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New iterative methods for fixed point approximation: application to the numerical solution of Fredholm integral equations

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Abstract This paper provides new insights to the search and approximation of fixed points of non-contractive operators. It analyzes the role of the concepts of demiclosedness and demicompactness in the existence of critical points, giving sufficient conditions for their existence. Both notions are related to the construction of approximation sequences of the fixed points, that provide iterative algorithms for the resolution of equations of all kinds. In this paper, two different algorithms are proposed, and their convergence is studied in the framework of normed and quasi-normed spaces. The sequences are composed of weighted averages of the variable and its image by the operator. They aim to be an alternative to Picard's method, when this procedure does not converge. These algorithms are applied to the numerical solution of a Fredholm integral equation of second kind, that appears in a great number of problems of physics, engineering and applied mathematics. The methods are checked in a particular case of Fredholm integral equation with exact solution, in order to compute the errors committed by the different approximations, and illustrate the convergence of the iterations to the real solution.

Mathematics Subject Classification 47H09 · 47H10 · 37C25 · 65R20

1 Introduction

In previous papers (see for instance [12,13]) we have given sufficient conditions to be satisfied by non-contractive operators defined in Banach and Hilbert spaces in order to have fixed points, and explored the way of finding these critical points by means of approximation sequences, that is to say, iterative methods. In this article we consider two important properties related to non-contractive operators, namely demiclosedness and demicompactness. The mappings studied are nonexpansive partial contractivities, defined in the reference [14], along with quasi-nonexpansive and nonexpansive operators.

The demiclosedness principle states that in a uniformly convex Banach space, a nonexpansive mapping T is such that $I-T$ is demiclosed, where I represents the identity (see for instance the reference [8]). In the second section of this article, a demiclosedness principle is proved for Banach spaces satisfying the Opial condition, concerning nonexpansive partial contractivities. A Hilbert space owns the Opial condition, and nonexpansive operators are particular cases of nonexpansive partial contractivities, consequently the result presented is very general. The demiclosedness property ensures an approximative sequence (in a sense defined in the text) to converge weakly to a fixed point of the operator T.

Another important class of operators are demicompact mappings. This family contains the usual compact operators, that have favorable properties regarding their approximation and spectra. We deal with demicompact self-maps in the third section of the paper. Both characteristics (demiclosedness and demicompactness) are

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closely related to the concept of approximative sequence, that enables the search of fixed points by means of an iterative algorithm.

It is well known that Picard's method to approach critical points is not useful in the case of non-contractive mappings and, starting in the articles [10,11], the mathematical community is looking for alternatives to this favorable procedure. In this paper two different algorithms for fixed point approximation are presented, called in the text N^*-iteration, and two-steps N-iteration. The latter is a particular case of the recurrence proposed in [14]. Several conditions for their convergence are given, when dealing with quasi-nonexpansive and nonexpansive operators.

In the last part of the paper, the proposed iterative procedures are applied to the numerical solution of a Fredholm integral equation of second kind, given by the expression:

$$F(x) = g(x) + \lambda \int_a^b k(x, y)F(y)dy, \tag{1.1}$$

where the function F is the search unknown, and the map g and the constant λ are given data. Fredholm integral equations appear in a great number of fields of physics, engineering and mathematics like heat transfer and radiation, fluid mechanics, signal and image processing, partial and ordinary differential equations, etc. This type of equation is closely related to the map defined as

$$T_k f(x) = \int_a^b k(x, y)f(y)dy.$$

The operator $T_k : \mathcal{L}^2[a, b] \rightarrow \mathcal{L}^2[a, b]$ is compact and of Hilbert–Schmidt type. The mapping $k : [a, b] \times [a, b] \rightarrow \mathbb{R}$ is the kernel of T_k. To solve the equation (1.1) is equivalent to find a fixed point of the operator

$$T_{g,\lambda} f = g + \lambda T_k f.$$

If $T_{g,\lambda}$ is contractive, one may use the Picard iteration to approach the fixed point, but otherwise an alternative procedure must be sought. Of course, there are several non-iterative methods for the resolution of the Fredholm integral equation like the Adomian decomposition method, quadrature formulae, series expansions, etc. (see for instance [3,7,18,19]). The methods proposed in this paper are simple and easy to implement with a good mathematical software, giving the approximate solution in closed-form. They can be applied to nonlinear integral equations as well.

To check these algorithms, an example of integral equation with exact solution has been chosen, in order to compute the error of the successive approximations. The example is illustrated with tables of errors and figures displaying the convergence of the iterations to the real solution.

2 Demiclosedness and fixed points

This section gives sufficient conditions for the existence of fixed points of nonexpansive partial contractivities defined in Banach spaces satisfying the Opial condition. These requirements are related to demiclosedness and approximating sequences.

