Available online at www.sciencedirect.com

ScienceDirect

Journal of Differential Equations 440 (2025) 113465

Journal of Differential Equations

www.elsevier.com/locate/jde

End-point maximal regularity for the discrete parabolic Cauchy problem and regularity of non-local operators in discrete Besov spaces ☆

Luciano Abadias [a,*], Marta De León-Contreras [b], Alejandro Mahillo [a]

[a] *Departamento de Matemáticas, Instituto Universitario de Matemáticas y Aplicaciones, Universidad de Zaragoza, 50009 Zaragoza, Spain*

[b] *Departamento de Análisis Matemático, Universidad de La Laguna, Avda. Astrofísico Sánchez, s/n 38206 La Laguna (Sta. Cruz de Tenerife), Spain*

Received 14 March 2024; revised 24 March 2025; accepted 20 May 2025
Available online 2 June 2025

Abstract

In this paper we prove both end-point maximal L^1-regularity for the discrete parabolic Cauchy problem and regularity of some non-local operators in discrete Besov spaces. To that aim, we prove characterizations of the discrete Besov spaces in terms of the heat and Poisson semigroups associated with the discrete Laplacian. Moreover, we provide new estimates for the derivatives of the discrete heat kernel and semigroup which are of independent interest.

MSC: 26A16; 35R11; 35B65; 35K08; 39A12; 47D07

☆ First and third authors acknowledge support from projects PID2022-137294NB-I00 funded by MICIU/AEI/10.13039/501100011033 and ERDF/EU and E48-23R funded by D. G. Aragón. The third author also acknowledges project PID2019-105979GB-I00 funded by MICIU/AEI/10.13039/501100011033 and ESF "Investing in your future". The second author acknowledges support from the project PID2023-148028NB-I00, funded by MICIU/AEI/10.13039/501100011033, and also from the grant FJC2020-044159-I, funded by MICIU/AEI/10.13039/501100011033 and the European Union NextGenerationEU/PRTR.

* Corresponding author.
E-mail addresses: labadias@unizar.es (L. Abadias), mleoncon@ull.edu.es (M. De León-Contreras), almahill@unizar.es (A. Mahillo).

https://doi.org/10.1016/j.jde.2025.113465

Keywords: Maximal L^1-regularity; Discrete Besov spaces; Discrete heat and Poisson semigroups; Regularity of discrete fractional operators

1. Introduction

In the study of partial differential equations, it is desirable to get some regularity results for the differential operators involved, as well as a priori estimates for the solutions, which provide information about the smoothness of the solutions before even solving the equation. Also, when we consider a parabolic Cauchy problem of the kind

$$\begin{cases} \partial_t u - Lu = g, & t \in (0, T), \\ u(0) = u_0, \end{cases} \tag{1.1}$$

where L is a certain differential operator and T is either positive or infinite, it is quite interesting to study for which operators L the loss of regularity phenomenon (that is, when the partial derivative, $\partial_t u$, of the solution, u, of the above Cauchy problem is less regular than the right-hand side term, g) does not hold. In this case we speak about maximal regularity. More precisely, suppose that X is a proper Banach space and $L : \mathrm{Dom}(L) \subset X \to X$ is a closed operator whose domain, $\mathrm{Dom}(L)$, is dense in X. Given $g \in L^q((0, T); X)$, $1 \le q \le \infty$, we say that L has maximal L^q-regularity if there exists a unique solution u to (1.1) with $u_0 = 0$ satisfying

$$\|\partial_t u\|_{L^q((0,T);X)} + \|Lu\|_{L^q((0,T);X)} \le C\|g\|_{L^q((0,T);X)}.$$

Maximal regularity for parabolic Cauchy problems as (1.1) has received great attention in the last years, see for instance [11,14,17,18,21,22,29–31] and the references therein. It is known that if the operator L generates a bounded analytic semigroup (or equivalently, L is sectorial of angle strictly less than $\pi/2$), $\{T_t\}_{t\ge 0}$, of operators in a Banach space X satisfying the Unconditional Martingale Differences (UMD) property, then maximal L^q-regularity holds for $q \in (1, \infty)$, see [22, Chapter 17]. However, in the cases when either $q = 1$ or $q = \infty$, or when considering Banach spaces that are not UMD, such as the non-reflexive Banach spaces, a different approach is needed in order to get maximal regularity. Moreover, if the operator L generates an analytic semigroup, $\{T_t\}_{t\ge 0}$, of operators in a Banach space X, then the solution of (1.1) can be written as

$$u(t) = T_t u_0 + \int_0^t T_{t-s} g(s, \cdot)\, ds, \quad t \in [0, T).$$

Therefore, when $u_0 = 0$, the maximal L^q-regularity is reduced to prove that the operator

$$R(g)(t) := \int_0^t \partial_t T_{t-s} g(s, \cdot)\, ds,$$

is bounded from $L^q((0, T); X)$ into itself (observe that $\partial_t T_t g = Lg$, $t > 0$), so in this case maximal L^q-regularity can be viewed as L^q-boundedness properties of certain Banach space valued singular integrals.

In this paper we shall consider as the operator L the discrete Laplacian, that is,

$$(\Delta_d f)(n) := f(n+1) - 2f(n) + f(n-1), \quad n \in \mathbb{Z},$$

for $f : \mathbb{Z} \to \mathbb{R}$. This operator has been well-studied in the last years and some of its main harmonic analysis properties can be found in [1,2,4,8,12,13]. In [11], the author proved weighted mixed norm estimates and end-point maximal L^1-regularity for (1.2) with $u_0 = 0$ and $t \in [0, T)$, $T \in (0, \infty)$, in the homogeneous discrete Besov spaces $\dot{B}^0_{p,1}(\mathbb{Z}^d)$. In that paper, the author says that he used a different method from the one employed in [29,30] because a characterization of discrete Besov spaces via the heat semigroup of Δ_d was not known.

The aim of this paper is twofold. On the one hand, we shall prove complete characterizations of the discrete Besov spaces in terms of the heat and Poisson semigroups associated with the discrete Laplacian, Δ_d. These characterizations, which are independently interesting, will help us achieve the second main objective of the paper: regularity results in discrete Besov spaces. In particular, we proved end-point maximal L^1-regularity for the solutions to the parabolic Cauchy problem

$$\begin{cases} \partial_t u(t,n) - \Delta_d u(t,n) = g(t,n), & n \in \mathbb{Z},\ t > 0, \\ u(0,n) = u_0(n), & n \in \mathbb{Z}, \end{cases} \tag{1.2}$$

and also we get regularity results for some fractional operators associated with Δ_d in the discrete Besov spaces.

Besov spaces are spaces of functions with certain smoothness degree which generalize Hölder spaces and play an important role in PDEs, mathematical physics and functional analysis. These spaces on $\mathbb{R}^n$ and on domains were introduced between 1959 and 1979 and can be viewed as real interpolation spaces in the scale of Triebel-Lizorkin spaces, see [38]. Providing suitable characterizations of these spaces in the continuous setting has been a central topic of study for many authors in the last 60 years, and there is an extensive literature in the topic, see for instance the impressive series of books [38–40], which collect most of the theory until 2006, and some more recent works like [6,7,9,10,25,27,41,42] and the references therein. In particular, obtaining characterizations of functional spaces in terms of the heat and/or Poisson semigroups is very convenient in order to get regularity results for non-local operators, see for instance [15,16,19,26,33–36].

Much less is known in the discrete case, and some main difficulties in this setting rely on the fact that the discrete heat kernel does not satisfy the usual hypotheses needed to obtain this type of characterizations, such as Gaussian estimates and Hölder continuity. Despite this fact, recently, a characterization of the discrete Hölder spaces was given in terms of semigroups associated to the discrete Laplacian, see [1], and this characterization allows the authors to get regularity results for the fractional powers of the discrete Laplacian in a more systematic way than in [13] and for a wider range of powers.

Before stating our results, we shall introduce the main definitions in our setting. For $f : \mathbb{Z} \to \mathbb{R}$, consider the discrete derivatives from the right and from the left,

$$\delta_{right} f(n) = f(n) - f(n+1), \qquad \delta_{left} f(n) = f(n) - f(n-1), \quad n \in \mathbb{Z}.$$

Observe that $\delta_{right}\delta_{left} f = \delta_{left}\delta_{right} f$, so every combination of these operators is not affected by the order when they are applied. We will use the notation δ^l_{right} and δ^l_{left} to denote the l-fold composition of the operator, $l \in \mathbb{N}$, being $\delta^0_{right} f = f$ and $\delta^0_{left} f = f$. Moreover, since the $\ell^p(\mathbb{Z})$-norms are invariant under translations, we have that for $f : \mathbb{Z} \to \mathbb{R}$, $\left\|\delta_{right} f\right\|_p = \left\|\delta_{left} f\right\|_p$. Therefore, we shall state and prove our results for the δ_{right} operator.

Let $\alpha > 0$ be a non-natural number, $l = [\alpha]$ the integer part of α, and $1 \le p, q \le \infty$. We define the *discrete Besov spaces*, also called discrete "generalized" Hölder spaces, as

$$C^{\alpha,p,q}(\mathbb{Z}) := \left\{ f : \mathbb{Z} \to \mathbb{R} \,:\, \sum_{j\neq 0} \left\| \frac{\delta^l_{right} f(\cdot + j) - \delta^l_{right} f(\cdot)}{|j|^{\alpha - l}} \right\|_p^q \frac{1}{|j|} < \infty \right\}, \quad 1 \le q < \infty,$$

and

$$C^{\alpha,p,\infty}(\mathbb{Z}) := \left\{ f : \mathbb{Z} \to \mathbb{R} \,:\, \sup_{j\neq 0} \left\| \frac{\delta^l_{right} f(\cdot + j) - \delta^l_{right} f(\cdot)}{|j|^{\alpha - l}} \right\|_p < \infty \right\}.$$

Observe that $\ell^p(\mathbb{Z})$ functions are trivially in $C^{\alpha,p,q}(\mathbb{Z})$. Furthermore, in Lemma 2.2 it will be proved that the functions belonging to these spaces satisfy $\frac{f}{1+|\cdot|^\alpha} \in \ell^q(\mathbb{Z}, \mu)$, being $\ell^q(\mathbb{Z}, \mu)$ the weighted ℓ^q-space with weight $\mu = \sum_{n\in\mathbb{Z}} \frac{1}{1+|n|}\delta_n$ and δ_n the Dirac measure in n. This size condition will be the starting point to define the following spaces of functions.

For $\alpha \in \mathbb{N}$ and $1 \le p, q \le \infty$, we introduce the *"generalized" Zygmund classes* by

$$Z^{\alpha,p,q}(\mathbb{Z}) := \Bigg\{ f : \mathbb{Z} \to \mathbb{R} \,:\, \frac{f}{1+|\cdot|^\alpha} \in \ell^q(\mathbb{Z}, \mu) \text{ and } \sum_{j\neq 0} \left\| \frac{\delta^{\alpha-1}_{right} f(\cdot - j) - 2\delta^{\alpha-1}_{right} f(\cdot) + \delta^{\alpha-1}_{right} f(\cdot + j)}{|j|} \right\|_p^q \frac{1}{|j|} < \infty \Bigg\}, \quad 1 \le q < \infty,$$

and

$$Z^{\alpha,p,\infty}(\mathbb{Z}) := \Bigg\{ f : \mathbb{Z} \to \mathbb{R} \,:\, \frac{f}{1+|\cdot|^\alpha} \in \ell^\infty(\mathbb{Z}) \text{ and } \sup_{j\neq 0} \left\| \frac{\delta^{\alpha-1}_{right} f(\cdot - j) - 2\delta^{\alpha-1}_{right} f(\cdot) + \delta^{\alpha-1}_{right} f(\cdot + j)}{|j|} \right\|_p < \infty \Bigg\}.$$

Observe that when $p = q = \infty$, $\ell^\infty(\mathbb{Z}, \mu) = \ell^\infty(\mathbb{Z})$ so $C^{\alpha,\infty,\infty}(\mathbb{Z})$ are the discrete Hölder spaces and $Z^{\alpha,\infty,\infty}(\mathbb{Z})$ are the discrete Zygmund spaces treated in [1].

For $1 \le p \le \infty$, we consider the mixed-norm spaces

$$L^q(((0,\infty), dt/t); \ell^p(\mathbb{Z})) = \left\{ f : (0,\infty) \to \ell^p(\mathbb{Z}) \,:\, \|f\|_{p,q} < \infty \right\},$$

where

$$\|f\|_{p,q} = \left(\int_0^\infty \|f(t)\|_p^q \, \frac{dt}{t} \right)^{1/q}, \quad 1 \le q < \infty,$$

and

$$\|f\|_{p,\infty} = \inf\left\{\alpha > 0 : \int\limits_{\{t>0\,:\,\|f(t)\|_p>\alpha\}} \frac{dt}{t} = 0\right\}.$$

It can be shown that the spaces $L^q(((0,\infty), dt/t); \ell^p(\mathbb{Z}))$, $1 \le p, q \le \infty$, are Banach spaces under this norm, see [5], and the norms $\|f\|_{p,\infty}$ and $\sup_{t>0}\|f(t)\|_p$ coincide.

Finally, we shall introduce Besov spaces in terms of semigroups associated with the discrete Laplacian. It is well-known that the heat semigroup associated with Δ_d is the solution to the discrete heat problem,

$$\begin{cases} \partial_t u(t,n) - \Delta_d u(t,n) = 0, & n \in \mathbb{Z},\ t \ge 0, \\ u(0,n) = f(n), & n \in \mathbb{Z}, \end{cases}$$

and it is given by the convolution $u(t,n) = e^{t\Delta_d} f(n) := \sum_{j\in\mathbb{Z}} G(t, n-j) f(j) = \sum_{j\in\mathbb{Z}} G(t, j) f(n-j)$, where

$$G(t,n) = e^{-2t} I_n(2t), \quad n \in \mathbb{Z},\ t > 0,$$

being I_n the modified Bessel function of the first kind and order $n \in \mathbb{Z}$, see Section 2 for more details. Moreover, by subordination (see [43, Chapter IX, Section 11]) we can define the Poisson semigroup associated with Δ_d as

$$e^{-y\sqrt{-\Delta_d}} f(n) := \frac{y}{2\sqrt{\pi}} \int_0^\infty \frac{e^{-\frac{y^2}{4t}}}{t^{3/2}} e^{t\Delta_d} f(n)\,dt, \quad y > 0,\ n \in \mathbb{Z},$$

which is the solution to the discrete Poisson problem

$$\begin{cases} \partial_y^2 v(y,n) - \Delta_d v(y,n) = 0, & n \in \mathbb{Z},\ y \ge 0, \\ v(0,n) = f(n), & n \in \mathbb{Z}. \end{cases}$$

For $\alpha > 0$ and $1 \le p, q \le \infty$, we define the discrete heat Besov spaces and discrete Poisson Besov spaces as

$$\Lambda_H^{\alpha,p,q} := \left\{ f : \mathbb{Z} \to \mathbb{R} \,:\, \frac{f}{1+|\cdot|^\alpha} \in \ell^q(\mathbb{Z},\mu) \text{ and } \left\| t^{k-\frac{\alpha}{2}} \partial_t^k e^{t\Delta_d} f \right\|_{p,q} < \infty, \text{ with } k = \left[\frac{\alpha}{2}\right] + 1,\ t > 0 \right\}$$

and

$$\Lambda_P^{\alpha,p,q} := \left\{ f : \mathbb{Z} \to \mathbb{R} \,:\, \sum_{n\in\mathbb{Z}} \frac{|f(n)|}{1+|n|^2} < \infty \text{ and } \left\| y^{l-\alpha} \partial_y^l e^{-y\sqrt{-\Delta_d}} f \right\|_{p,q} < \infty, \text{ with } l = [\alpha] + 1,\ y > 0 \right\}.$$

Our first main result is the following.

Theorem 1.1. *Let* $1 \le p, q \le \infty$.

(A) *Let* $\alpha > 0$.
  (A1) *If* $\alpha \notin \mathbb{N}$, *then* $C^{\alpha,p,q}(\mathbb{Z}) = \Lambda_H^{\alpha,p,q}$.
  (A2) *If* $\alpha \in \mathbb{N}$, *then* $Z^{\alpha,p,q}(\mathbb{Z}) = \Lambda_H^{\alpha,p,q}$.
(B) *Let* $f : \mathbb{Z} \to \mathbb{R}$ *such that* $\sum_{j\in\mathbb{Z}} \frac{|f(j)|}{1+|j|^2} < \infty$.
  (B1) *For every* $\alpha > 0$, $\alpha \notin \mathbb{N}$,

$$f \in C^{\alpha,p,q}(\mathbb{Z}) \iff f \in \Lambda_H^{\alpha,p,q} \iff f \in \Lambda_P^{\alpha,p,q}.$$

  (B2) *For every* $\alpha \in \mathbb{N}$,

$$f \in Z^{\alpha,p,q}(\mathbb{Z}) \iff f \in \Lambda_H^{\alpha,p,q} \iff f \in \Lambda_P^{\alpha,p,q}.$$

Observe that when $p = q = \infty$, we recover the results obtained in [1]. In order to prove Theorem 1.1, we will need to prove some refined estimates about the derivatives of the discrete heat kernel (see Lemma 2.1) and of the discrete heat semigroup (see Lemmata 2.4 and 2.5), and some mixed-norm estimates for the derivatives of the heat and Poisson semigroups (see Lemmata 2.6, 2.7 and 2.9). We believe that these results are also of independent interest.

Moreover, these characterizations of the discrete Besov spaces through the semigroup language will allow us to get some regularity results and end-point maximal L^1-regularity for the parabolic Cauchy problem (1.2) in a direct and systematic way.

First, we prove the regularity of some fractional operators associated with the discrete Laplacian, such as the Bessel potentials and the powers $(-\Delta_d)^{\pm\beta}$ in the discrete Besov spaces. For the appropriate definition of these operators, see Section 4.

Theorem 1.2. *Let* $\alpha, \beta > 0$, $1 \le p, q \le \infty$ *and* $f : \mathbb{Z} \to \mathbb{R}$ *such that* $f \in \Lambda_H^{\alpha,p,q}$, *then* $(I - \Delta_d)^{-\beta/2} f \in \Lambda_H^{\alpha+\beta,p,q}$.

In order to define the 'fractional' powers of order β of the discrete Laplacian, we need to consider the functions in the following spaces:

$$\ell_{\pm\beta} := \left\{ u : \mathbb{Z} \to \mathbb{R} : \sum_{m\in\mathbb{Z}} \frac{|u(m)|}{(1+|m|)^{1\pm 2\beta}} < \infty \right\}.$$

The choice of these spaces is justified since the fractional powers satisfy the following pointwise formula

$$(-\Delta_d)^{\pm\beta} f(n) = \sum_{m\in\mathbb{Z}} K_{\pm\beta}(n-m) f(m), \quad n \in \mathbb{Z},$$

where $K_\beta(m) \sim \frac{1}{|m|^{1+2\beta}}$ whenever $\beta > 0$ (with K_β being of compact support if $\beta \in \mathbb{N}$) and $K_{-\beta}(m) \sim \frac{1}{|m|^{1-2\beta}}$, for $0 < \beta < 1/2$, see [1,13].

Now we state our regularity results in the discrete Besov spaces for the discrete fractional Laplacian. The negative powers of the Laplacian are only well-defined for $0 < \beta < 1/2$, since the integral that defines it is not absolutely convergent for $\beta \ge 1/2$ (see Section 4).

