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The Exponential Versus the Complex Power e^z Function Revisited

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Abstract: The complex exponential function exp is a well-known entire function. In this paper, we recall its relation with the definition of the complex power of a complex number, which emanates that the complex power e^z may coincide with it at some complex values. Still, on most occasions, the power represents a much broader spectrum of complex values. We also outsight how the software *Mathematica* may become a valuable tool for evaluating and visualizing complex power functions, in some cases by introducing some specific commands that have not been implemented in the software.

Keywords: complex variable; complex exponential; complex logarithm; complex power of a complex number

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1. Introduction

Complex Analysis has widened its applications since its origins at the beginning of the nineteenth century. Today, it is the field of mathematics where researchers develop their findings in a variety of areas such as the geometric characteristics of these functions [1], differential and integral operators [2,3], differential and difference equations [4], some of which have a close relationship with Nevanlinna's value distribution theory of meromorphic functions [5,6], improper integrals evaluation [7–11], and fractal theory [12].

In addition to its mathematical applications, its use encompasses many engineering disciplines such as aerodynamics and elasticity [13–15], related to mechanical or movement phenomena [16], and others, which include Electrical Circuits and Signal Processing associated with electrical or electronic phenomena. In some of these applications, Euler's formula

$$e^{iy} = \cos y + i \sin y,\ y \in \mathbb{R}, \tag{1}$$

is essential. Beyond setting a key connection between exponential and trigonometric functions, it becomes one of the most remarkable formulas in mathematics, enabling the expression of a complex number z with modulus r and argument α, in trigonometric form

$$z = r\text{cis}\,(\alpha) := r(\cos\alpha + i\sin\alpha),$$

in its exponential representation, $z = re^{i\alpha}$.

Equation (1) provides the definition of the exponential of a pure complex number. The complex exponential function, exp, is standardly defined by means of its power series

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!},\ z \in \mathbb{C}, \tag{2}$$

which leads, by considering the real and imaginary parts of z, $x = \Re(z), y = \Im(z)$, to

$$\exp(x + iy) = e^x(\cos y + i \sin y), (x, y) \in \mathbb{R}^2. \tag{3}$$

We note that the right-hand side (RHS) term of Equation (3) contains the real exponential e^x times the exponential e^{iy}.

The exponential function, represented on the left-hand side (LHS) term of Equation (3), is commonly denoted as e^{x+iy}, and this notation appears in some undergraduate courses to define and represent the exponential of a complex number analogously to the real case. This may lead to confusion once the student is introduced to the complex power of a complex number. In this context, the following natural question arises.

Research Question (RQ): When does the power of the number e considered as a complex number to the power $z = x + iy$ consist of merely the expression given by Equation (3)?

To address this RQ, we revisit the definition of the generalized complex exponentiation, or the complex power of a complex number, to outsight why, despite $\exp(x + iy)$ coinciding with $e^x e^{iy}$, its representation as e^{x+iy} dismisses the fact that the generalized complex exponentiation of e, considered as a complex number, to the power $x + iy \in \mathbb{C}$ may take infinitely many different values, only one of which coincides with the value $\exp(z) = e^x e^{iy}$.

To develop our presentation, this paper is organized as follows. Since the definition of the complex power of a complex number relies on the logarithmic function, we will go through that prerequisite in Section 2 to facilitate its readership. In Section 3, we recall the concept of complex power of a complex number. In Section 4, we will pay attention to the case where the basis of the power is a non-zero real number to address our RQ in a more general setting. In Section 5, we will go through the MATHEMATICA commands that facilitate calculating and representing the corresponding expressions, even when they are not implemented. The paper ends with the conclusions in Section 6.

2. The Complex Logarithm Function Revisited

Since $e^x \neq 0$ for all $x \in \mathbb{R}$, Equation (3) implies that 0 does not belong to the range of the complex exponential, $R_{\exp}$. Hence, when searching for the complex logarithm function, log, providing all possible values $w = \log(z)$, such that $\exp(w) = z$, it follows that it cannot hold at $z = 0$.

Easy calculations lead to the well-known expression

$$\log(z) = \ln|z| + i \arg(z), \tag{4}$$

where "ln" stands for the real logarithm function [1,9,10,17,18].

Consequently, the complex logarithm is a multi-valued function. This follows from the fact that $\exp(2n\pi i) = 1\ \forall n \in \mathbb{Z}$, and given any complex number $z = x + iy$, then

$$\exp(z) = e^x e^{iy}$$

coincides with the exponential of all complex numbers $z_n = x + i(y + 2n\pi)$, making the complex exponential function not one-to-one.

The multivalued function log generates that

$$\exp(\log(z)) = z, z \in \mathbb{C} \setminus \{0\}; \;\; \log(\exp(z)) = z + 2n\pi i, n \in \mathbb{Z}, z \in \mathbb{C}.$$

To obtain a complex single-valued function, the inverse of the complex exponential, we should restrict the domain of the exp function so that it is one-to-one, and select an adequate value of $\arg(z)$. An option is to select the *principal argument* of z, $\mathrm{Arg}\,(z)$, on the RHS of Equation (4) as the value of $\arg(z)$, obtaining the logarithm function *principal value*, $\mathrm{Log}\,(z)$, but any other argument may work as we recall next.

Remark 1. *It is a matter of taste to choose the semi-open interval where to evaluate* $\mathrm{Arg}\,(z)$*, among* $[-\pi, \pi)$*,* $[0, 2\pi)$*, or* $(-\pi, \pi]$*. We will consider the last one. In any of them, the principal argument of any positive real number is 0.*

Definition 1. *Given* $z \in \mathbb{C} \setminus \{0\}$*,* $\alpha_0 \in \mathbb{R}$*, we denote by* $\arg_{\alpha_0}(z)$*, and call* α_0-*argument of* z*, the unique value* $\alpha \in \arg(z)$ *such that*

$$\alpha_0 - \pi < \alpha \leq \alpha_0 + \pi.$$

We consider the principal argument $\mathrm{Arg}\,(z)$ *to be defined as* $\arg_0(z)$*.*

Definition 2. *We call* α_0-*argumented* $\log(\cdot)$*, or* α_0-*arg of* $\log(\cdot)$ *for short, the function defined by taking* $\arg_{\alpha_0}(\cdot)$ *on the RHS of Equation (4) to evaluate* $\arg(\cdot)$*. It is represented by* $(\log(\cdot))_{\alpha_0}$*, i.e.,*

$$(\log(z))_{\alpha_0} = \ln|z| + i \arg_{\alpha_0}(z).$$

Remark 2. *Given* $\alpha_0 \in \mathbb{R}$*, the range of* $(\log(\cdot))_{\alpha_0}$ *coincides with*

$$R_{\alpha_0} := \{(x + iy), x \in \mathbb{R}, y \in (\alpha_0 - \pi, \alpha_0 + \pi]\}. \tag{5}$$

Restricting the domain of the exponential function to R_{α_0}*, it holds*

$$\exp((\log(z))_{\alpha_0}) = z, z \in \mathbb{C} \setminus \{0\}; \;\; (\log(\exp(z)))_{\alpha_0} = z, z \in R_{\alpha_0}.$$

Clearly, there is no $(\log(\cdot))_{\alpha_0}$ *that is continuous in* $\mathbb{C} \setminus \{0\}$ *as* $\arg_{\alpha_0}$ *is not continuous on the half-line* $Ray(\alpha_0) := \{re^{i(\alpha_0+\pi)}, r \geq 0\}$*.*

Restricting $(\log(\cdot))_{\alpha_0}$ *to* $\mathbb{C} \setminus Ray(\alpha_0)$*, it becomes a branch of the complex logarithm, defined in a connected open subset of the complex plane excluding zero, with derivative* $1/z$*.*

The principal value of the logarithm coincides with its 0-argument value,

$$\mathrm{Log}\,(z) = (\log(z))_0 = \ln|z| + i\,\mathrm{Arg}\,(z).$$

Special care must be taken with the logarithm functions. Some properties are true as a multi-valued function dealing with sets such as

$$\log(z_1 z_2) = \log(z_1) + \log(z_2), \;\; z_1, z_2 \in \mathbb{C} \setminus \{0\}. \tag{6}$$

This means that any α_0-arg of the logarithm considered on the LHS coincides with the sum of some α_1-arg considered on the first summand on the RHS and some α_2-arg on the second summand on the RHS, and vice versa.

