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Shadow demands for unobservable goods: an intertemporal incomplete demand approach

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Abstract

For goods whose quantities are not observed or available, there are no explicit, but shadow, demands for them. To analyze these shadow demands, this paper presents a novel incomplete demand model that allows for intertemporal consumption behavior, as summarized by the Euler equation. This equation is then estimated, using French data (1977–2020), jointly with the demands for observed goods. This allows to identify the parameters characterizing the shadow demands for unobservable goods and to properly measure the welfare effects of price changes. Results indicate that in the absence of available quantity data for some goods, the proposed shadow demand model provides sufficient information to adequately analyze observed consumer behavior.

Keywords Shadow demands · Incomplete demand systems · Indirect utility function · Weak integrability · Euler equation

JEL Classification D12 · D51 · E21

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1 Introduction

Empirical demand analysis entails a full set of data on the prices and quantities of all goods.[1] These data are available for most goods. For some goods, however, the quantities are observed but their prices are not. On the other hand, there are goods whose prices are observed but quantities are not. Nonmarket goods, such as public goods and environmental quality or amenities,[2] characterize the situation where the quantities are observed but the prices are not (Bockstael and McConnell 1993; Hanemann 1991). For these goods, there are no explicit markets, and so, they are not traded in markets and their prices are not directly observed. However, there are implied or implicit prices associated with given quantities. The implicit or shadow prices measure the consumer's marginal willingness to pay for them. This gives the inverse demand functions specifying the implicit prices as functions of the quantities and income (Hanemann 1991; Kim et al. 2020).

In incomplete demand systems, by contrast, while there are the goods of interest for which the prices and quantities are observed, for other goods the prices are observed but the quantities are not observable or readily available (Epstein 1982; LaFrance and Hanemann 1989). In such systems, therefore, the demands, expressed in quantities, for the goods of interest are observed, but the quantities for other goods are not directly observable, though total expenditure on them is observed.[3] When demand systems are complete, there is a full set of observed data on prices and quantities of all goods.[4] In incomplete demand analysis, the central focus is on the demands for the goods of interest based on available information present for these goods, such as prices and quantities, and other goods, such as prices and total expenditure on them, whereas the demands for other goods are treated of little or no importance.[5] Although there are no directly observable demands for other goods in incomplete demand systems due to the

[1] Throughout the paper, "demand" is used in the usual context in that it refers to the quantity of a good and service buyers are willing to purchase at different prices at a given period time. "Demands" refer to a system of demand functions for more than one good or service.

[2] We are concerned with public goods provided by the government, but there are public goods that are privately provided. We also assume that there are no markets for a clean air or water that allows to trade pollutants.

[3] There are four basic concepts—price, quantity, expenditure, and budget share—to characterize a demand system as complete or incomplete. For a complete demand system, we have all four information or data. For an incomplete system with the goods of interest, we also have these data for individual goods. However, for other goods, we have data on prices, but no information about quantities, expenditures, and budget shares for individual goods, though total expenditure on unobservable and observable goods are available. See Sect. 3.1 for a further discussion. For nonmarket goods, we have information on quantities only but no information expenditure and budget share.

[4] A note of clarification: a complete or incomplete demand system is not associated with a complete or incomplete market that describes the presence or absence of insurance markets or risk sharing arrangements to buffer idiosyncratic shocks that are peculiar to households, see Ahn et al. (2017).

[5] In empirical analysis, it is common to estimate a partial or conditional demand system for some subset of goods of interest, such as food, without considering other goods, by invoking weak separability in consumer preferences (Hanemann and Morey 1991). This assumption, however, is rarely satisfied in reality, and the partial demand system, by excluding the prices of all other goods, may provide biased information for policy or welfare analysis. An incomplete demand analysis can rectify this problem of the partial demand system by including the prices of other goods as well as income, which is more relevant than total expenditure on the goods of interest (Hanemann and Morey 1991).

absence of available quantities, there are the implied or implicit quantities the consumer would be willing to purchase at given prices and income. These quantities give rise to the unrealized “shadow” demands for unobservable goods, relating them to the prices of the goods of interest, the prices of other goods, and income, an idea similar to the shadow prices of nonmarket goods. The shadow demands can be indirectly measured, using available data on observable and unobservable goods.

Given the presence of unobservable prices or quantities of some goods, an important task of applied demand analysis is to extract the implicit prices or quantities for these goods from observed data. This information is essential for a full understanding of consumer behavior. There are studies on public goods that estimate the consumer’s willingness to pay for them using survey or hedonic pricing approaches (Bohm 1972; Hewitt 1985; Brookshire et al. 1982). Although the concept of the shadow demands is relevant in incomplete demand systems, it has not been formally analyzed in prior studies. The concern of these studies is with the analysis of the observed demands, but the demands for unobservable goods are not examined by treating these goods as a composite commodity (Beghin et al. 2004; von Haefen 2010; Zhen et al. 2014). Our focus in this paper is on the demands for unobservable goods, while taking into account the demands for observable goods.

To that end, we present an intertemporal incomplete demand model to formalize the shadow demands for unobservable goods for which the prices are observed but the quantities are not available. The model considers the quantities of these goods, though not directly observable, as indirectly measurable, and casts the incomplete demand system as a part of the consumer’s intertemporal optimization problem. Existing incomplete, and complete as well, demand analyses are static in nature and treat consumption expenditure, loosely referred to as income, as exogenously given; but this is unrealistic in an intertemporal environment. The proposed model endogenizes it in an intertemporal optimization framework, assuming that the consumer chooses consumption expenditure to optimally allocate wealth across periods. To characterize consumer preferences, we specify an indirect utility function, as a function of the prices of observable and unobservable goods and of total consumption expenditure, and derive the demands for observable goods as well as the shadow demands for unobservable goods. Then from the intertemporal optimization problem with the indirect utility function, we derive the Euler equation that governs intertemporal consumption behavior. This equation contains information necessary to identify all the parameters characterizing the indirect utility function and hence to identify the shadow demands for unobservable goods.[6] Importantly, this allows to recover the underlying preferences from the demand functions, the key issue of integrability for welfare analysis in incomplete demand analysis (Epstein 1982; LaFrance and Hanemann 1989). It should be noted, however, that the estimation of an incomplete demand system requires a special care because, unlike the complete demand system that utilizes the budget shares for all goods, the budget shares for unobservable goods are not available and thus cannot be included in estimation.

[6] This idea was first exploited by Kim et al. (2020) to measure the shadow price of a public goods; see Sect. 6 for a further discussion.

To implement the proposed model of the shadow demands for unobservable goods, we carry out an empirical analysis of it with a novel empirical methodology, to deal with unobservable goods. We use a flexible specification of the indirect utility function that places minimal restrictions on consumer preferences. We jointly estimate the system of the budget share equations for observable goods, together with the Euler equation for consumption. We do not include the budget share equations for unobservable goods in estimation but take into account the adding-up condition by imposing parameter restrictions on the model. While there are empirical studies on incomplete demand systems (Beghin et al. 2004; von Haefen 2010; Zhen et al. 2014), they are based on a simple framework and are limited in scope and analysis with no analysis of unobservable goods by treating them as a composite numeraire good. We provide a more complete empirical analysis of the incomplete demand system by estimating the shadow demands for unobservable goods.

Given a lack of readily available information about unobservable goods, to illustrate the proposed methodology, we use published French data for the period 1977–2020 that contain a full set of data on goods, by choosing a few goods considered to be reasonably unobservable, and estimate an incomplete demand system with these goods. Since we have actual data available for the chosen unobservable goods, we can assess the information loss associated with estimation of the incomplete demand system, relative to the complete demand system estimated with known information about the unobservable goods. Additionally, we examine the welfare effect associated with changes in the prices of unobservable goods. Our principal finding is that the estimated shadow demands from the incomplete demand system, in general, do not create a significant information loss in analysis relative to the complete demand system. This result is promising because even though quantity data are not readily available or not directly observed for some goods, the incomplete shadow demand framework provides sufficient information about these goods to understand consumer demand and welfare. This demonstrates the utility of the shadow demand framework for analyzing unobservable goods. The proposed intertemporal framework also brings a new perspective to the analysis of nonmarket goods and associated shadow prices, relative to previous studies in this area.

2 An Intertemporal incomplete demand system

In this section, we state the basic issue involved in incomplete demand analysis, present a review of existing studies, and formulate the proposed intertemporal model that forms the basis for empirical analysis.

2.1 The problem setup

We consider two group of goods–goods of interest and all other goods. The following notations are used: $\mathbf{x}_t$ is an n vector of the consumption levels for the goods of interest at period t, with the corresponding price vector denoted by $\mathbf{p}_t$; $\mathbf{z}_t$ is an m vector of the consumption levels of all other goods at period t, with the corresponding price vector denoted by $\mathbf{q}_t$. Total expenditure, M_t, is allocated to $\mathbf{x}_t$ and $\mathbf{z}_t$ at period t, and the budget constraint is $\mathbf{p}_t\prime\mathbf{x}_t + \mathbf{q}_t\prime\mathbf{z}_t \leq M_t$ (adding-up condition).

Given a direct utility function, $u(\mathbf{x}_t,\ \mathbf{z}_t)$, which is continuous, increasing, and quasi-concave in $(\mathbf{x}_t, \mathbf{z}_t)$, the indirect utility function, $v(\mathbf{p}_t, \mathbf{q}_t, M_t)$, is defined as

$$v(\mathbf{p}_t, \mathbf{q}_t, M_t) \equiv \max_{\mathbf{x}_t, \mathbf{z}_t}\{u(\mathbf{x}_t,\ \mathbf{z}_t)|\ \mathbf{p}_t\prime\mathbf{x}_t + \mathbf{q}_t\prime\mathbf{z}_t \leq M_t\} \quad (1)$$

This indirect utility function is well defined as a description of the consumer's within-period preferences under the following regularity conditions: It is continuous in $(\mathbf{p}_t, \mathbf{q}_t, M_t)$, decreasing in $(\mathbf{p}_t, \mathbf{q}_t)$, increasing in M_t, quasi-convex in $(\mathbf{p}_t, \mathbf{q}_t)$, homogeneous of degree zero in $(\mathbf{p}_t, \mathbf{q}_t)$ and M_t (Deaton and Muellbauer 1980a). The direct utility function, $u(\mathbf{x}_t,\ \mathbf{z}_t)$, can be recovered from the indirect utility function as the solution to the minimization problem:

$$u(\mathbf{x}_t,\ \mathbf{z}_t) \equiv \min_{\mathbf{p}_t, \mathbf{q}_t, M_t}\{v(\mathbf{p}_t, \mathbf{q}_t, M_t)|\ \mathbf{p}_t\prime\mathbf{x}_t + \mathbf{q}_t\prime\mathbf{z}_t \leq M_t\}. \quad (2)$$

Application of Roy's identity to the indirect utility function (1) yields the system of demand functions for $\mathbf{x}_t$ and $\mathbf{z}_t$:

$$x_{it} = g_i^{\mathbf{x}}(\mathbf{p}_t, \mathbf{q}_t, M_t) = -\frac{\partial v(\mathbf{p}_t, \mathbf{q}_t, M_t)/\partial p_{it}}{\partial v(\mathbf{p}_t, \mathbf{q}_t, M_t)/\partial M_t}, i = 1, ..., n. \quad (3)$$

$$z_{jt} = g_j^{\mathbf{z}}(\mathbf{p}_t, \mathbf{q}_t, M_t) = -\frac{\partial v(\mathbf{p}_t, \mathbf{q}_t, M_t)/\partial q_{jt}}{\partial v(\mathbf{p}_t, \mathbf{q}_t, M_t)/\partial M_t}, j = 1., , , m. \quad (4)$$

These demand functions for x and z have well-known properties, with consumption expenditure loosely referred to as income (Deaton and Muellbauer 1980a). They are homogenous of degree zero in $(\mathbf{p}_t, \mathbf{q}_t)$ and M_t, the matrix of the Slutsky terms is symmetric, negative semidefinite, and the adding-up condition hold for x and z.