Theorem 1.3 *(A priori estimates). Let $\alpha > 0$, $0 < \beta < 1/2$, $1 \le p, q \le \infty$ and $f : \mathbb{Z} \to \mathbb{R}$ such that $f \in \Lambda_H^{\alpha,p,q} \cap \ell_{-\beta}$, then $(-\Delta_d)^{-\beta} f \in \Lambda_H^{\alpha+2\beta,p,q}$.*

Theorem 1.4. *Let $\alpha, \beta > 0$, such that $0 < 2\beta < \alpha$, $1 \le p, q \le \infty$ and $f : \mathbb{Z} \to \mathbb{R}$.*

(1) If $f \in \Lambda_H^{\alpha,p,q} \cap \ell_\beta$, then $(-\Delta_d)^\beta f \in \Lambda_H^{\alpha-2\beta,p,q}$.
(2) If $\beta \in \mathbb{N}$ and $f \in \Lambda_H^{\alpha,p,q}$, then $\underbrace{(-\Delta_d) \circ (-\Delta_d) \circ \cdots \circ (-\Delta_d)}_{\beta \text{ times}} f \in \Lambda_H^{\alpha-2\beta,p,q}$.

Furthermore, as a consequence of our semigroup characterization of the discrete Besov spaces, we get the end-point maximal L^1-regularity for the parabolic Cauchy problem (1.2) in the homogeneous Besov spaces $\dot{\Lambda}_H^{\alpha,p,1}$, $\alpha > 0$, $1 \le p \le \infty$, which are the spaces of functions $f : \mathbb{Z} \to \mathbb{R}$ such that

$$\|f\|_{\dot{\Lambda}_H^{\alpha,p,1}} := \left\| t^{k-\frac{\alpha}{2}} \partial_t^k e^{t\Delta_d} f \right\|_{p,1} < \infty, \quad k = [\alpha/2] + 1.$$

The spaces $\dot{\Lambda}_H^{\alpha,p,1}$ generalize the spaces $\dot{B}^0_{p,1}(\mathbb{Z}^d)$ considered in [11].

Theorem 1.5 *(Endpoint maximal L^1-regularity). Let $\alpha > 0$, $1 \le p \le \infty$, and u the solution to (1.2) with T either $T > 0$ or $T = \infty$. Then, there exists a constant $C > 0$ such that*

$$\|\partial_t u\|_{L^1((0,T);\dot{\Lambda}_H^{\alpha,p,1})} + \|\Delta_d u\|_{L^1((0,T);\dot{\Lambda}_H^{\alpha,p,1})} \le C \left(\|u_0\|_{\dot{\Lambda}_H^{\alpha,p,1}} + \|g\|_{L^1((0,T);\dot{\Lambda}_H^{\alpha,p,1})} \right).$$

The paper is organized as follows. In Section 2, we establish all the results concerning pointwise and norm estimates of the discrete heat and Poisson kernels and semigroups. Section 3 is devoted to prove Theorem 1.1 and all the properties related to these spaces. In Section 4 we prove the regularity results for the fractional powers of the operators as well as the end-point maximal L^1-regularity theorem. Finally, in Section 5 we include the Hardy inequalities in their discrete and continuous versions, that will play a crucial role in our proofs.

Throughout this article, C and c always denote positive constants that can change in each occurrence.

2. Discrete heat and Poisson semigroups

2.1. Some known results

In this subsection we collect some known properties about gamma and Bessel functions that we will use along the paper.

For every $\gamma > 0$, and $\eta > 0$, it holds that

$$(1-r)^\eta r^\gamma \le \left(\frac{\gamma}{\gamma+\eta} \right)^\gamma, \quad 0 < r < 1. \tag{2.1}$$

This inequality was a key point in the proof of many results in [2] and [12]. We will also use the following estimates for the Euler's gamma function (see [37, Eq. (1)])

$$\frac{\Gamma(z+\alpha)}{\Gamma(z)} = z^{\alpha}\left(1 + O\left(\frac{1}{|z|}\right)\right), \quad \Re z > 0, \Re \alpha > 0. \tag{2.2}$$

We denote by I_n the Bessel function of imaginary argument (also called modified Bessel function of first kind) and order $n \in \mathbb{Z}$, given by

$$I_n(t) = \sum_{m=0}^{\infty} \frac{1}{m!\Gamma(m+n+1)} \left(\frac{t}{2}\right)^{2m+n}, \quad n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}, \ t \in \mathbb{C},$$

and $I_n = I_{-n}$ for $n \in \mathbb{N}$. It also has the following useful integral representation,

$$I_n(t) = \frac{t^n}{\sqrt{\pi}2^n\Gamma(n+1/2)} \int_{-1}^{1} e^{-ts}\left(1-s^2\right)^{n-1/2} \mathrm{d}s, \quad n \in \mathbb{N}_0, \ t \geq 0.$$

Likewise, for $l \in \mathbb{N}_0$ and $n \in \mathbb{N}_0$, the discrete derivatives $\delta^l_{right} I_n$ have the following representation, see [1, Proof of Lemma 2.4],

$$\begin{aligned} \delta^l_{right} I_n(t) = \frac{t^n}{\sqrt{\pi}2^n\Gamma(n+\frac{1}{2})} \int_{-1}^{1} e^{-ts}(1-s^2)^{n-\frac{1}{2}} \Bigg(&(s+1)^l \\ &+ \sum_{m=1}^{l-1} \frac{1}{t^m} \sum_{p=1}^{\min\{m,l-m\}} d_{p,m,l} s^p (s+1)^{l-m-p} \Bigg) ds, \end{aligned} \tag{2.3}$$

where $d_{p,m,l} \in \mathbb{R}$ are constants only depending on p, m and l, and the sum in the second line should only be interpreted when $l > 1$.

On the other hand, the generating function of the Bessel function I_n is given by

$$e^{t(x+x^{-1})/2} = \sum_{n \in \mathbb{Z}} x^n I_n(t), \quad x \neq 0, \ t \in \mathbb{C}.$$

From above identity, it was proved in [2, Theorem 3.3] that, for every $k \in \mathbb{N}_0$,

$$\sum_{n \in \mathbb{Z}} n^{2k} I_n(t) = e^t p_k(t), \quad \sum_{n \in \mathbb{Z}} n^{2k+1} I_n(t) = 0, \quad t > 0, \tag{2.4}$$

where each $p_k(t)$ is a polynomial of degree k with positive coefficients, $p_0(t) = 1$, and $p_k(0) = 0$ for all $k \in \mathbb{N}$.

2.2. *Discrete heat kernel*

The solution of the heat problem on $\mathbb{Z}$ is given by the function $e^{t\Delta_d} f(n) = \sum_{j \in \mathbb{Z}} G(t, n-j) f(j)$, where the discrete heat kernel is given by

$$G(t,n) = e^{-2t} I_n(2t), \quad n \in \mathbb{Z}, t > 0.$$

In the following we state a new general estimate for the heat kernel G and its discrete derivatives $\delta^l_{right}G$, refining the estimates obtained in [1].

Lemma 2.1. *Let $l \in \mathbb{N}_0$. Assume that one of the following statements holds*

(i) $-\left[\frac{l+1}{2}\right] - \frac{1}{2} \leq \beta$ *and* $t \geq 1$.
(ii) $-\left[\frac{l+1}{2}\right] - \frac{1}{2} \leq \beta \leq -\frac{1}{2}$ *and* $t \in (0, 1)$.

Then, there exists a positive constant $C_{\beta,l}$ only depending on β and l, such that

$$\left|\delta^l_{right}G(t,n)\right| \leq C_{\beta,l}\frac{t^\beta}{1+|n|^{l+2\beta+1}}, \quad n \in \mathbb{Z}.$$

Proof. Let $l \in \mathbb{N}_0$. First note that by (2.3), and performing the change of variables $1+s = \frac{u}{2t}$, we get for $n \in \mathbb{N}_0$ and $t > 0$ that

$$\begin{aligned}\left|\delta^l_{right}G(t,n)\right| \leq \frac{C_l}{\sqrt{t}\Gamma(n+\frac{1}{2})}\int_0^{4t} e^{-u}u^{n-\frac{1}{2}}\left(1-\frac{u}{4t}\right)^{n-\frac{1}{2}}\Bigg(\left(\frac{u}{4t}\right)^l \\ + \sum_{m=1}^{l-1}\frac{1}{t^m}\sum_{p=1}^{\min\{m,l-m\}}\left(\frac{u}{4t}\right)^{l-m-p}\Bigg)\,du.\end{aligned}$$

Next, we introduce the parameter β into this equation as follows,

$$\begin{aligned}\left|\delta^l_{right}G(t,n)\right| \leq \frac{C_{l,\beta}t^\beta}{\Gamma(n+\frac{1}{2})}\int_0^{4t} e^{-u}u^{n-1-\beta}\left(1-\frac{u}{4t}\right)^{n-\frac{1}{2}}\left(\frac{u}{4t}\right)^{\frac{1}{2}+\beta}\Bigg(\left(\frac{u}{4t}\right)^l \\ + \sum_{m=1}^{l-1}\frac{1}{t^m}\sum_{p=1}^{\min\{m,l-m\}}\left(\frac{u}{4t}\right)^{l-m-p}\Bigg)\,du =: I + II, \ \ t>0, \ n \in \mathbb{N}_0.\end{aligned}$$

We are interested in getting bounds that depend on $n \in \mathbb{Z}$. For this purpose, since we are assuming $\beta \geq -\left[\frac{l+1}{2}\right] - \frac{1}{2}$, in particular we have that $\beta + 2l + 1 > 0$, so we divide $\mathbb{Z}$ into the following three disjoint subsets

$$\begin{aligned}\mathbb{Z} &= \{n \in \mathbb{Z} : n > \beta+2l+1\} \cup \{n \in \mathbb{Z} : n < -(\beta+2l+1)\} \cup \{n \in \mathbb{Z} : |n| \leq \beta+2l+1\} \\ &=: A \cup B \cup D.\end{aligned}$$

First we consider that $n \in A$. Note also that, from the condition $\beta \geq -\left[\frac{l+1}{2}\right] - \frac{1}{2}$, we have in particular that $\beta + 1/2 + l \geq 0$. Thus, by using (2.1) and (2.2) we have that, for $\beta + 1/2 + l > 0$,

$$
\begin{aligned}
|I| &\le \frac{C_{l,\beta} t^{\beta}}{\Gamma(n+\frac{1}{2})} \left(\frac{\frac{1}{2}+\beta+l}{\beta+l+n} \right)^{\frac{1}{2}+\beta+l} \int_0^{4t} e^{-u} u^{n-1-\beta}\, du \\
&\le \frac{C_{l,\beta} t^{\beta}}{\Gamma(n+\frac{1}{2})} \frac{\Gamma(n-\beta)}{1+n^{\frac{1}{2}+\beta+l}} \le \frac{C_{l,\beta} t^{\beta}}{1+n^{1+2\beta+l}}.
\end{aligned}
$$

When $\beta+1/2+l=0$ the same inequality is obtained directly.

In order to get the desired bound in A for II, we need to make some observations. Let $l \ge 2$. Note that

$$
-\left[\frac{l+1}{2}\right] = \begin{cases} -l/2, & \text{if } l \text{ is even} \\ -l/2-1/2, & \text{if } l \text{ is odd,} \end{cases}
$$

and, for $1 \le m \le l-1$, it holds that

$$
\min\{m, l-m\} \le \begin{cases} l/2, & \text{if } l \text{ is even} \\ (l-1)/2, & \text{if } l \text{ is odd.} \end{cases} \tag{2.5}
$$

Therefore, the condition $\beta \ge -\left[\frac{l+1}{2}\right] - \frac{1}{2}$, implies that $\beta + 1/2 + l - p \ge 0$, for $1 \le p \le \min\{m, l-m\}$. In addition, if $n \in A$ then $n-\beta-m>0$, with $1 \le m \le l-1$. Therefore,

$$
\begin{aligned}
II &= \frac{C_{l,\beta} t^{\beta}}{\Gamma(n+\frac{1}{2})} \sum_{m=1}^{l-1} \sum_{p=1}^{\min\{m,l-m\}} \int_0^{4t} e^{-u} u^{n-1-\beta-m} \left(1-\frac{u}{4t}\right)^{n-\frac{1}{2}} \left(\frac{u}{4t}\right)^{\beta+\frac{1}{2}+l-p} du \\
&\le C_{l,\beta} t^{\beta} \sum_{m=1}^{l-1} \sum_{p=1}^{\min\{m,l-m\}} \frac{\Gamma(n-\beta-m)}{\Gamma(n+\frac{1}{2})(1+n^{\beta+\frac{1}{2}+l-p})},
\end{aligned}
$$

where we have used (2.1) whenever $\beta+1/2+l-p>0$, and a direct computation when $\beta+1/2+l-p=0$. Now by taking into account (2.5), we have that every p such that $1 \le p \le \min\{m, l-m\}$ satisfies $\beta+1/2+l-p=0$ if, and only if, $\beta+\left[\frac{l+1}{2}\right]+\frac{1}{2}=0$, and this holds whenever $p=l/2$ and $m=l/2$ (if l is even) or $p=(l-1)/2$ and is either $m=(l-1)/2$ or $m=(l+1)/2$ (if l is odd). Therefore, by using (2.2) we get that

$$
II \le C_{l,\beta} t^{\beta} \sum_{m=1}^{l-1} \sum_{p=1}^{\min\{m,l-m\}} \frac{1}{1+n^{l+2\beta+1+(m-p)}} \le \frac{C_{l,\beta} t^{\beta}}{1+n^{l+2\beta+1}}.
$$

Secondly, we consider $n \in B$, that is, $n < -\beta-2l-1$. Thus, $n<-l$, and then we can write $\left|\delta^l_{right} G(t,n)\right| = \left|\delta^l_{right} G(t,|n|-l)\right|$ for $n \in B$. Furthermore, if $n \in B$, we have $|n|-l > \beta+l+1$, and in particular $|n|-l > \beta+m$ for all $m=1,\dots,l-1$. So, one can repeat the same steps as in the case of the subset A but replacing n by $|n|-l$ and obtaining the same estimate.

Finally, we consider $n \in D$, i.e., $|n| \le \beta+2l+1$. Observe that $1 \le \frac{C_{\beta,l}}{1+|n|^{l+2\beta+1}}$. If $t \ge 1$, we use [1, Lemma 2.3] to get

$$\left|\delta^l_{right} G(t,n)\right| \le \frac{C_{\beta,l}}{t^{\left[\frac{l+1}{2}\right]+\frac{1}{2}}} \le C_{l,\beta} \frac{t^\beta}{1+|n|^{l+2\beta+1}}.$$

If $t \in (0,1)$ and $\beta \le -\frac{1}{2}$, we use that $\left|\delta^l_{right} G(t,n)\right| \le C t^{-\frac{1}{2}}$, which implies the desired bound. □

2.3. Heat and Poisson semigroups

Here we present some technical lemmata to prove our main results. Moreover, the following lemma will suggest the appropriate size condition that will be imposed to the functions so that the discrete heat semigroup is well-defined and satisfies the decay estimates necessary to work in $\Lambda_H^{\alpha,p,q}$ spaces.

Lemma 2.2. *Let $\alpha > 0$, $\alpha \notin \mathbb{N}$, $1 \le p, q \le \infty$ and $f \in C^{\alpha,p,q}(\mathbb{Z})$. Then, $\frac{f}{1+|\cdot|^\alpha} \in \ell^q(\mathbb{Z},\mu)$.*

Proof. We begin with the case $q = \infty$. Due to the $\ell^p(\mathbb{Z})$ embedding, if $f \in C^{\alpha,p,\infty}(\mathbb{Z})$ then $f \in C^{\alpha,\infty,\infty}(\mathbb{Z})$. Therefore, we can apply [1, Lemma 3.1] to obtain that $\frac{f}{1+|\cdot|^\alpha} \in \ell^\infty(\mathbb{Z})$.

Secondly we prove the case $1 \le q < \infty$, and we split the proof into several cases. Assume first that $\alpha \in (0,1)$. As $f \in C^{\alpha,p,q}(\mathbb{Z})$ we have that,

$$\left\|\frac{f}{1+|\cdot|^\alpha}\right\|_{q,\mu} \le \left\|\frac{f-f(0)}{1+|\cdot|^\alpha}\right\|_{q,\mu} + \left\|\frac{f(0)}{1+|\cdot|^\alpha}\right\|_{q,\mu},$$

where the second summand in the above expression is finite. For the first summand, since $|f(j) - f(0)| \le \|f(\cdot + j) - f(\cdot)\|_p$ for $j \in \mathbb{Z}$, we have

$$\left\|\frac{f-f(0)}{1+|\cdot|^\alpha}\right\|_{q,\mu} \le \left(\sum_{j\neq 0} \left\|\frac{f(\cdot+j)-f(\cdot)}{|j|^\alpha}\right\|_p^q \frac{1}{|j|}\right)^{1/q} < \infty.$$

Now, assume that $1 < \alpha < 2$. We have that

$$\left\|\frac{f}{1+|\cdot|^\alpha}\right\|_{q,\mu} \le |f(0)| + \left(\sum_{n=1}^\infty \left(\frac{|f(n)|}{1+n^\alpha}\right)^q \frac{1}{n}\right)^{\frac{1}{q}} + \left(\sum_{n=1}^\infty \left(\frac{|f(-n)|}{1+n^\alpha}\right)^q \frac{1}{n}\right)^{\frac{1}{q}} =: |f(0)| + A + B.$$

By using the fact that

$$|f(n)| \le |f(n) - f(n-1)| + \cdots + |f(1) - f(0)| + |f(0)| = \sum_{j=1}^n \left|\delta_{right} f(j-1)\right| + |f(0)|,$$

for $n \in \mathbb{N}$ and by the discrete Hardy inequality (see Lemma 5.3) we obtain that,

$$
\begin{aligned}
A &\leq \left(\sum_{n=1}^{\infty}\left(\sum_{j=1}^{n}\left|\delta_{right} f(j-1)\right|\right)^{q} \frac{1}{n^{\alpha q+1}}\right)^{\frac{1}{q}}+\left(\sum_{n=1}^{\infty}\left(\frac{|f(0)|}{1+n^{\alpha}}\right)^{q} \frac{1}{n}\right)^{\frac{1}{q}} \\
&\leq C\left(\sum_{j=1}^{\infty}\left(j\left|\delta_{right} f(j-1)\right|\right)^{q} \frac{1}{j^{\alpha q+1}}\right)^{\frac{1}{q}}+C \\
&\leq C\left(\sum_{j=1}^{\infty}\left(\frac{\left|\delta_{right} f(j-1)-\delta_{right} f(-1)\right|}{j^{\alpha-1}}\right)^{q} \frac{1}{j}\right)^{\frac{1}{q}}+C\left(\sum_{j=1}^{\infty}\left(\frac{\left|\delta_{right} f(-1)\right|}{j^{\alpha-1}}\right)^{q} \frac{1}{j}\right)^{\frac{1}{q}}+C \\
&\leq C\left(\sum_{j \neq 0}\left\|\frac{\delta_{right} f(\cdot+j)-\delta_{right} f(\cdot)}{|j|^{\alpha-1}}\right\|_{p}^{q} \frac{1}{|j|}\right)^{\frac{1}{q}}+C<\infty .
\end{aligned}
$$

For B we have to consider the following inequality,

$$
|f(-n)| \leq|f(-n)-f(n+1)|+\cdots+|f(-1)-f(0)|+|f(0)|=\sum_{j=1}^{n}\left|\delta_{right} f(-j)\right|+|f(0)|,
$$

and use the same techniques as for A to finish the proof for the case $1<\alpha<2$.