However $\log(z^2) = \log(z) + \log(z) \neq 2\log(z), z \in \mathbb{C} \setminus \{0\}$, e.g.,

$$2\log(i) = \{(\pi + 4k\pi)i, k \in \mathbb{Z}\} \subsetneq \{(\pi + 2k\pi)i, k \in \mathbb{Z}\} = \log(i) + \log(i).$$

On the other hand, the validity of Equation (6) becomes jeopardized when the same α_0-arg is considered in the three logarithms that appear in Equation (6). For instance the principal value, $\text{Log}\,(z_1 z_2)$ may coincide or not with $\text{Log}\, z_1 + \text{Log}\, z_2$.

Example 1. *The factorizations of* $-i = (-1)(i) = (2)(-\frac{1}{2}\, i)$ *show how the principal value of a product of two complex numbers may coincide or not with the sum of the principal value of the logarithm of the factors,*

$$\text{Log}\,(-i) = -\frac{\pi}{2}\, i \begin{cases} \neq \text{Log}\,(-1) + \text{Log}\,(i) = \pi i + \frac{\pi}{2}\, i = \frac{3\pi}{2}\, i. \\ = \text{Log}\,(2) + \text{Log}\left(\frac{-1}{2} i\right) = \ln 2 + (-\ln 2 - \frac{\pi}{2}\, i) = -\frac{\pi}{2}\, i. \end{cases}$$

Remark 3. *The two factorizations of Example 1 show that the sum of the* 0*-arg of the logarithm of the factors may differ from each other. Despite this, the output may change with another argumented value of the logarithms. For example, if we constrain each logarithm of Equation (6) to its* $(-\frac{\pi}{4})-$*argumented value,*

$$(\log(-i))_{-\frac{\pi}{4}} = -\frac{\pi}{2}\, i \begin{cases} = (\log(-1))_{-\frac{\pi}{4}} + (\log(i))_{-\frac{\pi}{4}} = -\pi i + \frac{\pi}{2}\, i = -\frac{\pi}{2}\, i. \\ = (\log(2))_{-\frac{\pi}{4}} + \left(\log(\frac{-1}{2} i)\right)_{-\frac{\pi}{4}} = \ln 2 + (-\ln 2 - \frac{\pi}{2}\, i) = -\frac{\pi}{2}\, i. \end{cases}$$

However, another factorization, such as $-i = (-1+i)\left(-\frac{1}{2} + \frac{1}{2} i\right)$*, may generate a different value. Indeed,*

$$(\log(-1+i))_{-\frac{\pi}{4}} + \left(\log\left(-\frac{1}{2} + \frac{1}{2} i\right)\right)_{-\frac{\pi}{4}} = \left(\ln\sqrt{2} + i\,\frac{3\pi}{4}\right) + \left(\ln\sqrt{\frac{1}{2}} + i\,\frac{3\pi}{4}\right) = \frac{3\pi}{2}\, i.$$

Consequently, special care must be taken with direct manipulations with the logarithm in the complex field as they do not always match the ones that are true in the real field. The same applies to the complex power of a complex number, which is based upon the log function and that we revisit in the next section. This reminds us of analogous constraints in operator theory, where disentanglement rules such as the Heisenberg–Weyl formula hold only under specific commutation relations [19–21].

3. Complex Power Function Revisited

The complex power function is a well-known multi-valued function, cf. [1,9,10,17,18]. We recall below its definition, which generalizes the relationship between the natural logarithm function and the real power of a positive number other than 1.

Definition 3. *Given* $z \in \mathbb{C} \setminus \{0\}$, $\omega \in \mathbb{C}$, *the power of* z *raised to the exponent* ω *is defined by*

$$\begin{aligned} z^{\omega} &:= \exp(\omega\, \log(z)) \\ &= \exp(\Re(\omega) \ln|z| - \Im(\omega) \arg z)\ \text{cis}\,(\Re(\omega) \arg z + \Im(\omega) \ln|z|). \end{aligned}$$

Remark 4. *If* $\omega = p \in \mathbb{Z}$, *the complex power* z^p *provides a unique number given by Moivre's formula, i.e., if* $z = r\text{cis}\,(\alpha)$, *then* $z^p = r^p\text{cis}\,(p\alpha)$.

In general, fixing a given α_0-argument as the only value considered in the logarithmic function, the complex power gives birth to a single-valued function.

Definition 4 ([10]). *Given $\alpha_0 \in \mathbb{R}$, and $z \in \mathbb{C} \setminus \{0\}$, $\omega \in \mathbb{C}$, we call α_0-log of z^ω the value obtained in Definition 3 considering the α_0-arg of* $\log(z)$. *We represent it by*

$$(z^\omega)_{\alpha_0} = \exp\Big(\omega(\log(z))_{\alpha_0}\Big).$$

Remark 5. *The Principal Value (PV) of the complex power z^ω becomes a particular case of Definition 4 by taking $\alpha_0 = 0$, i.e.,*

$$\mathrm{PV}(z^\omega) = (z^\omega)_0 = \exp(\omega \operatorname{Log}(z)). \tag{7}$$

When $\omega = \frac{p}{q} \in \mathbb{Q}$, with $p, q \in \mathbb{Z}$, $q \geq 1$, the principal value above is represented by $\sqrt[q]{z^p}$.

For convenience, we recall the proof of the following result, which includes as a particular case the well-known fact that for each $z \in \mathbb{C} \setminus \{0\}$, and each positive integer n, there are exactly n different complex numbers that raised to n give z, called the n^{th} roots of z, which are symmetrically located around the origin in the complex field, thereby conforming the vertices of a regular polygon of n sides centered at 0 when $n \geq 3$, or the extremes of a segment centered at 0 when $n = 2$.

Theorem 1. *Let $z \in \mathbb{C} \setminus \{0\}$ and $\omega \in \mathbb{Q} \setminus \mathbb{Z}$. If $\omega = \frac{p}{q}$ in irreducible form, the complex power z^ω takes q different complex values, each of which satisfies that when raised to the power q coincides with z^p. They are given by*

$$z^\omega = \sqrt[q]{|z|^p}\ \operatorname{cis}\left(\frac{p}{q}(\operatorname{Arg}(z) + 2\pi k)\right),\ k = 0, 1, \ldots, q-1.$$

Proof. By definition, and taking into account Equation (3),

$$\begin{aligned} z^\omega = \exp(\omega\ \log(z)) &= \exp\left(\frac{p}{q}\left(\ln|z| + i\arg(z)\right)\right) \\ &= e^{\frac{p}{q}\ln|z|}\ \operatorname{cis}\left(\frac{p\operatorname{Arg}(z)}{q} + p\frac{2\pi k}{q}\right),\ k \in \mathbb{Z}. \end{aligned}$$

This expression provides q different complex numbers, all of them with modulus $\sqrt[q]{|z|^p}$, and q different arguments for $k = 0, 1, \ldots, q-1$, each of them $\frac{2\pi}{q}$ radians greater than the previous one, after which the arguments generated by k differ from some of these q ones in a multiple of 2π radians.