In the above discussion, $\mathbf{x}_t$ and $\mathbf{z}_t$ are assumed to be observable, in that their prices and quantities are directly observed. Hence, the demands for $\mathbf{x}_t$ and $\mathbf{z}_t$ described in (3) and (4) constitute a complete demand system. An incomplete demand system is a subset of the demand functions in a complete demand system. In this demand system, while the demands for $\mathbf{x}_t$ are observable, the demands for $\mathbf{z}_t$ are not, in that their prices are observed but the quantities not. Accordingly, there are no explicit demand functions for $\mathbf{z}_t$ as described in (4), and only part of the consumer's budget is allocated to the consumption of $\mathbf{x}_t$. This indicates that the adding-up condition for the budget constraint holds for $\mathbf{x}_t$ only as an inequality restriction, that is, $\mathbf{p}_t\prime\mathbf{x}_t \leq M_t$. This has an important implication for integrability, namely the recoverability of the underlying preferences from the estimated demand functions satisfying the required regularity conditions. Strong integrability, which ensures the existence of a well-defined indirect, and hence direct, utility function across the goods of interest and other goods, is difficult to establish because of insufficient information on unobservable goods (Epstein 1982).[7]

[7] For ease of analysis, integrability is discussed with the indirect utility function, instead of expenditure function, which is more appropriate (Epstein 1982).

In particular, since the demands for $\mathbf{z}_t$ are unobservable, estimation of the demands for $\mathbf{x}_t$ only[8] does not provide sufficient information to identify all the parameters characterizing the indirect utility function $v(\mathbf{p}_t, \mathbf{q}_t, M_t)$ and hence to recover the direct utility function $u(\mathbf{x}_t, \mathbf{z}_t)$, as discussed with (4) for the complete demand system.

2.2 Previous approach

To circumvent the aforesaid issue, LaFrance and Hanemann (1989), instead, propose the concept of weak integrability that takes full advantage of all the information present for the observable goods of interest but is flexible for the unknown information for the unobservable goods. Weak integrability has provided the framework for empirical analysis of incomplete demand systems (Beghin et al. 2004; von Haefen 2010; Zhen et al. 2014). In this subsection, we summarize LaFrance and Hanemann's (1989) main results of the weakly integrable incomplete demand model as pertaining to our analysis, and draw some implications for our approach to be discussed in the next subsection.

Although individual elements of $\mathbf{z}_t$ are not observable, total expenditure on $\mathbf{z}_t$ (s_t), defined by $s_t \equiv \mathbf{q}_t{'}\mathbf{z}_t = M_t - \mathbf{p}_t{'}\mathbf{x}_t > 0$, is readily observable. This variable, indeed, plays a pivotal role in the weakly integrable incomplete demand model, and is utilized in a two-step optimization procedure.[9] A quasi-utility function is first defined as

$$\omega(\mathbf{x}_t, s_t, \ \mathbf{q}_t) \equiv \max_{\mathbf{z}_t}\{u(\mathbf{x}_t, \mathbf{z}_t)\big|\mathbf{q}_t\mathbf{z}_t \le s_t\}. \tag{5}$$

This function has the same properties of the direct utility function $u(\mathbf{x}_t, \ \mathbf{z}_t)$ conditional on $\mathbf{q}_t$ by treating s_t as a composite commodity with its price normalized to one. Given (5), a quasi-indirect utility function is then defined as

$$\phi(\mathbf{p}_t, \mathbf{q}_t, M_t) \equiv \max_{\mathbf{x}_t, s_t}\{\omega(\mathbf{x}_t, s_t, \ \mathbf{q}_t)\big|\mathbf{p}_t{'}\mathbf{x}_t + s_t \le M_t\}. \tag{6}$$

This function has the same properties of the indirect utility function (1). As with the complete demand system, the quasi-utility function (5) is dual to the quasi-indirect utility function (6), which provides the theoretical underpinnings for weak integrability of the incomplete demand system.

Applying Roy's identity to (6), we obtain the demands for $\mathbf{x}_t$:

$$x_{it} = h_i(\mathbf{p}_t, \mathbf{q}_t, M_t) = -\frac{\partial\phi(\mathbf{p}_t, \mathbf{q}_t, M_t)/\partial p_{it}}{\partial\phi(\mathbf{p}_t, \mathbf{q}_t, M_t)/\partial M_t}, i = 1, ..., n, \tag{7}$$

[8] Hanemann and Morey (1991) argue that it is possible to estimate the incomplete demand system (3) alone with data on $\mathbf{x}_t$, $\mathbf{p}_t$, and M_t if one is willing to assume that $\mathbf{q}_t$ does not vary across sample because, in this case, given values of $\mathbf{q}_t$ are subsumed in the coefficients on $\mathbf{p}_t$ and M_t. Other case is when $\mathbf{q}_t$ does not enter the demands for $\mathbf{x}_t$ under weak separability of $\mathbf{x}_t$ from $\mathbf{z}_t$.

[9] This is similar to the firm's two-stage optimization problem wherein capital is treated as a quasi-fixed input. In the first stage, we set up a cost or profit function conditional on given capital. Then capital is treated as a choice variable in the firm's second-stage problem.

which implies the total expenditure on unobservable goods, s_t, can be treated as a demand for a composite good expressed by

$$s_t = \sigma(\mathbf{p}_t, \mathbf{q}_t, M_t) = M_t - \sum_{i=1}^{n} p_{it} h_i(\mathbf{p}_t, \mathbf{q}_t, M_t). \tag{8}$$

Using available data on $(\mathbf{p}_t, \mathbf{q}_t, M_t)$ together with $\mathbf{x}_t$ and s_t, the demands for $\mathbf{x}_t$ and hence s_t can be estimated.

The quasi-indirect utility function (6) is related to the true indirect utility function (1) via the identity (Epstein 1982; LaFrance and Hanemann 1989)

$$\nu(\mathbf{p}_t, \mathbf{q}_t, M_t) \equiv \psi[\mathbf{q}_t, \phi(\mathbf{p}_t, \mathbf{q}_t, M_t)]. \tag{9}$$

The difference between Eqs. (1) and (6) is due to information loss with the incomplete demand system, which is associated with the structure of $\psi[\mathbf{q}_t, \phi(\mathbf{p}_t, \mathbf{q}_t, M_t)]$ with respect to $\mathbf{q}_t$. In particular, the direct and hence indirect utility functions are associated with the demands for $\mathbf{x}_t$ and $\mathbf{z}_t$ described in (3) and (4), while the quasi-utility and hence quasi-indirect utility functions are based on the demands for $\mathbf{x}_t$ only. Clearly, there is an information loss with the incomplete demand system, and, as LaFrance and Hanemann (1989) recognize, this information loss underlying weak integrability makes the incomplete demand system difficult to establish strong integrability, which is more desirable to achieve.

Weak integrability entails some parameter restrictions on the incomplete demand system (7) such as linear in parameters. LaFrance (1990) proposes several parametric forms to represent such incomplete demand systems (see also von Haefen 2002). He notes, however, the adopted parametric demand models are restrictive, in that they are not derived from flexible representations of consumer preferences that can be rationalized by utility maximization. More importantly, the primary concern in incomplete demand analysis is to estimate the demands for observed goods $\mathbf{x}_t$, with little or no attention paid to unobservable other goods $\mathbf{z}_t$; there is no information about individual demands for $\mathbf{z}_t$.

2.3 An intertemporal approach

Although the demands for $\mathbf{z}_t$ are unobservable, as argued in the Introduction, there are the implicit quantities associated with $(\mathbf{p}_t, \mathbf{q}_t)$ and M_t that can be indirectly measured using available data on observable and unobservable goods. Then the demand functions for $\mathbf{z}_t$ in (4) can be treated as forming the shadow demands for unobservable goods. These shadow demands for unobservable goods together with the demands for observable goods allows us to recover the underlying preferences and hence to establish strong integrability, resolving the key issue in incomplete demand analysis. To show this, we present an intertemporal approach, which allows us to address other problems of incomplete demand analysis as well. The key to this approach is to treat total consumption expenditure on $\mathbf{x}_t$ and $\mathbf{z}_t$, M_t, specified in the indirect utility function (1), as endogenous. In existing studies on incomplete demand systems, and on consumer demands in general, M_t is treated as exogenously given and is left unexplained. However, to the extent that the consumer chooses expenditure to optimally allocate wealth across periods, consumption expenditure is not exogenously given, but is endogenously determined in the consumer's optimization problem.[10]

In formulating an intertemporal optimization problem by endogenizing consumption expenditure, it is important to note that the direct, and hence indirect, utility function in (1) is ordinal. Thus, the intra-temporal allocation of consumption across goods as captured by the demand functions (3) and (4) is invariant to a monotonic transformation of the utility function (1). However, the intertemporal allocation of consumption, which underlies our approach to the incomplete demand system, is invariant with respect to a linear transformation of the utility function, but not to other transformations of this function. For intertemporal preferences, we therefore take a Box–Cox form for $v(\mathbf{p}_t, \mathbf{q}_t, M_t)$:

$$U_t = \frac{v(\mathbf{p}_t, \mathbf{q}_t, M_t)^{1-\zeta} - 1}{1-\zeta}, \tag{10}$$

where ζ is a Box–Cox parameter, with the marginal utility of M_t given by

$$\frac{\partial U_t}{\partial M_t} = \nu(\mathbf{p}_t, \mathbf{q}_t, M_t)^{-\zeta} v_M(\mathbf{p}_t, \mathbf{q}_t, M_t), \tag{11}$$

where $v_M(\mathbf{p}_t, \mathbf{q}_t, M_t) \equiv \partial \nu(\mathbf{p}_t, \mathbf{q}_t, M_t)/\partial M_t$. While the indirect utility function (3) as a representation of within-period preferences is well defined with its regularity conditions discussed above, we assume that the Box–Cox utility function (10) is continuous, increasing, and, more importantly, strictly concave in M_t for a given $\mathbf{p}_t$ and $\mathbf{q}_t$. Concavity of the Box–Cox utility function requires that $\partial^2 U_t/\partial M_t^2 < 0$, where

$$\frac{\partial^2 U_t}{\partial M_t^2} = \frac{\partial \nu_M(\mathbf{p}_t, \mathbf{q}_t, M_t)/\partial M}{\nu(\mathbf{p}_t, \mathbf{q}_t, M_t)^{\zeta}} - \zeta \frac{\left[\nu_M(\mathbf{p}_t, \mathbf{q}_t, M_t)\right]^2}{\nu(\mathbf{p}_t, \mathbf{q}_t, M_t)^{(\zeta+1)}}. \tag{12}$$

This condition ensures the existence of a solution to the intertemporal optimization problem and implies that the necessary conditions are indeed sufficient.