Lastly, for $\alpha>2$ the proof follows by writing f in terms of differences of order $[\alpha]$ and by iterating the previous arguments. □

Remark 2.3. Let $1 \leq q \leq \infty$ and $f: \mathbb{Z} \rightarrow \mathbb{R}$ satisfying $\frac{f}{1+|\cdot|^{\alpha}} \in \ell^{q}(\mathbb{Z}, \mu)$ for some $\alpha>0$. From the embedding of the $\ell^{q}(\mathbb{Z})$ spaces we have that $\frac{f}{1+|\cdot|^{\alpha+\frac{1}{q}}} \in \ell^{\infty}(\mathbb{Z})$. Therefore, [1, Lemma 2.12] gives that the heat semigroup $e^{t \Delta_{d}} f$ is well-defined for every $t>0$. Furthermore, from [1, Lemma 2.11] it follows that $\delta_{right}^{m} e^{t \Delta_{d}} f$ and $\partial_{t}^{l} e^{t \Delta_{d}} f$, $m, l \in \mathbb{N}$, are well-defined,

$$
\delta_{right} e^{t \Delta_{d}} f(n)=\sum_{j \in \mathbb{Z}}\left(\delta_{right} G(t, n-j)\right) f(j)=\sum_{j \in \mathbb{Z}} G(t, j) \delta_{right} f(n-j), \quad n \in \mathbb{Z},
$$

and for $t=t_{1}+t_{2}$, where $t, t_{1}, t_{2}>0$,

$$
\partial_{t} e^{t \Delta_{d}} f(n)|_{t=t_{1}+t_{2}}=\sum_{j \in \mathbb{Z}} \partial_{t_{1}} G(t_{1}, j) e^{t_{2} \Delta_{d}} f(n-j)=\sum_{j \in \mathbb{Z}} G(t_{1}, j) \partial_{t_{2}} e^{t_{2} \Delta_{d}} f(n-j), \quad n \in \mathbb{Z} .
$$

Next lemma shows an estimate for the size of the heat semigroup and some conditions under the derivatives of the heat semigroup vanish at infinity.

Lemma 2.4. *Let $1 \leq q \leq \infty$ and $f: \mathbb{Z} \rightarrow \mathbb{R}$ satisfying $\frac{f}{1+|\cdot|^{\alpha}} \in \ell^{q}(\mathbb{Z}, \mu)$, for certain $\alpha>0$. Then,*

(1) For every $t>0$, it holds that $\left\|\frac{e^{t \Delta_{d}} f}{1+|\cdot|^{\alpha}}\right\|_{\ell^{q}(\mathbb{Z}, \mu)} \leq C\left(1+t^{\frac{\alpha+1/q}{2}}\right)$.

(2) For every $n \in \mathbb{Z}$ and $m, l \in \mathbb{N}_0$ such that $\frac{m}{2} + l > \frac{\alpha}{2}$, we have that

$$\partial_t^l \delta_{right}^m e^{t\Delta_d} f(n) \to 0, \quad \text{as } t \to \infty.$$

Proof. First we prove *(1)*. The case $q = \infty$ was proved in [1, Lemma 2.12 A]. Let $1 \le q < \infty$, and $t > 0$. By using Minkowski's inequality we have that

$$\begin{aligned}
\left\| \frac{e^{t\Delta_d} f}{1 + |\cdot|^\alpha} \right\|_{\ell^q(\mathbb{Z},\mu)} &= \left(\sum_{n\in\mathbb{Z}} \left| \sum_{j\in\mathbb{Z}} G(t,j) \frac{f(n-j)}{(1+|n|^\alpha)(1+|n|)^{1/q}} \right|^q \right)^{1/q} \\
&\le \sum_{j\in\mathbb{Z}} G(t,j) \left(\sum_{n\in\mathbb{Z}} \left(\frac{|f(n-j)|}{(1+|n|^\alpha)(1+|n|)^{1/q}} \right)^q \right)^{1/q} \\
&\le CG(t,0) + \sum_{j\in\mathbb{Z}\setminus\{0\}} G(t,j)|f(-j)| \\
&+ \sum_{j\in\mathbb{Z}\setminus\{0\}} G(t,j) \frac{|f(0)|}{(1+|j|^\alpha)(1+|j|)^{1/q}} \\
&+ \sum_{j\in\mathbb{Z}\setminus\{0\}} G(t,j) \left(\sum_{n\in\mathbb{Z}\setminus\{0,j\}} \left(\frac{|f(n-j)|}{(1+|n|^\alpha)|n|^{1/q}} \right)^q \right)^{1/q} \\
&=: I + II + III + IV.
\end{aligned}$$

Next, we work with each summand above separately. Note that I is bounded.

Secondly, since $\frac{f}{1+|\cdot|^\alpha} \in \ell^q(\mathbb{Z},\mu)$ implies that $\frac{f}{1+|\cdot|^{\alpha+1/q}} \in \ell^\infty(\mathbb{Z})$, by taking m as the smallest integer such that $2m > \alpha + 1/q$, we have that

$$\begin{aligned}
II = \sum_{j\in\mathbb{Z}\setminus\{0\}} G(t,j)|f(-j)| &\le C \left\| \frac{f}{1+|\cdot|^{\alpha+1/q}} \right\|_\infty \sum_{j\in\mathbb{Z}\setminus\{0\}} G(t,j)(1+|j|^{\alpha+1/q}) \\
&\le C \left\| \frac{f}{1+|\cdot|^{\alpha+1/q}} \right\|_\infty \left(1 + \sum_{|j|\le\sqrt{t}} G(t,j)|j|^{\alpha+1/q} \right. \\
&\left. + \sum_{|j|>\sqrt{t}} G(t,j)|j|^{\alpha+1/q} \min\left\{ \frac{|j|}{\sqrt{t}}, |j| \right\}^{2m-\alpha-1/q} \right) \\
&\le C \left\| \frac{f}{1+|\cdot|^{\alpha+1/q}} \right\|_\infty \left(1 + t^{\frac{\alpha+1/q}{2}} + Cp_m(2t) \min\left\{ \frac{1}{t^{m-\frac{\alpha+1/q}{2}}}, 1 \right\} \right) \\
&\le C \left\| \frac{f}{1+|\cdot|^{\alpha+1/q}} \right\|_\infty (1 + t^{\frac{\alpha+1/q}{2}}),
\end{aligned}$$

where we have used that $\|G(t,\cdot)\|_1 = 1$ and that $|p_m(2t)| \le C$ for $0 < t < 1$, and $|p_m(2t)| \le Ct^m$ for $t \ge 1$, see (2.4).

For III, since $|j| \geq 1$, we get that $III \leq C\|G(t,\cdot)\|_1 = C$.

Finally, note that if $j \neq 0$ and $n \neq 0, j$, we have

$$\frac{(1+|n-j|^{\alpha})|n-j|^{1/q}}{(1+|n|^{\alpha})|n|^{1/q}} \leq C(1+|j|^{\alpha})|j|^{1/q},$$

so by proceeding as in the case II, we get

$$IV \leq C \sum_{j\in\mathbb{Z}\setminus\{0\}} G(t,j)(1+|j|^{\alpha})|j|^{1/q} \leq C\left(1+t^{\frac{\alpha+1/q}{2}}\right).$$

Now we shall prove *(2)*. The case $q=\infty$ was proved in [1, Lemma 2.13].

Let $1 \leq q < \infty$, $m, l \in \mathbb{N}_0$ such that $\frac{m}{2}+l > \frac{\alpha}{2}$ and $n \in \mathbb{Z}$. Since the semigroup is the solution to the heat equation, we can write

$$\left|\partial_t^l \delta_{right}^m e^{t\Delta_d} f(n)\right| = \left|\delta_{right}^{2l+m} e^{t\Delta_d} f(n-l)\right|.$$

Now we choose $\varepsilon > 0$ small enough such that $\alpha - 2l - m + \varepsilon < 0$. Then, by using Lemma 2.1 for $t \geq 1$ and $\beta = \frac{\alpha-2l-m+\varepsilon}{2}$, we obtain that

$$\begin{aligned}\left|\partial_t^l \delta_{right}^m e^{t\Delta_d} f(n)\right| &\leq C_{l,m,\alpha} \sum_{j\in\mathbb{Z}} \frac{t^{\beta}\,|f(n-l-j)|}{1+|j|^{\alpha+\varepsilon+1}} \\ &= C_{l,m,\alpha} t^{\beta} \sum_{j\in\mathbb{Z}} \frac{|f(n-l-j)|\,(1+|n-l-j|^{\alpha})}{(1+|n-l-j|^{\alpha})(1+|j|^{\alpha+\varepsilon+1})}.\end{aligned}$$

Hölder's inequality gives the convergence for the series above and since $\beta < 0$ it follows that $\left|\partial_t^l \delta_{right}^m e^{t\Delta_d} f(n)\right| \to 0$, as $t \to \infty$. $\square$

The following lemma provides the size condition for the sequence $e^{\cdot^2\Delta_d} f(\cdot) : \mathbb{Z} \to \mathbb{R}$, given by $e^{n^2\Delta_d} f(n) = \sum_{j\in\mathbb{Z}} G(n^2, n-j) f(j)$, $n \in \mathbb{Z}$, which will be a key point in the proof of one of our main results.

Lemma 2.5. *Let $1 \leq q < \infty$, $l \in \mathbb{N}_0$, $\alpha \geq l$ and $f : \mathbb{Z} \to \mathbb{R}$ such that $\frac{f}{1+|\cdot|^{\alpha}} \in \ell^q(\mathbb{Z},\mu)$. Then, $\frac{e^{\cdot^2\Delta_d}\delta_{right}^l f(\cdot)}{1+|\cdot|^{\alpha-l}} \in \ell^q(\mathbb{Z},\mu)$.*

Proof. First, recall that $I_j(0) = \delta_0(j)$ for $j \in \mathbb{Z}$, (where δ_0 denotes the Dirac delta sequence), so $\delta_{right}^l G(0,-j) = (-1)^j \binom{l}{j}$, for $j = 0, \ldots, l$, and $\delta_{right}^l G(0,-j) = 0$ in another case. Then,

$$\left\| \frac{e^{\cdot^2 \Delta_d} \delta^l_{right} f(\cdot)}{1+|\cdot|^{\alpha-l}} \right\|_{q,\mu} \leq \left(\sum_{n\in\mathbb{Z}} \left(\sum_{j\in\mathbb{Z}} \frac{\left|\delta^l_{right} G(n^2, n-j)\right| |f(j)|}{1+|n|^{\alpha-l}} \right)^q \mu(n) \right)^{\frac{1}{q}}$$

$$\leq C_l \max\{|f(j)| : j = 0, \dots, l\} + \left(\sum_{n\neq 0} \left(\sum_{j\in\mathbb{Z}} \frac{\left|\delta^l_{right} G(n^2, n-j)\right| |f(j)|}{1+|n|^{\alpha-l}} \right)^q \mu(n) \right)^{\frac{1}{q}}.$$

So, we have to prove the convergence of the last addend. For this purpose, we consider the next disjoint partition

$$\begin{aligned}
&\{(n, j) : n \in \mathbb{Z} \setminus \{0\}, j \in \mathbb{Z}\} \\
&\quad = \{(n, j) : n \geq 1, j \geq 1\} \cup \{(n, j) : n \geq 1, j \leq -1\} \cup \{(n, j) : n \geq 1, j = 0\} \\
&\quad \cup \{(n, j) : n \leq -1, j \geq 1\} \cup \{(n, j) : n \leq -1, j \leq -1\} \cup \{(n, j) : n \leq -1, j = 0\} \\
&\quad =: A.1 \cup \ldots \cup A.6.
\end{aligned}$$

By Lemma 2.1, we have that $\left|\delta^l_{right} G(n^2, n-j)\right| \leq C_{\beta,l} \frac{|n|^{2\beta}}{1+|n-j|^{2\beta+l+1}}$, for $n \in \mathbb{Z} \setminus \{0\}$, $j \in \mathbb{Z}$ and any $\beta \geq -[\frac{l+1}{2}] - \frac{1}{2}$.

On the one hand, we have that

$$\left|\delta^l_{right} G(n^2, n)\right| \leq C_{\beta,l} \, |n|^{-(l+1)}, \quad (n, 0) \in A.3, A.6,$$

so

$$\begin{aligned}
\left(\sum_{n\neq 0} \left(\frac{\left|\delta^l_{right} G(n^2, n)\right| |f(0)|}{1+|n|^{\alpha-l}} \right)^q \mu(n) \right)^{\frac{1}{q}} &\leq C_{\beta,l} \, |f(0)| \left(\sum_{n=1}^{\infty} \left(\frac{n^{-l-1}}{(1+n^{\alpha-l})} \right)^q \frac{1}{n} \right)^{\frac{1}{q}} \\
&\leq C_{\beta,l} \, |f(0)| .
\end{aligned}$$

Now, by taking $\beta = -\frac{l+1}{2}$, we get

$$\left|\delta^l_{right} G(n^2, n-j)\right| \leq C_{\beta,l} \, |n|^{-(l+1)}, \quad 1 \leq j \leq 2n.$$

Thus, by using Hardy's inequality (see Lemma 5.3) we have that

$$\begin{aligned}
\left(\sum_{n=1}^{\infty} \left(\sum_{j=1}^{n} \frac{\left|\delta^l_{right} G(n^2, n-j)\right| |f(j)|}{1+|n|^{\alpha-l}} \right)^q \mu(n) \right)^{\frac{1}{q}} &\leq C_{\beta,l} \left(\sum_{n=1}^{\infty} \left(\sum_{j=1}^{n} \frac{|f(j)|}{n^{\alpha+1}} \right)^q \frac{1}{n} \right)^{\frac{1}{q}} \\
&\leq C_{\beta,l} \left(\sum_{n=1}^{\infty} \left(\frac{|f(n)|}{n^{\alpha}} \right)^q \frac{1}{n} \right)^{\frac{1}{q}}
\end{aligned}$$

$$\leq C_{\beta,l}\left\|\frac{f(\cdot)}{1+|\cdot|^{\alpha}}\right\|_{q,\mu},$$

and by direct computations we get that

$$\begin{aligned}\left(\sum_{n=1}^{\infty}\left(\sum_{j=n+1}^{2n}\frac{\left|\delta^{l}_{right}G(n^2,n-j)\right||f(j)|}{1+|n|^{\alpha-l}}\right)^{q}\mu(n)\right)^{\frac{1}{q}} &\leq C_{\beta,l}\left(\sum_{n=1}^{\infty}\left(\sum_{j=n+1}^{2n}\frac{|f(j)|}{n^{\alpha+1}}\right)^{q}\frac{1}{n}\right)^{\frac{1}{q}}\\
&\leq C_{\beta,l}\left(\sum_{n=1}^{\infty}\left(\sum_{j=n+1}^{2n}\frac{|f(j)|n}{j^{\alpha+2}}\right)^{q}\frac{1}{n}\right)^{\frac{1}{q}}\\
&\leq C_{\beta,l}\left(\sum_{n=1}^{\infty}\left(\frac{|f(n)|n^2}{n^{\alpha+2}}\right)^{q}\frac{1}{n}\right)^{\frac{1}{q}}\\
&\leq C_{\beta,l}\left\|\frac{f(\cdot)}{1+|\cdot|^{\alpha}}\right\|_{q,\mu}.\end{aligned}$$

On the other hand, by using Lemma 2.1 with $\beta=\frac{\alpha-l+1}{2}$ one gets

$$\left|\delta^{l}_{right}G(n^2,n-j)\right|\leq C_{\beta,l}\frac{|n|^{\alpha-l+1}}{1+(j-n)^{\alpha+2}},\quad j\geq 2n+1.$$

Notice that in this case $j-n>\frac{j}{2}$, so by Hardy's inequality (see Lemma 5.3) we have that

$$\begin{aligned}&\left(\sum_{n=1}^{\infty}\left(\sum_{j=2n+1}^{\infty}\frac{\left|\delta^{l}_{right}G(n^2,n-j)\right||f(j)|}{1+|n|^{\alpha-l}}\right)^{q}\mu(n)\right)^{\frac{1}{q}}\\
&\leq C_{\beta,l}\left(\sum_{n=1}^{\infty}\left(\sum_{j=2n+1}^{\infty}\frac{|f(j)|n}{(j-n)^{\alpha+2}}\right)^{q}\frac{1}{n}\right)^{\frac{1}{q}}\\
&\leq C_{\beta,l}\left(\sum_{n=1}^{\infty}\left(\sum_{j=2n+1}^{\infty}\frac{|f(j)|n}{j^{\alpha+2}}\right)^{q}\frac{1}{n}\right)^{\frac{1}{q}}\\
&\leq C_{\beta,l}\left\|\frac{f(\cdot)}{1+|\cdot|^{\alpha}}\right\|_{q,\mu}.\end{aligned}$$

To finish the proof, we can proceed in the same way for the subsets $A.2$, $A.4$, $A.5$, by taking $|j|$, $|n|$ instead of j,n, and by taking into account that $|n-j|\geq ||n|-|j||$. $\square$

Now we include some lemmata which state mixed-norm estimates for the derivatives of the heat and Poisson semigroups that provide alternative semigroup conditions to characterize the discrete heat and Poisson Besov spaces.