Moivre's formula provides that each of these complex numbers raised to q coincides with $|z|^p$. □

Remark 6. *According to Theorem 1, the complex powers $i^{\frac{3}{6}}$, $i^{\frac{4}{8}}$, $i^{\frac{5}{10}}$, or $i^{\frac{6}{12}}$ represent the two square roots of $i^{\frac{1}{2}}$ since the irreducible form of all the exponents is $\frac{1}{2}$. It is worth noting that standard simplification rules in $\mathbb{R}$ do not hold in $\mathbb{C}$. For instance, the following expressions*

$$\left(i^3\right)^{\frac{1}{6}}, \left(i^4\right)^{\frac{1}{8}}, \left(i^5\right)^{\frac{1}{10}}, \left(i^6\right)^{\frac{1}{12}}$$

do not coincide with $i^{\frac{1}{2}}$ as they represent, respectively, the six 6th roots of $i^3 = -i$, the eight 8th roots of $i^4 = 1$, the ten 10th roots of $i^5 = i$, and the twelve 12th roots of $i^6 = -1$. They conform the q vertices of a regular polygon of q sides centered in the origin, with $q \in \{6, 8, 10, 12\}$, one of *which is their respective principal value, as shown in Figure 1. Note also that none of their principal values match each other in this case as they are, respectively,*

$$\mathrm{PV}((i^3)^{\frac{1}{6}}) = \mathrm{cis}\left(\frac{-\pi}{12}\right), \mathrm{PV}((i^4)^{\frac{1}{8}}) = 1, \mathrm{PV}((i^5)^{\frac{1}{10}}) = \mathrm{cis}\left(\frac{\pi}{10}\right), \mathrm{PV}((i^6)^{\frac{1}{12}}) = \mathrm{cis}\left(\frac{\pi}{12}\right),$$

and they differ from $\mathrm{PV}(i^{\frac{1}{2}}) = \sqrt[2]{i} = \mathrm{cis}\left(\frac{\pi}{4}\right)$, *too.*

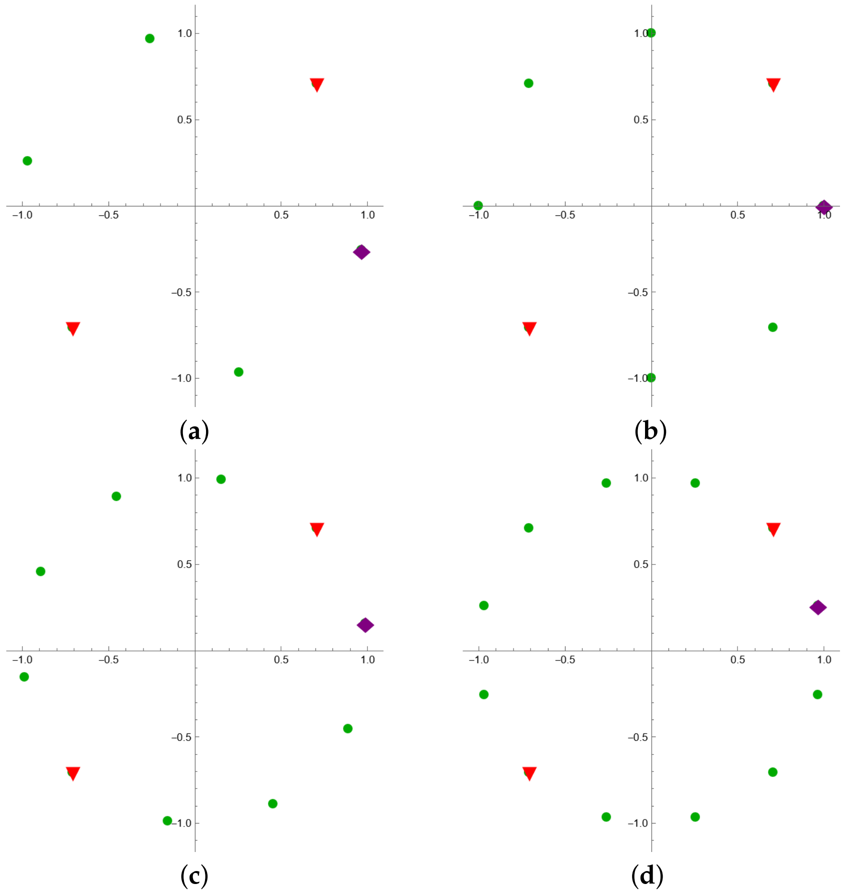

Figure 1. The q affices of $(i^p)^{\frac{1}{q}}$ for some $(p,q) \in \mathbb{N}^2$ (green dots) including the values of $i^{\frac{1}{2}}$ (red triangle), and $\sqrt[q]{i^p}$ (purple rhombus). (**a**) $(p,q) = (3,6)$. (**b**) $(p,q) = (4,8)$. (**c**) $(p,q) = (5,10)$. (**d**) $(p,q) = (6,12)$.

Remark 7. *Let* $z \in \mathbb{C} \setminus \{0\}$ *and* $\omega \in \mathbb{I} = \mathbb{R} \setminus \mathbb{Q}$. *Then, Definition 3 implies that the complex power* z^ω *takes a countably infinite set of different complex values, all of them with the same modulus* $\exp(\omega \ln |z|)$ *and their arguments are given by* $\omega \arg(z)$.

Remark 4, Theorem 1, and Remark 7 cover all the cases of the complex power z^ω, $z \in \mathbb{C} \setminus \{0\}$, when $\Im(\omega) = 0$. Adapting the considerations exposed in those results, it is easy to check the following result.

Remark 8. *Let* $z \in \mathbb{C} \setminus \{0\}$ *and* $\omega \in \mathbb{C} \setminus \mathbb{R}$, *i.e., with* $\Im(\omega) \neq 0$. *Then, Definition 3 provides that the complex power* z^ω *takes a countably infinite set of complex numbers with different modulus*

$$\exp(\Re(\omega) \ln |z| - \Im(\omega) \arg z) = \exp(\Re(\omega) \ln |z| - \Im(\omega)(Arg(z) + 2k\pi)), k \in \mathbb{Z}.$$

Hence, the modulus of each value of z^ω *changes in the factor of* $e^{-\Im(\omega)(2\pi)}$ *for consecutive values of* k. *Consequently, the sequence of values of* z^ω *generated by* $k \in \mathbb{Z}$, *tends to 0 (resp.* ∞*) when* k *increases if* $\Im(\omega)$ *is positive (resp. negative), and vice versa, when* k *decreases.*

We may also observe that if $|z| = 1$, the modulus of all values taken by its complex power z^{ω} coincide for all exponents ω with the same imaginary part.

Moreover, for any $z \in \mathbb{C} \setminus \{0\}$ and $\omega \in \mathbb{C} \setminus \mathbb{R}$, the following especial cases arise.