Definition 2.1 Let E be a normed space and $C \subseteq E$. $T : C \rightarrow E$ is a nonexpansive partial contractivity if there exists $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, $\psi(0) = 0$, such that for any $f, g \in C$,

$$||Tf - Tg|| \leq ||f - g|| + \psi(\min\{||f - Tf||, ||g - Tg||\}). \tag{2.1}$$

T is a strict nonexpansive partial contractivity if for $f \neq g$,

$$||Tf - Tg|| < ||f - g|| + \psi(\min\{||f - Tf||, ||g - Tg||\}). \tag{2.2}$$

Example 2.2 Let $T : [0, 1] \rightarrow \mathbb{R}$ be defined as $Tx = x/2$ for $x \in [0, 1/2)$, and $Tx = 0$ for $x \in [1/2, 1]$. The function T is a nonexpansive partial contractivity with respect to the map $\psi(t) = t$ for $t \geq 0$ since

- If $x, y \in [0, 1/2)$ then

$$|Tx - Ty| = |\frac{x}{2} - \frac{y}{2}| \leq |x - y| + \min\{|x - Tx|, |y - Ty|\}.$$

- If $x, y \in [1/2, 1]$ then

$$|Tx - Ty| = 0 \leq |x - y| + \min\{|x - Tx|, |y - Ty|\}.$$

- If $x \in [0, 1/2)$ and $y \in [1/2, 1]$ then

$$|Tx - Ty| = \frac{x}{2} \leq (y - x) + \min\{\frac{x}{2}, y\} \leq |x - y| + \min\{|x - Tx|, |y - Ty|\}.$$

The last inequality is due to the positions of x and y in the interval.

T is a strict nonexpansive partial contractivity since for $x \neq y$ the inequalities of the three cases are strict, that is to say,

$$|Tx - Ty| < |x - y| + \min\{|x - Tx|, |y - Ty|\}.$$

Example 2.3 Let $T : \mathbb{R} \to \mathbb{R}$ be defined as $Tx = 2x$. This map is not a nonexpansive partial contractivity since the inequality

$$|Tx - Ty| = 2|x - y| \leq |x - y| + \psi(\min\{|x|, |y|\})$$

should be true for any $x, y \in \mathbb{R}$. However, taking $x = 0$, we obtain

$$2|y| \leq |y|,$$

and this holds only for $y = 0$.

Remark 2.4 A straightforward consequence of this definition is the fact that if a strict nonexpansive partial contractivity has a fixed point, it is unique. In Example 2.2, the only fixed point is 0.

Remark 2.5 For $\psi(t) = 0$ for $t \geq 0$ we obtain a nonexpansive operator.

Remark 2.6 Unlike the nonexpansive mappings, a nonexpansive partial contractivity may be discontinuous, as shown in Example 2.2.

Definition 2.7 Let E, F Banach spaces. Then $S : E \to F$ is demiclosed at zero if $f_n \rightharpoonup f$ and $Sf_n \to 0$ implies that $Sf = 0$.

Remark 2.8 $f_n \rightharpoonup f$ denotes the weak convergence of (f_n) to f. The definition may be extended to $S : C \to C'$, where C, C' are subsets of E and F respectively.

The next concept was introduced in the reference [15].

Definition 2.9 A normed space E satisfies the Opial's condition if for any sequence $(f_n) \subseteq E$ such that (f_n) converges weakly to $f \in E$,

$$\liminf_{n\to\infty} ||f_n - f|| < \liminf_{n\to\infty} ||f_n - g||. \tag{2.3}$$

for any $g \neq f$.

Remark 2.10 A Hilbert space satisfies the Opial condition.

In the following we give a Demiclosedness Principle for nonexpansive partial contractivities.

Theorem 2.11 *Let E be a Banach space satisfying the Opial condition and C be a weakly closed subset of E. Let $T : C \to C$ be a nonexpansive partial contractivity, where ψ is right continuous at zero and non-decreasing. Then $I - T$ is demiclosed at zero.*

Proof Let us assume that $(f_n) \subseteq C$ is such that $f_n \rightharpoonup f$ and $(I - T)f_n \to 0$. If $Tf = f$ then $(I - T)f = 0$, and the result is proved. Otherwise, bearing in mind the Opial condition

$$\liminf_{n\to\infty} ||f_n - f|| < \liminf_{n\to\infty} ||f_n - Tf|| \leq \liminf_{n\to\infty} ||f_n - Tf_n|| + ||Tf_n - Tf|| = \liminf_{n\to\infty} ||Tf_n - Tf||.$$

Applying the definition of nonexpansive partial contractivity in the last limit,

$$\liminf_{n\to\infty} ||f_n - f|| < \liminf_{n\to\infty} ||f_n - f|| + \psi(||Tf_n - f_n||) = \liminf_{n\to\infty} ||f_n - f||.$$

Consequently $f = Tf$. □

Remark 2.12 The usual nonexpansive partial contractivities where $\psi(t) = Bt$, with $B \geq 0$, satisfy the conditions required in Theorem 2.11.

We consider in the next definition some types of sequences playing a key role in the approximation of fixed points of a self-map (see for instance the references [1,16]).

Definition 2.13 Let E be a normed space, $C \subseteq E$, $T : C \to C$ and a sequence $(f_n) \subseteq C$. Then (f_n) has the approximate fixed point property (AF property) if $\lim_{n\to\infty} ||f_n - Tf_n|| = 0$. The sequence (f_n) has the limit existence property (LE property) if $\lim_{n\to\infty} ||f_n - f^*||$ exists and is finite for any $f^* \in Fix(T)$, provided that $Fix(T) \neq \emptyset$.