With the transformed indirect utility function (10), the consumer's optimization problem is to choose M_s for $s \geq t$ so as to maximize

$$E_t\left[\sum_{s=t}^{\infty} (1+\rho)^{-(s-t)} \left(\frac{v(\mathbf{p}_t, \mathbf{q}_t, M_t)^{1-\zeta} - 1}{1-\zeta}\right)\right], \tag{13}$$

[10] This idea underlies intertemporal two-stage budgeting (Kim and McLaren 2024) where the level of consumption expenditure is chosen by optimally allocating wealth across periods in the first stage. Then, in the second stage, each period's optimal allocation of consumption expenditure is distributed across different goods (Kim et al. 2021: Kim et al. 2024).

where ρ is the constant rate of the consumer's time preference, subject to the intertemporal finance or budget constraint:

$$A_s = (1 + r_s)A_{s-1} + Y_s - M_s \text{ for all } s \geq t, \tag{14}$$

where A_s is the value of financial assets at the end of period s to be carried into the next period, r_s is the nominal interest rate on assets that can be both bought and sold between periods s and $s + 1$, and Y_s is labor income at period s.[11] The expectation operator E_t is taken over future variables, using information available at the beginning of period t. We assume that the consumer replans continuously when solving the above stochastic dynamic optimization problem. In other words, the consumer updates his plans continuously by reoptimizing the intertemporal problem at every period s,$s \geq t$, with the new information he has. This means that the calendar time τ solution for M_τ should be the successive time t solution for M_t as planning time s evolves through time, with the present always being time t.

For empirical analysis then, only the first-order conditions necessary for the above intertemporal optimization problem at the initial point in time ($s = t$) are relevant. Utilizing the Lagrange method, they are given by

$$M_t : v(\mathbf{p}_t, \mathbf{q}_t, M_t)^{-\zeta} v_M(\mathbf{p}_t, \mathbf{q}_t, M_t) = \lambda_t, \tag{15}$$

and

$$A_t : \lambda_t = E_t\left[\left(\frac{1 + r_{t+1}}{1 + \rho}\right)\lambda_{t+1}\right], \tag{16}$$

where λ_t is the Lagrange multiplier associated with the asset accumulation constraint (14) known at time t, which measures the marginal utility of wealth. Equation (15) indicates that the marginal utility of wealth is equated, at the optimum, to the marginal utility of consumption. Equation (16) is known as the Euler equation in consumption/saving studies, which describes an intertemporal allocation of wealth or consumption (Hall 1978; Ludvigson and Paxson 2001). While it is derived with the indirect utility function (1) under general conditions, previous studies on consumption/saving derive it using real consumption with restrictive utility functions based on homothetic preferences.

For empirical analysis, it is more convenient to work with (16) in a ratio form represented by

$$E_t\left[\left(\frac{1 + r_{t+1}}{1 + \rho}\right)\frac{\lambda_{t+1}}{\lambda_t}\right] = 1 \tag{17a}$$

[11] We consider leisure or labor supply as fixed and treat labor income as exogenous to the consumer's choice.

Using (15), this equation can be rewritten as

$$E_t\left[\left(\frac{1+r_{t+1}}{1+\rho}\right)\left(\frac{\nu(\mathbf{p}_{t+1},\mathbf{q}_{t+1},M_{t+1})^{-\zeta}\,v_M(\mathbf{p}_{t+1},\mathbf{q}_{t+1},M_{t+1})}{\nu(\mathbf{p}_t,\mathbf{q}_t,M_t)^{-\zeta}\,v_M(\mathbf{p}_t,\mathbf{q}_t,M_t)_t}\right)\right]=1. \quad (17b)$$

Equation (17b) plays a central role in our intertemporal analysis. Notably, it contains expressions for the indirect utility function, $\nu(\mathbf{p}_t, \mathbf{q}_t, M_t)$ and $\nu(\mathbf{p}_{t+1}, \mathbf{q}_{t+1}, M_{t+1})$, in two adjacent periods. With the specification of a functional form for the indirect utility function, joint estimation of the demands for $\mathbf{x}_t$ in (3) with the Euler Eq. (17b), using observed data on $(\mathbf{p}_t, \mathbf{q}_t, M_t)$, allows us to estimate the shadow demands for unobservable goods $\mathbf{z}_t$ in (4). Importantly, this provides sufficient information necessary to identify the underlying preferences as represented by the indirect utility function (1) (see Sect. 3.2 for a further discussion) and hence using the result in (2), to recover the direct utility function $u(\mathbf{x}_t, \mathbf{z}_t)$ and hence to establish strong integrability of the incomplete demand system.

Knowledge of the indirect utility function permits the calculation of exact measures for the welfare effects of changes in the prices of unobservable goods (see Sect. 5.4 for an empirical analysis). These measures can be considered more relevant than those derived from the quasi-indirect utility function (6) based on weak integrability (LaFrance and Hanemann 1989). Hence, we do not have to rely, in empirical analysis, on weak integrability based on the demands for observable goods. This is the gist of the intertemporal approach to the analysis of incomplete demand systems, i.e., the intertemporal incomplete demand model composed of the observable demand system (3) and the shadow demand system (4) together with the Euler Eq. (17).

3 Empirical model

In this section, we provide an empirical methodology of the proposed intertemporal incomplete demand model to demonstrate its working mechanism with data.

3.1 Specification of the indirect utility function

Empirical analysis requires the specification of an appropriate parametric form for the indirect utility function (1). To properly characterize consumer behavior, however, the chosen functional form should be flexible while satisfying the requisite regularity conditions for within-period, as well as intertemporal preferences. The PIGLOG (Price Independent Generalized Logarithmic) form, popularized by Deaton and Muellbauer's (1980b) Almost Ideal Demand System (AIDS), is widely used in applied demand analysis. In this study, we have adopted Cooper and McLaren's (1992) MPIGLOG (Modified Price Independent Generalized Logarithmic) form to characterize the indirect utility function. This form is a generalization of Deaton and Muellbauer's (1980b) PIGLOG form specified with the AIDS. Our choice for the MPIGLOG form is motivated by its property that regularity, at a base level of real consumption and at all higher levels of real consumption, can be ensured if parameter values satisfy certain simple inequality restrictions (Cooper and McLaren 1996; McLaren and Wong 2009). Other functional forms, such as AIDS or Translog, are only assured to be regular locally, in a region around a data point. Since we also analyze intertemporal consumption behavior, it is important that regularity be satisfied over a large range of values of consumption. The MPIGLOG form also gives a tractable Euler equation for consumption. In essence, the MPIGLOG form is parsimonious but, being effectively globally regular, provides an attractive compromise in tradeoff between regularity and flexibility. This form, however, is based on a complete demand system and does not allow for unobservable goods.

With the MPIGLOG specification allowing for unobservable goods, the indirect utility function (1) is of the following form:

$$\nu(\mathbf{p}_t,\ \mathbf{q}_t, M_t) = \ln\left[\frac{M_t}{P_A(\mathbf{p}_t, \mathbf{q}_t)}\right]\left[\frac{M_t}{P_B(\mathbf{p}_t, \mathbf{q}_t)}\right]^{\eta}, \tag{18}$$

where η is a nonnegative parameter with $0 \leq \eta \leq 1$, and $P_A(\mathbf{p}_t, \mathbf{q}_t)$ and $P_B(\mathbf{p}_t, \mathbf{q}_t)$ are price indexes that are positive, increasing, homogeneous of degree one, and concave in $(\mathbf{p}_\mathrm{t}, \mathbf{q}_\mathrm{t})$. To complete the specification in (18), we assume that the price indexes take the forms:

$$\mathrm{P}_A(\mathbf{p}_t,\ \mathbf{q}_t) = \left(\sum_{i=1}^{n} \alpha_i^x p_{it}^{\rho_A} + \sum_{j=1}^{m} \alpha_j^z q_{jt}^{\rho_A}\right)^{1/\rho_A},$$

$$\mathrm{P}_B(\mathbf{p}_t,\ \mathbf{q}_t) = \left(\prod_{i=1}^{n} p_{it}^{\beta_i^x}\right)\left(\prod_{j=1}^{m} q_{jt}^{\beta_j^z}\right) \text{ with } \sum_{i=1}^{n} \beta_i^x + \sum_{j=1}^{m} \beta_j^z = 1, \tag{19}$$

for $\rho_\mathrm{A} > 0$, and $\alpha_i^x \geq 0$, $\alpha_j^z \geq 0, \beta_i^x \geq 0$, and $\beta_j^z \geq 0\ \forall$ j, j, where $\mathrm{P}_A(\mathbf{p}_t,\ \mathbf{q}_t)$ is a constant elasticity of substitution (CES) function and $\mathrm{P}_B(\mathbf{p}_t,\ \mathbf{q}_t)$ is the product of two Cobb–Douglas functions.