Lemma 2.6. *Let $\beta > 0$, $1 \le p, q \le \infty$ and $f : \mathbb{Z} \to \mathbb{R}$.*

(i) *Suppose that $\frac{f}{1+|\cdot|^\alpha} \in \ell^q(\mathbb{Z}, \mu)$ for some $\alpha > 0$. If $m, l \in \mathbb{N}_0$, such that $\frac{m}{2} + l > \frac{\alpha}{2}$, then*

$$\left\| t^\beta \partial_t^l \delta_{right}^m e^{t\Delta_d} f \right\|_{p,q} \le \frac{1}{\beta} \left\| t^{\beta+1} \partial_t^{l+1} \delta_{right}^m e^{t\Delta_d} f \right\|_{p,q}, \quad t > 0.$$

(ii) *Suppose that f satisfies $\sum_{j\in\mathbb{Z}} \frac{|f(j)|}{1+|j|^2} < \infty$. If $m, l \in \mathbb{N}_0$, such that $m + l \ge 1$, then*

$$\left\| y^\beta \partial_y^l \delta_{right}^m e^{-y\sqrt{-\Delta_d}} f \right\|_{p,q} \le \frac{1}{\beta} \left\| y^{\beta+1} \partial_y^{l+1} \delta_{right}^m e^{-y\sqrt{-\Delta_d}} f \right\|_{p,q}, \quad y > 0.$$

Proof. The proof of this result runs parallel to the one of [35, Lemmata 4 c) and 4* c)]. In our case, we also have to use Lemma 2.4, so that $\partial_t^l \delta_{right}^m e^{t\Delta_d} f(n) \to 0$, as $t \to \infty$, and [1, Lemma 2.13], so that $\partial_y^l \delta_{right}^m e^{-y\sqrt{-\Delta_d}} f(n) \to 0$, as $y \to \infty$. □

Lemma 2.7. *Let $\beta > 0$, $1 \le p, q \le \infty$, $m, l \in \mathbb{N}_0$, and $f : \mathbb{Z} \to \mathbb{R}$.*

(i) *Suppose that $\frac{f}{1+|\cdot|^\alpha} \in \ell^q(\mathbb{Z}, \mu)$ for some $\alpha > 0$.*
  - *If $l \in \mathbb{N}$, there is $C > 0$ such that*

$$\left\| t^{\beta+1} \partial_t^l \delta_{right}^m e^{t\Delta_d} f \right\|_{p,q} \le C \left\| t^\beta \partial_t^{l-1} \delta_{right}^m e^{t\Delta_d} f \right\|_{p,q}, \quad t > 0.$$

  - *If $m \in \mathbb{N}$, there is $C > 0$ such that*

$$\left\| t^{\beta+\frac{1}{2}} \partial_t^l \delta_{right}^m e^{t\Delta_d} f \right\|_{p,q} \le C \left\| t^\beta \partial_t^l \delta_{right}^{m-1} e^{t\Delta_d} f \right\|_{p,q}, \quad t > 0.$$

(ii) *Suppose f satisfies $\sum_{j\in\mathbb{Z}} \frac{|f(j)|}{1+|j|^2} < \infty$.*
  - *If $l \in \mathbb{N}$, there is $C > 0$ such that*

$$\left\| y^{\beta+1} \partial_y^l \delta_{right}^m e^{-y\sqrt{-\Delta_d}} f \right\|_{p,q} \le C \left\| y^\beta \partial_y^{l-1} \delta_{right}^m e^{-y\sqrt{-\Delta_d}} f \right\|_{p,q}, \quad y > 0.$$

  - *If $m \in \mathbb{N}$, there is $C > 0$ such that*

$$\left\| y^{\beta+1} \partial_y^l \delta_{right}^m e^{-y\sqrt{-\Delta_d}} f \right\|_{p,q} \le C \left\| y^\beta \partial_y^l \delta_{right}^{m-1} e^{-y\sqrt{-\Delta_d}} f \right\|_{p,q}, \quad y > 0.$$

Proof. Again, the proof of this result runs parallel to the one of [35, Lemmata 4 a), b) and 4* a), b)]. In this case, we have used [1, Lemmata 2.6, 2.9 and 2.11, and Remarks 2.7 and 2.10]. □

Remark 2.8. From Lemmata 2.6 and 2.7 we deduce that if $f \in \Lambda_H^{\alpha,p,q}$ for some $\alpha > 0$, $1 \le p, q \le \infty$ and k, l are natural numbers such that $k, l \ge [\alpha/2] + 1$, then $\left\| t^{k-\frac{\alpha}{2}} \partial_t^k e^{t\Delta_d} f \right\|_{p,q} < \infty$ if, and only if, $\left\| t^{l-\frac{\alpha}{2}} \partial_t^l e^{t\Delta_d} f \right\|_{p,q} < \infty$. Analogously, if k, l are natural numbers such that $k, l \ge [\alpha] + 1$, then $\left\| y^{k-\alpha} \partial_y^k e^{-y\sqrt{-\Delta_d}} f \right\|_{p,q} < \infty$ if, and only if, $\left\| y^{l-\alpha} \partial_y^l e^{-y\sqrt{-\Delta_d}} f \right\|_{p,q} < \infty$.

Lemma 2.9. *Let $\beta > 0$, $1 \le p, q \le \infty$ and $f : \mathbb{Z} \to \mathbb{R}$.*

(i) *Suppose that $\frac{f}{1+|\cdot|^\alpha} \in \ell^q(\mathbb{Z}, \mu)$ for some $\alpha > 0$, and $m, l \in \mathbb{N}_0$ such that $\frac{m}{2} + l > \frac{\alpha}{2}$.*
- *If $l \in \mathbb{N}$, there is $C > 0$ such that*

$$\left\| t^{\beta+\frac{1}{2}} \partial_t^l \delta_{right}^m e^{t\Delta_d} f \right\|_{p,q} \le C \left\| t^\beta \partial_t^{l-1} \delta_{right}^{m+1} e^{t\Delta_d} f \right\|_{p,q}, \quad t > 0.$$

- *If $m \in \mathbb{N}$, there is $C > 0$ such that*

$$\left\| t^{\beta} \partial_t^l \delta_{right}^m e^{t\Delta_d} f \right\|_{p,q} \le C \left\| t^{\beta+\frac{1}{2}} \partial_t^{l+1} \delta_{right}^{m-1} e^{t\Delta_d} f \right\|_{p,q}, \quad t > 0.$$

(ii) *Suppose that f satisfies $\sum_{j\in\mathbb{Z}} \frac{|f(j)|}{1+|j|^2} < \infty$, and $m, l \in \mathbb{N}_0$.*
- *If $l \in \mathbb{N}$, there is $C > 0$ such that*

$$\left\| y^{\beta} \partial_y^l \delta_{right}^m e^{-y\sqrt{-\Delta_d}} f \right\|_{p,q} \le C \left\| y^\beta \partial_y^{l-1} \delta_{right}^{m+1} e^{-y\sqrt{-\Delta_d}} f \right\|_{p,q}, \quad y > 0.$$

- *If $m \in \mathbb{N}$, there is $C > 0$ such that*

$$\left\| y^{\beta} \partial_y^l \delta_{right}^m e^{-y\sqrt{-\Delta_d}} f \right\|_{p,q} \le C \left\| y^\beta \partial_y^{l+1} \delta_{right}^{m-1} e^{-y\sqrt{-\Delta_d}} f \right\|_{p,q}, \quad y > 0.$$

Proof. This result follows directly by using Lemmata 2.7 and 2.6 and the ideas presented in the proof of [35, Theorem 1]. □

The last lemma of this section states an inequality involving both, heat and Poisson semigroups. The proof follows similar steps than the ones appearing in [16, Theorem 4.1] and [15, Theorem 5.6].

Lemma 2.10. *Let $\alpha > 0$, $1 \le p \le \infty$ and $f : \mathbb{Z} \to \mathbb{R}$ such that $\frac{f}{1+|\cdot|^\alpha} \in \ell^\infty(\mathbb{Z})$ and $\sum_{n\in\mathbb{Z}} \frac{|f(n)|}{1+n^2} < \infty$. Then, for every $k \in \mathbb{N}$,*

$$\left\| \partial_y^{2k} e^{-y\sqrt{-\Delta_d}} f \right\|_p \le \int_0^\infty \frac{1}{2\sqrt{\pi}} \frac{y e^{\frac{-y^2}{4t}}}{t^{\frac{3}{2}}} \left\| \partial_t^k e^{t\Delta_d} f \right\|_p dt.$$

3. Characterization via semigroups of discrete Besov spaces

In the following we prove the characterization of $C^{\alpha,p,q}(\mathbb{Z})$ by using the spaces $\Lambda_H^{\alpha,p,q}$ and $\Lambda_P^{\alpha,p,q}$.

3.1. *Case* $0 < \alpha < 1$

Proposition 3.1. *Let* $0 < \alpha < 1$ *and* $1 \le p, q \le \infty$. *If* $f \in C^{\alpha,p,q}(\mathbb{Z})$ *then* $f \in \Lambda_H^{\alpha,p,q}$.

Proof. Let $f \in C^{\alpha,p,q}(\mathbb{Z})$. From Lemma 2.2 we have that $\frac{f}{1+|\cdot|^\alpha} \in \ell^q(\mathbb{Z},\mu)$. Now we have to prove that $\left\| t^{1-\frac{\alpha}{2}} \partial_t e^{t\Delta_d} f \right\|_{p,q} < \infty$. For that aim, we shall consider the p-norm of the derivative of the semigroup. Note that $\partial_t G(t,j) = \Delta_d G(t,j) = \delta^2_{right} G(t, j-1)$, for $j \in \mathbb{Z}$, and $\delta^2_{right} G(t, j-1) = \delta^2_{right} G(t, |j|-1)$, for $j \le -1$. Suppose that $1 \le p \le \infty$ and $1 \le q < \infty$. If $1 \le p < \infty$, by Minkowski's integral inequality one gets

$$
\begin{aligned}
\left\| \partial_t e^{t\Delta_d} f \right\|_p &= \left(\sum_{n\in\mathbb{Z}} \left| \sum_{j\in\mathbb{Z}} \partial_t G(t,j) f(n-j) \right|^p \right)^{\frac{1}{p}} \\
&\le \sum_{j\in\mathbb{Z}} \left(\sum_{n\in\mathbb{Z}} |\partial_t G(t,j)(f(n-j)-f(n))|^p \right)^{\frac{1}{p}} \\
&= \sum_{j\in\mathbb{Z}} |\partial_t G(t,j)| \, \|f(\cdot) - f(\cdot + j)\|_p \qquad (3.1) \\
&= 2\sum_{j\ge 1} \left| \delta^2_{right} G(t, j-1) \right| \|f(\cdot) - f(\cdot + j)\|_p \\
&= 2\sum_{j\ge 0} \left| \delta^2_{right} G(t, j) \right| \Omega_p(j+1),
\end{aligned}
$$

with $\Omega_p(j) = \|f(\cdot) - f(\cdot + j)\|_p$. If $p = \infty$, we also have

$$
\begin{aligned}
\left\| \partial_t e^{t\Delta_d} f \right\|_\infty &= \sup_{n\in\mathbb{Z}} \left| \sum_{j\in\mathbb{Z}} \partial_t G(t,j) f(n-j) \right| \le \sum_{j\in\mathbb{Z}} |\partial_t G(t,j)| \, \|f(\cdot) - f(\cdot + j)\|_\infty \\
&= 2\sum_{j\ge 0} \left| \delta^2_{right} G(t, j) \right| \Omega_\infty(j+1),
\end{aligned}
$$

with $\Omega_\infty(j) = \|f(\cdot) - f(\cdot + j)\|_\infty$. Then, we have that

$$
\begin{aligned}
\left\| t^{1-\frac{\alpha}{2}} \partial_t e^{t\Delta_d} f \right\|_{p,q} &\le \left(\int_0^1 \left| 2t^{1-\frac{\alpha}{2}} \sum_{j\ge 0} \left| \delta^2_{right} G(t,j) \right| \Omega_p(j+1) \right|^q \frac{dt}{t} \right)^{\frac{1}{q}} \\
&\quad + \left(\int_1^\infty \left| 2t^{1-\frac{\alpha}{2}} \sum_{j\ge 0} \left| \delta^2_{right} G(t,j) \right| \Omega_p(j+1) \right|^q \frac{dt}{t} \right)^{\frac{1}{q}} \\
&=: A + B.
\end{aligned}
$$

By using Lemma 2.1 with $\beta = -\frac{1}{2}$ we get that $\left|\delta^2_{right} G(t,j)\right| \leq \dfrac{C}{(1+j^2)t^{\frac{1}{2}}}$, $j \geq 0$, $t > 0$. Therefore,

$$A \leq C \left(\int_0^1 \left| t^{\frac{1}{2}-\frac{\alpha}{2}} \sum_{j\geq 1} \frac{\Omega_p(j)}{1+(j-1)^2} \right|^q \frac{dt}{t} \right)^{\frac{1}{q}}.$$

By taking into account that $\frac{1}{1+(j-1)^2} \leq \frac{2}{j^2}$, $j \geq 1$, and Minkowski's integral inequality, we obtain that

$$A \leq C \left(\int_0^1 \left| t^{\frac{1}{2}-\frac{\alpha}{2}} \sum_{j\geq 1} \frac{\Omega_p(j)}{j^2} \right|^q \frac{dt}{t} \right)^{\frac{1}{q}} \leq C \sum_{j\geq 1} \frac{\Omega_p(j)}{j^2} \left(\int_0^1 \left| t^{\frac{1}{2}-\frac{\alpha}{2}} \right|^q \frac{dt}{t} \right)^{\frac{1}{q}} \leq C \sum_{j\geq 1} \frac{\Omega_p(j)}{j^2}.$$

Notice that $2 = \alpha + 1/q + 2 - \alpha - 1/q$, so we can use Hölder's inequality to get

$$A \leq C \Bigg(\sum_{j\geq 1} \left\| \frac{f(\cdot + j) - f(\cdot)}{j^\alpha} \right\|_p^q \frac{1}{j} \Bigg)^{1/q} < \infty.$$

Regarding B, we split it into 2 different integrals using Minkowski's inequality as follows

$$\begin{aligned} B \leq & \left(\int_1^\infty \left| 2t^{1-\frac{\alpha}{2}} \sum_{0\leq j\leq \sqrt{t}} \left|\delta^2_{right} G(t,j)\right| \Omega_p(j+1) \right|^q \frac{dt}{t} \right)^{\frac{1}{q}} \\ & + \left(\int_1^\infty \left| 2t^{1-\frac{\alpha}{2}} \sum_{j > \sqrt{t}} \left|\delta^2_{right} G(t,j)\right| \Omega_p(j+1) \right|^q \frac{dt}{t} \right)^{\frac{1}{q}} \\ & =: B1 + B2. \end{aligned}$$

To compute the bound for $B1$, we apply Lemma 2.1 with $\beta = -\frac{3}{2}$ so that $\left|\delta^2_{right} G(t,j)\right| \leq \dfrac{C}{t^{\frac{3}{2}}}$, for $j \geq 0$ and $t > 0$, to get

$$B1 \leq C \left(\int_1^\infty \left| t^{-\frac{\alpha}{2}-\frac{1}{2}} \sum_{0\leq j\leq \sqrt{t}} \Omega_p(j+1) \right|^q \frac{dt}{t} \right)^{\frac{1}{q}}.$$

Now we define the function $g := \sum_{j=0}^\infty \Omega_p(j+1)\chi_{[j+1,j+2)}$. Then, we can write $\sum\limits_{0\leq j\leq \sqrt{t}} \Omega_p(j+1) = \int\limits_0^{[\sqrt{t}]+2} g(x)\,dx$. By using Hardy's inequality (see Lemma 5.1) one obtains that

$$B1 \leq C \left(\int_1^\infty \left| t^{-\frac{\alpha}{2}-\frac{1}{2}} \int_0^{[\sqrt{t}]+2} g(x)\,dx \right|^q \frac{dt}{t} \right)^{\frac{1}{q}} \leq C \left(\int_1^\infty \left| t^{-\frac{\alpha}{2}-\frac{1}{2}} \int_0^{3\sqrt{t}} g(x)\,dx \right|^q \frac{dt}{t} \right)^{\frac{1}{q}}$$

$$= C \left(\int_3^\infty \left(\int_0^u g(x)\,dx \right)^q u^{-q(\alpha+1)-1}\,du \right)^{\frac{1}{q}} \leq C \left(\int_0^\infty (xg(x))^q x^{-q(\alpha+1)-1}\,dx \right)^{\frac{1}{q}}$$

$$\leq C \left(\sum_{j=1}^\infty \frac{\|f(\cdot)-f(\cdot+j)\|_p^q}{j^{q\alpha+1}} \right)^{\frac{1}{q}} < \infty.$$

On the other hand, by using Lemma 2.1 with $\beta = 0$, we have that

$$B2 \leq C \left(\int_1^\infty \left| t^{1-\frac{\alpha}{2}} \sum_{j>\sqrt{t}} \frac{\Omega_p(j+1)}{j^3} \right|^q \frac{dt}{t} \right)^{\frac{1}{q}}.$$

We define the function $h := \sum_{j=0}^\infty \chi_{[j+1,j+2)} \frac{\Omega_p(j+1)}{j^3}$, so that we can write $\sum_{j>\sqrt{t}} \frac{\Omega_p(j+1)}{j^3} = \int_{[\sqrt{t}]+2}^\infty h(x)\,dx$. By using Hardy's inequality (Lemma 5.1) we have that

$$B \leq C \left(\int_1^\infty \left| t^{1-\frac{\alpha}{2}} \int_{[\sqrt{t}]+2}^\infty h(x)\,dx \right|^q \frac{dt}{t} \right)^{\frac{1}{q}} \leq C \left(\int_1^\infty \left| t^{1-\frac{\alpha}{2}} \int_{\sqrt{t}}^\infty h(x)\,dx \right|^q \frac{dt}{t} \right)^{\frac{1}{q}}$$

$$= C \left(\int_1^\infty \left(\int_u^\infty h(x)\,dx \right)^q u^{q(2-\alpha)-1}\,du \right)^{\frac{1}{q}} \leq C \left(\int_0^\infty (xh(x))^q x^{q(2-\alpha)-1}\,dx \right)^{\frac{1}{q}}$$

$$\leq C \left(\sum_{j=1}^\infty \frac{\|f(\cdot)-f(\cdot+j)\|_p^q}{j^{q\alpha+1}} \right)^{\frac{1}{q}} < \infty.$$

Therefore, we have proved that $f \in \Lambda_H^{\alpha,p,q}$, $1 \leq p \leq \infty$ and $1 \leq q < \infty$.

Suppose now that $f \in C^{\alpha,p,\infty}(\mathbb{Z})$, with $1 \leq p \leq \infty$. From equation (3.1), it follows that

$$\left\| t^{1-\frac{\alpha}{2}} \partial_t e^{t\Delta_d} f \right\|_{p,\infty} = \sup_{t>0} t^{1-\frac{\alpha}{2}} \left\| \partial_t e^{t\Delta_d} f \right\|_p \leq \sup_{t>0} t^{1-\frac{\alpha}{2}} \sum_{j\in\mathbb{Z}} |\partial_t G(t,j)| \, \|f(\cdot)-f(\cdot+j)\|_p$$

$$\leq \left(\sup_{j\neq 0} \frac{\|f(\cdot)-f(\cdot+j)\|_p}{|j|^\alpha} \right) \sup_{t>0} t^{1-\frac{\alpha}{2}} \left\| \partial_t G(t,\cdot)\,|\cdot|^\alpha \right\|_1.$$

By using the fact that $\|\partial_t G(t,\cdot)\,|\cdot|^\alpha\|_1 \leq Ct^{\frac{\alpha}{2}-1}$ (see [1, Proof of Theorem 3.3]) we conclude that $\left\| t^{1-\frac{\alpha}{2}} \partial_t e^{t\Delta_d} f \right\|_{p,\infty} < \infty$, so $f \in \Lambda_H^{\alpha,p,\infty}$. $\square$

Proposition 3.2. *Let $\alpha > 0$ and $f : \mathbb{Z} \to \mathbb{R}$ such that $\sum_{j\in\mathbb{Z}} \frac{|f(j)|}{1+|j|^2} < \infty$. If $f \in \Lambda_H^{\alpha,p,q}$, $1 \le p, q \le \infty$, then $f \in \Lambda_P^{\alpha,p,q}$.*

Proof. Let $f \in \Lambda_H^{\alpha,p,q}$, $1 \le p, q \le \infty$. By using Lemmata 2.6 and 2.7 (see also Remark 2.8), it is enough to prove that $\left\| y^{l-\alpha}\partial_y^l e^{-y\sqrt{-\Delta_d}} f \right\|_{p,q} < \infty$ for l the least even number such that $l > [\alpha]+1$ and $l/2 > [\alpha/2]+1$.