Theorem 2.14 *Let E be a Banach space satisfying the Opial condition and C be a weakly closed subset of E. Let $T : C \to C$ be a nonexpansive partial contractivity, where ψ is right continuous at zero and non-decreasing. If a sequence (f_n) has the AF property then any weakly convergent subsequence of (f_n) converges weakly to a fixed point of T.*

Proof Applying Theorem 2.11, $I - T$ is demiclosed at zero. The AF property of (f_n) is equivalent to the fact that $(I - T)f_n \to 0$. If a subsequence (f_{n_j}) is weakly convergent to $f \in C$, the definition of demiclosed operator implies that $(I - T)f = 0$, and $f \in Fix(T)$. □

Theorem 2.15 *Let E be a reflexive Banach space satisfying the Opial condition and C be a bounded, closed and convex subset of E. Let $T : C \to C$ be a nonexpansive partial contractivity, where ψ is right continuous at zero and non-decreasing. If a sequence (f_n) has the AF property then $Fix(T) \neq \emptyset$ and any weakly convergent subsequence of (f_n) converges weakly to a fixed point of T. If further (f_n) has the LE property, then all the sequence (f_n) converges weakly to a fixed point $f^* \in C$.*

Proof In a reflexive Banach space, a bounded, closed and convex subset is weakly compact. Consequently, a sequence $(f_n) \subseteq C$ has a weakly convergent subsequence. Applying Theorem 2.14, the limit of this subsequence is a fixed point f^* of T. If (f_n) has the LE property, let us assume that there exists another weakly convergent subsequence (f_{m_k}) whose limit is $g^* \in C$. If $g^* \neq f^*$, the Opial condition implies that

$$\lim_{n\to\infty} ||f_n - f^*|| = \lim_{n\to\infty} ||f_{n_j} - f^*|| < \lim_{n\to\infty} ||f_{n_j} - g^*|| = \lim_{n\to\infty} ||f_n - g^*||,$$

and

$$\lim_{n\to\infty} ||f_n - g^*|| = \lim_{n\to\infty} ||f_{m_k} - g^*|| < \lim_{n\to\infty} ||f_{m_k} - f^*|| = \lim_{n\to\infty} ||f_n - f^*||.$$

Consequently $f^* = g^*$. □

Remark 2.16 The results given are true, in particular, for nonexpansive mappings.

3 Demicompact operators and fixed points

This section proposes sufficient conditions for the existence of fixed points of demicompact operators, and studies the convergence of approximating sequences to the critical points.

Definition 3.1 Let E be a Banach space and $C \subseteq E$ be closed. $T : C \to E$ is demicompact if any bounded sequence $(f_n) \subseteq C$ such that $(Tf_n - f_n)$ is convergent has a convergent subsequence (f_{n_j}).

If any bounded sequence $(f_n) \subseteq C$ such that $(Tf_n - f_n)$ is convergent to zero has a convergent subsequence (f_{n_j}), then T is demicompact at zero.

Remark 3.2 All over the paper, "convergent sequence" means convergence in norm (strong convergence).

Definition 3.3 Let E, F Banach spaces. Then $S : C \subseteq E \to F$ is closed if $f_n \to f$ and $Sf_n \to g$ implies that $g = Sf$.

Theorem 3.4 *Let E be a Banach space and C be a nonempty closed subset of E. Let $T : C \to C$ be closed and demicompact at zero. If $(f_n) \subseteq C$ is bounded and it has the AF property then $Fix(T) \neq \emptyset$, and there exists a subsequence (f_{n_j}) convergent to $f^* \in Fix(T)$. If further the sequence $(||f_n - f^*||)$ is decreasing, then all the sequence (f_n) converges to the fixed point f^*.*

Proof If $(f_n) \subseteq C$ is bounded and it has the AF property, the demicompactness of T at zero implies that it has a convergent subsequence (f_{n_j}). Let $f^* = \lim_{j\to\infty} f_{n_j}$.

The AF property implies that $((I - T)f_{n_j})$ tends to zero. Since T is closed, then $0 = (I - T)f^*$ and $f^* \in Fix(T)$.

If $(||f_n - f^*||)$ is decreasing, let us see that $\lim_{n\to\infty} ||f_n - f^*|| = 0$:

For any $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $||f_{n_j} - f^*|| < \varepsilon$ for $n_j \geq n_0$. Then let $n \geq n_j \geq n_0$, the decreasing character of the norm sequence implies that $||f_n - f^*|| \leq ||f_{n_j} - f^*|| < \varepsilon$. Consequently (f_n) converges to a fixed point of T. □

Remark 3.5 A continuous operator is closed, consequently Theorem 3.4 is true for continuous and demicompact at zero operators.Then we have the next result.

Corollary 3.6 *Let E be a Banach space and C be a nonempty closed subset of E. Let $T : C \to C$ be nonexpansive and demicompact at zero. If $(f_n) \subseteq C$ is bounded and it has the AF property then $Fix(T) \neq \emptyset$, and there exists a subsequence (f_{n_j}) convergent to $f^* \in Fix(T)$. If further the sequence $(||f_n - f^*||)$ is decreasing, then the sequence (f_n) converges to the fixed point f^*.*

Proof A nonexpansive operator is continuous, and the hypotheses of Theorem 3.4 are fulfilled. □

Theorem 3.7 *Let E be a Banach space and let $C \subseteq E$ be nonempty and closed. Let $T : C \to C$ be closed, demicompact at zero and such that $Fix(T) \neq \emptyset$. If $(f_n) \subseteq C$ has the LE and AF properties then (f_n) converges strongly to a fixed point of T.*

Proof The LE property implies that (f_n) is bounded since

$$||f_n|| \leq ||f_n - f^*|| + ||f^*||.$$

According to Theorem 3.4, (f_n) has a convergent subsequence (f_{n_j}) and $\lim_{j\to\infty} f_{n_j} = f^* \in Fix(T)$. Then $\lim_{j\to\infty} ||f_{n_j} - f^*|| = 0$. The LE property implies that $\lim_{n\to\infty} ||f_n - f^*|| = 0$.