- *If $\Re(\omega) \in \mathbb{Z}$, then all the values of z^{ω} are located on one half-line of the complex field that starts at the origin as the argument of all of them is the real number $\Re(\omega)\mathrm{Arg}(z) + \Im(\omega) \ln |z|$.*
- *If $\Re(\omega) \in \mathbb{Q} \setminus \mathbb{Z}$ and $\omega = \frac{p}{q}$ in irreducible form, the complex power z^{ω} takes a countably infinite set of different complex values, located on q different half lines of the complex field that go through the origin with arguments*

$$\frac{p}{q}(\mathrm{Arg}(z) + 2\pi k) + \Im(\omega) \ln |z|, k = 0, 1, ..., q - 1.$$

- *If $\Re(\omega) \in \mathbb{I} = \mathbb{R} \setminus \mathbb{Q}$, then for each possible modulus of z^{ω}, there exists a countably infinite set of different complex values, each of them with a different argument.*

4. Confronting the Complex Power and Exponential Function

The fact that the complex power is a multi-valued function is a classical source of many fallacies that confront students to weird situations where their trust and confidence in their mathematical competencies is confronted. These fallacies usually start by considering some $z \in \mathbb{R} \setminus \{0\}$ and $\omega \in \mathbb{Q}$, dismiss that z^{ω} is multi-valued, and consider at some given moment the non-coinciding determinations of the argument.

Concerning the RQ, the complex power e^{ω} is generally a multi-valued function. It is single-valued when $\omega \in \mathbb{Z}$ by Remark 4, which is the only case in which the complex power e^{ω} and the complex exponential $\exp(\omega)$ coincide.

In any other case, if $\omega \in \mathbb{C} \setminus \mathbb{Z}$, e^{ω} is a multi-valued function by Theorem 1, Remarks 7 and 8, alas they do not coincide. However, $\exp(\omega)$ does not disappear from the scenario since it is always one of the values taken by the multi-evaluated e^{ω}. Indeed,

Remark 9. *The principal value of e^{ω} coincides with $\exp(\omega)$ as*

$$\mathrm{PV}(e^{\omega}) = \exp(\omega \,\mathrm{Log}\,(e)) = \exp(\omega), \; \omega \in \mathbb{C}.$$

Moreover, when the exponent ω is a real number, $\mathrm{PV}(e^{\omega})$ coincides with the real power e^{ω}. This is true when we consider any positive real number z in the base as the following, more general, result recalls.

Corollary 1. *Let $z \in \mathbb{R} \setminus \{0\}$ and $\omega \in \mathbb{R}$, i.e., with $\Im(\omega) = 0$. Then, the modulus of all the possible values of the complex power z^{ω} is the real power $|z|^{\omega}$, and the number of different values taken by z^{ω} is*

- *Unique for $\omega \in \mathbb{Z}$, and coincides with the standard real power.*
- *Finite for $\omega \in \mathbb{Q} \setminus \mathbb{Z}$, taking exactly q different values located in a symmetric form around the origin if $\frac{p}{q}$ is the irreducible fraction representing ω. One of these q values is the principal value $\sqrt[q]{z^p}$, and altogether conform the vertices of a regular polygon of q sides centered at 0 when $q \geq 3$, or the extremes of a segment centered at 0 when $q = 2$.*
- *Countably infinite for $\omega \in \mathbb{I}$.*

  The principal value PV (z^{ω}) coincides with
  (a) the real power z^{ω}, for $z \in \mathbb{R}^{+}$.
  (b) $|z|^{\omega} \,\mathrm{cis}(\omega\pi)$, for $z \in \mathbb{R}^{-}$.

The proof follows from Theorem 1, Remark 7, and Definition 3.

A consequence of Corollary 1 is that some of the values taken by the complex power z^ω are necessarily non-real when the base and the power are real, e.g., when $z \in \mathbb{R}^+$ and $\omega \in \mathbb{R} \setminus \mathbb{Q}$ or $\omega = \frac{p}{q}$ in irreducible form with $q > 2$.

However, there are some situations when the base z and the exponent ω are not real, and, amazingly, all the values of the complex power z^ω are real, as we point out next.

Remark 10. *Given $z, \omega \in \mathbb{C} \setminus \{0\}$, the complex power*

$$z^\omega = \exp(\Re(\omega) \ln|z| - \Im(\omega) \arg z) \ \text{cis}\,(\Re(\omega) \arg z + \Im(\omega) \ln|z|)$$

is real whenever $\Re(\omega) \arg z + \Im(\omega) \ln|z|$ is a multiple of π. This holds, for instance, when

$$\Re(\omega) = 0, |z| = e^{k\pi^r}, \Im(\omega) = n\pi^{1-r}, k, r, n \in \mathbb{Z},$$

simultaneously, or when simultaneously

$$z \in \mathbb{R}, \Re(\omega) \in \mathbb{Z}, \Im(\omega) \ln|z| = n\pi, n \in \mathbb{Z}.$$

Example 2. *Given $z, \omega \in \mathbb{C} \setminus \{0\}$, with $|z| = 1, \Re(\omega) = 0$,*

$$\begin{aligned} z^\omega &= \exp(-\Im(\omega) \arg(z))\,\text{cis}(0) \\ &= e^{-\Im(\omega)(Arg(z)+2k\pi)}, \; k \in \mathbb{Z}, \end{aligned}$$

which is a sequence of positive real numbers indexed in $\mathbb{Z}$ that converges to 0 on one side and, on the other one, is unbounded.

Example 3. *From Example 2, it follows that both the imaginary unit i and its opposite $-i$, raised to any pure imaginary number, generate only real numbers. The principal value is one of them, and when they both are considered as exponents, we obtain the particular cases:*

$$\begin{aligned} PV(i^i) &= e^{-\frac{\pi}{2}} = PV((-i)^{-i}), \\ PV(i^{-i}) &= e^{\frac{\pi}{2}} = PV((-i)^i). \end{aligned}$$

5. *Mathematica* Commands

5.1. Roots of a Complex Number

The software *Mathematica* (version 14.2, Wolfram Research, Champaign, IL, USA) can help us to calculate the values of the functions discussed previously when they are not implemented therein.

Firstly, let us note that given $z_0 \in \mathbb{C}$ and $\omega \in \mathbb{Q} \setminus \mathbb{Z}$, *Mathematica* provides a single output when we ask it to simplify z_0^ω, exactly the principal value $\sqrt[q]{z_0^p}$, where p, q stand for the two coprime integer numbers, with $q \geq 2$, that represent w in irreducible form.

Example 4. *If we introduce some given complex power, e.g.,*

$In[1] := (-1)\hat{}(1/5)$

$Out[1] = (-1)^{\frac{1}{5}}$

Its $a + bi$ form may be numerically approximated by introducing

$In[2] := (-1)\hat{}(1/5)//N$

$Out[2] = 0.809017 + 0.587785i.$

This value corresponds to the principal value of $(-1)^5$.

The easiest way to obtain all the possible values of the power $z_0^{\frac{p}{q}}$ with *Mathematica* is fin all the roots of the equation

$$z^q = z_0^p.$$

Remark 11. *It is worthwhile noting that in the above, the irreducible expression of the rational power must be considered. Hence, to obtain all the possible values of* $i^{0.4}$, *we must take into account that* $\frac{2}{5}$ *is the irreducible expression of the exponent and ask Mathematica to solve the equation* $z^5 = i^2 \ (= -1)$. *A fortiori, all the possible values of* $i^{0.4}$ *coincide with the five fith-roots of* -1.