Let l be the even number satisfying the conditions above and suppose that $1 \le p \le \infty$ and $1 \le q < \infty$. From Lemma 2.10 we have that

$$\left\|\partial_y^l e^{-y\sqrt{-\Delta_d}} f\right\|_p \le \int_0^\infty \frac{1}{2\sqrt{\pi}} \frac{y e^{\frac{-y^2}{4t}}}{t^{\frac{3}{2}}} \left\|\partial_t^{l/2} e^{t\Delta_d} f\right\|_p dt,$$

and

$$\left\|y^{l-\alpha}\partial_y^l e^{-y\sqrt{-\Delta_d}} f\right\|_{p,q} \le C_\alpha \left(\int_0^\infty \left(y^{l-\alpha}\int_0^\infty \frac{1}{2\sqrt{\pi}} \frac{y e^{\frac{-y^2}{4t}}}{t^{\frac{3}{2}}} \left\|\partial_t^{l/2} e^{t\Delta_d} f\right\|_p dt\right)^q \frac{dy}{y}\right)^{\frac{1}{q}},$$

where C_α is a positive constant depending on α. Notice that for every $\gamma \ge 0$ there is $C > 0$ such that $\left(\frac{y}{t^{1/2}}\right)^\gamma e^{\frac{-y^2}{4t}} \le C$, for every $t, y > 0$. So

$$\begin{aligned}\left\|y^{l-\alpha}\partial_y^l e^{-y\sqrt{-\Delta_d}} f\right\|_{p,q} \le C_\alpha &\left[\left(\int_0^\infty \left(y^{l-2-\alpha}\int_0^{y^2} \left\|\partial_t^{l/2} e^{t\Delta_d} f\right\|_p dt\right)^q \frac{dy}{y}\right)^{\frac{1}{q}}\right. \\ &\left. + \left(\int_0^\infty \left(y^{l-\alpha}\int_{y^2}^\infty \frac{\left\|\partial_t^{l/2} e^{t\Delta_d} f\right\|_p}{t} dt\right)^q \frac{dy}{y}\right)^{\frac{1}{q}}\right].\end{aligned}$$

For both integrals we perform the change of variables ($y = \sqrt{s}$) and we use Hardy's inequality (Lemma 5.1), which yields to

$$\left\|y^{l-\alpha}\partial_y^l e^{-y\sqrt{-\Delta_d}} f\right\|_{p,q} \le C_\alpha \left\|t^{l/2-\frac{\alpha}{2}}\partial_t^{l/2} e^{t\Delta_d} f\right\|_{p,q} < \infty.$$

The case $q = \infty$ follows the same steps. $\square$

Proposition 3.3. *Let $0 < \alpha < 1$, $1 \le p, q \le \infty$ and $f \in \Lambda_P^{\alpha,p,q}$. Then, $f \in C^{\alpha,p,q}(\mathbb{Z})$.*

Proof. Suppose that $1 \le q < \infty$. Then, we can write

$$\sum_{j\neq 0} \frac{\|f(\cdot+j) - f(\cdot)\|_p^q}{|j|^{\alpha q+1}} = 2\sum_{j=1}^\infty \frac{\|f(\cdot+j) - f(\cdot)\|_p^q}{j^{\alpha q+1}} = 2\int_1^\infty \frac{\|f(\cdot+[t]) - f(\cdot)\|_p^q}{[t]^{\alpha q+1}} dt.$$

Since for $t \geq 1$ it holds that $[t] \geq \frac{t}{2}$, then

$$\sum_{j \neq 0} \frac{\|f(\cdot + j) - f(\cdot)\|_p^q}{|j|^{\alpha q+1}} \leq C_{\alpha,q} \int_0^\infty (t^{-\alpha} \omega(t,p))^q \frac{dt}{t},$$

where $\omega(t,p) := \sup_{\substack{0 \leq j \leq t \\ j \in \mathbb{N}_0}} \|f(\cdot + j) - f(\cdot)\|_p$.

Moreover, for every $j \leq t$, we have that

$$\begin{aligned}\|f(\cdot + j) - f(\cdot)\|_p &\leq 2 \left\| f - e^{-t\sqrt{-\Delta_d}} f \right\|_p + \left\| e^{-t\sqrt{-\Delta_d}} f(\cdot) - e^{-t\sqrt{-\Delta_d}} f(\cdot + j) \right\|_p \\ &\leq 2 \int_0^t \left\| \partial_y e^{-y\sqrt{-\Delta_d}} f \right\|_p dy + t \left\| \delta_{right} e^{-t\sqrt{-\Delta_d}} f \right\|_p,\end{aligned}$$

where we have used the fact that f satisfying $\sum_{j \in \mathbb{Z}} \frac{|f(j)|}{1+|j|^2} < \infty$ implies that $\lim_{y \to 0} e^{-y\sqrt{-\Delta_d}} f(n) = f(n)$, for $n \in \mathbb{Z}$, (see [1, Lemma 2.12.B]) and either Minkowski's integral inequality (when $1 \leq p < \infty$) or a straightforward inequality (when $p = \infty$). Thus,

$$\begin{aligned}\left(\int_0^\infty \left(t^{-\alpha} \omega(t,p) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} &\leq 2 \left(\int_0^\infty \left(t^{-\alpha} \int_0^t \left\| \partial_y e^{-y\sqrt{-\Delta_d}} f \right\|_p dy \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \\ &\quad + \left(\int_0^\infty \left(t^{1-\alpha} \left\| \delta_{right} e^{-t\sqrt{-\Delta_d}} f \right\|_p \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \\ &=: I + II.\end{aligned}$$

On the one hand, by using Hardy's inequality (Lemma 5.1), we obtain that

$$I \leq C \left(\int_0^\infty \left(y^{1-\alpha} \left\| \partial_y e^{-y\sqrt{-\Delta_d}} f \right\|_p \right)^q \frac{dy}{y} \right)^{\frac{1}{q}} < \infty.$$

On the other hand, by using Lemma 2.9(ii),

$$II \leq C \left\| y^{1-\alpha} \partial_y e^{-t\sqrt{-\Delta_d}} f \right\|_{p,q} < \infty.$$

Therefore, the result is proved for $1 \leq q < \infty$. The proof for $q = \infty$ follows similarly. $\square$

Theorem 3.4. *Let $0 < \alpha < 1$ and $1 \leq p, q \leq \infty$. It holds that $C^{\alpha,p,q}(\mathbb{Z}) = \Lambda_H^{\alpha,p,q} = \Lambda_P^{\alpha,p,q}$.*

Proof. From Proposition 3.1 and Proposition 3.3 we know that $C^{\alpha,p,q}(\mathbb{Z}) \subset \Lambda_H^{\alpha,p,q}$ and $\Lambda_P^{\alpha,p,q} \subset C^{\alpha,p,q}(\mathbb{Z})$. It remains to prove that $\Lambda_H^{\alpha,p,q} \subset \Lambda_P^{\alpha,p,q}$.

In virtue of Proposition 3.2, it suffices to check that if $\frac{f}{1+|\cdot|^\alpha} \in \ell^q(\mathbb{Z},\mu)$ then $\sum_{j\in\mathbb{Z}} \frac{|f(j)|}{1+|j|^2} < \infty$. Indeed, if $\frac{f}{1+|\cdot|^\alpha} \in \ell^1(\mathbb{Z},\mu)$ we have that

$$\begin{aligned}\sum_{j\in\mathbb{Z}} \frac{|f(j)|}{1+|j|^2} &= |f(0)| + \sum_{j\neq 0} \frac{|f(j)|}{1+|j|^\alpha}\frac{1}{|j|}\left(\frac{(1+|j|^\alpha)\,|j|}{1+|j|^2}\right)\\ &\le |f(0)| + C\sum_{j\neq 0}\frac{|f(j)|}{1+|j|^\alpha}\frac{1}{|j|} < \infty.\end{aligned}$$

Now suppose that $\frac{f}{1+|\cdot|^\alpha} \in \ell^q(\mathbb{Z},\mu)$, with $1 < q < \infty$. By taking $\gamma = \frac{\alpha q+1}{2q} \in \left(\frac{1}{2},1\right)$, we have that

$$\begin{aligned}\sum_{j\in\mathbb{Z}} \frac{|f(j)|}{1+|j|^2} &= \sum_{j\in\mathbb{Z}} \frac{|f(j)|}{(1+|j|^2)^\gamma}\frac{1}{(1+|j|^2)^{1-\gamma}}\\ &\le \left(\sum_{j\in\mathbb{Z}} \frac{|f(j)|^q}{(1+|j|^2)^{\gamma q}}\right)^{\frac{1}{q}} \left(\sum_{j\in\mathbb{Z}} \frac{1}{(1+|j|^2)^{(1-\gamma)q'}}\right)^{\frac{1}{q'}} =: A\cdot B,\end{aligned}$$

where q' is the conjugate exponent of q. Observe that A is clearly finite and B is also finite because $2(1-\gamma)q' > 1$. Lastly, we know that if $\frac{f}{1+|\cdot|^\alpha} \in \ell^\infty(\mathbb{Z})$, then it follows that $\sum_{j\in\mathbb{Z}} \frac{|f(j)|}{1+|j|^2} < \infty$. $\square$

3.2. *Case* $0 < \alpha < 2$

Proposition 3.5. *Let $0 < \alpha < 2$ and $1 \le p, q \le \infty$. If $f \in \Lambda_P^{\alpha,p,q}$, then*

$$\sum_{j\neq 0}\left\|\frac{f(\cdot-j)-2f(\cdot)+f(\cdot+j)}{|j|^\alpha}\right\|_p^q \frac{1}{|j|} < \infty, \quad \text{if } 1 \le q < \infty,$$

and

$$\sup_{j\neq 0}\left\|\frac{f(\cdot-j)-2f(\cdot)+f(\cdot+j)}{|j|^\alpha}\right\|_p < \infty, \quad \text{if } q = \infty.$$

Proof. Let $f \in \Lambda_P^{\alpha,p,q}$. If $0 < \alpha < 1$ and $1 \le q < \infty$, from Proposition 3.3 we have that $f \in C^{\alpha,p,q}(\mathbb{Z})$, so

$$\left(\sum_{j\neq 0}\left\|\frac{f(\cdot-j)-2f(\cdot)+f(\cdot+j)}{|j|^\alpha}\right\|_p^q \frac{1}{|j|}\right)^q \le 2\left(\sum_{j\neq 0}\left\|\frac{f(\cdot+j)-f(\cdot)}{|j|^\alpha}\right\|_p^q \frac{1}{|j|}\right)^q < \infty.$$

The case $0 < \alpha < 1$ and $q = \infty$ is analogous.

Assume that $1 \le \alpha < 2$ and $1 \le q < \infty$. We can rewrite the sum similarly as we did in the proof of Proposition 3.3 to get

$$\left(\sum_{j\neq 0}\left\| \frac{f(\cdot - j) - 2f(\cdot) + f(\cdot + j)}{|j|^\alpha}\right\|_p^q \frac{1}{|j|}\right)^q \le C_{\alpha,q}\left(\int_0^\infty (t^{-\alpha}\omega(t,p))^q \frac{dt}{t}\right)^{\frac{1}{q}},$$

where $\omega(t,p) := \sup_{\substack{0\le j\le t\\ j\in\mathbb{N}_0}} \|f(\cdot + j) - 2f(\cdot) + f(\cdot - j)\|_p$. Let $j \in \mathbb{N}$ such that $j \le t$. We can write

$$\begin{aligned}
&\|f(\cdot + j) - 2f(\cdot) + f(\cdot - j)\|_p \\
&\le \left\| f(\cdot + j) - e^{-t\sqrt{-\Delta_d}} f(\cdot + j) - 2(f(\cdot) - e^{-t\sqrt{-\Delta_d}} f(\cdot)) + f(\cdot - j) - e^{-t\sqrt{-\Delta_d}} f(\cdot - j)\right\|_p \\
&\quad + \left\| e^{-t\sqrt{-\Delta_d}} (f(\cdot + j) - 2f(\cdot) + f(\cdot - j))\right\|_p = I + II.
\end{aligned}$$

By using Minkowski's integral inequality, [1, Lemma 2.12.B] and the fact that $\partial_u e^{-u\sqrt{-\Delta_d}} f = -\int_u^t \partial_w^2 e^{-w\sqrt{-\Delta_d}} f\, dw + \partial_t e^{-t\sqrt{-\Delta_d}} f$, we get that

$$\begin{aligned}
I &\le \int_0^t \left\| \partial_u e^{-u\sqrt{-\Delta_d}} (f(\cdot + j) - 2f(\cdot) + f(\cdot - j))\right\|_p du \\
&\le \int_0^t \int_u^t \left\| \partial_w^2 e^{-w\sqrt{-\Delta_d}} (f(\cdot + j) - 2f(\cdot) + f(\cdot - j))\right\|_p dw\, du \\
&\quad + \int_0^t \left\| \partial_t e^{-t\sqrt{-\Delta_d}} (f(\cdot + j) - 2f(\cdot) + f(\cdot - j))\right\|_p du, \\
&= I_1 + I_2.
\end{aligned}$$

On the one hand, we have that

$$\begin{aligned}
I_1 &\le 4\int_0^t \int_u^t \left\| \partial_w^2 e^{-w\sqrt{-\Delta_d}} f\right\|_p dw\, du = 4\int_0^t \int_0^w \left\| \partial_w^2 e^{-w\sqrt{-\Delta_d}} f\right\|_p du\, dw \\
&= 4\int_0^t w \left\| \partial_w^2 e^{-w\sqrt{-\Delta_d}} f\right\|_p dw.
\end{aligned}$$

On the other hand, we can write

$$
\begin{aligned}
I_2 &\le 2t \left\| \partial_t e^{-t\sqrt{-\Delta_d}} (f(\cdot + j) - f(\cdot)) \right\|_p \\
&= 2t \left\| \sum_{j'=1}^{j} \partial_t e^{-t\sqrt{-\Delta_d}} \big(f(\cdot + j') - f(\cdot + j' - 1)\big) \right\|_p \\
&\le 2t^2 \left\| \partial_t \delta_{right} e^{-t\sqrt{-\Delta_d}} f \right\|_p .
\end{aligned}
$$

Moreover, since $j \le t$,

$$
\begin{aligned}
II &= \left\| \sum_{j'=1}^{j} \delta_{right} e^{-t\sqrt{-\Delta_d}} (f(\cdot - j') - f(\cdot + j' - 1)) \right\|_p \\
&= \left\| \sum_{j'=1}^{j} \sum_{k=0}^{2j'-2} \delta^2_{right} e^{-t\sqrt{-\Delta_d}} f(\cdot - j' + k) \right\|_p \le \sum_{j'=1}^{j} \sum_{k=0}^{2j'-2} \left\| \delta^2_{right} e^{-t\sqrt{-\Delta_d}} f \right\|_p \\
&= j^2 \left\| \delta^2_{right} e^{-t\sqrt{-\Delta_d}} f \right\|_p \le t^2 \left\| \delta^2_{right} e^{-t\sqrt{-\Delta_d}} f \right\|_p .
\end{aligned}
$$

Therefore, we have got the following inequality

$$
\begin{aligned}
\| f(\cdot + j) - 2f(\cdot) + f(\cdot - j)\|_p \le 4 \int_0^t w \left\| \partial_w^2 e^{-w\sqrt{-\Delta_d}} f \right\|_p du + 2t^2 \left\| \partial_t \delta_{right} e^{-t\sqrt{-\Delta_d}} f \right\|_p \\
+ t^2 \left\| \delta^2_{right} e^{-t\sqrt{-\Delta_d}} f \right\|_p ,
\end{aligned}
$$

so

$$
\begin{aligned}
\left(\int_0^\infty \big(t^{-\alpha} \omega(t,p)\big)^q \frac{dt}{t} \right)^{\frac{1}{q}} &\le 4 \left(\int_0^\infty \left(t^{-\alpha} \int_0^t w \left\| \partial_w^2 e^{-w\sqrt{-\Delta_d}} f \right\|_p dw \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \\
&\quad + 2 \left(\int_0^\infty \left(t^{2-\alpha} \left\| \partial_t \delta_{right} e^{-t\sqrt{-\Delta_d}} f \right\|_p \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \\
&\quad + \left(\int_0^\infty \left(t^{2-\alpha} \left\| \delta^2_{right} e^{-t\sqrt{-\Delta_d}} f \right\|_p \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} .
\end{aligned}
$$

By applying Hardy's inequality (Lemma 5.1) in the first summand of above expression, and Lemma 2.9(ii) in the second and third ones, we get the result.

The case $1 \le \alpha < 2$ and $q = \infty$ can be proved by following the same ideas. □

Proposition 3.6. *Let* $0 < \alpha < 2$, $1 \le p, q \le \infty$ *and* $f : \mathbb{Z} \to \mathbb{R}$ *such that* $\frac{f}{1+|\cdot|^\alpha} \in \ell^q(\mathbb{Z}, \mu)$ *with*

$$\sum_{j \neq 0} \left\| \frac{f(\cdot - j) - 2f(\cdot) + f(\cdot + j)}{|j|^\alpha} \right\|_p^q \frac{1}{|j|} < \infty, \quad \text{if } 1 \le q < \infty,$$

and

$$\sup_{j \neq 0} \left\| \frac{f(\cdot - j) - 2f(\cdot) + f(\cdot + j)}{|j|^\alpha} \right\|_p < \infty, \quad \text{if } q = \infty.$$

Then $f \in \Lambda_H^{\alpha,p,q}$.

Proof. Since for every $t > 0$ it holds that $G(t, j) = G(t, -j)$, $j \in \mathbb{N}$, and $\partial_t e^{t\Delta_d} 1 = 0$ (where 1 denotes in this case the sequence with all entries equal to 1), we have that

$$\begin{aligned} \left\| \partial_t e^{t\Delta_d} f \right\|_p &= \left\| \frac{1}{2} \sum_{j \in \mathbb{Z}} \partial_t G(t, j)(f(n - j) - 2f(n) + f(n + j)) \right\|_p \\ &\le \sum_{j \ge 0} \left| \delta_{right}^2 G(t, j) \right| \Omega_p(j + 1), \end{aligned}$$

with $\Omega_p(j) = \| f(\cdot - j) - 2f(\cdot) + f(\cdot + j)\|_p$.

The rest of the proof follows from the same techniques used to prove Proposition 3.1. □

From Propositions 3.2, 3.5 and 3.6, we derive the following theorem, which is one of our main results: the complete characterization of the discrete Besov and Zygmund spaces for $0 < \alpha < 2$.

Theorem 3.7. *Let* $0 < \alpha < 2$, $1 \le p, q \le \infty$ *and* $f : \mathbb{Z} \to \mathbb{R}$ *be a function such that* $\frac{f}{1+|\cdot|^\alpha} \in \ell^q(\mathbb{Z}, \mu)$ *and* $\sum_{j \in \mathbb{Z}} \frac{|f(j)|}{1+|j|^2} < \infty$. *The following are equivalent:*

(1) $f \in \Lambda_H^{\alpha,p,q}$.
(2) $f \in \Lambda_P^{\alpha,p,q}$.
(3) f *satisfies*

$$\sum_{j \neq 0} \left\| \frac{f(\cdot - j) - 2f(\cdot) + f(\cdot + j)}{|j|^\alpha} \right\|_p^q \frac{1}{|j|} < \infty, \quad \text{if } 1 \le q < \infty,$$

and

$$\sup_{j \neq 0} \left\| \frac{f(\cdot - j) - 2f(\cdot) + f(\cdot + j)}{|j|^\alpha} \right\|_p < \infty, \quad \text{if } q = \infty.$$

Remark 3.8. Observe that the assumption $\sum_{j \in \mathbb{Z}} \frac{|f(j)|}{1+|j|^2} < \infty$ in the previous result is only needed for $1 \le \alpha < 2$. Indeed, we have proved the previous result for $0 < \alpha < 1$ in Theorem 3.4

without that assumption (in this case it is deduced from the hypothesis $\frac{f}{1+|\cdot|^\alpha} \in \ell^q(\mathbb{Z},\mu)$). In addition, by proceeding in an analogous way as in the proof of Proposition 3.5 (but performing the change of variables $\tilde{t}=\sqrt{t}$ in $\left(\int_0^\infty (t^{-\alpha}\omega(t,p))^q \frac{dt}{t}\right)^{\frac{1}{q}}$) we can prove that $f \in \Lambda_H^{\alpha,p,q} \implies f$ satisfies *(3)*, so it can be proved that *(1)* and *(3)* are equivalent for $0<\alpha<2$, without imposing the assumption $\sum_{j\in\mathbb{Z}} \frac{|f(j)|}{1+|j|^2} < \infty$.