□

Remark 3.8 This result is true for nonexpansive, and in general continuous, demicompact at zero operators.

4 A new three steps iteration for fixed point approximation

In this section we introduce a three-step algorithm for fixed point approximation. As per the knowledge of the author, this iterative scheme is new.

$$h_n = (1 - c_n)f_n + c_n T f_n, \tag{4.1}$$

$$g_n = (1 - b_n)f_n + b_n T h_n, \tag{4.2}$$

$$f_{n+1} = (1 - a_n)g_n + a_n T g_n, \tag{4.3}$$

for $a_n, b_n, c_n \in [0, 1]$, $n \geq 0$ and $f_0 \in E$. We will call this iterative scheme N^*-algorithm. We will prove that the sequence generated by this iterative procedure owns the LE and AF properties under some conditions on the underlying space when the operator T is quasi-nonexpansive.

Definition 4.1 Let E be a normed space and C be nonempty and closed. A self-map $T : C \to C$ is quasi-nonexpansive if $Fix(T) \neq \emptyset$ and

$$||Tf - f^*|| \leq ||f - f^*|| \tag{4.4}$$

for any $f \in C$ and $f^* \in Fix(T)$.

Proposition 4.2 *Let E be a normed space, and C be a nonempty, closed and convex subset of E. If $T : C \to C$ is quasi-nonexpansive, then the N^*-iteration has the LE property, is bounded and $(||f_n - f^*||)$ is decreasing for any $f^* \in Fix(T)$ and any $f_0 \in C$.*

Proof Let $f^* \in Fix(T)$, from (4.1) and the definition of quasi-nonexpansiveness,

$$||h_n - x^*|| = ||(1-c_n)(f_n - f^*) + c_n(Tf_n - f^*)|| \leq (1-c_n)||f_n - f^*|| + c_n||Tf_n - f^*|| \leq ||f_n - f^*|| \tag{4.5}$$

From (4.2) and (4.5),

$$||g_n - f^*|| \leq ||f_n - f^*||, \tag{4.6}$$

and using (4.3) and (4.6),

$$||f_{n+1} - f^*|| \leq (1-a_n)||g_n - f^*|| + a_n||g_n - f^*|| = ||f_n - f^*||. \tag{4.7}$$

Consequently the sequence $||f_n - f^*||$ is bounded and decreasing, and there exists $l := \lim_{n\to\infty} ||f_n - f^*|| \in \mathbb{R}$. The sequence (f_n) is bounded since

$$||f_n|| \leq ||f_n - f^*|| + ||f^*||,$$

and the first summand of the right-hand side is bounded. □

The next result is a consequence of concept of uniform convexity [17].

Lemma 4.3 *Let X be a uniformly convex Banach space, and a sequence $(\lambda_n) \subseteq X$ be such that there exist $p, q \in \mathbb{R}$ satisfying the condition $0 < p \leq \lambda_n \leq q < 1$ for all $n \in \mathbb{N}$. Let (x_n), (y_n) be sequences of X such that $\limsup_{n\to\infty} ||x_n|| \leq r$, $\limsup_{n\to\infty} ||y_n|| \leq r$, and $\limsup_{n\to\infty} ||\lambda_n x_n + (1-\lambda_n)y_n|| = r$ for some $r \geq 0$. Then $\lim_{n\to\infty} ||x_n - y_n|| = 0$.*

Theorem 4.4 *Let E be a uniformly convex Banach space, C be a closed and convex subset of C, and $T : C \to C$ be quasi-nonexpansive. Then the N^*-iteration such that $0 < \inf a_n \leq \sup a_n < 1$, $0 < \inf b_n \leq \sup b_n < 1$ and $0 < \inf c_n \leq \sup c_n < 1$, has the AF property for any $f_0 \in C$.*

Proof According to Proposition 4.2, the sequence (f_n) has the LE property. By the inequality (4.6),

$$\limsup_{n\to\infty} ||g_n - f^*|| \leq l := \lim_{n\to\infty} ||f_n - f^*|| \tag{4.8}$$

and

$$\limsup_{n\to\infty} ||Tg_n - f^*|| \leq l.$$

Moreover, by (4.3),

$$l = \lim_{n\to\infty} ||f_{n+1} - f^*|| = \lim_{n\to\infty} ||(1-a_n)(g_n - f^*) + a_n(Tg_n - f^*)||.$$

Lemma 4.3 implies that

$$\lim_{n\to\infty} ||Tg_n - g_n|| = 0. \tag{4.9}$$

From (4.3),

$$||f_{n+1} - f^*|| \leq ||g_n - f^*|| + a_n||Tg_n - g_n||.$$

Consequently

$$l \leq \liminf_{n\to\infty} ||g_n - f^*||$$

and bearing in mind (4.8),

$$l = \lim_{n\to\infty} ||g_n - f^*||. \tag{4.10}$$

Using (4.5),

$$||Th_n - f^*|| \leq ||h_n - f^*|| \leq ||f_n - f^*||$$

and thus

$$\limsup_{n\to\infty} ||Th_n - f^*|| \leq \limsup_{n\to\infty} ||h_n - f^*|| \leq \lim_{n\to\infty} ||f_n - f^*|| = l. \tag{4.11}$$

