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RESEARCH

New analytic representations of the Lerch transcendent

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Abstract

We consider an integral representation of the Lerch transcendent function $\Phi(z,s,a)$ of the form $\Phi(z,s,a)=\int_0^1 h(t,z)g(t,s,a)dt$, and two different analytical methods for the approximation of this integral transform to obtain new convergent expansions of the Lerch transcendent in the variable z. The first method uses multi-point Taylor expansions of $h(t,z)$ at certain appropriately selected base points that provides convergent expansions of the Lerch transcendent in terms of elementary functions of z uniformly valid in compact sets of the complex $z-$plane. The second method expands $g(t,s,a)$ in a Taylor series at a selected point in $[0,1]$ giving a uniform convergent expansion of $\Phi(z,s,a)$ in terms of elementary functions of z valid in a large unbounded region of the complex plane. We provide explicit and/or recursive algorithms for the computation of the coefficients of the expansions. Numerical experiments illustrate the accuracy of the new approximations.

Keywords Lerch transcendent function · Convergent expansions · Uniform convergent expansions · Multi-point Taylor expansions · Special functions

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1 Introduction

The Lerch transcendent, also known as the Hurwitz-Lerch zeta function, is defined by the power series [1, Eq. 25.14.1], [2, Section 1.11, Eq. (1)]

$$\Phi(z, s, a) := \sum_{n=0}^{\infty} \frac{z^n}{(a+n)^s}, \quad |z| < 1; \quad \Re s > 1, \ |z| = 1. \tag{1}$$

If s is not an integer then $|\arg a| < \pi$; if s is a positive integer then $a \neq 0, -1, -2, \ldots$; if s is a non-positive integer then a can be any complex number. For other values of the arguments z, s, a, $\Phi(z, s, a)$ is defined by means of analytic continuation, usually by means of a suitable integral representation (see, for example, [1, Eq. 25.14.5]).

This function was first investigated by Erdélyi et al. [2, Section 1.11], where functional relations, integral and series representations, limit relationships and connection with other special functions are provided by the authors. Using the notation $z = e^{2\pi i x}$, the function $\mathcal{R}(x, s, a) = \Phi\left(e^{2\pi i x}, s, a\right)$, known as the Lerch zeta function, was previously introduced by Lerch [3] and Lipschitz [4] in connection with Dirichlet's prime number theorem. More properties of the Lerch transcendent have been investigated by many authors. A comprehensive review of specific values, transformations, identities, differentation formulas or representations through more general functions can be found in [5].

The Lerch transcendent has important applications in particle physics, thermodynamics, statistical mechanics and quantum field theory (see [6] and references therein). Many sums of reciprocal powers can be expressed in terms of $\Phi(z, s, a)$ and also can be used to evaluate Dirichlet L−series (see [7]).

As special cases of the Lerch transcendent we can find, among others, the *Hurwitz zeta function* $\zeta(s, a)$ ([1, Eq. 25.11.1]),

$$\zeta(s, a) := \sum_{n=0}^{\infty} \frac{1}{(a+n)^s} = \Phi(1, s, a), \quad \Re s > 1, \ a \neq 0, -1, -2, \ldots, \tag{2}$$

the *polylogarithm* $\mathrm{Li}_s(z)$ ([1, Eq. 25.12.10]),

$$\mathrm{Li}_s(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^s} = z\Phi(z, s, 1), \quad |z| < 1; \quad \Re s > 1, \ |z| = 1, \tag{3}$$

the *Riemann zeta function* ([1, Eq. 25.2.1]),

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} = \Phi(1, s, 1), \quad \Re s > 1, \tag{4}$$

or the *Dirichlet beta function* $\beta(s)$ ([8, Eq. 1]),

$$\beta(s) := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s} = 2^{-s}\Phi\left(-1, s, \frac{1}{2}\right), \quad \Re s > 0. \tag{5}$$

The power series definition of the Lerch transcendent $\Phi(z, s, a)$ (1) is absolutely convergent inside the unit disk $|z| < 1$ under the constraints over s and a specified in the definition. On the unit circle $|z| = 1$ the series is convergent if $\Re s > 1$, $s \notin \mathbb{N}$ and $a \neq -1, -2, \ldots$, and if $\Re s > 1$, $s \in \mathbb{N}$ and $a \neq 0, -1, -2, \ldots$ For numerical computations we can use the right hand side of (1) to compute $\Phi(z, s, a)$ only in the disk $|z| \leq \rho \leq 1$, with ρ depending on numerical requirements, such as precision and efficiency.

We can find in the literature other convergent series representations of the Lerch transcendent. In [9, Eq. (22)], an expansion in powers of $z/(1-z)$ is given by

$$(1-z)\Phi(z, s, a) = \sum_{n=0}^{\infty} \left(\frac{-z}{1-z}\right)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} (a+k)^{-s}, \tag{6}$$

for $s, z \in \mathbb{C}$ with $\Re z < 1/2$. Erdélyi et al. [2, Section 1.11, Eq. (8)] obtained the series expansion

$$z^a \Phi(z, s, a) = \Gamma(1-s)(-\log(z))^{s-1} + \sum_{n=0}^{\infty} \zeta(s-n, a) \frac{\log^n z}{n!}, \tag{7}$$

which holds for $|\log z| < 2\pi$, $s \neq 1, 2, 3, \ldots$, $a \neq 0, -1, -2, \ldots$ and where $\zeta(s, a)$ is the Hurwitz zeta function (2). When $a \in (0, 1]$ the series is linearly convergent (see [10]).

Asymptotic expansions of $\Phi(z, s, a)$ for large a and fixed s and z with $|\arg(a)| < \pi$, $s \in \mathbb{C}$ and $z \in \mathbb{C} \setminus [1, \infty)$ if $\Re a > 0$ and $z \in \mathbb{C}$, $|z| < 1$ if $\Re a \leq 0$, including error bounds, have been investigated in [11, Theorem 1]. In [12], the authors study the special case $z \geq 1$, not covered in [11], deriving a complete asymptotic expansion for $z > 1$ and $\Re s > 0$ as $\Re a \to \infty$. In [11, Theorem 2], an asymptotic expansion of $\Phi(z, s, a) - a^{-s}$ for small a and fixed s and z with $s \in \mathbb{C}$ and $z \in \mathbb{C} \setminus [1, \infty)$ is obtained. In [13], Katsurada derives a power series and an asymptotic series for $\mathcal{R}(x, s, a) = \Phi\left(e^{2\pi i x}, s, a\right)$ in the parameter a using Mellin transform techniques, but error bounds are not provided. On the other hand, Paris studies in [14] the exponentially improved asymptotic expansion of the Lerch zeta function for large complex values of a (with s and z as parameters) from a Mellin-Barnes integral representation.

In [15, Theorem 3], the authors analyzed the asymptotic behaviour of $\Phi(z, s, a)$ as $\Re s \to -\infty$, obtaining an expansion valid as $\Re s \to -\infty$ with $|\Im s|$ bounded, $0 < |z| < 1$ and $a \in K$ with K a certain compact set in $\mathbb{C}$.

For $z \in \mathbb{C} \setminus [0, \infty)$, $|z| > 1$, $\Re a > 0$ and $\Re s > 0$ an asymptotic expansion of $\Phi(z, s, a)$ for large z and fixed a and s is derived in [11, Theorem 3],

$$\begin{aligned}\Phi(z, s, a) = \frac{1}{\Gamma(s)} \Bigg\{ & \left[\sum_{n=0}^{N-1} \frac{A_n(z, s, a)}{z^{n+1}} + R_N^1(z, s, a)\right] \\ & + \frac{(\log(-z))^s}{(-z)^a} \left[\sum_{n=0}^{M-1} \frac{B_n(s, a)}{(\log(-z))^{n+1}} + R_M^2(z, s, a)\right]\Bigg\},\end{aligned} \tag{8}$$

where $N, M = 1, 2, 3, \ldots,$

$$A_n(z, s, a) := \frac{\Gamma(s, (a-n-1)\log(-z)) - \Gamma(s)}{(a-n-1)^s}$$

and

$$B_n(s,a) := \frac{1}{2}\binom{s-1}{n}\begin{cases}\psi((a+1)/2) - \psi(a/2) & \text{if } n = 0,\\ n!2^{-n}[\zeta(n+1, a/2) - \zeta(n+1, (a+1)/2)] & \text{if } n > 0,\end{cases}$$

where $\Gamma(s, w)$ is the incomplete gamma function [1, Eq. 8.2.1], $\psi(a)$ is the digamma function ((3) with $s = 2$), $\zeta(s, a)$ is the Hurwitz zeta function (2), and $R_N^1(z, s, a) = \mathcal{O}(z^{-N-1})$ for $N \le a - 1$ and $R_M^2(z, s, a) = \mathcal{O}((\log z)^{-M-1})$ for $M = 1, 2, 3, \ldots,$ when $|z| \to \infty$ with fixed $\arg(z)$. In [16], the author uses a Mellin-Barnes integral representation for the Lerch transcendent $\Phi(z, s, a)$ to obtain large z asymptotic approximations. The asymptotic approximation is the sum of two series: the first one in powers of $(\log z)^{-1}$ and the second one in powers of z^{-1}. The second series converges, but it is completely hidden in the divergent tail of the first series. To make the second series visible, resummation and optimal truncation is considered.