It should be noted that the MPIGLOG specification of the indirect utility function in (18) exploits the rank of a demand system, the number of independent Engel curves, which is equivalent to the number of functions of prices in the indirect utility function (Lewbel 1991). Regularity of the indirect utility function depends on the properties of these functions of prices, and we can only prove regularity if these functions of prices are proper price indices or regular unit cost functions (Cooper and McLaren 1996; McLaren and Wong 2009). The MPIGLOG indirect utility function (18) is of rank 2 with the two price functions or indexes, $\mathrm{P}_A(\mathbf{p}_t,\ \mathbf{q}_t)$ and $\mathrm{P}_B(\mathbf{p}_t,\ \mathbf{q}_t)$, and characterizes nonhomothetic within-period preferences and places no a priori restrictions on the underlying consumer preferences.[12]

[12] Rank 1 demand systems are homothetic with one price index or function, so their budget shares are constant, violating basic empirical properties of demand. Rank 2 is sufficient to capture Engel behavior in aggregate data. The AIDS is a rank 2 demand system, but one of the price functions is a homogenous of degree *zero* Cobb Douglas, which is not a valid price index; hence the AIDS indirect utility function violates monotonicity in prices, allowing negative budget shares. The MPIGLOG, which is of rank 2 with the two price indexes, $\mathrm{P_A}(\mathbf{p}_\mathrm{t}, \mathbf{q}_\mathrm{t})$ and $\mathrm{P_B}(\mathbf{p}_\mathrm{t}, \mathbf{q}_\mathrm{t})$, resolves this problem, so it fits the data but can be effectively

Given (18) and (19), we obtain the following derivatives:

$$\frac{\partial \nu(\mathbf{p}_t, \mathbf{q}_t,\ M_t)}{\partial p_{\mathrm{it}}} = -\left[\frac{M_t}{\mathrm{P}_B(\mathbf{p}_t,\ \mathbf{q}_t)}\right]^{\eta} x \left[\frac{E^x_{\mathrm{Ait}} + \eta\beta^x_i R_t}{\mathrm{p}_{\mathrm{it}}}\right],\ i = 1, ..., n, \tag{20}$$

$$\frac{\partial \nu(\mathbf{p}_t, \mathbf{q}_t, M_t)}{\partial q_{jt}} = -\left[\frac{M_t}{\mathrm{P}_B(\mathbf{p}_t,\ \mathbf{q}_t)}\right]^{\eta} x \left[\frac{E^z_{\mathrm{Ajt}} + \eta\beta^z_j R_t}{\mathrm{q}_{jt}}\right], j = 1, ..., m, \tag{21}$$

and

$$\frac{\partial \nu(\mathbf{p}_t, \mathbf{q}_t, M_t)}{\partial M_t} = \frac{M_t^{\eta-1}}{\left[\mathrm{P}_B(\mathbf{p}_t,\ \mathbf{q}_t)\right]^{\eta}}(1 + \eta R_t), \tag{22}$$

with

$$\frac{\partial^2 \nu(M_t,\ \mathbf{p}_t, \mathbf{q}_t)}{\partial M_t^2} = \frac{M_t^{\eta-2}}{\left[P_B(\mathbf{p}_t, \mathbf{q}_t)\right]^{\eta}}[(2\eta - 1) + \eta(\eta - 1)R_t], \tag{23}$$

where $R_t \equiv \ln\left[\frac{M_t}{P_A(\mathbf{p}_t,\mathbf{q}_t)}\right]$, $E^x_{Ait} \equiv \frac{\partial \ln \mathrm{P}_A(\mathbf{p}_t, \mathbf{q}_t)}{\partial \ln p_{it}} = \frac{\alpha^x_i p^{\rho A}_{it}}{\sum_{i=1}^n \alpha^x_i p^{\rho A}_{it} + \sum_{j=1}^m \alpha^z_j q^{\rho A}_{jt}}$,

$E^z_{Ajt} \equiv \frac{\partial \ln \mathrm{P}_A(\mathbf{p}_t, \mathbf{q}_t)}{\partial \ln q_{jt}} = \frac{\alpha^z_j q^{\rho A}_{jt}}{\sum_{i=1}^n \alpha^x_i p^{\rho A}_{it} + \sum_{j=1}^m \alpha^z_j q^{\rho A}_{jt}}$.

Expressions (20) and (22) can be used to derive the demand functions for observable goods $\mathbf{x}_t$ via Roy's identity (3). In a budget or expenditure share form, we have

$$W^x_{it} = -\left(\frac{\partial \nu(\mathbf{p}_t, \mathbf{q}_t, M_t)/\partial p_{it}}{\partial \nu(\mathbf{p}_t, \mathbf{q}_t, M_t)/\partial M_t}\right)\frac{p_{it}}{M_t} = \frac{E^x_{Ait} + \eta\beta^x_i R_t}{1 + \eta R_t},\ i = 1, ..., n, \tag{24}$$

where $W^x_{it} \equiv p_{it/}x_{it}/M_t$ is the share of the ith ($i = 1, ..., n$) observable good in total expenditure.

Likewise, from (21) and (22), the shadow demand functions, in a budget share form, for unobservable goods $\mathbf{z}_t$ can be derived as

$$\widehat{W}^z_{jt} = -\left(\frac{\partial \nu(\mathbf{p}_t, \mathbf{q}_t, M_t)/\partial q_{jt}}{\partial \nu(\mathbf{p}_t, \mathbf{q}_t, M_t)/\partial M_t}\right)\frac{q_{jt}}{M_t} = \frac{E^z_{Ajt} + \eta\beta^z_j R_t}{1 + \eta R_t},\ j = 1, ..., m, \tag{25}$$

Footnote 12 continued
globally regular (Cooper and McLaren 1996; McLaren and Wong 2009). We also specified and estimated a rank 3 model, but it was rejected in favor of the rank 2 MPIGLOG model. This result is consistent with the finding that although demands for individual households appear to be of rank 3, there is evidence that aggregate demands, on which our study is based with time series data, can be adequately modeled as rank 2 (see Lewbel 1991). Moreover, the two price indexes are linked to both $\mathbf{p}_t$ and $\mathbf{q}_t$. This is for analytical tractability and an ease of estimation. A change in the price of $\mathbf{p}_t$ or $\mathbf{q}_t$ affects the indirect utility function (18) through the change in $\mathrm{P}_A(\mathbf{p}_t, \mathbf{q}_t)$ and $\mathrm{P}_B(\mathbf{p}_t, \mathbf{q}_t)$, We found that including $\mathbf{p}_t$ and $\mathbf{q}_t$ as separate subprice indexes creates an estimation problem.

where $\widehat{W}^z_{ij} \equiv q_{jt}/\widehat{z}_{jt}/M_t$ is the shadow share of the jth ($j = 1, ..., m$) unobservable good in total expenditure. The budget share equations in (24) and (25) are derived under nonhomothetic within-period preferences. If within-period preferences are homothetic, $\eta = 0$ and budget shares or Engel curves for observable and unobservable goods are independent of income or expenditure, yielding unitary expenditure elasticities for them. While Engel curves are linear in log income with PIGLOG (AIDS), MPIGLOG produces nonlinearity of Engel curves with nonzero η. The adding-up condition for the budget constraint implies that (24) and (25) satisfy that $\sum_{i=1}^{n} W^x_{it} + \sum_{j=1}^{m} \widehat{W}^z_{jt} = 1$. Appendix A provides a discussion of the price and expenditure elasticities derived from the above two budget share Eqs. (24) and (25).

Given (18), we can derive Euler Eq. (17b). Under rational expectations, it is written as

$$\frac{(1 + r_{t+1})}{(1 + \rho)} \frac{\frac{\partial \nu(\mathbf{p}_{t+1}, \mathbf{q}_{t+1}, M_{t+1})}{\partial M_{t+1}}}{\frac{\partial \nu(\mathbf{p}_t, \mathbf{q}_t, M_t)}{\partial M_t}} \left[\frac{\nu(\mathbf{p}_{t+1}, \mathbf{q}_{t+1}, M_{t+1})}{\nu(\mathbf{p}_t, \mathbf{q}_t, M_t)} \right]^{-\varsigma} = 1 + \varepsilon_{t+1}, \tag{26}$$

where $\frac{\partial \nu(\mathbf{p}_s, \mathbf{q}_s, M_s)}{\partial M_s}$($s = t$ and $t + 1$) is defined in (22), $\nu(\mathbf{p}_s, \mathbf{q}_s, M_s)$ ($s = t$ and $t + 1$) is defined in (18), and ε_{t+1} is an expectation error at time $t + 1$ with $E_t[\varepsilon_{t+1}] = 0$ and $Var_t(\varepsilon_{t+1}) = \sigma^2_{t+1}$. In our analysis, the error variance is assumed to be constant, i.e., $Var_t(\varepsilon_{t+1}) = \sigma^2$, because the representation of the Euler equation in the form of (17), rather than (16), should provide a more favorable disposition toward constant variance.

With the MPIGLOG indirect utility function (18), it is easy to check the regularity conditions for within-period and intertemporal preferences. Provided that the conditions on the two price indexes in (19) are satisfied over the region $M_t > P_A(\mathbf{p}_t, \mathbf{q}_t)$ for $0 < \eta \le 1$, then (18) will be effectively globally regular (Cooper and McLaren 1996; McLaren and Wong 2009). For intertemporal regularity with respect to M_t, for $\zeta > 0$, monotonicity of the Box–Cox utility function (10), i.e., $\partial U_t/\partial M_t > 0$, is satisfied if $\partial \nu(\mathbf{p}_s, \mathbf{q}_s, M_s)/\partial M_t > 0$ in (22). Concavity of the Box–Cox utility function, discussed in (12), is satisfied if $\partial^2 \nu(M_t, \mathbf{p}_t, \mathbf{q}_t)/\partial M_t^2 < 0$ in (23) for $\zeta > 0$.

For empirical analysis, the relevant equation system to analyze the shadow demands for unobservable goods comprises the budget share equations for observed goods in (24), the shadow demands for unobservable goods in (25), and the Euler equation for consumption (26). It is, however, important to note that the budget share equations for observed goods do not have all information to identify the parameters characterizing the shadow demands for unobservable goods, such as E^z_{Ajt} and β^z_j ∀ j. This is the essential problem underlying the analysis of the incomplete demand system. In the intertemporal model, however, this problem is resolved by utilizing the Euler equation in estimation. This equation contains an expression for the indirect utility function represented by the MIPGLOG form in (18) that has all information to identify the parameters of the demands for observable goods as well as the shadow demands for unobservable goods.

3.2 Estimation procedures

Estimation of an incomplete demand system is drastically different from that of a complete demand system. For the complete demand system, there is a full set of available data on $\mathbf{x}_t$ and $\mathbf{z}_t$. Therefore, the budget share equations for $\mathbf{x}_t$ and $\mathbf{z}_t$ in (24) and (25) can be jointly estimated to identify the parameters of the MPIGLOG indirect utility function (18) by imposing the adding-up condition that the sum of the budget shares for $\mathbf{x}_t$ and $\mathbf{z}_t$ is equal to one (Kim et al. 2024). However, this is not the case with estimation of an incomplete demand system. For the incomplete demand system, we have data on the budget shares for $\mathbf{x}_t$ only, and, because of the unobservable nature of $\mathbf{z}_t$, we cannot include the budget share equations for $\mathbf{z}_t$ (24) in estimation. This means that the sum of the included budget shares for $\mathbf{x}_t$ in (24) is not equal to unity, i.e., $\sum_{i=1}^{n} W_{it}^{x} \neq 1$, and we do not drop one budget share equation for $\mathbf{x}_t$ to satisfy the adding-up condition. The adding-up condition applies to both $\mathbf{x}_t$ and $\mathbf{z}_t$, and has to be imposed in the model by parameter restrictions without including the budget shares for $\mathbf{z}_t$. In our estimation, we include the Euler Eq. (26) and jointly estimate the system of the budget share equations for $\mathbf{x}_t$ (24), together with the Euler equation by allowing for the unique feature of the adding-up condition associated with unobservable goods. Inclusion of the Euler equation increases the efficiency in estimation with more degrees of freedom. More importantly, as discussed above, the Euler equation contains necessary information missing in the budget share equations for $\mathbf{x}_t$ to identify the parameters of the MPIGLOG indirect utility function (18) and hence to infer the shadow demands for unobservable goods in (25).