The equality

$$l = \lim_{n\to\infty} ||g_n - f^*|| = \lim_{n\to\infty} ||(1 - b_n)(f_n - f^*) + b_n(Th_n - f^*)||,$$

along with

$$l = \lim_{n\to\infty} ||f_n - f^*||$$

and

$$\limsup_{n\to\infty} ||Th_n - f^*|| \leq l$$

imply, by Lemma 4.3,

$$\limsup_{n\to\infty} ||Th_n - f_n|| = 0. \tag{4.12}$$

Then

$$\begin{aligned} ||g_n - f^*|| &\leq (1 - b_n)||f_n - f^*|| + b_n||h_n - f^*||, \\ ||g_n - f^*|| - ||f_n - f^*|| &\leq -b_n||f_n - f^*|| + b_n||h_n - f^*||, \\ ||g_n - f^*|| - ||f_n - f^*|| &\leq \frac{||g_n - f^*|| - ||f_n - f^*||}{b_n} \leq ||h_n - f^*|| - ||f_n - f^*||, \end{aligned}$$

and

$$||g_n - f^*|| \leq ||h_n - f^*||.$$

Then

$$l = \lim_{n\to\infty} ||g_n - f^*|| \leq \liminf_{n\to\infty} ||h_n - f^*||$$

and

$$l = \lim_{n\to\infty} ||h_n - f^*||,$$

since $\limsup_{n\to\infty} ||h_n - f^*|| \leq l$ due to (4.5). The equality

$$l = \lim_{n\to\infty} ||h_n - f^*|| = \lim_{n\to\infty} ||(1 - c_n)(f_n - f^*) + c_n(Tf_n - f^*)||,$$

along with

$$l = \lim_{n\to\infty} ||f_n - f^*||$$

and

$$\limsup_{n\to\infty} ||Tf_n - f^*|| \leq l$$

imply, by Lemma 4.3,

$$\lim_{n\to\infty} ||Tf_n - f_n|| = 0 \tag{4.13}$$

and (f_n) has the AF property for any $f_0 \in C$. □

Theorem 4.5 *Let E be a uniformly convex Banach space and $C \subseteq E$ nonempty, closed and convex. If $T : C \to C$ is closed, quasi-nonexpansive and demicompact at zero then the N^*-iteration such that $0 < \inf a_n \leq \sup a_n < 1$, $0 < \inf b_n \leq \sup b_n < 1$ and $0 < \inf c_n \leq \sup c_n < 1$, converges strongly to a fixed point of T.*

Proof It is a consequence of the properties of the N^*-iteration (boundedness, AF, LE properties, and the fact that $(||f_n - f^*||)$ is decreasing) along with Theorem 3.4. □

Theorem 4.6 *Let E be a uniformly convex Banach space and $C \subseteq E$ nonempty, bounded, closed and convex. If $T : C \to C$ is nonexpansive and demicompact at zero then the N^*-iteration such that $0 < \inf a_n \leq \sup a_n < 1$, $0 < \inf b_n \leq \sup b_n < 1$ and $0 < \inf c_n \leq \sup c_n < 1$, converges strongly to a fixed point of T.*

Proof With the hypotheses given, Browder's Theorem [4] ensures the existence of fixed point. Nonexpansive maps with fixed points are quasi-nonexpansive and continuous, and consequently closed. Thus we have the hypotheses of the previous theorem. □

Theorem 4.7 *Let E be a uniformly convex Banach space and $C \subseteq E$ nonempty, closed and convex. If $T : C \to C$ is a nonexpansive partial contractivity, closed/continuous, demicompact at zero and such that $Fix(T) \neq \emptyset$, then the N^*-iteration where $0 < \inf a_n \leq \sup a_n < 1$, $0 < \inf b_n \leq \sup b_n < 1$ and $0 < \inf c_n \leq \sup c_n < 1$, converges strongly to a fixed point of T.*

Proof Note that nonexpansive partial contractivities are quasi-nonexpansive since applying the definition for $g = f^* \in Fix(T)$

$$||Tf - f^*|| \leq ||f - f^*||.$$

Then we can apply Theorem 4.5. □

5 Two-steps N-iteration for fixed point approximation

By analogy with the concept of quasi-nonexpansive maps, we introduce the concept of Banach quasi-contractive map and remind the definition of partial contractivity. The rest of the section is devoted to study the convergence properties of a two-steps algorithm for fixed point approximation.

Definition 5.1 Let E be a normed space and $C \subseteq E$. $T : C \to E$ is Banach quasi-contractive if $Fix(T) \neq \emptyset$ and there exists $a \in \mathbb{R}$, $0 < a < 1$, such that

$$||Tf - f^*|| \leq a||f - f^*|| \tag{5.1}$$

for any $f \in C$ and $f^* \in Fix(T)$.

Remark 5.2 A Banach quasi-contractive map has a single fixed point due to its definition.