An equation relating $\Phi(z, n, a)$ and $\Phi(1/z, n, 1-a)$ is used in [6] to provide an expansion of the Lerch transcendent in powers of $1/z$, convergent for $|z| > 1$ and $z \notin (-\infty, -1) \cup (1, \infty)$, but just valid for $n \in \mathbb{N}$.

Different computational schemes for the numerical evaluation of $\Phi(z, n, a)$ have been examined in [10, 17]. The algorithms are based on Euler-MacLaurin expansions; convergent, asymptotic and uniform asymptotic series; Bernoulli-series and Riemann-splitting representation; functional relations, translation properties and doubling formulas; numerical integration, etc.

In this paper we investigate new convergent expansions of the Lerch transcendent $\Phi(z, s, a)$ in the variable z. The main objective is to extend the region of convergence of $\Phi(z, s, a)$ and obtain new analytical expansions given in terms of elementary functions. The starting point is the integral representation [1, Eq.25.14.5]:

$$\Phi(z, s, a) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{x^{s-1}e^{-ax}}{1 - ze^{-x}}\,\mathrm{d}x, \tag{9}$$

with $\Re s > 1, \Re a > 0$ if $z = 1$; $\Re s > 0, \Re a > 0$ if $z \in \mathbb{C} \setminus [1, \infty)$. After the change of variable $x \to t$ given by $x = -\log t$ in (9), the Lerch transcendent can be written as

$$\Phi(z, s, a) = \frac{1}{\Gamma(s)}\int_0^1 (-\log t)^{s-1}\, t^{a-1}\frac{1}{1 - zt}\,\mathrm{d}t. \tag{10}$$

That is, we can write $\Phi(z, s, a)$ in the form $\Phi(z, s, a) = \dfrac{1}{\Gamma(s)}F(z)$ where

$$F(z) = \int_0^1 h(t, z) g(t, s, a) dt, \tag{11}$$

with $h(t,z) = 1/(1-zt)$ and $g(t,s,a) = (-\log t)^{s-1}\, t^{a-1}$.

Then, we apply two different analytical methods for the approximation of $F(z)$: the first one considers a Taylor expansion of $h(t,z)$, and the second one of $g(t,s,a)$. In Section 2, we apply the first method at certain appropriately selected base points for the Taylor expansion, obtaining a series expansion valid in larger domains of convergence than (1) with the property of being uniformly valid in compact sets of the complex $z-$plane. In Section 3, we apply the second method, that provides a uniform convergent expansion of $F(z)$ in an unbounded region D of the complex plane. Numerical examples illustrate the properties of the new approximations and are compared with other existing ones.

2 Uniformly convergent expansions of $\Phi\,(z, s, a)$ on compact sets

In this section, we derive new expansions of the Lerch transcendent $\Phi\,(z,s,a)$ in terms of elementary functions of z that converge in other regions of the complex $z-$plane different from those previously considered in the literature and mentioned in the introduction. These new expansions are uniformly valid on any compact set included in those regions. For this purpose, we apply the method introduced in [18], and later successfully applied in [19, 20] in the approximation of generalized hypergeometric functions. We summarize the main ideas of this technique. Consider the multi-point Taylor expansion of $h(t,z)$ in (11), as a function of t, with base points $t_1, t_2, \ldots, t_m \in \mathbb{C}$ convergent in a Cassini oval $D_{m,r}$ of radius r (see [21]) and satisfying the following requirements:

(i) The interval of integration $(0,1)$ of (11) is completely contained in the domain of convergence $D_{m,r}$ of the multi-point Taylor expansion of $h(t,z)$.
(ii) The branch point of $h(t,z)$ as a function of the integration variable t, the point $t = 1/z$, must be located outside $D_{m,r}$, that is, z must be located in a region $S_r =$ the inverse to the exterior of $D_{m,r}$: $S_r = \left(D_{m,r}^{\rm EXT}\right)^{-1}$.

Then, replace $h(t,z)$ in (11) by the multi-point Taylor expansion and interchange sum and integral. We obtain an expansion of $F(z)$ convergent for $z \in S_r$. The larger S_r is, the better, and one expects that the smaller $D_{m,r}$ (satisfying (i)), the bigger S_r will be. See [18] for further details.

We next consider different selections of the base points which give rise to different expansions valid in different regions.

2.1 An expansion in the region $|1 - wz| > |z|\max\{|w|, |1-w|\}$ with arbitrary $w \in \mathbb{C}$

In this subsection, we consider the classical Taylor expansion of $h(t,z)$ at an arbitrary point $t = w \in \mathbb{C}$, that is, we consider only one base point $t_1 \equiv w$ for the Taylor expansion. The results are stated in the following theorem.

Theorem 1 *Consider any $w \in \mathbb{C}$, $W := \max\{|w|, |1-w|\}$ and the region $S = \{z \in \mathbb{C}, |1-wz| > W|z|\}$ (see Fig. 1). Then, for any $z \in S$, $\Re s > 0$, $\Re a > 0$ and*

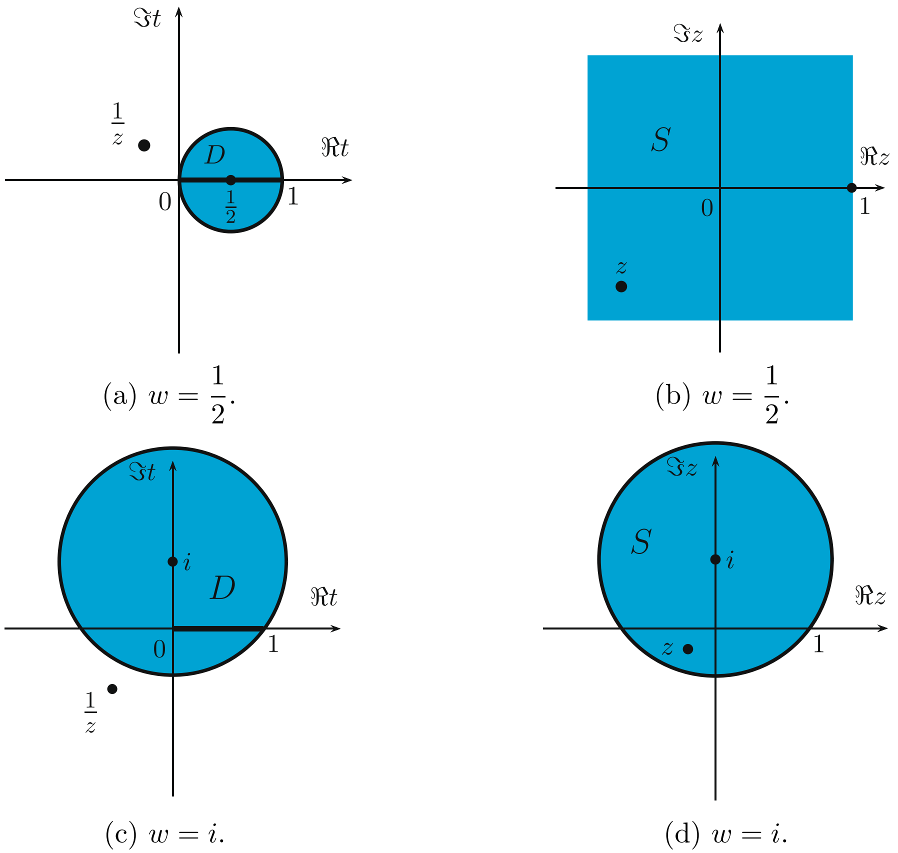

Fig. 1 The minimal domain of convergence D of the standard Taylor expansion of $h(t,z)$ at $t = w$ containing the interval $(0,1)$ is a disk of center at $t = w$ and radius $\max\{|w|, |1-w|\}$ (Figures (a) and (c)). The region S, inverse of the exterior of D is: the half-plane $S = \{z = x + iy;\, x, y \in \mathbb{R},\, 1 - 2\Re(wz) > 0\}$ if $\Re w \geq 1/2$ (Figure (b)) or the disk of center $\overline{w}/(2\Re w - 1)$ and radius $|w-1|/(1-2\Re w)$ if $\Re w < 1/2$ (Figure (d))