To estimate the systems of the budget share equations for $\mathbf{x}_t$ and the Euler equation, we add the stochastic error terms to the budget share equations for $\mathbf{x}_t$ (24) to allow for errors of optimization or taste shocks, which are uncorrelated with exogenous variables. The error in the Euler Eq. (26) is, by the nature of the first-order condition (17), uncorrelated with all information dated at time t. However, not all of the variables in (24) and (26) are strictly exogenous. In particular, total consumption M_t is endogenously determined in the consumer's optimization problem (see Sect. 2.2) and hence is correlated with the error terms. Also, there are variables dated $t+1$, such as $\mathbf{p}_{t+1}$, $\mathbf{q}_{t+1}$, M_{t+1}, and r_{t+1}, and the use of the realized values causes them to be correlated with the error terms. These facts suggest an instrumental variables approach to estimate the model.

In this study, we use the iterative three-stage least square (3SLS) procedure to estimate the model. As is common with estimation of the budget share equations, autocorrelation was clearly a problem in our initial estimation. We therefore introduce the first-order autoregressive scheme based on an order N parameterization of the autocovariance matrix using the full information maximum likelihood algorithm of Moschini and Moro (1994). To estimate the model, we consider the following set of instruments: a constant, the time trend, and the time trend squared, the first-order lags of all potentially nonexogenous variables excluding any current variables appearing in the equation system.

To summarize the estimation methodology, the budget share equations for $\mathbf{x}_t$ in (24) and the Euler Eq. (26) form a joint estimating system with the adding-up condition imposed. This system is estimated to obtain the required parameter estimates with 3SLS allowing for serial correlation, using chosen instruments because the variables—the right-hand side variables in (24) and the left-hand side variables in (26)—are not strictly exogenous. Then, with the estimated parameters, we check for a model fit by calculating R^2 and the J test of overidentifying restrictions to assess the relevance of the instruments.

4 Data

In incomplete demand systems, the goods of interest, $\mathbf{x}_t$, are observable with respect to their prices and quantities. For other goods, $\mathbf{z}_t$, however, there are no readily available data on their quantities unless we take a survey of households about the consumption of individual goods, which takes time. The partitioning of goods into$\mathbf{x}_t$ and $\mathbf{z}_t$ subgroups is done on a case-by-case basis, and reflects the analyst's interest in the welfare implications of price changes for the goods in $\mathbf{x}_t$ and/or a lack of consumption data for $\mathbf{z}_t$ (Beghin et al. 2004; von Haefen 2010). The incomplete demand model has been utilized to analyze the demand for outdoor recreation (von Haefen 2010) and the demand for disaggregate foods (Beghin et al. 2004; Zhen et al 2014), by treating unobservable goods as a composite numeraire good with their prices taken as the composite price. This is clearly undesirable, and the shadow demand model provides an answer to this problem. However, given a lack of readily available data on unobservable goods, to illustrate the proposed methodology, we use published data that contain a full set of data on goods and rely on our judgment to choose certain goods considered to be reasonably unobservable, and estimate the incomplete demand system.

We use annual French data on the consumption expenditure of households covering the period 1977–2020. These data are obtained from Eurostat online and come from the "Economy and Finance" statistics (code "nama_10_co3_p3"), where household consumption is classified in terms of the COICOP (Classification Of Individual Consumption Purpose) 3-digit classification. The website contains information on consumption expenditure in current prices (million Euros), percentage of total consumption, and the price index (2010 = 100), among others, for a range of European countries.[13] Estimation of the Euler Eq. (25) requires the interest rate, which is measured by a 3-month interest rate on short-term government bonds obtained from OECD data.[14]

We have constructed nine expenditure groups for goods and services consisting of (1) food and nonalcoholic beverages, (2) alcoholic beverages, tobacco and narcotics, (3) clothing and footwear, (4) housing services, (5) water and fuels, (6) health services, (7) transport services, (8) other nondurables and services, and (9) durable goods. For durable goods, we constructed durable stock and the user cost of durable goods (Kim et al. 2024). For observable goods, we take food and nonalcoholic beverages (x_1),

[13] See https://appsso.eurostat.ec.europa.eu/nui/show.do?dataset=nama_10_co3_p3&lang=en.

[14] See https://data.oecd.org/interest/short-term-interest-rates.htm.

alcoholic beverages, tobacco and narcotics (x_2), clothing and footwear (x_3), housing services (x_4), water and fuels (x_5), and transport services (x_6).

For possible unobservable goods, we consider health services (z_1), other nondurables and services (z_2), and durable goods (z_3). We acknowledge that the prices and quantities of all of these chosen unobservable goods are available or computable. Nevertheless, unlike other goods for which the measure of a quantity is clear, say the number of cars, there are some issues in quantifying these goods and services that could not be easily resolved. This is especially so for health care. For the services provided by medical doctors, to quantify these services, it is not clear to use the number of visits or the number of patients treated by them. For durable goods, unlike nondurable goods, consumers derive utility from the flow of services they provide over time. This implies that the relevant price of durable goods is not the purchase price, as in the case with nondurable goods, but the cost of using their services, i.e., the user cost or rental price of these goods. The service flow of durable goods can be assumed to be generated by the stock of these goods owned by consumers. While the user cost of durable goods is easy to measure, there are some issues measuring their stock (Deaton and Muellbauer 1980a). For these reasons, we treat health services and durable goods as unobservable goods, assuming that their quantities are not directly observable. We also include other nondurables and services as unobservable goods. They are an amalgamation of many goods and services, and we may not directly observe their quantities.

5 Estimation results

With the data described above, we jointly estimate an incomplete system of the six budget share equations for observable goods in (24), together with the Euler Eq. (26), using the 3SLS procedure in the TSP version 6.5 computer package. This procedure is well suited for the estimation of systems with complex cross-equation constraints; it also allows for heteroscedasticity in unknown form in the computation of the variance–covariance matrix. To assess information loss from estimation of the incomplete demand system relative to the complete demand system, we also estimate a complete demand system by including the three budget share equations for unobservable goods in (24), assuming that quantity data are made available for these goods.

5.1 Estimated model

Tables 1, 2 report joint estimation results for incomplete and complete demand systems, with Table 1 displaying model summary statistics and Table 2 presenting the parameter estimates. To ensure the estimated model is properly behaved, the regularity conditions for intra-temporal and intertemporal preferences, as discussed in Sect. 3.1, are checked and are satisfied for both incomplete and complete demand systems, with the inequality $M_t > P_A(\mathbf{p}_t, \mathbf{q}_t)$ holding, at every sample period. Table 1 shows that the J test for overidentifying restrictions for both incomplete and complete demand systems cannot reject the null hypothesis that the moment conditions for instruments are valid. This provides strong evidence for the relevance of the chosen instruments in our

Table 1 Joint estimation results of incomplete and complete budget share systems with Euler Equations (standard errors in parentheses): model summary statistics

	Incomplete system	Complete system
Log-likelihood value	1494.21	1993.380
J test (Overidentifying Restrictions)	test statistic = 59.633 df = 62 p value = 0.562	test statistic = 96.860 df = 88 p value = 0.243
Generalized R^2		
x_1	0.957	0.961
x_2	0.874	0.876
x_3	0.995	0.995
x_4	0.991	0.991
x_5	0.888	0.875
x_6	0.605	0.588
z_1	0.722	0.986
z_2	0.873	0.893
z_3	0.976	0.979
LM Test Statistics for Serial Correlation ($\chi^2_{2.5\%,4} = 11.143$)		
x_1	8.475	7.329
x_2	38.323	41.890
x_3	8.428	9.190
x_4	13.251	17.715
x_5	3.946	4.247
x_6	1.463	17.430
z_1	0.899	6.226
z_2	1.007	1.905
z_3	6.028	8.677

x_1 = food and nonalcoholic beverages, x_2 = alcoholic beverages, tobacco and narcotics, x_3 = clothing and footwear, x_4 = housing services, x_5 = water and fuels, x_6 = transport services, z_1 = health services, z_2 = other nondurables and services, and z_3 = durable stock. R^2 is calculated as the generalized R^2 for instrumental variables regressions (Pesaran and Smith 1994). LM test = Lagrange multiplier test of Breusch–Pagan

estimation. For both systems, the general fit of the budget share system, measured by the generalized R^2 for instrumental variables regressions (Pesaran and Smith 1994), is quite good. Autocorrelation diagnostic for both systems, revealed in the LM (Lagrange multiplier) χ^2 statistics, suggests that serial correlation in the error terms is no longer a concern for most goods.[15]

[15] Autocorrelation is tested by the modified version of the LM test since some of the regressors in the equation system may not be strictly exogenous (Fisher et. al. 2002). To perform this test, the residuals (e_{it})

Table 2 Joint estimation results of incomplete and complete budget share systems with Euler equations (standard errors in parentheses): parameter estimates

	Incomplete system	Complete system
α_1^x	0.132 (0.013)	0.161 (0.010)
α_2^x	0.023 (0.004)	0.032 (0.004)
α_3^x	0.052 (0.007)	0.060 (0.007)
α_4^x	0.181 (0.016)	0.225 (0.012)
α_5^x	0.060 (0.008)	0.075 (0.006)
α_6^x	0.050 (0.045)	0.024 (0.003)
α_1^z	0.099 (0.012)	0.120 (0.011)
α_2^z	0.167 (0.024)	0.184 (0.014)
α_3^z	0.138 (0.021)	0.119 (0.012)
β_1^x	0.058 (0.059)	0.058 (0.069)
β_2^x	0.087 (0.033)	0.083 (0.034)
β_3^x	0.030 (0.026)	0.003 (0.037)
β_4^x	0.076 (0.070)	0.164 (0.070)
β_5^x	0.030 (0.035)	0.000 (0.057)
β_6^x	0.262 (0.508)	0.069 (0.021)
β_1^z	0.094 (0.050)	0.134 (0.071)
β_2^z	0.363 (0.489)	0.393 (0.097)
β_3^z	0.000 (0.216)	0.095 (0.054)
τ_A	0.923 (0.088)	0.819 (0.049)
η	0.188 (0.087)	0.148 (0.061)
ξ	0.684 (0.108)	0.695 (0.082)
ρ	0.044 (0.005)	0.043 (0.005)

Estimation is based on the budget share equations with unobservable goods (24) (and without them) and the Euler Eq. (26) using nonlinear 3SLS

Table 2 presents the model parameter estimates for both incomplete and complete demand systems. The symbols a^x's and a^z's represent the estimated parameters that describe the price index $\mathrm{P}_A(\mathbf{p}_t,\ \mathbf{q}_t)$, and β^x's and β^z's are the estimated parameters for the price index $\mathrm{P}_B(\mathbf{p}_t,\ \mathbf{q}_t)$. Accordingly, these estimated parameters generate the two price indexes, $P_A(\mathbf{p}_t, \mathbf{q}_t)$ and $P_B(\mathbf{p}_t, \mathbf{q}_t)$, which form the MPIGLOG indirect utility function (18). They are used to estimate shadow demands for unobservable goods discussed in Sect. 5.2, expenditure and price elasticities for incomplete as well as complete demand systems presented in Sect. 5.3, and welfare effects of price changes

Footnote 15 continued
from each share equation are regressed on the current values of regressors ($\mathrm{v_{jt}}$) and the lagged residuals from the ith share equations; i.e., $e_{it} = \sum_{j=1} \gamma_{\mathrm{ij}} \mathrm{v_{jt}} + \sum_{j=1}^{p} \rho_{\mathrm{ij}} e_{\mathrm{it\text{-}j}} + \varepsilon_{\mathrm{it}}$ where γ_{j} and ρ_{j} are the coefficients and $\varepsilon_{\mathrm{it}}$ is the error term. Then, the LM test statistic is Chi-square distributed with p degrees of freedom. Failing to reject the null hypothesis $\mathrm{H_0}$: $\rho_{\mathrm{i1}} = \ldots = \rho_{\mathrm{ip}}$ indicates that there is no autocorrelation.

analyzed in Sect. 5.4. Looking at the incomplete demand system, the parameter estimates are reasonable in that they satisfy the regularity conditions for the price indices, and most of them are significant at conventional significance levels. The estimated value of η (0.188) is significantly different from zero at conventional significance levels, providing clear evidence against homotheticity for within-period preferences.[16] For the complete demand system, although the parameter estimates show some different values from those of the incomplete demand system, the two sets of estimates are, in large part, not substantially far off. In fact, given that they are expressed in decimals, they can be considered comparable. This result is very encouraging for our incomplete shadow demand model for unobservable goods because, in the absence of available quantity data for some goods, it provides sufficient information to adequately analyze observed consumer behavior in relation to the complete demand model. We investigate this finding further in the succeeding discussions below.