Definition 5.3 Let E be a normed space and $C \subseteq E$. $T : C \to E$ is called a partial contractivity if there exist constants $a, B \in \mathbb{R}$, $0 < a < 1$, $B \geq 0$ such that for any $f, g \in C$

$$||Tf - Tg|| \leq a||f - g|| + B \min\{||f - Tf||, ||g - Tg||\}. \tag{5.2}$$

Example 5.4 Let $T : [0, 1] \to \mathbb{R}$ be defined as $Tx = kx$ (where $0 < k < 1$) for $x \in [0, 1/2)$ and $Tx = 0$ for $x \in [1/2, 1]$. The mapping T is a partial contractivity with constants $a = k$ and $B = k/(1 - k)$ since:

- If $x, y \in [0, 1/2)$ then

$$|Tx - Ty| = |kx - ky| \leq k|x - y| + \frac{k}{1-k} \min\{|x - Tx|, |y - Ty|\}.$$

- If $x, y \in [1/2, 1]$ then

$$|Tx - Ty| = 0 \leq k|x - y| + \frac{k}{1-k} \min\{|x - Tx|, |y - Ty|\}.$$

- If $x \in [0, 1/2)$ and $y \in [1/2, 1]$ then

$$|Tx - Ty| = kx \leq k(y-x) + \frac{k}{1-k}\min\{(1-k)x, y\} \leq k|x-y| + \frac{k}{1-k}\min\{|x-Tx|, |y-Ty|\}.$$

The last inequality is due to the positions of x and y in the interval.

Example 5.5 Let $T : \mathbb{R} \to \mathbb{R}$ be defined as $Tx = 3x$. This map is not a partial contractivity since the inequality

$$|Tx - Ty| = 3|x-y| \leq a|x-y| + B\min\{2|x|, 2|y|\}$$

should be true for any $x, y \in \mathbb{R}$. However, taking $x = 0$, we would obtain

$$3|y| \leq a|y|,$$

where $a < 1$, and this holds only for $y = 0$.

Remark 5.6 Unlike the Banach contractive mappings, a partial contractivity may be discontinuous, as shown in Example 5.4.

Remark 5.7 A Banach contraction is quasi-contractive. A partial contractivity such that $Fix(T) \neq \emptyset$ is Banach quasi-contractive.

Remark 5.8 The concept of Banach quasi-contractive map should not be confused with the definition of quasi-contraction given by Ćirić [5]. They are independent notions.

We consider now an iterative procedure of two-steps for fixed point approximation. It corresponds to the N-iteration defined in [12] when the intermediate sequence of scalars is constantly equal to 1. It is given by the scheme:

$$g_n = (1-b_n)f_n + b_n Tf_n, \tag{5.3}$$

$$f_{n+1} = (1-a_n)g_n + a_n Tg_n, \tag{5.4}$$

where $a_n, b_n \in [0, 1]$, and $f_0 \in E$. If $b_n = 0$ for all n we obtain the Mann iteration. If further $a_n = \lambda$ the scheme agrees with the Krasnoselskii algorithm. If $a_n = 1$ and $b_n = 0$ we have the usual Picard iteration.

5.1 Convergence of the two-setps N-iteration for Banach quasi-contractive maps in quasi-normed spaces

Definition 5.9 If E is a real linear space, the mapping $|\cdot|_s : E \times E \to \mathbb{R}^+$ is a quasi-norm of index or modulus of concavity s if

(1) $|f|_s \geq 0$; $f = 0$ if and only if $|f|_s = 0$.
(2) $|\lambda f|_s = |\lambda||f|_s$.
(3) There exists $s \geq 1$ such that $|f+g|_s \leq s(|f|_s + |g|_s)$ for any $f, g \in E$.

The space $(E, |\cdot|_s)$ is a quasi-normed space. If E is complete with respect to the b-metric induced by the quasi-norm, then E is a quasi-Banach space. If $s = 1$ then E is a normed space.

Proposition 5.10 *Let E be a quasi-normed space with modulus of concavity s and quasi-norm $|\cdot|_s$, and $T : C \to C$ be a Banach quasi-contractive map with ratio a such that $as < 1$, and C a nonempty closed and convex subset of E. Let $r \in \mathbb{R}$ be such that $as < r < 1$. Choosing the constants a_n, b_n such that $\min\{a_n, b_n\} > (1 - rs^{-1})/(1-a)$ the two-steps N-iteration converges to the fixed point $f^* \in Fix(T)$, and it is asymptotically stable.*

Proof Let $f_0 \in C$, according to (5.3),

$$|g_n - f^*|_s = |(1-b_n)(f_n - f^*) + b_n(Tf_n - f^*)|_s \leq s(1 - b_n(1-a))|f_n - f^*|_s.$$

In the same way

$$|f_{n+1} - f^*|_s = |(1-a_n)(g_n - f^*) + a_n(Tg_n - f^*)|_s \leq s(1 - a_n(1-a))|g_n - f^*|_s,$$

and thus

$$|f_{n+1} - f^*|_s \leq s^2(1 - a_n(1 - a))(1 - b_n(1 - a))|f_n - f^*|_s \tag{5.5}$$

The conditions imposed on the scalars a_n, b_n imply that $(1 - a_n(1 - a)) < rs^{-1}$ and $(1 - b_n(1 - a)) < rs^{-1}$ and consequently

$$|f_{n+1} - f^*|_s \leq r^2|f_n - f^*|_s.$$

In general, we obtain that

$$|f_n - f^*|_s \leq r^{2n}|f_0 - f^*|_s.$$

Since $r < 1$ the algorithm converges with asymptotic stability. □

Remark 5.11 The condition $as < 1$ is necessary to obtain that $1 \geq \min\{a_n, b_n\} > (1 - rs^{-1})/(1 - a)$. Note that this condition is less demanding than the required for the Ishikawa iteration ($as^2 < 1$) (see the reference [13]).