3.3. The general case

Theorem 3.9. *Let $\alpha > 1$, $1 \le p,q \le \infty$ and $f:\mathbb{Z}\to\mathbb{R}$. Then, $f \in \Lambda_H^{\alpha,p,q}$ if, and only if $\delta_{right} f \in \Lambda_H^{\alpha-1,p,q}$.*

Proof. Let $f \in \Lambda_H^{\alpha,p,q}$. First, we shall prove that $\frac{\delta_{right} f}{1+|\cdot|^{\alpha-1}} \in \ell^q(\mathbb{Z},\mu)$. Observe that when $q=\infty$, in virtue of the embedding $\ell^p(\mathbb{Z}) \hookrightarrow \ell^\infty(\mathbb{Z})$, we have that $f \in \Lambda_H^{\alpha,\infty,\infty}$, so $\frac{\delta_{right} f}{1+|\cdot|^{\alpha-1}} \in \ell^\infty(\mathbb{Z})$ (see the proof of [1, Theorem 3.6]).

Let now $1 \le q < \infty$. We want to prove that $\left\| \frac{\delta_{right} f}{1+|\cdot|^{\alpha-1}} \right\|_{q,\mu} < \infty$. Since $\frac{f}{1+|\cdot|^\alpha} \in \ell^q(\mathbb{Z},\mu)$, then $\frac{f}{1+|\cdot|^{\alpha+\frac{1}{q}}} \in \ell^\infty(\mathbb{Z})$, so by Remark 2.3 and [1, Lemma 2.12.A.(iii)] we have $\lim_{t\to 0} e^{t\Delta_d} f(n) = f(n)$, for every $n \in \mathbb{Z}$. Then,

$$\begin{aligned}
\left\| \frac{\delta_{right} f}{1+|\cdot|^{\alpha-1}} \right\|_{q,\mu} \le & |\delta_{right} f(0)| + \left(\sum_{n\neq 0} \left(\frac{\left|\delta_{right} f(n)\right|}{1+|n|^{\alpha-1}} \right)^q \frac{1}{|n|} \right)^{\frac{1}{q}} \\
\le & |\delta_{right} f(0)| + \left(\sum_{n\neq 0} \left(\sup_{0<t<n^2} \frac{\left|e^{t\Delta_d}\delta_{right} f(n)\right|}{1+|n|^{\alpha-1}} \right)^q \frac{1}{|n|} \right)^{\frac{1}{q}} \\
\le & |\delta_{right} f(0)| + \left(\sum_{n\neq 0} \left(\sup_{0<t<n^2} \frac{\left|e^{t\Delta_d}\delta_{right} f(n) - e^{n^2\Delta_d}\delta_{right} f(n)\right|}{1+|n|^{\alpha-1}} \right)^q \frac{1}{|n|} \right)^{\frac{1}{q}} \\
& + \left(\sum_{n\neq 0} \left(\frac{\left|e^{n^2\Delta_d}\delta_{right} f(n)\right|}{1+|n|^{\alpha-1}} \right)^q \frac{1}{|n|} \right)^{\frac{1}{q}} \\
& =: |\delta_{right} f(0)| + A + B.
\end{aligned}$$

Since $\frac{f}{1+|\cdot|^\alpha} \in \ell^q(\mathbb{Z},\mu)$, from Lemma 2.5 we deduce that B is finite.

Now we study the sum A. Suppose that $\alpha \in (1,3)$. Then,

$$A \le \left(\sum_{n\neq 0} \left(\sup_{0<t<n^2} \int_t^{n^2} \left|\partial_u \delta_{right} e^{u\Delta_d} f(n)\right| du \right)^q |n|^{-q(\alpha-1)-1} \right)^{\frac{1}{q}}$$

$$\leq \left(\sum_{n\neq 0} \left(\int_0^{n^2} \left| \partial_u \delta_{right} e^{u\Delta_d} f(n) \right| du \right)^q |n|^{-q(\alpha-1)-1} \right)^{\frac{1}{q}}$$

$$\leq \left(\sum_{n\neq 0} \left(\int_0^{n^2} \left\| \partial_u \delta_{right} e^{u\Delta_d} f \right\|_p du \right)^q |n|^{-q(\alpha-1)-1} \right)^{\frac{1}{q}}$$

$$\leq C \left(\int_1^{\infty} \left(\int_0^{[t]^2} \left\| \partial_u \delta_{right} e^{u\Delta_d} f \right\|_p du \right)^q [t]^{-q(\alpha-1)-1} dt \right)^{\frac{1}{q}}.$$

Since $t > 1$, we have that $[t] > \frac{t}{2}$ and

$$A \leq C \left(\int_1^{\infty} \left(\int_0^{t^2} \left\| \partial_u \delta_{right} e^{u\Delta_d} f \right\|_p du \right)^q t^{-q(\alpha-1)-1} dt \right)^{\frac{1}{q}}$$

$$= C \left(\int_1^{\infty} \left(\int_0^{x} \left\| \partial_u \delta_{right} e^{u\Delta_d} f \right\|_p du \right)^q x^{\frac{-q(\alpha-1)}{2}-1} dx \right)^{\frac{1}{q}}.$$

By using Hardy's inequality (see Lemma 5.1) we get that

$$A \leq C \left(\int_0^{\infty} \left(u \left\| \partial_u \delta_{right} e^{u\Delta_d} f \right\|_p \right)^q u^{\frac{-q(\alpha-1)}{2}-1} \right)^{\frac{1}{q}} = C \left\| u^{\frac{3}{2}-\frac{\alpha}{2}} \partial_u \delta_{right} e^{u\Delta_d} f \right\|_{p,q}.$$

Thus, we use Lemma 2.9 and Remark 2.8 to obtain

$$A \leq C \left\| u^{\frac{3}{2}-\frac{\alpha}{2}} \partial_u \delta_{right} e^{u\Delta_d} f \right\|_{p,q} \leq C \left\| u^{2-\frac{\alpha}{2}} \partial_u^2 e^{u\Delta_d} f \right\|_{p,q} < \infty.$$

Now consider $\alpha \in [3, 5)$. Observe that the techniques that we will present in this part of the proof are enough to prove all the cases $\alpha \in [2k+1, 2k+3)$ with $k \in \mathbb{N}$, but we just prove the case $\alpha \in [3, 5)$. We start by using the fact that the semigroup is the solution to the heat equation, and splitting our sum into 2 different parts,

$$A \leq \left(\sum_{n\neq 0} \left(\int_0^{n^2} \left| e^{u\Delta_d} \delta_{right}^3 f(n-1) \right| du \right)^q |n|^{-q(\alpha-1)-1} \right)^{\frac{1}{q}}$$

$$\leq \left(\sum_{n\neq 0} \left(\int_0^{n^2} \int_u^{n^2} \left| \partial_w e^{w\Delta_d} \delta^3_{right} f(n-1) \right| dw\, du \right)^q |n|^{-q(\alpha-1)-1} \right)^{\frac{1}{q}}$$

$$+ \left(\sum_{n\neq 0} \left(\frac{\left| e^{n^2\Delta_d} \delta^3_{right} f(n-1) \right|}{|n|^{\alpha-3}} \right)^q \frac{1}{|n|} \right)^{\frac{1}{q}}$$

$$=: A_1 + A_2.$$

Since $\frac{f}{1+|\cdot|^\alpha} \in \ell^q(\mathbb{Z}, \mu)$ we have that $\frac{f(\cdot-1)}{1+|\cdot|^\alpha} \ell^q(\mathbb{Z}, \mu)$, so by Lemma 2.5 we get that A_2 is finite. For A_1 we interchange the integrals, apply Hardy's inequality and the techniques for the case $\alpha \in (1, 3)$, to get

$$A_1 \leq C \left(\sum_{n\neq 0} \left(\int_0^{n^2} \int_u^{n^2} \left\| \partial^2_w \delta_{right} e^{w\Delta_d} f \right\|_p dw\, du \right)^q |n|^{-q(\alpha-1)-1} \right)^{\frac{1}{q}}$$

$$\leq C \left(\int_1^\infty \left(\int_0^{t^2} w \left\| \partial^2_w \delta_{right} e^{w\Delta_d} f \right\|_p dw \right)^q t^{-q(\alpha-1)-1} dt \right)^{\frac{1}{q}}$$

$$= C \left(\int_1^\infty \left(\int_0^{x} w \left\| \partial^2_w \delta_{right} e^{w\Delta_d} f \right\|_p dw \right)^q x^{-\frac{q(\alpha-1)}{2}-1} dx \right)^{\frac{1}{q}}$$

$$\leq C \left(\int_0^\infty \left(w^{\frac{5}{2}-\frac{\alpha}{2}} \left\| \partial^2_w \delta_{right} e^{w\Delta_d} f \right\|_p \right)^q dw \right)^{\frac{1}{q}}$$

$$= C \left\| w^{\frac{5}{2}-\frac{\alpha}{2}} \partial^2_w \delta_{right} e^{w\Delta_d} f \right\|_{p,q}.$$

From Lemma 2.9 and Remark 2.8 (in case $\alpha \in [3, 4)$) we deduce that

$$A_1 \leq C \left\| w^{3-\frac{\alpha}{2}} \partial^3_w e^{w\Delta_d} f \right\|_{p,q} < \infty.$$

Finally, we prove the semigroup condition. First we shall consider $\alpha \in (1, 3)$. By using Lemma 2.9(i) and Remark 2.8 (in case $\alpha \in (1, 2)$) we get that

$$\left\| t^{1-\frac{\alpha-1}{2}} \partial_t e^{t\Delta_d} \delta_{right} f \right\|_{p,q} = \left\| t^{\frac{3}{2}-\frac{\alpha}{2}} \partial_t \delta_{right} e^{t\Delta_d} f \right\|_{p,q} \leq C \left\| t^{2-\frac{\alpha}{2}} \partial^2_t e^{t\Delta_d} f \right\|_{p,q} < \infty.$$

In general, if $\alpha \in [2k+1, 2k+3)$ with $k \in \mathbb{N}$, by using Lemma 2.9(i) and Remark 2.8 (in case $\alpha \in [2k+1, 2k+2)$) we get that

$$\left\| t^{k+1-\frac{\alpha-1}{2}} \partial_t^{k+1} e^{t\Delta_d} \delta_{right} f \right\|_{p,q} = \left\| t^{k+\frac{3}{2}-\frac{\alpha}{2}} \partial_t^{k+1} \delta_{right} e^{t\Delta_d} f \right\|_{p,q}$$
$$\leq C \left\| t^{k+2-\frac{\alpha}{2}} \partial_t^{k+2} e^{t\Delta_d} f \right\|_{p,q} < \infty.$$

Assume now that $\delta_{right} f \in \Lambda_H^{\alpha-1,p,q}$. By definition, we have that $\frac{\delta_{right} f}{1+|\cdot|^{\alpha-1}} \in \ell^q(\mathbb{Z}, \mu)$. Thus, the proof of Lemma 2.2 gives that

$$\left\| \frac{f}{1+|\cdot|^{\alpha}} \right\|_{q,\mu} \leq |f(0)| + C + C \left(\sum_{j=1}^{\infty} \left(\left| \delta_{right} f(j-1) \right| \right)^q \frac{1}{j^{(\alpha-1)q+1}} \right)^{\frac{1}{q}}$$
$$+ C \left(\sum_{j=1}^{\infty} \left(\left| \delta_{right} f(-j) \right| \right)^q \frac{1}{j^{(\alpha-1)q+1}} \right)^{\frac{1}{q}} < \infty,$$

so $\frac{f}{1+|\cdot|^{\alpha}} \in \ell^q(\mathbb{Z}, \mu)$. Suppose that $\alpha \in [2k+1, 2k+3)$ with $k \in \mathbb{N}_0$ (being $\alpha \neq 1$). By using Lemma 2.9(i) and Remark 2.8 (in case $\alpha \in [2k+1, 2k+2)$) we have that

$$\left\| t^{k+2-\frac{\alpha}{2}} \partial_t^{k+2} e^{t\Delta_d} f \right\|_{p,q} = \left\| t^{k+\frac{3}{2}-\frac{\alpha}{2}} \partial_t^{k+1} \delta_{right} e^{t\Delta_d} f \right\|_{p,q}$$
$$\leq C \left\| t^{k+1-\frac{\alpha-1}{2}} \partial_t^{k+1} e^{t\Delta_d} \delta_{right} f \right\|_{p,q} < \infty.$$

We conclude that $f \in \Lambda_H^{\alpha,p,q}$. $\square$

Theorem 3.10. *Let $\alpha > 1$, $1 \leq p, q \leq \infty$ and $f \in \Lambda_P^{\alpha,p,q}$. Then, $\delta_{right} f \in \Lambda_P^{\alpha-1,p,q}$.*

Proof. Since $f \in \Lambda_P^{\alpha,p,q}$ we have that $\sum_{n\in\mathbb{Z}} \frac{|f(n)|}{1+|n|^2} < \infty$, so it is clear that $\sum_{n\in\mathbb{Z}} \frac{\left|\delta_{right} f(n)\right|}{1+|n|^2} < \infty$.

Let $l_1 = [\alpha] + 1$ and $l_2 = [\alpha - 1] + 1 = [\alpha]$. Then, by using Lemma 2.9(ii) we have that

$$\left\| y^{l_2-(\alpha-1)} \partial_y^{l_2} e^{-y\sqrt{-\Delta_d}} \delta_{right} f \right\|_{p,q} = \left\| y^{l_2-(\alpha-1)} \partial_y^{l_2} \delta_{right} e^{-y\sqrt{-\Delta_d}} f \right\|_{p,q}$$
$$\leq C \left\| y^{l_1-\alpha} \partial_y^{l_1} e^{-y\sqrt{-\Delta_d}} f \right\|_{p,q} < \infty.$$

We conclude that $\delta_{right} f \in \Lambda_P^{\alpha-1,p,q}$. $\square$

Finally, we can prove our main theorem.

Proof of Theorem 1.1. We prove first *(A1)*. In Theorem 3.4 we have proved the result for $0 < \alpha < 1$. Let $k < \alpha < k+1$, for certain $k \in \mathbb{N}$, and assume that $f \in \Lambda_H^{\alpha,p,q}$. Then, by applying k times Theorem 3.9 we get that $\delta_{right}^k f \in \Lambda_H^{\alpha-k,p,q}$ and in virtue of Theorem 3.4 and the definition of $C^{\alpha-k,p,q}$ we get that $f \in C^{\alpha,p,q}(\mathbb{Z})$.

Conversely, suppose that $f \in C^{\alpha,p,q}(\mathbb{Z})$, $\alpha > 1$, $\alpha \notin \mathbb{N}$. Then, from Lemma 2.2 we know that $\frac{f}{1+|\cdot|^\alpha} \in \ell^q(\mathbb{Z}, \mu)$. Moreover, the definition of the space gives that $\delta^k_{right} f \in C^{\alpha-k,p,q}$, and Theorem 3.4 implies that $\delta^k_{right} f \in \Lambda_H^{\alpha-k,p,q}$. By applying k times Theorem 3.9 we conclude that $f \in \Lambda_H^{\alpha,p,q}$.

Regarding the proof of *(A2)*, we proceed as in the proof of *(A1)* but we use Theorem 3.7 (see Remark 2.3) instead of Theorem 3.4.

In virtue of Proposition 3.2 and *(A1)*, to establish *(B)* we only need to prove that if $f \in \Lambda_P^{\alpha,p,q}$ then $f \in C^{\alpha,p,q}(\mathbb{Z})$. Let $f \in \Lambda_P^{\alpha,p,q}$. By applying k times Theorem 3.10 we get that $\delta^k_{right} f \in \Lambda_P^{\alpha-k,p,q}$ and from Theorem 3.4 and the definition of $C^{\alpha-k,p,q}(\mathbb{Z})$ we conclude that $f \in C^{\alpha,p,q}(\mathbb{Z})$.

Regarding the proof of *(B2)*, we proceed as in the proof of *(B1)* but we use Theorem 3.7 instead of Theorem 3.4. $\square$

4. Applications

In this section, we shall prove regularity results for Bessel potentials and fractional powers of the discrete Laplacian in the Besov spaces defined through the heat semigroup and the maximal L^1-regularity theorem.

4.1. *Proofs of Theorems 1.2, 1.3 and 1.4*

Firstly, we recall the definition of the fractional powers of the discrete Laplacian by using the semigroup method, see [13].

Let I denote the identity operator. For good enough functions $f : \mathbb{Z} \to \mathbb{R}$, we define the following operators:

- The Bessel potential of order $\beta > 0$,

$$(I - \Delta_d)^{-\beta/2} f(n) = \frac{1}{\Gamma(\beta/2)} \int_0^\infty e^{-\tau(I-\Delta_d)} f(n) \tau^{\beta/2} \frac{d\tau}{\tau}, \quad n \in \mathbb{Z}.$$

- The positive fractional power of the Laplacian,

$$(-\Delta_d)^\beta f(n) = \frac{1}{c_\beta} \int_0^\infty \left(e^{\tau\Delta_d} - I\right)^{[\beta]+1} f(n) \frac{d\tau}{\tau^{1+\beta}}, \quad n \in \mathbb{Z}, \quad \beta > 0,$$

  where $c_\beta = \int_0^\infty \left(e^{-\tau} - 1\right)^{[\beta]+1} \frac{d\tau}{\tau^{1+\beta}}$.
- The negative fractional power of the Laplacian,

$$(-\Delta_d)^{-\beta} f(n) = \frac{1}{\Gamma(\beta)} \int_0^\infty e^{\tau\Delta_d} f(n) \frac{d\tau}{\tau^{1-\beta}}, \quad n \in \mathbb{Z}, \quad 0 < \beta < \frac{1}{2}.$$

The previous formulae come from the following gamma formulae, see [13],

$$\lambda^{-\beta} = \frac{1}{\Gamma(\beta)} \int_0^\infty e^{-\lambda t} t^{\beta} \frac{dt}{t}, \quad \text{ and } \lambda^{\beta} = \frac{1}{c_\beta} \int_0^\infty \left(e^{-\lambda t} - 1\right)^{[\beta]+1} \frac{dt}{t^{1+\beta}},$$

where $\beta > 0$ and λ is a complex number with $\Re\lambda \geq 0$.

As it was shown in [1, Theorem 1.2], Bessel potentials of order $\beta > 0$ are well defined for $f \in \Lambda_H^{\alpha,\infty,\infty}$, $\alpha > 0$. However, the fractional powers of the Laplacian, $(-\Delta_d)^{\pm\beta}$, are not well defined in general for $\Lambda_H^{\alpha,\infty,\infty}$ functions and an additional condition is needed. In [13], the authors assumed that the functions belong to the space

$$\ell_{\pm\beta} := \left\{ u : \mathbb{Z} \to \mathbb{R} : \sum_{m \in \mathbb{Z}} \frac{|u(m)|}{(1+|m|)^{1\pm 2\beta}} < \infty \right\},$$

in order to define $(-\Delta_d)^{\pm\beta} f$, where $0 < \beta < 1$ in the case of the positive powers and $0 < \beta < 1/2$ for the negative ones. The choice of these spaces is justified since the discrete kernel in the pointwise formula

$$(-\Delta_d)^{\pm\beta} f(n) = \sum_{m \in \mathbb{Z}} K_{\pm\beta}(n-m) f(m), n \in \mathbb{Z}, \tag{4.1}$$

satisfies $K_\beta(m) \sim \frac{1}{|m|^{1+2\beta}}$, whenever $0 < \beta < 1$ and $K_{-\beta}(m) \sim \frac{1}{|m|^{1-2\beta}}$, for $0 < \beta < 1/2$, see [13]. Observe that the negative powers of the Laplacian are only well defined for $0 < \beta < 1/2$, since the integral that defines it is not absolutely convergent for $\beta \geq 1/2$.