$n = 1, 2, 3, \ldots$,

$$\Phi\left(z, s, a\right) = \sum_{k=0}^{n-1} \frac{z^k}{(1-wz)^{k+1}} \phi_k(s, a, w) + R_n(z, s, a, w), \tag{12}$$

with

$$\phi_k\left(s, a, w\right) := \sum_{j=0}^{k} \binom{k}{j} \frac{(-w)^{k-j}}{(a+j)^s}. \tag{13}$$

For $k = 0, 1, 2, \ldots$, and $a \neq 1$, the terms of the expansion satisfy the recurrence relation

$$\phi_{k+1}\left(s, a, w\right) = \phi_k\left(s, a+1, w\right) - w\phi_k\left(s, a, w\right), \quad \phi_0\left(s, a, w\right) = \frac{1}{a^s}. \tag{14}$$

The remainder term is bounded by

$$|R_n(z,s,a,w)| \le \frac{\Gamma(\Re s)}{|\Gamma(s)|}\frac{M(z)}{\Re a^{\Re s}}\left|\frac{z}{1-wz}\right|^n, \tag{15}$$

with $M(z)$ defined below in (21). Bound (15) shows that expansion (12) is uniformly valid in compact sets of S.

Proof Consider the Taylor expansion of the function $h(t,z)=(1-zt)^{-1}$, as a function of the variable t, at a generic point $w=u+vi\in\mathbb{C}$, $u,v\in\mathbb{R}$,

$$h(t,z)=\sum_{k=0}^{n-1}\frac{z^k(t-w)^k}{(1-wz)^{k+1}}+r_n(t,w,z), \tag{16}$$

with

$$r_n(t,w,z):=\frac{z^n(t-w)^n}{(1-zt)(1-wz)^n}. \tag{17}$$

Expansion (16) satisfies the following conditions: (i) The interval of integration (0, 1) is completely contained in the convergence disk of this expansion $D:=\{t\in\mathbb{C},\ |t-w|<W\}$; (ii) When $z\in S$, the singularity $1/z\notin D$. For $u=\Re w\ge 1/2$ the domain S is the half-plane $S=\{z=x+iy;\, x,y\in\mathbb{R},\, 2\Re(wz)=2ux-2vy<1\}$. For $u<1/2$ it is the disk $S=\{z\in\mathbb{C},\ |z+(1-2u)^{-1}\overline{w}|<(1-2u)^{-1}|w-1|\}$ (see Fig. 1).

Then, for $z\in S$, we can substitute expansion (16) into (10) to obtain (12), with

$$\phi_k(s,a,w)=\frac{1}{\Gamma(s)}\int_0^1(-\log t)^{s-1}t^{a-1}(t-w)^k\,\mathrm{d}t, \tag{18}$$

and

$$R_n(z,s,a,w):=\frac{1}{\Gamma(s)}\left(\frac{z}{1-wz}\right)^n\int_0^1(-\log t)^{\Re s-1}t^{a-1}\frac{(t-w)^n}{1-zt}\,\mathrm{d}t. \tag{19}$$

Expanding the factor $(t-w)^k$ in (18), we obtain the explicit representation given in (13). Moreover, after straightforward manipulations in the integral (18), it is readily seen that the coefficients $\phi_k(s,a,w)$ satisfy the recurrence relation (14) for $a\neq 1$.

On the other hand,

$$\begin{aligned}|R_n(z,s,a,w)| &\le \frac{1}{|\Gamma(s)|}\left|\frac{z}{1-wz}\right|^n\int_0^1(-\log t)^{\Re s-1}t^{\Re a-1}\frac{|t-w|^n}{|1-zt|}\,\mathrm{d}t\\ &\le \frac{1}{|\Gamma(s)|}\left|\frac{zW}{1-wz}\right|^n\int_0^1(-\log t)^{\Re s-1}t^{\Re a-1}\frac{1}{|1-zt|}\,\mathrm{d}t.\end{aligned} \tag{20}$$

Now, if $z \in \mathbb{C} \setminus [1, \infty)$, we can bound

$$\frac{1}{|1-zt|} = \frac{1}{\sqrt{1-2\Re zt+|z|^2t^2}} \le M(z) := \begin{cases} 1 & \text{if } \Re z \le 0 \text{ (Region 1)}, \\ |1-z|^{-1} & \text{if } \Re(1/z) \ge 1 \text{ (Region 2)}, \\ |\sin(\arg(z))|^{-1} & \text{if } 0 < \Re(1/z) < 1 \text{ (Region 3)}. \end{cases} \tag{21}$$

In Region 1, the maximum of $|(1-zt)|^{-1}$ is attained at $t = 0$, in Region 2 at $t = 1$, and in Region 3 at $t = \frac{\Re z}{|z|^2}$. Figure 2 shows the three different regions defined in (21).

Finally, introducing the bound (21) into (20) and taking into account that $\Re s > 0$ and $\Re a > 0$, we obtain (15). □

We consider now the particular case of the polylogarithm $\mathrm{Li}_s(z) = z\Phi(z, s, 1)$ defined in (3). The results stated for the Lerch transcendent $\Phi(z, s, 1)$ in Theorem 1 remain true replacing $a \to 1$, although the recurrence relation (14) becomes useless in this case. We obtain the following corollary.

Corollary 2 *Consider any arbitrary $w \in \mathbb{C}$, $W := \max\{|w|, |1-w|\}$ and the region $S = \{z \in \mathbb{C}, |1-wz| > W|z|\}$ (see Fig. 1). Then, for any $z \in S$, $\Re s > 0$ and $n = 1, 2, 3, \ldots$,*

$$\mathrm{Li}_s(z) = \sum_{k=0}^{n-1} \left(\frac{z}{1-wz}\right)^{k+1} \sum_{j=0}^{k} \binom{k}{j} \frac{(-w)^{k-j}}{(1+j)^s} + R_n(z, s, w). \tag{22}$$

The remainder term is bounded by

$$|R_n(z, s, w)| \le M(z) \frac{\Gamma(\Re s)}{|\Gamma(s)|} \left|\frac{z}{1-wz}\right|^n, \tag{23}$$

with $M(z)$ defined in (21). Bound (23) shows that expansion (22) is uniformly valid in compact sets of S.

Remark 1 Observe that formula (6), valid for $\Re(z) < 1/2$, corresponds to the case $w = 1$ of Theorem 1, although formula (12) in Theorem 1 has been derived here under additional restrictions on the parameters s and a. Then, indeed (6) is valid for more

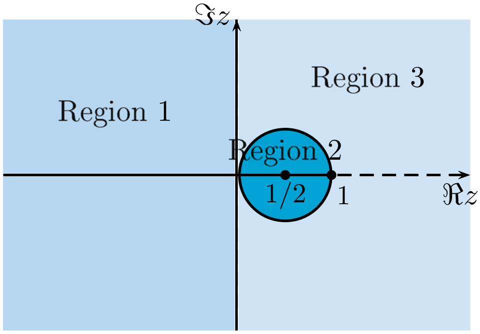

Fig. 2 Different regions considered in formula (21). Region 1 is the half plane $\Re z \le 0$, region 2 is the open disk of radius $1/2$ with center at $z = 1/2$, and region 3 is the intersection of the half plane $\Re z > 0$ with the exterior of this disk

general values of the parameters s and a than formula (12), on the other hand it does not provide information about uniform convergence properties.

In Tables 1 and 2 we illustrate the accuracy of the approximations of $\Phi(z,s,a)$ and $\mathrm{Li}_s(z)$ given by (12) and (22) respectively, for different values of z, s and a and two different base points $w = 1/2$ and $w = (1+i)/2$. All the computations have been carried out by using the software *Wolfram Mathematica 14.0*. In particular, we have used the HurwitzLerchPhi[z,s,a] function for the *exact* evaluation of the Lerch zeta function $\Phi(z,s,a)$. The numerical results confirm the convergence of the expansions.

2.2 An expansion in the region $|(1-qz)(1+qz-z)| > |(1/2-q)z|^2$ with $0 \le q < (2-\sqrt{2})/4$

In this section, we use a two-point Taylor expansion with base points in $[0,1]$ to better avoid the singularity $t = 1/z$ of $h(t,z)$ in its domain of convergence. In this way, we can obtain new expansions of the Lerch transcendent valid in larger regions of convergence. Thus, we consider the two-point Taylor expansion of $h(t,z) = (1-zt)^{-1}$ at the base points $t = q$, and $t = 1-q$, for a certain positive number $q \in [0,q_0]$, with $q_0 := (2-\sqrt{2})/4 < 1$. The result is given in the following theorem.