Corollary 5.12 *If T is a partial contractivity such that $Fix(T) \neq \emptyset$ the algorithm converges for the scalars chosen in the previous proposition.*

5.2 Convergence of the two-steps N-iteration for quasi-nonexpansive maps in normed spaces

Theorem 5.13 *Let E be a normed space, $T : C \to C$, where C is nonempty, closed and convex. If T is quasi-nonexpansive, then the two-steps N-iteration has the LE property, is bounded and $(||f_n - f^*||)$ is decreasing for any $f^* \in Fix(T)$ and any $f_0 \in C$ If further E is uniformly convex then the N-iteration such that $0 < \inf a_n \leq \sup a_n < 1$ and $0 < \inf b_n \leq \sup b_n < 1$ has the AF property.*

Proof The arguments are similar to the given in the proof of Theorem 4.3 of the reference [12]. □

Theorem 5.14 *Let E be a uniformly convex Banach space and $C \subseteq E$ nonempty, closed and convex. If $T : C \to C$ is closed, quasi-nonexpansive and demicompact at zero then the two-steps N-iteration such that $0 < \inf a_n \leq \sup a_n < 1$ and $0 < \inf b_n \leq \sup b_n < 1$ converges strongly to a fixed point of T.*

Proof It is a consequence of the properties of the two-steps N-iteration (boundedness, AF, LE properties, and the fact that $(||f_n - f^*||)$ is decreasing) and Theorem 3.4. □

Remark 5.15 Note that the Picard iteration does not satisfy the conditions given for convergence.

Theorem 5.16 *Let E be a uniformly convex Banach space and $C \subseteq E$ nonempty, bounded, closed and convex. If $T : C \to C$ is nonexpansive and demicompact at zero then the two-steps N-iteration such that $0 < \inf a_n \leq \sup a_n < 1$ and $0 < \inf b_n \leq \sup b_n < 1$ converges strongly to a fixed point of T.*

Proof The arguments are similar to the given in Theorem 4.6. □

Theorem 5.17 *Let E be a uniformly convex Banach space and $C \subseteq E$ nonempty, closed and convex. If $T : C \to C$ is a nonexpansive partial contractivity, closed/continuous, demicompact at zero and $Fix(T) \neq \emptyset$, then the two-steps N-iteration such that $0 < \inf a_n \leq \sup a_n < 1$ and $0 < \inf b_n \leq \sup b_n < 1$ converges strongly to a fixed point of T.*

Proof The arguments are similar to the given in Theorem 4.7. □

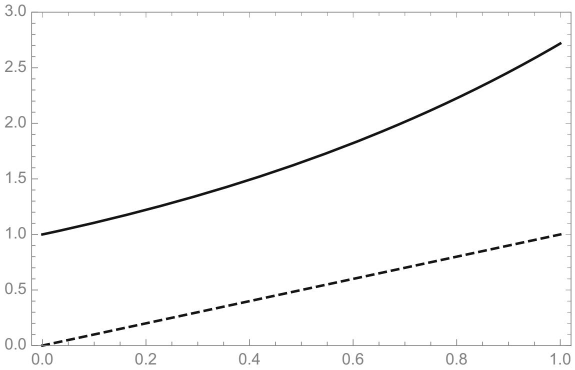

Fig. 1 Exact solution (solid line) with starting function f_0 (dashed line)

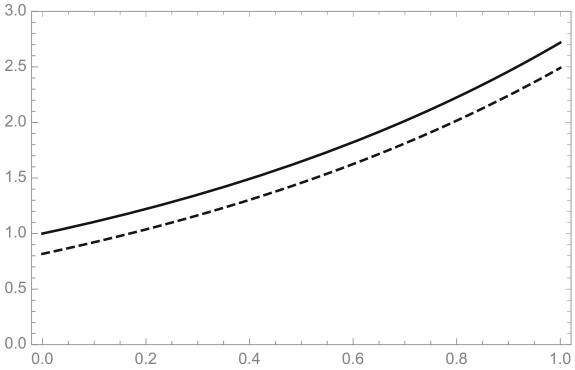

Fig. 2 Exact solution (solid line) with second approximation f_2 (dashed line)

6 Application to the numerical solution of Fredholm integral equations of second kind

This section is devoted to the numerical solution of Fredholm integral equations of second kind, expressed as

$$F(x) = g(x) + \lambda \int_a^b k(x, y)F(y)dy, \tag{6.1}$$

where g and k are given functions, and $\lambda \leq 1$. The problem consists of finding the map F satisfying the equality (6.1). For it, we define two operators T_k and $T_{g,\lambda}$ as

$$T_k f(x) = \int_a^b k(x, y)f(y)dy, \tag{6.2}$$

and

$$T_{g,\lambda} f(x) = g(x) + \lambda T_k f(x), \tag{6.3}$$

for $x \in [a, b]$. These operators are defined on the Hilbert space $\mathcal{L}^2[a, b]$. The function k is the kernel of T_k.

Let us start recalling the definition of a special class of operators.

Definition 6.1 Let E be a normed space, and $T : E \to E$ be linear. T is compact if any bounded sequence $(f_n) \subseteq E$ has a subsequence (f_{n_j}) such that $(T(f_{n_j}))$ is convergent.

If $k \in \mathcal{L}^2([a, b] \times [a, b])$, T_k is a well known linear compact operator on $\mathcal{L}^2[a, b]$. T_k is also a Hilbert–Schmidt operator [6], and the Hilbert–Schmidt norm of T agrees with $||k||_2$. For the 2-norm, $||T_k|| \leq ||k||_2$.