5.2 Shadow and actual quantity indexes for unobservable goods

Using the parameter estimates of the incomplete demand system in Tables 2, we can generate the fitted shadow quantity series for the three unobservable goods considered in our analysis: health services (z_1), other nondurables and services (z_2), and durable goods (z_3). These series are derived from the shadow budget share equations in (25) as

$$\widehat{z}_{jt} = \widehat{W}^{z}_{jt} \frac{M_t}{q_{jt}} = \left\{ \frac{E^{z}_{Ajt} + \eta\beta^{z}_{j} R_t}{1 + \eta R_t} \right\} \frac{M_t}{q_{jt}}, \ j = 1, 2, 3. \tag{27}$$

To see whether the estimated shadow quantity series $\widehat{z}_{jt}$ under-, or over-, estimate the actual quantity series z_{jt} in the data, Fig. 1 plots the time paths of the two quantity series for the three unobservable goods. The quantity indexes can be considered real expenditure on these goods. Looking at the actual quantity series, other goods and durable goods show an increasing trend over time, while health services show a stable pattern within the sample period. A comparison of the actual and shadow quantity series reveals that there is no measurable difference between them for durable goods. However, the shadow quantity index appears to underestimate between them for durable goods. However, the shadow quantity index appears to underestimate the actual quantity series for health services; on the other hand, the shadow quantity index appears to overestimate the actual quantity series for other goods.

[16] Although it is not a main concern of our analysis, it is worth noting that the estimated time preference rate of 0.044 means that consumers discount the utility of future consumption at an annual rate of 4.4% with the discount factor (1/(1 +)) of 0.96. The intertemporal stability condition for consumers with a fixed interest rate requires that the interest rate is greater than the time preference rate for a growing economy (Kim 2017). With the average interest of 5.33% during the sample period, this condition is satisfied. However, since 1997, the interest rate in France has been falling much below the estimated time preference rate. This may imply that French consumers, to a large extent, appear to be impatient, in the sense that they have a high time preference for present consumption relative to the risk-free interest rate.

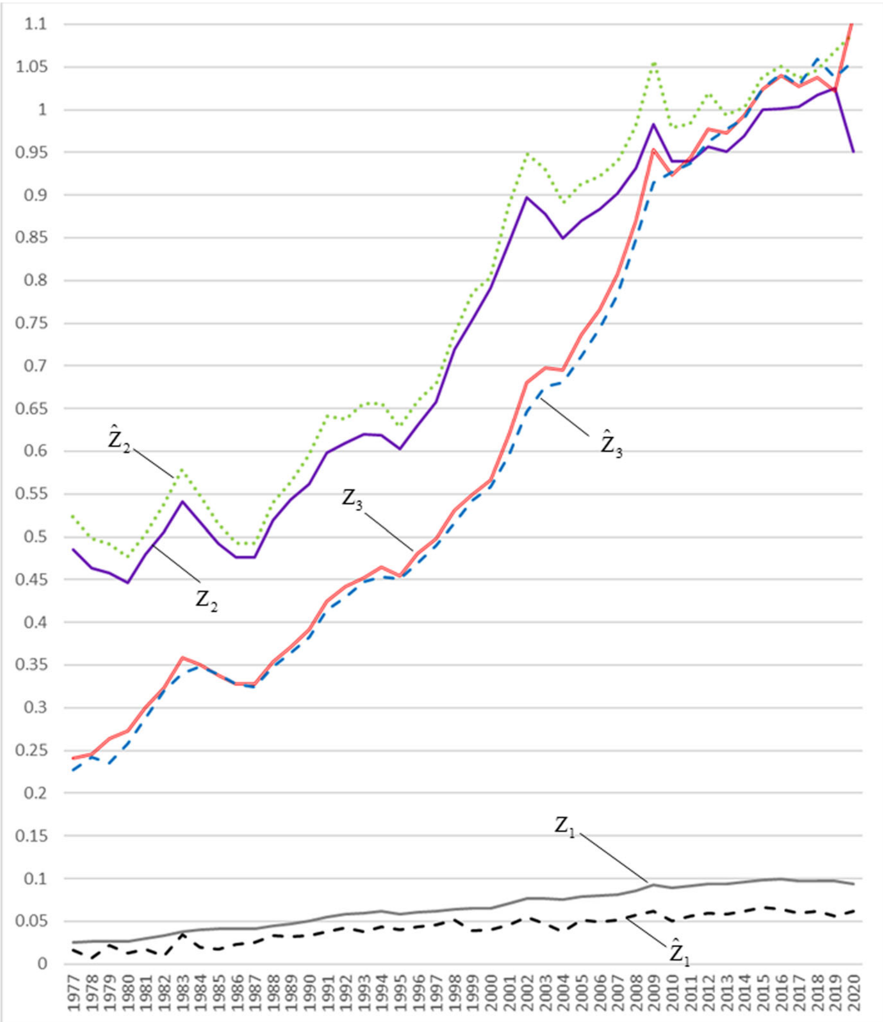

Fig. 1 Actual versus shadow quantity indexes of unobserved Goods, 1977–2020. *Notes*——— z_1 = Actual quantity index of health services. – – – – – $\widehat{z}_1$ = Fitted shadow quantity index of health services. ——— z_2 = Actual quantity index of other goods and services. ··········· $\widehat{z}_2$ = Fitted shadow quantity index of other goods and services. ═══ z_3 = Actual quantity index of durable goods. – – – – – $\widehat{z}_3$ = Fitted shadow quantity index of durable goods

While these results are informative, they do not provide conclusive evidence whether the two series are statistically different. Accordingly, we conduct two formal tests—t and χ^2 tests—to substantiate them. For the$t$ test, the null hypothesis is no significant difference between the actual and shadow quantity series for each unobservable variable at each period. The χ^2 test is a goodness of fit test, and the null hypothesis is no significant difference between the observed and shadow quantity series for each unobservable variable for the whole sample period.

The two test results are summarized in Table 3 with the relevant test statistics. The *t* test reveals that there is no significant difference between the actual and shadow quantity series for health services and others at conventional significance levels, but for durable goods, there is some evidence for the difference between the two series for some sample years. These results are also corroborated by the χ^2 test. It confirms the *t* test for health services and others, but for durable goods, there is a significant difference between the actual and shadow quantity series for the whole sample period. We found that for durable goods, the standard error for some years (1996–2009) are surprisingly small implying that the *t* ratios and hence the χ^2 test statistics are surprising high in that period. These results suggest that, while there is no information loss in treating the quantities of health services and others as unobservable, it is not so for durable goods.

- The t-ratio is defined as $t = \frac{\left(z_j - \widehat{z}_j\right)}{se\left(\hat{z}_j\right)}$, where $se\left(\widehat{z}_j\right)$ is standard error of the shadow quantity index of unobservable good j.
- The test statistic is $\chi^2 = \sum_{i=1}^{N} \frac{\left(z_j - \widehat{z}_j\right)^2}{Var\left(\widehat{z}_j\right)}$, where $Var\left(\widehat{z}_j\right)$ is variance of the shadow quantity index of unobservable good j, which is distributed as Chi-square $\left(\chi^2\right)$ with degree of freedom $= N$ (number of observations).

5.3 Estimated demand elasticities for incomplete and complete demand systems

Price and income or expenditure elasticities of goods and services are important summary measures characterizing consumer behavior. The derivations of these elasticities are presented in Appendix A. Table 4 displays the estimated expenditure and price elasticities for the incomplete and complete demand systems evaluated at the sample means of the variables, using the parameter estimates of Table 2. The complete demand system is based on the premise that health services (z_1), other nondurables and services (z_2), and durable goods (z_3) are observable, but the incomplete demand system is not. The estimated expenditure and price elasticities are significant at conventional significance levels for both incomplete and complete demand systems. Overall, there is no essential difference in the two elasticities between the incomplete and complete demand systems. Given that there is no substantial difference found between the parameter estimates for the incomplete and complete demand systems presented in Table 2, this is as expected. All of the goods are found to be price-inelastic for both incomplete and complete demand systems. There is no clear pattern for the estimated expenditure elasticities, but alcoholic beverages, tobacco, and narcotics (x_2) and health services (z_1) are weakly income-elastic, especially for the incomplete demand system.

There are studies estimating income and price elasticities of demand for French consumers (see Appendix B for a literature review). However, they do not consider durable goods and use different groupings of nondurable goods. These results are, thus, not directly comparable to the ones reported in Table 3. Yet, some of those findings for

Table 3 Tests for unobservable quantity indexes

Period (Health Services)	$z_1 - \widehat{z}_1$	p-value	Period (Others)	$z_2 - \widehat{z}_2$	p-value	Period (Durables)	$z_3 - \widehat{z}_3$	p-value
1977	0.009	0.333	1977	– 0.038	0.044	1977	0.014	0.011
1981	0.012	0.292	1981	– 0.024	0.116	1981	0.012	0.016
1985	0.024	0.292	1985	– 0.023	0.261	1985	– 0.001	0.431
1989	0.015	0.323	1989	– 0.021	0.226	1989	0.008	0.056
1993	0.022	0.318	1993	– 0.036	0.192	1993	0.004	0.166
1997	0.016	0.374	1997	– 0.022	0.299	1997	0.008	0.081
2001	0.025	0.316	2001	– 0.043	0.202	2001	0.024	0.000
2005	0.028	0.326	2005	– 0.043	0.245	2005	0.026	0.000
2009	0.032	0.318	2009	– 0.075	0.141	2009	0.039	0.000
2014	0.035	0.326	2014	– 0.043	0.302	2014	– 0.005	0.110
2018	0.038	0.329	2018	– 0.034	0.351	2018	– 0.001	0.426
2020	0.032	0.346	2020	– 0.138	0.487	2020	0.052	0.000
1977–2020								
χ^2 Test Statistic = 9.246			χ^2 Test Statistic = 35.212			χ^2 Test Statistic = 382.272		
$\chi^2_{5\%,41} = 60.481$, p -value =1.000			$\chi^2_{2.5\%,41} = 64.201$, p -value =0.725			$\chi^2_{1\%,41} = 68.710$, p -value =0.000		

z_j ($j = 1$–3) is the observed quantity index of unobservable good j, and $\widehat{z}_j$ is its shadow quantity index.