Theorem 3 *For an arbitrary* $q \in [0,q_0]$, *consider the region*

$$S_q := \{z \in \mathbb{C};\ 4|(1-qz)(1+qz-z)| > (1-2q)^2|z|^2\}. \tag{24}$$

Table 1 Relative errors in the computation of $\Phi(z,s,a)$ by using expansion (12) for different values of z, s and a and two different base points $w = 1/2$ and $w = (1+i)/2$

z	$n=0$	$n=5$	$n=10$	$n=15$	$n=20$
$w=1/2,\ s=1.2,\ a=2.1$					
$e^{i\frac{\pi}{4}}$	$0.21e{-}00$	$0.10e{-}01$	$0.72e{-}03$	$0.72e{-}04$	$0.74e{-}05$
$e^{i\frac{2\pi}{3}}$	$0.75e{-}01$	$0.27e{-}03$	$0.85e{-}06$	$0.53e{-}08$	$0.26e{-}10$
-1	$0.60e{-}01$	$0.13e{-}03$	$0.20e{-}06$	$0.71e{-}09$	$0.18e{-}11$
$-1-i$	$0.77e{-}01$	$0.72e{-}03$	$0.49e{-}05$	$0.75e{-}07$	$0.81e{-}09$
-5	$0.56e{-}01$	$0.12e{-}01$	$0.53e{-}03$	$0.13e{-}03$	$0.10e{-}04$
$w=(1+i)/2,\ s=1.2,\ a=2.1$					
$e^{i\frac{\pi}{4}}$	$0.37e{-}00$	$0.51e{-}02$	$0.17e{-}03$	$0.79e{-}05$	$0.39e{-}06$
$e^{i\frac{2\pi}{3}}$	$0.20e{-}00$	$0.70e{-}03$	$0.37e{-}05$	$0.35e{-}07$	$0.37e{-}09$
-1	$0.32e{-}00$	$0.11e{-}02$	$0.80e{-}05$	$0.95e{-}07$	$0.15e{-}08$
$-1-i$	$0.53e{-}00$	$0.19e{-}01$	$0.13e{-}02$	$0.13e{-}03$	$0.21e{-}04$
-5	$0.58e{-}00$	$0.40e{-}01$	$0.42e{-}02$	$0.11e{-}02$	$0.48e{-}03$

Table 2 Relative errors in the computation of $\mathrm{Li}_s(z)$ by using expansion (22) for different values of z and s and two different base points $w = 1/2$ and $w = (1+i)/2$

z	$n = 0$	$n = 5$	$n = 10$	$n = 15$	$n = 20$
$w = 1/2, s = 1.2$					
$e^{i\frac{\pi}{4}}$	$0.14e{-}00$	$0.11e{-}01$	$0.63e{-}03$	$0.92e{-}04$	$0.74e{-}05$
$e^{i\frac{2\pi}{3}}$	$0.87e{-}01$	$0.44e{-}03$	$0.13e{-}05$	$0.11e{-}07$	$0.47e{-}10$
-1	$0.79e{-}01$	$0.22e{-}03$	$0.36e{-}06$	$0.16e{-}08$	$0.36e{-}11$
$-1-i$	$0.12e{-}00$	$0.13e{-}02$	$0.99e{-}05$	$0.18e{-}06$	$0.19e{-}08$
-5	$0.28e{-}00$	$0.29e{-}01$	$0.29e{-}02$	$0.46e{-}03$	$0.61e{-}04$
$w = (1+i)/2, s = 1.2$					
$e^{i\frac{\pi}{4}}$	$0.33e{-}00$	$0.28e{-}02$	$0.13e{-}03$	$0.11e{-}04$	$0.49e{-}06$
$e^{i\frac{2\pi}{3}}$	$0.28e{-}00$	$0.81e{-}03$	$0.52e{-}05$	$0.68e{-}07$	$0.75e{-}09$
-1	$0.32e{-}00$	$0.15e{-}02$	$0.12e{-}04$	$0.20e{-}06$	$0.34e{-}08$
$-1-i$	$0.55e{-}00$	$0.27e{-}01$	$0.22e{-}02$	$0.31e{-}03$	$0.52e{-}04$
-5	$0.62e{-}00$	$0.82e{-}01$	$0.18e{-}01$	$0.57e{-}02$	$0.18e{-}02$

Then, for any $z \in S_q$ and $n = 1, 2, 3, \ldots$,

$$\begin{aligned}\Phi(z, s, a) = \sum_{n=0}^{N-1}\sum_{j=0}^{n}\sum_{k=0}^{n}\binom{n}{j}\binom{n}{k}(-q)^{n-j}(q-1)^{n-k}\\ \times\left[\frac{A_n(z,q)}{(a+k+j)^s}+\frac{B_n(z,q)}{(a+k+j+1)^s}\right]+R_N(z,s,a,q).\end{aligned} \tag{25}$$

For $n = 0, 1, 2, \ldots$, the coefficients $A_n(z,q)$ and $B_n(z,q)$ are defined by

$$\begin{aligned}A_{n+1}(z,q) &:= \frac{M_{11}(n,z,q)A_n(z,q)+M_{21}(n,z,q)B_n(z,q)}{(n+1)(4q^2-4q+1)(z-1-q(1-q)z^2)},\\ B_{n+1}(z,q) &:= \frac{M_{21}(n,z,q)A_n(z,q)+M_{22}(n,z,q)B_n(z,q)}{(n+1)(4q^2-4q+1)(z-1-q(1-q)z^2)},\end{aligned} \tag{26}$$

$$\begin{aligned}M_{11}(n,z,q) &:= (2n+1)z[1-(2q^2-2q+1)z],\\ M_{12}(n,z,q) &:= (-n+2q(q-1))z^2+(2q(1-q)+3n+1)z-2n-1,\\ M_{21}(n,z,q) &:= (2n+1)z(z-2),\\ M_{22}(n,z,q) &:= [n(1+4q-4q^2)+4q(1-q)]z^2-3(2n+1)z+2(2n+1),\end{aligned} \tag{27}$$

and

$$A_0(z,q) := \frac{1-z}{(1-qz)(1+qz-z)}, \quad B_0(z,q) := \frac{z}{(1-qz)(1+qz-z)}. \tag{28}$$

The remainder term is bounded by

$$|R_N(z,s,a,q)| \leq \frac{C(z)}{\Re a^{\Re s}} \left| \frac{(1-2q)^2 z^2}{4(1-qz)(1+qz-z)} \right|^N, \tag{29}$$

where $C(z)$ is a certain function of z uniformly bounded on compact sets in S_q and independent of N, and then the rate of convergence is of power type.

Proof From Theorem 1 of [22], we have that $h(t,z)$ admits the following two-point Taylor expansion at the base points $t=q$ and $t=1-q$:

$$h(t,z) = \sum_{n=0}^{N-1} [A_n(z,q) + B_n(z,q)t][(t-q)(t+q-1)]^n + r_N(t), \quad t \in D_q, \tag{30}$$

where $D_q := \{t \in \mathbb{C},\ |(t-q)(t+q-1)| < r\}$ is a Cassini oval with foci at $t=q$ and $t=1-q$ and a certain convergence radius $r>0$ that we determine below. Here, $r_N(t)$ is the two-point Taylor remainder [22, Theorem 1].

An explicit formula for the coefficients $A_n(z,q)$ and $B_n(z,q)$ may be derived from [22], but we omit it here for simplicity. Instead, we derive the recurrence relation (26), (27) and (28) from the differential equation satisfied by $h(t,z)$ in the variable t: $(1-zt)h' = zh$. To this end we substitute expansion (30) and

$$\begin{aligned} h'(t,z) = \sum_{n=0}^{\infty} \{ & [(2n+1)B_n(z,q) - (n+1)(A_{n+1}(z,q) + 2q(1-q)B_{n+1}(z,q))] \\ & + (n+1)[2A_{n+1}(z,q) + B_{n+1}(z,q)]t\}(t-q)^n(t+q-1)^n, \end{aligned} \tag{31}$$

into the differential equation $(1-zt)h' = zh$. Equating coefficients of $(t-q)^n(t+q-1)^n$ and $t(t-q)^n(t+q-1)^n$, we obtain (26), (27) and (28).