In this section, in order to study regularity properties for positive powers larger than 1, we proceed as in [1], by extending the ℓ_β spaces for $\beta > 0$, and working with the following extended kernel,

$$K_\beta(n) := \begin{cases} 0, & |n| - \beta - 1 \in \mathbb{N}_0, \\ \frac{(-1)^{|n|}\Gamma(2\beta+1)}{\Gamma(1+\beta+|n|)\Gamma(1+\beta-|n|)}, & \text{otherwise.} \end{cases}$$

For any $\beta > 0$, the kernel satisfies the same asymptotic estimates, that is, $K_\beta(m) \sim \frac{1}{|m|^{1+2\beta}}$. Note that when $\beta \in \mathbb{N}_0$, then $K_\beta(n) = 0$ for all $|n| \geq \beta + 1$. In fact, in [1, Lemma 4.1] it was proved that if $f \in \ell_\beta$ then $(-\Delta_d)^\beta f$ is well defined for $\beta > 0$ and the identity (4.1) holds. The cases $-1/2 < \beta < 0$ and $0 < \beta < 1$ had been proved previously in [13].

Now, we prove our main results of this section.

Proof of Theorem 1.2. Let $f \in \Lambda_H^{\alpha,p,q}$ and $k = \left[\frac{\alpha+\beta}{2}\right] + 1$. Notice that since $f \in \Lambda_H^{\alpha,p,q}$ we have that $\frac{f}{1+|\cdot|^{\alpha+\frac{1}{q}}} \in \ell^\infty(\mathbb{Z})$, so by [1, Lemma 2.12] we have that

$$\left|(I-\Delta_d)^{-\beta/2} f(n)\right| \leq C \int_0^\infty e^{-\tau} \left(1 + |n|^{\alpha+1/q} + \tau^{\frac{\alpha+1/q}{2}}\right) \tau^{\beta/2} \frac{d\tau}{\tau} \leq C(1+|n|^{\alpha+1/q}), \quad n \in \mathbb{Z}.$$

This proves that the Bessel potential is well-defined. Moreover, from Lemma 2.4 *(1)* and Minkowski's integral inequality we have

$$\left\| \frac{(I-\Delta_d)^{-\beta/2} f}{1+|\cdot|^{\alpha}} \right\|_{q,\mu} \leq C \int_0^{\infty} e^{-\tau} \left\| \frac{e^{\tau\Delta_d} f}{1+|\cdot|^{\alpha}} \right\|_{q,\mu} \tau^{\beta/2} \frac{d\tau}{\tau} \leq C \int_0^{\infty} e^{-\tau} (1+\tau^{\frac{\alpha+1/q}{2}}) \tau^{\beta/2} \frac{d\tau}{\tau} < \infty.$$

Since $\left\| \frac{(I-\Delta_d)^{-\beta/2} f}{1+|\cdot|^{\alpha}} \right\|_{q,\mu} < \infty$ implies that $\left\| \frac{(I-\Delta_d)^{-\beta/2} f}{1+|\cdot|^{\alpha+\beta}} \right\|_{q,\mu} < \infty$, the size condition is satisfied.

Finally, we prove the semigroup condition. First note that

$$\begin{aligned}
\left\| \partial_t^k e^{t\Delta_d} (I-\Delta_d)^{-\beta/2} f \right\|_p &= \left(\sum_{n\in\mathbb{Z}} \left| \frac{1}{\Gamma(\beta/2)} \int_0^{\infty} e^{-\tau} \partial_t^k e^{t\Delta_d} (e^{\tau\Delta_d} f)(n) \tau^{\beta/2} \frac{d\tau}{\tau} \right|^p \right)^{\frac{1}{p}} \\
&\leq \frac{1}{\Gamma(\beta/2)} \int_0^{\infty} e^{-\tau} \left\| \partial_w^k e^{w\Delta_d} f \Big|_{w=t+\tau} \right\|_p \tau^{\beta/2} \frac{d\tau}{\tau} \\
&\leq \frac{1}{\Gamma(\beta/2)} \int_t^{\infty} \left\| \partial_u^k e^{u\Delta_d} f \right\|_p (u-t)^{\beta/2} \frac{du}{u-t}, \qquad \text{for } 1 \leq p < \infty,
\end{aligned}$$

and the same inequality holds for $p=\infty$ directly (one can intertwine the operator $\partial_t^k e^{t\Delta_d}$ and the integral in previous estimates as it is shown in [1, Proof of Theorem 1.2]).

Now we use Lemma 5.2 for $1 \leq q < \infty$ and Remark 2.8 to obtain

$$\begin{aligned}
\left\| t^{k-\frac{\alpha+\beta}{2}} \partial_t^k e^{t\Delta_d} (I-\Delta_d)^{-\beta/2} f \right\|_{p,q} &\leq C \left(\int_0^{\infty} \left(t^{k-\frac{\alpha+\beta}{2}} \int_t^{\infty} \left\| \partial_u^k e^{u\Delta_d} f \right\|_p \frac{(u-t)^{\beta/2}}{u-t} du \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \\
&\leq C \left(\int_0^{\infty} \left(t^{k-\frac{\alpha}{2}} \left\| \partial_t^k e^{t\Delta_d} f \right\|_p \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \\
&= C \left\| t^{k-\frac{\alpha}{2}} \partial_t^k e^{t\Delta_d} f \right\|_{p,q} < \infty.
\end{aligned}$$

The case $q=\infty$ follows analogously, that is,

$$\begin{aligned}
\left\| t^{k-\frac{\alpha+\beta}{2}} \partial_t^k e^{t\Delta_d} (I-\Delta_d)^{-\beta/2} f \right\|_{p,\infty} &\leq C \sup_{t>0} t^{k-\frac{\alpha+\beta}{2}} \int_t^{\infty} \left\| \partial_u^k e^{u\Delta_d} f \right\|_p \frac{(u-t)^{\beta/2}}{u-t} du \\
&\leq C \sup_{t>0} t^{k-\frac{\alpha}{2}} \left\| \partial_t^k e^{t\Delta_d} f \right\|_p \int_0^1 (1-y)^{\frac{\beta}{2}-1} y^{k-\frac{\alpha+\beta}{2}-1} dy \\
&\leq C \sup_{t>0} t^{k-\frac{\alpha}{2}} \left\| \partial_t^k e^{t\Delta_d} f \right\|_p = C \left\| t^{k-\frac{\alpha}{2}} \partial_t^k e^{t\Delta_d} f \right\|_{p,\infty} < \infty. \qquad \square
\end{aligned}$$

Proof of Theorem 1.3. Notice that $f \in \ell_{-\beta}$ implies that $(-\Delta_d)^{-\beta} f$ is well defined and (4.1) holds (see [13]). Now we check that $\frac{(-\Delta_d)^{-\beta} f}{1+|\cdot|^{\alpha+2\beta}} \in \ell^q(\mathbb{Z}, \mu)$.

If $q = \infty$, then $f \in \Lambda_H^{\alpha,p,\infty} \subset \Lambda_H^{\alpha,\infty,\infty}$ and therefore $\frac{(-\Delta_d)^{-\beta} f}{1+|\cdot|^{\alpha+2\beta}} \in \ell^\infty(\mathbb{Z})$, see [1].

Let $1 \le q < \infty$. In virtue of (4.1), we have to prove that

$$S := \left(\sum_{n\in\mathbb{Z}} \left| \sum_{j\in\mathbb{Z}} \frac{K_{-\beta}(n-j) f(j)}{1+|n|^{\alpha+2\beta}} \right|^q \frac{1}{1+|n|} \right)^{\frac{1}{q}} < \infty.$$

By using the bounds for $K_{-\beta}$ and the inverse triangle inequality we can split our sum into 3 different parts,

$$\begin{aligned} S &\le C \left(\sum_{n=1}^{\infty} \left(\sum_{j=1}^{\infty} \frac{|f(j)|}{n^{\alpha+2\beta}\left(1+|n-j|^{1-2\beta}\right)} \right)^q \frac{1}{n} \right)^{\frac{1}{q}} + C \left(\sum_{n=1}^{\infty} \left(\frac{|f(0)|}{n^{\alpha+1}} \right)^q \frac{1}{n} \right)^{\frac{1}{q}} \\ &+ C \left(\sum_{n=1}^{\infty} \left(\sum_{j=1}^{\infty} \frac{|f(-j)|}{n^{\alpha+2\beta}\left(1+|n-j|^{1-2\beta}\right)} \right)^q \frac{1}{n} \right)^{\frac{1}{q}} + C \sum_{j\in\mathbb{Z}} \frac{|f(j)|}{1+|j|^{1-2\beta}}. \\ &= S_1 + S_2 + S_3 + S_4. \end{aligned}$$

It is clear that $S_2, S_4 < \infty$. We will estimate S_1 and S_3 together by naming as a_j both $|f(j)|$ and $|f(-j)|$. On the one hand, $\frac{f}{1+|\cdot|^\alpha} \in \ell^q(\mathbb{Z}, \mu)$ implies that $\frac{f}{1+|\cdot|^{\alpha+1/q}} \in \ell^q(\mathbb{Z})$. Thus, by using the $\ell^q(\mathbb{N}_0)$-boundedness of the discrete Cesàro operator, see [3, Theorem 7.2], for $1 < q < \infty$ we get

$$\begin{aligned} \left(\sum_{n=1}^{\infty} \left(\sum_{j=1}^{n-1} \frac{a_j}{n^{\alpha+2\beta}(n-j)^{1-2\beta}} \right)^q \frac{1}{n} \right)^{\frac{1}{q}} &\le \left(\sum_{n=1}^{\infty} \left(\frac{1}{n^{2\beta}} \sum_{j=1}^{n-1} \frac{a_j}{j^{\alpha+1/q}(n-j)^{1-2\beta}} \right)^q \right)^{\frac{1}{q}} \\ &\le C \left\| \frac{a_j}{|\cdot|^{\alpha+1/q}} \right\|_q < \infty. \end{aligned}$$

For the case $q = 1$ we use Tonelli's theorem to obtain

$$\begin{aligned} \sum_{n=1}^{\infty} \sum_{j=1}^{n-1} \frac{a_j}{n^{\alpha+2\beta+1}(n-j)^{1-2\beta}} &= \sum_{j=1}^{\infty} a_j \sum_{n=j+1}^{\infty} \frac{1}{n^{\alpha+2\beta+1}(n-j)^{1-2\beta}} \\ &\le \sum_{j=1}^{\infty} \frac{a_j}{j^{\alpha+2\beta+1}} \sum_{n=j+1}^{2j} \frac{1}{(n-j)^{1-2\beta}} + \sum_{j=1}^{\infty} \frac{a_j}{j^{1-2\beta}} \sum_{n=2j+1}^{\infty} \frac{1}{n^{\alpha+2\beta+1}} \\ &\le \sum_{j=1}^{\infty} \frac{a_j}{j^{\alpha+2\beta+1}} \int_0^j \frac{dx}{x^{1-2\beta}} + C < \infty, \end{aligned}$$

where we have used that $\frac{f}{1+|\cdot|^\alpha} \in \ell^1(\mathbb{Z}, \mu)$ and $f \in \ell_{-\beta}$.

On the other hand, by using the $\ell^q(\mathbb{N}_0)$-boundedness of the discrete adjoint Cesàro operator, see [3, Theorem 7.2], for $1 < q < \infty$, we get

$$
\begin{aligned}
\left(\sum_{n=1}^{\infty}\left(\sum_{j=n}^{2n}\frac{a_j}{n^{\alpha+2\beta}(1+(j-n)^{1-2\beta})}\right)^q\frac{1}{n}\right)^{\frac{1}{q}} &\leq \left(\sum_{n=1}^{\infty}\left(\frac{a_n}{n^{\alpha+2\beta+1/q}}\right)^q\right)^{1/q} \\
&\quad + C\left(\sum_{n=1}^{\infty}\left(\sum_{j=n+1}^{2n}\frac{a_j}{j^{\alpha+1/q}}\frac{1}{j^{2\beta}(j-n)^{1-2\beta}}\right)^q\right)^{\frac{1}{q}} \\
&\leq C\left\|\frac{a_j}{|\cdot|^{\alpha+1/q}}\right\|_q < \infty.
\end{aligned}
$$

When $q = 1$ we use again Tonelli's theorem so that

$$
\begin{aligned}
\sum_{n=1}^{\infty}\sum_{j=n}^{2n}\frac{a_j}{n^{\alpha+2\beta+1}(1+(j-n)^{1-2\beta})} &= \sum_{n=1}^{\infty}\frac{a_n}{n^{\alpha+2\beta+1}} + \sum_{j=2}^{\infty}a_j\sum_{\frac{j}{2}\leq n\leq j-1}\frac{1}{n^{\alpha+2\beta+1}(j-n)^{1-2\beta}} \\
&\leq C + C\sum_{j=2}^{\infty}\frac{a_j}{j^{\alpha+1}}\sum_{1\leq m\leq\frac{j}{2}}\frac{1}{m^{1-2\beta}(j-m)^{2\beta}} \\
&\leq C + C\sum_{j=2}^{\infty}\frac{a_j}{j^{\alpha+2\beta+1}}\sum_{1\leq m\leq\frac{j}{2}}\frac{1}{m^{1-2\beta}} \\
&\leq C + C\sum_{j=2}^{\infty}\frac{a_j}{j^{\alpha+2\beta+1}}\int_0^{\frac{j}{2}}\frac{dx}{x^{1-2\beta}} < \infty,
\end{aligned}
$$

where we have used that $\frac{f}{1+|\cdot|^\alpha} \in \ell^1(\mathbb{Z}, \mu)$.

Finally, observe that if $j \geq 2n+1$ we have that $\frac{1}{j-n} < \frac{2}{j}$, so by using the fact that $a_j \in \ell_{-\beta}$ we get

$$
\left(\sum_{n=1}^{\infty}\left(\sum_{j=2n+1}^{\infty}\frac{a_j}{1+(j-n)^{1-2\beta}}\right)^q\frac{1}{n^{q(\alpha+2\beta)+1}}\right)^{\frac{1}{q}} \leq C\left(\sum_{n=1}^{\infty}\frac{1}{n^{q(\alpha+2\beta)+1}}\right)^{\frac{1}{q}} < \infty.
$$

It remains to check that $\left\|t^{k-\frac{\alpha+2\beta}{2}}\partial_t^k e^{t\Delta_d}(-\Delta_d)^{-\beta}f\right\|_{p,q} < \infty$ with $k = \left[\frac{\alpha+2\beta}{2}\right]+1$. As in the proof of Theorem 1.2, we observe that

$$
\begin{aligned}
\left\| \partial_t^k e^{t\Delta_d} (-\Delta_d)^{-\beta} f \right\|_p &= \left(\sum_{n\in\mathbb{Z}} \left| \frac{1}{\Gamma(\beta)} \int_0^\infty \partial_w^k e^{w\Delta_d} f \Big|_{w=t+\tau} (n) \frac{d\tau}{\tau^{1-\beta}} \right|^p \right)^{\frac{1}{p}} \\
&\leq \frac{1}{\Gamma(\beta)} \int_t^\infty \left\| \partial_t^k e^{u\Delta_d} f \right\|_p \frac{(u-t)^\beta}{(u-t)} du, \quad 1 < q < \infty.
\end{aligned}
$$

By using Lemma 5.2 and Remark 2.8 we have that

$$
\left\| t^{k-\frac{\alpha+2\beta}{2}} \partial_t^k e^{t\Delta_d} (-\Delta_d)^{-\beta} f \right\|_{p,q} \leq C \left\| t^{k-\frac{\alpha}{2}} \partial_t^k e^{t\Delta_d} f \right\|_{p,q} < \infty.
$$

The case $q = \infty$ follows similarly. $\square$

Proof of Theorem 1.4. Notice that if $\beta \in \mathbb{N}$ the integral definition of the fractional Laplacian coincides with the β times composition of the operator (see [1, Remark 4.2]), and as $(-\Delta_d) f(n) = -\delta_{right}^2 f(n-1)$, $n \in \mathbb{Z}$, we can apply Theorem 3.9 to prove epigraph *(2)* and epigraph *(1)* when $\beta \in \mathbb{N}$.

Now consider $\beta \notin \mathbb{N}$ and let $k = \left[\frac{\alpha-2\beta}{2}\right] + 1$. Observe that the study of the size condition follows the same steps as in the last proof but in this case with 2β instead of -2β. It remains to prove that $\left\| t^{k-\frac{\alpha-2\beta}{2}} \partial_t^k e^{t\Delta_d} (-\Delta_d)^{\beta} f \right\|_{p,q} < \infty$. If $1 \leq p < \infty$, by using Minkowski's integral inequality twice, we get

$$
\begin{aligned}
\left\| \partial_t^k e^{t\Delta_d} (-\Delta_d)^{\beta} f \right\|_p &= \left\| \frac{1}{c_\beta} \partial_t^k e^{t\Delta_d} \left(\int_0^\infty \int_{[0,\tau]^l} \partial_\nu^l e^{\nu\Delta_d} f \Big|_{\nu=s_1+\cdots+s_l} d(s_1,\ldots,s_l) \frac{d\tau}{\tau^{1+\beta}} \right) \right\|_p \\
&\leq \frac{1}{c_\beta} \int_0^\infty \int_{[0,\tau]^l} \left\| \partial_\nu^{k+l} e^{\nu\Delta_d} f \Big|_{\nu=t+s_1+\cdots+s_l} \right\|_p d(s_1,\ldots,s_l) \frac{d\tau}{\tau^{1+\beta}},
\end{aligned}
$$

where $l = [\beta] + 1$. In an analogous way the inequality holds for $p = \infty$. Now we are going to focus on the integral over $[0,\tau]^l$. If $\beta \in (0,1)$, we apply Tonelli's theorem to obtain

$$
\begin{aligned}
\frac{1}{c_\beta} \int_0^\infty \int_0^\tau \left\| \partial_\nu^{k+1} e^{\nu\Delta_d} f \Big|_{\nu=t+s_1} \right\|_p ds_1 \frac{d\tau}{\tau^{1+\beta}} &= \frac{1}{c_\beta} \int_0^\infty \left\| \partial_\nu^{k+1} e^{\nu\Delta_d} f \Big|_{\nu=t+s_1} \right\|_p \int_{s_1}^\infty \frac{d\tau}{\tau^{1+\beta}} ds_1 \\
&= \frac{1}{c_\beta} \int_0^\infty \left\| \partial_\nu^{k+1} e^{\nu\Delta_d} f \Big|_{\nu=t+s_1} \right\|_p \frac{s_1^{1-\beta}}{s_1} ds_1 \\
&= \frac{1}{c_\beta} \int_t^\infty \left\| \partial_u^{k+1} e^{u\Delta_d} f \right\|_p \frac{(u-t)^{1-\beta}}{u-t} du.
\end{aligned}
$$