Table 4 Estimated Expenditure and Price Elasticities (p-value in parentheses)

Commodities	Expenditure elasticities		Own price elasticities	
	Incomplete	Complete	Incomplete	Complete
x_1	0.916 (0.000)	0.923 (0.000)	– 0.245 (0.002)	– 0.323 (0.000)
x_2	1.237 (0.000)	1.163 (0.000)	– 0.496 (0.000)	– 0.439 (0.000)
x_3	0.934 (0.000)	0.879 (0.000)	– 0.209 (0.021)	– 0.228 (0.000)
x_4	0.916 (0.000)	0.974 (0.000)	– 0.274 (0.000)	– 0.380 (0.000)
x_5	0.926 (0.000)	0.873 (0.000)	– 0.204 (0.017)	– 0.232 (0.000)
x_6	1.278 (0.009)	1.164 (0.000)	– 0.544 (0.165)	– 0.445 (0.000)
z_1	0.989 (0.000)	1.018 (0.000)	– 0.282 (0.000)	– 0.358 (0.000)
z_2	1.100 (0.000)	1.105 (0.000)	– 0.402 (0.000)	– 0.437 (0.000)
z_3	0.847 (0.000)	0.979 (0.000)	– 0.203 (0.005)	– 0.346 (0.000)

x_1 = food and nonalcoholic beverages, x_2 = alcoholic beverages, tobacco and narcotics, x_3 = clothing and footwear, x_4 = housing services, x_5 = water and fuels, x_6 = transport services, z_1 = health services, z_2 = other nondurables and services, and z_3 = durable stock. Estimation is based on the price and expenditure elasticities derived in Appendix A, evaluated at the sample means of the variables using the parameter estimates of Table 2

the estimated expenditure elasticities are noteworthy. Health services or health care are typically considered a necessity. The finding of health services as a luxury, though not strong, for both incomplete and complete demand systems means that the share of income spent on health expenditures rises with income, and it is believed that this is a consequence of rising income or living standards (Hall and Jones 2007). Durable goods are found to be a necessity for both incomplete and complete demand systems. This result is markedly different from previous finding that durable goods are luxuries (see Clements et al. 2020). Traditional demand studies, however, estimate the demand for durable goods in a flow form, but our analysis is based on a stock form with the user cost. There are also studies for the United States based on a stock form finding durable goods as luxuries, but they are limited in scope and analysis with restrictive utility functions (Mankiw 1985). There might be some expensive or large durable goods that are luxuries, but most of them are small and can be treated as necessities.

It is important to note that treating health services, other goods, and durable goods as unobservable, in large part, does not produce much difference in their expenditure and price elasticities from those estimated by treating them as observable. This result is different from what we found from Table 3 where there is no information loss in treating the quantities of health services and others as unobservable, but it is not so for durable goods. Thus, while the results for health services and others are corroborated with the demand elasticities, they are not born out for durable goods. However, the expenditure and price elasticities are more relevant to understand consumer behavior. Our finding here, then, suggests that even though quantity data are not readily available or not directly observed for health services, other goods, and durable goods, the estimated incomplete shadow demand model provides sufficient information about them to enable us to understand consumer behavior.

5.4 Estimated welfare effects of changes in prices

Since we can recover the indirect utility function (18) from estimation of the demands for observable goods (24) and the Euler Eq. (26), we can utilize it to assess the impacts of changes in the prices of unobservable goods and alternative economic policies on the welfare of the consumer, using exact measures of the compensating and equivalent variations (Deaton and Muellbauer 1980a).

The consumer currently faces total expenditure denoted by M_t^0 and the prices of unobservable goods denoted by $\mathbf{q}_t^0$. Suppose now that there is a change in policy or circumstances that cause the prices to $\mathbf{q}_t^1$, with no change in total expenditure. The compensating variation (CV) is defined by

$$v(\mathbf{p}_t,\mathbf{q}_t^1,M_t^0+CV_t)=v(\mathbf{p}_t,\mathbf{q}_t^0,M_t^0)\equiv u_t^0, \tag{28}$$

for given $\mathbf{p}_t$. The CV is the amount of additional money necessary for the consumer to enjoy the utility level u_t^0 while facing $\mathbf{q}_t^1$ with M_t^0. A positive value for CV indicates that the consumer is worse off when facing $\mathbf{q}_t^1$ than they were with $\mathbf{q}_t^0$.

The equivalent variation (EV) is defined by

$$u_t^1\equiv v(\mathbf{p}_t,\mathbf{q}_t^1,M_t^0)=v(\mathbf{p}_t,\mathbf{q}_t^0,M_t^0-EV_t), \tag{29}$$

for given $\mathbf{p}_t$. The EV is the amount of additional money that would enable the consumer to maintain the new utility level u^1 while facing $\mathbf{q}_t^0$ with M_t^0. As with the CV, a positive EV value indicates that the consumer is made worse off under $\mathbf{q}_t^1$ than under $\mathbf{q}_t^0$.[17]

The MPIGLOG indirect utility function (18) is highly nonlinear in M_t, so it is not feasible to derive a closed form for the CV in (28) and the EV in (29). We, therefore, rely on a numerical method. This is appropriate because the indirect utility function $\nu(\mathbf{p}_t,\ \mathbf{q}_t,\ M_t)$ is strictly increasing in M_t, as can be seen from $\partial\nu(\mathbf{p}_s,\ \mathbf{q}_s,\ M_s)/\partial M_t>0$ in (22) for the MPIGLOG indirect utility function, which allows us to numerically solve (28) and (29) for the CV and the EV, respectively, using (18). To carry out the calculations of the CV and the EV, we utilize the GAUSS program language (the NLSYS module of GAUSS version 11.0).

Table 5 presents the welfare estimates, as measured by the CV and the EV, from increases in period, for the incomplete and complete demand systems using the parameter estimates in Table 2. To calculate them, we assume an arbitrary 10% increase in the prices of food and beverages, water and fuel, health services, and durable goods

[17] The CV and EV can also be defined in terms of the expenditure function defined by $e(\mathbf{p}_t,\ \mathbf{q}_t,\ u_t)$ by relating to the indirect utility function. The expenditure function can be derived from the indirect utility function by solving $u_t=\nu(\mathbf{p}_t,\ \mathbf{q}_t,\ M_t)$ for M_t, which gives $M_t=e(\mathbf{p}_t,\ \mathbf{q}_t,\ u_t)$. Now, from (28), we get $M_t^0+CV=e(\mathbf{p}_t,\ \mathbf{q}_t^1,\ u_t^0)$ and $M_t^0=e(\mathbf{p}_t,\ \mathbf{q}_t^0,u_t^0)$. Combining the two expressions gives $CV_t=e(\mathbf{p}_t,\ \mathbf{q}_t^1,\ u_t^0)-e(\mathbf{p}_t,\ \mathbf{q}_t^0,\ u_t^0)$. Similarly, from (29), we get $M_t^0-EV_t=e(\mathbf{p}_t,\ \mathbf{q}_t^0,\ u_t^1)$ and $M_t^0=e(\mathbf{p}_t,\ \mathbf{q}_t^1,\ u_t^1)$, which gives $EV_t=e(\mathbf{p}_t,\ \mathbf{q}_t^1,\ u_t^1)-e(\mathbf{p}_t,\ \mathbf{q}_t^0,u_t^1)$. Wiling (1976) uses the income compensation function, but this function plays the same role as the expenditure function.

Table 5 Estimates of compensated and equivalent variations for 10% increases in prices of selected commodities

Commodity	Incomplete system				Complete system			
	CV (€)	%CV	EV (€)	%EV	CV (€)	%CV	EV (€)	%EV
1977								
Food	36.315	1.397	35.888	1.381	41.669	1.603	41.086	1.581
Fuel	16.559	0.527	16.509	0.526	15.648	0.602	15.593	0.600
Health Services	26.739	0.852	26.473	0.843	25.683	0.988	25.448	0.979
Durables	64.261	2.046	63.209	2.054	44.484	1.712	43.802	1.685
1987								
Food	79.280	1.258	78.368	1.244	91.465	1.452	90.244	1.432
Fuel	43.676	0.539	43.442	0.536	38.295	0.608	38.088	0.604
Health Services	67.145	0.829	66.398	0.820	65.491	1.039	64.791	1.028
Durables	156.537	1.933	154.020	1.938	103.282	1.639	101.727	1.614
1998								
Food	123.020	1.272	121.606	1.257	138.966	1.437	137.170	1.418
Fuel	63.158	0.539	62.846	0.537	56.597	0.585	56.380	0.583
Health Services	110.130	0.941	108.848	0.930	113.467	1.173	112.145	1.160
Durables	146.981	1.255	145.436	1.258	112.249	1.161	111.035	1.148
2009								
Food	176.339	1.302	174.335	1.287	196.294	1.449	193.644	1.430
Fuel	94.725	0.585	94.304	0.582	83.846	0.619	83.381	0.616
Health Services	159.494	0.985	157.676	0.973	172.749	1.275	170.544	1.259
durables	133.098	0.822	132.261	0.823	115.014	0.849	114.045	0.842
2020								
Food	201.463	1.346	199.154	1.330	221.966	1.483	219.058	1.463
Fuel	121.505	0.700	120.897	0.694	109.361	0.730	108.784	0.727
Health services	168.993	0.970	167.154	0.960	196.922	1.315	194.480	1.299
Durables	108.546	0.623	108.134	0.625	103.684	0.693	102.990	0.688
Sample mean								
Food	126.080	1.290	124.625	1.275	141.687	1.459	139.837	1.440
Fuel	70.621	0.583	70.249	0.579	62.594	0.633	62.275	0.630
Health services	111.196	0.916	109.921	0.906	119.325	1.172	117.895	1.159
Durables	137.987	1.345	136.467	1.349	105.593	1.221	104.431	1.206

The column entitled %CV (or %EV) denotes compensating (or equivalent) variation as a percent of total consumption expenditure (including durable stocks) per capita. Calculation is based on the compensating variation, CV, in (28) and the equivalent variation, EV, in (29), using the parameter estimates of Tables 1, 2 in each year. The following comments are in order. First, overall, there is no substantial difference in the welfare estimates between the incomplete and complete demand systems. Although the incomplete demand system tends to overestimate the welfare estimates for durable goods relative to the complete demand system, the difference cannot be considered drastically substantial. Second, the estimated CVs are greater than the EVs. This is expected because of nonnegligible income effects (see Willig 1976), as indicated by significant nonzero expenditure elasticities found in Table 3. However, the numerical differences between the CV and EV estimates are rather small, amounting no more than €3 in all cases. Third, the estimated CV and EV indicate that French consumers are made worse off after the increase in prices. During the sample period, on average, when we look at the CV for the complete demand model, the largest welfare loss they experience comes from the increases in the prices of food and beverages, while the smallest welfare loss is from the increases in the prices of water and fuel. Fourth, in general, there is an increase in welfare loss over time, but as a percentage of total expenditure (%CV and %EV), it remains about the same except for durable goods. It is however, important to note that French consumers suffer the most from the increase in the price of food in recent years. Lastly, when the effects of all selected price increases are aggregated, we find that the average CV and EV estimates range from €424.4 to €445.9, which is about 4.1–4.5% of per capita total consumption expenditure.