Now we determine the radius r. The interval $[0,1]$ is contained in the Cassini oval D_q when its radius $r \geq \sup_{t\in[0,1]}\{|(t-q)(t-q-1)|\} = \max\left\{q(1-q), (1/2-q)^2\right\}$. This happens for $r \geq (1/2-q)^2$ when $0 \leq q \leq q_0 := (2-\sqrt{2})/4$, where q_0 is the solution of the equation $q(1-q) = (1/2-q)^2$. Then, expansion (30) satisfies condition (i) for $r > (1/2-q)^2$ and $q \in [0,q_0]$. On the other hand, it satisfies condition (ii) if $1/z \notin D_q$ (see [22]), that is, for any

$$r < \left| \left(\frac{1}{z} - q\right)\left(\frac{1}{z} + q - 1\right) \right|.$$

The smallest r that we can take is $r = (1/2-q)^2 + \varepsilon$ for arbitrarily small $\epsilon > 0$ and then, the largest S_q we can choose is (24). Then, for $z \in S_q$, we can substitute expansion (30) into (10) and interchange sum and integral to obtain (25), with

$$R_N(z,s,a,q) := \frac{1}{\Gamma(s)} \int_0^1 (-\log(t))^{s-1} t^{a-1} r_N(t)\, dt. \tag{32}$$

From [22, Theorem 1], we have that the remainder $r_N(t)$ in the expansion (30) may be bounded by $|r_N(t)| \leq C(z)|(t-q)(t+q-1)/r|^N$, $t \in D_q$, where $C(z)$ is bounded on compact sets of S_q and independent of t and N. When t belongs to the domain of convergence of $h(t,z)$, we have that $|r_N(t)| \leq C(z)|(1/2-q)^2/r|^N$. Introducing this bound in (32) and after straightforward computations, we obtain (29). □

Two particularly interesting corollaries of Theorem 3 are obtained for $q = 0$ and for $q = q_0$. In the first case, the analytic form of (25) is the simplest possible one; in the second case, the convergence region S_q is the largest possible one. They are analyzed in the following two subsections.

2.2.1 Case $q = 0$: An expansion for $|z|^2 < 4|1-z|$

From Theorem 3, we see that the simplest form of the analytic expansion (25) is obtained for $q = 0$, that is, when we consider the two-point Taylor expansion of the function $h(t,z)$ at the end points of the $t-$domain: $t = 0$ and $t = 1$. In this case, a simple explicit formula for the coefficients $A_n(z,q)$ and $B_n(z,q)$ is given in [22]. We formulate the result in the form of a corollary.

Corollary 4 *Consider the region*

$$\begin{aligned} S_0 :=& \{z \in \mathbb{C};\ |z|^2 < 4|1-z|\} \\ =& \{x+iy;\, x,y \in \mathbb{R},\ y^4 + (2x^2-16)y^2 + (x^4 - 16x^2 + 32x - 16 < 0)\} \end{aligned} \tag{33}$$

(Fig. 3) for any $z \in S_0$ and $n = 1, 2, 3, \ldots$,

$$\Phi(z,s,a) = \sum_{n=0}^{N-1}\sum_{j=0}^{n}\binom{n}{j}(-1)^{n-j}\left[\frac{A_n(z)}{(a+n+j)^s} + \frac{B_n(z)}{(a+n+j+1)^s}\right] + R_N(z,s,a), \tag{34}$$

where for $n = 0, 1, 2, \ldots$, the coefficients $A_n(z)$ and $B_n(z)$ are defined by

$$\begin{aligned} A_{n+1}(z) &:= \frac{M_{11}(z,n)A_n(z) + M_{21}(z,n)B_n(z)}{(n+1)(z-1)}, \\ B_{n+1}(z) &:= \frac{M_{21}(z,n)A_n(z) + M_{22}(z,n)B_n(z)}{(n+1)(z-1)}, \end{aligned} \tag{35}$$

$$\begin{aligned} &M_{11}(n,z) := (2n+1)z(1-z), \quad M_{12}(n,z) := -nz^2 + (3n+1)z - 2n - 1, \\ &M_{21}(n,z) := (2n+1)z(z-2), \quad M_{22}(n,z) := nz^2 - 3(2n+1)z + 2(2n+1), \end{aligned} \tag{36}$$

and

$$A_0(z) := 1, \quad B_0(z) := \frac{z}{1-z}. \tag{37}$$

The remainder term is bounded by

$$|R_n(z,s,a)| \leq \frac{C(z)}{\Re a^{\Re s}}\left|\frac{z^2}{4(1-z)}\right|^N, \tag{38}$$

with $C(z)$ independent on N and the rate of convergence is of power type.

2.2.2 Case $q = (2-\sqrt{2})/4$: An expansion for $|z^2-8z+8| > |z|^2$

When we want to maximize the size of the convergence region S_q, the optimal choice of the base points q and $1-q$ is the one that minimizes the region D_q. In the minimal region D_q, the end points of the interval $[0, 1]$, as well as the middle point $t = 1/2$ are on the boundary of the Cassini oval D_q (see Fig. 4a). As we have seen above, this happens for $q = q_0 := (2-\sqrt{2})/4$. With this choice, the Cassini oval D_{q_0} of convergence of the two-point Taylor expansion is the smallest possible two-point Cassini oval satisfying the necessary conditions. We formulate the result in the following corollary.

Corollary 5 *Consider the region*

$$\begin{aligned} S_{q_0} &:= \{z \in \mathbb{C};\ |z^2-8z+8| > |z|^2\} \\ &= \{x+iy;\, x, y \in \mathbb{R},\ -x^3 - xy^2 + 5x^2 + 3y^2 - 8x + 4 > 0\} \end{aligned} \tag{39}$$

(Fig. 3) for any $z \in S_{q_0}$ and $n = 1, 2, 3, \ldots$,

$$\begin{aligned} \Phi(z, s, a) = &\sum_{n=0}^{N-1} \sum_{j=0}^{n} \sum_{k=0}^{n} \binom{n}{j}\binom{n}{k} (-q_0)^{n-j} (q_0-1)^{n-k} \\ &\times \left[\frac{A_n(z, q_0)}{(a+k+j)^s} + \frac{B_n(z, q_0)}{(a+k+j+1)^s} \right] + R_N(z, s, a, q_0), \end{aligned} \tag{40}$$

where for $n = 0, 1, 2, \ldots$, the coefficients $A_n(z, q_0)$ and $B_n(z, q_0)$ are defined by

$$A_{n+1}(z, q_0) := \frac{M_{11}(z, n, q_0) A_n(z, q_0) + M_{21}(z, n, q_0) B_n(z, q_0)}{(n+1)(4q^2-4q+1)(-z^2+8z-8)},$$

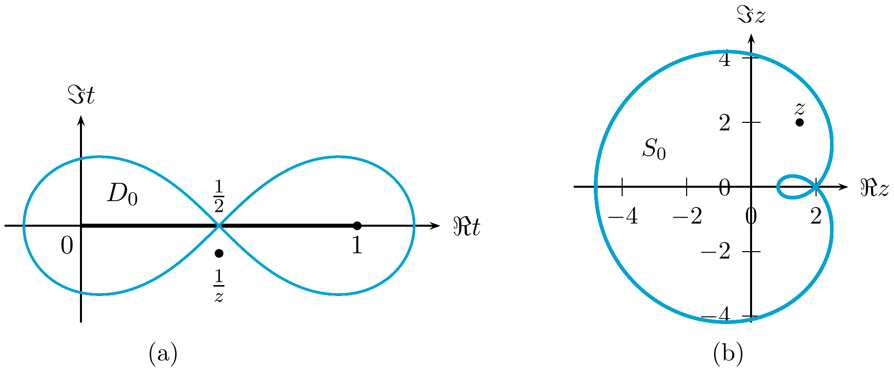

Fig. 3 The minimal domain of convergence D_0 of the two-point Taylor expansion of $h(t, z)$ at $t = 0$ and $t = 1$ containing the interval $(0, 1)$ is a Cassini oval of radius $1/4$ and foci $t = 0$ and $t = 1$ (Figure (a)). The region S_0, inverse of the exterior of D_0 is the region shown in Figure (b): $S = \{z \in \mathbb{C};\ \ |z|^2 < 4|1-z|\}$

