1, \dots, 0$ that:

$$\psi_r(x_1,\dots,x_n,y_1,\dots,y_{r-1}) \leftrightarrow \psi_r^X(x_1,\dots,x_n,y_1,\dots,y_{r-1}),$$

for $x_1,\dots,x_n,y_1,\dots,y_{r-1} \in X$; and for $r = 0$ this is

$$\phi(x_1,\dots,x_n) \leftrightarrow \phi^X(x_1,\dots,x_n).$$

That is, $V_{f(\zeta)}$ reflects ϕ. So R_0 is proved for ϕ. □

Now, since $V_\kappa \models$ ZFC for every inaccessible cardinal κ, we can expect that inaccessible cardinals themselves have some interesting reflection properties. We shall make some definitions first:

Definition 10.3. Let $\mathcal{A} \subset \mathcal{B}$ be two $\mathcal{L}$-structures for some language $\mathcal{L}$. We say that $\mathcal{A}$ is an *elementary substructure* of $\mathcal{B}$ (and $\mathcal{B}$ is an *elementary extension* of $\mathcal{A}$) if every formula is absolute between $\mathcal{A}$ and $\mathcal{B}$. Write $\mathcal{A} \prec \mathcal{B}$ for this.

Definition 10.4 (Skolem function)**.** Suppose that $\phi(x, y_1, \ldots, y_n)$ is an $\mathcal{L}$-formula for a structure $\mathcal{M}$ with domain M, with (at most) $x, y_1, \ldots, y_n$ free. Then a function $f : M^n \to M$ is a *Skolem function* for $\phi(x, y_1, \ldots, y_n)$ in $\mathcal{M}$ if for any $a_1, \ldots, a_n \in M$, if there is any $a \in M$ such that $\mathcal{M} \vDash \phi(a, a_1, \ldots, a_n)$, then $\mathcal{M} \vDash \phi(f(a_1, \ldots, a_n), a_1, \ldots, a_n)$.

Skolem functions will be useful to prove Theorem 10.3, thanks to the following lemma (we ommit its proof for being somewhat technical; it can be found in [3, page 86]):

Lemma 10.2. *Suppose that $\mathcal{A} \subseteq \mathcal{B}$ are two $\mathcal{L}$-structures, and for some collection F of Skolem functions, one for each formula of $\mathcal{B}$, A (the domain of $\mathcal{A}$) is closed under each member of F. Then $\mathcal{A} \prec \mathcal{B}$.*

Theorem 10.3. *If κ is strongly inaccessible, then there is $\alpha < \kappa$ such that $V_\alpha \prec V_\kappa$. Moreover, the set $\{\alpha < \kappa : V_\alpha \prec V_\kappa\}$ is closed unbounded on κ.*

Proof. Let $\beta < \kappa$ and let F be a set of Skolem functions for V_κ. Now, construct by ordinary induction the following sequence:

$$\alpha_0 = \beta,$$
$$\alpha_{n+1} = \min\{\delta < \kappa : F(V_{\alpha_n}) \subseteq V_\delta\},$$

where by $F(X)$ we mean $\{f(x_1, \ldots, x_n) : f \in F \wedge x_1, \ldots, x_n \in X\}$, which is a set in V_κ for any $X \in V_\kappa$ since $V_\kappa \vDash$ Replacement (as κ is inaccessible), so there exists $\delta < \kappa$ such that $F(X) \subseteq V_\delta$, and the sequence above is well defined. So let

$$\alpha = \sup_{n<\omega} \alpha_n.$$

Now, $\alpha < \kappa$ (as κ is regular) and $F(V_\alpha) \subseteq V_\alpha$: $x \in F(V_\alpha) \implies x = f(x_1, \ldots, x_n)$ for some $f \in F$ and $x_1, \ldots, x_n \in V_\alpha$, and therefore $x_1, \ldots, x_n \in V_{\alpha_m}$ for some $m < \omega$, so $x \in V_{\alpha_{m+1}} \subseteq V_\alpha \implies x \in V_\alpha$.
That is, V_α is closed under a set of Skolem functions for V_α. Hence, by lemma 10.2, $V_\alpha \prec V_\kappa$.

Now, let us see that $C = \{\alpha < \kappa : V_\alpha \prec V_\kappa\}$ is closed and unbounded on κ.

- C is unbounded since we can argue as before starting from any $\beta < \kappa$, so $\forall \beta < \kappa \exists \alpha > \beta (V_\alpha \prec V_\kappa)$.
- C is closed: for any sequence of ordinals in C of length $\gamma < \kappa$, $\langle \alpha_\xi \in C : \xi < \gamma \rangle$, we have to check that $\alpha = \sup_{\xi<\gamma} \alpha_\xi \in C$, that is, $V_\alpha \prec V_\kappa$. Let $f \in F$ and $x_1, \ldots, x_n \in V_\alpha$. Then, $x_1, \ldots, x_n \in V_{\alpha_\xi}$ for some $\xi < \gamma$, and since $V_{\alpha_\xi} \prec V_\kappa$, also $f(x_1, \ldots, x_n) \in V_{\alpha_\xi}$, so $f(x_1, \ldots, x_n) \in V_\alpha$. Thus $F(V_\alpha) \subseteq V_\alpha$ and therefore $V_\alpha \prec V_\kappa$, as we wanted to prove. ☐

What this theorem means is that for any strongly inaccessible cardinal κ, we can find some $V_\alpha \subset V_\kappa$ such that V_α *behaves just like* V_κ: every formula has the same truth value in V_κ as in V_α when restricting the variables to V_α. And not only that, the collection of such αs is infinite and unbounded.

That is, we can find in V_κ infinite models of itself, and as big as we want them to be. In particular, there are infinite models of ZFC inside V_κ.

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