Despite this being a straightforward method, it is enriching to use the principal value given by *Mathematica* and, following the proof of Theorem 1, obtain the others by rotating $\frac{2kp}{q}\pi$ radians around the origin the previously obtained value.

This is easily achieved taking into account that if $z \in \mathbb{C}$ and $\alpha \in \mathbb{R}$, then $z \cdot e^{i\alpha}$ is a complex number whose affix is the result of a counterclockwise rotation of α radians around the origin.

Example 5. *To write down the five values of* $(-1)^{\frac{1}{5}}$ *in* $a + bi$ *form, we may start from the principal value and obtain the rest by adequate rotations around the origin.*

We start by obtaining the principal value

$In[1] := (-1)\hat{}(1/5)$

$Out[1] = (-1)^{\frac{1}{5}}$

Now, we calculate the other powers by making five rotations of $\frac{2\pi}{5}$ radians around the origin, the fifth one returning the initial affix

$In[2] := Table[\%1 * Exp[2 * k * Pi * I/5], \{k, 1, 5\}]$

$Out[2] = \{(-1)^{1/5} e^{\frac{2i\pi}{5}}, (-1)^{1/5} e^{\frac{4i\pi}{5}}, (-1)^{1/5} e^{-\frac{4i\pi}{5}}, (-1)^{1/5} e^{-\frac{2i\pi}{5}}, (-1)^{1/5}\}$

Their $a + bi$ numerically approximated values are given by

$In[3] := \%2//N$

$Out[3] = \{-0.309017 + 0.951057i, -1. + 1.11022 \times 10^{-16} i, -0.309017 - 0.951057i, 0.809017 - 0.587785i, 0.809017 + 0.587785i\}.$

The polygon, whose vertices are the given affices, is easily drawn with the command

$$\mathbf{Graphics}[\mathbf{Line}[points], Axes \to True]$$

which generates the polygon whose vertices have the coordinates given in *points*. We just need to start and finish in one of the vertices.

$In[4] := vertaffix = (-1)^{1/5} * Exp[2 * k * Pi * I/5];$

$In[5] := vertices = Table[ReIm[vertaffix], \{k, 0, 5\}];$

$In[6] := Graphics[Line[vertices], Axes \to True]$

$Out[6] =$

The Output [6] of Example 5 is displayed as Figure 2 and its caption recalls the fact outlined in Remark 11. On the other hand, we note that the graphs displayed in Figure 1 have been generated by using the argument *Points* instead of *Lines* within the *Graphics* command.

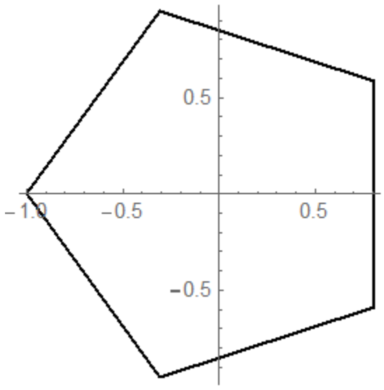

Figure 2. The five values of $i^{0.4}$ conform the vertices of a regular pentagon centered at O.

Remark 12. *From Theorem 1, it follows that the outputs may change drastically when some numerical approximation is realized. For instance, the complex power* $i^{\frac{1}{3}}$ *generates three different complex numbers. In contrast, any numerical approximation of 1/3 may generate many more, e.g.,* $i^{0.3}$, $i^{0.33}$, *and* $i^{0.333}$ *generate 10, 100, and 1000 different complex numbers, respectively, none of which coincides with the three values of* $i^{\frac{1}{3}}$. *Their affices are depicted in Figure 3.*

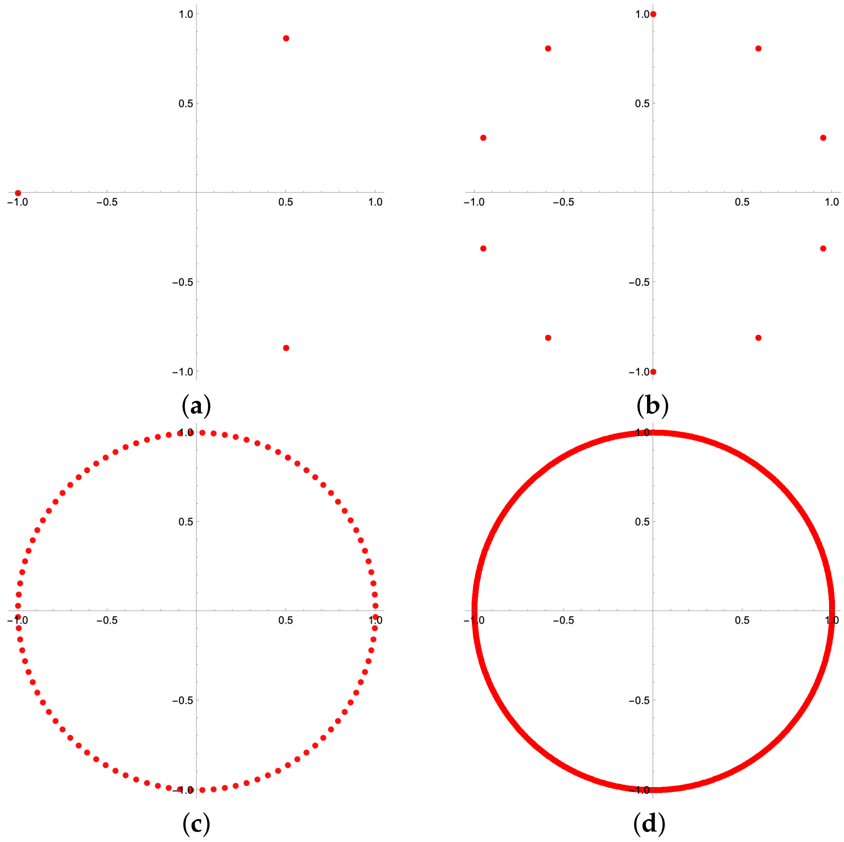

Figure 3. Affices (in red) of $(-1)^{\omega}$ when ω is (**a**) $\frac{1}{3}$. (**b**) 0.3. (**c**) 0.33. (**d**) 0.333.

5.2. α_0-*Argument of a Complex Number*

Mathematica does not have a command calculating a given α_0-argument of a complex number. However, its command *FixedPoint* will support us in finding the number of turns around the origin that we must take from the principal argument to evaluate a given α_0-argument.

With this purpose, let us define

$$circs[a\_, z\_] := FixedPoint[If[a - Pi < Arg[z] + 2\#Pi <= a + Pi, \#, \# + Sign[a]]\&, 0]$$

which is negative when a number of $|circs|$ full turns must be considered clockwise, and positive when counterclockwise.