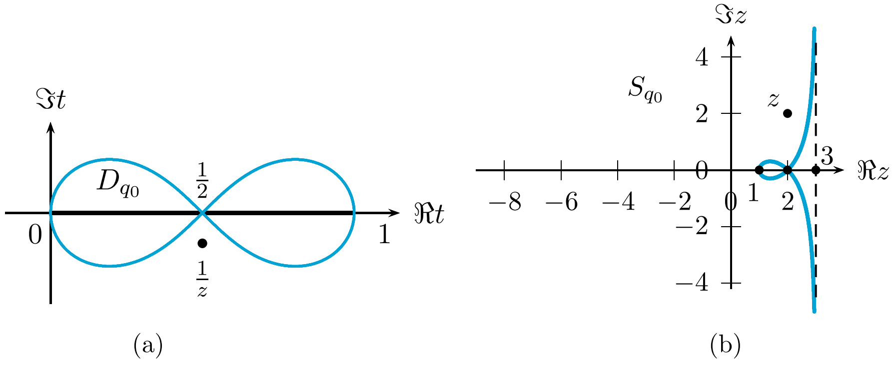

Fig. 4 The minimal domain of convergence D_{q_0} of the two-point Taylor expansion of $h(t,z)$ at $t = q_0 = (2-\sqrt{2})/4$ and $t = 1 - q_0 = (2+\sqrt{2})/4$ containing the interval $(0,1)$ is a Cassini oval of radius $1/8$ and foci $t = q_0$ and $t = 1 - q_0$ (Figure (a)). The region S, inverse of the exterior of D is the region shown in Figure (b): $S = \{z \in \mathbb{C};\ \ |z|^2 < 4|1-z|\}$

$$B_{n+1}(z,q_0) := \frac{M_{21}(z,n,q_0)A_n(z,q_0) + M_{22}(z,n,q_0)B_n(z,q_0)}{(n+1)(4q^2-4q+1)(-z^2+8z-8)}, \tag{41}$$

$$\begin{aligned} M_{11}(n,z,q) &:= 4(2n+1)z(4-3z),\\ M_{12}(n,z,q) &:= -4(4n+1)z^2 + 4(12n+5)z - 16(2n+1),\\ M_{21}(n,z,q) &:= 16(2n+1)z(z-2),\\ M_{22}(n,z,q) &:= 8(3n+1)z^2 - 48(2n+1)z + 32(2n+1), \end{aligned} \tag{42}$$

and

$$A_0(z,q) := \frac{8(1-z)}{z^2-8z+8}, \quad B_0(z,q) := \frac{8z}{z^2-8z+8}. \tag{43}$$

The remainder term is bounded by

$$|R_N(z,s,a,q)| \le \frac{C(z)}{\Re a^{\Re s}} \left| \frac{z^2}{z^2-8z+8} \right|^N, \tag{44}$$

with $C(z)$ independent on N and the rate of convergence is of power type.

In Table 3 we illustrate the accuracy in the approximation of $\Phi(z,s,a)$ by using expansions (34) and (40) respectively, for different values of z, s and a. The first column in the table represents the number n of terms used in either of the expansions. The remaining columns represent the relative error obtained with the indicated approximations. The numerical results confirm the convergence of the expansions. Besides, we observe a faster convergence for the optimal value $q = (2-\sqrt{2})/4$ than for $q = 0$.

In Table 4, we compare the accuracy of the new approximations (12) and (40) obtained in Theorem 1 and Corollary 5 respectively, with the power series definition (1) and formula (6). The numerical experiments suggest that, for moderate values of $|z|$, the approximations provided by the new approximations are more accurate than the power series definition (1) and formula (6). We also observe that, approximations

Table 3 Relative errors in the computation of $\Phi\,(z, s, a)$ for different values of z, s and a by using (34) and (40) respectively, that is, considering $q = 0$ and $q = (2-\sqrt{2})/4$

	$z = \frac{1}{2}e^{i\frac{7\pi}{6}}$, $s = 1.2$, $a = 2.1$			$z = -\frac{1}{5}(1+i)$, $s = 1.2$, $a = 2.1$	
	$q = 0$	$q = (2-\sqrt{2})/4$		$q = 0$	$q = (2-\sqrt{2})/4$
$n = 0$	$0.31e{-}01$	$0.96e{-}02$	$n = 0$	$0.12e{-}01$	$0.36e{-}02$
$n = 2$	$0.41e{-}04$	$0.32e{-}07$	$n = 2$	$0.23e{-}06$	$0.19e{-}08$
$n = 4$	$0.63e{-}09$	$0.12e{-}10$	$n = 4$	$0.51e{-}11$	$0.10e{-}12$
$n = 6$	$0.10e{-}11$	$0.48e{-}14$	$n = 6$	$0.12e{-}14$	$0.78e{-}17$

derived from a two-point Taylor expansions (40) are uniformly more accurate than the approximations derived from the standard Taylor expansion (12).

We also observe in all the tables (Tables 1–4) that, for any of the new uniform expansions (12), (34) and (40), the accuracy improves when z moves away from the boundary of the corresponding convergence regions (considering the infinity as part of the boundary of the convergence region when it is unbounded).

Table 4 Relative errors in the computation of $\Phi\,(z, s, a)$ for different values of z, $s = 1.2$ and $a = 2.1$ by using the new approximations (12) with $w = 1/2$ and (40), the power series definition (1) and formula (6)

	0	2	4	6
Approximations $z = \frac{-1+i}{5}$				
Power series definition (1)	$0.17e{-}00$	$0.75e{-}02$	$0.40e{-}03$	$0.24e{-}04$
Formula (6)	$0.89e{-}01$	$0.16e{-}02$	$0.42e{-}04$	$0.14e{-}05$
Theorem 1: formula (12)	$0.29e{-}01$	$0.27e{-}03$	$0.31e{-}05$	$0.37e{-}07$
Corollary 5: formula (40)	$0.36e{-}02$	$0.19e{-}06$	$0.10e{-}10$	$0.77e{-}15$
Approximations $z = -2 + I$				
Power series definition (1)	–	–	–	–
Formula (6)	$0.31e{-}00$	$0.58e{-}01$	$0.16e{-}01$	$0.48e{-}02$
Theorem 1: formula (12)	$0.74e{-}01$	$0.13e{-}01$	$0.27e{-}02$	$0.60e{-}03$
Corollary 5: formula (40)	$0.80e{-}01$	$0.17e{-}02$	$0.41e{-}04$	$0.10e{-}05$
Approximations $z = e^{i\pi/4}$				
Power series definition (1)	$0.64e{-}00$	$0.36e{-}00$	$0.24e{-}00$	$0.18e{-}00$
Formula (6)	–	–	–	–
Theorem 1: formula (12)	$0.21e{-}00$	$0.52e{-}01$	$0.16e{-}02$	$0.53e{-}03$
Corollary 5: formula (40)	$0.88e{-}01$	$0.25e{-}02$	$0.76e{-}04$	$0.24e{-}05$
Approximations $z = \frac{1}{2}e^{i7\pi/6}$				
Power series definition (1)	$0.30e{-}00$	$0.40e{-}01$	$0.67e{-}02$	$0.12e{-}03$
Formula (6)	$0.14e{-}00$	$0.54e{-}02$	$0.32e{-}03$	$0.23e{-}04$
Theorem 1: formula (12)	$0.44e{-}01$	$0.10e{-}02$	$0.29e{-}04$	$0.92e{-}06$
Corollary 5: formula (40)	$0.96e{-}02$	$0.32e{-}05$	$0.12e{-}08$	$0.48e{-}12$

3 A uniformly convergent expansion of $\Phi\,(z, s, a)$ on an unbounded set

In this section we derive a convergent expansion of $\Phi(z, s, a)$ in terms of elementary functions of z that is uniformly valid for z in the extended (unbounded) sector (see Fig. 5):

$$S_\theta = \{\theta \le |\arg z| \le \pi\} \cup \left\{ z \in \mathbb{C};\ \left| z - \frac{1}{2} \right| \le \frac{1}{2} \text{ and } |z-1| \ge \sin\theta \right\}, \tag{45}$$

with arbitrary $0 < \theta \le \pi/2$. For that purpose, we apply the method introduced in [23]. The technique considers (11) with $z \in \mathcal{D}$, which is a certain unbounded region in $\mathbb{C}$ that contains the point $z = 0$ (the method seeks expansions valid for both, large and small values of the complex *uniform* variable z). We assume the following three hypotheses for the functions $h(t, z)$ and $g(t, s, a)$:

(H1) $g(t, s, a)$ is analytic in an open region Ω that contains the interval $(0, 1)$ and the function $f(t, s, a) := t^{1-\sigma}(1-t)^{1-\gamma} g(t, s, a)$, with $0 < \sigma \le 1$ and $0 < \gamma \le 1$, is bounded in Ω.

(H2) $|h(t, z)| \le H t^\alpha (1-t)^\beta$ for $(t, z) \in [0, 1] \times \mathcal{D}$, with $H > 0$ independent of z and t and $\alpha + \sigma > 0$, $\beta + \gamma > 0$.