A compact operator is demicompact: If (f_n) is bounded and $(f_n - T f_n)$ is convergent, the compactness of T implies that there exists (f_{n_j}) such that $(T(f_{n_j}))$ is convergent. Then

$$f_{n_j} = (f_{n_j} - T f_{n_j}) + T f_{n_j}.$$

Table 1 Root mean square errors of the first N^*-iterations

Iteration	Error
0	1.236070
1	0.660805
2	0.347547
3	0.181017
4	0.093803
5	0.048485
6	0.025030
7	0.012913
8	0.006660
9	0.003435
10	0.001771

Then f_{n_j} is the sum of two convergent sequences and thus convergent.

Consequently T_k is demicompact, and particularly is demicompact at zero, and the results obtained in Sects. 3, 4 and 5 are applicable to it. Moreover, the linear compact operators are bounded and consequently continuous. Let us see that the affine operator $T_{g,\lambda}$ is also demicompact:

If (f_n) is bounded and $(f_n - T_{g,\lambda} f_n)$ is convergent, since T_k is compact there is a convergent subsequence $(T_k(f_{n_j}))$. As $(f_{n_j} - g - \lambda T_k f_{n_j})$ is convergent then (f_{n_j}) is.

Let k be such that $||k||_2 \leq 1$. Then $||T_{g,\lambda} f - T_{g,\lambda} f'||_2 \leq \lambda ||f - f'||_2$.

If $\lambda < 1$, $T_{g,\lambda}$ is a Banach contraction and its unique fixed point is the unique solution of the equation (6.1). If $\lambda = 1$ then $T_{g,\lambda}$ is nonexpansive. In both cases $T_{g,\lambda}$ is continuous. The results of Sect. 3, regarding general approximation sequences, and the theorems concerning the N^*-iteration and the two-steps N-iteration of Sects. 4 and 5, are applicable to this problem. In the following, an example of iterative resolution of a Fredhlom integral equation of second kind with exact solution is presented, in order to check the methods, and to compute approximation errors.

Example 6.2 Let us consider the Fredholm integral equation of the second kind:

$$F(x) = (e^x - 1) + \int_0^1 yF(y)dy, \tag{6.4}$$

whose exact solution is $F(x) = e^x$. The operator in this case is $T_{g,\lambda} f(x) = (e^x - 1) + \int_0^1 yf(y)dy$, with $g(x) = e^x - 1$ and $\lambda = 1$.

The N^*-iteration has been implemented, corresponding to the scheme:

$$h_n(x) = (1 - c_n) f_n(x) + c_n(e^x - 1) + c_n \int_0^1 y f_n(y)dy.$$

$$g_n(x) = (1 - b_n) f_n(x) + b_n(e^x - 1) + b_n \int_0^1 y h_n(y)dy.$$

$$f_{n+1}(x) = (1 - a_n) g_n(x) + a_n(e^x - 1) + a_n \int_0^1 y g_n(y)dy.$$

Ten steps of the algorithm were performed, starting at the function $f_0(x) = x$. The scalars have been chosen as $a_n = b_n = c_n = 1/2$ for all n.

For every approximation, the root mean square error was computed as $||f_n - f^*||_2$, where $f^*(x) = e^x$ is the exact solution. Table 1 represents the error for the different approximations.

For the two-steps N-iteration the scheme is

$$g_n(x) = (1 - b_n) f_n(x) + b_n(e^x - 1) + b_n \int_0^1 y f_n(y)dy.$$

$$f_{n+1}(x) = (1 - a_n) g_n(x) + a_n(e^x - 1) + a_n \int_0^1 y g_n(y)dy.$$

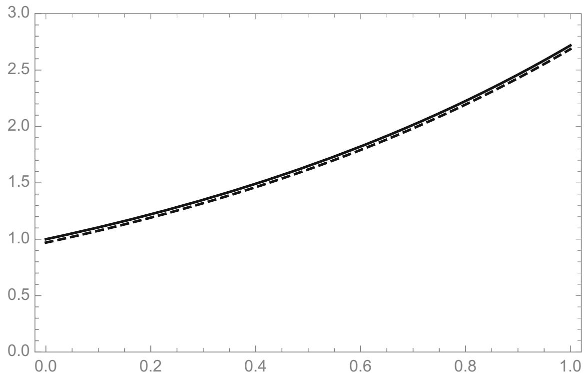

Fig. 3 Exact solution (solid line) with fourth approximation f_4 (dashed line)

Table 2 Root mean square errors of the first two-steps N-iterations

Iteration	Error
0	1.236070
1	0.723126
2	0.414890
3	0.235530
4	0.133037
5	0.074973
6	0.042207
7	0.023750
8	0.013362
9	0.007516
10	0.004228

The number of iterations was 10, the scalars chosen were $a_n = b_n = 1/2$ and the starting function $f_0(x) = x$, as in the previous case. For every approximation, the root mean square error was computed as $||f_n - f^*||_2$, where $f^*(x) = e^x$, from $n = 0$ to 10. Table 2 represents the error for the different approximations.

We may note that the two-steps algorithm is lightly slower than the N^*- procedure, but we should bear in mind that the latter involves one more operator evaluation at each iteration and consequently is computationally more expensive.

Figures 1, 2 and 3 represent the graph of the exact solution of the equation (6.4) along with the zero-th, second and fourth approximation of the two-steps N-iteration, respectively.

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Conflict of interest The author declares that she has no conflict of interest.

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