Therefore, the α_0-argument of z is obtained by defining

$$arg[\alpha_0\_, z\_] := Arg[z] + 2Pi * circs[\alpha_0, z].$$

Example 6. *To find the* $7\pi-$, $(-50)-$, *the* $2-$*argument of* $1+i$, *and the number of full turns around the origin run from its principal value , we introduce the following.*

$In[1] := circs[a\_, z\_] := FixedPoint[If[a - Pi < Arg[z] + 2\#Pi <= a + Pi, \#, \# + Sign[a]]\&, 0]$

$In[2] := arg[a\_, z\_] := Arg[z] + 2Pi * circs[a, z]$

$In[3] := \{circs[7Pi, 1 + I], arg[7Pi, 1 + I]\}$

$Out[3] = \{3, \frac{25\pi}{4}\}$

$In[4] := \{circs[-50, 1 + I], arg[-50, 1 + I]\}$

$Out[4] = \{-8, -\frac{63\pi}{4}\}$

$In[5] := \{circs[2, 1 + I], arg[2, 1 + I]\}$

$Out[5] = \{0, \frac{\pi}{4}\}$

Hence, the requested determinations of the argument are $\frac{25\pi}{4}$, $-\frac{63\pi}{4}$, *and* $\frac{\pi}{4}$, *respectively. The number of circumferences run is three counterclockwise, eight clockwise, and zero, respectively.*

In this case, the $2-$*argument of* $1+i$ *coincides with* $\mathrm{Arg}(1+i)$.

5.3. Complex Logarithm Function

Mathematica has obtained the command **Log**[_], which just provides the principal value $\mathrm{Log}(z) = \ln|z| + i\,\mathrm{Arg}(z)$.

Therefore, to represent all the possible values of $\ln z$, we must write down

$$Log[Abs[z]] + I(Arg[z] + 2k\ \mathrm{Pi})\ (\text{or } Log[z] + 2k\ \mathrm{Pi}\ I),\ k \in \mathbb{Z}.$$

In general, the α_0-argumented $\log(\cdot)$ given by

$$(\log(z))_{\alpha_0} = \ln|z| + i\,\arg_{\alpha_0}(z)$$

cannot be obtained directly from *Mathematica*.

This is easily sorted by defining and using the α_0-argument as shown in the previous subsection, i.e., by introducing

$$Log[Abs[z]] + I * arg[\alpha_0, z].$$

Example 7. *To calculate* $Log(1+i)$ *and* $(\log(1+i))_{-50}$ *in* $a+bi$ *form, we shall introduce the described commands.*

$In[1] := Log[1 + I]//N$

$Out[1] = 0.346574 + 0.785398i$

$In[2] := Log[Abs[1 + I]] + I * arg[-50, 1 + I]$

$Out[2] = -\frac{63i\pi}{4} + \frac{Log[2]}{2}.$

Mathematica has provided the requested values, approximated in the former case as we have added //N at the end of the input, and exact in the latter.

5.4. Power Complex Function with Complex Base and Exponent

Similarly to the argument and complex logarithm cases, given $z \in \mathbb{R} \setminus \{0\}$ and $\omega \in \mathbb{R}$, *Mathematica* just provides the principal value of the complex power z^{ω}.

Example 8. *If we try to find the values of the power i^i, Mathematica just provides $PV(i^i)$*

$In[1] := I\hat{}I$

$Out[1] = i^i$

$In[2] := I\hat{}I//N$

$Out[2] = 0.20788 + 0.\,i$

which corresponds to the numerical approximation of $e^{-\frac{\pi}{2}}$ calculated in Example 3.

Consequently, to represent all the possible values of z^{ω}, we must write down the whole set of complex powers $\exp(\omega \ln z)$ generated by Definition 3, i.e.,

$$z^{\omega} = \{\mathrm{Exp}[\omega * (\mathrm{Log}[z] + 2k\ \mathrm{Pi}\ I))],\ k \in \mathbb{Z}\} \tag{8}$$

Remark 13. *When the exponent $\omega \in \mathbb{Q}$ is expressed as $\omega = \frac{p}{q}$ in irreducible form, the affices of z^{ω} conform the vertices of a regular polygon of q sides centered at O.*

When $p > 1$, we must keep in mind that the polygonal chain generated connecting the orderly generated sequence of values of z^{ω} does not conform to a regular polygon since, in this case, any two consecutive values of k in Equation (8), produce a twist of $\frac{2p}{q}\pi$ radians, and the coincidence with some previously generated value happens after q turns around the origin as shown in Figure 4 with the example cases discussed in Remark 12.

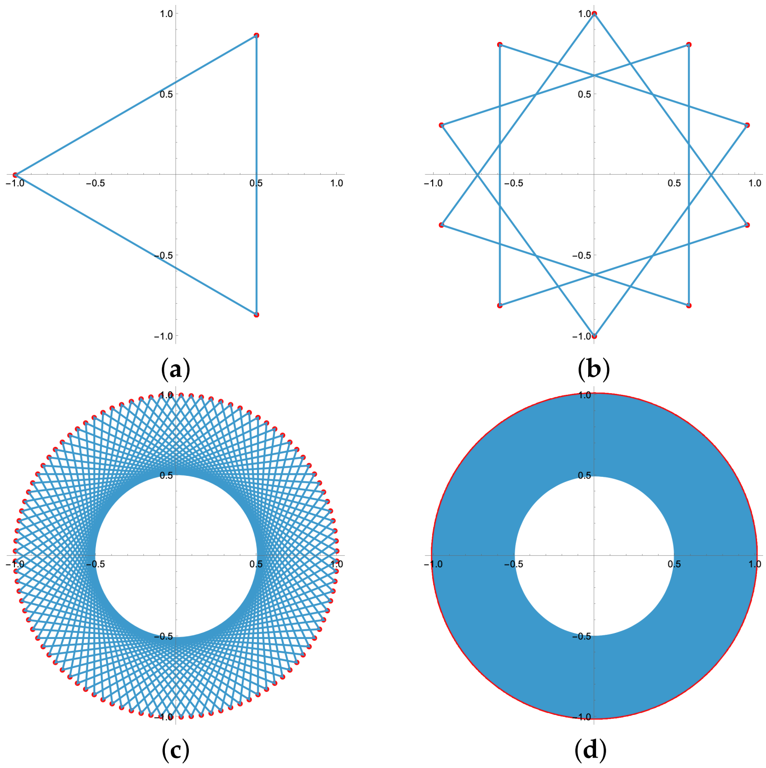

Figure 4. Polygonal chain generated by $(-1)^{\omega}$ when ω is **(a)** $\frac{1}{3}$. **(b)** 0.3. **(c)** 0.33. **(d)** 0.333.

Remark 14. *When the exponent $\omega \in \mathbb{C} \setminus \mathbb{R}$ is concerned, it is difficult to represent the countably infinite different values taken by the complex power z^{ω} as there is a sequence of points converging to 0 on one side and another tending to ∞ on the other when the Equation (8) is used to generating them.*

Example 9. *The values taken by the complex power i^{ω} for non-real ω change their modulus in a factor that makes it increase and decrease. For this reason, to visualize some of the situations discussed in Remark 8, we have constructed Tables 1 and 2, whose data have been generated with Mathematica.*

Table 1. Modulus $r(k)$ and argument $\alpha(k)$ of 21 values of $i^{\omega} = \exp(\omega(\frac{\pi}{2} + 2k\pi)), -10 \le k \le 10$ for.