For $\beta > 1$, instead of integrating over $[0,\tau]^l$, we will compute the volume integral under the hyperplane which will be indeed bigger than our original integral, but easier to calculate. In order to do that we introduce some notation. We denote by $s = (s_1, s_2, \dots, s_l) \in \mathbb{R}^l$ and by $s' = (s_2, \dots, s_l) \in \mathbb{R}^{l-1}$ with $s_i \geq 0, \forall i = 1, \dots, l$. We define the set $K_l(\theta)$ as $K_l(\theta) = \{s \in \mathbb{R}^l : 0 \leq s_1 + \dots + s_l \leq \theta\}$. Hence, using Tonelli's theorem we have that

$$\begin{aligned}
\int_{[0,\tau]^l} \left\| \partial_\nu^{k+l} e^{\nu\Delta_d} f \Big|_{\nu=t+s_1+\dots+s_l} \right\|_p ds &\leq \int_{s\in K_l(l\tau)} \left\| \partial_\nu^{k+l} e^{\nu\Delta_d} f \Big|_{\nu=t+s_1+\dots+s_l} \right\|_p ds \\
&= \int_{s'\in K_{l-1}(l\tau)} \int_0^{l\tau-(s_2+\dots+s_l)} \left\| \partial_\nu^{k+l} e^{\nu\Delta_d} f \Big|_{\nu=t+s_1+\dots+s_l} \right\|_p ds_1 ds' \\
&= \int_{s'\in K_{l-1}(l\tau)} \int_{s_2+\dots+s_l}^{l\tau} \left\| \partial_\nu^{k+l} e^{\nu\Delta_d} f \Big|_{\nu=t+u} \right\|_p du ds' \\
&= \int_0^{l\tau} \left\| \partial_\nu^{k+l} e^{\nu\Delta_d} f \Big|_{\nu=t+u} \right\|_p \int_{s'\in K_{l-1}(u)} ds' \, du \\
&\leq C \int_0^{l\tau} \left\| \partial_\nu^{k+l} e^{\nu\Delta_d} f \Big|_{\nu=t+u} \right\|_p u^{l-1} \, du.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left\| \partial_t^k e^{t\Delta_d} (-\Delta_d)^\beta f \right\|_p &\leq C \int_0^\infty \int_0^{l\tau} \left\| \partial_\nu^{k+l} e^{\nu\Delta_d} f \Big|_{\nu=t+u} \right\|_p u^{l-1} \, du \frac{d\tau}{\tau^{1+\beta}} \\
&= C \int_0^\infty \left\| \partial_\nu^{k+l} e^{\nu\Delta_d} f \Big|_{\nu=t+u} \right\|_p \int_{\frac{u}{l}}^\infty \frac{d\tau}{\tau^{1+\beta}} u^{l-1} \, du \\
&= C \int_0^\infty \left\| \partial_\nu^{k+l} e^{\nu\Delta_d} f \Big|_{\nu=t+u} \right\|_p \frac{u^{l-\beta}}{u} du \\
&= C \int_t^\infty \left\| \partial_\nu^{k+l} e^{\nu\Delta_d} f \right\|_p \frac{(\nu - t)^{l-\beta}}{\nu - t} \, d\nu.
\end{aligned}$$

When $1 \leq q < \infty$, we can use Lemma 5.2 and Remark 2.8 to get

$$\left\| t^{k-\frac{\alpha-2\beta}{2}} \partial_t^k e^{t\Delta_d} (-\Delta_d)^\beta f \right\|_{p,q} \leq C \left\| t^{k+l-\frac{\alpha}{2}} \partial_t^{k+l} e^{t\Delta_d} f \right\|_{p,q} < \infty.$$

The case $q = \infty$ follows by using similar ideas to the ones in the proof of Theorem 1.2. $\square$

4.2. Proof of Theorem 1.5

Finally, we demonstrate the maximal L^1-regularity for (1.2).

Proof of Theorem 1.5. We write the proof in the case $T = \infty$. For $T < \infty$ is analogous, with the obvious changes.

Let $u_0 \in \dot{\Lambda}_H^{\alpha,p,1}$, $f \in L^1((0,\infty); \dot{\Lambda}_H^{\alpha,p,1})$ and $k = [\alpha/2]+1$. Now, by using that the semigroup satisfies the discrete heat equation, we can write

$$\begin{aligned}
\|\Delta_d e^{t\Delta_d} u_0\|_{L^1((0,\infty);\dot{\Lambda}_H^{\alpha,p,1})} &= \int_0^\infty \|\nu^{k-\alpha/2}\partial_\nu^k e^{\nu\Delta_d}(\Delta_d e^{t\Delta_d}u_0)\|_{p,1}\,dt \\
&= \int_0^\infty\int_0^\infty \nu^{k-\alpha/2}\|e^{(\nu+t)\Delta_d}\Delta_d^{k+1}u_0\|_p \frac{d\nu}{\nu}\,dt \\
&= \int_0^\infty\int_t^\infty (w-t)^{k-\alpha/2}\|e^{w\Delta_d}\Delta_d^{k+1}u_0\|_p \frac{dw}{w-t}\,dt \\
&= \frac{1}{k-\alpha/2}\int_0^\infty w^{k+1-\alpha/2}\|e^{w\Delta_d}\Delta_d^{k+1}u_0\|_p \frac{dw}{w} \\
&= \frac{1}{k-\alpha/2}\int_0^\infty w^{k+1-\alpha/2}\|\partial_w^{k+1}e^{w\Delta_d}u_0\|_p \frac{dw}{w}.
\end{aligned}$$

Now, by taking into account Remark 2.3, the discrete Young's convolution inequality and [1, Lemma 2.6], we get that

$$\begin{aligned}
\|\partial_w^{k+1}e^{w\Delta_d}u_0\|_p = \|\Delta_d^k\partial_w e^{w\Delta_d}u_0\|_p &= \left\|\sum_{j\in\mathbb{Z}}\partial_{\frac{w}{2}}G\left(\frac{w}{2},j\right)\Delta_d^k e^{\frac{w}{2}\Delta_d}u_0(n-j)\right\|_p \\
&\le \left\|\partial_{\frac{w}{2}}G\left(\frac{w}{2},\cdot\right)\right\|_1\left\|\partial_{\frac{w}{2}}^k e^{\frac{w}{2}\Delta_d}u_0\right\|_p \le \frac{C}{w}\left\|\partial_{\frac{w}{2}}^k e^{\frac{w}{2}\Delta_d}u_0\right\|_p. \qquad (4.2)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|\Delta_d e^{t\Delta_d}u_0\|_{L^1((0,\infty);\dot{\Lambda}_H^{\alpha,p,1})} &\le C\int_0^\infty w^{k-\alpha/2}\|\partial_w^k e^{w\Delta_d}u_0\|_p\frac{dw}{w} \\
&= C\|w^{k-\alpha/2}\partial_w^k e^{w\Delta_d}u_0\|_{p,1} = C\|u_0\|_{\dot{\Lambda}_H^{\alpha,p,1}}.
\end{aligned}$$

On the other hand, by applying Fubini's Theorem we obtain that

$$
\begin{aligned}
\|R(g)\|_{L^1((0,\infty);\dot\Lambda_H^{\alpha,p,1})} &= \int_0^\infty \|v^{k-\alpha/2}\partial_v^k e^{v\Delta_d} R(g)\|_{p,1}\,dt \\
&\le \int_0^\infty\int_0^t \|v^{k-\alpha/2}\partial_v^k e^{v\Delta_d}\,\partial_t e^{(t-s)\Delta_d} g(s,\cdot)\|_{p,1}\,ds\,dt \\
&= \int_0^\infty\int_0^t\int_0^\infty v^{k-\alpha/2}\|e^{(v+t-s)\Delta_d}\Delta_d^{k+1} g(s,\cdot)\|_p\frac{dv}{v}\,ds\,dt \\
&= \int_0^\infty\int_s^\infty\int_0^\infty v^{k-\alpha/2}\|e^{(v+t-s)\Delta_d}\Delta_d^{k+1} g(s,\cdot)\|_p\frac{dv}{v}\,dt\,ds \\
&= \int_0^\infty\int_s^\infty\int_{t-s}^\infty (v+s-t)^{k-\alpha/2-1}\|e^{v\Delta_d}\Delta_d^{k+1} g(s,\cdot)\|_p\,dv\,dt\,ds \\
&= \frac{1}{k-\alpha/2}\int_0^\infty\int_0^\infty v^{k+1-\alpha/2}\|\partial_v^{k+1} e^{v\Delta_d} g(s,\cdot)\|_p\frac{dv}{v}\,ds.
\end{aligned}
$$

Then, by proceeding as in (4.2), we get that

$$
\begin{aligned}
\|R(g)\|_{L^1((0,\infty);\dot\Lambda_H^{\alpha,p,1})} &\le C\int_0^\infty\int_0^\infty v^{k-\alpha/2}\|\partial_v^k e^{v\Delta_d} g(s,\cdot)\|_p\frac{dv}{v}\,ds \\
&= C\int_0^\infty \|v^{k-\alpha/2}\partial_v^k e^{v\Delta_d} g(s,\cdot)\|_{p,1}\,ds = C\|g\|_{L^1((0,\infty);\dot\Lambda_H^{\alpha,p,1})}. \qquad \square
\end{aligned}
$$

5. Appendix

In this appendix we collect some known Hardy type inequalities that we use several times along the paper.

Next inequalities can be found in [20, Chapter IX, Section 9.9, Theorem 329, Eq. (9.9.8) and (9.9.9)], [28] and [32, A.4, Appendix A].

Lemma 5.1. *Let $1 \le p < \infty$, $r > 0$, and f be a non-negative measurable function f. Then*

$$
\left(\int_0^\infty \left(\int_0^x f(y)\,dy\right)^p x^{-r-1}\,dx\right)^{\frac{1}{p}} \le \frac{p}{r}\left(\int_0^\infty (yf(y))^p\,y^{-r-1}\,dy\right)^{\frac{1}{p}},
$$

$$\left(\int_0^\infty \left(\int_x^\infty f(y)\,dy\right)^p x^{r-1}\,dx\right)^{\frac{1}{p}} \le \frac{p}{r}\left(\int_0^\infty (yf(y))^p\, y^{r-1}\,dy\right)^{\frac{1}{p}}.$$

The following Hardy type convolution inequality appears in [20, Chapter IX, Section 9.9, Theorem 329, Eq. (9.9.7)].

Lemma 5.2. *Let $1 \le p < \infty$, $\alpha, \beta > 0$, and f be a non-negative measurable function f. Then*

$$\left(\int_0^\infty \left(\int_x^\infty f(y)\frac{(y-x)^\beta}{(y-x)}\,dy\right)^p x^{\alpha-1}\,dx\right)^{\frac{1}{p}} \le \frac{\Gamma(\beta)\Gamma\left(\frac{\alpha}{p}\right)}{\Gamma\left(\beta+\frac{\alpha}{p}\right)}\left(\int_0^\infty \left(y^\beta f(y)\right)^p y^{\alpha-1}\,dy\right)^{\frac{1}{p}}.$$

Finally we include the following discrete weighted Hardy inequalities. They can be found in [24, Section 2, Theorem 1, Eq. (2.14)], see also [23, Eq. (1') and (2'')].

Lemma 5.3. *Let $1 \le p < \infty$ and $r > 0$. There is $C > 0$ such that for every positive sequence $\{a_n\}_{n\in\mathbb{N}}$, it holds that*

$$\left(\sum_{n=1}^\infty \left(\sum_{j=1}^n a_j\right)^p n^{-r-1}\right)^{\frac{1}{p}} \le C\left(\sum_{n=1}^\infty (na_n)^p\, n^{-r-1}\right)^{\frac{1}{p}},$$

$$\left(\sum_{n=1}^\infty \left(\sum_{j=n+1}^\infty a_j\right)^p n^{r-1}\right)^{\frac{1}{p}} \le C\left(\sum_{n=1}^\infty (na_n)^p\, n^{r-1}\right)^{\frac{1}{p}}.$$

Data availability

No data was used for the research described in the article.

References

[1] L. Abadias, M. De León-Contreras, Discrete Hölder spaces and their characterization via semigroups associated with the discrete Laplacian and kernel estimates, J. Evol. Equ. 22 (4) (2022) 91.
[2] L. Abadias, J. González-Camus, P.J. Miana, J.C. Pozo, Large time behaviour for the heat equation on $\mathbb{Z}$, moments and decay rates, J. Math. Anal. Appl. 500 (2021) 125137.
[3] L. Abadias, P.J. Miana, Generalized Cesàro operators, fractional finite differences and gamma functions, J. Funct. Anal. 274 (5) (2018) 1424–1465.
[4] Víctor Almeida, Jorge J. Betancor, Discrete Hardy spaces and heat semigroup associated with the discrete Laplacian, Mediterr. J. Math. 16 (4) (2019) 91 (ISSN 1660-5446, 1660-5454).
[5] A. Benedek, R. Panzone, The space L^p, with mixed norm, Duke Math. J. 28 (1961) 301–324.
[6] T. Bruno, Homogeneous algebras via heat kernel estimates, Trans. Am. Math. Soc. 375 (10) (2022) 6903–6946.
[7] T. Bruno, M.M. Peloso, M. Vallarino, Besov and Triebel-Lizorkin spaces on Lie groups, Math. Ann. 377 (1–2) (2020) 335–377.
[8] H.-Q. Bui, X.T. Duong, Hardy spaces, Besov spaces and Triebel–Lizorkin spaces associated with a discrete Laplacian and applications, Ann. Sc. Norm. Super. Pisa, Cl. Sci. (2023).
[9] H.-Q. Bui, X.T. Duong, L. Yan, Calderón reproducing formulas and new Besov spaces associated with operators, Adv. Math. 229 (4) (2012) 2449–2502.

[10] T.A. Bui, Besov and Triebel-Lizorkin spaces for Schrödinger operators with inverse-square potentials and applications, J. Differ. Equ. 269 (1) (2020) 641–688.
[11] T.A. Bui, Maximal regularity of parabolic equations associated with a discrete Laplacian, J. Differ. Equ. 375 (2023) 277–303.
[12] Ó. Ciaurri, T.A. Gillespie, L. Roncal, J.L. Torrea, J.L. Varona, Harmonic analysis associated with a discrete Laplacian, J. Anal. Math. 132 (2017) 109–131.
[13] Ó. Ciaurri, L. Roncal, P.R. Stinga, J.L. Torrea, J.L. Varona, Nonlocal discrete diffusion equations and the fractional discrete Laplacian, regularity and applications, Adv. Math. 330 (2018) 688–738.
[14] Thierry Coulhon, Xuan Thinh Duong, Maximal regularity and kernel bounds: observations on a theorem by Hieber and Prüss, Adv. Differ. Equ. (ISSN 1079-9389) 5 (1–3) (2000) 343–368.
[15] M. De León-Contreras, J.L. Torrea, Lipschitz spaces adapted to Schrödinger operators and regularity properties, Rev. Mat. Complut. 34 (2) (2021) 357–388.
[16] M. De León-Contreras, J.L. Torrea, Parabolic Hermite Lipschitz spaces: regularity of fractional operators, Mediterr. J. Math. 17 (6) (2020) 205.
[17] Hongjie Dong, Doyoon Kim, On L_p-estimates for elliptic and parabolic equations with A_p weights, Trans. Am. Math. Soc. 370 (7) (2018) 5081–5130 (ISSN 0002-9947, 1088-6850).
[18] Hongjie Dong, Tuoc Phan, Regularity for parabolic equations with singular or degenerate coefficients, Calc. Var. Partial Differ. Equ. 60 (1) (2021) 44 (ISSN 0944-2669, 1432-0835).
[19] A.E. Gatto, R. Urbina, O. Wilfredo, On Gaussian Lipschitz spaces and the boundedness of fractional integrals and fractional derivatives on them, Quaest. Math. 38 (1) (2015) 1–25.
[20] G.H. Hardy, J.E. Littlewood, G. Pólya, Inequalities, 2d ed., University Press, Cambridge, 1952.
[21] Matthias Hieber, Jan Prüss, Heat kernels and maximal L^p-L^q estimates for parabolic evolution equations, Commun. Partial Differ. Equ. 22 (9–10) (1997) 1647–1669 (ISSN 0360-5302, 1532-4133).
[22] Tuomas Hytönen, Jan van Neerven, Mark Veraar, Lutz Weis, Analysis in Banach spaces. Vol. III. Harmonic Analysis and Spectral Theory, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 76, Springer, Cham, 2023.
[23] L. Leindler, Generalization of inequalities of Hardy and Littlewood, Acta Sci. Math. (Szeged) 31 (1970) 279–285.
[24] J.E. Littlewood, G.H. Hardy, Elementary theorems concerning power series with positive coefficients and moment constants of positive functions, J. Reine Angew. Math. 157 (1927) 141–158.
[25] C. Liu, D.J. Prömel, J. Teichmann, Characterization of nonlinear Besov spaces, Trans. Am. Math. Soc. 373 (1) (2020) 529–550.
[26] T. Ma, P.R. Stinga, J.L. Torrea, C. Zhang, Regularity properties of Schrödinger operators, J. Math. Anal. Appl. 388 (2) (2012) 817–837.
[27] S.D. Moura, On some characterizations of Besov spaces of generalized smoothness, Math. Nachr. 280 (9–10) (2007) 1190–1199.
[28] B. Muckenhoupt, Hardy's inequality with weights, Stud. Math. 44 (1972) 31–38.
[29] Takayoshi Ogawa, Senjo Shimizu, End-point maximal L^1-regularity for the Cauchy problem to a parabolic equation with variable coefficients, Math. Ann. 365 (1–2) (2016) 661–705 (ISSN 0025-5831, 1432-1807).
[30] Takayoshi Ogawa, Senjo Shimizu, The drift-diffusion system in two-dimensional critical Hardy space, J. Funct. Anal. 255 (5) (2008) 1107–1138 (ISSN 0022-1236, 1096-0783).
[31] Jan Prüss, Gieri Simonett, Moving Interfaces and Quasilinear Parabolic Evolution Equations, Monographs in Mathematics, vol. 105, Birkhäuser/Springer, Cham, 2016.
[32] E.M. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton Mathematical Series, vol. 30, Princeton University Press, Princeton, N.J., 1970.
[33] P.R. Stinga, J.L. Torrea, Regularity theory and extension problem for fractional nonlocal parabolic equations and the master equation, SIAM J. Math. Anal. 49 (5) (2017) 3893–3924.
[34] M.H. Taibleson, On the theory of Lipschitz spaces of distributions on Euclidean n-space. II. Translation invariant operators, duality, and interpolation, J. Math. Mech. 14 (1965) 821–823.
[35] M.H. Taibleson, On the theory of Lipschitz spaces of distributions on Euclidean n-space. I. Principal properties, J. Math. Mech. 13 (1964) 407–479.
[36] M.H. Taibleson, On the theory of Lipschitz spaces of distributions on Euclidean n-space. III. Smoothness and integrability of Fourier transforms, smoothness of convolution kernels, J. Math. Mech. 15 (1966) 973–981.
[37] F.G. Tricomi, A. Erdélyi, The asymptotic expansion of a ratio of gamma functions, Pac. J. Math. 1 (1951) 133–142.
[38] H. Triebel, Theory of Function Spaces, Mathematik und ihre Anwendungen in Physik und Technik (Mathematics and its Applications in Physics and Technology), vol. 38, Akademische Verlagsgesellschaft Geest & Portig K.-G, Leipzig, 1983.

[39] H. Triebel, Theory of Function Spaces. II, Monographs in Mathematics, vol. 84, Birkhäuser Verlag, Basel, 1992.
[40] H. Triebel, Theory of Function Spaces. III, Monographs in Mathematics, vol. 100, Birkhäuser Verlag, Basel, 2006.
[41] D. Yang, W. Yuan, Characterizations of Besov-type and Triebel-Lizorkin-type spaces via maximal functions and local means, Nonlinear Anal. 73 (12) (2010) 3805–3820.
[42] D. Yang, W. Yuan, Pointwise characterizations of Besov and Triebel-Lizorkin spaces in terms of averages on balls, Trans. Am. Math. Soc. 369 (11) (2017) 7631–7655.
[43] K. Yosida, Functional Analysis, second edition, Die Grundlehren der Mathematischen Wissenschaften, vol. 123, Springer-Verlag, New York Inc., New York, 1968.