6 Summary and conclusion

For unobservable goods whose prices are observed but quantities are not available, there are no explicit demands for them, but we can find the shadow demands—the implicit quantities the consumer would be willing to purchase at given prices and income—for a better understanding of consumer behavior. These demands have not been formally analyzed in previous studies. In this paper, we present an intertemporal incomplete demand model to formalize the shadow optimization problem. To illustrate the proposed methodology, we estimate the incomplete demand system model composed of the observable demand system and the shadow demand system demands for unobservable goods, by deriving the Euler equation from an intertemporal together with the Euler equation, using the French data with a full set of data on goods, choosing a few goods as possible unobservable goods. Then we evaluate information loss from estimating the incomplete, rather than complete, demand systems when quantity data are assumed available for unobservable goods. We find that the shadow demands for unobservable goods estimated from the incomplete demand system, in general, do not create much information loss in analysis relative to the complete demand system. This indicates that the shadow demands can be analyzed without much difficulty to understand consumer behavior even though quantity data are not readily available or directly observed for some goods.

These results are illuminating and reveal the utility of the proposed intertemporal shadow demand framework for unobservable goods. However, due to a lack of available information about real unobservable goods, we treated some goods with data on quantities as possible unobservable goods. Our hope is to find relevant data on real unobservable goods and conduct a more realistic analysis of the intertemporal incomplete demand model.

In closing, our study demonstrates the importance of an intertemporal framework in incomplete demand analysis. This framework can also allow for nonmarket goods to analyze the shadow prices of these goods, in sharp contrast to existing studies on nonmarket goods (see the Introduction). The issue here is whether we can recover underlying consumer preferences, defined over market and nonmarket goods, from a demand system for market goods that is observable and depends on nonmarket goods. This, in turn, allows us to make welfare measurement for changes in nonmarket goods. Although it is widely believed that this is possible under certain special conditions such as weak complementarity or weak neutrality, Ebert (1998) shows that if the marginal willingness to pay functions for nonmarket goods can be estimated from surveys and questionnaires, this information together with the estimated demands for market goods can be used to recover underlying consumer preferences and hence to derive welfare measures for nonmarket goods. However, the intertemporal framework can be utilized to analyze nonmarket goods when there is no available information about the marginal willingness to pay for these goods, without making any assumption about preferences. This is based on the idea that the intertemporal first-order condition captured by the Euler equation for consumption, derived with a Box–Cox transformation of the indirect utility function, contains all information to identify underlying consumer preferences over market and nonmarket goods. In our framework, this can be done by treating the prices of unobservable goods as nonmarket goods and the shadow demand Eqs. (25) as the shadow price equations. Kim, et al. (2020) applied this procedure to public goods to measure the shadow price or marginal willingness to pay for them using national defense as an example.

Additionally, by specifying an indirect utility function that incorporates the prices of unobservable market goods as well as nonmarket goods, the demands for observable market goods are derived and estimated together with the Euler equation that contains all the information necessary to identify the specified indirect utility function. This allows to infer the shadow demands for unobservable market goods as well as the shadow prices of nonmarket goods, and hence to recover the indirect utility function and in turn to conduct welfare analysis of unobservable market goods and of nonmarket goods. This suggests that unobservability of the quantities or prices of some goods cannot be a hindrance to analyze consumer behavior in empirical demand analysis, when properly conducted.[18] Our analysis can also be utilized when there are insufficient or missing observations for the prices and quantities of some variables by treating them as unobservable. Thus, the intertemporal framework provides a new perspective into demand analysis by allowing us to tackle some relevant issues that cannot be addressed in previous studies. Moreover, this framework allows to jointly accounts for the consumer's intra-temporal and intertemporal choices, with an integrated analysis of consumer demand and consumption, which helps us better understand consumer behavior (Kim et al. 2021, 2024).

[18] LaFrance and Hanemann (1989), however, hold that although the welfare measures (CV and EV) associated with a change in the price of observable market goods can be derived from an incomplete demand system, for a change in nonmarket goods, these measures cannot, in general, be derived from any incomplete demand system (Hanemann and Morey 1991; Ebert 1998).

Appendix A: Expenditure and Price Elasticities of Demand

From the budget share Eqs. (24) and (25), we can derive the expenditure (or income) and price elasticities of the demands for observable goods and of the shadow demands for unobservable goods. Let $\mathbf{I} = \{1,\ldots, n\}$ and $\mathbf{J} = \{1,\ldots, m\}$ be two index sets for the observed and unobserved goods respectively. Then we have.

Expenditure elasticities

$$\frac{\partial \ln x_{it}}{\partial \ln M_t} \equiv 1 + \frac{\eta\beta_i^x}{E_{A_{it}}^x + \eta\beta_i^x R_t} - \frac{\eta}{1+\eta R_t}, \quad i \in \mathbf{I},$$

$$\frac{\partial \ln z_{jt}}{\partial \ln M_t} = 1 + \frac{\eta\beta_j^z}{E_{A_{jt}}^z + \eta\beta_j^z R_t} - \frac{\eta}{1+\eta R_t}, \quad j \in \mathbf{J},$$

where $E_{A_{it}}^x$, $E_{A_{jt}}^z$, and R_t are defined in expressions (20) and (21).

Own/cross-price elasticities

$$\frac{\partial \ln x_{it}}{\partial \ln p_{i't}} = -\delta_{ii'} + \frac{E_{Aii't}^x - \eta\beta_i^x E_{A_{i't}}^x}{E_{A_{it}}^x + \eta \cdot \beta_i^x R_t} + \frac{\eta E_{A_{i't}}^x}{1+\eta R_t}, \quad i \text{ and } i' \in \mathbf{I},$$

$$\frac{\partial \ln x_{it}}{\partial \ln q_{jt}} = \frac{E_{Aijt} - \eta\beta_i^x E_{A_{jt}}^z}{E_{A_{it}}^x + \eta\beta_i^x R_t} + \frac{\eta E_{A_{jt}}^z}{1+\eta R_t}, \quad i \in \mathbf{I}, \text{ and } j \in \mathbf{J},$$

where $\delta_{ii'}$ is the Kronecker delta, $E_{Aii\prime t}^x \equiv \frac{\partial E_{Ait}^x}{\partial \ln p_{i\prime t}} = E_{Ait}^x\left(\delta_{ii'} - E_{Ai\prime t}^x\right)$, and $E_{Aijt} \equiv \frac{\partial E_{Ait}^x}{\partial \ln q_{jt}} = -E_{Ait}^x E_{Ajt}^z$.

$$\frac{\partial \ln z_{jt}}{\partial \ln q_{j\prime t}} = -\delta_{jj'} + \frac{E_{Ajj\prime t}^z - \eta\beta_j^z E_{A_{j\prime t}}^z}{E_{A_{jt}}^z + \eta\beta_j^z R_t} + \frac{\eta E_{A_{j\prime t}}^z}{1+\eta R_t}, \quad \text{j} \in \text{ and j}' \in \mathbf{J}$$

$$\frac{\partial \ln z_{jt}}{\partial \ln p_{it}} = \frac{E_{Ajit} - \eta\beta_j^z E_{A_{it}}^x}{E_{A_{jt}}^z + \eta\beta_j^z R_t} + \frac{\eta E_{A_{it}}^x}{1+\eta R_t}, \quad \text{i} \in \mathbf{I}, \text{ and j} \in \mathbf{J},$$

where $\delta_{jj'}$ is the Kronecker delta, $E_{Ajj\prime t}^z \equiv \frac{\partial E_{Ajt}^z}{\partial \ln q_{j\prime t}} = E_{Ajt}^z\left(\delta_{jj'} - E_{Aj\prime t}^z\right)$, and $E_{Ajit} \equiv \frac{\partial E_{Ajt}^z}{\partial \ln p_{it}} = -E_{Ajt}^z E_{Ait}^x$.

Appendix B: Brief literature review on consumer demand studies on France

The analysis of demand in France has traditionally taken two approaches. First, it focused on food expenditures using time series, with Fulponi (1989) being the first paper applied to food and meat groups. He estimated an AIDS model and found that meat and dairy products are necessities, but the rest of the food categories are classified as luxury goods. The literature has been extensive in the following decades, estimating demand systems for very specific foods (e.g., red meat, beef, seafood, salmon, vegetables, walnuts, alcohol, drugs, psychotropics, etc.) and other goods (e.g., energy, gas, etc.). All of these studies are concerned with the intra-temporal allocation of consumption expenditure to different goods. Molina (1995) is the only paper which estimated the intertemporal behavior of French consumers using time series. He tested the intertemporal separability hypothesis in France, estimating the SNAP model with annual time series of nondurable goods expenditures for the period 1964–1992 (OECD). The results indicated that intertemporal separability is not accepted, that is to say, French consumers consider both current exogenous variables, and one-period leading and one-period lagged prices, to be important in their consumption decisions.

The second approach uses micro data (French Household Budget Survey—FHBS) in the collective context, *à la* Chiappori, which is based on the efficient hypothesis. The "French School" of the collective framework has allowed the exploration of different intra-household issues, consumption, labor supply, and welfare. Specific analyses of intra-household inequalities are attractive because they calculate gender differences, which can be used for policy purposes. The following three papers explore different issues of intra-household expenditures. Bourguignon et al. (1993) used the FHBS-1984/85 to confirm that the intra-household composition of family income seems to influence household behavior with respect to expenditures, even when total income is fixed. Additionally, this first empirical implementation of the collective model in France suggests that behavior may prove to be consistent with the cooperative hypothesis.

Couprie et al. (2010) employed the FHBS-2000 to test the concavity of the public and private sharing functions of expenditures. They do not reject this double concavity condition and, consequently, indicate that the bargaining power within couples does not appear to change with household income, so the structure of intra-household allocations can be ignored in welfare comparisons across individuals. The authors conclude that French couples seem to behave in a highly egalitarian way. Bargain and Donni (2012) also use the FHBS-2000 in the context of the collective model, to estimate that the parents' expenditures on children living in the household amount to about 23–27% of household total expenditure. The authors also show that expenditures made by parents for boys seem to be larger than for girls in the case of one-child families.

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