(a)	$\omega = i$		**(b)**	$\omega = 1 + i$	
k	$r(k)$	$\alpha(k)$	k	$r(k)$	$\alpha(k)$
-10	$e^{\frac{39\pi}{2}}$	0	-10	$e^{\frac{39\pi}{2}}$	$\frac{\pi}{2}$
-9	$e^{\frac{35\pi}{2}}$	0	-9	$e^{\frac{35\pi}{2}}$	$\frac{\pi}{2}$
-8	$e^{\frac{31\pi}{2}}$	0	-8	$e^{\frac{31\pi}{2}}$	$\frac{\pi}{2}$
-7	$e^{\frac{27\pi}{2}}$	0	-7	$e^{\frac{27\pi}{2}}$	$\frac{\pi}{2}$
-6	$e^{\frac{23\pi}{2}}$	0	-6	$e^{\frac{23\pi}{2}}$	$\frac{\pi}{2}$
-5	$e^{\frac{19\pi}{2}}$	0	-5	$e^{\frac{19\pi}{2}}$	$\frac{\pi}{2}$
-4	$e^{\frac{15\pi}{2}}$	0	-4	$e^{\frac{15\pi}{2}}$	$\frac{\pi}{2}$
-3	$e^{\frac{11\pi}{2}}$	0	-3	$e^{\frac{11\pi}{2}}$	$\frac{\pi}{2}$
-2	$e^{\frac{7\pi}{2}}$	0	-2	$e^{\frac{7\pi}{2}}$	$\frac{\pi}{2}$
-1	$e^{\frac{3\pi}{2}}$	0	-1	$e^{\frac{3\pi}{2}}$	$\frac{\pi}{2}$
0	$e^{-\frac{\pi}{2}}$	0	0	$e^{-\frac{\pi}{2}}$	$\frac{\pi}{2}$
1	$e^{-\frac{5\pi}{2}}$	0	1	$e^{-\frac{5\pi}{2}}$	$\frac{\pi}{2}$
2	$e^{-\frac{9\pi}{2}}$	0	2	$e^{-\frac{9\pi}{2}}$	$\frac{\pi}{2}$
3	$e^{-\frac{13\pi}{2}}$	0	3	$e^{-\frac{13\pi}{2}}$	$\frac{\pi}{2}$
4	$e^{-\frac{17\pi}{2}}$	0	4	$e^{-\frac{17\pi}{2}}$	$\frac{\pi}{2}$
5	$e^{-\frac{21\pi}{2}}$	0	5	$e^{-\frac{21\pi}{2}}$	$\frac{\pi}{2}$
6	$e^{-\frac{25\pi}{2}}$	0	6	$e^{-\frac{25\pi}{2}}$	$\frac{\pi}{2}$
7	$e^{-\frac{29\pi}{2}}$	0	7	$e^{-\frac{29\pi}{2}}$	$\frac{\pi}{2}$
8	$e^{-\frac{33\pi}{2}}$	0	8	$e^{-\frac{33\pi}{2}}$	$\frac{\pi}{2}$
9	$e^{-\frac{37\pi}{2}}$	0	9	$e^{-\frac{37\pi}{2}}$	$\frac{\pi}{2}$
10	$e^{-\frac{41\pi}{2}}$	0	10	$e^{-\frac{41\pi}{2}}$	$\frac{\pi}{2}$

Table 2. Modulus $r(k)$ and argument $\alpha(k)$ of 21 values of $i^{\omega} = \exp(\omega(\frac{\pi}{2} + 2k\pi)), -10 \le k \le 10$ for.

(a)	$\omega = \frac{5}{7} + i$		**(b)**	$\omega = \sqrt{2} + i$	
k	$r(k)$	$\alpha(k)$	k	$r(k)$	$\alpha(k)$
-10	$e^{\frac{39\pi}{2}}$	$\frac{\pi}{14}$	-10	$e^{\frac{39\pi}{2}}$	$28\pi - \frac{39\pi}{\sqrt{2}}$
-9	$e^{\frac{35\pi}{2}}$	$-\frac{\pi}{2}$	-9	$e^{\frac{35\pi}{2}}$	$24\pi - \frac{35\pi}{\sqrt{2}}$
-8	$e^{\frac{31\pi}{2}}$	$\frac{13\pi}{14}$	-8	$e^{\frac{31\pi}{2}}$	$22\pi - \frac{31\pi}{\sqrt{2}}$
-7	$e^{\frac{27\pi}{2}}$	$\frac{5\pi}{14}$	-7	$e^{\frac{27\pi}{2}}$	$20\pi - \frac{27\pi}{\sqrt{2}}$
-6	$e^{\frac{23\pi}{2}}$	$-\frac{3\pi}{14}$	-6	$e^{\frac{23\pi}{2}}$	$16\pi - \frac{23\pi}{\sqrt{2}}$
-5	$e^{\frac{19\pi}{2}}$	$-\frac{11\pi}{14}$	-5	$e^{\frac{19\pi}{2}}$	$14\pi - \frac{19\pi}{\sqrt{2}}$
-4	$e^{\frac{15\pi}{2}}$	$\frac{9\pi}{14}$	-4	$e^{\frac{15\pi}{2}}$	$10\pi - \frac{15\pi}{\sqrt{2}}$
-3	$e^{\frac{11\pi}{2}}$	$\frac{\pi}{14}$	-3	$e^{\frac{11\pi}{2}}$	$8\pi - \frac{11\pi}{\sqrt{2}}$

Table 2. *Cont.*

(a)	$\omega = \frac{5}{7} + i$		**(b)**	$\omega = \sqrt{2} + i$	
k	$r(k)$	$\alpha(k)$	k	$r(k)$	$\alpha(k)$
-2	$e^{\frac{7\pi}{2}}$	$-\frac{\pi}{2}$	-2	$e^{\frac{7\pi}{2}}$	$4\pi - \frac{7\pi}{\sqrt{2}}$
-1	$e^{\frac{3\pi}{2}}$	$\frac{13\pi}{14}$	-1	$e^{\frac{3\pi}{2}}$	$2\pi - \frac{3\pi}{\sqrt{2}}$
0	$e^{-\frac{\pi}{2}}$	$\frac{5\pi}{14}$	0	$e^{-\frac{\pi}{2}}$	$\frac{\pi}{\sqrt{2}}$
1	$e^{-\frac{5\pi}{2}}$	$-\frac{3\pi}{14}$	1	$e^{-\frac{5\pi}{2}}$	$\frac{5\pi}{\sqrt{2}} - 4\pi$
2	$e^{-\frac{9\pi}{2}}$	$-\frac{11\pi}{14}$	2	$e^{-\frac{9\pi}{2}}$	$\frac{9\pi}{\sqrt{2}} - 6\pi$
3	$e^{-\frac{13\pi}{2}}$	$\frac{9\pi}{14}$	3	$e^{-\frac{13\pi}{2}}$	$\frac{13\pi}{\sqrt{2}} - 10\pi$
4	$e^{-\frac{17\pi}{2}}$	$\frac{\pi}{14}$	4	$e^{-\frac{17\pi}{2}}$	$\frac{17\pi}{\sqrt{2}} - 12\pi$
5	$e^{-\frac{21\pi}{2}}$	$-\frac{\pi}{2}$	5	$e^{-\frac{21\pi}{2}}$	$\frac{21\pi}{\sqrt{2}} - 14\pi$
6	$e^{-\frac{25\pi}{2}}$	$\frac{13\pi}{14}$	6	$e^{-\frac{25\pi}{2}}$	$\frac{25\pi}{\sqrt{2}} - 18\pi$
7	$e^{-\frac{29\pi}{2}}$	$\frac{5\pi}{14}$	7	$e^{-\frac{29\pi}{2}}$	$\frac{29\pi}{\sqrt{2}} - 20\pi$
8	$e^{-\frac{33\pi}{2}}$	$-\frac{3\pi}{14}$	8	$e^{-\frac{33\pi}{2}}$	$\frac{33\pi}{\sqrt{2}} - 24\pi$
9	$e^{-\frac{37\pi}{2}}$	$-\frac{11\pi}{14}$	9	$e^{-\frac{37\pi}{2}}$	$\frac{37\pi}{\sqrt{2}} - 26\pi$
10	$e^{-\frac{41\pi}{2}}$	$\frac{9\pi}{14}$	10	$e^{-\frac{41\pi}{2}}$	$\frac{41\pi}{\sqrt{2}} - 28\pi$