(H3) The moments of h, $M[h(\cdot, z); k] := \displaystyle\int_0^1 h(t, z) t^k dt$, are more elementary functions of z than the function $F(z)$.

Roughly speaking, the method proceeds as follows: consider the Taylor expansion of the function $g(t, s, a)$ at a certain point $t_0 \in \Omega$ such that the open disk of convergence $D_{t_0}(r)$, centered at t_0 with radius $r > 0$, contains inside it the interval of integration $(0, 1)$. We replace $g(t)$ in (11) by its Taylor expansion at that point t_0. For $n =$

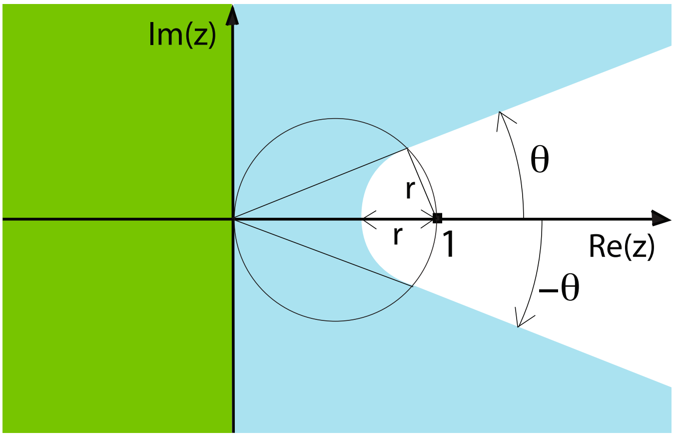

Fig. 5 The blue and green regions comprise the extended sector S_θ defined in (45), with $r := \sin\theta$. In particular, $S_{\pi/2}$ is just the half plane $\Re z \le 0$ and $\lim_{\theta\to 0} S_\theta = \mathbb{C} \setminus [1, \infty)$

1, 2, 3, . . .,

$$g(t,s,a)=\sum_{k=0}^{n-1}c_k(s,a)(t-t_0)^k+r_n(t,s,a),\quad t\in D_{t_0}(r)\subset\Omega,\quad (0,1)\subset D_{t_0}(r), \tag{46}$$

where $r_n(t,s,a)$ is the Taylor remainder. Then, replace $g(t,s,a)$ in (11) by the right hand side of the above equality and interchange sum and integral to obtain the expansion

$$F(z)=\sum_{k=0}^{n-1}c_k(s,a)\Phi_k(z)+R_n(z,s,a), \tag{47}$$

where

$$\Phi_k(z):=\int_0^1 h(t,z)(t-t_0)^k dt,\quad R_n(z,s,a):=\int_0^1 h(t,z)r_n(t,s,a)dt. \tag{48}$$

It is shown in [23] that expansion (47) has the following three properties:

(P1) The expansion is convergent in $\mathcal{D}$. More precisely, when $n\to\infty$, $R_n(z)=\mathcal{O}(n^{-\sigma-\alpha}+n^{-\gamma-\beta})$.
(P2) Expansion (47) is valid uniformly for all $z\in\mathcal{D}$. That is, for any $z\in\mathcal{D}$ and any order n of the approximation, the absolute error satisfies the bound $|R_n(z)|\le C_n$, with $C_n>0$ independent of z.
(P3) The terms of the expansion $\Phi_k(z)$ are functions of z more elementary than $F(z)$.

See [23] for further details.
The main result is given in the following theorem.

Theorem 6 *For $\Re s>0$, $\Re a>0$, $z\in S_\theta$ with $0<\theta\le\pi/2$, and $n=1,2,3,\ldots$,*

$$\Phi(z,s,a)=\frac{1}{\Gamma(s)}\sum_{k=0}^{n-1}c_k(s,a)\phi_k(z)+R_n(z,s,a), \tag{49}$$

with

$$c_k(s,a):=\sum_{j=0}^{k}\binom{k-j}{a-1}\frac{1}{2^{a-k+j-1}}b_j(s), \tag{50}$$

and for $k=0,1,2,\ldots$, coefficients b_k satisfy the recurrence relation

$$\frac{\log(2)(k+1)}{2}b_{k+1}=[1-s+k(1-\log(2))]b_k-\sum_{j=0}^{k-2}\frac{(-2)^{k-j-1}(j+1)}{(k-j)(k-j-1)}b_{j+1},\ b_0=[\log(2)]^{s-1} \tag{51}$$

(empty sums must be understood as zero). The functions $\phi_k(z)$ are the elementary functions

$$\phi_k(z) := \sum_{j=0}^{k} (-1)^{k-j+1} \binom{k}{j} \frac{1}{2^{k-j}} \left(\frac{1}{z^{j+1}} \log|1-z| + \sum_{m=0}^{j-1} \frac{1}{(j-m)z^{m+1}} \right). \quad (52)$$

Define $\sigma := \min\{1, \Re(a) - \varepsilon\}$ with $\varepsilon > 0$ as close to zero as we wish and $\gamma := \min\{1, \Re(s)\}$. Then, the remainder term is of the following order as $n \to \infty$ uniformly in $|z|$ in the extended sector S_θ: $R_n(z) = \mathcal{O}(n^{-\sigma} + n^{-\gamma})$.

Proof Consider the integral representation of $\Phi(z, s, a)$ given in (10) with $g(t, s, a) = (-\log t)^{s-1} t^{a-1}$ and $h(t, z) = 1/(1 - zt)$. Then, (H1) is satisfied by $0 < \sigma \leq \min\{1, \Re(a) - \varepsilon\}$ with $\varepsilon > 0$ as close to zero as we wish and $0 < \gamma \leq \min\{1, \Re(s)\}$. For hypothesis (H2), we observe that, for $t \in [0, 1]$, we have previously proved that $|(1 - zt)^{-1}| \leq M(z)$ with $M(z)$ given by (21). The regions of the complex $z-$plane considered in this formula are depicted in Fig. 2. On the other hand, for $z \in S_\theta$, we have that $M(z) \leq [\sin(\theta)]^{-1}$. This inequality may be proved using the following geometrical arguments: (i) at the points of the circle $|z - 1/2| = 1/2$ we have that $|1 - z| = |\sin(\arg(z))|$; (ii) the closest points of the sector $\theta \leq |\arg(z)| < \pi/2$ to the point $z = 1$ are just the two points obtained from the intersection of the rays $\arg z = \pm\theta$ with the circle $|z - 1/2| = 1/2$; (iii) the closest points of the region $\{z \in \mathbb{C}; |z - 1/2| \leq 1/2 \text{ and } |z - 1| \geq \sin(\theta)\}$ to the point $z = 1$ are those of the portion of the circle $|z - 1| = \sin(\theta)$ contained inside this region. Then, we can take $\alpha = \beta = 0$ in (H2).

We now consider the truncated Taylor series expansion of the factor $g(t, s, a)$ at the middle point $t = 1/2$ of the integration interval[1], that is,

$$g(t, s, a) = \sum_{k=0}^{n-1} c_k(s, a) \left(t - \frac{1}{2}\right)^k + r_n(t, s, a), \quad t \in (0, 1), \quad (53)$$

where $r_n(t, s, a)$ is the Taylor remainder

$$r_n(t, s, a) := \sum_{k=n}^{\infty} c_k(s, a) \left(t - \frac{1}{2}\right)^k, \quad t \in (0, 1). \quad (54)$$

For the computation of the coefficients $c_k(s, a)$, we consider $g(t, s, a) = g_1(t, s) g_2(t, a)$ with $g_1(t, s) = (-\log t)^{s-1}$ and $g_2(t, a) = t^{a-1}$ and their corresponding Taylor series expansions at $t = 1/2$. We derive the recurrence relation (51) for the coefficients b_k of the Taylor expansion of $g_1(t, s)$ from the differential equation satisfied by this function in the variable t: $t \log(t) g_1'(t, s) = (s-1) g_1(t, s)$, by substituting the Taylor expansion

[1] Instead of the point $t = 1/2$, we may take any other point $t \in [1/2, 1)$ such that the convergence disk of the Taylor series of $g(t)$ contains the interval $(0, 1)$. But we have proved in previous papers (in different contexts) that the optimal choice is the middle point of the interval.