In them, we consider i^ω for four different complex values of ω. A column contains the values given to $k \in \mathbb{Z}$, between -10 and 10 on the RHS of Equation (8), the adjacent columns containing the corresponding modulus $r(k)$ and argument $\alpha(k)$ of the value i^ω obtained for that k, i.e.,

$$r(k) = Exp\Big[-Im[\omega] * \Big(\frac{\pi}{2} + 2 * k * \ Pi\Big)\Big],\ \alpha(k) = Re[\omega] * \Big(\frac{\pi}{2} + 2 * k * \ Pi\Big).$$

Hence, the ratio of change in the modulus of each value of i^ω will be $e^{-\Im(\omega)(2\pi)}$ for consecutive values of k, as pointed out in Remark 8, and the sequence of modulus will tend to 0 when k increases and to ∞ when k decreases since in our cases we have considered values of ω with $\Im(\omega) = 1 > 0$.

- *In Table 1(a), $\omega = i$. As pointed out in Example 3, all the values of z^ω at $z = i$ are real.*
- *In Table 1(b), $\omega = 1 + i$. As pointed out in Remark 8, for each $z \in \mathbb{C} \setminus \{0\}$, the countably infinite set of complex numbers with different modulus defined by z^{1+i} are located on one half-line that starts at the origin as $\Re(\omega) = 1 \in \mathbb{Z}$. Their argument at $z = i$ is defined by*

$$\Re(\omega)\mathrm{Arg}(z) + \Im(\omega)\ln|z| = \frac{\pi}{2} \ \textit{radians}.$$

- *In Table 2(a), $\omega = \frac{5}{7} + i$. As pointed out in Remark 8, for each $z \in \mathbb{C} \setminus \{0\}$, the countably infinite set of complex numbers with different modulus defined by $z^{\frac{5}{7}+i}$ are located on seven half-lines that start at the origin as $\Re(\omega) = \frac{5}{7} \in \mathbb{Q}$ in irreducible form. At $z = i$, the arguments corresponding to the 7 half-lines are defined by*

$$\frac{5}{7}(\mathrm{Arg}(z) + 2\pi k) + \Im(\omega)\ln|z| = \frac{5}{7}\Big(\frac{\pi}{2} + 2\pi k\Big) \ \textit{radians},\ k = 0, 1, \ldots, 6.$$

  Every seven values of k in a row generate a complex number on the same half-line, closer to 0 when k is increasing, and farther from 0 when k is decreasing in $\mathbb{Z}$.
- *In Table 2(b), $\omega = \sqrt{2} + i$. Now, Remark 8 recalls that for any given $z \in \mathbb{C} \setminus \{0\}$, there are no two different values of $z^{\sqrt{2}+i}$ located on any half-line that starts at the origin. The results*

shown in the table confirm that this is the case. Here, $z = i$, and the arguments of any two values generated by $k_1, k_2 \in \mathbb{Z}$, with $k_1 \neq k_2$, are

$$\Re(\omega)(\text{Arg}(z) + 2\pi k) + \Im(\omega)\ln|z| = \sqrt{2}\left(\frac{\pi}{2} + 2\pi k\right),\ k \in \{k_1, k_2\},$$

respectively. Hence, they differ in $2\pi(k_1 - k_2)\sqrt{2}$ radians, which is not a multiple of 2π and, consequently, are never aligned with the origin.

In these four examples, as noted in Remark 8, the modulus of the complex powers z^ω coincide for each given $k \in \mathbb{Z}$ as we have considered the imaginary unit, which is a complex number of modulus 1, in the base and different values of ω in the exponent, but all of them with the same $\Im(\omega)$.

If some α_0-log is to be picked up among all the possible values of a complex power, we will use the commands we defined in the previous subsections.

Example 10. *To evaluate the principal value and the $(-50)-$log of the complex power $(1+i)^{(2+i)}$, in $a + bi$ form, we proceed as follows*

$In[1] = Exp[((2+I)Log[1+I])]//N$

$Out[1] = -0.309744 + 0.857658i$

$In[2] = circs[a\_, z\_] := FixedPoint[If[a - Pi < Arg[z] + 2\#Pi <= a + Pi, \#, \# + Sign[a]]\&, 0];$

$In[3] = arg[a\_, z\_] := Arg[z] + 2Pi * circs[a, z];$

$In[4] = log[a\_, z\_] := Log[Abs[z]] + I\, arg[a, z]$

$In[5] = Exp[(2+I)log[0, 1+I]]//N$

$Out[5] = -0.309744 + 0.857658i$

This output confirms that the result given in $Out[1]$ is the principal value of $(1+i)^{(2+i)}$. The $(-50)-$log of the complex power comes out by simplifying the following command,

$In[6] = Exp[(2+I)log[-50, 1+I]]$

$Out[6] = e^{(2+i)\left(-\frac{63i\pi}{4} + \frac{Log[2]}{2}\right)}$

$In[7] = Exp[(2+I)log[-50, 1+I]]//ComplexExpand$

$Out[7] = 2ie^{63\pi/4}\, Cos[\frac{Log[2]}{2}] - 2e^{63\pi/4}\, Sin[\frac{Log[2]}{2}]$

$In[8] = \%//N$

$Out[8] = -2.09423 \times 10^{21} + 5.79877 \times 10^{21} i.$

6. Conclusions

In this paper, we revisited the concept of the complex power of a complex number and the problems of notation of the exponential function exp as the power of the number e that are relevant to avoid occasional misconceptions that may arise between early learners and users of complex functions.

The complex exponential function is defined by $\exp(z) = e^{\Re(z)}\text{cis}(\Im(z)),\ \forall z \in \mathbb{C}$. To outline the exact difference between $\exp(z)$ and the complex power e^z function, we revisited the complex logarithmic function log on which the latter relies.

By doing this, we recalled that expressions such as $\log(z_1 z_2)$ versus $\log(z_1) + \log(z_2)$ do not always allow straightforward manipulations as in the real field. This has reminded us of analogous constraints in operator theory, where disentanglement rules such as the Heisenberg–Weyl formula only hold under specific commutation relations. A similar degree of caution must be taken when manipulating complex powers and logarithms, particularly when visualizing or computing with software systems that default to principal values.

From Section 4, it follows that $\exp(z)$ and the complex power function e^z coincide when $z \in \mathbb{Z}$. In any other case, e^z is multivalued, Remark 9, recalling that one of those values, the principal value, coincides with $\exp(z)$. Corollary 1 summarizes the broader picture that we find when we look at the real exponential from a complex perspective.

Several examples have been exposed showing how the complex power of a complex number gives birth to a sequence of complex numbers, indexed in $\mathbb{Z}$, tending to 0 and to infinity when the exponent is not real, and that when the exponent is real, all the values taken by the complex power are located in a single circumference.

Moreover, we have shown how the capabilities of the software *Mathematica* allow the definition of adequate functions that facilitate the calculation of any argumented value of the complex logarithm and complex power functions, in addition to the principal value that is implemented, to generate some graphs that showcase the different values taken by the complex power function.

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