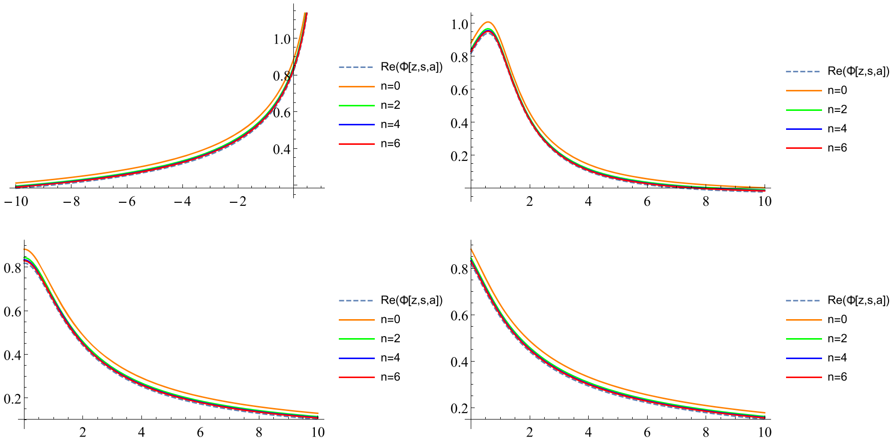

Fig. 6 Graphics of the real part of $\Phi(z,s,a)$ (dashed) for $s = 1.1$, $a = 1.2$ and the approximations given in Theorem 6 for $n = 0$ (orange), $n = 2$ (green), $n = 4$ (blue) and $n = 6$ (red) in several intervals: $[-10, 1)$, $[0, 10e^{i\pi/4}]$, $[0, 10e^{i\pi/2}]$ and $[0, 10e^{2i\pi/3}]$

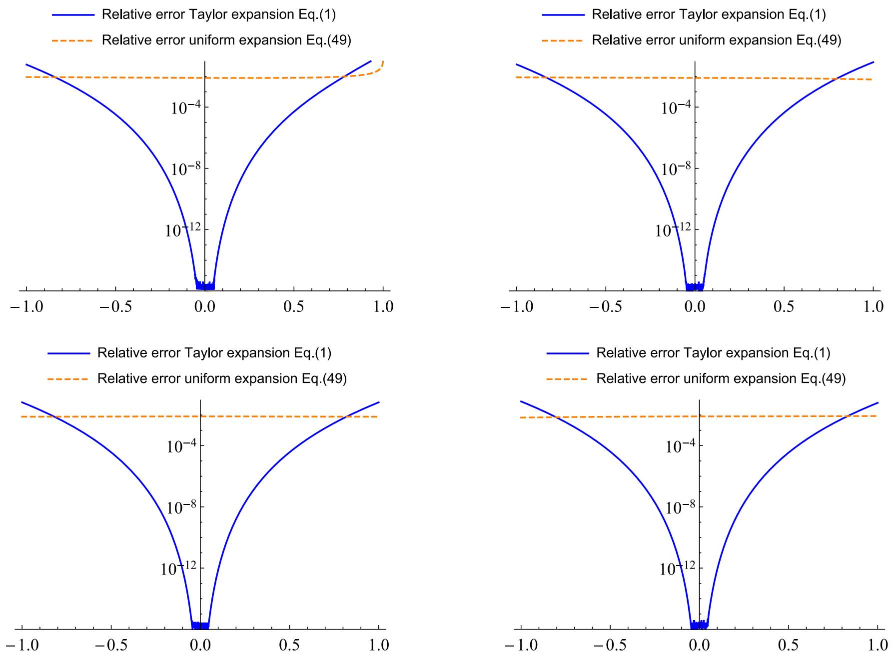

Fig. 7 Graphics in a logarithmic scale on the $y-$axis of the absolute value of the relative errors in the approximation of $\Phi(z,s,a)$ for $s = 1.9 + i/4$, $a = 5.8 - i/2$ by using the uniform expansion given in Theorem 6 (orange and dashed) and the Taylor expansion (1) (blue) in several intervals: $[-1, 1]$, $[-e^{i\pi/4}, e^{i\pi/4}]$, $[-e^{i\pi/2}, e^{i\pi/2}]$ and $[-e^{2i\pi/3}, e^{2i\pi/3}]$. In both cases, $n = 10$ is considered

of $g_1(t,s)$ at $t = 1/2$ in the differential equation and equating the coefficients of $(t-1/2)^n$. On the other hand,

$$g_2(t,a) = \sum_{n=0}^{\infty} \binom{n}{a-1} \frac{1}{2^{a-n-1}} \left(t - \frac{1}{2}\right)^n .$$

Replacing (53) in the integral representation (10) and interchanging sum and integral, we obtain (49) with

$$\phi_k(z) = \int_0^1 \frac{(t-1/2)^k}{1-zt}\, dt \tag{55}$$

and

$$R_n(z,s,a) := \frac{1}{\Gamma(s)} \sum_{k=n}^{\infty} c_k(s,a)\phi_k(s). \tag{56}$$

Expanding the first factor of the integrand in (55) in powers of t, we find

$$\phi_k(z) = \sum_{j=0}^{k} \binom{k}{j} \frac{(-1)^{k-j}}{2^{k-j}} \int_0^1 \frac{t^j}{1-zt}\, dt,$$

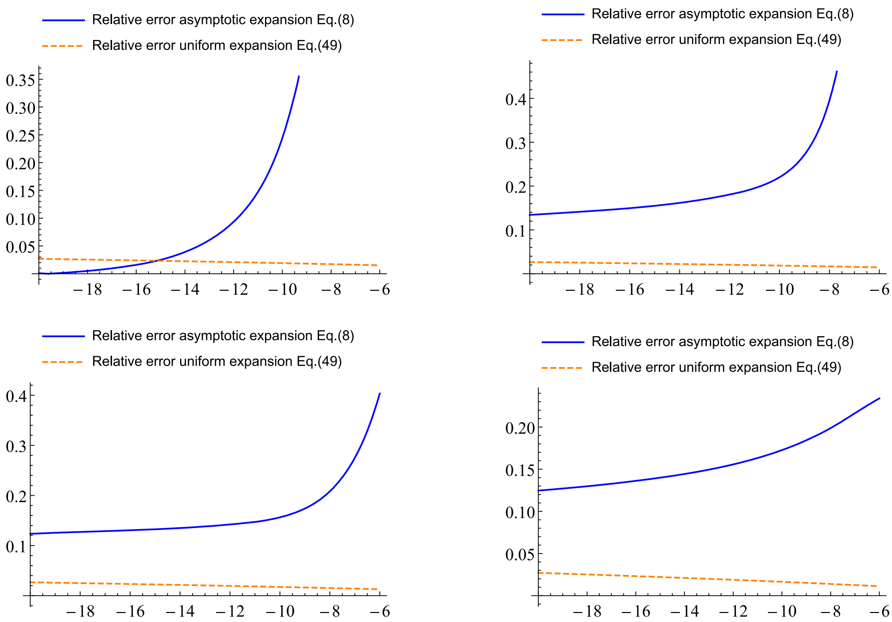

Fig. 8 Graphics of the absolute value of the relative errors in the approximation of $\Phi(z,s,a)$ for $s = 1.9 + i/4$, $a = 5.8 - i/2$ by using the uniform expansion given in Theorem 6 (orange and dashed) and the asymptotic expansion (8) (blue) in the interval $[-20,-6]$, $[-20e^{i\pi/4}, -6e^{i\pi/4}]$, $[-20e^{i\pi/2}, -6e^{i\pi/2}]$ and $[-20e^{2i\pi/3}, -6e^{2i\pi/3}]$. In both cases, $n = 10$ is considered

and using [24, Section 1.2.4, Eq. 12] for the integral in the previous formula, we obtain (52).

Therefore, expansion (49) is uniformly convergent for $z \in S_\theta$ with $0 < \theta \leq \pi/2$. Moreover, property (P1) assures that $R_n(z) = \mathcal{O}(n^{-\sigma} + n^{-\gamma})$. $\square$

Figure 6 shows some numerical experiments regarding the approximation of the real part of $\Phi(z, s, a)$ given by Theorem 6; the graphics for the imaginary part are similar.

In Fig. 7 we compare the accuracy of expansion (49) and the power series (1) valid for $|z| \leq 1$. The power series is more competitive for small $|z|$ than expansion (49), but we observe the existence of a region, close to the boundary of the region of convergence $|z| = 1$, where the uniformly convergent expansion performs better. The uniform character of (49) is also shown in the graphics.

Finally, Fig. 8 shows the accuracy of expansion (49) and the asymptotic series (8) valid for $|z| > 1$. In this case, the asymptotic series is more competitive for large $|z|$ than expansion (49), but (49) is more competitive for intermediate values of $|z|$ and it is uniform in $|z|$.

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Data Availability No datasets were generated or analysed during the current study.

Declarations

Competing Interests The authors declare no competing interests.

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